

Scalar Tensor Theories of Gravity and Their Relevance in Cosmology

Lorenzo Fabbri

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Department of Physics
Theoretical Physics Curriculum
University of Bologna
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INTRODUCTION

General Relativity (GR) – the gravitational framework underlying the Λ CDM cosmological model – remains one of the most elegant and thoroughly tested theories in modern science. More than a century after Einstein introduced his field equations, GR still offers the most accurate description of both cosmological dynamics and astrophysical phenomena.

Despite these triumphs, there are good reasons to explore extensions or modifications of GR. The most important regarding cosmology is probably the unclear nature of dark matter, i.e. components that interact only gravitationally yet dominate the energy budget in Λ CDM. It is natural to ask whether altered gravitational laws on galactic or cosmological scales might account for this dark sector. Furthermore, deviations from GR in strong-gravity or quantum-gravity regimes could remove the Big-Bang singularity or those inside black holes [1].

An historical milestone in this direction came from Dirac’s proposal that fundamental constants may vary with time; a notion later formalized in the Brans–Dicke (BD) theory, where Newton’s constant becomes a dynamical quantity controlled by a time-dependent scalar field. BD theory is the prototype of modern scalar–tensor theories (STT) of gravity, which are the focus of this essay.

Chapter 1 introduces the basics of BD theory, along with a discussion clarifying the relations between Jordan and Einstein frames.

Chapter 2 concerns $f(R)$ theories, their equivalence with STTs and their ability to generate accelerated cosmic expansion without the need for exotic matter or dark energy. The essay concludes with few words on the current status of STTs and related open problems.

In the whole essay the metric signature is $(-, +, +, +)$. We use natural units with $c = 1$ and $\hbar = 1$. The Einstein gravitational constant is denoted by G .

CHAPTER 1

BRANS–DICKE THEORY

In STTs, gravitational phenomena are partly due to the curvature of spacetime and partly due to the scalar field that sets the strength of the gravitational interactions at each point in spacetime.

We make this statement more clear introducing the simplest STT, the Brans-Dicke theory (BD). The action of the BD theory is given by

$$S_{BD} = \int d^4x \sqrt{-g} \left(\phi R - \frac{\omega_{BD}}{\phi} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - 2V(\phi) + 2\mathcal{L}_m(\psi, \partial\psi, g_{\mu\nu}) \right) \quad (1.1)$$

where ϕ is the BD scalar field, $V(\phi)$ is the self-interaction potential for ϕ and ω_{BD} is a free constant called the *BD parameter*. It is convenient to define the Lagrangian density for the scalar field ϕ : $\mathcal{L}_\phi \equiv -\frac{\omega_{BD}}{\phi} (\partial\phi)^2 - 2V(\phi)$.

Let us compare S_{BD} with the Einstein-Hilbert action complemented with a matter piece in the form of a self-interacting scalar field φ .

$$S_{EH} = \int d^4x \sqrt{-g} \left(\frac{1}{8\pi G} R + 2\mathcal{L}_\varphi \right) \quad (1.2)$$

with $\mathcal{L}_\varphi = -(\partial\varphi)^2 - 2V(\varphi)$. This is simply Einstein's GR with a matter piece in the form of a perfect fluid [2]. We can see that in (1.1) ϕ plays the role of the gravitational coupling constant, which is now dynamical:

$$\phi(x) = \frac{1}{8\pi G(x)} \quad (1.3)$$

1.1 Brans-Dicke Dynamics

By varying S_{BD} with respect to the metric we obtain the Einstein-Brans-Dicke equation of motion:

$$G_{\mu\nu} = \frac{1}{\phi} [T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(m)}] + \frac{1}{\phi} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \square) \phi \quad (1.4)$$

with the energy-momentum tensors (EMT) defined as:

$$T_{\mu\nu}^{(\phi)} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_\phi)}{\delta g^{\mu\nu}} = \frac{\omega_{BD}}{\phi} \left[\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2 \right] - g_{\mu\nu} V(\phi) \quad (1.5)$$

$$T_{\mu\nu}^{(m)} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}} \quad (1.6)$$

