1. $\bar{A} = \begin{pmatrix} 1 & -2 & 3 & -1 & -1 & 2 \\ 1 & 1 & -1 & 1 & -2 & 1 \\ 2 & -1 & 1 & 0 & -2 & 2 \\ 2 & 2 & -t & 2 & -1 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & -2 & 3 & -1 & -1 & 2 \\ 0 & 3 & -4 & 2 & -1 & -1 \\ 0 & 3 & -5 & 2 & 0 & -2 \\ 0 & 6 & -11 & 4 & 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & -2 & 3 & -1 & -1 & 2 \\ 0 & 1 & -\frac{4}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & -3 & 0 & 3 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & -2 & 3 & -1 & -1 & 2 \\ & 1 & -\frac{4}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ & & 1 & 0 & -1 & 1 \\ & & 1 & 0 & -1 & -1 \end{pmatrix}$

从最后两条可以看出，此题无解.

2. $\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 & -3 & a \\ 0 & 1 & 2 & 2 & 6 & 3 \\ 5 & 4 & 3 & 3 & -1 & b \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -2 & -6 & a-3 \\ 0 & 1 & 2 & 2 & 6 & 3 \\ 0 & -1 & -2 & -2 & -6 & b-5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & -1 & -1 & -5 & -2 \\ & 1 & 2 & 2 & 6 & 3 \\ & & & & & -a \\ & & & & & 2-b \end{pmatrix}$

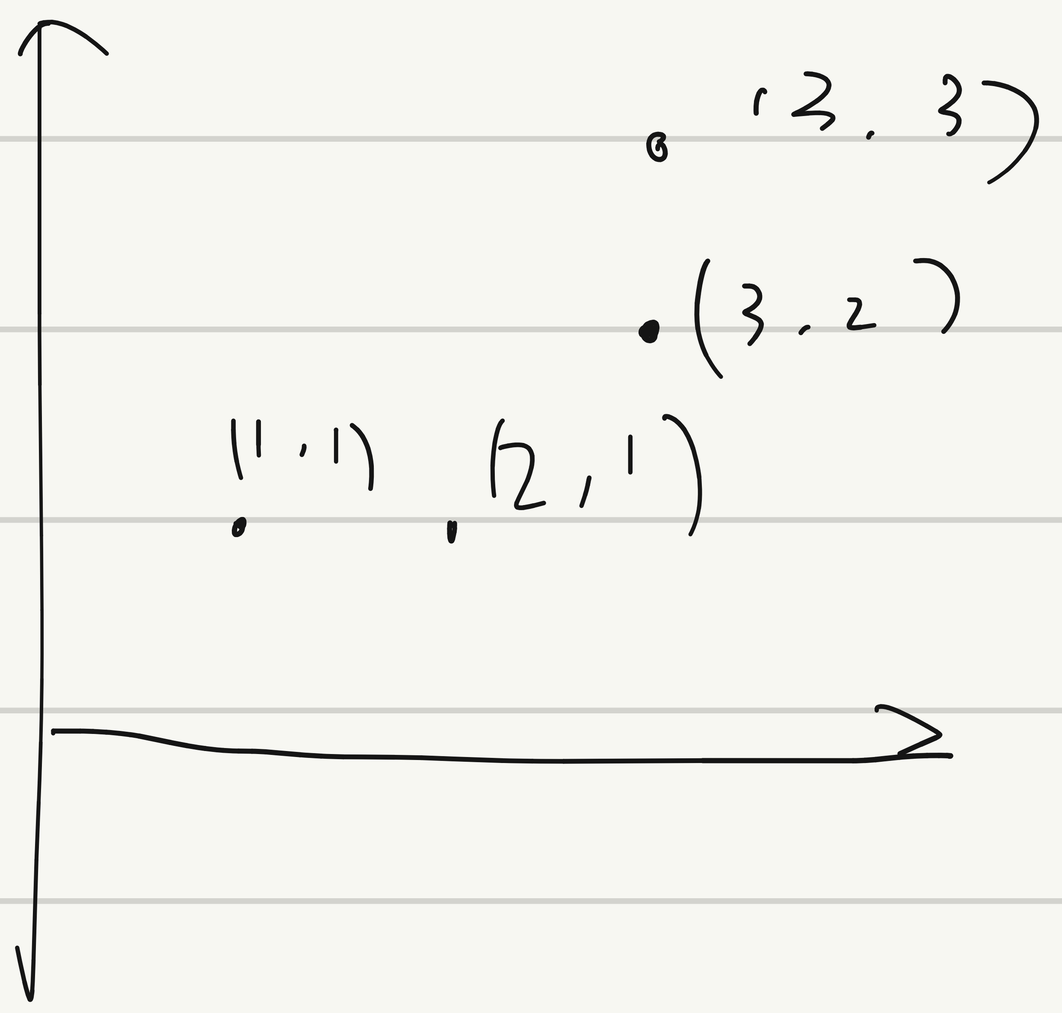
有解 $\Rightarrow \begin{cases} a=0 \\ b=2 \end{cases}$ ，且一般解为 $\begin{cases} x_1 = -2 + x_3 + x_4 + 5x_5 \\ x_2 = 3 - 2x_3 - 2x_4 - 6x_5 \\ x_3 = x_5 \\ x_4 = x_4 \\ x_5 = x_5 \end{cases}$

取 $y_0 = (-2, 3, 0, 0, 0)^T$ 为特解，通解为

$$y = \begin{pmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + k_1 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} 5 \\ -6 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

k_1, k_2, k_3 为任意常数.

