Notes on Inflationary Cosmology

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These notes follow the lectures of the *Quantum Cosmology* course held by Professor F.G. Pedro at the University of Bologna in the a.y. 2024–2025. The notes were written for personal use and are meant to be a useful summary for reviewing the basics of the inflationary framework in cosmology. External sources from which some parts are taken can be found in References section. The metric convention is mostly minus through the whole document, units with $c = 1 = h$ are employed.

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Problems of the HBB Model

One obtains Friedmann equations by $\delta \int d^4x \sqrt{|g|} \left(\frac{R}{16\pi G} + \mathcal{L}_m(\psi, g_{\mu\nu}) \right) = 0$, with the metric being the FRW one:

$$ds^2_{\text{FRW}} = dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + d\Omega^2 \right] \tag{1}$$

$$G^0_0 = 8\pi G T^0_0 \quad \Rightarrow \quad H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{2}$$

$$G^i_i = 8\pi G T^i_i \quad \Rightarrow \quad 3H^2 + 2\dot{H} = -8\pi G P - \frac{k}{a^2} \tag{3}$$

Combining (2) and (3), or just by imposing covariant conservation of T^μ_ν, continuity equation follows:

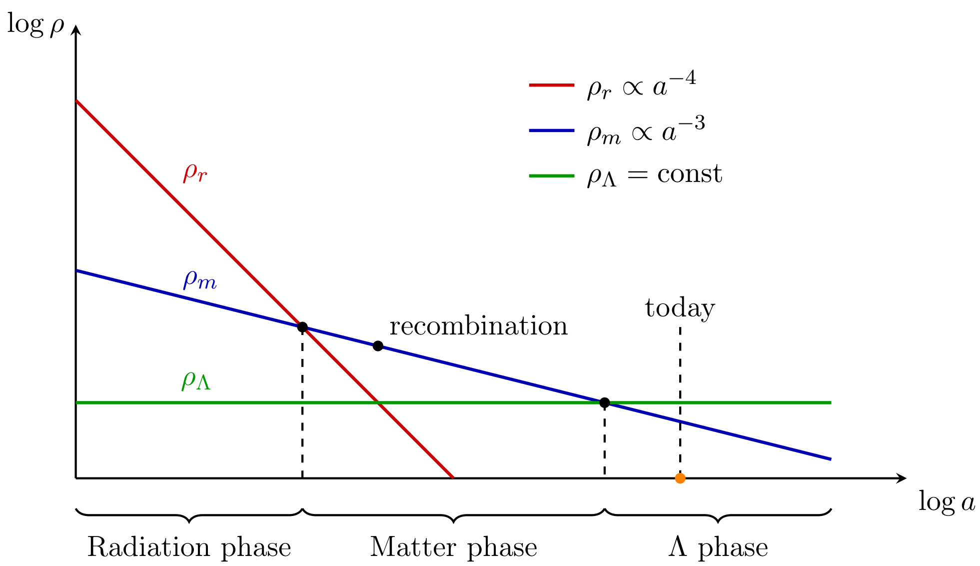
$$\dot{\rho} + 3H(\rho + P) = 0 \tag{4}$$

$$H \equiv \frac{\dot{a}}{a} \qquad \tau \equiv \int \frac{dt}{a} \qquad a \equiv \frac{1}{1+z} \qquad a_{\text{now}} \equiv 1 \tag{5}$$

$$\omega \equiv \frac{P}{\rho} \qquad M_P^2 \equiv (8\pi G)^{-1} \qquad \Omega_i \equiv \frac{8\pi G \rho_i}{3H^2} \qquad \Omega_k \equiv \frac{-k}{(aH)^2} \tag{6}$$

Dust has $P = 0$. For radiation one inspects the pressure components $T^{ii} = \rho/3$, as $T^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu p^\nu}{p^0} f(\mathbf{p})$ and $p^0 \equiv E = |\mathbf{p}|$ for radiation. Analogue for the cosmological constant: $\rho = T^{00} = \Lambda/8\pi G = -T^{ii} = -P$. Hence

$$\omega_m = 0 \qquad \omega_\gamma = \frac{1}{3} \qquad \omega_\Lambda = -1 \tag{7}$$



Fine Tuning Problems in HBB

Horizion problem:
Observations tell us that the Universe is highly homogeneous and isotropic.

$$\frac{\delta T}{T_{now}} \sim 10^{-4} - 10^{-5} \qquad T_{now} \approx 2.73K \tag{8}$$

Assume $k = 0$ in the metric. Rewrite (2) (3). Consider a radial photon. Assume an initial singularity: $t_i = 0$, $a(t_i) = 0$. Find *comoving particle horizon*

$$\Delta r = \int_0^t \frac{t'}{a(t')} = \tau(t) - \tau(0) \tag{9}$$

Use (2) (4) to relate proportionally a and t. See that for $\omega > -1/3$ (SEC) Δr is finite $\Rightarrow$ *there were regions not in causal contact in the past!* Then compute the angle corresponding to the comoving horizon at recombination[a] $\theta = (\tau_{rec} - \tau_i)/(\tau_{now} - \tau_{rec})$. Rewrite a general τ difference integrating and change variables to make H appear so you can substitute with (2). $\theta \approx 1.15°$.

[a]Because CMB is emitted at that time! That's when the universe became transparent and photons were able to travel freely.

Flatness problem:
From current observations $\Omega_k < 0.02 \Rightarrow$ one expects $\sum_i \Omega_i - 1 \equiv \Omega_{tot} - 1 \lesssim 10^{-2}$
Compare present day to Plank epoch:

$$\frac{\Omega_{tot}(t_{now}) - 1}{\Omega_{tot}(t_P) - 1} = \frac{(aH)^2|_{t_P}}{(aH)^2|_{t_{now}}} \tag{10}$$

employ

$$a \propto T^{-1}\,^{a} \qquad H(t_P) \sim M_P \qquad T_P \sim M_P \qquad H_{now} \sim 10^{-60} M_P \tag{11}$$

This means $\Omega_{tot}(t_P) - 1 \lesssim 10^{-60}$ *which would be highly fine-tuned.*

[a]Holds for radiation dominated Universe only, where $a \propto \rho^{-1/4}$ and $\rho \propto T^4$ for black body radiation

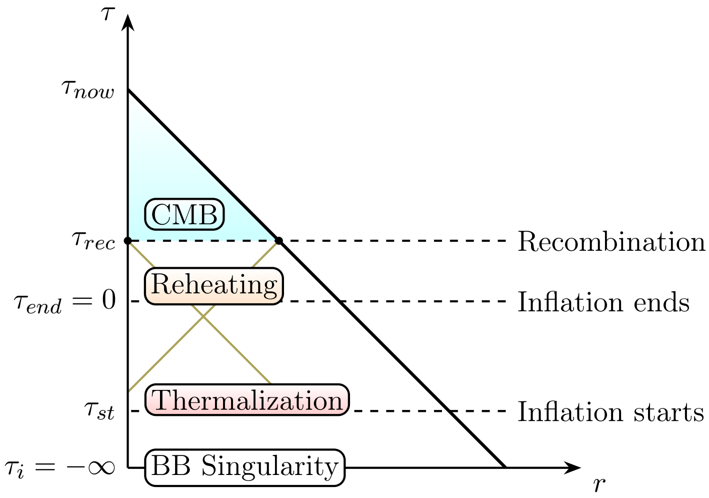
The actual problem: *aH(t)*

It all comes down to aH being a decreasing function of time. ($1/aH$ is the *Hubble radius*)

Use (2) (4) to relate proportionally a and t. Then relate aH and t, include multiplicative *omega* factors!. Finally, relate aH and a. See that SEC ($\omega > -1/3$) must be violated to have aH increasing. $\tau \in [0, \infty[$, violating SEC we can push the singularity to $\tau_i = -\infty$.

Equivalent conditions to $\omega < -1/3$:

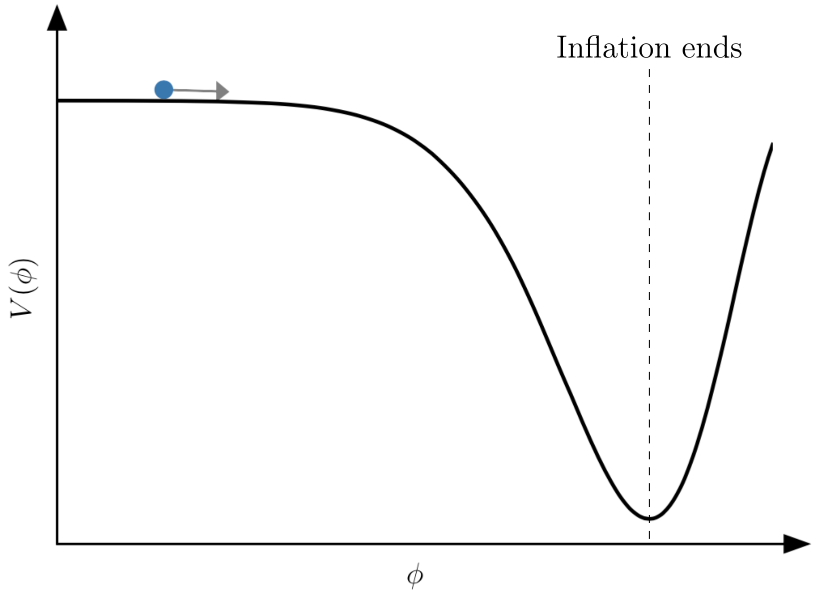
1. Decreasing comoving horizon
2. Accelerated expansion: $\ddot{a} > 0$
3. Slowly varying H $\qquad \epsilon_H \equiv -\frac{\dot{H}}{H^2} = 3/2(1+\omega) \quad \Rightarrow \quad \epsilon < 1$
4. Negative pressure



How much do we need to push back in time τ_i so that the whole observable universe could have been in causal contact and thus thermalize?

At the onset of inflation assume an universe with far from flat initial conditions: $\Omega_k \sim 1 \Rightarrow \Omega_k(t_{st})/\Omega_k(t_{now}) \gg 1$, rewrite and insert $aH(t_{end})/aH(t_{end})$. H is nearly constant during inflation, $N \equiv \log a$. With $\log\left(\frac{a_{end}}{a_{st}}\right) = N_{tot} > 66 + \log HT^{-1}(t_{end})$ the flatness problem is solved. Usually, *one requires inflation to last 50-60 e-foldings*.

1. **What caused inflation?** Whatever it is that drives inflation it must have negative pressure. Thus, it cannot be matter or radiation. Neither can be a cosmological constant because it would lead to perpetual rapid inflation (a cosmological constant does not diminish or evolve with time) [1].
   Simplest possibility is via the potential energy of a scalar field ϕ (there is no known scalar field that can drive inflation). It must have negative pressure and this requirement leads to potential energy dominating over kinetic energy.
2. **Why is H const during inflation?** The equation of state for the scalar field gives $\omega \sim -1$. This leads to H being constant. When ϕ finishes its roll down the potential, ω becomes positive and H starts to decrease. This is the end of inflation.
3. **Is $a \propto T^{-1}$ valid in every epoch?** It always holds for T being the temperature of radiation. Beware that the latter is not the temperature of the Universe in every epoch.



A scalr field rolling down a potential [1].