Proof. Let us split the action S_{BD} in two parts: $S_{BD} = S^{(\text{grav})} + S^{(m)}$, where

$$S^{(\text{grav})} \equiv \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega_{BD}}{\phi} (\partial \phi)^2 - 2V(\phi) \right], \quad S^{(m)} \equiv 2 \int d^4x \sqrt{-g} \mathcal{L}_m$$

The interesting part of the variation is the first term, which we can write as

$$\frac{1}{\sqrt{-g}} \frac{\delta S^{(\text{grav})}}{\delta g^{\mu\nu}} = \phi \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right] + \frac{\omega_{BD}}{2\phi} g_{\mu\nu} (\nabla \phi)^2 - \frac{\omega_{BD}}{\phi} \nabla_\mu \phi \nabla_\nu \phi + g_{\mu\nu} V + \frac{1}{\sqrt{-g}} \frac{\delta \bar{S}}{\delta g^{\mu\nu}}$$

One can easily recognize the Einstein tensor $G_{\mu\nu}$ and the EMT for ϕ , while the last term is highly non trivial.

$$\delta \bar{S} = \int_{\mathcal{M}} d^4x \sqrt{|g|} \phi g^{\mu\nu} \delta R_{\mu\nu}$$

To compute this variation we employ the typical requirement that any fields variations, as well as variations of their first derivatives, vanish on the integration boundary $\partial \mathcal{M}$; this, along with some algebra which can be found in [Appendix A](#), leads to

$$\frac{\delta \bar{S}}{\delta g^{\mu\nu}} = (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu \phi) \phi$$

and thus completes the proof. ■

Action S_{BD} can be also varied with respect to the scalar field ϕ to get the Klein-Gordon equations for Brans-Dicke theory:

$$2\omega_{BD}\frac{\square\phi}{\phi} - \omega_{BD}\left(\frac{\partial\phi}{\phi}\right)^2 + R = 2V_{,\phi} \quad (1.7)$$

Proof. It is easy to see that $\delta_\phi S_{BD}$ can be cast as

$$\delta_\phi S_{BD} = \int_{\mathcal{M}} d^4x \sqrt{|g|} \left[\delta\phi R - 2\frac{\omega_{BD}}{\phi} \nabla^\mu \phi \nabla_\mu (\delta\phi) + \omega_{BD} \left(\frac{\partial\phi}{\phi}\right)^2 \delta\phi - 2V_{,\phi} \delta\phi \right]$$

Now notice that we can construct a total derivative term

$$\nabla_\mu \left(\frac{\nabla^\mu \phi}{\phi} \delta\phi \right) = \frac{\square\phi}{\phi} \delta\phi - \left(\frac{\partial\phi}{\phi}\right)^2 \delta\phi + \frac{\nabla^\mu \phi}{\phi} \nabla_\mu (\delta\phi)$$

which vanishes at the boundary of the integration domain. This allows us to substitute directly inside $\delta_\phi S_{BD}$ the following expression:

$$-2\frac{\omega_{BD}}{\phi} \nabla^\mu \phi \nabla_\mu (\delta\phi) = 2\omega_{BD} \left[\frac{\square\phi}{\phi} \delta\phi - \left(\frac{\partial\phi}{\phi}\right)^2 \delta\phi \right]$$

Doing this gives us

$$\delta_\phi S_{BD} = \int_{\mathcal{M}} d^4x \sqrt{|g|} \delta\phi \left[R + 2\omega_{BD}\frac{\square\phi}{\phi} - \omega_{BD}\left(\frac{\partial\phi}{\phi}\right)^2 - 2V_{,\phi} \right]$$

From which equation (1.7) follows after imposing $\delta_\phi S_{BD}/\delta\phi = 0$. ■

It can be shown [2] that combining the equations of motions (1.4) and (1.7) gives the covariant conservation of the EMT for matter:

$$\nabla^\nu T_{\mu\nu}^{(m)} = 0 \quad (1.8)$$

meaning that the matter fields follow geodesics of the metric while the scalar field ϕ just modulates the strength of the interaction through equation (1.3).

