Discrete Ricci Flow applied to Facebook Ego Network

Subtitle

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Abstract

This project uses Ricci Flow as a geometric approach to detect community structures in networks. It applies Ollivier-Ricci curvature to adjust the weights of edges in a graph, iterating the process to shrink intra-community edges and stretch inter-community edges. Planar graphs with different community structures are used as datasets, with the goal of identifying pre-labelled communities using the Ricci Flow method.

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Chapter 1

Overview of the Project

The study of networks has gained considerable attention in various fields, ranging from sociology to biology, and beyond. One of the central problems in network science is community detection, where the objective is to identify groups of nodes (communities) that are more densely connected internally than with the rest of the network. Traditional methods for community detection often rely on statistical or combinatorial approaches. However, recent developments in geometric methods have introduced new ways to approach this problem by leveraging concepts from differential geometry [7].

A powerful geometric tool, the Ricci Flow, originally developed in the context of smooth Riemannian manifolds, can be adapted to discrete network structures. In its original formulation, the Ricci Flow evolves the metric of a manifold according to the curvature (represented by Ricci tensor), leading to a smoothing process over time. Ollivier-Ricci curvature, a discretization of Ricci curvature for graphs, provides a framework to extend this idea to networks, where the "curvature" of edges encodes structural information about node connectivity. Specifically, positive curvature tends to shrink intra-community edges, while negative curvature expands inter-community edges [7].

In this project, we explore the application of Ricci Flow in community detection by focusing on planar graphs, i.e. graphs that can be drawn on a plane without any edges crossing each other. By applying Ollivier-Ricci curvature to this network, we aim to assign and iteratively modify the curvature of edges, enhancing intra-community cohesion while stretching inter-community connections. Starting from graphs in which all the edges have weights set to 1, the flow will introduce weight diversity based on the topological characteristics of the graph, allowing observation of the natural curvature-driven "deformation" of the network.

The results will be compared to the predefined community labels, allowing for a direct comparison between the detected communities and the actual community structures.

The developed code can be accessed in the corresponding GitHub repository: *RicciFlowNetwork*.

CHAPTER 2

INTRODUCTION AND MOTIVATION

2.1 Poincaré Conjecture

To understand the origins of Ricci Flow and its relevance in the mathematical context, one has to get an insight on its application on geometrical and topological problems. Therefore, to begin with the discussion, we'll introduce the Poincaré conjecture and Perelman's proof.

In the 1985 paper *Analysis Situs* [8], Poincaré laid the foundations for what we call today as *topology*. Its purpose is clearly underlined in the articles, through:

> *...geometry is the art of reasoning well from badly drawn figures; however, these figures, if they are not to deceive us, must satisfy certain conditions; the proportions may be grossly altered, but the relative positions of the different parts must not be upset.*

Since there's no reason to follow the historical development, we'll furnish the modern definitions and intuition about topology.

Definition 1 (Topological space)**.** Let X be a non-empty set. A *topology* on X is a family $\mathcal{A}$ of subsets $\mathcal{A} \ni A \subseteq X$, called *open sets*, such that

- The empty set and X are open sets,

$$\emptyset \in \mathcal{A}, \quad X \in \mathcal{A}. \tag{2.1}$$

- The union of any collection of open sets is again an open set,

$$A_i \in \mathcal{A}, i \in J \implies \bigcup_{i \in J} A_i \in \mathcal{A}, \tag{2.2}$$

with J a set of indices.

- The intersection of any finite number of open sets is again an open set,

$$A_i \in \mathcal{A}, i = 1, 2 \ldots, N \implies \bigcap_{i=1}^{N} A_i \in \mathcal{A}. \tag{2.3}$$

The set X, with the topology $\mathcal{A}$, is called *topological space* $(X, \mathcal{A})$.

We can then define straightforwardly what a closed set is. Notice that we could have defined the topology with closed sets as well.

Definition 2 (Closed sets)**.** Given $A \in \mathcal{A}$, a subset $C \subseteq X$ is *closed* if is of the form

$$C \text{ closed } \iff X \setminus C \text{ open}. \tag{2.4}$$

There are some crucial properties of topological spaces which will be extensively used throughout this works. The first, concerns compactness, which is a subtle concept to identify finiteness in this context.

Definition 3 (Compact)**.** A topological space $(X, \mathcal{A})$ is called *compact* is every open cover,

$$X \in \bigcup_{\alpha \in C} U_\alpha, \tag{2.5}$$

contains a finite subcover,

$$X \in \bigcup_{i=1}^{N} U_i. \tag{2.6}$$

Further, another characterizing property of some topological spaces is the connectedness. The intuition is that a connect space can't be cut up into parts that have nothing to do with each other. To make this sentence more precise, we need first to define a path on a topological space.

Definition 4 (Path)**.** Given a topological space $(X, \mathcal{A})$ and two points $x, y \in X$, a *path* is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. A *path-component* of X is an equivalence class of X under the equivalence relations which makes x equivalent to y if and only if there is a path from x to y

Definition 5 (Connected)**.** A topological space $(X, \mathcal{A})$ is said to be *connected* if it can't be represented as the union of two or more disjoint non-empty open subsets.

Definition 6 (Path connected)**.** A topological space $(X, \mathcal{A})$ is said to be *path connected* if there is exactly one path-component. For non-empty spaces, this is equivalent to the statement that there is a path joining any two points in X.

Definition 7 (Simply connected)**.** A topological space $(X, \mathcal{A})$ is said to be *simply connected* if it is path connected and every path between two points can be continuously transformed into any other such path while preserving the two endpoints in question.

Other than these properties, we need maps from a topological space to another, both for relating these abstract concepts to the Euclidean space, for which we have intuition, and for studying the algebraic properties of the structures. This gives rise to the necessity of defining a homeomorphism.