Theories of Inflation

Slow Roll Inflation

We will assume that the field is homogeneous to zeroth order, consisting of $\phi(t) + \delta\phi(t, \mathbf{x})$. Action for a scalar field ϕ with potential $V(\phi)$ in a FLRW universe

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R + \underbrace{\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V}_{\mathcal{L}}\right] \tag{12}$$

Knowing that, with mostly minus convention (then assume $k = 0$):

$$R_{00} = \partial_0\Gamma^k{}_{0k} - \partial_k\Gamma^k{}_{00} + \Gamma^\sigma{}_{0k}\Gamma^k{}_{\sigma 0} - \Gamma^k{}_{k\sigma}\Gamma^\sigma{}_{00} = 3\frac{\ddot{a}}{a} \qquad R = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) \tag{13}$$

Its equations of motion are given by the Einstein equations and the Klein-Gordon equation

$$H^2 = \frac{1}{3M_P^2}\left(\frac{\dot{\phi}^2}{2} + V\right) \qquad \ddot{\phi} + 3H\dot{\phi} = -V_{,\phi} \tag{14}$$

From KG eq you can get $\dot{H} = -\dot{\phi}^2/2M_P^2$. For a perfect fluid $T^{\mu\nu} = (\rho + P)u^\mu u^\nu - Pg^{\mu\nu}$,

$$T^{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S_\phi}{\delta g_{\mu\nu}} \quad \Rightarrow \quad \omega = \frac{\frac{\dot{\phi}^2}{2} - V(\phi)}{\frac{\dot{\phi}^2}{2} + V(\phi)} \tag{15}$$

For $V(\phi) \gg \dot{\phi}^2/2 \to \omega \sim -1 \to \epsilon \ll 1$ and one has quasi de Sitter expansion. To quantify how long inflation lasts [1], define $\eta \equiv d\log\epsilon/dN \quad \Rightarrow \quad \eta \ll 1$.

[1] ϵ should stay $\ll 1$ for a relevant physical time: $\dot{\epsilon} \ll \epsilon H$

Rewrite (14) imposing slow roll approx for potential and field acceleration (vacuum energy dominates H^2 and terminal velocity). Then define

$$\epsilon_V \equiv \frac{M_P^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \qquad \eta_V \equiv M_P^2 \frac{V_{,\phi\phi}}{V} \tag{16}$$

$\epsilon = \epsilon_V$ and $\eta = -2\eta_V + 4\epsilon_V$. Hence, V must be flat enough to have $V_{,\phi}/V$, $V_{,\phi\phi}/V \ll 1$.

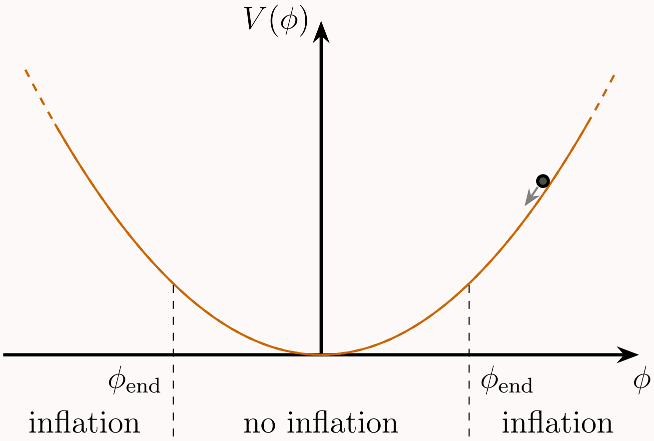
For each model calculate ϵ_V, η_V and impose $\ll 1$. Then find $\phi(N)$ to get the field value for 60 e-foldings. Rewrite $\epsilon_V(N), \eta_V(N)$

Large Fields Models:

$$V(\phi) = \lambda_n \phi^n \qquad n > 0 \tag{17}$$

$N_{tot} = \int dN = \int_{\phi(end)}^{\phi(start)} \frac{d\phi}{\sqrt{2\epsilon}}$. $\phi(end) \sim \sqrt{2} M_P$ and $\phi(start) \equiv \phi(N_{tot}) \sim 15 M_P$.

$$\epsilon_V = \frac{n}{4N} \qquad \eta_V = \frac{n-1}{2N} \tag{18}$$

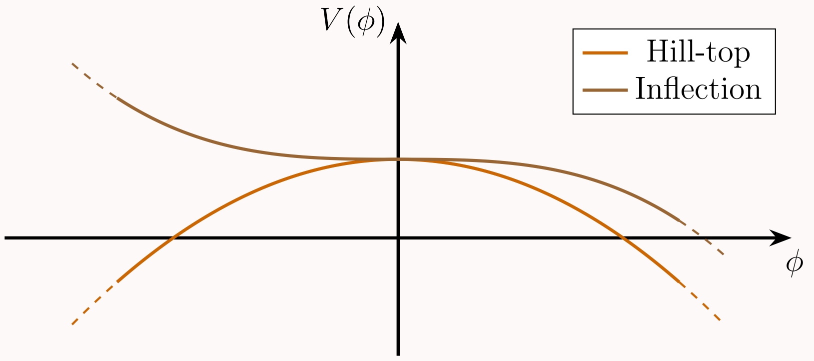


Small Fields Hill-top & Inflection Models:

$$V(\phi) = V_0 - \lambda_n \phi^n \qquad n > 0 \tag{19}$$

$\lambda_n \phi^n \ll V_0$, $\epsilon_V, \eta_V \ll 1 \Rightarrow \phi \ll M_P$ $N(\phi = 0) \to \infty$ so we can have as much expansion as needed.

$$\epsilon_V = \frac{n^2 \lambda^2}{2V_0^2}\left[n(n-2)N\frac{\lambda}{V_0}\right]^{\frac{2n-2}{2-n}} \qquad \eta_V = -\frac{n-1}{n-2}\frac{1}{N} \tag{20}$$

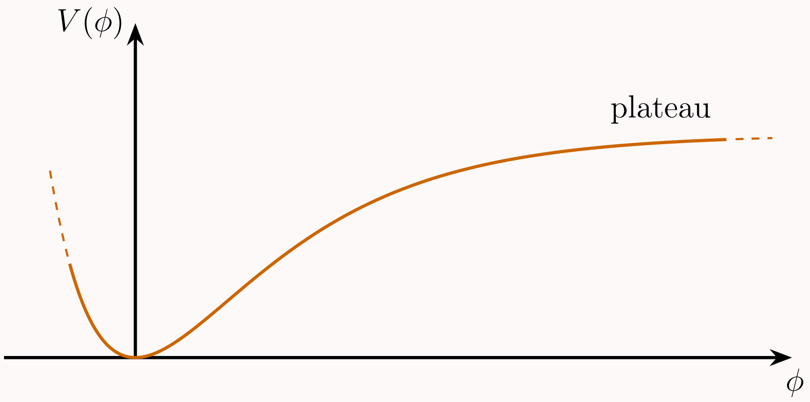


Plateau Models:

$$V(\phi) = V_0\left(1 - \alpha e^{-\kappa\phi}\right)^2 \tag{21}$$

Neglect α^2 terms in V_ϕ and $1 - \alpha e^{-\kappa\phi} \sim 1$ in ϵ_V, η_V

$$\epsilon_V \sim \frac{1}{2k^2N^2} \qquad \eta_V \sim -\frac{1}{N} \tag{22}$$



$P(X,\phi)$ **theories**

$X \equiv 1/2(\partial_\mu\phi)^2 \qquad E \equiv 2XP_X - P$. Action for $P(X,\phi)$ theories is

$$S = \int d^4x\sqrt{g}\left[\frac{M_P^2}{2}R + P(X,\phi)\right] \tag{23}$$

Equations of motion for $g^{\mu\nu}$ and ϕ are

$$H^2 = \frac{1}{3M_P^2}E \qquad \dot{E} = -3H\left(E + P\right) \tag{24}$$

Find expression for ϵ and condition for $\epsilon \ll 1$.

In fluid mechanics the speed of sound is defined as a derivative of the pressure with respect to the density at fixed entropy per particle: $c_s^2 = \frac{\partial P}{\partial \rho}\Big|_{s/n}$. It is the velocity at which adiabatic compression and rarefaction waves propagate in a medium [2].

$$c_s^2 = \frac{P_X dX + P_\phi d\phi}{\rho_X dX + \rho_\phi d\phi}\Bigg|_{s/n} = \frac{P_X}{\rho_X} = \frac{P_X}{P_X + 2XP_{XX}} \tag{25}$$

As one can prove that $d(s/n) = 0$ is equivalent to $d\phi = 0$. In our context, c_s is the speed of scalar perturbations.

K-Inflation models:
Effective kinetic Lagrangian can be written as

$$\mathcal{L} \supset \sum_{n\geq 1} f(\phi) \frac{X^n}{\Lambda^{4n-4}} \tag{26}$$

Λ is the cut-off scale of the effective field theory because $\Lambda^4 \gg X$ kills everything execpt the first term, which is the one of slow roll inflation. But there are potential problems with ghosts and radiative stability.

Dirac-Born-Infeld Inflation:
It is a model of inflation based on the Dirac-Born-Infeld (DBI) action, which describes the dynamics of a D-brane in string theory.

$$S = -\int d^4x \sqrt{g} \left[\frac{1}{f(\phi)} \sqrt{1 + f(\phi) \partial_\mu \phi \partial^\mu \phi} - \frac{1}{f(\phi)} + V(\phi) \right] \tag{27}$$

The DBI action includes a kinetic term that is non-linear in the field velocity, leading to a speed limit on the field's motion (due to $\sqrt{\ldots}$). This non-linear kinetic term can give rise to an effective sound speed that is less than one (which is constrained by observations!), allowing for a more gradual roll of the inflaton field and potentially addressing issues related to the generation of primordial perturbations.

Quantum Inflation

The inflaton field ϕ couples to gravity and maybe also to other fields inducing quantum corrections to the tree level potential; the latter can always be written to obey slow roll conditions $V_\phi, V_{\phi\phi} \ll V$. Now we require $V_{tot} = V_{tree} + \delta V$ to obey these conditions.

Consider two scalar fields ϕ and ψ (with a trilinear coupling) with masses m and $M \gg m$.

$$\mathcal{L} \supset -\frac{m^2}{2}\phi^2 - \frac{M^2}{2}\psi^2 - g\phi^2\psi \tag{28}$$

Eliminate ψ in terms of ϕ using the equations of motion to find $V_{\text{eff}}(\phi) = \overbrace{\frac{m}{2}\phi^2}^{\text{tree level}} + \overbrace{\frac{g^2}{M^2}\phi^4}^{\text{quant corr}}$
In general,

$$V_{\text{eff}}(\phi) = V_{\text{tree}} + \sum_{q=1} g_q \left(\frac{\phi}{M}\right)^q M^4 \tag{29}$$

Assume $M \sim M_P$ and $V_{\text{tree}} = m^2\phi^2/2$ and fix $\phi = 15M_P$ such that $N_{tot} = 60$. To make sure V_{tot} obeys slow roll conditions evaluate $\frac{\delta V}{V_{tree}}$ and ask it to be $\ll 1$ so that $V_{tot} \sim V_{tree}$. *Problem of UV sensitivity of inflation: coupling g_q needs to be very small otherwise quantum corrections are not negligible. This is a problem of the theory, not of the model. It means is hard to keep inflation light.*

Scalar field in dS space-time

Use conformal time in the metric. Cosnider a real, minimally coupled scalar field $\phi(x^\mu)$:

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) \tag{30}$$

Rescale with $\chi \equiv a\phi$ and eventually define $m^2_{\text{eff}} \equiv m^2a^2 - \frac{a''}{a}$ Action is explicitly time dependent $\Rightarrow$ energy not conserved $\Rightarrow$ particle creation (with energy supplied by the classical gravitational field). Get EOM for χ, then expand in Fourier modes:

$$\chi(\mathbf{x},\tau) = \int \frac{d^3k}{(2\pi)^{3/2}}\chi_{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}} \tag{31}$$

Rewrite EOM in terms of $\chi_{\mathbf{k}}$. Note that χ is a real number $\Rightarrow \chi^*_{\mathbf{k}} = \chi_{-\mathbf{k}}$. EOM just depends on $|\mathbf{k}|$, general solution is

$$\chi_{\mathbf{k}}(\tau) = \frac{1}{\sqrt{2}}\left(a^-_{\mathbf{k}}\, v^*_k(\tau) + a^+_{-\mathbf{k}}\, v_k(\tau)\right) \tag{32}$$

