Freddie Perez
MATH 611
HW1
9/8/2019

1 HW W_i is the wait time for the ith Person

1a) Determine $P(W_2 \geq C)$ for $C>0$. $T_i \sim Exp(\lambda)$ iid, $S_i \sim Exp(\mu)$ iid

$W_i = D_{i-1} - A_i$, we A_i is the Arrival time of ith Person and

D_i is the Departure time of the ith person

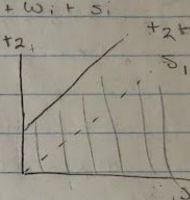
We know $W_2 = \begin{cases} 0, & \text{if } A_2 > D_1 \\ D_1 - A_2, & o/w \end{cases}$

$P(D_1 - A_2 > C)$ since $A_i = \sum T_i$, $D_i = A_i + W_i + S_i$

$= P(A_1 + S_1 + W_1 - A_2 \geq C)$, we Assume $W_1 = 0$

$= P(\cancel{T_1} + S_1 + 0 - \cancel{T_1} - T_2 \geq C)$

$= P(S_1 - T_2 \geq C)$



Joint pdf is $f(t_2, s_1) = \lambda e^{-\lambda t_2} \mu e^{-\mu s_1}$

$\therefore P(S_1 \geq C + T_2) = \int_0^\infty \int_{C+T_2}^\infty \lambda e^{-\lambda t_2} \mu e^{-\mu s_1}\, ds_1\, dt_2$

$= \int_0^\infty \lambda e^{-\lambda t_2} \int_{C+T_2}^\infty \mu e^{-\mu s_1}\, ds_1\, dt_2$

$= \int_0^\infty \lambda e^{-\lambda t_2} \left[-e^{-\mu s_1}\right]\Big|_{C+T_2}^\infty dt_2 = \int_0^\infty \lambda e^{-\lambda t_2} \left[0 - (-e^{-\mu(C+T_2)})\right] dt_2$

$= \int_0^\infty \lambda \exp\{-\lambda t_2 - \mu c - \mu t_2\}\, dt_2$

$= \int_0^\infty \lambda \exp\{-t_2(\mu+\lambda)\} \exp\{-\mu c\}\, dt_2$

$= \lambda \exp\{-\mu c\} \int_0^\infty \exp\{-t_2(\mu+\lambda)\}\, dt_2$

$= \lambda \exp\{-\mu c\} \left\{ \left(\frac{-1}{\mu+\lambda}\right) \exp\{t_2(\mu+\lambda)\} \right\}\Big|_0^\infty$

$= \frac{\lambda \exp\{-\mu c\}}{(\mu+\lambda)} = P(W_2 \geq C), \quad \forall C > 0$

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1b) $P(\omega_3 \geq C)$, where $\omega_3 = D_2 - A_3$, $A_3 = T_1 + T_2 + T_3$

$D_2 = A_2 + S_2 + \omega_2$, $A_2 = T_1 + T_2$

$\omega_2 = D_1 - A_2$, $A_2 = T_1 + T_2$

$D_1 = A_1 + S_1 + \cancel{\omega_1}^{0}$, $A_1 = T_1$

$\Rightarrow D_2 - A_3 = A_2 + S_2 + \omega_2 - A_3 = \cancel{T_1} + \cancel{T_2} + S_2 + \omega_2 - \cancel{T_1} - \cancel{T_2} - T_3$

$= S_2 + \omega_2 - T_3$, if $A_2 > D_1$, then $\omega_2 = 0$ such that

$P(\omega_3 \geq C) = P(S_2 - T_3 \geq C, A_2 \geq D_1) = P(S_2 - T_3 \geq C, T_2 > S_1)$ ①

Else if $A_2 < D_1$, then $\omega_2 = D_1 - A_2$ such that

$\Rightarrow S_2 + \omega_2 - T_3 = S_2 + D_1 - A_2 - T_3 = S_2 + A_1 + S_1 + 0 - A_2 - T_3$

$= S_2 + \cancel{T_1} + S_1 - \cancel{T_1} - T_2 - T_3 = S_2 + S_1 - T_2 - T_3$ st

$P(\omega_3 \geq C) = P(S_2 + S_1 - T_2 - T_3 \geq C, A_2 < D_1)$

$= P(S_2 + S_1 - T_2 - T_3 \geq C, T_2 < S_1)$ ②

① $P(S_2 - T_3 > C, T_2 > S_1) = P(S_2 > C + T_3) \cdot P(T_2 > S_1)$ due to independence

$$= \left(\int_0^\infty \int_{C+T_3}^\infty \mu \exp(-\mu s_2) \lambda \exp(-\lambda t_3)\, ds_2\, dt_3\right)\left(\int_0^\infty \int_{s_1}^\infty \lambda \exp(-\lambda t_2) \mu \exp(-\mu s_1)\, dt_2\, ds_1\right)$$

② $P(S_2 + S_1 - T_2 - T_3 \geq C, T_2 < S_1) = P(S_2 + S_1 \geq C + T_2 + T_3, T_2 < S_1)$

$$= \int_0^\infty ds_1 \int_0^{s_1} dt_2 \int_0^\infty ds_2 \int_0^{s_2 + s_1 - C - T_2} dt_3$$

$$= \int_0^\infty \int_0^{s_1} \int_0^\infty \int_0^{s_2 + s_1 - C - T_2} \left(\mu e^{-\mu s_1}\right)\left(\lambda e^{-\lambda t_2}\right)\left(\mu e^{-\mu s_2}\right)\left(\lambda e^{-\lambda t_3}\right) ds_1\, dt_2\, ds_2\, dt_3$$

$P(\omega_3 \geq C) = ① + ②$

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(1e)

Using the Monte Carlo Approach, we will Assume

$$V_n = \begin{cases} 0, & \text{if } w_n < 1 \\ 1, & \text{if } w_n \geq 1 \end{cases} = I(w_n \geq 1) \text{ As an indicator function}$$

such that $E(V_n) = 0 \cdot P(V_n < 1) + 1 \cdot P(V_n \geq 1) = P(V_n \geq 1)$

in order to calculate the $E(V_n)$, we will use the Law of Large Numbers such that the

$$P(w_n \geq 1) \cong \frac{1}{m} \sum_{i=1}^{m} I \cdot (w_n^{(i)} \geq 1) \approx E(V_n)$$

As we Are only taking the number samples where $w_n^{(i)} \geq 1$ for the m^{th} person, and dividing By the total number of samples for the m^{th} person.