Lorenzo Fabbri

This essay is submitted for the exam of
Quantum Cosmology

Department of Physics
Theoretical Physics Curriculum
University of Bologna
May 28, 2025

Contents

Introduction

General Relativity (GR) – the gravitational framework underlying the ΛCDM cosmological model – remains one of the most elegant and thoroughly tested theories in modern science. More than a century after Einstein introduced his field equations, GR still offers the most accurate description of both cosmological dynamics and astrophysical phenomena.

Despite these triumphs, there are good reasons to explore extensions or modifications of GR. The most important regarding cosmology is probably the unclear nature of dark matter, i.e. components that interact only gravitationally yet dominate the energy budget in ΛCDM. It is natural to ask whether altered gravitational laws on galactic or cosmological scales might account for this dark sector. Furthermore, deviations from GR in strong-gravity or quantum-gravity regimes could remove the Big-Bang singularity or those inside black holes [1].

An historical milestone in this direction came from Dirac's proposal that fundamental constants may vary with time; a notion later formalized in the Brans–Dicke (BD) theory, where Newton's constant becomes a dynamical quantity controlled by a time-dependent scalar field. BD theory is the prototype of modern scalar–tensor theories (STT) of gravity, which are the focus of this essay.

Chapter 1 introduces the basics of BD theory, along with a discussion clarifying the relations between Jordan and Einstein frames.

Chapter 2 concerns $f(R)$ theories, their equivalence with STTs and their ability to generate accelerated cosmic expansion without the need for exotic matter or dark energy.

Chapter 3 discusses a particular screening mechanism for STTs: *spontaneous scalarization*, which allows the STT to reduce to GR in the low-gravity regimes and hence satisfy the Solar-System tests.

In the whole essay the metric signature is $(-,+,+,+)$. We use natural units with $c = 1$ and $\hbar = 1$. The Einstein gravitational constant is denoted by G.

CHAPTER 1

BRANS–DICKE THEORY

In STTs, gravitational phenomena are partly due to the curvature of spacetime and partly due to the scalar field that sets the strength of the gravitational interactions at each point in spacetime.

We make this statement more clear introducing the simplest STT, the Brans-Dicke theory (BD). The action of the BD theory is given by

$$S_{BD} = \int d^4x\sqrt{-g}\left(\phi R - \frac{\omega_{BD}}{\phi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - 2V(\phi) + 2\mathcal{L}_m(\psi, \partial\psi, g_{\mu\nu})\right) \tag{1.1}$$

where ϕ is the BD scalar field, $V(\phi)$ is the self-interaction potential for ϕ and ω_{BD} is a free constant called the *BD parameter*. It is convenient to define the Lagrangian density for the scalar field ϕ: $\mathcal{L}_\phi \equiv -\frac{\omega_{BD}}{\phi}(\partial\phi)^2 - 2V(\phi)$.

Let us compare S_{BD} with the Einstein-Hilbert action complemented with a matter piece in the form of a self-interacting scalar field φ.

$$S_{EH} = \int d^4x\sqrt{-g}\left(\frac{1}{8\pi G}R + 2\mathcal{L}_\varphi\right) \tag{1.2}$$

with $\mathcal{L}_\varphi = -(\partial\varphi)^2 - 2V(\varphi)$. This is simply Einstein's GR with a matter piece in the form of a perfect fluid [2]. We can see that in (1.1) ϕ plays the role of the gravitational coupling constant, which is now dynamical:

$$\phi(x) = \frac{1}{8\pi G(x)} \tag{1.3}$$

1.1 Brans-Dicke Dynamics

By varying S_{BD} with respect to the metric we obtain the Einstein-Brans-Dicke equation of motion:

$$G_{\mu\nu} = \frac{1}{\phi}\left[T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu}\right] + \frac{1}{\phi}\left(\nabla_\mu \nabla_\nu \phi - g_{\mu\nu}\Box\right)\phi \tag{1.4}$$

with the energy-momentum tensors (EMT) defined as:

$$T^{(\phi)}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\,\mathcal{L}_\phi)}{\delta g^{\mu\nu}} = \frac{\omega_{BD}}{\phi}\left[\nabla_\mu\phi\,\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2\right] - g_{\mu\nu}V(\phi) \tag{1.5}$$

$$T^{(m)}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\,\mathcal{L}_m)}{\delta g^{\mu\nu}} \tag{1.6}$$

