Geometry and Statics of Optimal Freeform Gridshells

Baustatik – Baupraxis 14, l' 2020, Universität Stuttgart

Geometry and Statics of Optimal Freeform Gridshells

Davide Pellis[1], **Helmut Pottmann**[2]

[1]*TU Wien*
[2]*King Abdullah University of Science and Technology*

Abstract: We present our recent results on the geometric and static optimization of freeform load-bearing architectural skins. An efficient design strategy is the use of planar cladding panels supported by a prismatic framework substructure with optimized static performance. We show how these structures can be achieved discretizing membranes where principal stress and curvature directions coincide, and the absolute principal stresses are minimized. We outline then a design workflow and we provide some architectural examples.

1 Introduction

Our work tackles a prominent class of structures in freeform architecture, namely surface-like skins realized with cladding panels and supported by a framework substructure. These kind of structures are often referred to as *gridshells*. The actual manufacturing of freeform gridshells easily leads to unsustainable costs. This is mainly due to the high geometric complexity of building components and to the high mechanical demand.

It is well known that an efficient strategy to realize a gridshell is to use as subdivision layout a so-called *principal mesh*. This is a quadrilateral mesh whose edges follow the principal curvature directions of an implicit underlying surface. With this solution, we achieve flat cladding panels and a substructure with simplified connections. On the other hand, it is well known that an optimal static performance is attained when the substructure is subject to axial forces only. Principal meshes that are also in axial force equilibrium are then an efficient solution for the design of gridshells. However, principality and equilibrium are generally conflicting goals, and proper design tools for such meshes are currently missing.

In section 2, we show that indeed these meshes can only be achieved discretizing a surface in membrane equilibrium where principal stress and curvature directions coincide. In section 3 we go then further and wonder which is, among all frameworks in axial force equilibrium,

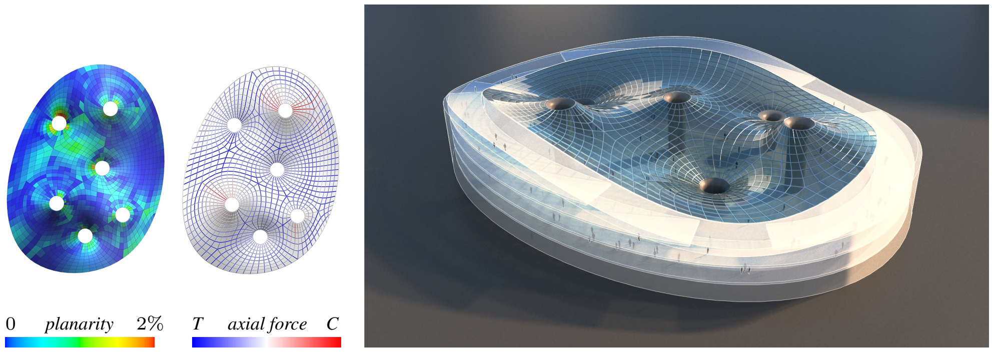

Figure 1: A freeform gridshell designed with our procedure. Cladding panels are planar (a), the substructure joints are torsion-free, and the beams are stressed only by axial forces (T = tension, C = compression) (b). Moreover, the volume of the load-bearing structure is minimized. Planarity is here computed as the distance between the face diagonals, expressed as percentage of the mean diagonal length. Values below 2% are compatible with flat glass cladding.

the most efficient one to span over a given boundary. To address this problem, we combine the works of J. C. Maxwell [6] and A. G. M. Michell [7] together with the recent results of architectural geometry on the equilibrium of self-supporting structures [1, 14, 15]. It turns out that such optimal solution is a quadrilateral mesh that follows the principal stress directions of a membrane that minimizes a certain total stress.

We present then a method for the design of surfaces where the total stress is minimal, together with stress and curvature alignment. Hence, we outline a design workflow. Thanks to this procedure, we achieve gridshells with planar cladding panels and a torsion-free substructure, which also minimize the use of structural material. We emphasize that with this method both the gridshell shape and the beam connectivity layout are part of the solution. More details can be found in [5, 10], while for a wider introduction to this topic we refer to [8].

2 Principal meshes in equilibrium

In this section, we start introducing principal meshes in architecture. We show then the condition for a principal mesh to be in axial equilibrium as well. This condition is found through a refinement process, where at the limit the mesh approaches a network of curves on a smooth surface, called *limit surface*. Accordingly, the gridshell becomes mechanically a fibrous membrane.

