Estruturas de Informação

Priority Queues - Heaps

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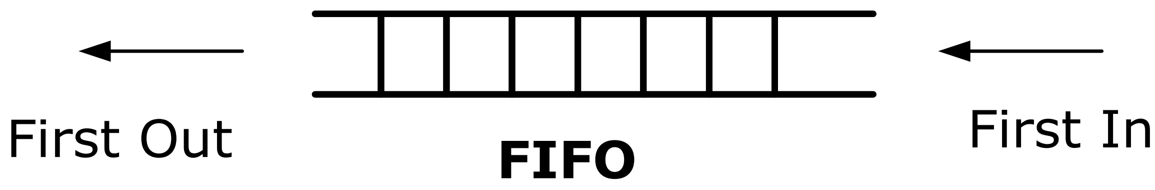
Queue

- Elements are inserted at one end, and removed from another
- Elements are in the queue in order of arrival



Queue interface

- addBack(newElement) O(?)
- front()
- removeFront() O(?)
- isEmpty()
- size()

Priority Queue

Associates a "priority" with each object:

– First element has the highest priority (typically, lowest value)

Examples of priority queues:

- to-do list with priorities
- hospital emergency queue
- air-traffic control
- active processes in an OS
- device controller for a shared printer

Priority Queue

- A priority queue is an abstract data type for storing a collection of prioritized elements
- The elements in a priority queue have a priority provided by its associated Key - each entry is a pair (key, value)
- The element with the minimal key will be the next to be removed from the queue

Priority queue interface

- insert(k,v)
- min()
- removeMin()
- size()
- isEmpty()

Priority Queue - Implementations

	Array	**Sorted Array**	**Sorted List**
insert(k,v)	O(1)	O(n)	O(n)
min()	O(n)	O(1)	O(1)
removeMin()	$O(n^2)$	O(n)	O(1)

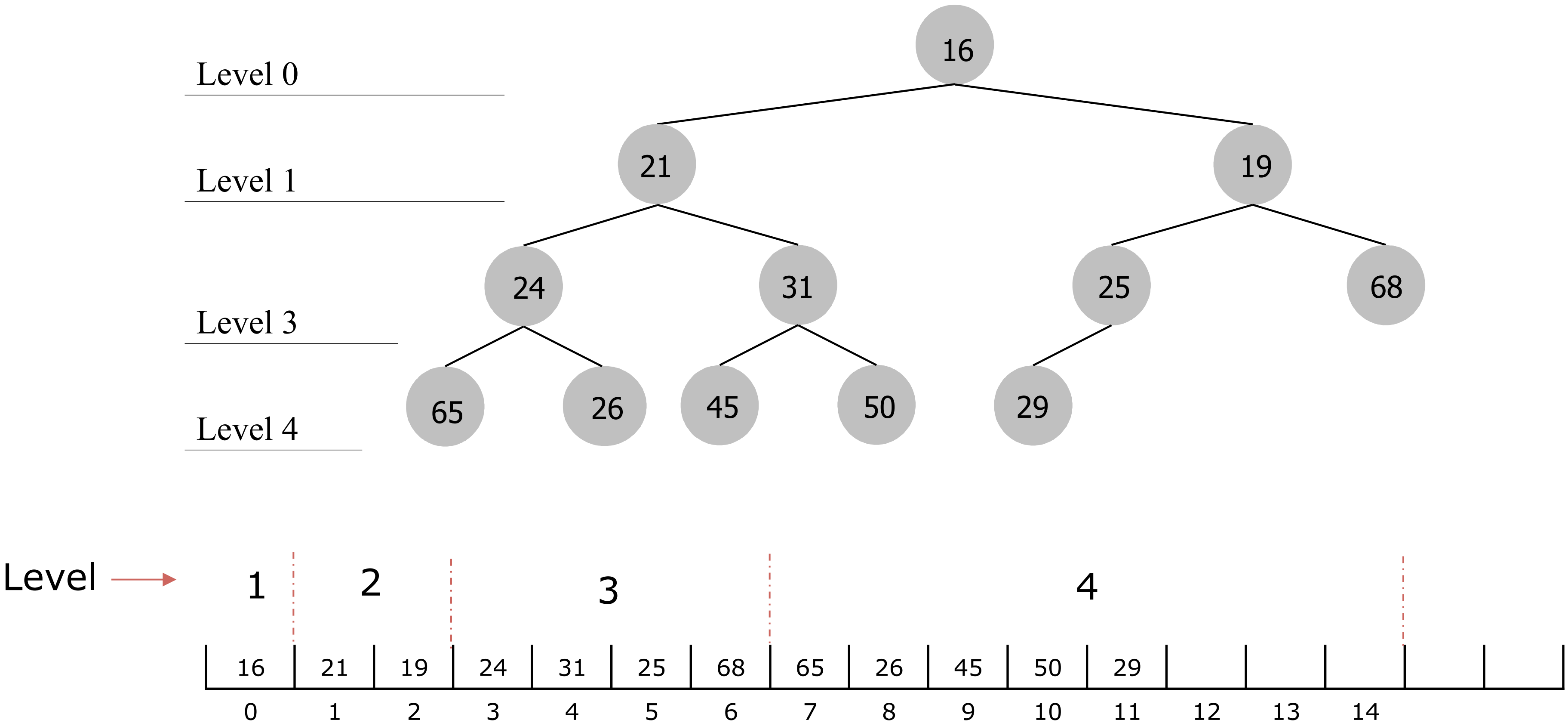
Any implementation using a sequential data structure implies operations with linear complexity

