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DATA SCIENCE AND ECONOMICS (LM-91)

TERM STRUCTURE OF INTEREST RATES

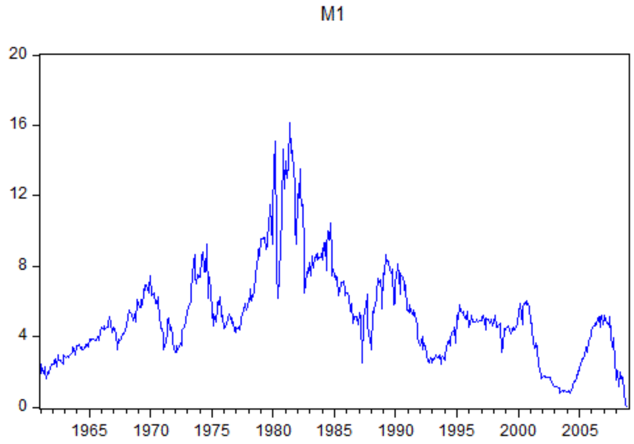
1. INTRODUCTION

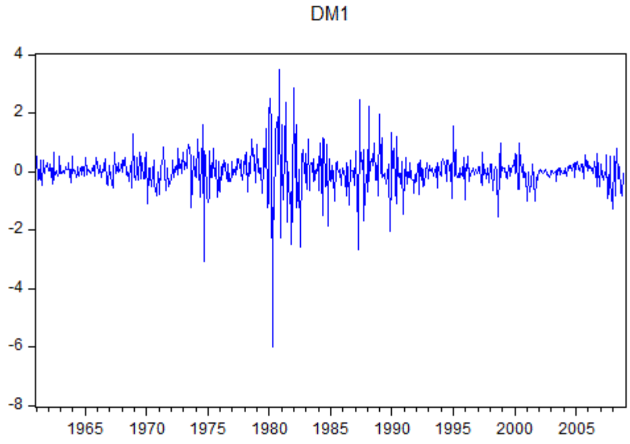
We begin defining formally our data. Let $i_{k,t}$ be the continuously compounded interest rates of a k period zero-coupon bond at time t. Then, our available data consist of:

$i_{1,t}$	for $t \in [1,576]$	stored in *m1*
$i_{2,t}$	for $t \in [1,576]$	stored in *m2*
$i_{3,t}$	for $t \in [1,576]$	stored in *m3*
$i_{6,t}$	for $t \in [1,576]$	stored in *m6*
$i_{12,t}$	for $t \in [1,576]$	stored in *y1*
$i_{24,t}$	for $t \in [1,576]$	stored in *y2*
$i_{32,t}$	for $t \in [1,576]$	stored in *y3*
$i_{48,t}$	for $t \in [1,576]$	stored in *y4*
$i_{60,t}$	for $t \in [1,576]$	stored in *y5*

where periods are measured in months, that is, any integer k, t in $i_{k,t}$ is a number of months.

We shall then proceed with a plot of the raw data and begin our first observations. At first glance, all time series show a non-stationary trajectory, which suggest they might be $I(1)$. As a matter of fact, plotting any first difference, conversely suggests that it might be mean-reverting, possibly with 0 mean. The latter claim is supported by a non-trending behavior of interest rates: inspecting our data along the whole period 1961-2008 we can't find a general upward/downward sloping trend. We will consider this when testing for unit roots. Here is, as an example, the graphs and correlograms of $m1$, $dm1 = m1 - m1(-1)$, $y1$ and $dy1 = y1 - y1(-1)$.





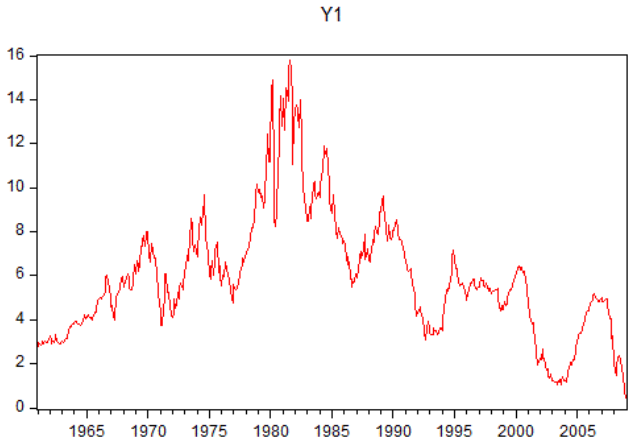
Date: 12/11/19 Time: 12:15
Sample: 1961M01 2008M12
Included observations: 576