2.1 The geometry of gridshells

From a geometric point of view, a gridshell can be effectively described by a mesh. A mesh is a collection of *vertices*, *edges*, and *faces*, arranged together in such a way to form a polyhedral surface. The faces of the mesh can correspond to cladding panels, the edges to beam axes, and the vertices to beam joints. Particularly interesting are then those meshes whose faces are planar; in this case we can materialize the cladding panels with flat elements, simplifying considerably the manufacturing. Regarding the substructure, in case we use prismatic beams we shall want their symmetry planes to be approximately normal to the reference surface. It is then desirable that the symmetry planes meet in the nodes along a common axis. We achieve in this way a *torsion-free substructure*. This simplifies the manufacturing of nodes and improves their aesthetic. For a wide introduction to meshes in architecture, see [11, 12].

It is well known that every surface can be discretized by a mesh with planar faces and a torsion-free substructure if the edges of the mesh follow its *principal curvature network* (see [13]). This is a network of curves on the surface that at each point is tangent to the directions of maximal and minimal curvature, called *principal curvature directions*; differential geometry tells us that these two directions are orthogonal (see [4]). The resulting mesh, called *principal mesh*, has then a quadrilateral connectivity.

We shall see in section 3 that a quadrilateral connectivity arises also from static optimization. Furthermore, it turns out [9] that principal meshes are even the ones with the 'smoothest' visual appearance. This can be a desirable quality when panels are realized with reflective materials. For these reasons, when it comes to optimization, the choice falls on principal meshes.

2.2 Gridshells in axial force equilibrium

We focus now on the load-bearing behavior of a quadrilateral gridshell. As assumption, we consider the static loads as dominant and neglect dynamic and buckling effects, leaving them to subsequent structural verification. It is well known that the most efficient way to bear loads within a framework structure is through strictly axial forces. In this way, the structural material is stressed in the most efficient way and offers the highest stiffness. The mechanical model of our gridshell is therefore that of a *truss*. Note that a quadrilateral gridshell truss is a mechanism in equilibrium. However, even if the actual gridshell will be realized with rigid joints, the use of a truss model will minimize bending effects.

Let us now imagine to refine a quadrilateral gridshell truss until it approaches, at the limit, a network of fibers on the limit surface. Since these fibers cannot support bending, this limit structure can carry loads only in the tangent plane of the limit surface. It is therefore mechanically a fibrous membrane. The stress state is now described by a 2D stress tensor defined at each point in the tangent plane, and computed through membrane equilibrium equations. We call such a structure *truss-like membrane*.

2.3 Stress and curvature alignment

We now look for the in-plane condition of equilibrium for the two families of fibers in a quadrilateral truss-like membrane. For this end, let us first think of it as a continuous membrane. At a point, let us then imagine to do an infinitesimal line cut, and to substitute the continuum along this cut with a first infinitesimal fiber. To maintain equilibrium, a second infinitesimal fiber must be disposed along the direction of the two opposite forces exchanged across the cut. To insert this fiber, we shall do a second infinitesimal cut along its direction. If the two opposite forces exchanged across this second cut are aligned with the first fiber, then the two fibers locally equilibrate the in-plane stress state of the membrane.

Let us now consider a principal mesh in equilibrium. At the limit of refinement, this mesh will converge to a truss-like membrane with fibers aligned to the principal curvature directions of the limit surface. Since principal curvature directions are orthogonal, the in-plane fiber equilibrium requires now the forces exchanged across the two cuts to be orthogonal to the cuts themselves. This is exactly the definition of *principal stress directions* (a mathematical description can be found in [5, 10]). We can then state the following.

At the limit of refinement, principal meshes in equilibrium are discretizations of surfaces in membrane equilibrium where principal stress and principal curvature directions agree. There, they follow these directions.

3 Material-minimizing gridshells

We now go further and look, among all possible layouts in axial force equilibrium that span over a given boundary, for the most efficient one. To address this problem, we rely on the groundbreaking result of the engineer A. G. M. Michell, dating back to 1904. We extend then his result to freeform gridshells. For a more detailed treatment of this topic, see [5, 8].

3.1 Michell trusses

A. G. M. Michell in [7] solves the following problem: *Find the framework structure that requires the minimum amount of structural material to safely support a given set of loads*. He solves this problem under two main assumptions. First, the self-weight of the framework is negligible. Second, the optimal framework is only subject to axial forces in its members, and then it is a *truss*. A first remarkable result on truss optimization was given by J. C. Maxwell in 1872 [6]. He showed that:

Under a set of given forces, if a statically determinate truss is in equilibrium through only compressive or only tensile forces in its members, then its volume is minimal.