$a_{\mathbf{k}}$ coefficients are not yet identified as annihilation/creation operators, v_k are called mode functions. Normalization is $\text{Im}(v'v^*) = 1/2$. Put back in (31)

$$\chi(\mathbf{x},\tau) = \frac{1}{\sqrt{2}}\int \frac{d^3k}{(2\pi)^{3/2}}\left[a^-_{\mathbf{k}}v^*_k(\tau)e^{i\mathbf{k}\cdot\mathbf{x}} + a^+_{-\mathbf{k}}v_k(\tau)e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \tag{33}$$

EOM for v_k is

$$v''_k + \overbrace{\left(k^2 + m_{\text{eff}^2}\right)}^{\omega_k^2} v_k = 0 \tag{34}$$

Define canonical momentum and Hamiltonian:

$$\Pi = \frac{\partial \mathcal{L}}{\partial \chi'} \qquad \hat{\mathcal{H}} = -\mathcal{L} + \hat{\Pi}\chi' = \frac{1}{2}\int d^3x \left(\hat{\Pi}^2 + \left(\nabla\hat{\chi}\right)^2 + m_{\text{eff}}^2\hat{\chi}^2\right) \tag{35}$$

Impose equal time commutation relations: $[\hat{\chi}(x,\tau), \hat{\Pi}(y,\tau)] = i\delta^3(x-y)$

Here one may show that, promoting the coefficients $a_{\mathbf{k}}$ to operators, they obey commutation relations of annihilation/creation operators; with which one can construct quantum states of the theory.

$$|m_{\mathbf{k_1}}, n_{\mathbf{k_2}}, \ldots\rangle = \frac{1}{\sqrt{m!n!\ldots}}(\hat{a}^\dagger_{\mathbf{k}_1})^m(\hat{a}^\dagger_{\mathbf{k}_2})^n \ldots |0\rangle \tag{36}$$

Bogoliubov transformation:

$$u_k(\tau) = \alpha_k v_k(\tau) + \beta_k v_k^*(\tau) \tag{37}$$

Impose normalization condition on u_k: $u_k' u_k^* - u_k'^* u_k = i$ to get $|\alpha_k|^2 - |\beta_k|^2 = 1$.

$$\chi(\mathbf{x},\tau) = \frac{1}{\sqrt{2}}\int \frac{d^3k}{(2\pi)^{3/2}}\left[b^-_{\mathbf{k}} u_k^*(\tau)e^{i\mathbf{k}\cdot\mathbf{x}} + b^+_{-\mathbf{k}} u_k(\tau)e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \tag{38}$$

To find relation between $b_{\mathbf{k}}$ and $a_{\mathbf{k}}$, equate (33) and (38) A generic state is

$$|\psi\rangle = \sum_{m,n,\ldots} C^A_{mn\ldots}\,|m_{\mathbf{k_1}}, n_{\mathbf{k_2}}, \ldots\rangle_A = \sum_{m,n,\ldots} C^B_{mn\ldots}\,|m_{\mathbf{k_1}}, n_{\mathbf{k_2}}, \ldots\rangle_B \tag{39}$$

with $|m_{\mathbf{k_1}}, n_{\mathbf{k_2}}, \ldots\rangle_A \equiv (m!n!\ldots)^{-1/2}\left|(\hat{a}^\dagger_{\mathbf{k_1}})^m(\hat{a}^\dagger_{\mathbf{k_2}})^n \ldots\right\rangle$

Show that ${}_B\langle 0|\,\hat{N}^A_{\mathbf{k}}\,|0\rangle_B = |\beta_{\mathbf{k}}|^2\delta^{(3)}(0)$, divergence is due to infinite spatial volume.

Flat background:

Hamiltonian is time independent. The vacuum is the eigenstate of the $\hat{H}$ with lowest energy. Substitute (32) in $\hat{H}$ (here $a = 1 \Rightarrow m_{\text{eff}} = m$, set it to zero for convenience, since we are only interested in the fact that it is not time dependent) and define

$$F_k \equiv v_k'^2 + k^2 v_k^2 \qquad E_k \equiv |v_k'|^2 + k^2|v_k|^2 \tag{40}$$

Write the mode functions as $v_k = r_k(\tau)e^{i\alpha_k(\tau)}$. For $|0\rangle_v$ to be vacuum, ${}_v\left\langle 0|\hat{H}|0\right\rangle_v = \frac{\delta(0)}{2}\int d^3k E_k$ must be minimized. Impose normalization condition on v_k to rewrite E_k and then minimize it: $dE_K/dr' = 0$, $dE_K/dr = 0$. Find mode functions that minimize energy:

$$v_k = \frac{1}{\sqrt{2k}}e^{ik\tau} \tag{41}$$

dS background: $\rho = T^{00} = \Lambda/8\pi G = -T^{ii} = -P \Rightarrow H^2 = \text{const}$

Let $k^2 \to \omega_k^2(\tau) = k^2 + m_{\text{eff}}^2(\tau)$

$$v_k(\tau_0) = \frac{1}{\sqrt{2\omega_k(\tau_0)}} e^{i\omega_k(\tau_0)\tau_0} \tag{42}$$

To these one can associate a vacuum state $|0\rangle_{\tau_0}$*, which is the eigenstate of the annihilation operator* $\hat{a}_{\mathbf{k}}$ *at time* τ_0*. It will not be the lowest energy state at later times* $\to$ *time dependence of the effective mass gives rise to particle creation.*
In dS spacetime $a \propto e^{Ht} \;\Rightarrow\; \tau = \int_{-\infty}^{t} dt e^{-Ht} \;\Rightarrow\; \tau = -1/aH \;\Rightarrow\; \tau \in \,]-\infty, 0[\;\Rightarrow\; m_{\text{eff}} = (aH)^2\left[(m/H)^2 - 2\right]$ EOM for v_k becomes

$$v_k'' + \left[k^2 - \frac{1}{\tau^2}\overbrace{\left(2 - \frac{m^2}{H^2}\right)}^{\nu^2 - 1/4}\right] v_k = 0 \tag{43}$$

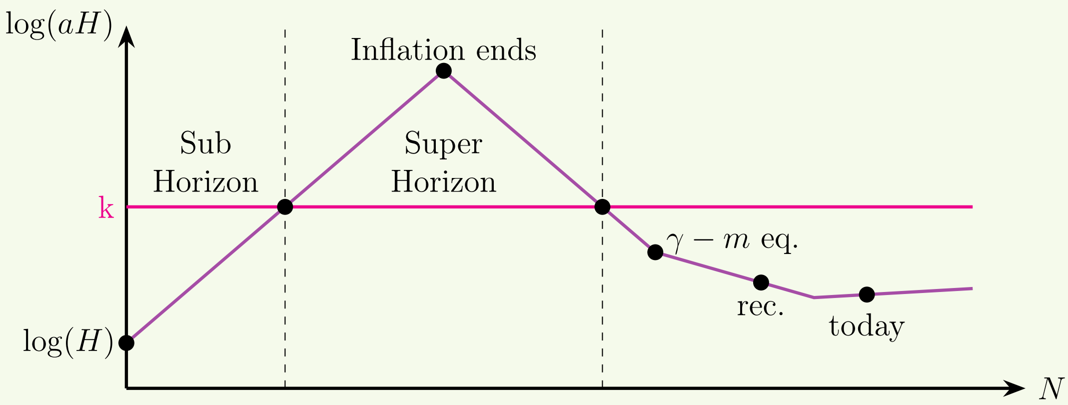
Solution is in terms of Hankel functions:

$$v_k(\tau) = \sqrt{-\tau}\left[C_1 H_\nu^{(1)}(-k\tau) + C_2 H_\nu^{(2)}(-k\tau)\right] \tag{44}$$

with $\nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$. <u>Consider massless scalar field</u> $\Rightarrow \nu = 3/2$. This leads to

$$v_k(\tau) = \left[C_1 \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) + C_2 \frac{e^{ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right)\right] \tag{45}$$

Now impose initial conditions at $\tau \ll 0$ to determine C_1 and C_2: *ask the scalar to be in the instantaneous vacuum state at early times* $\Rightarrow C_1 = 0$ and $C_2 = 1$ (due to limit forms of $H_\nu^{(1,2)}$) to recover the flat space limit (41). Note that, at early times, $k \gg 1/\tau$ as k is a fixed mode [a].



The mode functions in (45) define the *Bunch-Davies vacuum* $|0\rangle_{BD}$, i.e. the vacuum state at τ_0. The physical wavelength is defined as $\lambda_{\text{phys}} = a\lambda$. $|k\tau| \propto (H)^{-1}/\lambda_{\text{phys}}$, imposing $\gg 1$ we find that *background curvature (Hubble radius) is much grater than the physical wavelength of the mode* $\Rightarrow$ *at early times modes behave as they were in Minkowski spacetime.*
One can then calculate observables like the *power spectrum* ${}_{BD}\langle 0|\hat{\chi}_{\mathbf{k}}\,\hat{\chi}_{\mathbf{k}'}|0\rangle_{BD} = P_{v_k}\delta(\mathbf{k} + \mathbf{k}')$, which is $\approx (aH)^2/(2k^3)$ for $|k\tau| \ll 1$.

[a] A mode k is associated with a wavelength $\lambda = 2\pi/k$, so when $k < / > aH$, it is said to be super/sub-horizon meaning that the λ is larger/smaller than the Hubble radius. We compare with the Hubble radius instead of the comoving particle horizon because the former is a local, instantaneous length scale over which causal processes can operate at that moment, while the latter is a cumulative distance that light could have traveled (monotonically increasing function).

Cosmological Perturbation Theory

The zeroth-order scheme outlined in Section 2.1 ensures that the universe will be close to uniform on all scales of interest today. There are perturbations about this zeroth-order scheme, though, and these perturbations—produced early on when the scales are causally connected—persist long after inflation has terminated.

In cosmology, we always work in terms of statistics, such as the correlation function and power spectrum, because no known theory predicts the overdensity in a given spot on the sky. In the inflationary scenario, this uncertainty is fundamental: inflation erases all traces of what came before it, and replaces those with quantum-mechanical vacuum fluctuations, which cannot be predicted in principle. What inflation predicts then is precisely the statistical distributions from which the perturbations are drawn.

We are most interested in scalar perturbations to the metric since these couple to the density of matter and radiation and ultimately are responsible for the structure we observe in the universe. Inflation also generates tensor fluctuations in the metric, that is, gravitational waves. These are not coupled to the density and so are not responsible for the large-scale structure of the universe, but they do induce anisotropies in the CMB [1].

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \qquad T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu} \tag{46}$$

Expanding the Einstein equations to first order in perturbations, we have:

$$\delta G_{\mu\nu} = 8\pi G\, \delta T_{\mu\nu}, \tag{47}$$

Write down the most general linear perturbation that is compatible with the spatial symmetries of an FRW background, and then decompose every spatial object into irreducible pieces. Background is homogeneous and isotropic $\rightarrow$ spatial rotations (plus translations) act as a symmetry group. Irreducible representations of $SO(3)$ are labelled by spin: 0 scalars, 1 vectors and 2 tensors. Use conformal time in the metric.

$$\frac{ds^2}{a^2} = (1+2\Phi)\, d\tau^2 + 2\,(B_{,i} + S_i)\, d\tau dx^i - [(1-2\Psi)\delta_{ij} - 2E_{,ij} - F_{i,j} - F_{j,i} - h_{ij}]\, dx^i dx^j \tag{48}$$

$$\delta g_{00} = 2a^2\Phi \qquad \delta g_{0i} = a^2\,(B_{,i} + S_i) \qquad \delta g_{ij} = a^2\,(2\Psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij}) \tag{49}$$

S_i, F_i and h_{ij} are transverse because they come from the decomposition of $V_i = \partial_i V + V_i^{\perp}$ and $W_{ij} = W\delta_{ij} + \partial_i\partial_j W + \partial_i W_j + \partial_j W_i + W_{ij}^{\perp} = 0$ (Helmholtz theorem):

$$S^i_{,i} = 0 \qquad F^i_{,i} = 0 \qquad h^i_{j,i} = 0 = h^i_i \tag{50}$$

$\delta g_{\mu\nu}$ is symmetric thus has 10 d.o.f., indeed we have 4 scalar, 2 vector (with 2 d.o.f. each) and 1 tensor (with $3(3+1)/2 - 1 - 3 = 2$ d.o.f.) perturbations: 10 functions specifying the perturbation. Not all are independent, though. The gauge freedom of the metric allows us to set some of them to zero[2].