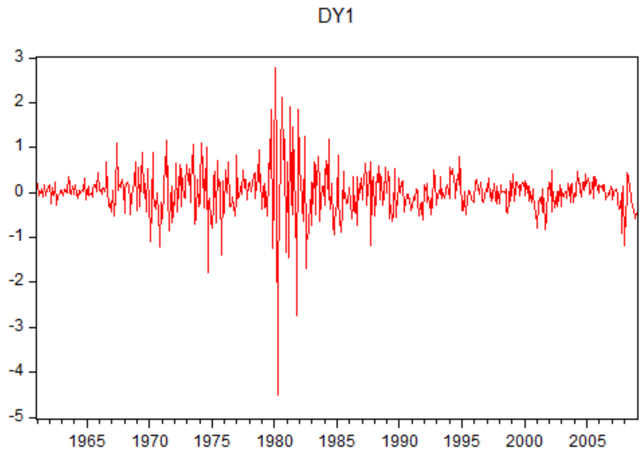
Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.962	0.962	535.34	0.000
		2	0.930	0.071	1036.9	0.000
		3	0.901	0.027	1508.6	0.000
		4	0.876	0.049	1955.6	0.000
		5	0.859	0.099	2386.1	0.000
		6	0.843	0.029	2801.2	0.000
		7	0.823	-0.043	3197.6	0.000
		8	0.812	0.115	3583.9	0.000

Date: 12/11/19 Time: 12:16
Sample: 1961M01 2008M12
Included observations: 575

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.095	-0.095	5.1975	0.023
		2	-0.041	-0.050	6.1631	0.046
		3	-0.051	-0.061	7.6808	0.053
		4	-0.092	-0.107	12.637	0.013
		5	-0.020	-0.048	12.880	0.025
		6	0.054	0.033	14.574	0.024
		7	-0.120	-0.130	22.947	0.002
		8	0.132	0.100	33.120	0.000

Figure 1 (from top left to bottom right) : **plot of *m1* (interest rates with maturity 1 month); plot of *dm1* (first differences of m1); correlogram of *m1*; correlogram of *dm1*.**





Date: 12/11/19 Time: 12:21
Sample: 1961M01 2008M12
Included observations: 576

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.979	0.979	554.95	0.000
		2	0.954	-0.095	1083.4	0.000
		3	0.933	0.066	1589.0	0.000
		4	0.914	0.041	2075.3	0.000
		5	0.899	0.079	2546.4	0.000
		6	0.881	-0.078	3000.0	0.000
		7	0.865	0.048	3437.6	0.000
		8	0.853	0.107	3864.5	0.000

Date: 12/11/19 Time: 12:21
Sample: 1961M01 2008M12
Included observations: 575

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.128	0.128	9.3976	0.002
		2	-0.069	-0.087	12.160	0.002
		3	-0.067	-0.048	14.766	0.002
		4	-0.102	-0.095	20.869	0.000
		5	0.082	0.103	24.791	0.000
		6	-0.035	-0.081	25.519	0.000
		7	-0.152	-0.139	39.098	0.000
		8	0.104	0.144	45.406	0.000

Figure 2(from top left to bottom right) : **plot of *y1* (interest rates with maturity 1 year); plot of *dy1* (first differences of y1); correlogram of *y1*; correlogram of *dy1*.**

2. UNIT ROOT TESTING OF INTEREST RATES

Claims such as the preceding, "interest rates are $I(1)$", must be properly tested. We then run an ADF unit root test in level (Case "no intercept and no trend") for each of the nine time series given and immediately get that the null hypothesis $H_0 = \{\rho = 1\}$ is never rejected. Alongside tests on levels, we run the same ADF unit root tests for the 1st differences ($dm1, \dots, dm6$, $dy1, \dots, dy5$): the null hypothesis here is strongly rejected, supporting our initial insight that interest rates are integrated of order 1 ($I(1)$). In the following table we report the values of the t-statistics in each test against the 5% critical value:

	t-statistic	c.v. 5%
m1	-1.43	-1.94
m2	-1.09	-1.94
m3	-1.17	-1.94
m6	-1.21	-1.94
y1	-1.10	-1.94
y2	-0.99	-1.94
y3	-0.89	-1.94
y4	-0.78	-1.94
y5	-0.72	-1.94
dm1	-26.33	-1.94
dm2	-17.76	-1.94
dm3	-20.90	-1.94
dm6	-25.67	-1.94
dy1	-21.04	-1.94
dy2	-20.20	-1.94

dy3	-20.98	-1.94
dy4	-22.09	-1.94
dy5	-21.92	-1.94

Table 1: ADF test statistics and c.v. for interest rates

Having found that first differences of interest rates are $I(0)$, one may wonder which ARMA process they follow. Considering, for example, series *dm1, dm3* and *dy3* we could perform model estimation tests with Conditional Least Squares method, write down each Bayes Information Criteria and select the model which minimizes it. Outcomes are summarized in the following tables:

dy3	iid	MA(1)	MA(2)
iid	1.09	1.08*	1.09
AR(1)	1.10	1.11	1.11
AR(2)	1.10	1.11	1.12

Table 2: BIC's for *dy3*

dm1	iid	MA(1)	MA(2)
iid	2.11	2.10*	2.12
AR(1)	2.11	2.11	2.12
AR(2)	2.12	2.12	2.11

Table 3: BIC's for *dm1*

dm3	iid	MA(1)	MA(2)
iid	1.52	1.51* (1.509)	1.52
AR(1)	1.51 (1.511)	1.52	1.53
AR(2)	1.52	1.53	1.54

Table 4: BIC's for *dm3*

As a result, model MA(1) is selected for all three series, but before jumping to conclusions we should check that residuals aren't correlated. We do so performing a Portmanteau test in each case:

Date: 12/10/19 Time: 18:50
Sample: 1961M01 2008M12
Included observations: 575
Q-statistic probabilities adjusted for 1 ARMA term

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.006	0.006	0.0217	
		2	-0.047	-0.047	1.3125	0.252
		3	-0.067	-0.066	3.9097	0.142

Table 5: Residuals correlogram and Portamanteau test: d*m1* as MA(1) process

Date: 12/10/19 Time: 18:50
Sample: 1961M01 2008M12
Included observations: 575
Q-statistic probabilities adjusted for 1 ARMA term

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.004	-0.004	0.0105	
		2	-0.029	-0.029	0.4890	0.484
		3	-0.021	-0.021	0.7387	0.691

Table 6: Residuals correlogram and Portamanteau test: d*m3* as MA(1) process

Date: 12/10/19 Time: 18:51
Sample: 1961M01 2008M12
Included observations: 575
Q-statistic probabilities adjusted for 1 ARMA term

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.007	-0.007	0.0303	
		2	-0.060	-0.060	2.0831	0.149
		3	-0.078	-0.079	5.5806	0.061

Table 7: Residuals correlogram and Portamanteau test: dy*1* as MA(1) process

Since p-values are rather high, the null hypothesis that there is no residual autocorrelation is never rejected, so we can confirm model *MA(1)* has been selected consistently. Furthermore, looking at the estimation output (here omitted) null hypothesis that the intercepts are 0 is never rejected and we can conclude that variables *dm1*, *dm3*, *dy3* follow the process

$X_t = \varepsilon_t + \theta\varepsilon_{t-1}$, with θ given by each test.

This results explain that both short-term and long-term rates change according to shocks occurred in the previous month but not to those which date back to more than one month.

3. COINTEGRATION TESTING OF INTEREST RATES

Next, we wish to show a very useful result, namely that interest rates are cointegrated and that the cointegrating vector for each pair $(i_{j,t}, i_{k,t})$ is $[-1,1]$. If this holds, we move one step forward in validating the Expectation Hypothesis and Rational Expectation assumptions, which will allow to use spreads at present time to predict the evolution of long and short rates in the future. To test it, we reframe the first statement by saying that the spreads $S_t^{(k,j)} = i_{k,t} - i_{j,t} \in I(0)$. Actually, we may check that only $S_t^{(k,1)} = i_{k,t} - i_{1,t} \in I(0)$, for each k, and that's what we will do.