Proof. Let us split the action S_{BD} in two parts: $S_{BD} = S^{(\text{grav})} + S^{(m)}$, where

$$S^{(\text{grav})} \equiv \int d^4x\sqrt{-g}\left[\phi R - \frac{\omega_{BD}}{\phi}(\partial\phi)^2 - 2V(\phi)\right] \quad , \quad S^{(m)} \equiv 2\int d^4x\sqrt{-g}\mathcal{L}_m$$

The interesting part of the variation is the first term, which we can write as

$$\frac{1}{\sqrt{-g}}\frac{\delta S^{(\text{grav})}}{\delta g^{\mu\nu}} = \phi\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right] + \frac{\omega_{BD}}{2\phi}g_{\mu\nu}(\nabla\phi)^2 - \frac{\omega_{BD}}{\phi}\nabla_\mu\phi\nabla_\nu\phi + g_{\mu\nu}V + \frac{1}{\sqrt{-g}}\frac{\delta\bar{S}}{\delta g^{\mu\nu}}$$

One can easily recognize the Einstein tensor $G_{\mu\nu}$ and the EMT for ϕ, while the last term is highly non trivial.

$$\delta\bar{S} = \int_{\mathcal{M}} d^4x\,\sqrt{|g|}\,\phi\, g^{\mu\nu}\,\delta R_{\mu\nu}$$

To compute this variation we employ the typical requirement that any fields variations, as well as variations of their first derivatives, vanish on the integration boundary $\partial\mathcal{M}$; this, along with some algebra which can be found in Appendix A, leads to

$$\frac{\delta\bar{S}}{\delta g^{\mu\nu}} = \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\phi\right)\phi$$

and thus completes the proof. ■

Action S_{BD} can be also varied with respect to the scalar field ϕ to get the Klein-Gordon equations for Brans-Dicke theory:

$$2\,\omega_{\rm BD}\frac{\Box\phi}{\phi} - \omega_{\rm BD}\left(\frac{\partial\phi}{\phi}\right)^2 + R = 2\,V_{,\phi} \tag{1.7}$$

Proof. It is easy to see that $\delta_\phi S_{BD}$ can be cast as

$$\delta_\phi S_{\rm BD} = \int_{\mathcal{M}} d^4x\,\sqrt{|g|}\left[\delta\phi\,R - 2\,\frac{\omega_{\rm BD}}{\phi}\,\nabla^\mu\phi\,\nabla_\mu(\delta\phi) + \omega_{\rm BD}\left(\frac{\partial\phi}{\phi}\right)^2\,\delta\phi - 2\,V_{,\phi}\,\delta\phi\right]$$

Now notice that we can construct a total derivative term

$$\nabla_\mu\!\left(\tfrac{\nabla^\mu\phi}{\phi}\,\delta\phi\right) = \frac{\Box\phi}{\phi}\,\delta\phi - \left(\frac{\partial\phi}{\phi}\right)^2\delta\phi + \frac{\nabla^\mu\phi}{\phi}\,\nabla_\mu(\delta\phi)$$

which vanishes at the boundary of the integration domain. This allows us to substitute directly inside $\delta_\phi S_{\rm BD}$ the following expression:

$$-2\frac{\omega_{BD}}{\phi}\nabla^\mu\phi\,\nabla_\mu(\delta\phi) = 2\omega_{BD}\left[\frac{\Box\phi}{\phi}\,\delta\phi - \left(\frac{\partial\phi}{\phi}\right)^2\delta\phi\right]$$

Doing this gives us

$$\delta_\phi S_{BD} = \int_{\mathcal{M}} d^4x\,\sqrt{|g|}\,\delta\phi\,\left[R + 2\,\omega_{\rm BD}\frac{\Box\phi}{\phi} - \omega_{\rm BD}\left(\frac{\partial\phi}{\phi}\right)^2 - 2\,V_{,\phi}\right]$$

From which equation (1.7) follows after imposing $\delta_\phi S_{BD}/\delta\phi = 0$. ■

It can be shown [2] that combining the equations of motions (1.4) and (1.7) gives the covariant conservation of the EMT for matter:

$$\nabla^\nu T^{(m)}_{\mu\nu} = 0 \tag{1.8}$$

meaning that the matter fields follow geodesics of the metric while the scalar field ϕ just modulates the strength of the interaction through equation (1.3).