Moving from this result, Michell contrives a method far ahead of his times. He addresses the discrete problem of finding an optimal truss through a refinement process where, at the limit, the truss became a fibrous continuum, called *truss-like continuum*. This procedure relies on the fact that the refinement process increases the nodes of the truss, and then the degrees of freedom available for optimization; the optimum is thus reached at the limit by a truss-like continuum, while an optimal truss is an approximation of this optimum at a reasonable scale. Thanks to this procedure, Michell found out that:

In a volume minimal truss-like continuum, if a tension member meets a compression member, they must do so at right angles.

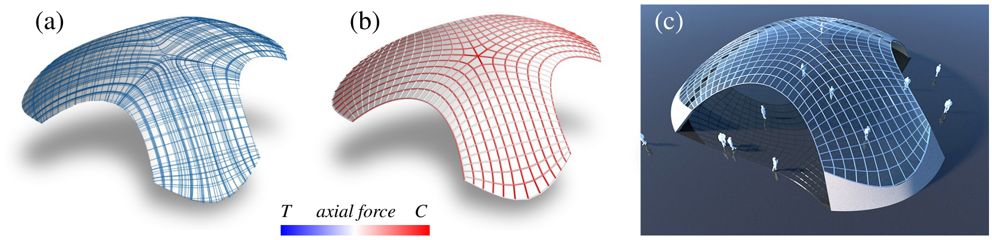

Figure 2: Material-minimizing gridshell. An optimal truss-like membrane with minimized total absolute stress, and with fibers aligned with principal stress directions (a). An optimal gridshell is a quad mesh approximating the stress lines of such optimal truss-like membrane (b). In (c) a possible architectural application is shown, together with boundary conditions.

3.2 Optimal gridshells

We now use Maxwell's and Michell's results to find volume-minimal gridshells. In this case, the assumption of neglecting the self-weight of the load-bearing framework is in general acceptable, in particular for steel-glass structures. The assumption of axial force equilibrium, as seen in section 2, implies that the truss-like continuum is now a truss-like membrane. Regarding the fiber layout, Michell's result implies that in tension-compression areas an optimal gridshell is an orthogonal quad network. According to Maxwell's result, we can keep the orthogonal quad network layout for tension-tension and compression-compression areas as well (note that this is not strictly necessary but convenient for practical reasons). As seen in section 2, axial force equilibrium together with orthogonality implies that the fiber layout is aligned with the principal stress directions of the membrane.

We compute now the volume of such a truss-like membrane. Let x_1, x_2 be a curvilinear coordinate system following the principal stress directions of the membrane. Let σ_1 and σ_2 be then the corresponding principal stresses. The axial force acting in an infinitesimal fiber in x_1 direction is given by $\sigma_1 \mathrm{d}x_2$, while the fiber's cross section area is directly proportional to the absolute value of the axial force. To get the infinitesimal volume, we multiply the cross section area by the infinitesimal length $\mathrm{d}x_1$. The volume of an infinitesimal fiber in x_1 direction is then directly proportional to $|\sigma_1|\, \mathrm{d}x_1 \mathrm{d}x_2$. The same holds for the infinitesimal fiber in x_2 direction. On a truss-like membrane, the minimal volume is then reached when

$$\iint (|\sigma_1| + |\sigma_2|)\, \mathrm{d}x_1 \mathrm{d}x_2 \mapsto \min.$$

We call this quantity *total absolute stress*. A volume optimal gridshell can then be found in the following way. First, we find the membrane that minimizes total absolute stress for the given boundary conditions, and then we discretize it with a quad mesh following its principal stress lines. If we want flat cladding panels and a torsion-free substructure as well, in addition we must ask the membrane to have its principal stress directions aligned with those of curvature.

