MPC for linear constrained systems

Dynamics

$$x^+ = Ax + B_1 u + B_2 \delta + B_3 z$$

State Evolution

$$\begin{aligned}
x_0 &= x_0 \\
x_1 &= Ax_0 + B_1 u_0 + B_2 \delta_0 + B_3 z_0 \\
x_2 &= A^2 x_0 + A(B_1 u_0 + B_2 \delta_0 + B_3 z_0) + B_1 u_1 + B_2 \delta_1 + B_3 z_1 \\
&\vdots \\
x_N &= A^N x_0 + \sum_{i=0}^{N-1} A^{N-1-i} B_1 u_i + \sum_{i=0}^{N-1} A^{N-1-i} B_2 \delta_i + \sum_{i=0}^{N-1} A^{N-1-i} B_3 z_i
\end{aligned}$$

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ A \\ A^2 \\ A^3 \\ \vdots \\ A^N \end{bmatrix}}_{\Gamma_x} x_0 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ B_1 & 0 & 0 & \dots & 0 \\ AB_1 & B_1 & 0 & \dots & 0 \\ A^2 B_1 & AB_1 & B_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^{N-1} B_1 & A^{N-2} B_1 & A^{N-3} B_1 & \dots & B_1 \end{bmatrix}}_{\Gamma_u} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix} +$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ B_2 & 0 & 0 & \dots & 0 \\ AB_2 & B_2 & 0 & \dots & 0 \\ A^2 B_2 & AB_2 & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^{N-1} B_2 & A^{N-2} B_2 & A^{N-3} B_2 & \dots & B_2 \end{bmatrix}}_{\Gamma_\delta} \begin{bmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \\ \vdots \\ \delta_{N-1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ B_3 & 0 & 0 & \dots & 0 \\ AB_3 & B_3 & 0 & \dots & 0 \\ A^2 B_3 & AB_3 & B_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^{N-1} B_3 & A^{N-2} B_3 & A^{N-3} B_3 & \dots & B_3 \end{bmatrix}}_{\Gamma_\delta} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{N-1} \end{bmatrix}$$

$$\mathbf{x} = \Gamma_x x_0 + \Gamma_u \mathbf{u} + \Gamma_\delta \delta + \Gamma_z \mathbf{z}$$

where $\mathbf{x} = [x_0^{\mathrm{T}} \; x_1^{\mathrm{T}} \; \dots x_N^{\mathrm{T}}]^{\mathrm{T}}$; $\mathbf{u} = [u_0^{\mathrm{T}} \; u_1^{\mathrm{T}} \; \dots u_{N-1}^{\mathrm{T}}]^{\mathrm{T}}$; $\delta = [\delta_0^{\mathrm{T}} \; \delta_1^{\mathrm{T}} \; \dots \delta_{N-1}^{\mathrm{T}}]^{\mathrm{T}}$; and $\mathbf{z} = [z_0^{\mathrm{T}} \; z_1^{\mathrm{T}} \; \dots z_{N-1}^{\mathrm{T}}]^{\mathrm{T}}$.

Sizes: $\mathbf{x} \in \mathbb{R}^{(N+1)n_x}$, $\mathbf{u} \in \mathbb{R}^{Nn_u}$, $\delta \in \mathbb{R}^{Nn_\delta}$, $\delta \in \mathbb{R}^{Nn_z}$, $\Gamma_x \in \mathbb{R}^{(N+1)n_x \times n_x}$, $\Gamma_u \in \mathbb{R}^{(N+1)n_x \times Nn_u}$, $\Gamma_\delta \in \mathbb{R}^{(N+1)n_x \times Nn_\delta}$, and $\Gamma_z \in \mathbb{R}^{(N+1)n_x \times Nn_z}$.

Quadratic Program Structure

$$\begin{aligned}
\min \; & \frac{1}{2} x^T P x + q^T x && (1) \\
\text{subj. to } & Gx \le h && (2) \\
& Ax = b && (3) \\
& lb \le x \le ub && (4)
\end{aligned}$$

Cost Function

For the microgrid system:

$$C_{\text{total}}(k) = \delta_{\text{grid}}(k)c_{\text{buy}}(k)P_{\text{grid}}(k) + (1-\delta_{\text{grid}}(k))c_{\text{sell}}(k)P_{\text{grid}}(k) + c_{\text{prod}}(k)\sum_{i=1}^{N_{\text{gen}}} P_i^{\text{dis}}(k)$$

$$C_{\text{total}}(k) = c_{\text{buy}}(k)z_{\text{grid}}(k) - c_{\text{sell}}(k)z_{\text{grid}}(k) + c_{\text{sell}}(k)P_{\text{grid}}(k) + c_{\text{prod}}(k)\sum_{i=1}^{N_{\text{gen}}} P_i^{\text{dis}}(k)$$

$$C_{\text{total}}(k) = \begin{bmatrix} 0 & c_{\text{sell}}(k) & c_{\text{prod}}(k) & c_{\text{prod}}(k) & c_{\text{prod}}(k) & 0 & c_{\text{buy}}(k)-c_{\text{sell}}(k) \end{bmatrix} \begin{bmatrix} P_{\text{b}}(k) \\ P_{\text{grid}}(k) \\ P_1^{\text{dis}}(k) \\ P_2^{\text{dis}}(k) \\ P_3^{\text{dis}}(k) \\ z_{\text{b}}(k) \\ z_{\text{grid}}(k) \end{bmatrix}$$

$$\sum_{i=0}^{N-1} C_{\text{total}}(k) = [c_0^{\text{T}},\ ...,\ c_{\text{N}-1}^{\text{T}}]^{\text{T}} \begin{bmatrix} u_0^a \\ \vdots \\ u_{\text{N}-1}^a \end{bmatrix}$$

where

$$u_k^a = \begin{bmatrix} u(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} P_{\text{b}}(k) \\ P_{\text{grid}}(k) \\ P_1^{\text{dis}}(k) \\ P_2^{\text{dis}}(k) \\ P_3^{\text{dis}}(k) \\ z_{\text{b}}(k) \\ z_{\text{grid}}(k) \end{bmatrix}$$