1.2 Conformal Transformations

The action for Brans–Dicke theory in (1.1) is said to be given in the *Jordan frame*. However, this formulation is not unique as one can always perform a conformal trans-

formation on the metric, moving to a different frame; usually to facilitate calculations or to get more physical insights from a certain theory. Hence, it is important to try to clarify the relationships between different frames and discuss the two most common ones.

Invariance of physics under re-definition of units of measurements implies invariance under Weyl transformations of the metric. Usually, the frame with the original metric tensor $g_{\mu\nu}$ is the abovementioned *Jordan frame* (JF) while the frame obtained after a conformal transformation is the *Einstein frame* (EF) [3].

After a Weyl transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, in 4 dimensions, we have the following rescalings [4]:

$$\sqrt{-\tilde{g}} = \Omega^4 \sqrt{-g} \quad , \quad \tilde{R} = \Omega^{-2} \left[R - 6 \frac{\square \Omega}{\Omega} \right] \quad (1.9)$$

Considering a conformal transformation in which $\Omega(x) \sim \phi^n(x)$, we would have that a term $\sqrt{-g}\phi R$ becomes $\sim \sqrt{-\tilde{g}}\phi^{-4n}\tilde{R}\phi^{2n} + \dots = \sqrt{-\tilde{g}}\phi^{-2n+1}\tilde{R} + \dots$ so we can break the coupling between R and ϕ choosing $\Omega^2(x) \sim \phi(x)$.

For instance in Brans–Dicke theory in JF the choice of $\Omega = \sqrt{16\pi G\phi}$ leads to

$$S_{BD}^{(E)} = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{\tilde{R}}{16\pi G} - \frac{1}{2}(\tilde{\partial}\tilde{\phi})^2 + \frac{e^{-8\sqrt{\frac{\pi G}{2\omega+3}}\tilde{\phi}}}{(16\pi G)^2} \left(-2V(\tilde{\phi}) + 2\mathcal{L}_m(\tilde{g}) \right) \right\} \quad (1.10)$$

Notice that to obtain the correct form of the kinetic term in Einstein frame we have also performed a redefinition of the field:

$$d\tilde{\phi} = \sqrt{\frac{2\omega+3}{16\pi G}} \frac{d\phi}{\phi} \quad (1.11)$$

Proof. The Ricci tensor in JF can be written using (1.9) as

$$R = \Omega^2 \tilde{R}_{\mu\nu} + 6 \frac{\square \Omega}{\Omega} = 16\pi G \phi \tilde{R}_{\mu\nu} + \sqrt{\phi} \frac{16\pi G}{2} \left[\frac{\square \phi}{\sqrt{\phi}} - \frac{(\tilde{\partial}\phi)^2}{2\phi^{3/2}} \right]$$

Using this expression we can rewrite the action in EF as

$$S_{BD}^{(E)} = \int d^4x \frac{\sqrt{-\tilde{g}}}{16\pi G} \left\{ \tilde{R} + 3 \left[\frac{\square \phi}{\phi} - \frac{(\tilde{\partial}\phi)^2}{2\phi^2} \right] - \frac{\omega_{BD}}{\phi^2} (\tilde{\partial}\phi)^2 - \frac{2V(\phi)}{16\pi G \phi^2} + \frac{2\mathcal{L}_m}{16\pi G \phi^2} \right\}$$

Then, we transform ϕ in order to obtain the correct form of the kinetic term, i.e. we ask

$$-\frac{3}{2} \frac{1}{16\pi G} \frac{(\tilde{\partial}\phi)^2}{\phi^2} - \frac{2\omega_{BD}}{16\pi G} \frac{(\tilde{\partial}\phi)^2}{\phi^2} \stackrel{!}{=} -\frac{1}{2}(\tilde{\partial}\tilde{\phi})^2$$

This induces the transformation (1.11) for the field's derivatives, which can be integrated to obtain the transformation for ϕ :

$$\phi = \exp \left\{ \sqrt{\frac{16\pi G}{2\omega_{BD} + 3}} \tilde{\phi} \right\}$$

Inserting this in the transformed action reproduces (1.10).