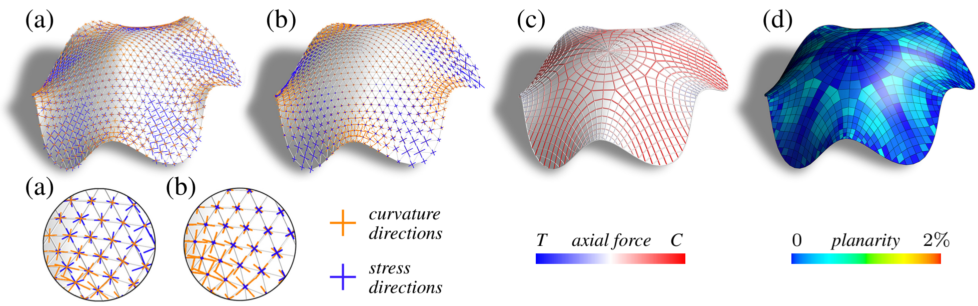

Figure 3: Design workflow. An initial shape is given as triangular mesh and the equilibrium is enforced on the edges. The estimated curvature and stress directions, in general, are not aligned (a). After optimization, we reach the alignment with a change in the shape. In this step, total absolute stress can also be minimized (b). We remesh the resulting shape with mixed integer quadrangulation along the principal curvature directions. After a post-optimization, the structure is in equilibrium under axial forces (c), and panels are close to planar (d).

4 Design workflow

We have now the elements to address the following problem: *Given a set of loads and boundary conditions, find a gridshell with planar faces, torsion-free layout, and with minimal need of load-bearing material to span over the given boundary.*

To implement this procedure, we model the reference membrane as triangular mesh. The membrane behavior can be achieved by modeling the mesh as a truss, with members given by the edges, with loads and supports applied to the vertices, and asking axial force equilibrium at each unsupported vertex. To estimate the principal stress directions, we compute at each vertex an equivalent stress tensor from the incoming beam forces, as outlined in [10]. The principal curvature directions can be derived at each vertex as well through the dihedral angles between faces, as described in [3]. Our design workflow is then the following (see figure 3).

- *Step 1*: A starting triangular mesh in membrane equilibrium is optimized for stress and curvature alignment and, at the same time, total absolute stress is minimized. If required, we can ask for closeness to a given reference surface, for interpolation of given points, and for other geometric constraints (see figure 5). Implementations of this step can be found in [5, 10].
- *Step 2*. From the resulting triangular mesh, we extract a principal mesh at an appropriate scale, chosen according to manufacturing considerations. At this purpose, the *mixed integer quadrangulation* method [2] can be used.
- *Step 3*. We have now the optimal mesh connectivity and form. This quad mesh can be post-optimized for axial force equilibrium and planar faces to improve the result. Thanks to steps 1 and 2, we can expect convergence with small changes in the shape. We can simultaneously optimize the layout for fairness of its polylines, to improve the aesthetic quality. For this step, the *guided projection* method [14] can be used.

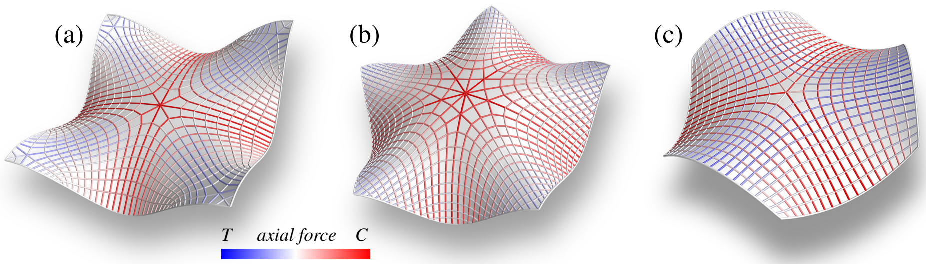

Figure 4: Grishells designed with our procedure. In designs (a) and (c) an additional constraint for alignment of the beam layout with the boundary has been superimposed in step 1. Mechanically, all boundaries are here supported.

5 Limitations

The main limitation of this approach is due to the lack of design control over the mesh layout. In some cases, the stress-curvature network could be not suitable for the extraction of architectural meshes. Indeed, the network layout may yield a mesh with a large variation of cell size, numerous or bad positioned singularities, or more generally, the resulting mesh may not possess the desired aesthetic qualities. However, the principal curvature network layout is highly sensible to shape changes; small modifications in the shape can often solve the issue. On the static side, the presented method does not consider buckling effects in the form finding criteria. Nevertheless, the buckling of a gridshell can be considered as a phenomenon connected with out-of-plane stiffness. While our method minimizes the cross sectional area of beams, the out-of-plane stiffness can be controlled, up to a certain extent, with an appropriate distribution of this area among the cross section to maximize the moment of inertia.