ADF unit root tests for any spread $S_t^{(k,1)}$ are performed (in eViews, we've generated such series taking differences of interest rates with different maturities, by definition). Values of t-statistics and c.v. at 5% are presented in the following table (this time we allow for an intercept in the model):

	t-statistic	c.v. 5%
$S^{(m2,m1)}$	-2.52	-2.87
$S^{(m3,m1)}$	-2.28	-2.87
$S^{(m6,m1)}$	-11.35	-2.87
$S^{(y1,m1)}$	-8.22	-2.87
$S^{(y2,m1)}$	-6.65	-2.87
$S^{(y3,m1)}$	-6.07	-2.87
$S^{(y4,m1)}$	-5.74	-2.87
$S^{(y5,m1)}$	-5.51	-2.87

Table 8: Unit root test for spreads

What can be read from the table is that the null hypothesis of unit root is rejected whenever spreads are over more than 5 months. It follows that spreads

$$S_t^{(m6,m1)},\ S_t^{(y1,m1)},\ S_t^{(y2,m1)},\ S_t^{(y3,m1)},\ S_t^{(y4,m1)},\ S_t^{(y5,m1)} \in I(0).$$

Nevertheless, there is no statistical evidence of non-integration of the first two spreads, if we keep the confidence level at 5% (but even at 10% confidence the null hypothesis would not be rejected, as c.v. is -2.57).

This doesn't mean series $i_{1,t}, i_{2,t}$ and $i_{3,t}$ are not whatsoever cointegrated: their cointegrating vector might be different from $[-1,1,0]$ and $[-1,0,-1]$, and hereafter we're going to prove it with two kinds of test. The first test we perform is an Engle-Granger cointegration test for both couples $(i_{1,t}, i_{2,t})$ and $(i_{1,t}, i_{3,t})$. Null hypotheses of non-cointegration are rejected (with E-G test we can rely on given p-values as they take into account that the residual are used):

Date: 12/10/19 Time: 17:36
Series: M1 M2
Sample: 1961M01 2008M12
Included observations: 576
Null hypothesis: Series are not cointegrated
Cointegrating equation deterministics: C
Automatic lags specification based on Schwarz criterion (maxlag=18)

Dependent	tau-statistic	Prob.*	z-statistic	Prob.*
M1	-3.706958	$\boxed{0.0188}$	-36.33291	0.0015
M2	-3.372923	0.0468	-29.06319	0.0077

*MacKinnon (1996) p-values.

Intermediate Results:

	M1	M2
Rho - 1	-0.342896	-0.293243
Rho S.E.	0.092501	0.086940
Residual variance	0.097529	0.103270
Long-run residual variance	0.003442	0.003189
Number of lags	11	11
Number of observations	564	564
Number of stochastic trends**	2	2

**Number of stochastic trends in asymptotic distribution

Table 9: E-G test of cointegration: couple $i_{1,t}$, $i_{2,t}$

Date: 12/10/19 Time: 17:40
Series: M1 M3
Sample: 1961M01 2008M12
Included observations: 576
Null hypothesis: Series are not cointegrated
Cointegrating equation deterministics: C
Automatic lags specification based on Schwarz criterion (maxlag=18)

Dependent	tau-statistic	Prob.*	z-statistic	Prob.*
M1	-4.234602	$\boxed{0.0035}$	-55.57150	0.0000
M3	-3.711761	0.0186	-38.86840	0.0008

*MacKinnon (1996) p-values.

Intermediate Results:

	M1	M3
Rho - 1	-0.383715	-0.310488
Rho S.E.	0.090614	0.083650
Residual variance	0.132252	0.144630
Long-run residual variance	0.008751	0.007151
Number of lags	12	12
Number of observations	563	563
Number of stochastic trends**	2	2

**Number of stochastic trends in asymptotic distribution

Table 10: E-G test of cointegration: couple $i_{1,t}$, $i_{3,t}$

Secondly, we estimate a cointegrated model for couples $(i_{1,t}, i_{2,t})$ and $(i_{1,t}, i_{3,t})$ using Vector Error Correction. We set the number of cointegrating equation to 1, allowing for an intercept, and select 2 lags for differences in VAR without intercept. The outputs are:

Vector Error Correction Estimates
Date: 12/10/19 Time: 14:47
Sample (adjusted): 1961M04 2008M12
Included observations: 573 after adjustments
Standard errors in ( ) & t-statistics in [ ]