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \xi^\mu(x) \qquad \text{gives} \qquad \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}(x) \tag{51}$$

[2]The gauge freedom of the metric is a consequence of the diffeomorphism invariance of GR. We can always choose a coordinate system in which the metric takes a certain form. This is called a gauge choice.

From this and equation (46) you can see that the perturbation transforms as

$$\delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} - \bar{g}_{\mu\nu,\sigma}\xi^\sigma - \bar{g}_{\mu\sigma}\xi^\sigma_{,\nu} - \bar{g}_{\nu\rho}\xi^\rho_{,\mu} \tag{52}$$

Expand $\xi^\mu = (\xi^0, \xi^i_\perp + \xi^{,i})$ using Helmholtz decomposition. Find $\delta\tilde{g}_{00}, \delta\tilde{g}_{0i}, \delta\tilde{g}_{ii}$.

Hydrodynamical Perturbations

Hydrodynamical perturbations are not gauge invariant, they are scalar perturbations:

$$ds^2 = a^2\left\{(1+2\Phi)\,d\tau^2 + 2B_{,i}dx^i d\tau - \left[(1-2\Psi)\,\delta_{ij} - 2E_{,ij}\right]dx^i dx^j\right\}$$

Use $\delta\tilde{g}_{00}, \delta\tilde{g}_{0i}, \delta\tilde{g}_{ii}$ to find $\tilde{\Phi}, \tilde{B}, \tilde{\Psi}, \tilde{E}$. Then choose ξ, ξ^0 so that only two perturbations remain. *Conformal Newtonian gauge* is E=B=0.

EMT of a perfect fluid:

Expand ρ, P, u^μ in terms of the background and perturbations.
$u^0 = \bar{u}^0 + \delta u^0, \quad u^i = \frac{dx^i}{ds} = a^{-1}\delta v^i$, as $(\delta v^i = \frac{dx^i}{d\tau})$. Impose $\bar{u}^\mu\bar{u}_\mu = 1$, and $u^\mu u_\mu = 1$ to get u_0 and u^0. At first order in perturbations obtain

$$\delta T^0_0 = \delta\rho \qquad \delta T^0_i = -\delta v_i\left(\bar{\rho} + \bar{P}\right) \qquad \delta T^i_j = -\delta^i_j\delta P \tag{53}$$

Solve conservation equation $\nabla^\mu T_{\mu\nu} = 0$ (employ zeroth order $\bar{\nabla}^\mu\bar{T}_{\mu\nu} = 0$) in $\delta g_{0i} = 0$ gauge (because $B = 0$ and I ignore S_i as I am considering scalar perturbations); to do so expand $\Gamma^\rho_{\mu\nu} = \bar{\Gamma}^\rho_{\mu\nu} + \delta\Gamma^\rho_{\mu\nu}$[a] and define $h_{\mu\nu} \equiv a^{-2}g_{\mu\nu} \Rightarrow g_{\mu\nu} = \bar{g}_{\mu\nu} + a^2 h_{\mu\nu}$
E.g. $\delta\Gamma^0_{00} \simeq \frac{1}{2}\bar{g}^{00}\left(\bar{\nabla}_0\delta g_{00}\right) = \frac{1}{2}\delta g'_{00}\frac{1}{a^2} - \delta g_{00}\frac{a'}{a^3} = \frac{1}{2}h'_{00}$

$$\delta\rho' + 3\frac{a'}{a}(\delta P + \delta\rho) + (\bar{P} + \bar{\rho})(\partial_i\delta v^i - \frac{1}{2}h^{i}_{i}{}') = 0 \tag{54}$$

$$\partial_i\delta P + (\bar{P} + \bar{\rho})\left(4\frac{a'}{a}\delta v_i + \frac{1}{2}\partial_i h_{00}\right) + \left[\delta v_i(\bar{P} + \bar{\rho})\right]' = 0 \tag{55}$$

If the Universe contains multiple non-interacting fluids, each one is conserved separately and these equations hold for each fluid component.

[a] $\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\left(\nabla_\mu\delta g_{\alpha\nu} + \nabla_\nu\delta g_{\alpha\mu} - \nabla_\alpha\delta g_{\mu\nu}\right)$

At first order the Einstein equations are

$$\delta G^0_0 = \frac{2}{a^2}\left(-\Delta\Psi + 3\frac{a'}{a}\Psi' - 3\left(\frac{a'}{a}\right)^2\Phi\right) \stackrel{!}{=} 8\pi G\delta T^0_0 = 8\pi G\,\delta\rho_{\text{tot}} \tag{56}$$

$$\delta G^0_i = \frac{2}{a^2}\left(-\partial_i\Psi' + \frac{a'}{a}\partial_i\Phi\right) \stackrel{!}{=} 8\pi G\delta T^0_i = -8\pi G\,\delta v^{\text{tot}}_i(\bar{\rho}_{\text{tot}} + \bar{P}_{\text{tot}}) \tag{57}$$

$$\begin{aligned}\delta G^i_j = \frac{1}{a^2}\partial^i\partial_j(\Phi + \Psi) - \frac{2}{a^2}\delta^i_j\Bigg[&-\Psi'' + \frac{1}{2}\Delta(\Phi + \Psi) + \frac{a'}{a}(\Phi' - 2\Psi') + \frac{2a''}{a}\Phi \\ &- \left(\frac{a'}{a}\right)^2\Phi\Bigg] \stackrel{!}{=} 8\pi G\delta T^i_j = -8\pi G\,\delta^i_j\,\delta P_{\text{tot}}\end{aligned} \tag{58}$$

From (58) taking a component with $i \neq j$ we find $\Phi = -\Psi$. Use this and $\delta v_i = \partial_i \delta v$[3] to simplify the equations. Recalling $h_{\mu\nu}$ is related to Φ by (49), *you have 5 single fluid component (λ) equations* (54) - (58) *and 4 variables: $\delta\rho_\lambda, \delta P_\lambda, \delta v_\lambda, \Phi$; not all are independent.* Below they are rewritten (sometimes use the trick $\partial_i A = \partial_i B \Rightarrow A = B$)

$$\Delta\Phi - 3\frac{a'}{a}\Phi' - 3\left(\frac{a'}{a}\right)^2 \Phi = 4\pi G\, a^2\, \delta\rho_{\text{tot}} \tag{59}$$

$$\Phi' + \frac{a'}{a}\Phi = -4\pi G\, a^2 \left[(\bar\rho + \bar P)\delta v\right]_{\text{tot}} \tag{60}$$

$$\Phi'' + 3\frac{a'}{a}\Phi' + 2\frac{a''}{a}\Phi - \left(\frac{a'}{a}\right)^2 \Phi = 4\pi G\, a^2\, \delta P_{\text{tot}} \tag{61}$$

$$\delta\bar\rho_\lambda' + 3\frac{a'}{a}(\delta\rho_\lambda + \delta P_\lambda) + (\bar\rho_\lambda + \bar P_\lambda)\left(\Delta\delta v_\lambda - 3\Phi'\right) = 0 \tag{62}$$

$$\delta P_\lambda + \left[(\bar\rho_\lambda + \bar P_\lambda)\delta v_\lambda\right]' + 4\frac{a'}{a}(\bar\rho_\lambda + \bar P_\lambda)\delta v_\lambda + (\bar\rho_\lambda + \bar P_\lambda)\Phi = 0 \tag{63}$$

Notice that $\Delta \equiv \nabla_i\nabla^i$ is the Laplacian operator in 3D space. For flat FRW spacetime $\Delta = \partial_i\partial^i$ because the connection is trivial as no metric terms depend on x^i.

Single Component Fluid

Take (61) - c_s^2 (59) to isolate Φ:

$$\Phi'' + 3\frac{a'}{a}(1 + c_s^2)\Phi' - c_s^2\Delta\Phi + \left[2\frac{a''}{a} - \left(\frac{a'}{a}\right)^2 (1 - 3c_s^2)\right]\Phi = 0 \tag{64}$$

Note that, in a single fluid Universe, $c_s^2 = \omega$ because ω is constant! (except during transition periods). Combining this with the zeroth order equations (2) - (3) (substitute in conformal time) results in

$$2\frac{a''}{a} - \left(\frac{a'}{a}\right)^2 (1 - 3c_s^2) = -8\pi G a^2 \left(\bar P - c_s^2\bar\rho\right) = 0 \tag{65}$$

and thus leads to the equation of state for a single component fluid:

$$\Phi'' + 3\frac{a'}{a}(1 + c_s^2)\Phi' - c_s^2\Delta\Phi = 0 \tag{66}$$

Note that (65) gives

$$a \propto \tau^{2/(1+3\omega)} \tag{67}$$

Non-relativistic matter: $\bar P = 0$

From (67) you have $a \propto \tau^2 \Rightarrow$ (66) becomes

$$\Phi = C_1 + C_2\tau^{-5} \tag{68}$$

C_1 and C_2 of course can be function of x^i. Using (59) and dividing by (2) (remember

[3] You still have residual gauge freedom on $\xi_i^\perp$ to set $\delta v_i^\perp = 0$

to transform in conformal time) we find

$$\frac{\delta\rho}{\bar{\rho}} = \frac{1}{6}\left[\left(\Delta C_1\tau^2 - 12C_1\right) + \left(\Delta C_2\tau^2 - 18C_2\right)\frac{1}{\tau^5}\right] \tag{69}$$

Using $C_{1,2} \propto e^{i\mathbf{kx}} \Rightarrow \Delta C_{1,2} = -k^2 C_{1,2}$,

- Super-horizon scales $k\tau \ll 1$: neglect decaying mode $\Rightarrow \frac{\delta\rho}{\bar{\rho}} \sim -2C_1 \sim -2\Phi$
- Sub-horizon scales $k\tau \gg 1$: $\frac{\delta\rho}{\bar{\rho}} \sim -\frac{k^2}{6}\left(C_1\tau^2 + \frac{C_2}{\tau^3}\right)$

Ultra relativistic matter:
Use (67) to rewrite (66), then go to Fourier space $\Phi = \int \frac{d^3k}{(2\pi)^{3/2}}\Phi_k(\tau)e^{i\mathbf{kx}}$:

$$\Phi_k'' + \frac{6}{1+3\omega}\frac{1}{\tau}\Phi_k' + k^2\omega\Phi_k = 0 \tag{70}$$

Solution is in terms of (ordinary) Bessel functions:

$$\Phi_k = \tau^{-\nu}\left[C_1\, J_\nu\!\left(\sqrt{\omega}\,k\,\tau\right) + C_2\, Y_\nu\!\left(\sqrt{\omega}\,k\,\tau\right)\right] \quad \text{with} \quad \nu = \frac{1}{2}\frac{5+3w}{1+3w} \tag{71}$$

$\omega_\gamma = \frac{1}{3} \Rightarrow \nu > 0$, with this condition one can evaluate the limits for J and $Y \to 0$:

- Super-horizon scales $k\tau \ll 1$: $J_\nu(x\to 0) \sim x^\nu$, $Y_\nu(x \to 0) \sim x^{-\nu}$
  $\tau^{-\nu}$ negligible in this regime. Hence $\Phi_k \sim const \Rightarrow \Phi = \Phi(\mathbf{x})$, use (59) and divide by (2) $\Rightarrow \frac{\delta\rho}{\bar{\rho}} \sim -2\,\Phi$
- Sub-horizon scales $k\tau \gg 1$: Use (59), in LHS $\Delta\Phi$ dominates $\Rightarrow \frac{\delta\rho}{\rho} \propto \Phi_k a^2/a'^2$
  $J_\nu(x\to\infty) \sim x^{-1/2}\cos x$, $Y_\nu(x\to\infty) \sim x^{-1/2}\sin x$
  $\Rightarrow\ \Phi_k \propto \tau^{-\nu-\frac{1}{2}}\, e^{\pm i\sqrt{\omega}k\,\tau} \ \Rightarrow\ \frac{\delta\rho}{\bar{\rho}} \propto \tau^{\frac{3}{2}-\nu}\, e^{\pm i\sqrt{\omega}k\,\tau}$. In a radiation dominated background $\nu = 2/3 \Rightarrow \frac{\delta\rho}{\bar{\rho}} \propto e^{\pm i\,\frac{k}{\sqrt{3}}\,\tau}$, i.e. acoustic oscillations.