Alternative → Heap

Binary tree representation with a vector

A binary tree can be represented by a vector:

- Fill up the vector with the tree items ordered by level



Binary tree representation with a vector

How to access the elements?

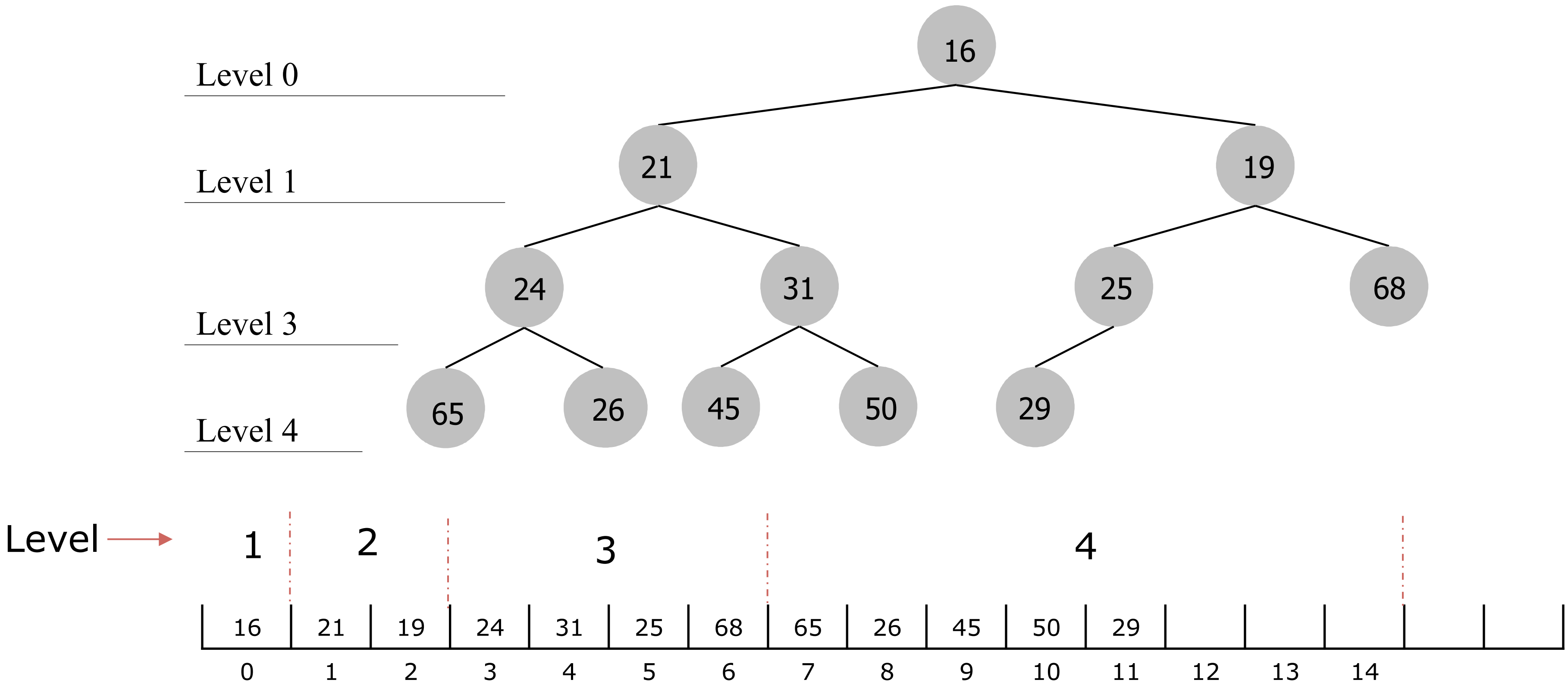
- Being the root at index 0
- Its direct descendants are in the indices 1 and 2
- The descendants of the element at index 3 are at indexes: 7 and 8
- ....