Cointegrating Eq:	CointEq1	
M1(-1)	1.000000	
M2(-1)	-0.946332	
	(0.00734)	
	[-128.881]	
C	-0.057884	
	(0.04478)	
	[-1.29255]	
Error Correction:	**D(M1)**	**D(M2)**
CointEq1	-0.935185	-0.302364
	(0.11256)	(0.09981)
	[-8.30833]	[-3.02932]
D(M1(-1))	-0.102497	0.155019
	(0.09985)	(0.08854)
	[-1.02652]	[1.75080]
D(M1(-2))	-0.004704	0.128343
	(0.07533)	(0.06680)
	[-0.06244]	[1.92130]
D(M2(-1))	0.317330	0.005668
	(0.10953)	(0.09712)
	[2.89722]	[0.05836]
D(M2(-2))	-0.040576	-0.247288
	(0.09068)	(0.08041)
	[-0.44748]	[-3.07543]
R-squared	0.283813	0.054607
Adj. R-squared	0.278769	0.047950
Sum sq. resids	195.9957	154.1164
S.E. equation	0.587421	0.520895
F-statistic	56.27213	8.202139
Log likelihood	-505.6966	-436.8265
Akaike AIC	1.782536	1.542152
Schwarz SC	1.820502	1.580118
Mean dependent	-0.003745	-0.003939
S.D. dependent	0.691691	0.533851
Determinant resid covariance (dof adj.)		0.028319
Determinant resid covariance		0.027827
Log likelihood		-599.9280
Akaike information criterion		2.139365
Schwarz criterion		2.238076
Number of coefficients		13

Table 11: VECM on interest rates $i_{1,t}$ and $i_{2,t}$

Vector Error Correction Estimates
Date: 12/10/19 Time: 17:11
Sample (adjusted): 1961M04 2008M12
Included observations: 573 after adjustments
Standard errors in ( ) & t-statistics in [ ]

Cointegrating Eq:	CointEq1	
M1(-1)	1.000000	
M3(-1)	-0.928226	
	(0.00853)	
	[-108.817]	
C	-0.033138	
	(0.05325)	
	[-0.62230]	
Error Correction:	**D(M1)**	**D(M3)**
CointEq1	-0.806303	-0.156319
	(0.08773)	(0.07534)
	[-9.19121]	[-2.07486]
D(M1(-1))	-0.009425	0.070506
	(0.08053)	(0.06916)
	[-0.11704]	[1.01946]
D(M1(-2))	0.104799	0.075158
	(0.06431)	(0.05523)
	[1.62960]	[1.36082]
D(M3(-1))	0.247386	0.071275
	(0.09137)	(0.07847)
	[2.70754]	[0.90832]
D(M3(-2))	-0.169791	-0.118100
	(0.08082)	(0.06941)
	[-2.10080]	[-1.70146]
R-squared	0.271006	0.030071
Adj. R-squared	0.265872	0.023240
Sum sq. resids	199.5005	147.1430
S.E. equation	0.592649	0.508974
F-statistic	52.78893	4.402463
Log likelihood	-510.7745	-423.5606
Akaike AIC	1.800260	1.495848
Schwarz SC	1.838226	1.533814
Mean dependent	-0.003745	-0.004036
S.D. dependent	0.691691	0.514994
Determinant resid covariance (dof adj.)		0.038084
Determinant resid covariance		0.037423
Log likelihood		-684.8137
Akaike information criterion		2.435650
Schwarz criterion		2.534361
Number of coefficients		13

Table 12: VECM on interest rates $i_{1,t}$ and $i_{3,t}$

Then, the estimated error correction model for $i_{1,t}$ and $i_{2,t}$ is

$$ECM_t = i_{1,t} - 0.95 \cdot i_{2,t} - 0.06$$

which is rewritten as

$$i_{2,t} = 1.06 \cdot i_{1,t} - 0.06 + \widehat{w}_t$$

with $w_t \in I(0)$. We immediately get that the difference between this model and model in which $i_{2,t} - i_{1,t} \in I(0)$ is very little (standard errors are low). A similar argument shows that

$$i_{3,t} = 1.08 \cdot i_{1,t} - 0.04 + \widehat{w}_t$$

and again cointegrating vector of $(i_{3,t}, i_{1,t})$ don't significantly differ from $[1, -1]$.