Lastly, for later convenience, we notice that the Klein-Gordon equation (1.7) can be rewritten in the form [2]

$$(3 + 2\omega_{\rm BD})\,\Box\phi = 2\phi V_{,\phi} - 4V + T^{(m)} \tag{1.9}$$

by taking the trace of (1.4).

1.2 Conformal Transformations

The action for Brans–Dicke theory in (1.1) is said to be given in the *Jordan frame*. However, this formulation is not unique as one can always perform a conformal transformation on the metric, moving to a different frame; usually to facilitate calculations or to get more physical insights from a certain theory. Hence, it is important to try to clarify the relationships between different frames and discuss the two most common ones.

Invariance of physics under re-definition of units of measurements implies invariance under Weyl transformations of the metric. Usually, the frame with the original metric tensor $g_{\mu\nu}$ is the abovementioned *Jordan frame* (JF) while the frame obtained after a conformal transformation is the *Einstein frame* (EF) [3].

After a Weyl transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, in 4 dimensions, we have the following rescalings [4]:

$$\sqrt{-\tilde{g}} = \Omega^4\sqrt{-g} \quad , \quad \tilde{R} = \Omega^{-2}\left[R - 6\frac{\Box\Omega}{\Omega}\right] \tag{1.10}$$

Considering a conformal transformation in which $\Omega(x) \sim \phi^n(x)$, we would have that a term $\sqrt{-g}\phi R$ becomes $\sim \sqrt{-\tilde{g}}\phi^{-4n}\phi\tilde{R}\phi^{2n} + \cdots = \sqrt{-\tilde{g}}\phi^{-2n+1}\tilde{R} + \ldots$ so we can break the coupling between R and ϕ choosing $\Omega^2(x) \sim \phi(x)$.

For instance in Brans–Dicke theory in JF the choice of $\Omega = \sqrt{16\pi G\phi}$ leads to

$$S_{BD}^{(E)} = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{\tilde{R}}{16\pi G} - \frac{1}{2}(\tilde{\partial}\tilde{\phi})^2 + \frac{e^{-8\sqrt{\frac{\pi G}{2\omega+3}}\tilde{\phi}}}{(16\pi G)^2}\left(-2V(\tilde{\phi}) + 2\mathcal{L}_{\mathrm{m}}(\tilde{g})\right)\right\} \tag{1.11}$$

Notice that to obtain the correct form of the kinetic term in Einstein frame we have also performed a redefinition of the field:

$$d\tilde{\phi} = \sqrt{\frac{2\omega+3}{16\pi G}}\frac{d\phi}{\phi} \tag{1.12}$$

Proof. The Ricci tensor in JF can be written using (1.10) as

$$R = \Omega^2\tilde{R}_{\mu\nu} + 6\frac{\Box\Omega}{\Omega} = 16\pi G\phi\tilde{R}_{\mu\nu} + \sqrt{\phi}\frac{16\pi G}{2}\left[\frac{\tilde{\Box}\phi}{\sqrt{\phi}} - \frac{(\tilde{\partial}\phi)^2}{2\phi^{3/2}}\right]$$

Using this expression we can rewrite the action in EF as

$$S_{BD}^{(E)} = \int d^4x \frac{\sqrt{-\tilde{g}}}{16\pi G} \left\{ \tilde{R} + 3\left[\frac{\tilde{\Box}\phi}{\phi} - \frac{(\tilde{\partial}\phi)^2}{2\phi^2} \right] - \frac{\omega_{BD}}{\phi^2}(\tilde{\partial}\phi)^2 - \frac{2V(\phi)}{16\pi G\phi^2} + \frac{2\mathcal{L}_{\mathrm{m}}}{16\pi G\phi^2} \right\}$$

Then, we transform ϕ in order to obtain the correct form of the kinetic term, i.e. we ask

$$-\frac{3}{2}\frac{1}{16\pi G}\frac{(\tilde{\partial}\phi)^2}{\phi^2} - \frac{2\omega_{BD}}{16\pi G}\frac{(\tilde{\partial}\phi)^2}{\phi^2} \stackrel{!}{=} -\frac{1}{2}(\tilde{\partial}\tilde{\phi})^2$$

This induces the transformation (1.12) for the field's derivatives, which can be integrated to obtain the transformation for ϕ:

$$\phi = \exp\left\{ \sqrt{\frac{16\pi G}{2\omega_{BD} + 3}}\tilde{\phi} \right\}$$

Inserting this in the transformed action reproduces (1.11).