Matter and radiation:
In this case, ω is not constant, but it is a function of time because $p(\rho)$. It can be shown that, in a *generic background* on *super-horizon scales*, (66) is solved by

$$\Phi = A\frac{d}{dt}\left(\frac{1}{a}\int a dt\right) \tag{72}$$

Find scale factor evolution in matter-radiation Universe solving the Cauchy problem of (2) (in conformal time). To do this, write $\rho_{tot} = \rho_m + \rho_\gamma = \rho_{m_{eq}}(a_{eq}/a)^3 + \rho_{\gamma_{eq}}(a_{eq}/a)^4$, $\Omega_{m_{eq}} = \Omega_{\gamma_{eq}} \simeq 1/2$, $\tau_* \equiv \tau_{eq}/(\sqrt{2}-1)$. You get $a = a_{\rm eq}\left[(\tau/\tau_*)^2 + 2\,(\tau/\tau_*)\right]$. Plug this into (72) and define $x \equiv \tau/\tau_*$. Keep constants A, B. Setting $B = 0$:

$$\Phi = \frac{x+1}{(x+2)^3}A\left(\frac{3}{5}x^2 + 3x + \frac{1}{x+1} + \frac{13}{3}\right) \tag{73}$$

Notice that $x \to 0$ is radiation domination, while $x \to \infty$ is matter domination. Hence, taking the limits, $\Phi_{\rm matter}/\Phi_{\rm rad} = 9/10$. Now compute $\delta\rho_{tot}/\bar{\rho}_{tot}$ using (59). Use $\Phi' \approx 0$ (it's non-zero only around τ_{eq}). $\delta\rho_{tot}/\bar{\rho}_{tot} = -2\Phi$

Perturbations During Inflation

Scalar Perturbations

With $\mathcal{L}_\phi = \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V$, EMT of a scalar field $(\frac{2}{\sqrt{-g}}\delta S_\phi/\delta g^{\mu\nu})$ is of the perfect fluid form $\Rightarrow$ we can use hydrodynamical approach of Section 3.3 to study perturbations during inflation. Using Newtonian gauge, (48) becomes (recall found $\Psi = -\Phi$ from Einsteins eq for hydrodynamical perturbations)

$$ds^2 = a^2(\tau)\left\{(1+2\Phi)\, d\tau^2 - (1+2\Phi)\delta_{ij}dx^i dx^j\right\} \tag{74}$$

Expand $\phi = \bar{\phi}(\tau) + \delta\phi(\tau, \bar{x})$. Evaluate δT^0_0, δT^0_i and δT^i_j. To do this expand $(1+2\phi)^{-1}$, $V(\phi)$ and employ KG equation in conformal time for $\bar{\phi}$. Impose equalities with correspondent variational quantities for perfect fluid (53) and also non variational ones.

$$\delta\rho = a^{-2}\left[-\Phi\bar{\phi}'^2 + \bar{\phi}'\delta\phi' - \delta\phi\left(\bar{\phi}'' + 2\frac{a'}{a}\bar{\phi}'\right)\right] \qquad \delta P = a^{-2}\left(\bar{\phi}'\delta\phi' - \Phi\bar{\phi}'^2\right) - V_{,\phi}\delta\phi \tag{75}$$

$$-(\bar{\rho} + \bar{P})\delta v_i = -\frac{\bar{\phi}'^2}{a^2} \overset{!}{=} \frac{\bar{\phi}'}{a^2}\partial_i\delta\phi \;\Rightarrow\; \delta v_i = -\frac{\partial_i\delta\phi}{\bar{\phi}'} \;\Rightarrow\; \delta v = -\frac{\delta\phi}{\bar{\phi}'} \tag{76}$$

Thus in the comoving frame, in which $\delta v = 0$, $\delta\phi = 0$ $\Rightarrow$ inflation is constant in space. Equations (59) (61) can then be rewritten as:

$$\Delta\Phi - 3\frac{a'}{a}\Phi' - 3\left(\frac{a'}{a}\right)^2\Phi = -4\pi G\,\bar{\phi}'^2\Phi + 4\pi G\left[\bar{\phi}'\delta\phi' - \delta\phi\left(\bar{\phi}'' + 2\frac{a'}{a}\bar{\phi}'\right)\right] \tag{77}$$

$$\Phi' + \frac{a'}{a}\Phi = 4\pi G\,\bar{\phi}'\,\delta\phi \tag{78}$$

$$\Phi'' + 3\frac{a'}{a}\Phi' + 2\frac{a''}{a}\Phi - \left(\frac{a'}{a}\right)^2\Phi = 4\pi G\, a^2\left[a^{-2}\left(\bar{\phi}'\,\delta\phi' - \bar{\phi}'^2\Phi\right) - V_{,\phi}\,\delta\phi\right] \tag{79}$$

3 equations, 2 unknowns $(\Phi, \delta\phi)$. Work with the first two. In flat Universe, (2) - (3) (or simply manipulation of KG equation, see Section 2.1) gives

$$\dot{H} = -4\pi G\dot{\bar{\phi}}^2 \;\Rightarrow\; -4\pi G\bar{\phi}'^2 = \frac{a''}{a} - 2\frac{a'^2}{a^2} \tag{80}$$

(77) can thus be rewritten as (use (78) to rewrite Φ)

$$\Delta\Phi = 4\pi G\frac{a}{a'}\bar{\phi}'^2\frac{d}{d\tau}\left(\Phi + \frac{a'}{a\bar{\phi}'}\delta\phi\right) \tag{81}$$

Define the *Mukhanov-Sasaki variable*

$$u \equiv z\Phi + z\frac{a'}{a\bar{\phi}'}\delta\phi \qquad z \equiv \frac{a^2\bar{\phi}'}{a'} = \sqrt{2\epsilon}a \quad\Rightarrow\quad u = z\Phi + a\delta\phi \tag{82}$$

So that

$$\Delta\Phi = 4\pi G\frac{\bar{\phi}'}{a}z\frac{d}{d\tau}\left(\frac{u}{z}\right) \tag{83}$$

Now rewrite (78) in terms of u:

$$\frac{a'}{a^2}\frac{d}{d\tau}\left(\frac{a^3}{a'}\Phi\right) = 4\pi G\bar{\phi}'u \tag{84}$$

(83) and (84) can be combined to get *Mukhanov-Sasaki equation*

$$u'' - \frac{z''}{z}u - \Delta u = 0 \tag{85}$$

Scalar perturbations behave as an harmonic oscillator with mass $m^2_{eff} = -z''/z$.

MS equation can be equivalently derived by expanding $S = \int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V\right)$ up to second order in Φ and $\delta\phi$. Then, when requiring a canonical form for the kinetic term, one is lead to define u and z to obtain

$$S = \frac{1}{2}\int d\tau d^3x\left(u'^2 - \frac{z''}{z}u^2 - (\partial_i u)^2\right) \tag{86}$$

$\delta S/\delta u = 0$ reproduces (85).

Notice that MS equation has a similar form to (43) in dS spacetime. In order to make this similarity more explicit evaluate z'/z in terms of η and ϵ, then same for z''/z (recall that here the results are not expanded, they are exact). So that you can define $\nu^2 - 1/4 \equiv -\tau^2 m_{\text{eff}}$ (knowing the expression for ν in terms of ϵ, η) with the same intent as in dS space. So that, expanding u in fuourier modes $\int \frac{d^3k}{(2\pi)^{3/2}}u_{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}}$ and then $u(\mathbf{x},\tau) = \frac{1}{\sqrt{2}}\int\frac{d^3k}{(2\pi)^{3/2}}\left[a^-_k v^*_k(\tau)e^{i\mathbf{k}\cdot\mathbf{x}} + a^+_k v_k(\tau)e^{-i\mathbf{k}\cdot\mathbf{x}}\right]$

$$v''_k + \left(k^2 - \frac{\nu^2 - 1/4}{\tau^2}\right)v_k = 0 \quad \textit{with} \quad \nu = \frac{3}{2} + \epsilon + \frac{\eta}{2} \tag{87}$$

Again, as in (3.1), Hankel functions are solution, and you impose Bunch-Davies vacuum at $|k\tau| \gg 1$ obtaining as a consequence the Bunch-Davies mode functions

$$v_k(\tau) = \frac{\sqrt{\pi}}{2}\sqrt{-\tau}\,H^{(2)}_\nu(-k\tau) \overset{k\ll aH}{\approx} 2^{\nu-1}\frac{i}{\sqrt{aH}}\left(\frac{k}{aH}\right)^{-\nu}\frac{\Gamma(\nu)}{\sqrt{\pi}} \tag{88}$$

Because we have the following limit form for Hankel functions (which fixes C_1 and C_2):

$$H^{(1,2)}_\nu(x\to\infty) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{x}}e^{\pm ix}e^{\pm i\pi(\nu+1/2)/2} \tag{89}$$

Define the *comoving curvature* $\mathcal{R} \equiv -\Phi - a\delta\phi/z = -u/z$; it is the *spatial curvature on constant inflation hypersurfaces.* Define also

$$P_{\mathcal{R}_k} \equiv z^{-2}P_{v_k} = |v_k|^2/(2\epsilon a^2) \overset{k\ll aH}{\approx} \frac{2^{2\nu-3}}{\pi\,\epsilon\,a^2k}\left(\frac{k}{aH}\right)^{1-2\nu}\Gamma^2(\nu) \propto k^{-3} \tag{90}$$

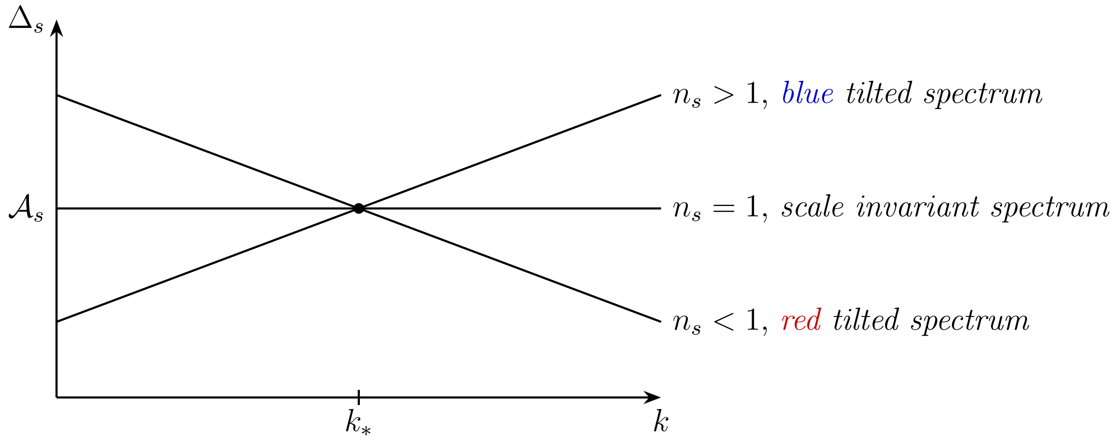
This in turn leads to the definition of the *dimensionless power spectrum of curvature perturbations* (in terms of a pivot scale k_*)

$$\Delta^2_s \equiv \frac{k^3}{2\pi^2}P_{\mathcal{R}_k} \overset{\text{slow roll}}{\approx} \frac{H^2}{8\pi^2\epsilon}\left(\frac{k}{aH}\right)^{3-2\nu} = \mathcal{A}_s\left(\frac{k}{k_*}\right)^{n_s-1} \tag{91}$$

Where we defined also the *amplitude of scalar perturbations* and the *spectral index*

$$\mathcal{A}_s \equiv \frac{H^2}{8\pi^2\epsilon} \qquad n_s - 1 \equiv 3 - 2\nu = -2\epsilon - \eta \tag{92}$$

Observations of CMB constrain $\mathcal{A}_s \simeq 2 \cdot 10^{-9}$ and $n_s \simeq 0.96$. Thus our Universe is said to be slightly *red tilted*.



