Elasticity
Stress

stress tensors, stress-strain relation, Hooke‘s law, elastic tensor, elastic parameters, Lame parameters, shear modulus, compressibility, stress drop, stress on faults

What are the forces when an elastic body deforms?

Questions

What is ***stress*** in the Earth?

What are the **physical units**?

How does stress vary **inside the Ear**th?

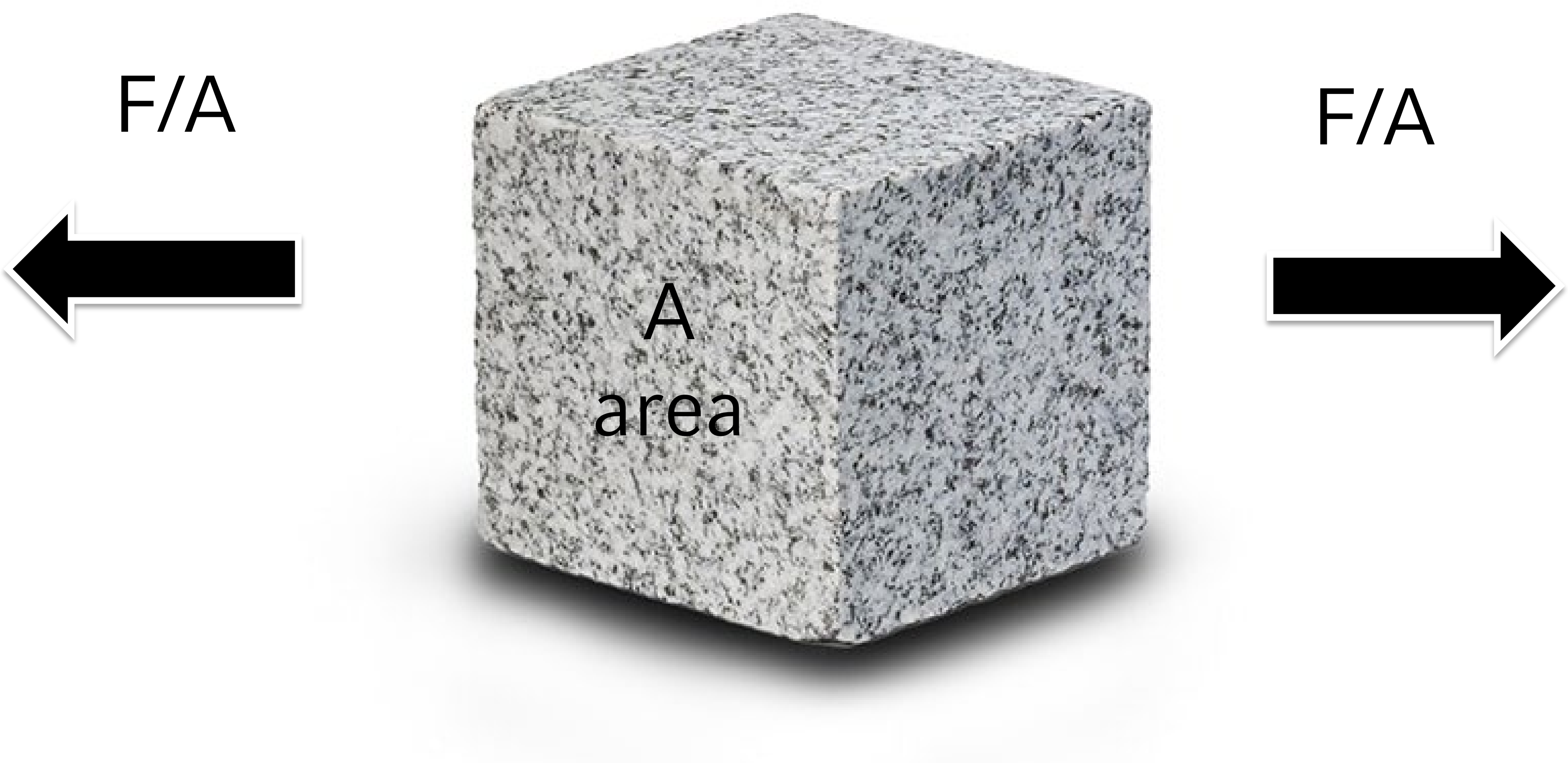
Can stress be **directly observed**?

What is the **stress change (stress drop)** during an earthquake?

What is the stress of **seismic waves**?

What **types** of stress are there?