The $(\tilde{\Box}\phi)/\phi$ term is usually omitted because it is compensated by the inclusion of boundary terms in $S_{BD}^{(E)}$ [2]. After this cancellation, the final EF action contains the standard Gibbs-Hawking-York boundary term[1]: $2\int_{\partial\mathcal{M}} d^3x\sqrt{|\tilde{h}|}K$. ■

While both formulations of the BD theory discussed above are in a relationship of mathematical equivalence, their physical equivalence is questionable. In fact, while the JF Brans–Dicke theory (1.1) is a STT of gravity in the sense that the gravitational interactions are carried by the metric field of geometric origin, together with the non-geometric BD scalar field, the EFBD theory (1.11) is a purely geometric theory of gravity indistinguishable from GR except for the presence of an additional non-gravitational universal interaction (fifth-force) between the scalar and the remaining matter fields through the interaction term $\propto e^{-(\dots)\tilde{\phi}}\mathcal{L}_m$ in the action. When comparing these two different frames, as long as the physical laws are not invariant under the equivalence relationship, we are comparing two different theories with their own set of measurable quantities. Hence, it is natural to get different predictions for a given quantity when computed in terms of the measurable quantities of one or another frame. In this regard, looking for evidence on the non-equivalence of the different conformal frames reduces to looking for evidence in favor of one or the other framework [2].

[1] h is the determinant of the induced metric on $\partial\mathcal{M}$, K is the trace of the extrinsic curvature of the boundary. The derivation of the discussed boundary terms is rather technical and goes beyond the scope of this essay but can be found in [5].

Chapter 2

$f(R)$ Theories

Stability issues are central in the study of higher-order modifications of general relativity. In particular, the *Ostrogradsky instability*[1] constrains modifications of S_{EH} to a substitution $R \to f(R)$, from which this new kind of theories are named [2].

Including the matter degrees of freedom, the action for the $f(R)$ theories reads:

$$S_{f(R)} = \int d^4x\sqrt{-g}\,[f(R) + \mathcal{L}_m] \tag{2.1}$$

It may not be immediately clear that the above action is equivalent to a STT, but starting from (here we focus only on the gravitational part of the action)

$$S = \int d^4x\sqrt{-g}\,[f(\psi) + (\partial_\psi f)(R - \psi)] \tag{2.2}$$

one can show that this action is dynamically equivalent to both $S_{f(R)}$ and S_{BD}.

Proof. Using the equation of motion for the scalar field ψ: $(\partial^2_\psi)f(R-\psi) = 0$ one gets $R = \psi$, as in general $\partial^2_\psi f \neq 0$. Substituting this into (2.2), we obtain $S_{f(R)}$. Now define

$$\phi = \partial_\psi f \quad \text{and} \quad V(\phi) = \psi(\phi)\phi - f(\psi(\phi))$$

Inserting these into (2.2) gives S_{BD} with $\omega_{BD} = 0$:

$$S = \int d^4x\sqrt{-g}\,[\psi\phi - V(\phi) + \phi(R-\psi)] = \int d^4x\sqrt{-g}\,[\phi R + V(\phi)]$$

∎

[1] The Ostrogradski theorem states that for Lagrangians which depend on derivatives of the fields of order higher than the first one, the associated Hamiltonian is unbounded from below [2].

The equations of motion for $f(R)$ theories are obtained by varying (2.1) with respect to the metric tensor, which gives

$$R_{\mu\nu} - \frac{f(R)}{2\partial_R f} g_{\mu\nu} = \frac{1}{\partial_R f} \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box \right) \partial_R f + \frac{T^{(m)}_{\mu\nu}}{\partial_R f} \tag{2.3}$$

The derivation of this equation is carried out in an analogous fashion to the one of Brans–Dicke theory, again removing boundary terms coming from $\delta R_{\mu\nu}$.

2.1 Cosmic Expansion in $f(R)$ Theories

The main ingredient of the inflationary models is a scalar field, called inflaton, that enters Einstein's equations as an exotic matter field. In this section, we will show how the scalar field can be replaced by a modified gravity theory, in this case $f(R)$ theory, which can produce an accelerated cosmic expansion without the need for an unknown form of dark energy.