Generally to an element at index n:

- its left child is at index: 2n+1
- its right child is at index: 2n+2
- its parent is at index: (n-1)/2

These formulas allow transit between the elements of the different levels of the tree, using an alternative way to references

Binary tree representation with a vector



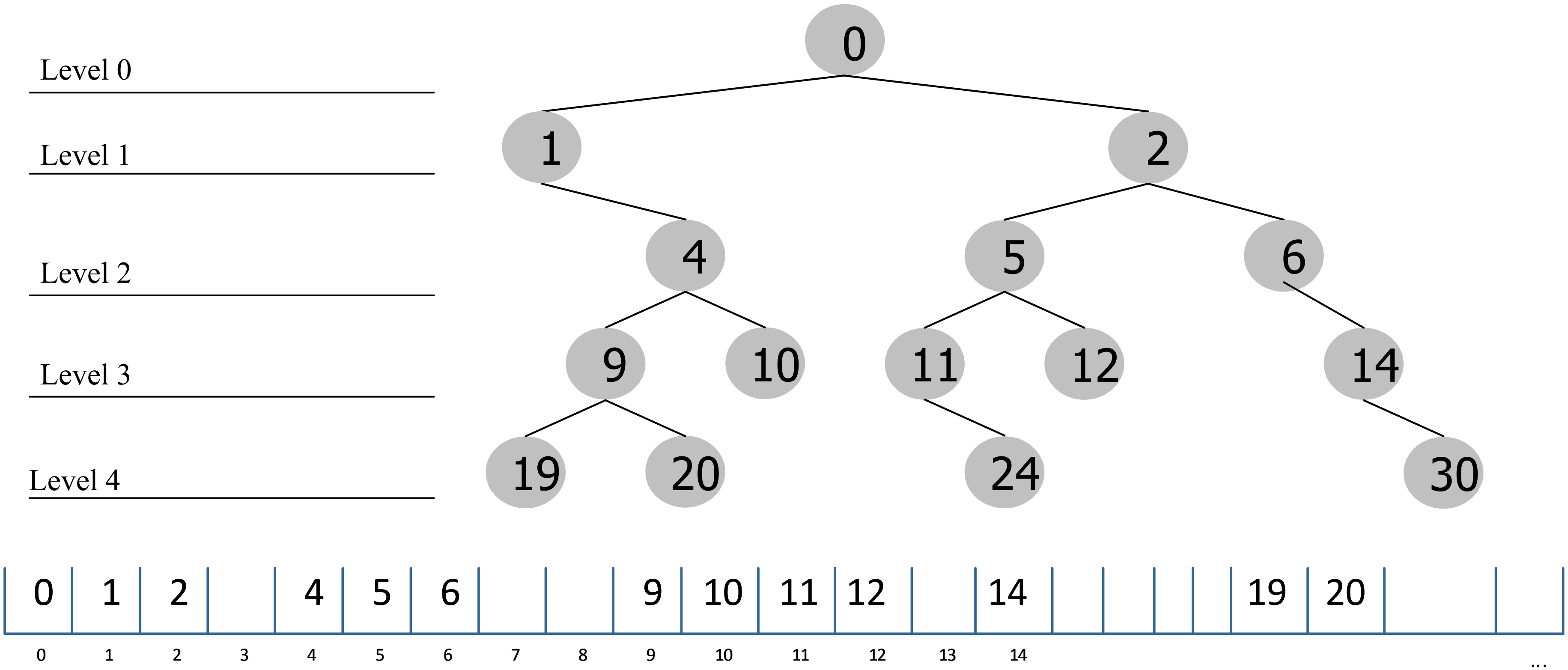
Predecessors of the node 50

50 → n=10 → (n - 1)/2 = 4 → vect[4] = 31

n= 4 → (n - 1)/2 = 1 → vect[1] = 21

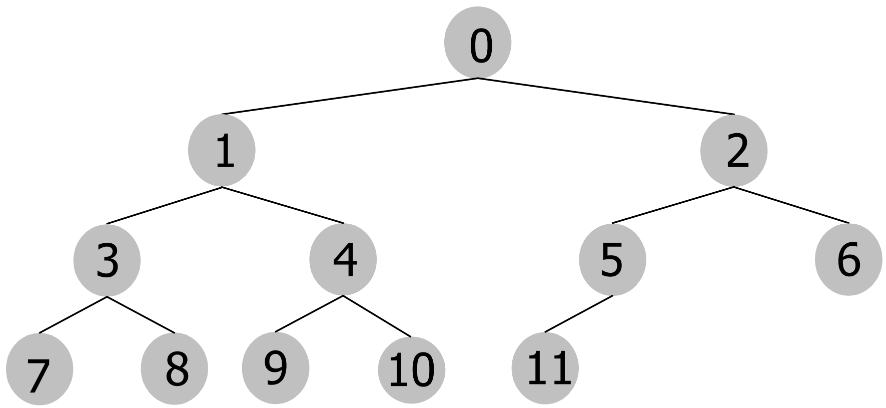
n= 1 → (n - 1)/2 = 0 → vect[0] = 16

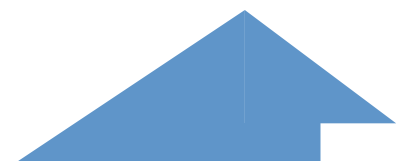
Vector representation problem



Complete binary tree

A complete binary tree is a tree where every leaf is at the same depth and the bottom level is filled from left to right