Tensor Perturbations

They are gauge invariant as you can prove that $\tilde{h}_{ij} = h_{ij}$ (see Section 3.2),

$$ds^2 = a^2(\tau)\left[d\tau^2 - (\delta_{ij} - h_{ij})\, dx^i dx^j\right] = a^2(\tau)\,(\eta_{\mu\nu} + h_{\mu\nu}) \quad \text{with} \quad h_{\mu 0} = 0 \tag{93}$$

$$\delta G^0_0 = 0 = \delta G^0_i \qquad \delta G^i_j = \frac{1}{2a^2}\left(h^i_j{}'' + 2\frac{a'}{a}h^i_j{}' - \Delta h^i_j\right) \tag{94}$$

In Section 3.2 we evaluated δT^i_j of a perfect fluid/scalar field and found it was $\propto \delta^i_j$; you can check that adding h_{ij} in the perturbed metric does not change this result. This implies $\delta G^1_1 = \delta G^2_2 = \delta G^3_3 \equiv A$, but due to the fact that h is traceless (and also transverse), we have $3A = 0 \Rightarrow$[4]

$$h_{ij}{}'' + 2\frac{a'}{a}h_{ij}{}' - \Delta h_{ij} = 0 \tag{95}$$

A perfect fluid does not source gravitational waves.

h_{ij} has $3(3+1)/2 - 1 - 3 = 2$ independent components $\Rightarrow$ 2 polarizations of the graviton. For propagation along x^3 ($h_{ij} = \epsilon_{ij} e^{ikz - i\omega\tau}$) we have

$$h_{ij} = \begin{pmatrix} h_+ & h_x & 0 \\ h_x & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{96}$$

Each polarization h_+, h_x follows (95) and propagates at the speed of light, as there is just $c = 1$ in front of Δh_{ij}. Therefore, the action for gravitational waves can be written as:

$$S_{TT} = \sum_A \frac{M_p^2}{8} \int d^4x\, a^2 \left[(h^{(A)})'^2 - (\partial_k h^{(A)})^2\right] \tag{97}$$

[4] One can reach (95) also by expanding S_{EH} action at second order in h_{ij}: $\frac{M_p^2}{8}\int d^4x\, a^2\left[h'_{ij}{}^2 - (\partial_k h_{ij})^2\right]$

Now, if we define the canonical variable

$$v^{(A)} \equiv a\, h^{(A)} \frac{M_P}{2} \tag{98}$$

we can rewrite the action as

$$S_{TT} = \sum_A \frac{1}{2} \int d\tau\, d^3x \big[(\partial_\tau v^{(A)})^2 + \tfrac{a''}{a}\,(v^{(A)})^2 - (\partial_k v^{(A)})^2\big] \tag{99}$$

Compare it with (86), this is MS equation for tensor perturbations. The "mass" term here is a''/a, which is $2/\tau^2$ in dS background and $(2+3\epsilon)/\tau^2$ in slow roll background.

As in scalar case, $\int \frac{d^3k}{(2\pi)^{3/2}} v_{\mathbf{k}}^{(A)}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}}$ and then solving MS equation in Fourier space leads to $v^{(A)}(\mathbf{x},\tau) = \frac{1}{\sqrt{2}} \int \frac{d^3k}{(2\pi)^{3/2}} \left[a_k^- v_k^{(A)*}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}} + a_k^+ v_k^{(A)}(\tau)e^{-i\mathbf{k}\cdot\mathbf{x}}\right]$, we find a solution in the form of Hankel functions but with an index $\nu_T = 3/2 - \epsilon$. Imposing BD vacuum fixes constants ($C_1 = 0$, $C_2 = \sqrt{\pi}/2$) and then as before we define

$$\Delta_T \equiv \frac{k^3}{2\pi^2} \sum_A |h_k^{(A)}|^2 \qquad \mathcal{A}_T \equiv \frac{2H^2}{\pi^2 M_p^2}\bigg|_{k=aH} \qquad n_T = 3 - 2\nu_T = -2\epsilon \tag{100}$$

as, for slow roll ($\nu \approx 3/2$),

$$\Delta_T = \frac{k^3}{2\pi^2} 2 \left(\frac{2}{aM_P}\right)^2 |v_k|^2 \approx 2\frac{H^2}{\pi^2}\left(\frac{k}{aH}\right)^{3-2\nu} \tag{101}$$

One can then define the *tensor to scalar ratio* (in 4.1 I had $M_P = 1$)

$$r \equiv \frac{\mathcal{A}_T}{\mathcal{A}_S} = 16\epsilon = -8\eta_T \tag{102}$$

The latter is true for all single field models and it is called *consistency relation*. If observations say $r \neq -8\eta_T$ then inflation was not single field.

Lyth Bound

Start from the fact that in flat FRW space-time $\dot{H} = -\frac{\dot{\phi}}{2M_P^2} \Rightarrow \frac{d\phi}{dN} = \sqrt{2M_P^2\epsilon}$.

$$\Delta\phi = \int dN \sqrt{2M_P^2\epsilon} \approx \sqrt{2M_P^2\epsilon}\Delta N = M_P\sqrt{\frac{r}{0.01}}\sqrt{\frac{0.01}{8}}\Delta N \tag{103}$$

$$\frac{\Delta\phi}{M_P} = \sqrt{\frac{r}{0.01}}\,O(1) \tag{104}$$

Thus, for $r \gtrsim 10^{-2}$, $\Delta\phi > M_P$ $\rightarrow$ large field inflation gives observable tensor modes, meaning that the primordial gravitational-wave signal it produces has an amplitude high enough to be distinguished from noise in current experiments. $r \leq 7 \cdot 10^{-2}$

Observables from Inflation

In single-field slow-roll inflation the key "observables" are all evaluated at the moment each Fourier mode exits the Hubble radius during inflation. No need to know when inflation "begins" in some absolute sense. All that matters for observables is how many e-folds remain before inflation ends, because that determines when a given comoving scale k crossed the Hubble radius.

In practice one picks a "pivot" wavenumber $k^* = a^* H$ (corresponding to $N^* \sim 50 - 60$) and defines $\mathcal{A}_s^*$, n_s^*, r^*, where $*$ means that are calculated at the pivot scale

$$\log \frac{a_{end}}{a(\phi)} \equiv N^* \sim \log \frac{a_{end} H}{k^*} \tag{105}$$

Once a given mode has left the Hubble radius it "freezes in" (i.e. its curvature perturbation becomes conserved on superhorizon scales), so nothing in the microphysics of reheating or recombination changes the primordial values of $\mathcal{A}_s^*$, n_s^*, r^* that were set at exit. *What does happen between inflation and recombination is simply linear evolution through radiation and matter domination (via the transfer functions), but that evolution is completely determined by the background cosmology and does not alter the primordial spectral shape or amplitude.* Recall $\mathcal{A}_s \simeq 2 \cdot 10^{-9}$, $n_s \simeq 0.96$, $r \leq 7 \cdot 10^{-2}$.

As in Section 2.1, evaluate ϵ_V, η_V for each model. Then, get $H^* = 8\pi^2 \epsilon A_s$ then n_s and r.

Large Fields/Monomial Models:

$$H^* \approx 2.7 \cdot 10^{-5} \sqrt{n}\, M_P \sim 10^5 GeV \tag{106}$$

Duration of Inflation:
Approximating H as constant during inflation, and using $H^* \approx 2.7 \cdot 10^{-5} \sqrt{n}\, M_P$

$$\Delta t = \frac{1}{H} \int_{a_{st}}^{a_{end}} \frac{da}{a} \quad \Rightarrow \quad \Delta t \approx \frac{N}{2.7 \cdot 10^{-5} \sqrt{n} M_P} \sim 10^{-36} s \tag{107}$$

Small Fields/Hill-top & Inflection Models:
Work in the regime $V_0 \gg \lambda_n \phi^n$, where you get $\eta_V \gg \epsilon_V$. Do the calculations for $n = 4$. For slow roll models recall EOMs, in this case

$$H^2 = \frac{1}{3M_P^2} \left(\frac{\dot{\phi}^2}{2} + V \right) \sim \frac{V_0}{3} \tag{108}$$

With this you can evaluate A_s and the fix λ, you have only V_0 as free parameter.

Plateau Models:
For n_s discard $1/N^2$ term. For Starobinsky model ($\alpha = 1, k = \sqrt{2/3}$) we have excellent agreement ($r \approx 0.004$).

The last scattering surface

When inflation ends the inflaton dumps its energy into standard model particles: this is the process called *reheating*. After reheating the Universe is still radiation dominated until recombination at t_{eq}. The latter time can be estimated, in terms of redshift z_{eq}, by $\rho_m^{eq} = \rho_\gamma^{eq}$ and then employing $\Omega_{\rho_i}^{\text{now}}$.

After radiation-matter equivalence the Universe remains a coupled baryon-photon plasma until recombination, when the already existing free photons can propagate freely; resulting in Plankian spectrum we observe: the CMB.

To estimate z_{rec} start from the number density of a certain species

$$n = \frac{g}{(2\pi)^3}\int f(\mathbf{p})d^3p = \frac{g}{2\pi^2}\int_m^\infty f(\mathbf{E})E\sqrt{E^2-m^2}dE \qquad f(\mathbf{p}) = \frac{1}{e^{\frac{E-\mu}{T}}\pm 1} \tag{109}$$

Approx $e^{\frac{E-\mu}{T}} \pm 1 \approx e^{\frac{E-\mu}{T}}$ and consider non-relativistic particles, for which $T \ll m$:

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{\frac{\mu-m}{T}} \tag{110}$$

$g = 2S+1$-factors are the usual internal degeneracy (spin, etc.) of each species. The *proton-electron capture* to form neutral hydrogen is the process happening at recombination. In the early Universe before recombination we have equilibrium: $p + e \leftrightarrow H + \gamma$. Assume $n_B = n_p + n_H$, i.e. all baryons are in the form of protons and hydrogen atoms. Assume thermal and chemical ($\mu_H = \mu_p + \mu_e$) equilibrium. Furthermore, $n_p = n_e$ due to charge neutrality. We are interested in the ratio $n_H/(n_p n_e)$ as it tells us how neutral (and thus transparent for photons) is the Universe at temperature T. Define *hydrogen biding energy* $B \equiv m_p + m_e - m_H$. Employing (110)

$$\frac{n_H}{n_p n_e} \overset{m_H \simeq m_p}{\approx} \underbrace{\frac{g_H}{g_p g_e}}_{=1}\left(\frac{2\pi}{m_e T}\right)^{3/2} e^{B/T} \overset{!}{=} \frac{n_p/n_B^2}{n_e^2/n_B^2} = \frac{1}{n_B}\frac{1-n_e/n_B}{n_e^2/n_B^2} \tag{111}$$

Define the *fractional ionization* $\chi_e \equiv n_e/n_B$ and the *baryon to photon ratio* $\eta \equiv n_B/n_\gamma \simeq 6\cdot 10^{-10}$. Where $n_\gamma = \frac{2}{\pi^2}\zeta(3)T^3$ comes from direct integration of (109). We can now rewrite (111) in the form called *Saha equation*

$$\left.\frac{1-\chi_e}{\chi_e^2}\right|_{eq} = \frac{4\sqrt{2}}{\sqrt{\pi}}\zeta(3)\eta\left(\frac{T}{m_e}\right)^{3/2} e^{B/T} \tag{112}$$

Universe becomes transparent for $\chi_e \to 0$, as free protons and electrons diminish and the former start having a high mean free path. For $\chi_e = 0.01$ we have from (112) $T \approx 3500K$ $\Rightarrow z_{rec} = \frac{T_{rec}}{T_{now}} - 1 \simeq 1200$.