Stress and strain



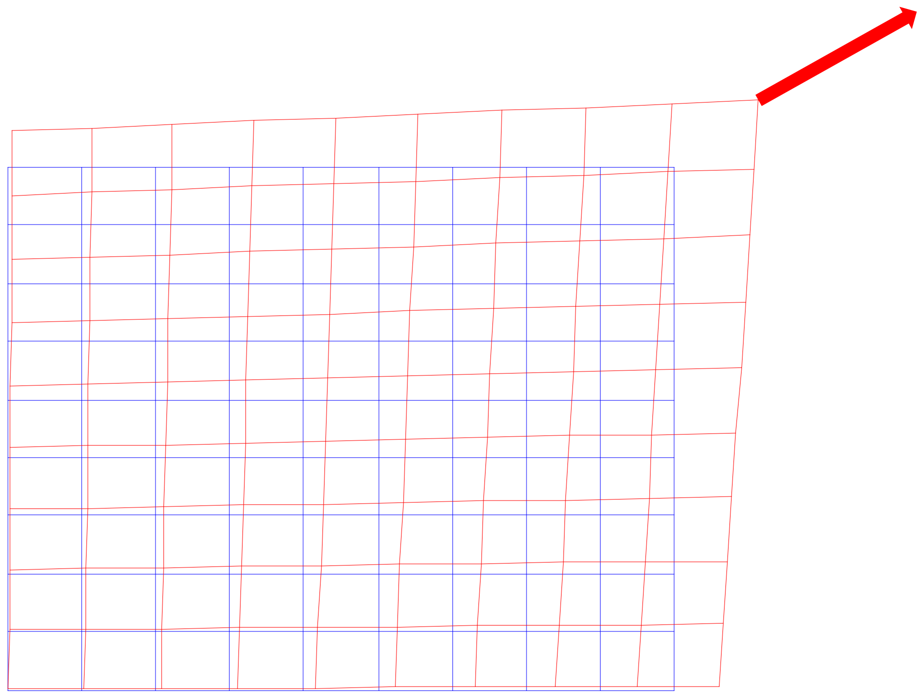
Recall deformation

$$\varepsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i \right)$$

$$i,j = 1,2,3$$

$$u_i \rightarrow u_i(x,y,z,t)$$

$$\varepsilon_{ij} \rightarrow \varepsilon_{ij}(x,y,z,t)$$

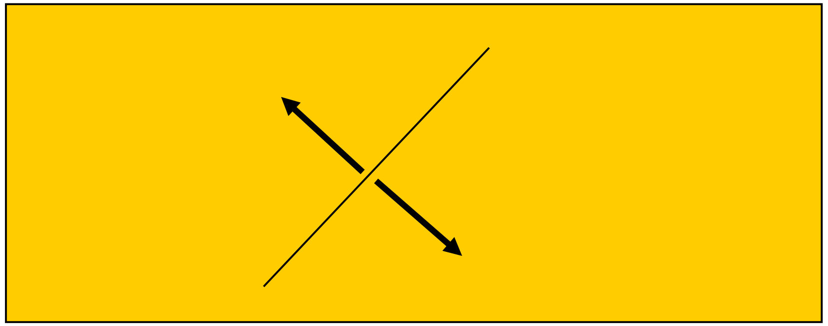


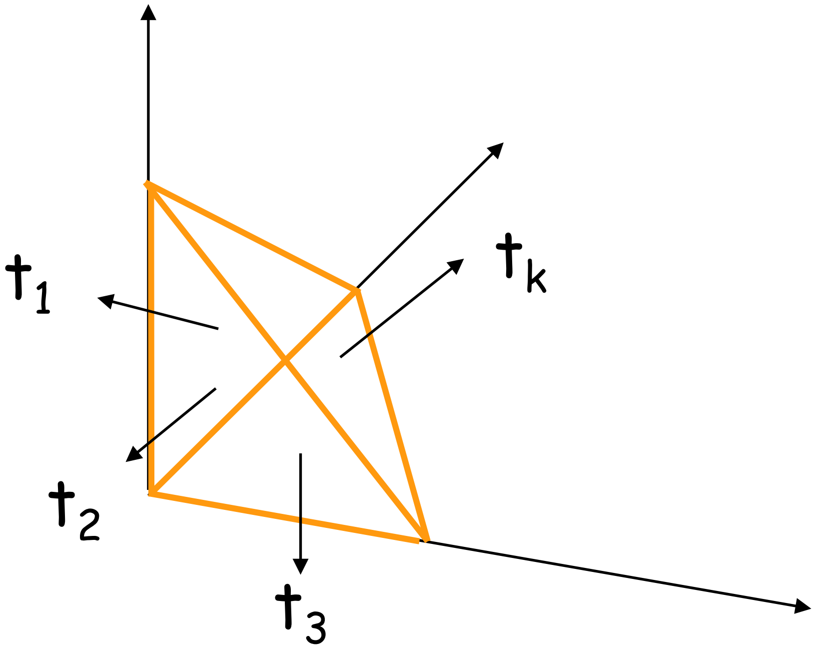
Tractions

Force divided by an area is called stress.

Forces acting on planes are called **tractions**

The **tractions $\mathbf{t}_k$** along axis **k** are





$$\mathbf{t}_k = \begin{pmatrix} t_{k1} \\ t_{k2} \\ t_{k3} \end{pmatrix}$$

... and along an arbitrary direction

$$\mathbf{t} = \mathbf{t}_i n_i$$

... which – using the summation convention yields ..

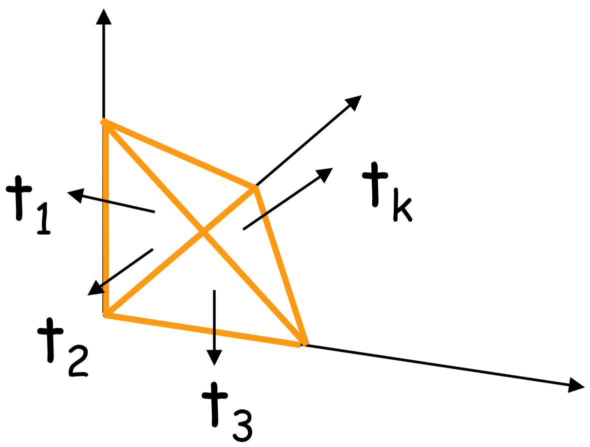
$$\mathbf{t} = \mathbf{t}_1 n_1 + \mathbf{t}_2 n_2 + \mathbf{t}_3 n_3$$

Traction -> stress

Arranged in matrix form

$$\sigma = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} \\ t_{2x} & t_{2y} & t_{2z} \\ t_{3x} & t_{3y} & t_{3z} \end{pmatrix}$$

the stress tensor



The stress tensor

Properly written as

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

compression – decompression

shear stresses

Stress and traction

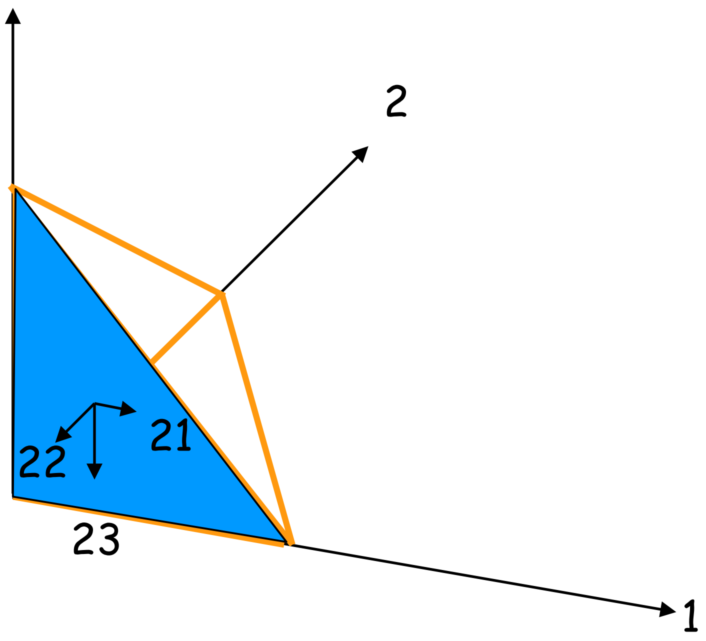
... in components we can write this as

$$\boxed{t_i = \sigma_{ij} n_j}$$

where σ_{ij} ist the stress tensor and n_j is a surface normal.
The stress tensor describes the forces acting on planes within a body. Due to the **symmetry condition**

$$\boxed{\sigma_{ij} = \sigma_{ji}}$$

there are only **six independent elements**.



$$\sigma_{ij}$$

- i: The vector normal to the corresponding surface
- j: The direction of the force vector acting on that surface