3.



一般解法 $\min f(a,b) = \sum_{i=1}^{4} (ax_i + b - y_i)^2$

$$\begin{cases} \dfrac{\partial f}{\partial a} = 2\sum_{i=1}^{n} (ax_i + b - y_i)x_i = 0 \\ \dfrac{\partial f}{\partial b} = 2\sum_{i=1}^{n} (ax_i + b - y_i) = 0 \end{cases}$$

化简可得解为
$$\begin{cases} a = \dfrac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \\ b = \dfrac{1}{n}\sum_{i=1}^{n} y_i - \dfrac{a}{n}\sum_{i=1}^{n} x_i \end{cases}$$

$$a = \frac{4\cdot(1+2+6+9) - (1+2+3+3)(1+1+2+3)}{4\cdot(1^2+2^2+3^2+3^2) - (1+2+3+3)^2}$$

$$= \frac{4\cdot 18 - 9\cdot 7}{4\cdot 23 - 9\cdot 9} = \frac{72-63}{92-81} = \frac{9}{11}$$

$$b = \frac{1}{4}(1+1+2+3) - \frac{\frac{9}{11}}{4}(1+2+3+3)$$

$$= \frac{7}{4} - \frac{\frac{81}{11}}{4} = \frac{\frac{-4}{11}}{4} = -\frac{1}{11}$$

矩阵解法

$$c = (X^T X)^{-1} X^T y$$

$$X = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 3 & 1 \end{pmatrix} \qquad y = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 3 & 1 \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 23 & 9 \\ 9 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 18 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{4}{11} & 0 \\ -\frac{9}{11} & \frac{23}{11} \end{pmatrix} \begin{pmatrix} 18 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{11} & -\frac{1}{11} \end{pmatrix}$$

$$\begin{pmatrix} 23 & 9 & 1 & \\ 9 & 4 & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{9}{23} & \frac{1}{23} & \\ 0 & \frac{11}{23} & -\frac{9}{23} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{9}{23} & \frac{1}{23} & 0 \\ 0 & 1 & -\frac{9}{11} & \frac{23}{11} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \frac{4}{11} & -\frac{9}{11} \\ 0 & 1 & -\frac{9}{11} & \frac{23}{11} \end{pmatrix}$$

4

$$X = \begin{pmatrix} 1 & 5 & 3 & 1 \\ 4 & 2 & 6 & 3 \\ 1 & 4 & 3 & 2 \\ 4 & 4 & 1 & 1 \\ 5 & 5 & 2 & 3 \end{pmatrix}$$

$m=5$, $n=4$

5条 4维的数据.

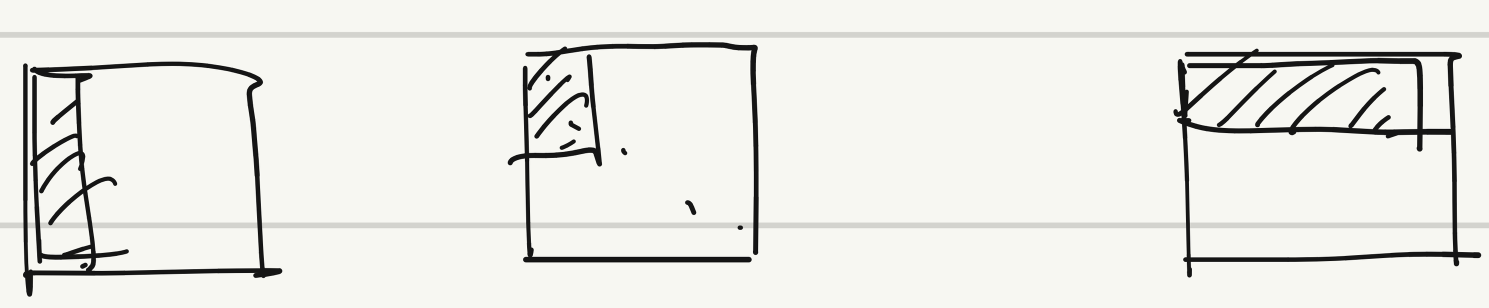
$$A = \frac{1}{m} X X^T = \begin{pmatrix} \frac{36}{5} & 7 & \frac{32}{5} & \frac{28}{5} & \frac{39}{5} \\ 7 & 13 & \frac{36}{5} & \frac{33}{5} & \frac{51}{5} \\ \frac{32}{5} & \frac{36}{5} & 6 & 5 & \frac{37}{5} \\ \frac{18}{5} & \frac{33}{5} & 5 & \frac{34}{5} & 9 \\ \frac{39}{5} & \frac{51}{5} & \frac{37}{5} & 9 & \frac{63}{5} \end{pmatrix}$$

$svd(A) = U \Sigma V^T$. (对应矩阵省略)

降至2维.

$$A_2 = U_2 \Sigma_2 V^T_2$$

$$= U[:, :2] \; \Sigma[:2, :2] \; V^T[:2, :]$$



$$= \begin{pmatrix} 6.01 & 7.36 & 5.58 & 6.09 & 8.52 \\ 7.36 & 12.57 & 7.50 & 6.40 & 10.01 \\ 5.58 & 7.50 & 5.30 & 5.47 & 7.84 \\ 6.09 & 6.40 & 5.47 & 6.46 & 8.76 \\ 8.52 & 10.01 & 7.84 & 8.76 & 12.14 \end{pmatrix}$$