With flat FRW metric,

$$ds^2 = -dt^2 + a(t)^2 \left(dx^2 + dy^2 + dz^2 \right) \tag{2.4}$$

the 00 and ii equations of motion (2.3) can be written respectively as:

$$3H^2 = \frac{1}{f_{,R}} \left(\rho_m + \rho_{\text{eff}} \right) \tag{2.5}$$

$$2\dot{H} = -\frac{1}{f_{,R}} \left(\rho_m + p_m + \rho_{\text{eff}} + p_{\text{eff}} \right) \tag{2.6}$$

where ρ_m and p_m are the energy density and pressure of the matter fields, respectively, and ρ_{eff} and p_{eff} are the correspondent ones of the modified gravity sector.

$$\rho_{\text{eff}} = \tfrac{1}{2}(R f_{,R} - f) - 3H\dot{R} f_{,RR} \tag{2.7}$$

$$p_{\text{eff}} = f_{,RR} \left(\ddot{R} + 2H\dot{R} + \frac{f_{,RRR}}{f_{,RR}} \dot{R}^2 \right) + \tfrac{1}{2}(f - R f_{,R}) \tag{2.8}$$

For convenience, we defined $f_{,R} \equiv \partial_R f$, $f_{,RR} \equiv \partial_{RR} f$ and $f_{,RRR} \equiv \partial_{RRR} f$. The calculations to get (2.5) and (2.6) from (2.3) quite lengthy, but straightforward. We just clarify that (2.6) is obtained by taking $ii - 00$ components of (2.3).

For simplicity, we assume $\rho_m = p_m = 0$. Thus, combining (2.5) and (2.6), we get:

$$\frac{\ddot{a}}{a} = -\frac{1}{2f_{,R}}\left(\rho_{\text{eff}} + p_{\text{eff}}\right) + \frac{1}{3f_{,R}}\rho_{\text{eff}} = -\frac{1}{6f_{,R}}\left(\rho_{\text{eff}} + 3p_{\text{eff}}\right) \tag{2.9}$$

As $f_{,R} > 0$ in order to a have a positive gravitational coupling, we can have an accelerated expansion if $\rho_{\text{eff}} + 3p_{\text{eff}} < 0$, which is precisely the violation of the *strong energy condition*. This requirement can be satisfied by choosing a suitable form of $f(R)$[2], which shall respect

$$3f_{,RR}\big(\ddot{R} + H\dot{R}\big) + 3f_{,RRR}\dot{R}^2 + f - Rf_{,R} < 0 \tag{2.10}$$

It is also noteworthy that $f(R)$ theories can mimic cosmological constant models, as $\rho_{\text{eff}} = -p_{\text{eff}}$ holds for

$$f_{,RR}\big(\ddot{R} - H\dot{R}\big) + f_{,RRR}\dot{R}^2 = 0 \tag{2.11}$$

[2]Besides the already mentioned $f_{,R} > 0$ restriction, one also has $f_{,RR} > 0$ due to Dolgov-Kawasaki instability, i.e. the appearance of a tachyonic (negative-mass-squared) scalaron that would drive the Ricci curvature to grow explosively inside high-density matter [6].

CHAPTER 3

SPONTANEOUS SCALARIZATION

STTs are a generalization of the Brans-Dicke theory to allow the BD coupling to be a function of the scalar field [2]: $\omega_{BD} \to \omega(\phi)$. The resulting action is

$$S_{\text{ST}} = \int d^4x\,\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}(\partial\phi)^2 - 2V(\phi) + 2\mathcal{L}_m\right] \tag{3.1}$$

and it is simple to show that it is equivalent to

$$S_{\text{ST}} = \int d^4x\,\sqrt{-g}\left[f(\phi)\,R - \omega(\phi)(\partial\phi)^2 - 2V(\phi) + 2\mathcal{L}_m\right] \tag{3.2}$$

by a redefinition of $f(\phi) \to \phi$ and $\omega(\phi) \to \omega(\phi)f(\phi)/\left(\partial_\phi f\right)^2$. STTs can also be further generalized to include higher-order derivatives of the scalar field and thus many other models can be constructed.

As shown in Section 2.1, scalar-tensor frameworks are flexible enough to replace the cold-dark-matter fluid with either the scalar itself or a scalar-induced modification of gravity. However, in practical terms, the price is tight constraints from laboratory, Solar-System and cosmological observations. Therefore, every successful model must contain a screening or suppression mechanism in order to reduce to Einstein's GR in the low gravity regime.