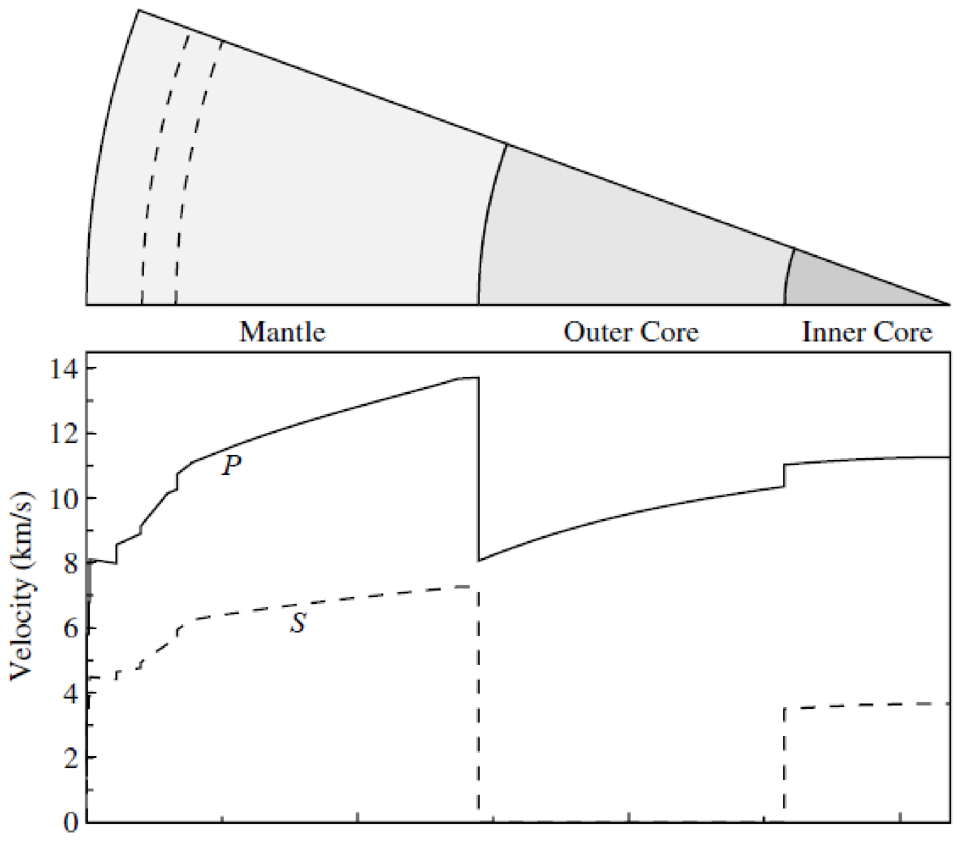
Physical units of stress

Stress units	bars (10^6dyn/cm^2), 1N=10^5 dyn (cm g/s^2) 10^6Pa=1MPa=10bars **1 Pa=1 N/m^2** At sea level p=1bar At depth 3km p=1kbar
maximum compressive stress	the direction perpendicular to the minimum compressive stress, near the surface mostly in horizontal direction, linked to tectonic processes.
principle stress axes	the direction of the eigenvectors of the stress tensor

Stress inside the Earth

Depth (km)	Region	Pressure (GPa)
0—24	Crust	0—0.6
24—400	Upper mantle	0.6—13.4
400—670	Transition zone	13.4—23.8
670—2891	Lower mantle	23.8—135.8
2891—5150	Outer core	135.8—328.9
5150—6371	Inner core	328.9—363.9

Table 2.1: Pressure versus depth inside Earth.



Stress values: Examples

Earthquake stress drop ca. 5 MPa

$F_G = mg$

$m = V\rho$

$h \approx 192\ \mathrm{m}$

Column height?

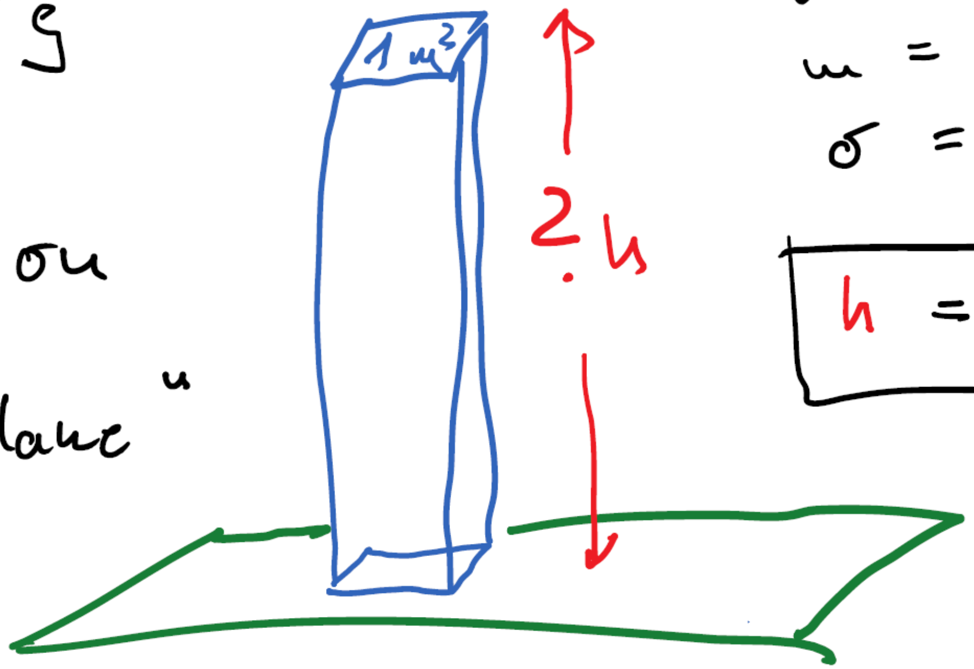
$V = h \quad [\mathrm{m}^3]$

$m = V\rho$

$\sigma = \frac{F_G}{A} = \frac{h\rho g}{A}$

$$\boxed{h = \frac{\sigma A}{g\rho}}$$

"Weight on fault plane"



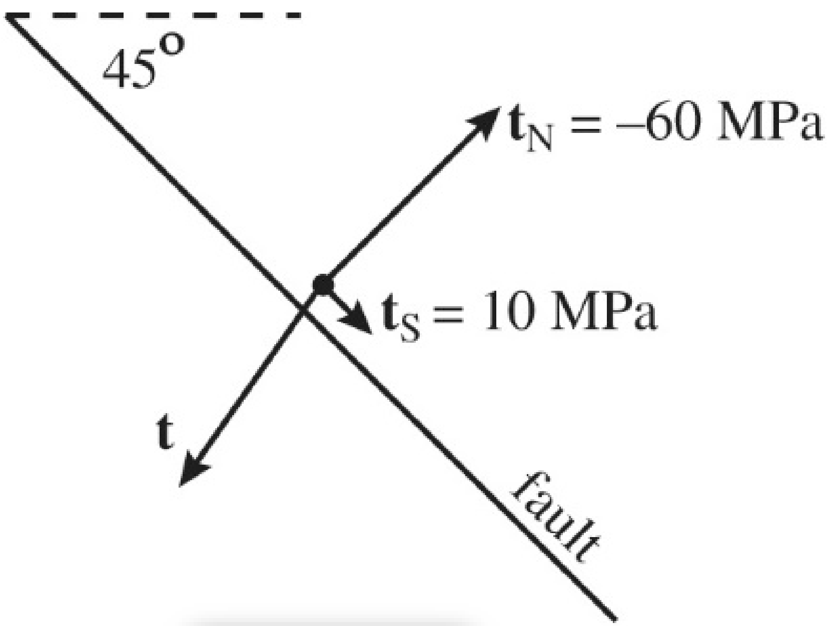
fault plane

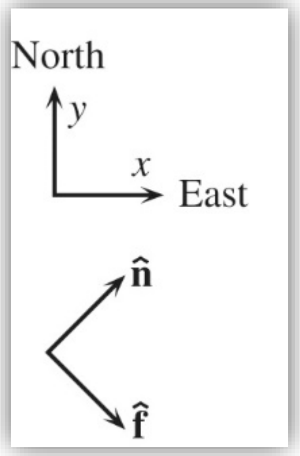
Traction on faults

Traction on faults (2D)

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} = \begin{pmatrix} -40 & -10 \\ -10 & -60 \end{pmatrix} \text{MPa}$$

$$n = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0.701 \\ 0.701 \end{pmatrix}$$