4. IMPULSE-RESPONSE OF INTEREST RATES

We devote our last part of the project to analyzing vector autoregression of interest rates, trying to determine how they influence each other. We shall focus on the causality between short-term rates and long-term rates by estimating VAR models of such series and plotting structuralized impulse-response function to display how much shocks of short rates weigh on long ones. For taking into account enough information without complicating the model, we decided to use the following vectors for the VAR:

$$\bar{Y}_t^1 = \begin{bmatrix} i_{1,t} \\ i_{2,t} \\ i_{3,t} \\ i_{12,t} \end{bmatrix} \quad , \quad \bar{Y}_t^2 = \begin{bmatrix} i_{1,t} \\ i_{2,t} \\ i_{3,t} \\ i_{24,t} \end{bmatrix} \quad , \quad \bar{Y}_t^5 = \begin{bmatrix} i_{1,t} \\ i_{2,t} \\ i_{3,t} \\ i_{60,t} \end{bmatrix}$$

First of all, the Shwarz criterion for lag length select in all three regressions 1 lag, so we run VAR estimations indicating 1 lag for endogenous variables, allowing for an exogenous intercept. Results for $\bar{Y}_t^1$ are shown here below:

Vector Autoregression Estimates
Date: 12/11/19 Time: 16:29
Sample (adjusted): 1961M02 2008M12
Included observations: 575 after adjustments
Standard errors in ( ) & t-statistics in [ ]

	M1	M2	M3	Y1
M1(-1)	-0.025478	-0.088105	-0.093087	-0.066075
	(0.07719)	(0.06505)	(0.06618)	(0.06518)
	[-0.33005]	[-1.35452]	[-1.40648]	[-1.01369]
M2(-1)	0.720162	0.033301	0.023280	-0.237243
	(0.18541)	(0.15623)	(0.15896)	(0.15656)
	[3.88425]	[0.21316]	[0.14644]	[-1.51538]
M3(-1)	0.301333	1.094347	0.968143	0.437735
	(0.17965)	(0.15137)	(0.15403)	(0.15169)
	[1.67737]	[7.22942]	[6.28558]	[2.88565]
Y1(-1)	-0.063982	-0.078947	0.082528	0.842019
	(0.06113)	(0.05151)	(0.05241)	(0.05162)
	[-1.04670]	[-1.53273]	[1.57467]	[16.3132]
C	0.120464	0.086973	0.038671	0.135085
	(0.06295)	(0.05305)	(0.05398)	(0.05316)
	[1.91352]	[1.63956]	[0.71646]	[2.54117]
R-squared	0.949839	0.967107	0.967190	0.967891
Adj. R-squared	0.949487	0.966876	0.966960	0.967665
Sum sq. resids	202.7994	143.9907	149.0801	144.5990
S.E. equation	0.596480	0.502609	0.511414	0.503669
F-statistic	2698.337	4189.760	4200.698	4295.468
Log likelihood	-516.2707	-417.8111	-427.7973	-419.0230
Akaike AIC	1.813116	1.470647	1.505382	1.474863
Schwarz SC	1.850980	1.508511	1.543246	1.512727
Mean dependent	5.208259	5.442382	5.575271	5.977223
S.D. dependent	2.653952	2.761602	2.813528	2.800988

Determinant resid covariance (dof adj.)	3.59E-05
Determinant resid covariance	3.46E-05
Log likelihood	-310.8060
Akaike information criterion	1.150630
Schwarz criterion	1.302086
Number of coefficients	20

Although standard errors reach 18% in *M2(-1)* and *M3(-1)* coefficients, we see that in *M1(-1)* and *Y1(-1)* they don't go over 7%. We may plot impulse response for these latter components, as they also represent most short-term rate and most long-term rate in the vector $\bar{Y}_t^1$. Analogous argument is applied to $\bar{Y}_t^2$ and $\bar{Y}_t^5$. Finally, we must choose the variables ordering for Cholesky representation and since economic theory suggests that short term interest rate determines the long term rate at simultaneous stage, we choose the ordering "*m1 m2 m3 y1*".

IRF's are presented below and in the following page. What is relevant is that long term rates responses to short rates always depart from 0: specifically, responses of $i_{12,t}$ to shocks $\varepsilon_{1,t-j}$ are around 0.30 for any lag; responses of $i_{24,t}$ to shocks $\varepsilon_{1,t-j}$ are around 0.24 for any lag; responses of $i_{60,t}$ to shocks $\varepsilon_{1,t-j}$ are around 0.16 for the first 4 lags and then increase. On the contrary, short term rate $i_{1,t}$ doesn't nearly respond at all to shocks $\varepsilon_{12,t-j}$, $\varepsilon_{24,t-j}$, $\varepsilon_{60,t-j}$ for $j = 1,2,3,4$ and respond very moderately for further lags.