References

[1] BLOCK, P. ; OCHSENDORF, J. : Thrust network analysis: A new methodology for three-dimensional equilibrium. In: *Journal of International Association for Shell and Spatial Structures* 48 (2007), Nr. 3, S. 167–173

[2] BOMMES, D. ; ZIMMER, H. ; KOBBELT, L. : Mixed-integer quadrangulation. In: *ACM Transactions on Graphics* 28 (2009), Nr. 3, S. 77:1–77:10

[3] COHEN-STEINER, D. ; MORVAN, J. M.: Restricted Delaunay Triangulations and Normal Cycle. In: *Proceedings of the Nineteenth Annual Symposium on Computational Geometry*. New York : ACM, 2003, S. 312–321

[4] DO CARMO, M. : *Differential Geometry of Curves and Surfaces*. Prentice-Hall, 1976

[5] KILIAN, M. ; PELLIS, D. ; WALLNER, J. ; POTTMANN, H. : Material-minimizing forms and structures. In: *ACM Transactions on Graphics* 36 (2017), Nr. 6, S. 173:1–173:12

[6] MAXWELL, J. C.: On reciprocal figures, frames, and diagrams of forces. In: *Transactions of the Royal Society of Edinburgh* 26 (1872), S. 1–40

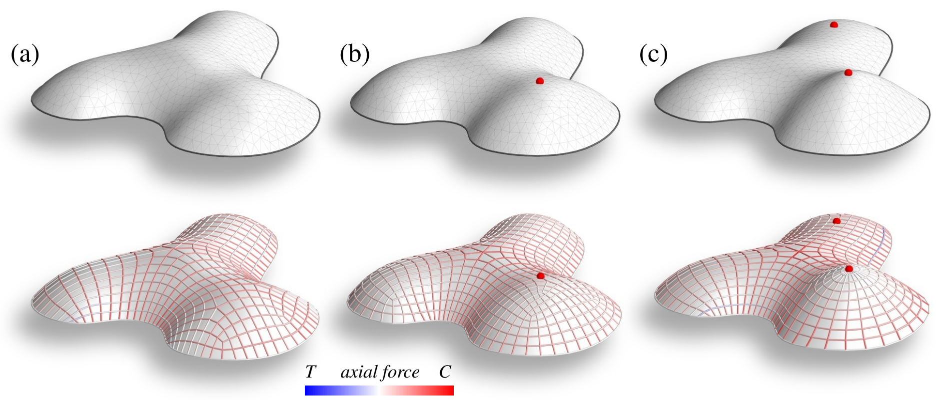

Figure 5: Additional constraints. In the bottom, an optimal gridshell spanning over a given boundary (a), and sub-optimal gridshells with one (b) and two (c) prescribed interpolation points (in red). On top, the corresponding triangulated membrane is shown.

[7] MICHELL, A. G. M.: The limits of economy of material in frame-structures. In: *Philosophical Magazine Series 6* 8 (1904), Nr. 47, S. 589–597

[8] PELLIS, D. : *Quad Meshes as Optimized Architectural Freeform Structures*, TU Wien, Diss., 2019

[9] PELLIS, D. ; KILIAN, M. ; DELLINGER, F. ; WALLNER, J. ; POTTMANN, H. : Visual Smoothness of Polyhedral Surfaces. In: *ACM Transactions on Graphics* 38 (2019), Nr. 4, S. 260:1–260:11

[10] PELLIS, D. ; POTTMANN, H. : Aligning principal stress and curvature directions. In: HESSELGREN, L. (Hrsg.): *Advances in Architectural Geometry 2018*, Klein Publishing, 2018

[11] POTTMANN, H. ; ASPERL, A. ; HOFER, M. ; KILIAN, A. : *Architectural geometry*. Exton : Bentley Institute Press, 2007

[12] POTTMANN, H. ; EIGENSATZ, M. ; VAXMAN, A. ; WALLNER, J. : Architectural Geometry. In: *Computers & Graphics* 47 (2015), S. 145–164

[13] POTTMANN, H. ; LIU, Y. ; WALLNER, J. ; BOBENKO, A. ; WANG, W. : Geometry of multi-layer freeform structures for architecture. In: *ACM Transactions on Graphics* 26 (2007), Nr. 3, S. 65:1–65:11

[14] TANG, C. ; SUN, X. ; GOMES, A. ; WALLNER, J. ; POTTMANN, H. : Form-finding with Polyhedral Meshes Made Simple. In: *ACM Transactions on Graphics* 33 (2014), Nr. 4, S. 70:1–70:9

[15] VOUGA, E. ; HÖBINGER, M. ; WALLNER, J. ; POTTMANN, H. : Design of self-supporting surfaces. In: *ACM Transactions on Graphics* 31 (2012), Nr. 4, S. 87:1–87:11