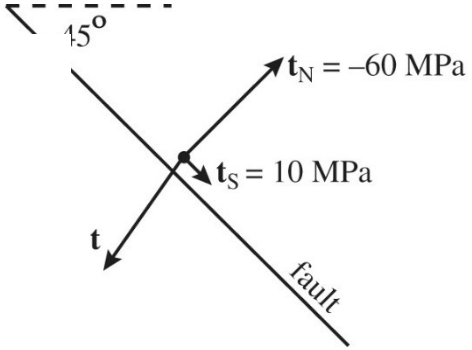


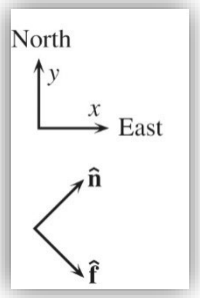


Traction on faults (2D)

$$\underline{t} = \underline{\underline{\sigma}}\;\underline{n} = \begin{pmatrix} -40 & -10 \\ -10 & -60 \end{pmatrix} \begin{pmatrix} 0.7071 \\ 0.7071 \end{pmatrix}$$

$$= \begin{pmatrix} -35.4 \\ -49.4 \end{pmatrix} \text{ MPa}$$

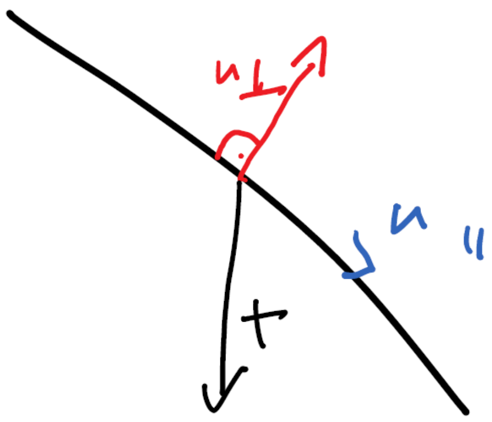


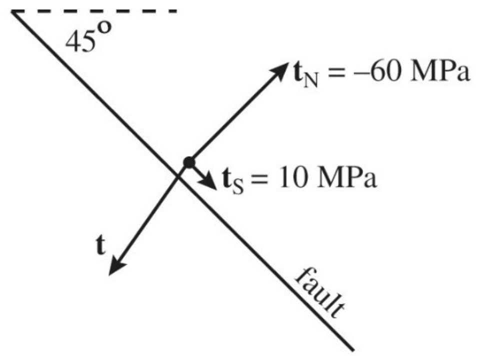


Traction on faults (2D)

Unit vectors:

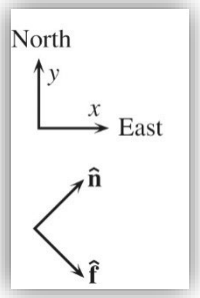
$$n_\perp = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad n_\parallel = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$





$$t_\perp = t\, n_\perp \qquad (\text{scalar!})$$

$$t_\parallel = t\, n_\parallel \qquad (\text{scalar!})$$



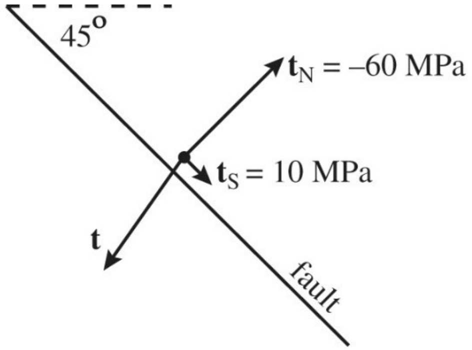
Traction on faults (2D)

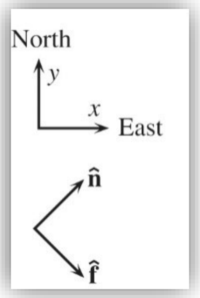
In our example:

$$t_{\perp} = (-35.4, -49.4) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -60 \text{ MPa}$$

$$t_{\parallel} = (-35.4, -49.4) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 10 \text{ MPa}$$

scalar product!





Traction on faults - Exercise

Assume that the horizontal components of the 2-D stress tensor are

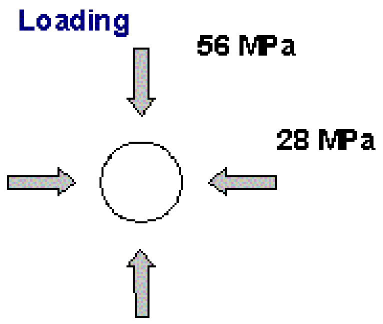
$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix} = \begin{bmatrix} -30 & -20 \\ -20 & -40 \end{bmatrix} \text{MPa}$$

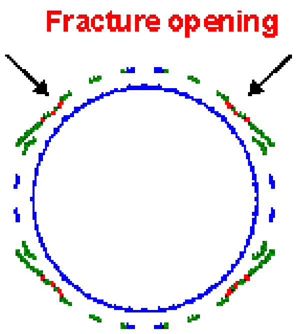
(a) Compute the normal and shear stresses on a fault that strikes 10° east of north.

- Make a drawing
- Define normal and tangential vectors
- Calculate traction
- Calculate normal and tangential stresses