Most screening mechanisms (e.g. chameleon or symmetron) rely on the local density of the environment: the scalar's effective mass increases in high-density regions, suppressing its fifth force there, but it remains light in low-density regions. In what follows we present instead a screening process triggered by gravity's strength only, the so called *spontaneous scalarization*[1]; which is currently the only known mechanism that

[1]The name comes from an analogy with the spontaneous magnetization of ferromagnets below the Curie temperature.

could make appear (in a strong gravity regime) fields that remain dormant at small curvature [7].

3.1 Tachyonic Instability

The standard way to trigger spontaneous scalarization is via a *tachyonic instability* at the linear level, which is eventually quenched due to the effect of nonlinear terms. Consider for instance a scalar ϕ^n theory with $n > 2$ in Minkowski space-time:

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\mu^2\phi^2 - \frac{1}{n}\lambda\phi^n \tag{3.3}$$

the field equation $\Box_\eta\phi - \mu^2\phi - \lambda\phi^{n-1} = 0$ is linearized considering small perturbations $\delta\phi$ around $\phi = 0$:

$$\Box_\eta\delta\phi - \mu^2\delta\phi = 0 \tag{3.4}$$

A tachyonic instability arises for imaginary bare mass with $\mu^2 < 0$ and $k^2 < |\mu^2|$, as the dispersion relation $\omega^2 = k^2 - |\mu^2|$ leads to

$$\delta\phi \sim e^{\pm i\mathbf{kx}}e^{\pm i\sqrt{k^2-|\mu^2|}t} \tag{3.5}$$

we see that perturbations with small wave number exhibit exponential growth.

However, as ϕ grows, the nonlinear self-interaction $\lambda\phi^n$ becomes dominant. Looking for a constant solution for the equation of motion of (3.3) we get $\phi_{min}^{n-2} = -\frac{\mu^2}{\lambda}$ (it corresponds to the minimum of the potential). Therefore, nonlinear interactions eventually quench the instability and drive the field to a different, stable configuration. The process is called *tachyonic condensation* and is associated to a phase transition of the system [7].

Notice that in BD theory we cannot have spontaneous scalarization as in the Klein Gordon equation (1.9) the coupling with matter is constant and thus does not survive in the linearized equation of motion (which in turn implies that no tachyonic instability can occur). Even with $V \neq 0$, thus having a term $V_{,\phi\phi}\delta\phi$ in the linearized equation, spontaneous scalarization cannot take place because for $V_{,\phi\phi} < 0$ the tachyonic instability is *global*, i.e. $\delta\phi$ grows indefinitely and tachyonic condensation never occurs.

3.2 Damour-Esposito-Farese Model

The (hystorically) first model for spontaneous scalarization is the Damour-Esposito-Farese (DEF) model [8], which is usually presented in the Einstein frame

$$S^{(\mathrm{E})}_{\mathrm{DEF}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R - 2\nabla_\mu\phi\nabla^\mu\phi) + S_m[\Psi_m; A^2(\phi)g^{(J)}_{\mu\nu}] \tag{3.6}$$

with $A^2(\phi)$ being the conformal factor that relates this frame with Jordan one (for notation clarity here we used $g_{\mu\nu}$ and $g^{(J)}_{\mu\nu}$ for Einstein/Jordan frame metric); in addition to a redefinition of the scalar field. Writing down the equations of motion we obtain [8]

$$G_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi - (\partial\phi)^2\, g_{\mu\nu} + 8\pi G T^{(m)}_{\mu\nu} \tag{3.7}$$

$$\Box\phi = 4\pi G\,\alpha(\phi)\,T^{(m)} \tag{3.8}$$

Where $\alpha(\phi) \equiv \frac{\partial \ln A(\phi)}{\partial\phi}$ and $T_{\mu\nu} \equiv -2/\sqrt{-g}\,\delta S_m/\delta g^{\mu\nu}$. Therefore, the possible tachyonic instability is governed by $\alpha(\phi)$, which can be expanded around $\phi = \phi_0$ as

$$\alpha(\phi) = \alpha_0 + \beta_0\,(\phi - \phi_0) + \textit{higher order terms} \tag{3.9}$$

producing a linearized Klein Gordon equation

$$\Box\delta\phi = 4\pi G\,\beta_0\,\delta\phi\,T^{(m)} \tag{3.10}$$

Usually stars have $T > 0$, thus for $\beta(\phi_0) < 0$ they can develop tachyonic instability. The higher order terms in (3.9) will eventually lead to tachyonic condensation.