Definition 8 (Homeomorphism)**.** A *homeomorphism*, also known as *topological isomorphism*, is a bijective and continuous between topological spaces that has a continuous inverse function.

We should have defined what a continuous map is, but the definition is not too different from the one used in multivariate calculus.

In particular, we're interested in applying those properties to some particular types of topological spaces, that is, topological manifolds. To apply the Ricci Flow to a topological context, the manifold must be further equipped with a wider structure, i.e., must be a Riemannian Manifold. While this topic will be covered in the next section, for now let's just define a Manifold.

Definition 9 (Manifold)**.** A *topological manifold* is a topological space which locally resembles the real n-dimensional Euclidean space.

Without delving into the details of the Manifolds with boundaries, we make use of our intuition to understand this concept, and directly define a closed manifold.

Definition 10 (Closed manifold)**.** A *closed manifold* is a manifold without boundary that is also compact.

A simple example is provided by the circle, for one-dimensional manifolds, and the sphere for two-dimensional ones. We're interested in the 3-sphere, which is a straightforward generalization of the previous examples.

Definition 11 (3-sphere)**.** In coordinates[1], a 3-sphere with center (C_0, C_1, C_2, C_3) and radius r is the set of all points (x_0, x_1, x_2, x_3) in $\mathbb{R}^4$ such that

$$\sum_{i=0}^{3}(x_i - C_i)^2 = (x_0 - C_0)^2 + (x_1 - C_1)^2 + (x_2 - C_2)^2 + (x_3 - C_3)^2 = r^2. \tag{2.7}$$

[1] We'll define a chart on a manifold later on, for now let's use our intuitive understanding of a set of coordinates in $\mathbb{R}^n$.

In particular, the 3-sphere centred at the origin with radius 1 is called the *unit* 3-*sphere* and is usually denoted ad S^3

$$S^3 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}. \tag{2.8}$$

As a Manifold, the 3-sphere is a compact, connected, 3-dimensional Manifold without boundary. We're now able to understand the formulation of the Poincaré conjecture in its work:

A simply-connected closed manifold is homeomorphic to a sphere.

To be fair, at the beginning it didn't think of it as a conjecture, since it considered it a trivial statement. It turned out to be one of the biggest unresolved problems in mathematics. After some refinements during the following years cite, he came up to the final form of its conjecture [9]:

Every three-dimensional topological manifold which is closed, connected, and has trivial fundamental group is homeomorphic to the three-dimensional sphere.

To overcome the complexity of fundamental groups and homologies, we'll just quote the following theorem, which allows us to understand the necessity of the presence of the trivial fundamental group in the latter formulation of the conjecture.

Theorem T.1. *A path-connected topological space is simply connected if and only if its fundamental group is trivial.*

To understand how to tackle the problem of Poincaré conjecture proof, we first need to investigate the manifold structure of some particular topological spaces.

2.2 Basics of Differential Geometry

In order to understand the Ricci Flow applications, it's necessary to study the properties of Riemannian Manifolds, in particular how to define a flow on a Manifold and how this is related to some differential equations. We'll see that the Ricci Flow is nothing but a differential equation for some class of different metrics on a Riemannian Manifold.

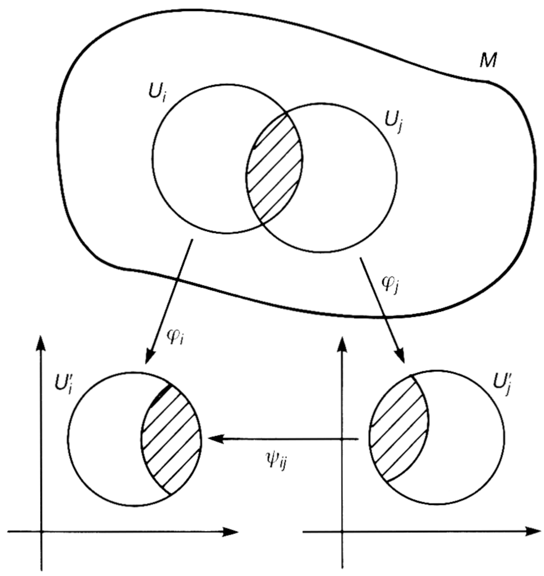

Figure 2.1 φ_i is a homeomorphism from U_i onto an open subset U_i' of $\mathbb{R}^n$.

2.2.1 Differentiable Manfifolds

Let's first refine the definition of a differentiable Manifold 9.

Definition 12 (Differentiable manifold)**.** $\mathcal{M}$ is an n-dimensional differentiable manifold if

- $\mathcal{M}$ is a topological space.
- $\mathcal{M}$ is provided with a family of pairs $\{(U_i, \varphi_i)\}$.
- $\{U_i\}$ is a family of open sets which covers $\mathcal{M}$, that is, $\cup_i U_i = \mathcal{M}$. φ_i is a homeomorphism from U_i onto open subsets U_i' of $\mathbb{R}^n$. See fig. 2.1
- Given U_i and U_j such that $U_i \cap U_j \neq \emptyset$, the map $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cap U_j)$ is infinitely differentiable.

The pair (U_i, φ_i) is called a *chart*, while the whole family $\{(U_i, \varphi_i)\}$ is an *atlas*. The subset U_i is called the *coordinate neighbourhood*, while φ_i is the *coordinate function*. This homeomorphism is represented by n functions $\{x_1(p), \dots, x_n(p)\}$, which are called *coordinates*. However, notice that a point $p \in \mathcal{M}$ is independent of its coordinates. In each coordinate neighbourhood U_i, $\mathcal{M}$ looks like an open subset of $\mathbb{R}_n$ whose element is $\{x^1, \dots x^n\}$, and this explains our previous definition 9.