0	1	2	3	4	5	6	7	8	9	10	11					
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14		

A complete binary tree can be efficiently stored in a vector - the vector does not contain empty cells and all elements are contiguous

Array-based representation of binary trees

An array-based structure is adequate for representing a complete binary tree

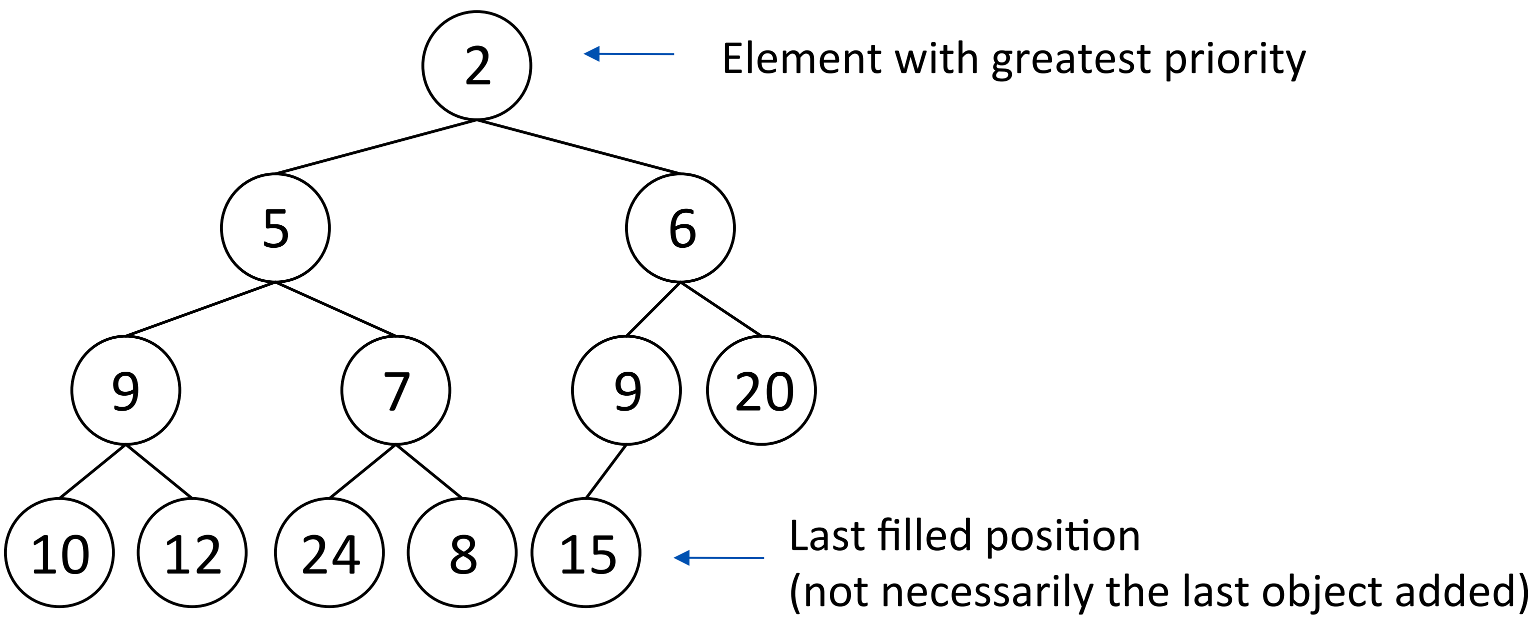
With an array implementation

- Operations: size, isEmpty, replace, root, parent, children, left, right, hasLeft, hasRight, isInternal, isExernal, isRoot take O(1) time
- Operations: elements, positions are O(n) time

Heap

Heap is a complete binary tree in which every node's value is less or equal its children values