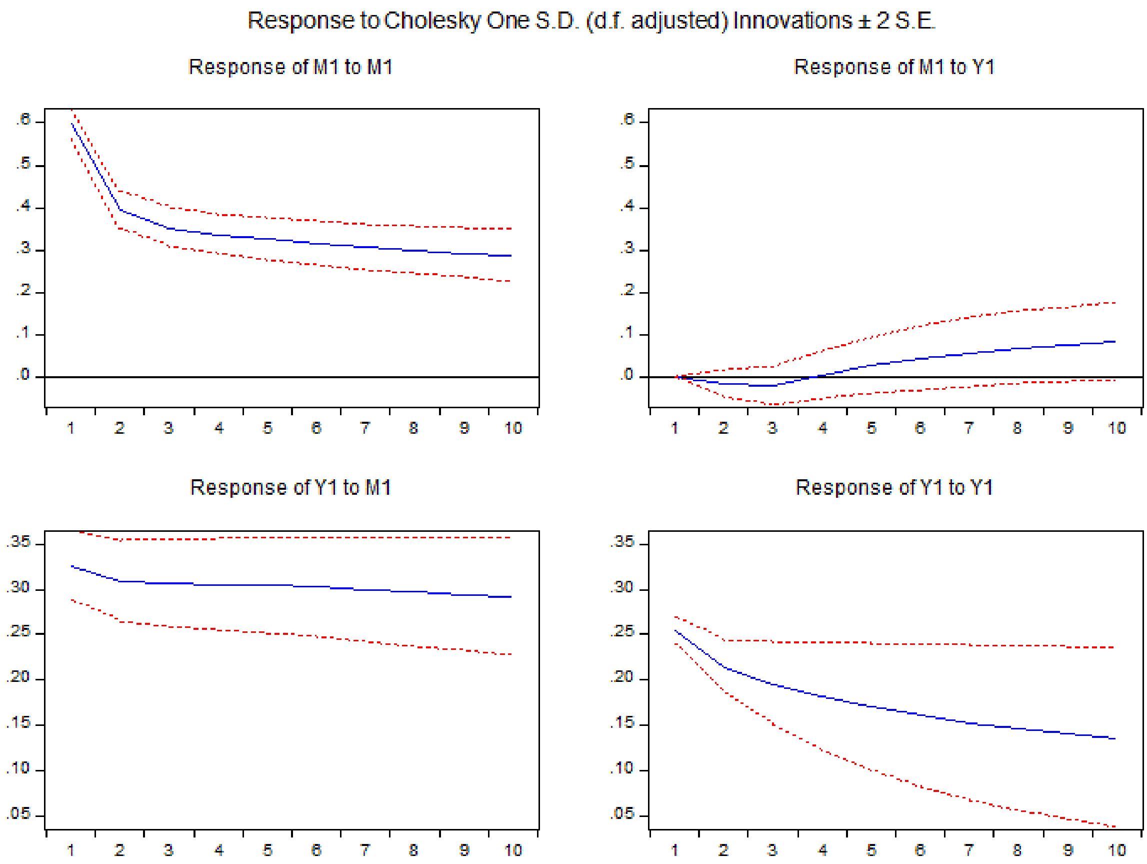


Figure 3: Structuralized IRF's of $i_{1,t}$ and $i_{12,t}$

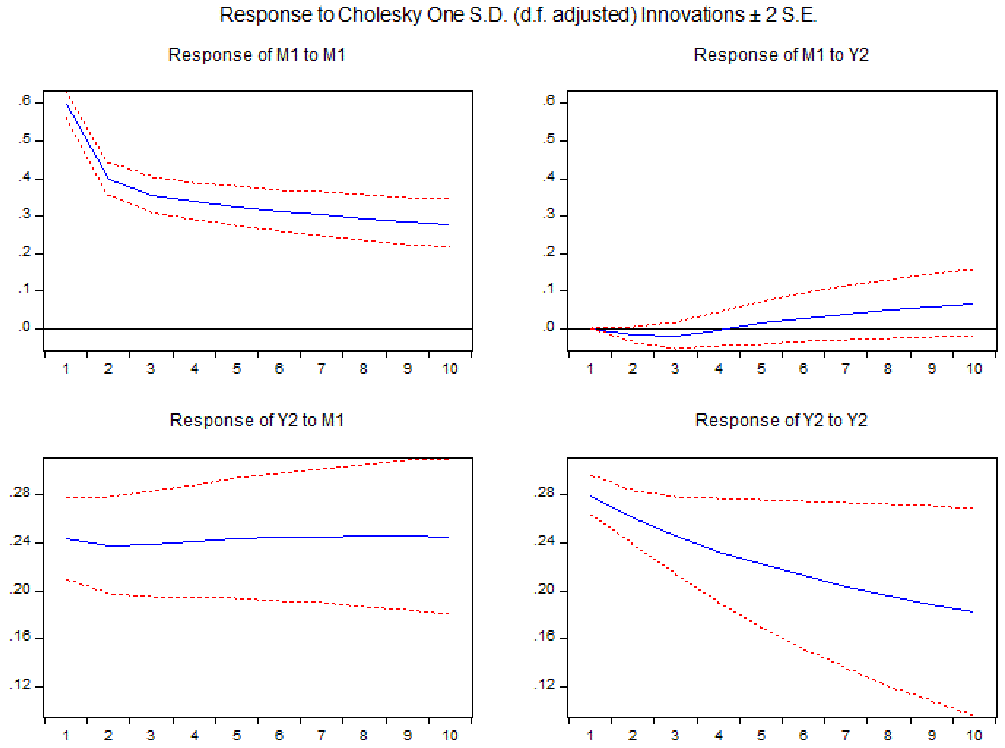