This procedure can be applied also to a wide range of scalar tensor theories with an analogue Klein Gordon equation to (3.8). Considering BD theory, in 1.2 we found $\alpha = \textit{const}$; as for BD A is proportional to the scalar field of the Jordan frame, which can be obtained integrating (1.12). This confirms the result obtained in the previous section inspecting the equation of motion in the Jordan frame: BD theory does not exhibit spontaneous scalarization, with or without a potential V.

Appendix A

We derive here the full expression of $\delta\bar{S}$ used in *proof* 1.1.
In our conventions, the components of the Ricci tensor are defined as it follows

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\mu\nu}\Gamma^\tau_{\lambda\tau} - \Gamma^\tau_{\mu\lambda}\Gamma^\lambda_{\nu\tau} \tag{A.1}$$

and the Christoffel symbols are given by

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}\, g^{\sigma\lambda}\big(\partial_\nu g_{\mu\lambda} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\mu\nu}\big) \tag{A.2}$$

We can then compute the variation of R as

$$\begin{aligned} \delta R_{\mu\nu} &= \partial_\lambda \delta\Gamma^\lambda_{\mu\nu} - \partial_\nu \delta\Gamma^\lambda_{\mu\lambda} + \delta\big(\Gamma^\lambda_{\mu\nu}\big)\Gamma^\tau_{\lambda\tau} + \Gamma^\lambda_{\mu\nu}\delta\big(\Gamma^\tau_{\lambda\tau}\big) - \delta\big(\Gamma^\tau_{\mu\lambda}\big)\Gamma^\lambda_{\nu\tau} - \Gamma^\tau_{\mu\lambda}\delta\big(\Gamma^\lambda_{\nu\tau}\big) \\ &= \nabla_\lambda\big(\delta\Gamma^\lambda_{\mu\nu}\big) - \nabla_\nu\big(\delta\Gamma^\lambda_{\mu\lambda}\big) \end{aligned} \tag{A.3}$$

One can show the last equality by direct computation of the covariant derivative of the variations of the Christoffel symbols, noting that

$$\nabla_\nu\big(\delta\Gamma^\lambda_{\mu\lambda}\big) = \partial_\nu\big(\delta\Gamma^\lambda_{\mu\lambda}\big) + \cancel{\Gamma^\lambda_{\tau\nu}\delta\Gamma^\tau_{\mu\lambda}} - \Gamma^\tau_{\tau\nu}\delta\Gamma^\lambda_{\tau\lambda} - \cancel{\Gamma^\tau_{\lambda\nu}\delta\Gamma^\lambda_{\mu\tau}} \tag{A.4}$$

Therefore, we can rewrite $\delta\bar{S}$ as

$$\delta\bar{S} = \int_{\mathcal{M}} d^4x\,\sqrt{|g|}\,\phi\,g^{\mu\nu}\,\delta R_{\mu\nu} = \int_{\mathcal{M}} d^4x\,\sqrt{|g|}\,\phi\,g^{\mu\nu}\Big[\nabla_\lambda\big(\delta\Gamma^\lambda_{\mu\nu}\big) - \nabla_\nu\big(\delta\Gamma^\lambda_{\mu\lambda}\big)\Big] \tag{A.5}$$

Employing the chain rule,

$$\phi g^{\mu\nu}\nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}) = \nabla_\lambda\big(\phi g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\big) - (\nabla_\lambda\phi)\, g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} \tag{A.6}$$

$$\phi g^{\mu\nu}\nabla_\nu(\delta\Gamma^\lambda_{\mu\lambda}) = \nabla_\nu\big(\phi g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}\big) - (\nabla_\nu\phi)\, g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda} \tag{A.7}$$

and we can see that the first term of the RHS of (A.6) vanishes upon integration (analogue reasoning for (A.7)), due to the requirement of the stationary action principle:

$$\int_{\mathcal{M}} d^4x\,\sqrt{|g|}\,\nabla_\lambda\big(\phi g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\big) = \int_{\partial\mathcal{M}} d\sigma_\lambda\,\phi g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} = 0 \tag{A.8}$$

Using this result, equation (A.5) becomes

$$\delta\bar{S} = \int_{\mathcal{M}} d^4x\,\sqrt{|g|}\,\nabla_\lambda\phi\Big[g^{\mu\lambda}\delta\Gamma^\nu_{\mu\nu} - g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\Big] \tag{A.9}$$