Recall from the previous section that we're interested in Manifolds with boundaries. We're now able to define them.

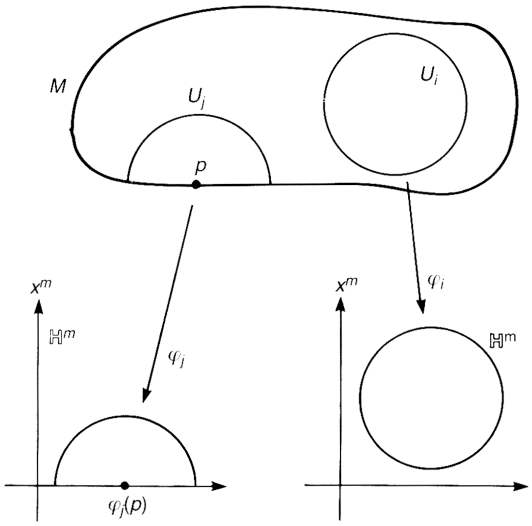

Figure 2.2 A Manifold $\mathcal{M}$ with boundary. Here, $p \in \partial\mathcal{M}$.

Definition 13 (Manifold with boundary)**.** A *Manifold with boundary* (see fig. 2.2) is a topological space $\mathcal{M}$ which is covered by a family of open sets $\{U_i\}$, each of which is homeomorphic to an open set of H^n, where

$$H^n := \{(x^1, \dots, x^n) \in R^n | x^n \geq 0\}. \tag{2.9}$$

The set of points which are mapped to points with $x_n = 0$ is called the *boundary* of $\mathcal{M}$, denoted by $\partial\mathcal{M}$. The coordinates of $\partial\mathcal{M}$ may be given by $n-1$ numbers $(x^1, \dots, x^{n-1}, 0)$. One must then be careful to define smoothness. Indeed, the map $\psi_{ij} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is defined on an open set of H^n in general, and φ_{ij} is said to be smooth if it's C^∞ in an open set of $\mathbb{R}^n$ which contains $\varphi_j(U_i \cap U_j)$.

2.2.2 Differentiable Maps

Due to the presence of the differentiable structure on a Manifold, we're able to use the calculus techniques developed for $\mathbb{R}^n$. We can define a map between manifolds.

Let $\mathcal{M}$ and $\mathcal{N}$ be an m-dimensional and an n-dimensional Manifold, respectively. Let's then consider a map between them, i.e., $f : \mathcal{M} \to \mathcal{N}$, where $\mathcal{M} \ni p \mapsto f(p) \in \mathcal{N}$, see fig. 2.3 Taking the charts (U, φ) on $\mathcal{M}$ and (V, ψ) on $\mathcal{N}$,the coordinate representation of f will be

$$\varphi \circ f \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n. \tag{2.10}$$

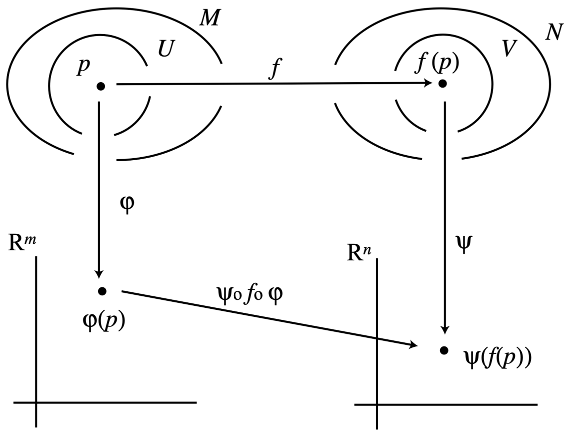

Figure 2.3 A map $f\colon \mathcal{M} \to \mathcal{N}$ has coordinates representation $\psi \circ f \circ \varphi^{-1}\colon \mathbb{R}^m \to \mathbb{R}^n$.

We can write $\varphi(p) = \{x^\mu\}$ and $\psi(f(p)) = \{y^\alpha\}$, which makes explicit that $y \equiv \psi \circ f \circ \varphi^{-1}(x)$ is an m-variables, vector-valued, usual function. Rendering the coordinate charts implicit, we may write $y^\alpha = f^\alpha(x^\mu)$, which makes clear why f is said to be *differentiable* at p if y is C^∞. Notice, however, that the differentiability of f is independent of the chosen coordinates.

A very important type of maps between manifold is given by diffeomorphisms.

Definition 14 (Diffeomorphism)**.** Let $f\colon \mathcal{M} \to \mathcal{N}$ be a homeomorphism and ψ and φ the same coordinate functions as before. Then, if $\psi \circ f \circ \varphi^{-1}$ is invertible and both $y \equiv \psi \circ f \circ \varphi^{-1}(x)$ and $x \equiv \varphi \circ f^{-1} \circ \psi^{-1}(y)$ are C^∞, f is called a *diffeomorphism* and $\mathcal{M}$ is said to be *diffeomorphic* to $\mathcal{N}$, $\mathcal{M} \equiv \mathcal{N}$.

Taking a diffeomorphism from a Manifold into itself, we can implement the concept of change of coordinates, in two different interpretation. First, the set of diffeomorphisms $f\colon \mathcal{M} \to \mathcal{M}$ form a group denoted by Diff($\mathcal{M}$). Considering a particular $f \in$ Diff($\mathcal{M}$), and a chart (U, φ), such that, for $p \in U$ and $f(p) \in U$, we get $\varphi(p) = x^\mu(p)$ and $\varphi(f(p)) = y^\mu(f(p))$, then y is a differentiable function of x and the above diffeomorphism can be thought as an *active transformation* for a change of coordinates.