Figure 4: Structuralized IRF's of $i_{1,t}$ and $i_{24,t}$

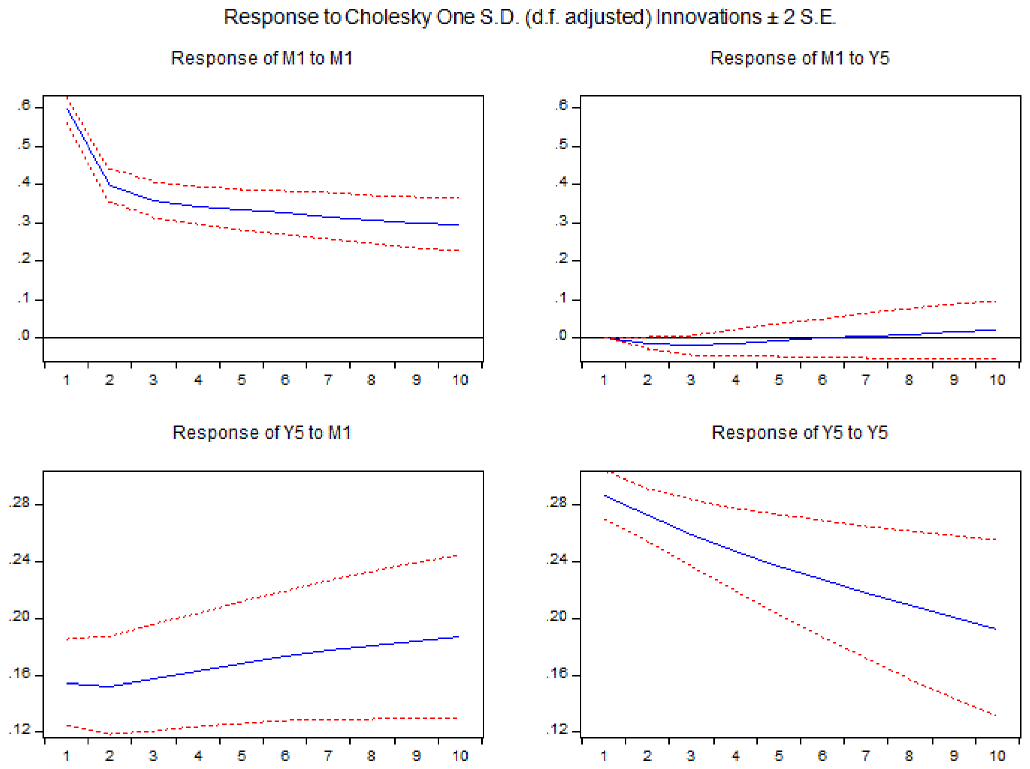


Figure 5: Structuralized IRF's of $i_{1,t}$ and $i_{60,t}$