By direct computation we can see that

$$\begin{aligned} g^{\mu\lambda}\,\delta\Gamma^\nu_{\mu\nu} - g^{\nu\mu}\,\delta\Gamma^\lambda_{\mu\nu} &= -\frac{1}{2}g^{\mu\lambda}g_{\nu\tau}\nabla_\mu\Big(\delta g^{\nu\tau}\Big) + \frac{1}{2}g^{\mu\nu}\Big[g_{\nu\tau}\nabla_\mu\Big(\delta g^{\lambda\tau}\Big) \\ &\quad + g_{\mu\tau}\nabla_\nu\Big(\delta g^{\lambda\tau}\Big) - g_{\mu\tau}g_{\mu\sigma}\nabla^\lambda\Big(\delta g^{\tau\sigma}\Big)\Big] \\ &= \nabla_\mu(\delta g^{\mu\lambda}) - g_{\mu\nu}\nabla^\lambda(\delta g^{\mu\nu}) \end{aligned} \tag{A.10}$$

Finally, we employ a similar chain rule trick as before to cancel out some boundary terms:

$$\nabla_\mu\big(\nabla_\nu\phi\,\delta g^{\mu\nu}\big) = \nabla_\mu\nabla_\nu\phi\,\delta g^{\mu\nu} + \nabla_\nu\phi\,\nabla_\mu(\delta g^{\mu\nu}) \tag{A.11}$$

$$\nabla^\lambda\big(\nabla_\lambda\phi\, g_{\mu\nu}\,\delta g^{\mu\nu}\big) = \Box\phi\, g_{\mu\nu}\,\delta g^{\mu\nu} + \nabla_\lambda\phi\, g_{\mu\nu}\,\nabla^\lambda(\delta g^{\mu\nu}) \tag{A.12}$$

And we are left with

$$\delta_g\bar{S} = \int_{\mathcal{M}} d^4x\,\sqrt{|g|}\,\delta g^{\mu\nu}\big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\big)\phi \tag{A.13}$$

References

[1] Matteo Braglia. "Scalar-tensor theories in light of cosmological tensions". PhD thesis. Bologna U., 2021.

[2] Israel Quiros. "Selected Topics in Scalar-Tensor Theories and Beyond". In: *International Journal of Modern Physics D* 28.07 (Jan. 2019), p. 1930012. ISSN: 0218-2718, 1793-6594. DOI: 10.1142/S021827181930012X. arXiv: 1901.08690 [gr-qc]. (Visited on 04/28/2025).

[3] Gabriele Gionti S. J and Matteo Galaverni. "On the Canonical Equivalence between Jordan and Einstein Frames". In: *Eur. Phys. J. C* 84.3 (Mar. 12, 2024), p. 265. DOI: 10.1140/epjc/s10052-024-12586-z. arXiv: 2310.09539 [gr-qc]. URL: http://arxiv.org/abs/2310.09539 (visited on 04/28/2025).

[4] Valerio Faraoni, Edgard Gunzig, and Pasquale Nardone. "Conformal Transformations in Classical Gravitational Theories and in Cosmology". In: 20 (1999), p. 121. DOI: 10.48550/arXiv.gr-qc/9811047. arXiv: gr-qc/9811047. (Visited on 04/28/2025).

[5] James E. Lidsey, David Wands, and E.J. Copeland. "Superstring cosmology". In: *Physics Reports* 337.4–5 (Oct. 2000), 343–492. ISSN: 0370-1573. DOI: 10.1016/s0370-1573(00)00064-8. URL: http://dx.doi.org/10.1016/S0370-1573(00)00064-8.

[6] A.D. Dolgov and M. Kawasaki. "Can modified gravity explain accelerated cosmic expansion?" In: *Physics Letters B* 573 (Oct. 2003), 1–4. ISSN: 0370-2693. DOI: 10.1016/j.physletb.2003.08.039. URL: http://dx.doi.org/10.1016/j.physletb.2003.08.039.

[7] Daniela D. Doneva et al. "Spontaneous scalarization". In: *Reviews of Modern Physics* 96.1 (Mar. 2024). ISSN: 1539-0756. DOI: 10.1103/revmodphys.96.015004. URL: http://dx.doi.org/10.1103/RevModPhys.96.015004.

[8] Gilles Esposito-Farese. "Nonperturbative strong field effects in tensor - scalar gravity". In: *28th Rencontres de Moriond: Perspectives in Neutrinos, Atomic Physics and Gravitation*. 1993, pp. 525–532.