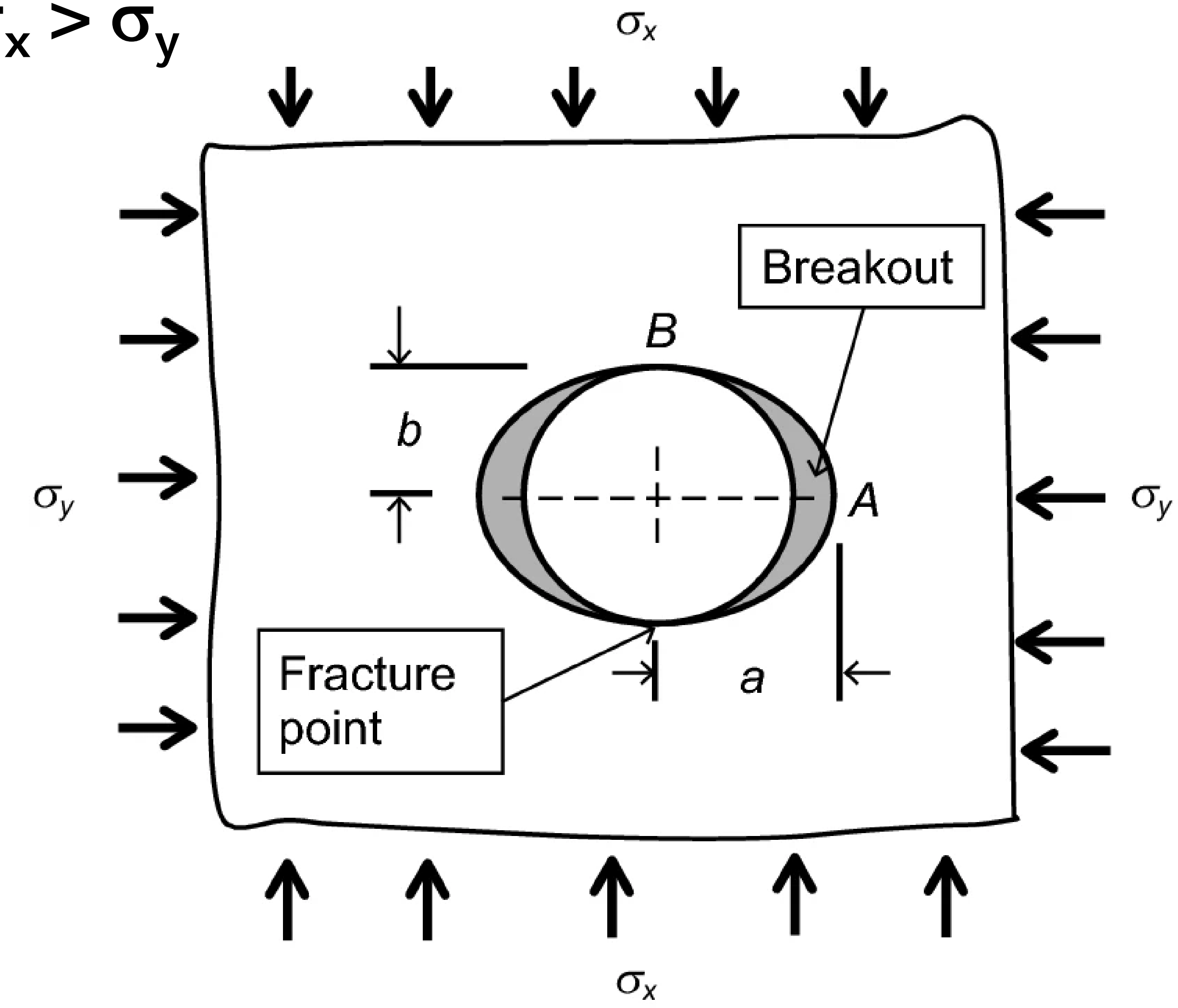
Approx. 20 mins

Stress in boreholes





$\sigma_x > \sigma_y$



Eigenvalue Problems

Assume we have a square matrix A

This matrix operates on a vector x

$$A x = y$$

y has different length + direction

Are there variations of A that lead to y pointing in same direction as x?

$$A x = \lambda x$$

Eigenvalues

A bit of maths:

$$\underline{\underline{A}}\,\underline{x} - \lambda\,\underline{x} = 0$$

λ: scalar "eigenvalue"

$\underline{x}$: eigenvector

How to calculate λ and $\underline{x}$?

→ Eigenvalue problem

Eigenvalues

Simple example:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

we seek

$$\det |A - \lambda| = 0$$

$$\det \left| \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} \right| = 0$$

This is done like:

$$\begin{matrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{matrix}$$

$$(1-\lambda)^2 - 4 = 0$$

$$\lambda_1 = 3$$

$$\lambda_2 = -1$$

Eigenvectors

Eigenvectors:

Put $\lambda_{1,2}$ into $\underline{A}\,\underline{x} - \lambda_{1,2}\underline{x} = 0$

and solve for $\underline{x}$ (twice in this case)

We obtain 1. $x_1 = x_2 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underline{e}_1$

2. $x_1 = -x_2 \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \underline{e}_2$

Eigenvectors are orthogonal!

$$\underline{e}_1 \cdot \underline{e}_2 \overset{!}{=} 0$$

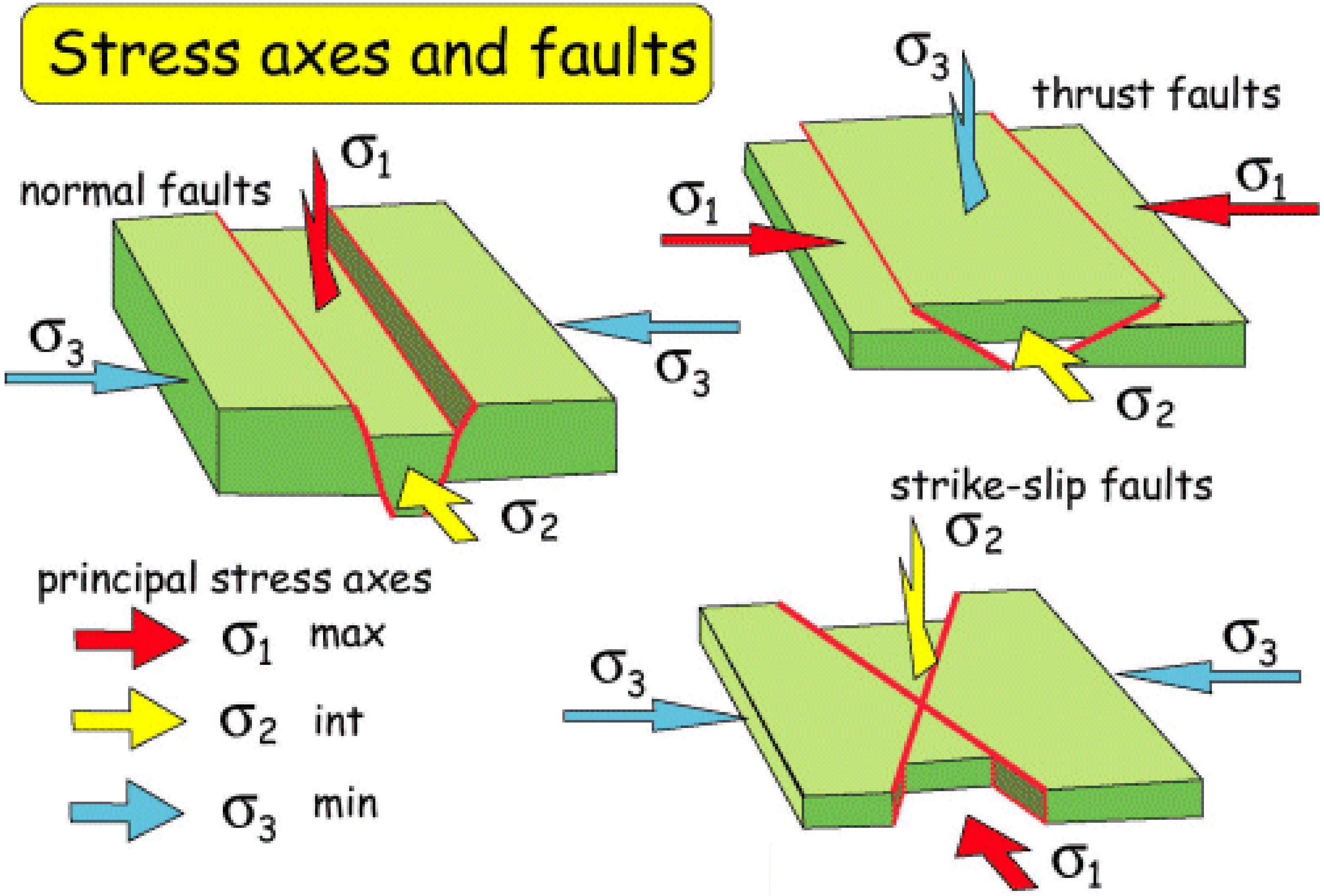
Principal stresses

$$\sigma^{rot} = \underline{\underline{N}}^T \sigma \underline{\underline{N}} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ & \sigma_2 & 0 \\ & & \sigma_3 \end{pmatrix}$$