The $(\tilde{\square}\phi)/\phi$ term is usually omitted because it is compensated by the inclusion of boundary terms in $S_{BD}^{(E)}$ [2]. After this cancellation, the final EF action contains the standard Gibbs-Hawking-York boundary term¹: $2 \int_{\partial\mathcal{M}} d^3x \sqrt{|\tilde{h}|} K$. ■

While both formulations of the BD theory discussed above are in a relationship of mathematical equivalence, their physical equivalence may be, at least, questionable. In fact, while the JF Brans–Dicke theory (1.1) is a STT of gravity in the sense that the gravitational interactions are carried by the metric field of geometric origin, together with the non-geometric BD scalar field, the EFBD theory (1.10) is a purely geometric theory of gravity indistinguishable from GR except for the presence of an additional non-gravitational universal interaction (fifth-force) between the scalar and the remaining matter fields through the interaction term $\propto e^{-(\dots)\tilde{\phi}} \mathcal{L}_m$ in the action. When one compares two different frames, even when these are related by a mathematical relationship of equivalence, as long as the physical laws are not invariant under the equivalence relationship, what one is comparing is two different theories with their own set of measurable quantities. Hence, it is natural to get different predictions for a given quantity when computed in terms of the measurable quantities of one or another frame. In this regard, looking for evidence on the non-equivalence of the different conformal frames reduces to looking for evidence in favor of one or the other framework. A different thing is to search for a physical conformal frame among the conformally related ones. This would be a task inevitably doomed to failure. Actually, if the conformally related frames are physically equivalent, then all (or none) of them are physical [2].

¹ h is the determinant of the induced metric on $\partial\mathcal{M}$, K is the trace of the extrinsic curvature of the boundary. The derivation of the discussed boundary terms is rather technical and goes beyond the scope of this essay but can be found in [5].

CHAPTER 2

$f(R)$ THEORIES

Stability issues are central in the study of higher-order modifications of general relativity. In particular, the *Ostrogradsky instability*¹ constrains modifications of S_{EH} to a substitution $R \rightarrow f(R)$, from which this new kind of theories are named [2].

Including the matter degrees of freedom, the action for the $f(R)$ theories reads:

$$S_{f(R)} = \int d^4x \sqrt{-g} [f(R) + \mathcal{L}_m] \quad (2.1)$$

It may not be immediately clear that the above action is equivalent to a STT, but starting from (here we focus only on the gravitational part of the action)

$$S = \int d^4x \sqrt{-g} [f(\psi) + (\partial_\psi f)(R - \psi)] \quad (2.2)$$

one can show that this action is dynamically equivalent to both $S_{f(R)}$ and S_{BD} .

Proof. Using the equation of motion for the scalar field ψ : $(\partial_\psi^2)f(R - \psi) = 0$ one gets $R = \psi$, as in general $\partial_\psi^2 f \neq 0$. Substituting this into (2.2), we obtain $S_{f(R)}$. Now define

$$\phi = \partial_\psi f \quad \text{and} \quad V(\phi) = \psi(\phi)\phi - f(\psi(\phi))$$

Inserting these into (2.2) gives S_{BD} with $\omega_{BD} = 0$:

$$S = \int d^4x \sqrt{-g} [\psi\phi - V(\phi) + \phi(R - \psi)] = \int d^4x \sqrt{-g} [\phi R + V(\phi)]$$

■

¹The Ostrogradski theorem states that for Lagrangians which depend on derivatives of the fields of order higher than the first one, the associated Hamiltonian is unbounded from below [2].