Heap-order implies that each path in the tree is sorted



Heaps and Priority Queues

A heap can be used to efficiently implement a priority queue

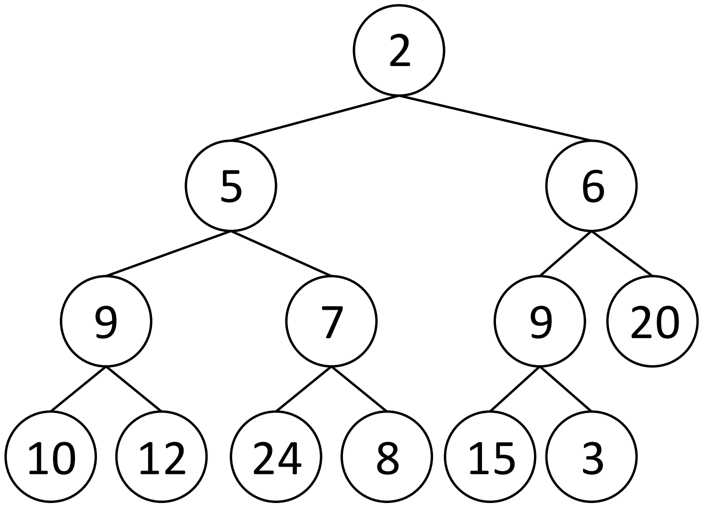
Each entry (key, element) is inserted at each node

- Two distinct entries in a priority queue can have the same key
- Keys in a priority queue can be arbitrary objects on which an order is defined
- A generic priority queue uses an external comparator object

Heap – Insertion

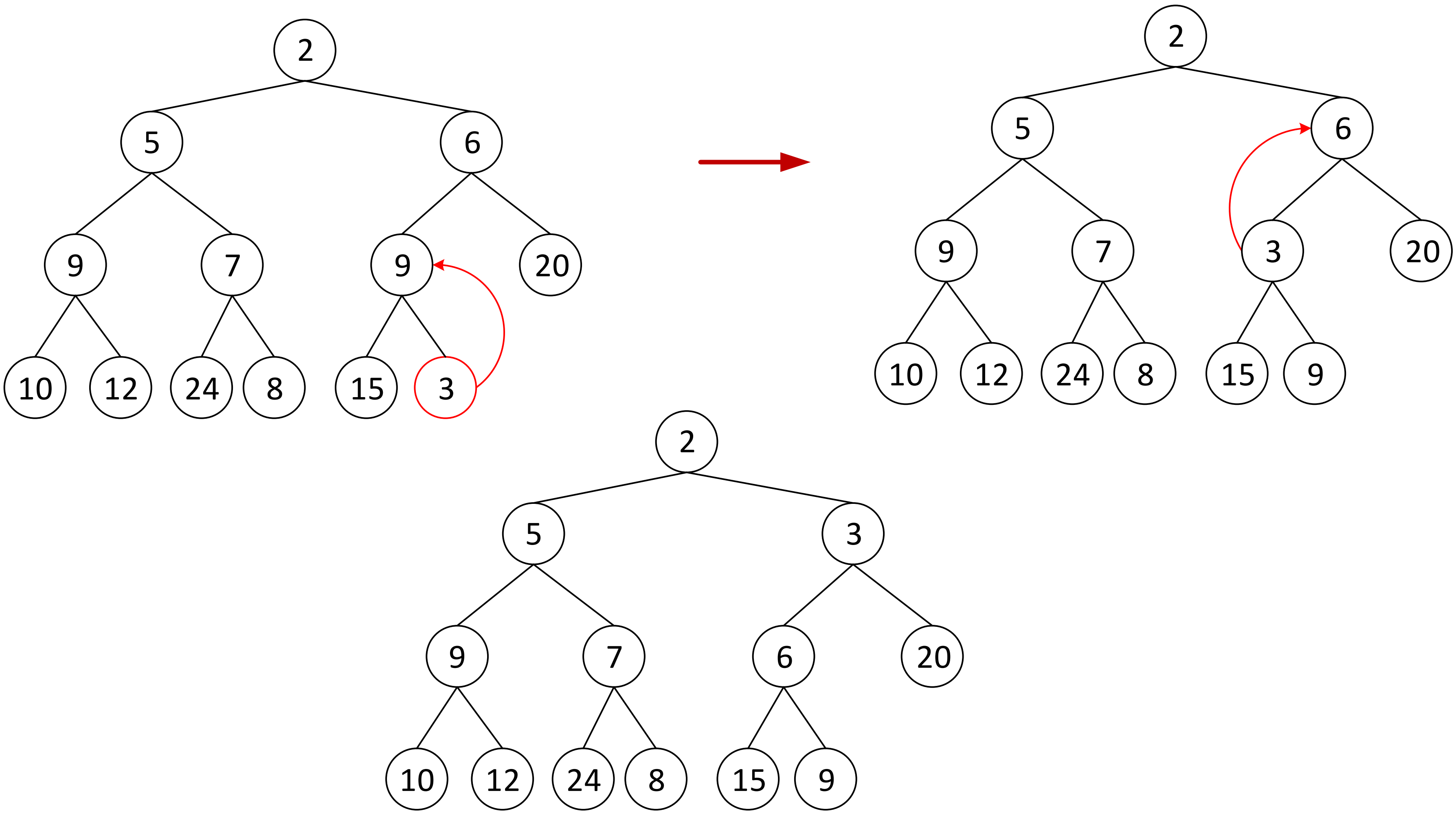
The insertion of an element in a heap must guarantee two properties:

1. the tree remains complete - the new element is inserted on the last level of the tree the rightmost possible
2. the tree remains orderly - Fix it by percolating up

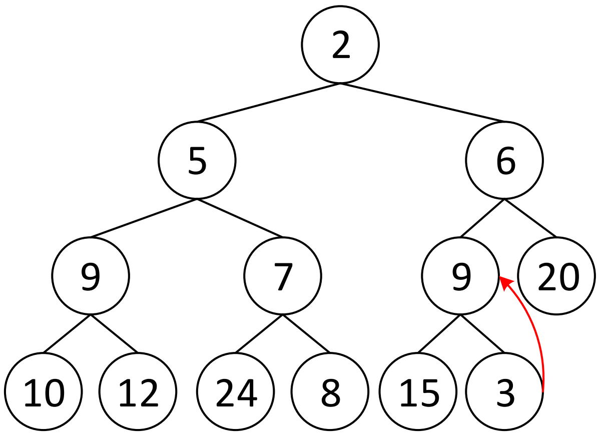


Heap – Insertion: Percolate up

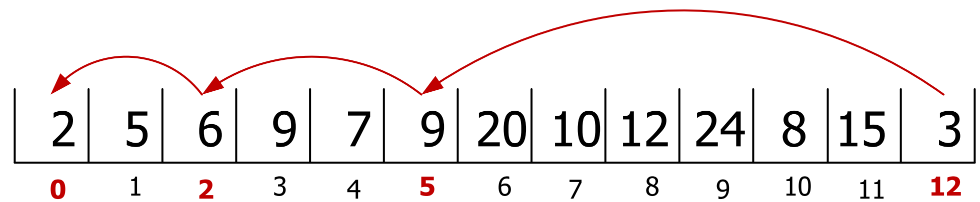
To maintain the tree ordered the new element must be put in the correct place



Percolate up



Parent's node at index n: (n-1)/2



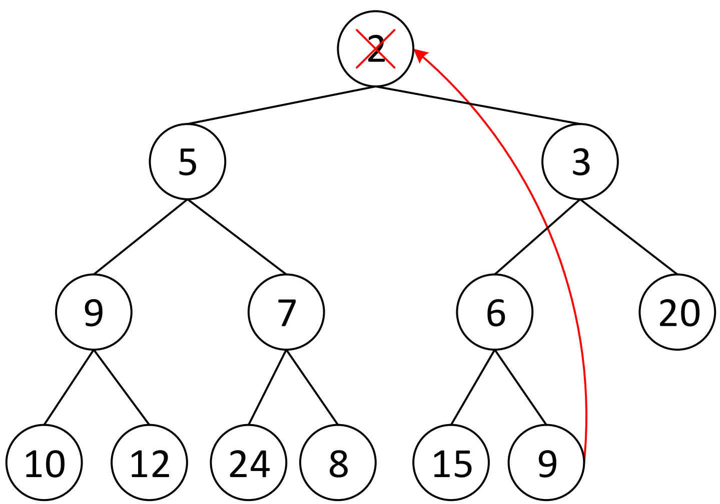

```
Algorithm percolateUp (int i){
     ind = (i-1)/2;
     while (ind>=0 && vector[i] < vector[ind]){
       swap(vector[i],vector[ind])
       i = ind
       ind = (i-1)/2
     }
}
```

Complexity(?)