matrix with normalized eigenvectors

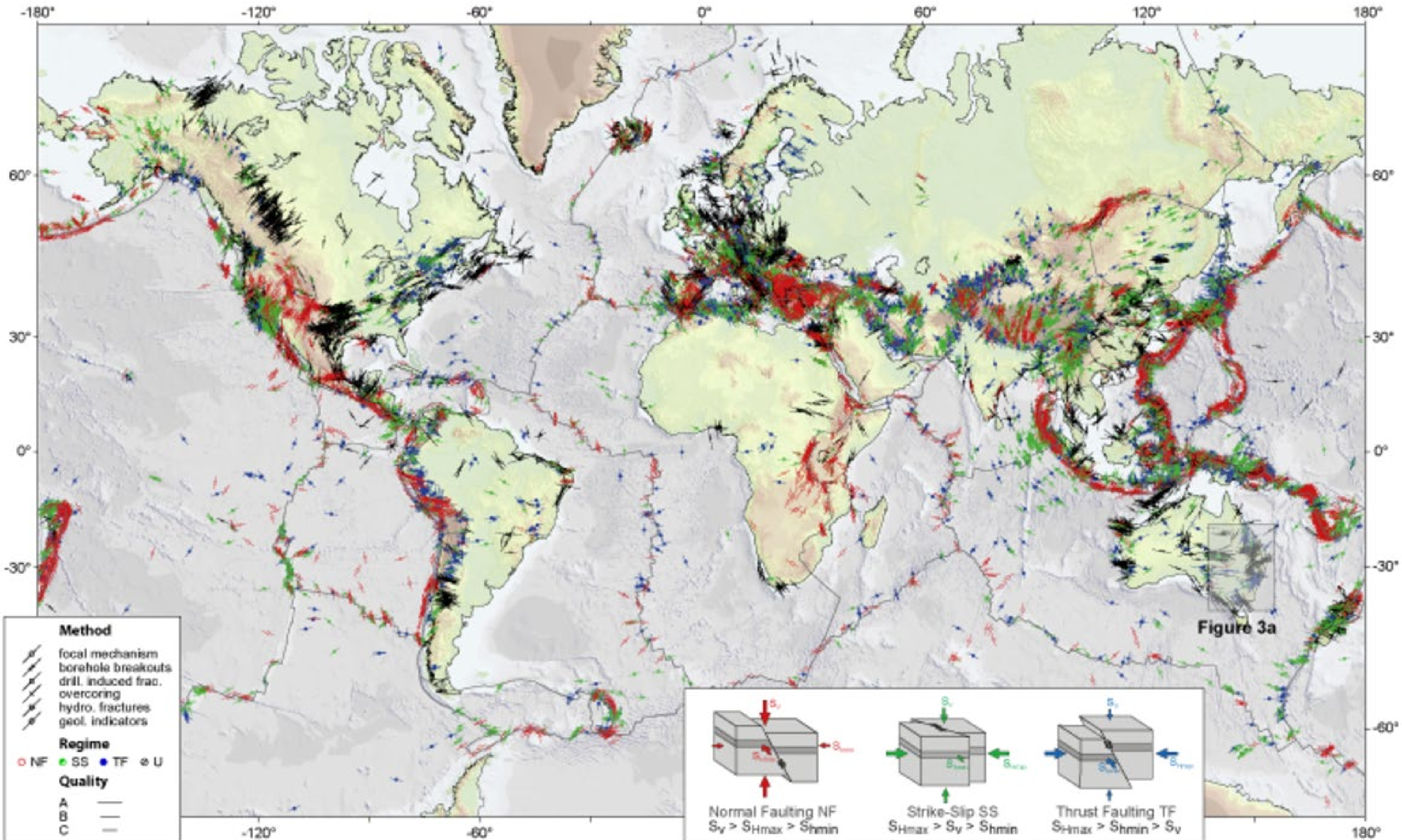
Principal stresses

Why does this matter?

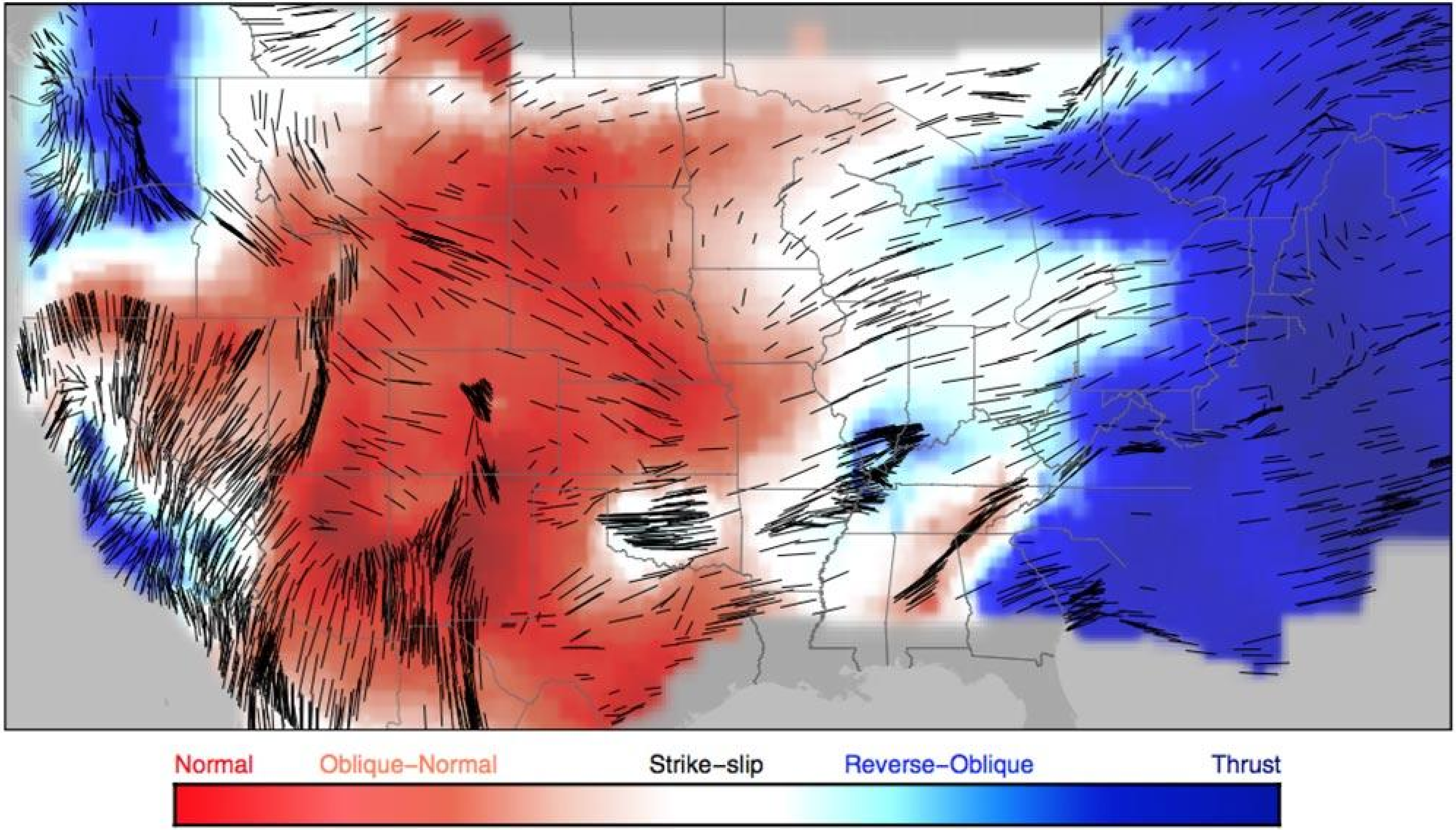


The **eigenvectors** give you the **directions of the principle stresses** that are of central interest in earthquake physics and tectonics

The world stress map



US stress map



Hydrostatic stress

$$\sigma^{H} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}$$

P is equal pressure in all direction.
This is the case in fluids.