However, if (U, φ) and (V, ψ) are overlapping charts, for a point $p \in U \cap V$, there are two coordinates values, i.e., $x^\mu = \varphi(p)$ and $y^\mu = \psi(p)$. Then, the map $x \mapsto y$ is differentiable, and it represents a *passive transformation* for a change of coordinates.

Two important kinds of maps are curves and functions.

Definition 15 (Curve)**.** An *open curve* in an n-dimensional Manifold $\mathcal{M}$ is a map $c\colon (a, b) \to \mathcal{M}$, where (a, b) is an open interval such that $a < 0 < b$. A *closed curve* is

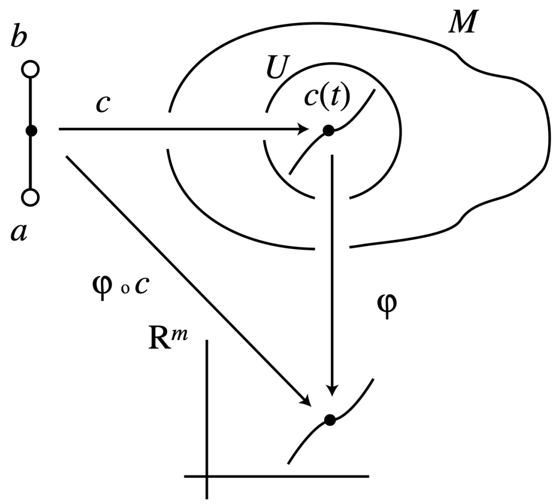

Figure 2.4 A curve c in $\mathcal{M}$ and its coordinate representation $\psi \circ c$.

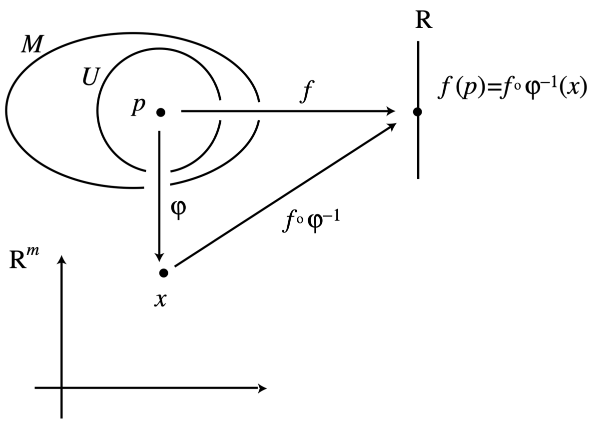

Figure 2.5 A function $f\colon \mathcal{M} \to \mathbb{R}$ and its coordinate representation $f \circ \varphi^{-1}$.

a map $c\colon S^1 \to \mathcal{M}$. On a chart (U, φ), a curve $c(t)$ ha the coordinate representation $x = \varphi \circ c\colon \mathbb{R} \to \mathbb{R}^n$. See fig. 2.4

Definition 16 (Function)**.** A *function* f on $\mathcal{M}$ is a smooth map from $\mathcal{M}$ to $\mathbb{R}$. On a chart (U, φ), the coordinate representation of f is given by $f \circ \varphi^{-1}\colon \mathbb{R}^n \to \mathbb{R}$, which is a real-valued function of n variables. The set of functions is denoted by $\mathfrak{F}(\mathcal{M})$. See fig. 2.5

2.2.3 Vectors

The curves allow us to define a vector as a differential operator. So understand this more deeply, let's take a Manifold $\mathcal{M}$, a curve $c\colon (a, b) \to \mathcal{M}$ and a function $f\colon \mathcal{M} \to \mathbb{R}$,

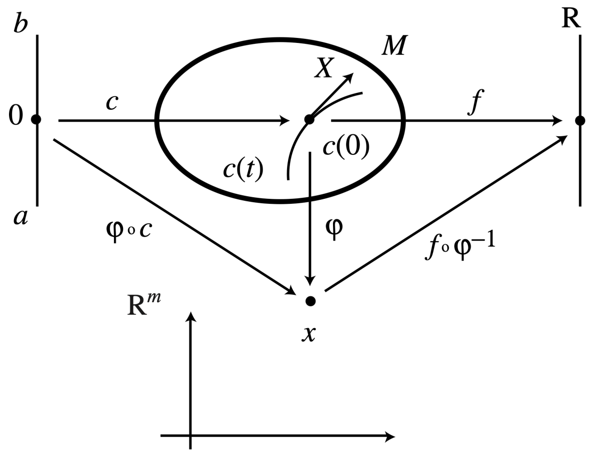

Figure 2.6 A curve c and a function f define a tangent vector along the curve in terms of directional derivatives.

where (a, b) is an open interval containing $t = 0$, as showed in fig. 2.6. The tangent vector at $c(0)$ is defined to be the directional derivative of a function $f(c(t))$ along the curve $c(t)$ at $t = 0$. In particular,

$$X[f] \coloneqq \left.\frac{\mathrm{d}f(c(t))}{\mathrm{d}t}\right|_{t=0} = \frac{\partial f}{\partial x^\mu}\left.\frac{\mathrm{d}x^\mu(c(t))}{\mathrm{d}t}\right|_{t=0} = X^\mu\left(\frac{\partial f}{\partial x^\mu}\right), \tag{2.11}$$

where we defined

$$X = X^\mu\left(\frac{\partial}{\partial x^\mu}\right), \quad X^\mu = \left.\frac{\mathrm{d}x^\mu(c(t))}{\mathrm{d}t}\right|_{t=0} \tag{2.12}$$

Notice that we used abuse of notation, that is $\partial f/\partial x^\mu$ really means $\partial(f \circ \varphi^{-1}(x))/\partial x^\mu$.