Thermal equilibrium is no longer a good approximation during inflation, indeed a more accurate result can be found using kinetic theory: $z_{rec} \simeq 1100$.

Introducing CMB temperature anisotropies

To study how photons propagate between last scattering surface (LSS) and today consider a flat Universe subject to scalar perturbations (here $\Psi \neq -\Phi$ as we are not working in single perfect fluid approximation), change sign of Ψ for convenience:

$$ds^2 = a^2(\tau)\left[(1+2\Phi)\,d\tau^2 - (1+2\Psi)\,\delta_{ij}dx^i dx^j\right] \tag{113}$$

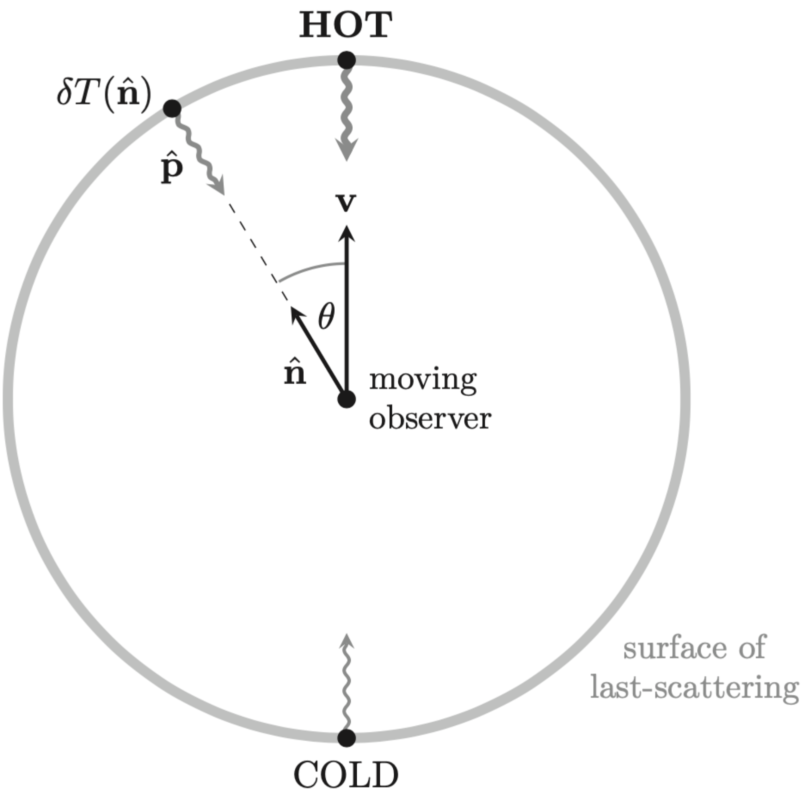
Write down geodesic equation for photons, first with affine parameter λ, then in conformal time.

$$\dot{P}^\beta + \Gamma^\beta_{\mu\nu}P^\mu P^\nu = 0 \qquad P^\mu = \frac{dx^\mu}{d\lambda} \qquad \dot{P}^\mu = \frac{dP^\mu}{d\lambda} = \frac{dP^\mu}{d\tau}P^0 \tag{114}$$

Look at 0 component and then compute full perturbed connections at first order, in doing so expand $(1+2\Phi)^{-1}$:

$$\frac{dP^0}{d\tau} + \Gamma^0_{\mu\nu}\frac{P^\mu P^\nu}{P^0} = 0 \tag{115}$$

Plug the expressions in and use the null condition $P^\mu P_\nu = 0$ to get $P^i = P^0\,(1+\Phi-\Psi)\,n^i$ Where n^i is any unit vector on the sphere (pointing along photon's direction in the comoving spatial frame). This leads to[5]



Motion of the Solar System relative to CMB rest frame [3].

$$\frac{dP^0}{d\tau} + 2\frac{a'}{a}P = -P^0\,(\Psi' + \Phi') - 2\Phi_{,i}n^i P^0 \tag{116}$$

[5]Recall $'$ are *partial* derivatives with respect to τ.

Define the *conformal momenta* $\Theta^\mu \equiv a^2 P^\mu$ to notice that Θ^0 is conserved at zeroth order! This means that *different photons of different frequency get red-shifted in the same way* $\Rightarrow$ the spectrum keeps its shape (at zeroth order).

Notice that the *total* derivative of τ on $\Phi(\tau, x^i) = \Phi' + \Phi_{,i}\frac{P^i}{P^0}$, so you can rewrite (116) as

$$\frac{d\Theta^0}{d\tau} = \Theta^0 \left[(\Phi' - \Psi') - 2\frac{d\Phi}{d\tau}\right] \tag{117}$$

This can be integrated and then expanded around $\Theta^0(\tau'') \sim \Theta^0(\tau')$ (motivated by the consideration above).

$$\frac{\Theta^0(\tau'') - \Theta^0(\tau')}{\Theta^0(\tau')} = -2\left[\Phi(\tau'') - \Phi(\tau')\right] + \int_{\tau'}^{\tau''} (\Phi' - \Psi')\, d\tau \tag{118}$$

The four momentum components $P^\mu = (P^0, P^i)$ are defined in the coordinate frame, while what an observer actually measures as the photon energy and momentum are the components $P^{\hat{\mu}} = (E, P^{\hat{i}})$ in their local inertial frame. <u>Assume that the observer is comoving with the cosmic fluid.</u> To get the relative frequency shift from (118), consider $E = U_\mu P^\mu$ and get U_μ from $U^\mu U_\mu = 1$.

$$E = aP^0\,(1 + \Phi - \mathbf{n}\cdot\delta\mathbf{v}) \tag{119}$$

So that, if we multiply both sides by $a(\tau)$ we construct Θ:

$$\frac{\Theta^0(\tau'') - \Theta^0(\tau')}{\Theta^0(\tau')} = \frac{Ea(\tau'')}{Ea(\tau')}\Big[1 - \Phi(\tau'') + \mathbf{n}\cdot\delta\mathbf{v}(\tau'') + \Phi(\tau') - \mathbf{n}\cdot\delta\mathbf{v}(\tau')\Big] - 1 \tag{120}$$

Now we wish to expand $E(\tau'')$, to do so first notice that $E(\tau'')a(\tau'') = E(\tau')a(\tau')$ at zeroth order. Hence $E(\tau'')a(\tau'') \sim \left[E(\tau') + \Delta E(\tau'', \tau')\right] a(\tau')$. The relative frequency shift for a photon emitted at τ' and detected at τ'' is thus

$$\frac{\Delta E(\mathbf{n}, \tau'', \tau')}{E(\tau')} = \Phi(\tau') - \Phi(\tau'') + \mathbf{n}\cdot\delta\mathbf{v}(\tau') - \mathbf{n}\cdot\delta\mathbf{v}(\tau'') + \int_{\tau'}^{\tau''} (\Phi' - \Psi')\, d\tau \tag{121}$$

Notice that *so far we only computed ΔE, which is the change in the energy of a single photon along its geodesic, we are now interested in δT: the change in the local blackbody temperature of a whole ensemble of photons at last scattering.*

Consider an ensemble of photons with temperature T and set $\tau'' = \tau_{now}$, $\tau' = \tau_{rec}$. Temperature and energy redshift the same way, therefore a fractional change in photon energy is exactly the same as a fractional change in the blackbody temperature you would assign to the beam. Now we have also to add an “intrinsic” piece due to local density perturbations in the ensemble (recalling $\rho_\gamma \propto T^4$). You can absorb the *monopole term* $\Phi(\tau_{now})$ into the average temperature[6] and neglect the *dipole term* $\mathbf{n}\cdot\delta\mathbf{v}(\tau_{now})$ as it accounts for the Doppler effect due to the relative motion of the observer with respect to the CMB rest frame[7].

$$\frac{\delta T}{T}(\mathbf{n}, \tau_{now}) = \frac{\delta\rho_\gamma}{4\rho_\gamma}(\tau_{rec}) + \Phi(\tau_{\rm rec}) + n^i v_i(\tau_{\rm rec}) + \int_{\tau_{\rm rec}}^{\tau_{now}} (\Phi' - \Psi')\, d\tau \tag{122}$$

[6]CMB experiments only measure differences in temperature: $\Delta T(\mathbf{n}) = T(\mathbf{n}) - T_{avg}$, thus, adding a constant to every photon's energy (or to the mean temperature) simply shifts the zero-point of T.

[7]It comes from our local motion, not from fluctuations at recombination or along the line of sight; we subtract it in data processing: experiments fit and remove the dipole to isolate the cosmological signal.

This is the *Sachs-Wolfe equation*, which states that the observed temperature anisotropies originate from

- Intrinsic density fluctuations at LSS
- Fluctuations of gravitational potential (gravitational red/blue shift) at LSS
- Doppler effects at LSS
- Contributions from time varying gravitational potential between LSS and detection

CMB pattern is generated by sending photons from the last-scattering surface (at recombination) to us, through an inhomogeneous universe. The shift in energy of (121) is direction dependent (due to Φ, Ψ and $\mathbf{n}\cdot\delta\mathbf{v}$), so it imprints a spatial anisotropy in the observed photon energies.

$$\frac{T(\mathbf{n}) - T_{now}^{avg}}{T_{now}^{avg}} \sim 10^{-4} - 10^{-5} \tag{123}$$

CMB temperature anisotropies: multipoles

The temperature anisotropy in the $\mathbf{n}$ direction in the sky is a square-integrable function on the sphere and thus can be expanded in terms of spherical harmonics

$$\frac{\delta T_{now}}{T_{now}}(\mathbf{n}) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_{lm}(\mathbf{n}) \tag{124}$$

Using common conventions, we introduce spherical harmonics and (associated) Legendre polinomials as follows:

$$Y_l^m(\theta,\varphi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,e^{im\phi}\,P_l^m(\cos\theta) \qquad P_l^{-m} = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m \tag{125}$$

It follows that $Y_l^{m*} = (-1)^m Y_l^{-m}$ and, from (124), $a_{lm}^* = (-1)^m a_{l-m}$. It also follows that

$$\int_0^{\pi} d\theta \int_0^{2\pi} Y_l^m Y_{l'}^{-m'} sin(\theta) d\theta d\phi = \delta_{ll'}\delta_{mm'} \tag{126}$$

We can use this orthonormality to determine the *multipole moments* a_{lm}:

$$a_{lm} = \int d\mathbf{n}\frac{\delta T_{now}}{T_{now}}(\mathbf{n})Y_{lm}^*(\mathbf{n}) \tag{127}$$

The CMB *angular power spectrum* C_l is the ensemble two-point function of the a_{lm}; it *measures amplitude of temperature fluctuations as a function of wavelength:* $\lambda = 2\pi/l$. One considers an average over all possible realizations of the Universe (a hypothetical average over infinitely many CMB skies) [8].

$$\langle a_{lm}a_{l'm'}^*\rangle = C_l\,\delta_{ll'}\,\delta_{mm'} \qquad \hat{C}_l = \frac{1}{2l+1}\sum_{m=-l}^{l}|a_{lm}|^2 \approx C_l \tag{128}$$

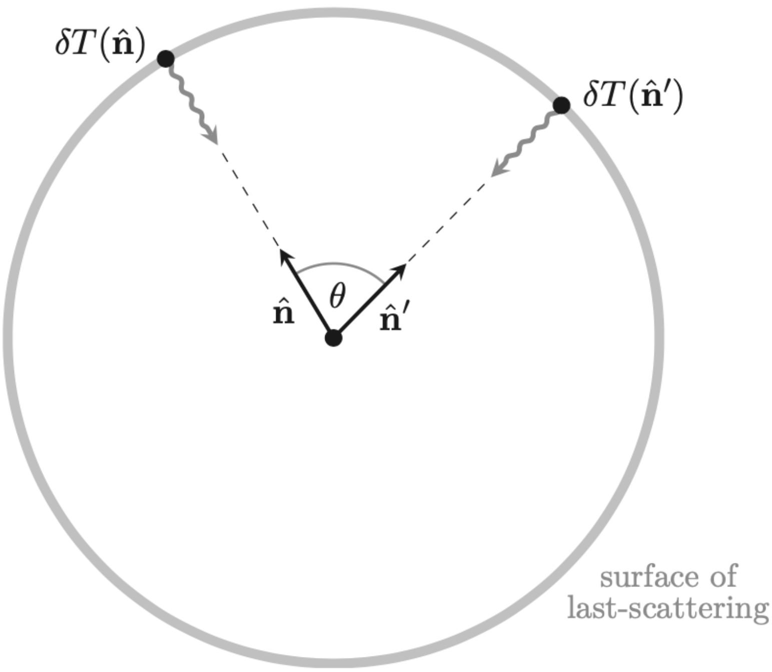
[8] Not to be confused with the inner product on the 2-sphere, which decomposes *one* sky map into modes.