Deviatoric stress

σ_m is trace of σ: $\sigma_m = \underbrace{\sigma_1 + \sigma_2 + \sigma_3}_{\text{principal stresses}}$

Deviatoric stress:

$$\sigma_D = \begin{pmatrix} \sigma_{11} - \sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma_m & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma_m \end{pmatrix}$$

Stress separation

In other words:

$$\underline{\underline{\sigma}} = \sigma_m \underline{\underline{I}} + \underline{\underline{\sigma_D}}$$

hydrostatic stress

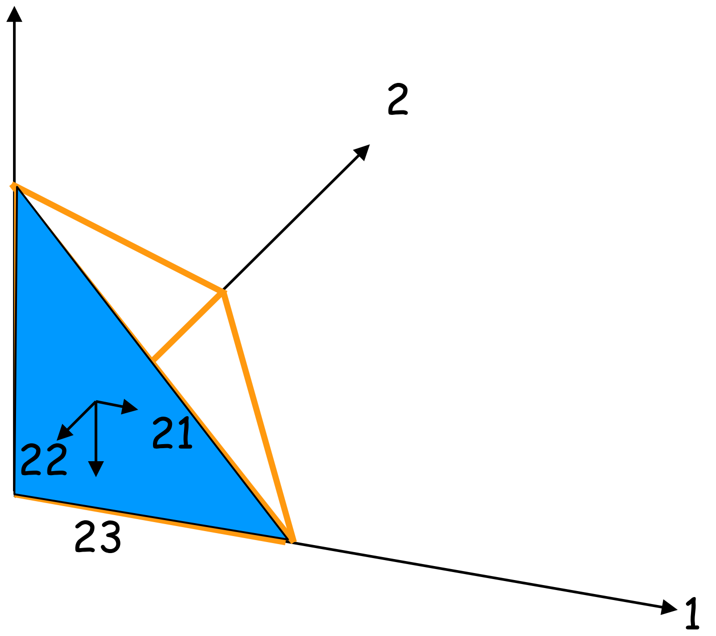
deviatoric stress

$$\underline{\underline{\sigma}} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \begin{pmatrix} \sigma_{11}+p & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}+p & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}+p \end{pmatrix}$$

$$p = -\sigma_m$$

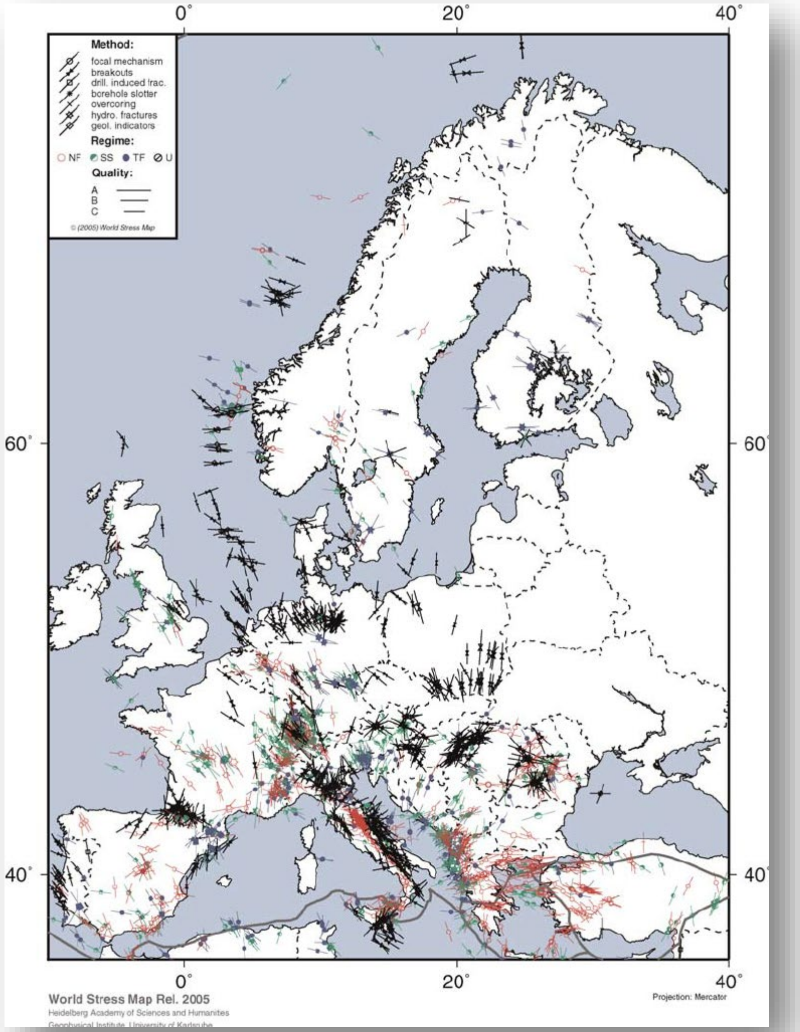
Summary

- Stress is **force per area**
- The stress units are **Pa (Pascal – N/m²)**
- Stress is a **tensor** (just like strain)
- The stress tensor elements correspond to **forces acting on faces**
- The Earth's surface is (in principal) **stress free**
- Stress **cannot** be **directly measured**



Summary (cont‘d)

- **Principal stress directions** are extremely important for tectonics and earthquake physics
- The principal stress directions and values can be obtained by **eigenvector analysis**
- The stress tensor can be synthesized by an **isotropic hydrostatic stress (volume changes)** and a **deviatoric stress (shape changes)**
- Stress directions can be estimated by **borehole breakouts** and **shear-wave splitting (anisotropy)**
- Stresses can reach up to **364 GPa** (inner core)



