Notes pullback attractor

1 Problem definition

Let's look at the modified double well potential

$$V(x, y, \mu, t) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + y^2 - \mu x t \tag{1}$$

with the corresponding SDE

$$\mathrm{d}\mathbf{X}_t = -\nabla V(\mathbf{X}_t; \mu, t)\ \mathrm{d}t + g(\mathbf{X}_t; \mu)\ \mathrm{d}\mathbf{W}_t, \tag{2}$$

with $g(\mathbf{X}_t; \mu) = 0.1$. If we take $\mu = 0$ or fix $t = 0$, this corresponds to the standard 2D double well potential that I always look at.

Our starting point when computing transitions will be $(-1, 0)$. The score function will be a linear one $\phi(x, y) = (x + 1)/2$. Note that this is not a correct score function, especially for large μ, but it works well enough for the purpose of these notes.

I will also plot several transients together with the bifurcation diagram shown in black.

2 Bifurcation diagrams

For fixed values of t that are being passed to V, we find the following bifurcation diagrams

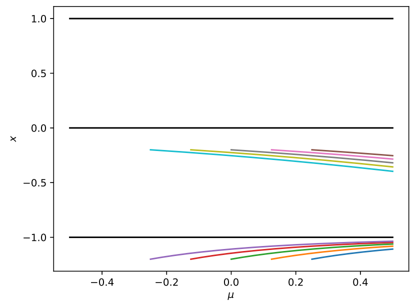

Figure 1: Fixed $t = 0$

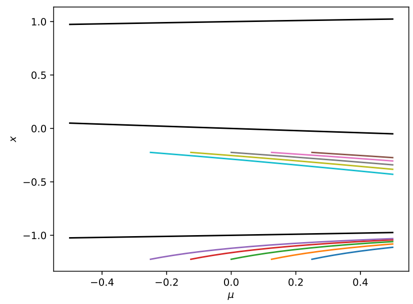

Figure 2: Fixed $t = 0.1$

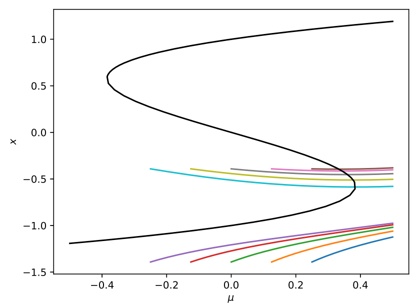

Figure 3: Fixed $t = 1$

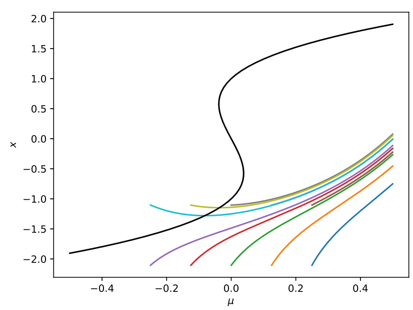

Figure 4: Fixed $t = 10$

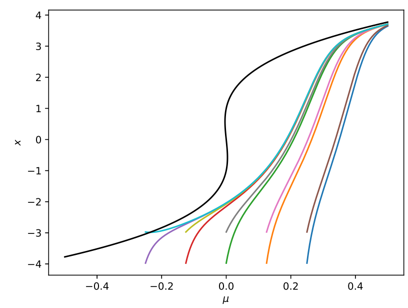

Figure 5: Fixed $t = 100$

So from this we can see that for t going to infinity, the saddle node bifurcations disappear. This is what I tried to sketch on the blackboard. From this we can see that for $t = 100$, μ needs to be extremely small to not have guaranteed (rate-induced) transitions. So if we look at the bifurcation diagram at our end time, we should get an idea of how small the bifurcation parameter (μ) needs to be for us to still be in a noise-induced transition regime.

On the other hand, we can also use t as a bifurcation parameter for fixed μ. This gives us an idea of when transitions start being guaranteed. Since the relation I used is linear, you could just imagine t being on the x-axis and μ being fixed in the above plots. But since this is more interesting for small μ and large t, here are some more plots.

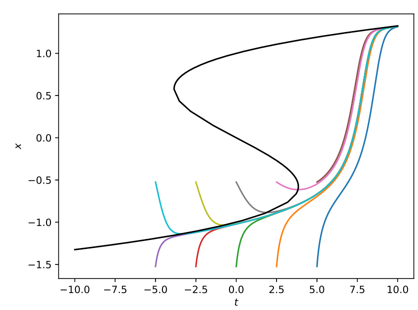

Figure 6: Fixed $\mu = 0.1$

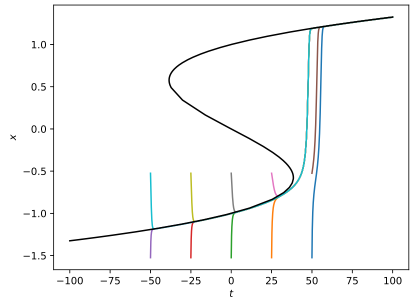

Figure 7: Fixed $\mu = 0.01$

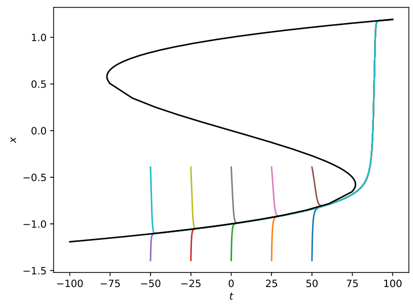

Figure 8: Fixed $\mu = 0.005$

So here we can see that for $\mu = 0.01$, transitions and inevitable after $t =$ 40 and for $\mu = 0.005$ this is the case after $t =$ 80.