Inequality constraints

MLD constraints:

$$\begin{aligned}
&E_2\delta_k + E_3 z_k \leq E_1 u_k + E_4 x_k + E_5 \\
&E_2\delta_k - E_1 u_k + E_3 z_k \leq E_4 x_k + E_5 \\
&E_2\delta_k + \begin{bmatrix} -E_1 & E_3 \end{bmatrix} \begin{bmatrix} u_k \\ z_k \end{bmatrix} \leq E_4 x_k + E_5 \\
&E_2\delta_k + E_1^a u_k^a \leq E_4 x_k + E_5
\end{aligned}$$

where

$$E_1^a = \begin{bmatrix} -E_1 & E_3 \end{bmatrix}$$

The constraints have to be valid for all time steps of the prediction horizon, then:

$$\begin{aligned}
&E_2^{\text{blk}}\delta + E_1^{a,\text{blk}}\mathbf{u}^a \leq E_4^{\text{blk}}\mathbf{x} + E_5^{\text{conc}} \\
&E_2^{\text{blk}}\delta + E_1^{a,\text{blk}}\mathbf{u}^a \leq E_4^{\text{blk}}(\Gamma_x x_0 + \Gamma_u \mathbf{u} + \Gamma_\delta \delta + \Gamma_z \mathbf{z}) + E_5^{\text{conc}} \\
&E_1^{a,\text{blk}}\mathbf{u}^a - E_4^{\text{blk}}\Gamma_u\mathbf{u} - E_4^{\text{blk}}\Gamma_z\mathbf{z} \leq -E_2^{\text{blk}}\delta + E_4^{\text{blk}}\Gamma_x x_0 + E_4^{\text{blk}}\Gamma_\delta\delta + E_5^{\text{conc}} \\
&E_1^{a,\text{blk}}\mathbf{u}^a - E_4^{\text{blk}}\begin{bmatrix} \Gamma_u & \Gamma_z \end{bmatrix}\begin{bmatrix} \mathbf{u} \\ \mathbf{z} \end{bmatrix} \leq -E_2^{\text{blk}}\delta + E_4^{\text{blk}}\Gamma_x x_0 + E_4^{\text{blk}}\Gamma_\delta\delta + E_5^{\text{conc}} \\
&E_1^{a,\text{blk}}\mathbf{u}^a - E_4^{\text{blk}}\Gamma_{u^a}\mathbf{u}^a \leq -E_2^{\text{blk}}\delta + E_4^{\text{blk}}\Gamma_x x_0 + E_4^{\text{blk}}\Gamma_\delta\delta + E_5^{\text{conc}} \\
&(E_1^{a,\text{blk}} - E_4^{\text{blk}}\Gamma_{u^a})\mathbf{u}^a \leq -E_2^{\text{blk}}\delta + E_4^{\text{blk}}\Gamma_x x_0 + E_4^{\text{blk}}\Gamma_\delta\delta + E_5^{\text{conc}}
\end{aligned}$$

where

$$E_i^{\text{blk}} = \begin{bmatrix} E_i & 0 & \dots & 0 \\ 0 & E_i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & E_i \end{bmatrix}; \qquad \Gamma_{u^a} = \begin{bmatrix} \Gamma_u & \Gamma_z \end{bmatrix}$$

Equality constraints

Power balance: $P_b(k) - P_1^{\mathrm{dis}}(k) - P_2^{\mathrm{dis}}(k) - P_3^{\mathrm{dis}}(k) - P_{\mathrm{res}}(k) - P_{\mathrm{grid}}(k) + P_{\mathrm{load}}(k) = 0$

$$\begin{aligned} \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{\mathrm{b}}(k) \\ P_{\mathrm{grid}}(k) \\ P_1^{\mathrm{dis}}(k) \\ P_2^{\mathrm{dis}}(k) \\ P_3^{\mathrm{dis}}(k) \\ z_{\mathrm{b}}(k) \\ z_{\mathrm{grid}}(k) \end{bmatrix} &= 0 \\ \underbrace{\begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 0 & 0 \end{bmatrix}}_{G\mathrm{eq}} u_k^a - P_{\mathrm{res}}(k) + P_{\mathrm{load}}(k) &= 0 \end{aligned}$$

For all time steps:

$$G_{\mathrm{eq}}^{\mathrm{blk}} \mathbf{u}^a = P_{\mathrm{res}} - P_{\mathrm{load}}$$

where

$$G_{\mathrm{eq}}^{\mathrm{blk}} = \begin{bmatrix} G_{\mathrm{eq}} & 0 & \dots & 0 \\ 0 & G_{\mathrm{eq}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & G_{\mathrm{eq}} \end{bmatrix}$$

If the power balance constraints are formulated as inequality constraints, then:

$$\begin{bmatrix} G_{\mathrm{eq}}^{\mathrm{blk}} \\ -G_{\mathrm{eq}}^{\mathrm{blk}} \end{bmatrix} \mathbf{u}^a \leq \begin{bmatrix} P_{\mathrm{res}} - P_{\mathrm{load}} \\ -(P_{\mathrm{res}} - P_{\mathrm{load}}) \end{bmatrix}$$