Trying to be more precise, as known from multivariate analysis, whenever we have a curve, we can define an equivalence class. This case is no different. Indeed, given two curves $c_1(t)$ and $c_2(t)$ on $\mathcal{M}$, if they satisfy

$$c_1(0) = c_2(0) = p, \tag{2.13a}$$

$$\left.\frac{\mathrm{d}x^\mu(c_1(t))}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}x^\mu(c_2(t))}{\mathrm{d}t}\right|_{t=0}, \tag{2.13b}$$

then they yield the same differential operator X at p. Then, the above conditions define an equivalence relation $\sim$ such that $c_1(t) \sim c_2(t)$. This allows us to define

Definition 17 (Vector). A *tangent vector* X is the *equivalence class of curves*

$$[c(t)] = \left\{ \tilde{c}(t) \;\middle|\; \tilde{c}(0) = c(0) \;\wedge\; \left.\frac{\mathrm{d}x^\mu(\tilde{c}(t))}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}x^\mu(c(t))}{\mathrm{d}t}\right|_{t=0} \right\}. \tag{2.14}$$

All the tangent vectors at a particular point $p \in \mathcal{M}$ form a vector space called *tangent space* of $\mathcal{M}$ at p, denoted by $T_p\mathcal{M}$. By means of equation (2.12), it's straightforward to see that the basis vectors of this vector space are

$$\left\{ e_\mu = \frac{\partial}{\partial x^\mu} \right\}, \quad \mu = 1, \ldots, n. \tag{2.15}$$

Clearly, $\dim T_p\mathcal{M} = \dim \mathcal{M}$. Further, for each $V \in T_p\mathcal{M}$, we can expand it as $V = V^\mu e_\mu = V^\mu \partial_\mu$, and we call V^μ the components of V with respect to the basis.

Obviously, $\{e_\mu\}$ are not the only possible basis for the vector space $T_p\mathcal{M}$. Indeed, as known from linear algebra, an arbitrary linear combination $\hat{e}_i := A_i^\mu e_\mu$, with $A = (A_i^\mu) \in GL(n, \mathbb{R})$, is a basis as well. In this case, $\{\hat{e}_i\}$ is called a *non-coordinate basis*.

Further, given that a vector exists independently of its coordinates, we can take two coordinate charts, with a point in the intersection of their domains, i.e., $p \in U_i \cap U_j$, with $x = \varphi_i(p)$ and $y = \varphi_j(p)$. A generic vector $X \in T_p\mathcal{M}$ can be equivalently expanded as

$$X = X^\mu \frac{\partial}{\partial x^\mu} = \tilde{X}^\mu \frac{\partial}{\partial y^\mu}, \tag{2.16}$$

which shows the relation of the components of a vector in two different charts, i.e,

$$\tilde{X}^\mu = X^\nu \frac{\partial y^\mu}{\partial x^\nu}. \tag{2.17}$$

2.2.4 One-Forms

Given that $T_p\mathcal{M}$ is a vector space, we can define its dual vector space, whose elements are linear functional from $T_p\mathcal{M}$ to $\mathbb{R}$. This space is called *cotangent space*, is denoted by $T_p^*\mathcal{M}$, and their elements $\omega\colon T_p\mathcal{M} \to \mathbb{R} \in T_p^*\mathcal{M}$ are called *one-forms*.

The simplest one-form is the *differential* of a function $f \in \mathfrak{F}(\mathcal{M})$. Indeed, for a vector $V \in T_p\mathcal{M}$, its action on f is

$$V[f] = V^\mu \frac{\partial f}{\partial x^\mu} \in \mathbb{R}. \tag{2.18}$$

So, we define the action of $\mathrm{d}f \in T_p^*\mathcal{M}$ on a vector $V \in T_p\mathcal{M}$ by

$$\langle \mathrm{d}f, V \rangle \equiv V[f] = V^\mu \frac{\partial f}{\partial x^\mu} \in \mathbb{R}. \tag{2.19}$$

In coordinates $x = \varphi(p)$, the differential can be expanded as

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu} \mathrm{d}x^\mu, \tag{2.20}$$

which clearly shows that $\{\mathrm{d}x^\mu\}$ is a basis of $T_p^*\mathcal{M}$. In particular, it's the dual basis of $\{\partial_\mu\}$, since

$$\langle \mathrm{d}x^\mu, \partial_\mu \rangle = \frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu. \tag{2.21}$$

It's now easy to consider generic one-forms $\omega \in T_p^*\mathcal{M}$ and generic vectors $V \in T_p\mathcal{M}$. Picking a coordinate basis for the tangent space and its dual basis for the cotangent one, we can expand $\omega = \omega_\mu \,\mathrm{d}x^\mu$ and $V = V^\mu \partial_\mu$. Then, as usual, we can define an *inner product* $\langle ., . \rangle \colon T_p^*\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ by

$$\langle \omega, V \rangle = \omega_\mu V^\mu \langle \mathrm{d}x^\mu, \partial_\nu \rangle = \omega_\mu V^\nu \delta^\mu_\nu = \omega_\mu V^\mu. \tag{2.22}$$

Finally, analogously than before, we can pick two charts and a point in their domain's intersection, $p \in U_i \cap U_j$, so that

$$\omega = \omega_\mu \,\mathrm{d}x^\mu = \tilde{\omega}_\nu \,\mathrm{d}y^\nu, \tag{2.23}$$

with $x = \varphi_i(p)$ and $y = \varphi_j(p)$. Then, we can easily infer the transformation rule of a one-form under a change of coordinates, indeed,

$$\mathrm{d}y^\nu = \frac{\partial y^\nu}{\partial x^\mu} \,\mathrm{d}x^\mu \implies \tilde{\omega}_\nu = \omega_\mu \frac{\partial x^\mu}{\partial y^\nu}. \tag{2.24}$$