The equations of motion for $f(R)$ theories are obtained by varying (2.1) with respect to the metric tensor, which gives

$$R_{\mu\nu} - \frac{f(R)}{2\partial_R f} g_{\mu\nu} = \frac{1}{\partial_R f} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) \partial_R f + \frac{T_{\mu\nu}^{(m)}}{\partial_R f} \quad (2.3)$$

The derivation of this equation is carried out in an analogous fashion to the one of Brans–Dicke theory, again removing boundary terms coming from $\delta R_{\mu\nu}$.

2.1 Cosmic Expansion in $f(R)$ Theories

The main ingredient of the inflationary models is a scalar field, called inflaton, that enters Einstein's equations as an exotic matter field. In this section, we will show how the scalar field can be replaced by a modified gravity theory, in this case $f(R)$ theory, which can produce an accelerated cosmic expansion without the need for an unknown form of dark energy or anti-gravitational matter.

With flat FRW metric,

$$ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2) \quad (2.4)$$

the 00 and ii equations of motion (2.3) can be written respectively as:

$$3H^2 = \frac{1}{f_{,R}} (\rho_m + \rho_{\text{eff}}) \quad (2.5)$$

$$2\dot{H} = -\frac{1}{f_{,R}} (\rho_m + p_m + \rho_{\text{eff}} + p_{\text{eff}}) \quad (2.6)$$

where ρ_m and p_m are the energy density and pressure of the matter fields, respectively, and ρ_{eff} and p_{eff} are the correspondent ones of the modified gravity sector.

$$\rho_{\text{eff}} = \frac{1}{2}(Rf_{,R} - f) - 3H\dot{R}f_{,RR} \quad (2.7)$$

$$p_{\text{eff}} = f_{,RR} \left(\ddot{R} + 2H\dot{R} + \frac{f_{,RRR}}{f_{,RR}} \dot{R}^2 \right) + \frac{1}{2}(f - Rf_{,R}) \quad (2.8)$$

For convenience, we defined $f_{,R} \equiv \partial_R f$, $f_{,RR} \equiv \partial_{RR} f$ and $f_{,RRR} \equiv \partial_{RRR} f$. The calculations to get (2.5) and (2.6) from (2.3) are quite lengthy, but straightforward. We just clarify that (2.6) is obtained by taking $ii - 00$ components of (2.3).

2.1 Cosmic Expansion in $f(R)$ Theories

For simplicity, we assume $\rho_m = p_m = 0$. Thus, combining (2.5) and (2.6), we get:

$$\frac{\ddot{a}}{a} = -\frac{1}{2f_{,R}}(\rho_{\text{eff}} + p_{\text{eff}}) + \frac{1}{3f_{,R}}\rho_{\text{eff}} = -\frac{1}{6f_{,R}}(\rho_{\text{eff}} + 3p_{\text{eff}}) \quad (2.9)$$

As $f_{,R} > 0$ in order to have a positive gravitational coupling, we can have an accelerated expansion if $\rho_{\text{eff}} + 3p_{\text{eff}} < 0$, which is precisely the violation of the *strong energy condition*. This requirement can be satisfied by choosing a suitable form of $f(R)$ ², which shall respect

$$3f_{,RR}(\ddot{R} + H\dot{R}) + 3f_{,RRR}\dot{R}^2 + f - Rf_{,R} < 0 \quad (2.10)$$

It is also noteworthy that $f(R)$ theories can mimic cosmological constant models, as $\rho_{\text{eff}} = -p_{\text{eff}}$ holds for

$$f_{,RR}(\ddot{R} - H\dot{R}) + f_{,RRR}\dot{R}^2 = 0 \quad (2.11)$$

²Besides the already mentioned $f_{,R} > 0$ restriction, one also has $f_{,RR} > 0$ due to Dolgov-Kawasaki instability, i.e. the appearance of a tachyonic (negative-mass-squared) scalaron that would drive the Ricci curvature to grow explosively inside high-density matter [6].