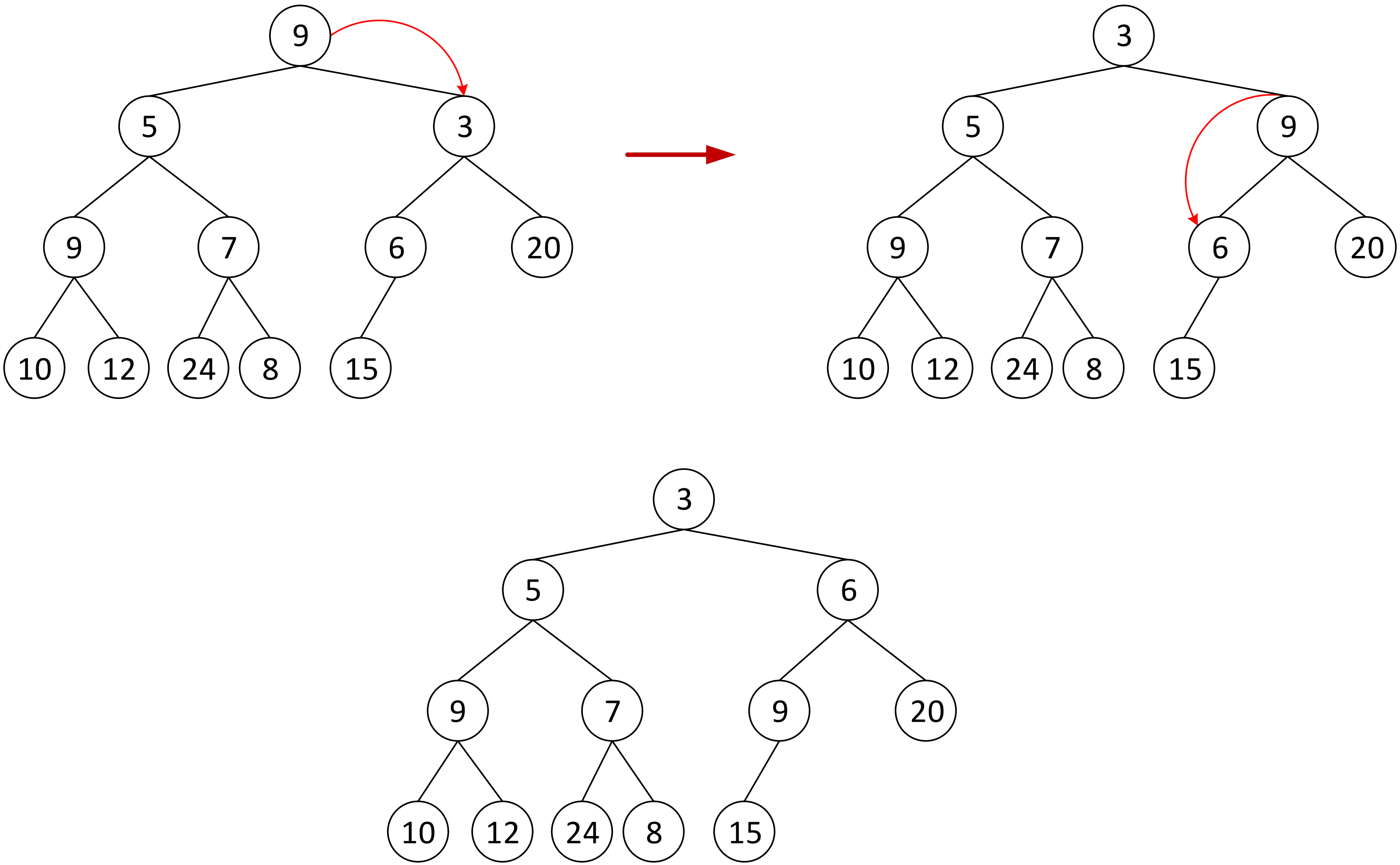
Heap – RemoveMin

The RemoveMin of an element in a heap must also guarantee the two properties:

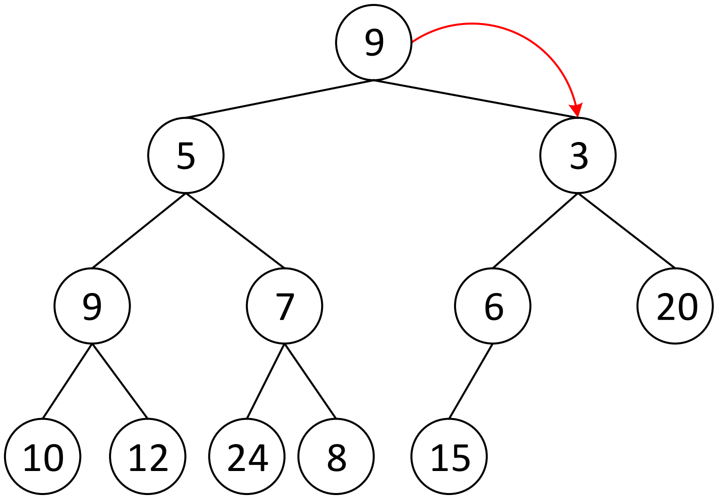
1. the tree remains complete – root is replaced by the rightmost element in the last level
2. the tree remains orderly – fix it by percolating down