2.2.5 Tensors and Tensor Fields

From multilinear algebra, it should be familiar that a *tensor* of type (p, q) is a multilinear map which maps q elements of $T_p^*\mathcal{M}$ and r elements of $T_p\mathcal{M}$ to a real number:

$$T^{(q,r)} \colon \underbrace{T_p^*\mathcal{M} \otimes \cdots \otimes T_p^*\mathcal{M}}_{q \text{ times}} \otimes \underbrace{T_p\mathcal{M} \otimes \cdots \otimes T_p\mathcal{M}}_{r \text{ times}} \longrightarrow \mathbb{R}. \tag{2.25}$$

Picking a coordinate basis and its dual basis,

$$T^{(q,r)} = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_q}} \,\mathrm{d}x^{\nu_1} \dots \mathrm{d}x^{\nu_r}. \tag{2.26}$$

The set of type (q, r) tensors at $p \in \mathcal{M}$ is denoted by $\mathcal{T}^q_{r;p}(\mathcal{M})$. Further, taking r-vectors $V_a = V_a^\mu \partial_\mu$ and q one-forms $\omega_a = \omega_{a\mu} \,\mathrm{d}x^\mu$,the action of $T^{(q,r)}$ on them is given by

$$T(\omega_1, \dots, \omega_q; V_1, \dots, V_r) = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \,\omega_{1\mu_1} \dots \omega_{q\mu_q} V_1^{\nu_1} \dots V_r^{\nu_r}. \tag{2.27}$$

In order to probe a Manifold structure, it's necessary to have fields defined in it, i.e., structures which are smoothly defined for each point of $\mathcal{M}$. Knowing the tensor structure at a point, it's easy to generalize it to the entire Manifold. To begin with, let's consider the simplest tensor, i.e., a vector, which is a $(1,0)$ tensor.

A *vector field* is a vector assigned smoothly to each point of $\mathcal{M}$. In other words, V is a vector field if $V[f] \in \mathfrak{F}(\mathcal{M})$, for any $f \in \mathfrak{F}(\mathcal{M})$. Then, each component of a vector field is itself a smooth function from $\mathcal{M}$ to $\mathbb{R}$. We denote the set of vector fields with $\mathfrak{X}(\mathcal{M})$. Then, a vector $X \in \mathfrak{X}(\mathcal{M})$ computed at a point $p \in \mathcal{M}$ is a vector at $T_p\mathcal{M}$, i.e., $X|_p \in T_p\mathcal{M}$.

Similarly, a *tensor field* of type (q,r) is a smooth assignment of an element of $\mathcal{T}^q_{r;p}(\mathcal{M})$ at each point $p \in \mathcal{M}$. The set of tensor fields of type (q,r) on $\mathcal{M}$ is denoted by $\mathcal{T}^q_r(\mathcal{M})$.

An important tensor field we're interested in is the metric tensor, which is tightly related to the definition of a Ricci Flow, in the context of Riemannian Manifolds. However, let's first study an important property of vector fields: they can generate a flow on a Manifold. Studying further the relation among flows and differential equation, allows us to better grasp the definition of a Ricci Flow.

2.2.6 Flow generated by a vector field

Let X be a vector field in $\mathcal{M}$, $X \in \mathfrak{X}(\mathcal{M})$. An *integral curve* $x(t)$ of X is a curve in $\mathcal{M}$, whose tangent vector at $x(t)$ is $X|_x$. In a chart (U,φ), this is equivalent to

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = X^\mu(x(t)), \tag{2.28}$$

where $x^\mu(t)$ is the μ-th component of $\varphi(x(t))$ and $X = X^\mu\partial_\mu$. Pay attention to the abuse of notation. Indeed, we used x to denote both a point on $\mathcal{M}$ and its coordinates.

Then, to finding the integral curve of a vector field X is equivalent to solving the system of ODEs (2.28), with initial condition $x_0^\mu = x^\mu(0)$, which are the coordinates of the integral curve at $t = 0$.

The existence and uniqueness theorem of ODEs guarantees that there is a unique solution to (2.28), at least locally, with the initial condition x_0^μ.

Let, then, $\sigma(t, x_0)$ be an integral curve of X passing through a point x_0 at $t = 0$ and let's denote its coordinates by $\sigma^\mu(t, x_0)$. Then, eq. (2.28) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma^\mu(t, x_0) = X^\mu(\sigma(t, x_0)), \tag{2.29}$$

with the initial condition

$$\sigma^\mu(0, x_0) = x_0^\mu. \tag{2.30}$$

The map $\sigma \colon \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ si called *flow* generated by $X \in \mathfrak{X}(\mathcal{M})$. We can prove it satisfies the condition

$$\sigma(t, \sigma^\mu(s, x_0)) = \sigma(t + s, x_0), \tag{2.31}$$

for $s, t \in \mathbb{R}$.

Proof. Since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sigma^\mu(t, \sigma^\mu(s, x_0)) &= X^\mu(\sigma(t, \sigma^\mu(s, x_0))),\\ \sigma(0, \sigma(s, x_0)) &= \sigma(s, x_0),\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sigma^\mu(t + s, x_0) &= \frac{\mathrm{d}}{\mathrm{d}(t + s)}\sigma^\mu(t + s, x_0) = X^\mu(\sigma(t + s, x_0)),\\ \sigma(0 + s, x_0) &= \sigma(s, x_0),\end{aligned}$$

both sides of eq. (2.31) satisfy the same ODE and the same initial condition. Then, from the uniqueness theorem for the solution of ODEs, they must be the same. ■

This leads to the following theorem.