Appendix

a few words on vector fields

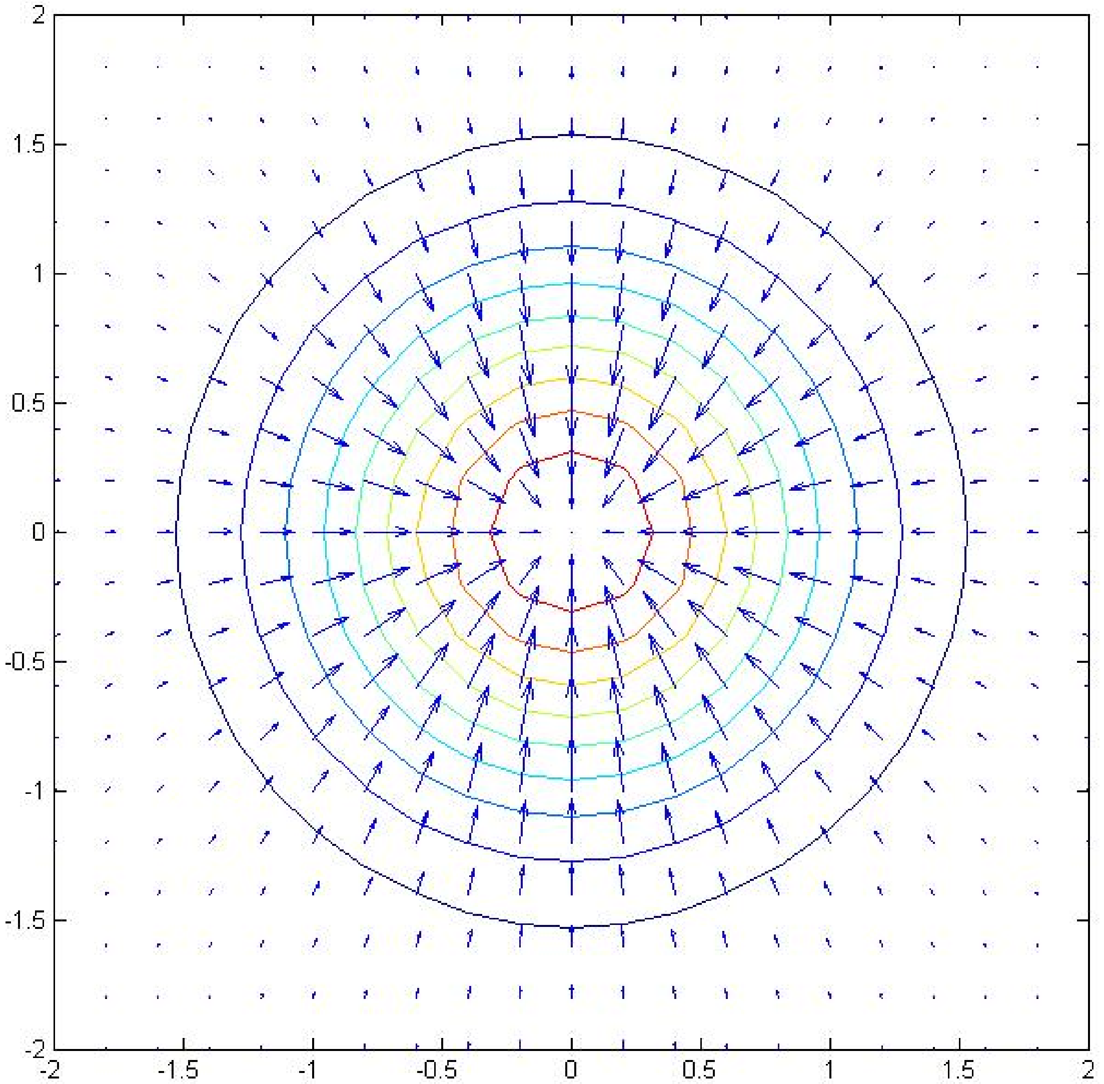
Some operations on vector fields

Gradient of a vector field

$$\nabla \mathbf{u} = \begin{pmatrix} \partial_{\mathbf{x}} \\ \partial_{\mathbf{y}} \\ \partial_{\mathbf{z}} \end{pmatrix} \mathbf{u} = \begin{pmatrix} \partial_{\mathbf{x}} \\ \partial_{\mathbf{y}} \\ \partial_{\mathbf{z}} \end{pmatrix} \begin{pmatrix} u_{\mathbf{x}} \\ u_{\mathbf{y}} \\ u_{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \partial_{\mathbf{x}} u_{\mathbf{x}} & \partial_{\mathbf{y}} u_{\mathbf{x}} & \partial_{\mathbf{z}} u_{\mathbf{x}} \\ \partial_{\mathbf{x}} u_{\mathbf{y}} & \partial_{\mathbf{y}} u_{\mathbf{y}} & \partial_{\mathbf{z}} u_{\mathbf{y}} \\ \partial_{\mathbf{x}} u_{\mathbf{z}} & \partial_{\mathbf{y}} u_{\mathbf{z}} & \partial_{\mathbf{z}} u_{\mathbf{z}} \end{pmatrix}$$

What is the meaning of the gradient?

Gradient of 2D Gaussian function



Some operations on vector fields

Divergence of a vector field

$$\nabla \bullet \mathbf{u} = \begin{pmatrix} \partial_{\mathbf{x}} \\ \partial_{\mathbf{y}} \\ \partial_{\mathbf{z}} \end{pmatrix} \bullet \mathbf{u} = \begin{pmatrix} \partial_{\mathbf{x}} \\ \partial_{\mathbf{y}} \\ \partial_{\mathbf{z}} \end{pmatrix} \bullet \begin{pmatrix} u_{\mathbf{x}} \\ u_{\mathbf{y}} \\ u_{\mathbf{z}} \end{pmatrix} = \partial_{\mathbf{x}} u_{\mathbf{x}} + \partial_{\mathbf{y}} u_{\mathbf{y}} + \partial_{\mathbf{z}} u_{\mathbf{z}}$$

When u is the displacement what is its divergence?

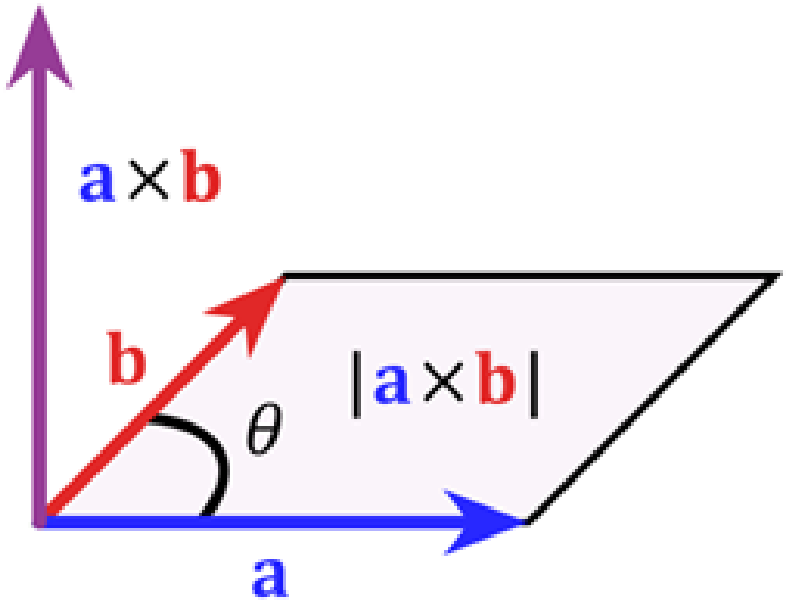
Some operations on vector fields

Curl of a vector field

$$\nabla \times \mathbf{u} = \begin{pmatrix} \partial_\mathbf{x} \\ \partial_\mathbf{y} \\ \partial_\mathbf{z} \end{pmatrix} \times \mathbf{u} = \begin{pmatrix} \partial_\mathbf{x} \\ \partial_\mathbf{y} \\ \partial_\mathbf{z} \end{pmatrix} \times \begin{pmatrix} u_\mathbf{x} \\ u_\mathbf{y} \\ u_\mathbf{z} \end{pmatrix} = \begin{pmatrix} \partial_\mathbf{y} u_\mathbf{z} - \partial_\mathbf{z} u_\mathbf{y} \\ \partial_\mathbf{z} u_\mathbf{x} - \partial_\mathbf{x} u_\mathbf{z} \\ \partial_\mathbf{x} u_\mathbf{y} - \partial_\mathbf{y} u_\mathbf{x} \end{pmatrix}$$

Can we observe it?

Vector product



$$A = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$