Heap – RemoveMin: Percolate down



Percolate down



9	5	3	9	7	6	20	10	12	24	8	15	
0	1	**2**	3	4	5	6	7	8	9	10	11	12

3	5	9	9	7	6	20	10	12	24	8	15	
0	1	2	3	4	**5**	6	7	8	9	10	11	12

3	5	6	9	7	9	20	10	12	24	8	15	
0	1	2	3	4	5	6	7	8	9	10	11	12

Percolate down

```
Algorithm percolateDown (int i) {
  indLeft = 2×i+1
  indRight = 2×i+2
  swaps=true
  while (indLeft < vector.size() && swaps) {
       smallindex = indLeft
       if (indRight < vector.size())
          if (vector[indRight] < vector[indLeft])
            smallindex = indRight

       if (vector[i] > vector[smallindex]) {
          swap(vector[i],vector[smallindex]) //change the elem by the
          i = smallindex                     //child with the highest
          indLeft = 2×i+1                    //priority
          indRight = 2×i+2
      }
      else
          swaps=false;
  }
}
```

Complexity(?)

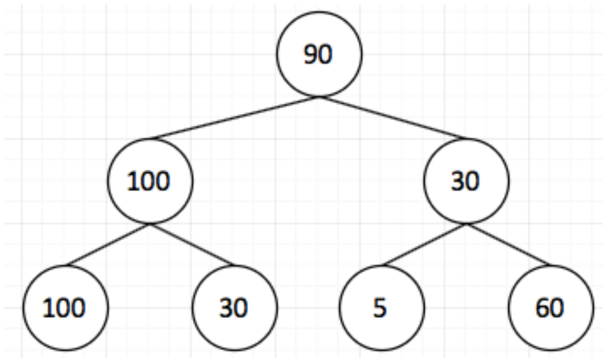
Batch Bottom-Up Heap Construction

- If we start with an initially empty heap, n successive calls to the insert operation will run in O(nlog n) time, in the worst case

- However, if all n key-value pairs to be stored in the heap **are given in advance**, there is an alternative batch bottom-up construction method more efficient

- Intuitively, the **bottom-up heap construction** performs a single percolate-down operation at each internal node of the tree, rather than a single percolate-up operation from each

Batch Bottom-Up Heap Construction



90	100	30	20	10	5	60
0	**1**	**2**	**3**	**4**	**5**	**6**

Start index

The bottom-up heap construction:

- Starts at the parent of last entry
- Performs percolate-down operation of each internal node of the tree until reaches the root

Asymptotic Analysis of Bottom-Up Heap Construction

- Bottom-up heap construction is asymptotically faster than incrementally inserting n entries into an initially empty heap
- The primary cost of the bottom-up heap construction is due to the percolate-down steps performed at each non-leaf position
- Since more nodes are closer to the bottom of a tree than the top, the sum of the percolate-down paths is linear
- Bottom-up construction of a heap with n entries takes O(n) time, assuming two keys can be compared in O(1) time

Priority Queue - Performance Evaluation

	Array	Sorted Array	Sorted List	Heap
insert(k,v)	$O(1)$	$O(n)$	$O(n)$	$O(\log n)$
min()	$O(n)$	$O(1)$	$O(1)$	$O(1)$
removeMin()	$O(n^2)$	$O(n)$	$O(1)$	$O(\log n)$

The main purpose of a priority queue is rapidly accessing and removing the smallest element!

HeapSort Algorithm

Sort a collection using a Heap:

1. Build a heap using the elements of the collection
2. Extract all elements from heap and insert them into the collection

```
Algorithm heapSort (ArrayList<E> vector) {
   for (int i = 0; i < vector.size(); i++)
      insert(i,v[i]);
   for (int i = 0; i < vector.size(); i++)
       v[i] = removeMin()
}
```

1| 28 | 14 | 5 | 20 | 6

1| 5 | 6 | 28 | 20 | 14

1| 5 | 6 | 14 | 20 | 28

Complexity(?)

Comparison of Sorting Algorithms

Algorithm	Best Case	Worst case
SelectionSort	$O(n^2)$	$O(n^2)$
BubbleSort	$O(n)$	$O(n^2)$
InsertionSort	$O(n)$	$O(n^2)$
MergeSort	$O(n\log n)$	$O(n\log n)$
QuickSort	$O(n\log n)$	$O(n^2)$
HeapSort	$O(n\log n)$	$O(n\log n)$

Priority Queue Application: Simulation

Original, and one of most important, applications

Discrete event driven simulation:

- Actions represented by "events" – things that have (or will) happen at a given time
- Priority queue maintains list of pending events → Highest priority is the next event
- Event pulled from list is executed → often spawns more events, which are inserted into priority queue
- Loop until everything happens, or until fixed time is reached

Priority Queue Application: Example

Example: Ice cream store

– People arrive

– People order

– People leave

Simulation algorithm:

1. Determine time of each event using random number generator with some distribution
2. Put all events in priority queue based on when it happens
3. Simulation framework pulls minimum (next to happen) and executes the event