CONCLUSIONS

STTs are a generalization of the Brans-Dicke theory to allow the BD coupling to be a function of the scalar field [2]: $\omega_{BD} \rightarrow \omega(\phi)$. The resulting action is

$$S_{\text{ST}} = \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} (\partial\phi)^2 - 2V(\phi) + 2\mathcal{L}_m \right] \quad (2.12)$$

and it is simple to show that it is equivalent to

$$S_{\text{ST}} = \int d^4x \sqrt{-g} \left[f(\phi) R - \omega(\phi) (\partial\phi)^2 - 2V(\phi) + 2\mathcal{L}_m \right] \quad (2.13)$$

by a redefinition of $f(\phi) \rightarrow \phi$ and $\omega(\phi) \rightarrow \omega(\phi)f(\phi)/(\partial_\phi f)^2$. STTs can also be further generalized to include higher-order derivatives of the scalar field and thus many other models can be constructed.

As shown in Section 2.1, scalar-tensor frameworks are flexible enough to replace the cold-dark-matter fluid with either the scalar itself or a scalar-induced modification of gravity. However, in practical terms, the price is tight constraints from laboratory, Solar-System and cosmological observations. Therefore, every successful model must contain a screening or suppression mechanism in order to reduce to Einstein's GR in the low gravity regime.

APPENDIX A

We derive here the full expression of $\delta\bar{S}$ used in *proof 1.1*.
In our conventions, the components of the Ricci tensor are defined as it follows

$$R_{\mu\nu} = \partial_\lambda \Gamma_{\mu\nu}^\lambda - \partial_\nu \Gamma_{\mu\lambda}^\lambda + \Gamma_{\mu\nu}^\lambda \Gamma_{\lambda\tau}^\tau - \Gamma_{\mu\lambda}^\tau \Gamma_{\nu\tau}^\lambda \quad (\text{A.1})$$

and the Christoffel symbols are given by

$$\Gamma_{\mu\nu}^\sigma = \frac{1}{2} g^{\sigma\lambda} (\partial_\nu g_{\mu\lambda} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\mu\nu}) \quad (\text{A.2})$$

We can then compute the variation of R as

$$\begin{aligned} \delta R_{\mu\nu} &= \partial_\lambda \delta \Gamma_{\mu\nu}^\lambda - \partial_\nu \delta \Gamma_{\mu\lambda}^\lambda + \delta(\Gamma_{\mu\nu}^\lambda) \Gamma_{\lambda\tau}^\tau + \Gamma_{\mu\nu}^\lambda \delta(\Gamma_{\lambda\tau}^\tau) - \delta(\Gamma_{\mu\lambda}^\tau) \Gamma_{\nu\tau}^\lambda - \Gamma_{\mu\lambda}^\tau \delta(\Gamma_{\nu\tau}^\lambda) \\ &= \nabla_\lambda (\delta \Gamma_{\mu\nu}^\lambda) - \nabla_\nu (\delta \Gamma_{\mu\lambda}^\lambda) \end{aligned} \quad (\text{A.3})$$

One can show the last equality by direct computation of the covariant derivative of the variations of the Christoffel symbols, noting that

$$\nabla_\nu (\delta \Gamma_{\mu\lambda}^\lambda) = \partial_\nu (\delta \Gamma_{\mu\lambda}^\lambda) + \cancel{\Gamma_{\tau\nu}^\lambda \delta \Gamma_{\mu\lambda}^\tau} - \Gamma_{\tau\nu}^\tau \delta \Gamma_{\mu\lambda}^\lambda - \cancel{\Gamma_{\lambda\nu}^\tau \delta \Gamma_{\mu\tau}^\lambda} \quad (\text{A.4})$$