Theorem T.2. *For any point $x \in \mathcal{M}$, there exists a differentiable map $\sigma \colon \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ such that*

- *$\sigma(0, x) = x$;*
- *$t \mapsto \sigma(t, x)$ is a solution of (2.29) and (2.30);*
- *$\sigma(t, \sigma^\mu(s, x)) = \sigma(t + s, x)$.*

We're now ready to introduce Riemannian Manifold and their most relevant properties.

2.2.7 Riemannian Manifolds

A Riemannian Manifold is defined as a smooth Manifold which admits a Riemannian metric.

Definition 18. Let $\mathcal{M}$ be a differentiable Manifold. A *Riemannian metric* g on $\mathcal{M}$ is a type $(0,2)$ tensor field on $\mathcal{M}$ such that, at each point $p \in \mathcal{M}$:

- $g_p(U,V) = g_p(V,U)$,
- $g_p(U,V) \geq 0$, where the equality holds only when $U = 0$.

Here, $U, V \in T_p\mathcal{M}$ and $g_p = g|_p$. Basically, g_p is a symmetric positive-definite bilinear form.

Recall the previous definition of inner product (2.22) between vectors and dual forms. For $V \in T_p\mathcal{M}$ and $\omega \in T_p^*\mathcal{M}$, then $\langle .,. \rangle \colon T_p^*\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$. If there exists a metric tensor g, then, we can use it to define the inner product between two vectors $U, V \in T_p\mathcal{M}$, specifically by $g_p(U,V)$. Since $g_p \colon T_p\mathcal{M} \otimes T_p\mathcal{M} \to \mathbb{R}$, we may define a linear map $g_p(U,.) \colon T_p\mathcal{M} \to \mathbb{R}$ by $V \mapsto g_p(U,V)$. Then, it's straightforward that $g_p(U,.) \in T_p^*\mathcal{M}$ is a one-form. Thus, the metric g_p gives rise to an isomorphism between $T_p\mathcal{M}$ and $T_p^*\mathcal{M}$.

Picking a chart (φ, U), with coordinates $\{x^\mu\}$, we can expand the metric tensor in coordinates as

$$g_p = g_{\mu\nu}(p)\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu, \quad g_{\mu\nu}(p) = g_p(\partial_\mu, \partial_\nu) = g_{\nu\mu}(p), \quad p \in \mathcal{M}. \tag{2.32}$$

It's usual convention to forget about the point p, denote the inverse metric as $g^{\mu\nu}$, and the determinant as $\det(g_{\mu\nu}) \coloneqq g$ and $\det(g^{\mu\nu}) \coloneqq g^{-1}$. Then, the isomorphism between $T_p\mathcal{M}$ and $T_p^*\mathcal{M}$ can be expressed as

$$\omega_\mu = g_{\mu\nu} U^\nu, \quad U^\mu = g^{\mu\nu} \omega_\nu. \tag{2.33}$$

The purpose of the next chapter will be to rigorously define the Ricci Flow. However, to conclude this motivation part, let's take a brief detour onto string theory, in which, with a simple example in the context of non-linear sigma models, it's possible to grasp the meaning of the Ricci Flow application to understand topological properties of a Manifold through renormalization group flows in a Riemannian Manifold.

2.2.8 Curvature and Ricci tensor

2.2.9 Covariant Derivative

2.3 Non-Linear Sigma Models and String Theory

The standard starting point of string theory is *Polyakov action*, which describes a bosonic classical, one-dimensional, string, which describes a two-dimensional world-sheet Σ on a 26-dimensional spacetime described by the spacetime coordinates $X^\mu(\xi)$, $\mu = 0, \ldots 25$, where $\xi^a = (\tau, \sigma)$, $a = 1, 2$, are the intrinsic coordinates on Σ. The metric on spacetime is denoted by $g_{\mu\nu}$, while the metric on the worldsheet is γ_{ab}. For a flat spacetime, with Minkowski metric $g_{\mu\nu} \equiv \eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$, the action reads

$$S_P[X^\mu(\xi), \gamma_{ab}(\xi)] = -\frac{T}{2} \int_\Sigma \mathrm{d}\tau \, \mathrm{d}\sigma \sqrt{-\det(\gamma)} \gamma^{ab} \partial_a X_\mu(\xi) \partial_b X^\mu(\xi), \tag{2.34}$$

where T is a characteristic parameter of the string, related to the string length l.

The symmetries of this action allow us to consider a flat worldsheet metrix, $\gamma_{ab} = \eta_{ab}$, considering the so-called *unit gauge*. Then, reintroducing explicitly the metric $g_{\mu\nu}$, even if it's flat in this case, we obtain

$$S_P = -T \int_\Sigma \mathrm{d}^2\xi g_{\mu\nu}(X) \partial_a X^\mu \partial^a X^\nu. \tag{2.35}$$

Basically, it represents a 2-dimensional field theory on the worldsheet, where the coordinates X^μ are 26 dynamical fields in there. This allows us to quantize the theory with the usual quantization prescription, based on the substitution of the classical Poisson brackets defined on a symplectic manifold with the commutators of operators acting on a Hilbert space.

After quantization, one notice that, for a closed string, defined by the periodicity condition $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + l)$, the particle spectrum contains a *graviton* $\gamma_{\mu\nu}$, which resembles a gravitational wave at low energies, a scalar field φ called *dilaton* and an antisymmetric two-tensor $b_{\mu\nu}$ called *Kalb-Ramond tensor*.