3 Transitions

I implemented both a standard Monte-Carlo method (MC) and TAMS for computing the probabilities. Below I will state the probabilities and variances for different values of μ, along with some plots of the trajectories. From the above we know that transitions are inevitable after a certain time.

Let's first use Monte-Carlo to look at this. First with very small noise ($\sigma = \sqrt{0.001}$).

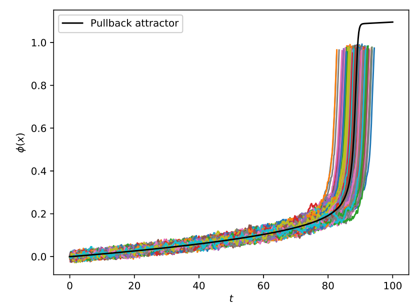

Figure 9: Small noise. Fixed $\mu = 0.005$

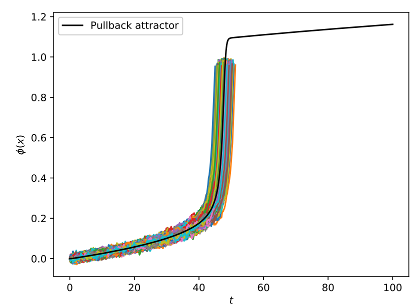

Figure 10: Small noise. Fixed $\mu = 0.01$

From this it seems like we could use the pullback attractor to determine a good score function?!

And then with large noise ($\sigma = \sqrt{0.1}$).

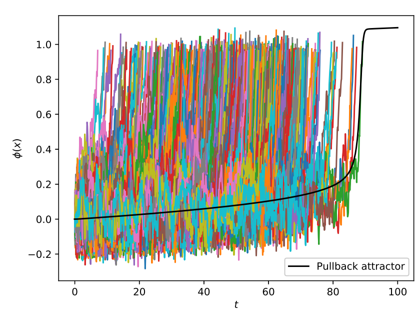

Figure 11: Large noise. Fixed $\mu = 0.005$

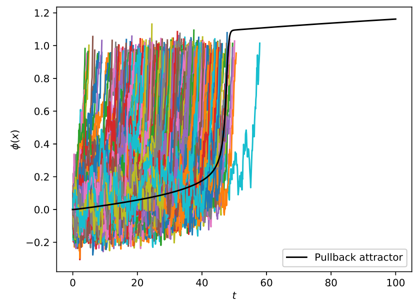

Figure 12: Large noise. Fixed $\mu = 0.01$

This indeed confirms our findings in the previous section. Of course this is not very useful for testing TAMS, since transitions are guaranteed. Now let's use a very small μ.

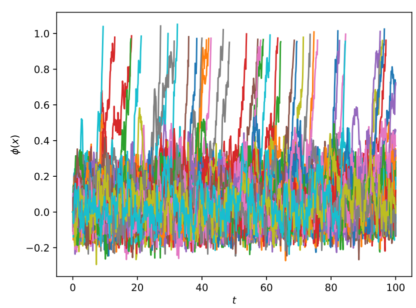

Figure 13: Large noise. Fixed $\mu = 0.001$

We can see that in this case transitions are not guaranteed. Now let's look at probabilities and variances. I used 500 samples and 500 trajectories. First large noise ($\sigma = \sqrt{0.1}$).

μ	Method	Probability	Variance
0.01	MC	1	0
	TAMS	1	0
0.005	MC	0.99992	0.00041
	TAMS	0.99992	0.00039
0.001	MC	0.308	0.020
	TAMS	0.306	0.0153
0	MC	0.127	0.0143
	TAMS	0.128	0.0093

Table 1: $\sigma = \sqrt{0.1}$

So it seems like TAMS indeed works in this case as well. Let's try again with smaller noise to make sure ($\sigma = \sqrt{0.05}$).

μ	Method	Probability	Variance
0.01	MC	1	0
	TAMS	1	0
0.005	MC	0.99909	0.00133
	TAMS	0.99932	0.00115
0.001	MC	0.00872	0.00424
	TAMS	0.00886	0.00104
0	MC	0.00096	0.00136
	TAMS	0.00095	0.00014

Table 2: $\sigma = \sqrt{0.05}$

So it indeed seems like TAMS has a lower variable for lower probabilities, and MC and TAMS are very close to each other.

4 Conclusion

We can generate bifurcation diagrams in both μ and t to get an idea of whether transitions will be noise or rate induced for this problem. We can also still use TAMS, even if the transitions are mostly rate induced. It also seems that when we can compute (an estimate of) the pullback attractor, this can be used to get a good score function.