Therefore, we can rewrite $\delta\bar{S}$ as

$$\delta\bar{S} = \int_{\mathcal{M}} d^4x \sqrt{|g|} \phi g^{\mu\nu} \delta R_{\mu\nu} = \int_{\mathcal{M}} d^4x \sqrt{|g|} \phi g^{\mu\nu} [\nabla_\lambda (\delta \Gamma_{\mu\nu}^\lambda) - \nabla_\nu (\delta \Gamma_{\mu\lambda}^\lambda)] \quad (\text{A.5})$$

Employing the chain rule,

$$\phi g^{\mu\nu} \nabla_\lambda (\delta \Gamma_{\mu\nu}^\lambda) = \nabla_\lambda (\phi g^{\mu\nu} \delta \Gamma_{\mu\nu}^\lambda) - (\nabla_\lambda \phi) g^{\mu\nu} \delta \Gamma_{\mu\nu}^\lambda \quad (\text{A.6})$$

$$\phi g^{\mu\nu} \nabla_\nu (\delta \Gamma_{\mu\lambda}^\lambda) = \nabla_\nu (\phi g^{\mu\nu} \delta \Gamma_{\mu\lambda}^\lambda) - (\nabla_\nu \phi) g^{\mu\nu} \delta \Gamma_{\mu\lambda}^\lambda \quad (\text{A.7})$$

and we can see that the first term of the RHS of (A.6) vanishes upon integration (analogue reasoning for (A.7)), due to the requirement of the stationary action principle:

$$\int_{\mathcal{M}} d^4x \sqrt{|g|} \nabla_\lambda (\phi g^{\mu\nu} \delta \Gamma_{\mu\nu}^\lambda) = \int_{\partial\mathcal{M}} d\sigma_\lambda \phi g^{\mu\nu} \delta \Gamma_{\mu\nu}^\lambda = 0 \quad (\text{A.8})$$

Using this result, equation (A.5) becomes

$$\delta \bar{S} = \int_{\mathcal{M}} d^4x \sqrt{|g|} \nabla_\lambda \phi \left[g^{\mu\lambda} \delta \Gamma_{\mu\nu}^\nu - g^{\mu\nu} \delta \Gamma_{\mu\nu}^\lambda \right] \quad (\text{A.9})$$

By direct computation we can see that

$$\begin{aligned} g^{\mu\lambda} \delta \Gamma_{\mu\nu}^\nu - g^{\nu\mu} \delta \Gamma_{\mu\nu}^\lambda &= -\frac{1}{2} g^{\mu\lambda} g_{\nu\tau} \nabla_\mu (\delta g^{\nu\tau}) + \frac{1}{2} g^{\mu\nu} \left[g_{\nu\tau} \nabla_\mu (\delta g^{\lambda\tau}) \right. \\ &\quad \left. + g_{\mu\tau} \nabla_\nu (\delta g^{\lambda\tau}) - g_{\mu\tau} g_{\mu\sigma} \nabla^\lambda (\delta g^{\tau\sigma}) \right] \\ &= \nabla_\mu (\delta g^{\mu\lambda}) - g_{\mu\nu} \nabla^\lambda (\delta g^{\mu\nu}) \end{aligned} \quad (\text{A.10})$$

Finally, we employ a similar chain rule trick as before to cancel out some boundary terms:

$$\nabla_\mu (\nabla_\nu \phi \delta g^{\mu\nu}) = \nabla_\mu \nabla_\nu \phi \delta g^{\mu\nu} + \nabla_\nu \phi \nabla_\mu (\delta g^{\mu\nu}) \quad (\text{A.11})$$

$$\nabla^\lambda (\nabla_\lambda \phi g_{\mu\nu} \delta g^{\mu\nu}) = \square \phi g_{\mu\nu} \delta g^{\mu\nu} + \nabla_\lambda \phi g_{\mu\nu} \nabla^\lambda (\delta g^{\mu\nu}) \quad (\text{A.12})$$

And we are left with

$$\delta_g \bar{S} = \int_{\mathcal{M}} d^4x \sqrt{|g|} \delta g^{\mu\nu} (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) \phi \quad (\text{A.13})$$

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