Therefore, due to the presence of the graviton, one could wonder what happens for a non-flat spacetime. Then, after redefining the coordinates as a constant X_0^μ plus some other arbitrary fields Y^μ, i.e.,

$$X^\mu(\xi) = X_0^\mu(\xi) + \sqrt{\alpha'} Y^\mu(\xi), \tag{2.36}$$

we can expand the term in the Lagrangian as

$$
\begin{aligned}
&g_{\mu\nu}(X)\partial_a X^\mu \partial^a X^\nu \\
&= \alpha' \left[g_{\mu\nu}(X_0) + \sqrt{\alpha'} g_{\mu\nu,\rho}(X_0) Y^\rho(\xi) + \frac{\alpha'}{2} g_{\mu\nu,\rho\sigma}(X_0) Y^\rho(\xi Y^\sigma(\xi)) + \dots \right] \partial_a Y^\mu \partial^a Y^\nu,
\end{aligned}
\tag{2.37}
$$

where $g_{\mu\nu,\rho} \equiv \partial_\rho g_{\mu\nu}$.

We obtained an expansion in α', where each term is an interaction term for the fields Y^μ, with couplings given by the derivatives of the metric. However, a crucial symmetry of the Polyakov action (2.34) is the invariance under *conformal transformations*. Those are diffeomorphisms on a Riemannian Manifold which preserve the metric up to rescaling, i.e.,

$$g(x) \to \tilde{g}(\tilde{x}) = e^{2\omega(\tilde{x})} g(\tilde{x}). \tag{2.38}$$

Without going into the details, the presence of this symmetry is considered as a consistency condition for the theory, as it allows for a perturbative interpretation of the interactions. In addition, the above interacting quantum field theory must undergo renormalization, in order to cure the divergences.

However, after renormalization, the conformal symmetry may be anomalous[2]. Indeed, a particular example of conformal transformation (2.38) is provided by scale-invariance. That is to say, the theory can't depend on a scale to be conformal invariant. But, as the methods of *renormalization group* teach us, after renormalization the couplings run with the energy scale M of the system, dependence which is encapsulated into the *β-function*

$$\beta(g_{\mu\nu}) = M \frac{\partial}{\partial M} g_{\mu\nu}. \tag{2.39}$$

As a consequence, for a quantum string theory to make sense, it must be conformal invariance, so the β-function associated to the metric, in the action (2.35) with the expansion (2.37), must vanish

$$\beta(g_{\mu\nu}) \overset{!}{=} 0. \tag{2.40}$$

A similar argument can be pursued for the other two particles in the closed string spectrum, i.e., the dilaton φ and the Kalb-Ramond form $b_{\mu\nu}$. The corresponding action

[2]An anomaly is a classical symmetry which is not preserved at the quantum level. In particular, it is due to a non-invariance of the measure of the path integral.

will be

$$S_\sigma = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\xi \sqrt{-\det(\gamma)}\left[\left(\gamma^{ab} g_{\mu\nu}(X) + i\varepsilon^{ab} b_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b^\nu + \alpha' \mathcal{R}\varphi(X)\right], \tag{2.41}$$

where ε^{ab} is the 2d Levi-Civita symbol, while $\mathcal{R} = \mathcal{R}(\gamma)$ is the Ricci scalar on the worldsheet.

Defining the Field Strength $H_{\mu\nu\rho} = \partial_\mu b_{\nu\rho} + \partial_\nu b_{\rho\mu} + \partial_\rho b_{\mu\nu}$, the vanishing of the beta functions reads

$$\beta(g_{\mu\nu}) = \alpha'\left(R_{\mu\nu} - \frac{1}{4}H_{\mu\lambda\rho}H^{\mu\lambda\rho} + 2\nabla_\mu\nabla_\nu\varphi\right) + O(\alpha'^2) \stackrel{!}{=} 0, \tag{2.42a}$$

$$\beta(b_{\mu\nu}) = \alpha'\left(\frac{1}{2}\nabla^\rho H_{\rho\mu\nu} + \nabla^\rho\varphi H_{\rho\mu\nu}\right) + O(\alpha'^2) \stackrel{!}{=} 0, \tag{2.42b}$$

$$\beta(\varphi) = \alpha'\left(\frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi - \frac{1}{2}\nabla^2\varphi - \frac{1}{24}H_{\mu\nu\rho}H^{\mu\nu\rho}\right) + O(\alpha'^2) \stackrel{!}{=} 0. \tag{2.42c}$$

The above equations are constraints for the spacetime fields (g, b, φ), imposed to preserve conformal invariance of the quantum string. However, since those fields should be dynamical on spacetime, those must also be their equations of motion. This leads to the following *low-energy effective action*, which has (2.42) as equations of motion

$$S_{26} = \frac{1}{k_0^2}\int \mathrm{d}^{26}x\sqrt{\det(g)}e^{-2\varphi}\left(\mathcal{R}(g) - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\nabla_\mu\varphi\nabla^\mu\varphi\right). \tag{2.43}$$

To set this problem to a more general ground, and understand how this model is related to Ricci Flow, let's define more accurately what a σ-model is in field theory. A σ-model is a field theory for a field $\Phi\colon \Sigma \to \mathcal{M}$ that takes values in a manifold $\mathcal{M}$. Traditionally, Σ is the spacetime on which the field theory lives, and $\mathcal{M}$ is called the target space. If the target space carries some linear structure, like a vector space, then the whole physical system is called a linear σ-model. For general manifolds such as generic Riemannian ones, it is then called a non-linear σ-model. What we did above is to study the σ-model spacetime renormalization effects on the target space $\mathcal{M}$, and this is indeed an instance of Ricci Flow.

2.4 Ricci Flow to Tackle Topological Problems

Chapter 3

Ricci Flow and Oliver-Ricci Curvature

Chapter 4

Application to Complex Networks

CHAPTER 5

IMPLEMENTATION OF THE CODE

Chapter 6

Conclusions and Future Directions

Appendix A

Metric of a 2-sphere

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