The introduction of an *estimator* $\hat{C}_l$ for C_l is motivated due to the fact that, for fixed l, we have $2l+1$ different a_{lm}, allowing for $2l+1$ independent estimates of C_l. Because the process is assumed to be statistically isotropic, all the $2l+1$ different m-modes at fixed l are statistically equivalent, hence averaging over them in our one sky gives an unbiased estimator of the ensemble-average power [3]. $\hat{C}_l = C_l$ in the infinite ensemble limit, i.e. $\langle \hat{C}_l \rangle = C_l$. We have a nonzero *cosmic variance*

$$\frac{\Delta C_l}{C_l} = \sqrt{\frac{2}{2l+1}} \tag{129}$$

which can be proved using Wick's theorem in quite a long calculation starting from $\Delta C_l \equiv \sqrt{\langle (C_l - \hat{C}_l)^2 \rangle}$. We thus have large errors for small l values. What we obtain experimentally is $\hat{C}_l$: the observed spectrum of a given realization. However, the physics is given by the ensemble averaged power spectrum C_l.

The CMB temperature fluctuations are analyzed statistically by measuring the correlations between hot and cold spots as a function of their angular separation. Employing the



2-point correlations $\langle \frac{\delta T}{T}(\mathbf{n}) \frac{\delta T}{T}(\mathbf{n}') \rangle$ between temperature fluctuations on LSS [3].

addition theorem,

$$\sum_{m=-l}^{l} Y_l^m(\mathbf{n}) Y_l^{m*}(\mathbf{n}') = \frac{2l+1}{4\pi} P_l Y_l^m(\mathbf{n} \cdot \mathbf{n}') \tag{130}$$

It is easy to show that

$$\langle \frac{\delta T}{T}(\mathbf{n}) \frac{\delta T}{T}(\mathbf{n}') \rangle = \sum_{ll'} \sum_{mm'} \langle a_{lm} a^*_{l'm'} \rangle Y_l^m(\mathbf{n}) Y_{l'}^{m'}(\mathbf{n}') = \sum_l C_l \frac{2l+1}{4\pi} P_l(\mathbf{n} \cdot \mathbf{n}') \tag{131}$$

For large l, setting $\mathbf{n} = \mathbf{n}'$ ($P_l(1) = 1 \quad \forall\, l$),

$$\langle \frac{\delta T}{T}(\mathbf{n}) \frac{\delta T}{T}(\mathbf{n}') \rangle \approx \int d \log l \frac{l^2+1}{2\pi} C_l \tag{132}$$

One usually plots the CMB power spectrum as

$$\mathcal{D}_l \equiv T_0^2 \frac{l(l+1)}{2\pi} C_l \tag{133}$$

We now want to express the angular power spectrum in terms of perturbations during inflation. Using (128) (124) (130) we get

$$C_l = \frac{1}{4\pi} \int d\mathbf{n} d\mathbf{n}' \langle \frac{\delta T}{T}(\mathbf{n}) \frac{\delta T}{T}^*(\mathbf{n}') \rangle P_l(\mathbf{n} \cdot \mathbf{n}') \tag{134}$$

Then assume instantaneous recombination, i.e. infinitely small LSS, and expand $\frac{\delta T}{T}$ in Fourier space with a further expansion of a plane wave in terms of Legendre polinomials. A rather technical derivation brings to

$$C_l = \frac{2}{\pi} \int dk k^2 P_\Phi(k) \underbrace{\frac{\tilde{\delta T}}{T}(k) \frac{\tilde{\delta T}}{T}^*(k) \left[j_l(kr_*)\right]^2}_{\text{transfer function}} \tag{135}$$

where $P_\Phi(k) \delta^{(3)}(\mathbf{k}+\mathbf{k}') \equiv \left\langle \Phi(\mathbf{k}) \Phi^*(\mathbf{k}') \right\rangle$ is a primordial spectrum contribution coming from the power spectrum of scalar perturbations at recombination, while $\frac{\tilde{\delta T}}{T}(k)$ is defined to be independent of the gravitational field: $\frac{\delta T}{T} = \frac{\tilde{\delta T}}{T}\Phi$; j_l is a spherical Bessel function (r_* is the comoving radial distance from us to LSS). The transfer function is determined by (122).

CMB temperature anisotropies: Sachs-Wolfe plateau

Consider perturbations that are superhorizon at recombination, $k\tau_{rec} < 1$. Recall that in Section 3.4 we showed that $\delta\rho_m / \bar{\rho}_m = -2\Phi$ on superhorizon scales in a matter dominated Universe. Assume that the perturbation is adiabatic; i.e. the same for matter, radiation, cold dark matter and baryons:

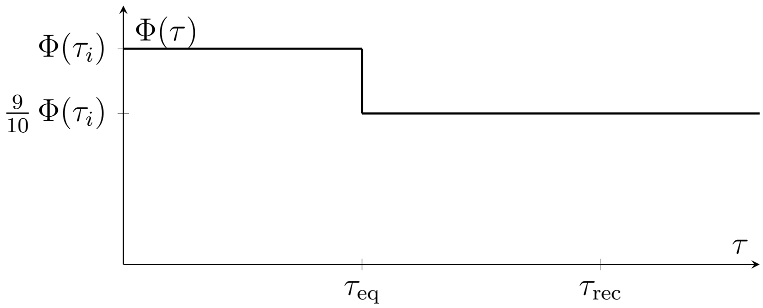
$$\frac{\delta\rho_m}{(\rho_m + p_m)} = \frac{\delta\rho_\gamma}{(\rho_\gamma + p_\gamma)} = \frac{3}{4} \frac{\delta\rho_\gamma}{\rho_\gamma} \tag{136}$$

and thus, at recombination (when Universe is matter dominated), $\frac{\delta\rho_\gamma}{\rho_\gamma} = -\frac{8}{3}\Phi$.

For these (high $\lambda \to$ small l) modes, Sachs-Wolfe equation can be approximated by ignoring Doppler and time varying potential contribution:

$$\frac{\Delta T_{now}}{T_{now}}(\mathbf{n}) \approx \frac{\delta\rho_\gamma}{4}(\tau_{rec}) + \Phi(\tau_{rec}) = \frac{1}{3}\Phi(\tau_{rec}) \tag{137}$$

Again from Section 3.4, recall that $\Phi_{\text{matter}} = (9/10)\Phi_{\text{rad}} \Rightarrow \Phi(\tau_{rec}) = (9/10)\Phi(\tau_{infl})$, as $\Phi(\tau_{infl}) \equiv \Phi(\tau_i)$ is constant on superhorizon scales.



Therefore,

$$\frac{\Delta T_{now}}{T_{now}}(\mathbf{n}) \approx \frac{3}{10}\Phi(\tau_i) = \frac{3}{10}\int \frac{d^3k}{(2\pi)^{3/2}}\Phi(\mathbf{k},\tau_i)e^{i\mathbf{kn}r_*} \tag{138}$$

As before expand the plane wave in Legendre polinomials to reach

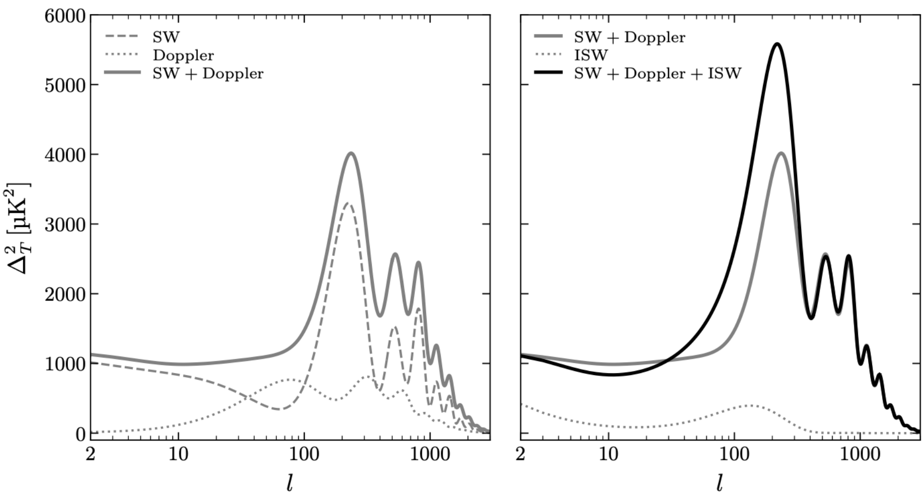
$$C_l = \frac{36\pi}{100}\int \frac{dk}{k}\Delta_\Phi(k)j_l^2(kr_*) \tag{139}$$

with $\Delta_\Phi(k) \equiv \frac{k^3}{2\pi^2}P_\Phi(k)$. Then define $\Delta_\Phi(k) \equiv \left(\frac{k}{k_*}\right)^{n_s-1}\mathcal{A}_\Phi$. With $n_s \approx 1$, (139) gives

$$C_l = \frac{18\pi}{100}\frac{\mathcal{A}_\Phi}{l(l+1)} \tag{140}$$

For small l values, CMB data yelds

$$\frac{l(l+1)C_l}{2\pi} \sim 10^{-10} \quad \Rightarrow \quad \mathcal{A}_\Phi \approx 10^{-9} \tag{141}$$



Sachs-Wolfe plateau characterizes the CMB angular spectrum for small l values [3].

We can use this bound to constrain the scale of inflation. On superhorizon scales, the comoving curvature perturbation $\mathcal{R} = -u/z$ is $\approx \xi \equiv -\Phi + \delta\rho/3(\rho+p)$. The latter being the *uniform density curvature perturbation*, i.e. the curvature perturbation measured on hypersurfaces where the total energy density is unperturbed. In a radiation dominated Universe, as it is during inflation, $\xi = -\frac{3}{2}\Phi$.

$$\mathcal{R} \approx \xi = -\frac{3}{2}\Phi \quad \Rightarrow \quad \Delta_s^2 = \frac{9}{4}\Delta_\Phi \quad \Rightarrow \quad \mathcal{A}_s = \frac{9}{4}\mathcal{A}_\Phi \approx 2\cdot 10^{-9} \tag{142}$$

This implies $H^2/(8\pi^2\epsilon) \approx 2\cdot 10^{-9} \;\Rightarrow\; H \simeq 10^{-4}\sqrt{\epsilon}\,M_P$.

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