Summary: Mathematical Foundations for Finance

Fabian MARBACH, Autumn Semester 2015/16

1 Financial Markets in Discrete Time

1.1 Basic setting

Basic setting of the market

- **Probability space:** $(\Omega, \mathcal{F}, \mathbb{P})$
- Finite discrete time horizon: $k = 0, 1, \ldots, T$
- Flow of information over time: **filtration** $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\ldots,T}$.
  This is a family of σ-fields $\mathcal{F}_k \subseteq \mathcal{F}$ which is increasing.
- An ($\mathbb{R}^d$-valued) **stochastic process** in discrete time: $X = (X_k)_{k=0,1,\ldots,T}$ of ($\mathbb{R}^d$-valued) RVs which are all defined on the same probability space.
  This describes the random evolution over time of d quantities.
- A stochastic process is called **adapted** (w.r.t. $\mathbb{F}$) if each X_k is $\mathcal{F}_k$-measurable, i.e. observable at time k.
- A stochastic process is called **predictable** (w.r.t. $\mathbb{F}$) if each X_k is even $\mathcal{F}_{k-1}$-measurable.

Frictionless financial market

- no transaction cost
- short-selling allowed
- investors are small, i.e. their trading does not affect stock prices

1.2 Basic processes

Trading strategy φ

- A trading stragey is an $\mathbb{R}^d$-valued stochastic process $\varphi = (\varphi^0, \vartheta)$.
- $\varphi^0 = (\varphi^0_k)_{k=0,1,\ldots,T}$ denotes the riskless asset.
  φ^0 is $\mathbb{R}$-valued and adapted.
- $\vartheta = (\vartheta_k)_{k=0,1,\ldots,T}$ denotes the d risky assets.
  ϑ is $\mathbb{R}^d$-valued and predictable.
- Initial value: $\varphi = (\varphi^0, \vartheta_0 \equiv 0)$
  (since there is no trading before time 0, i.e. investors start out without any shares)
- A trading strategy describes a dynamically evolving portfolio in the $d+1$ basic assets available for trade.

Value process V

- $V(\varphi) = (V_k(\varphi))_{k=0,1,\ldots,T}$ denotes the discounted value process of the strategy φ and is given by

$$V_k(\varphi) := \underbrace{\varphi^0_k S^0_k}_{\text{bank account}} + \underbrace{\vartheta_k \cdot S_k}_{\text{portfolio}} = \varphi^0_k + \sum_{i=1}^{d} \vartheta^i_k S^i_k$$

- V is $\mathbb{R}$-valued and adapted.
- Initial value: $V_0(\varphi) = \varphi^0_0 = C_0(\varphi)$.

Cost process C

- $(C_k(\vartheta))_{k=0,1,\ldots,T}$ denotes the discounted cost process associated to φ:

$$C_k(\varphi) := V_k(\varphi) - G_k(\varphi)$$

- By construction, $C_k(\varphi)$ describes the cumulative (total) costs for the strategy φ on the time interval $[0, k]$.
- Incremental cost:

$$\Delta C_{k+1}(\varphi) := \underbrace{\varphi^0_{k+1} - \varphi^0_k}_{\text{bank account}} + \underbrace{\sum_{i=1}^{d} \left(\vartheta^i_{k+1} - \vartheta^i_k\right) S^i_k}_{\text{portfolio}}$$

Gains process G

- $(G_k(\vartheta))_{k=0,1,\ldots,T}$ denotes the discounted gains process associated to ϑ:

$$G_k(\vartheta) := \sum_{j=1}^{k} \vartheta_j \cdot \Delta S_j$$

- G is $\mathbb{R}$-valued and adapted.

Self-financing strategy

- A trading strategy $\varphi = (\varphi^0, \vartheta)$ is called self-financing if its cost process $C(\varphi)$ is constant over time.
- A self-financing strategy $\varphi = (\varphi^0, \vartheta)$ is uniquely determined by its initial wealth V_0 and its risky asset component ϑ.
  In particular, any pair (V_0, ϑ) specifies in a unique way a self-financing strategy.
  If $\varphi = (\varphi^0, \vartheta)$ is self-financing, then $(\varphi^0_k)_{k=1,\ldots,T}$ is automatically predictable.
- It then holds for the corresponding (incremental) **cost process:**

$$\Delta C_{k+1}(\varphi) = \varphi^0_{k+1} - \varphi^0_k + (\vartheta_{k+1} - \vartheta_k) \cdot S_k = 0$$
$$C(\varphi) = C_0(\varphi) = V_0(\varphi) = \varphi^0_0$$

- It then holds for the corresponding **value process:**

$$V_k(\varphi) = V_0(\varphi) + G_k(\vartheta) = \varphi^0_0 + G_k(\vartheta)$$
$$= \varphi^0_0 + \sum_{j=1}^{k} \vartheta_j \cdot \Delta S_j$$

Remarks:

- The notion of a strategy being self-financing is a kind of economic budget constraint.
- The notion of self-financing is numeraire irrelevant, i.e. it does not depend on the units in which the calculations are done.

Admissibility

- For $a \in \mathbb{R}, a \geq 0$, a trading strategy φ is called **a-admissible** if its value process $V(\varphi)$ is uniformly bounded from below by $-a$, i.e.

$$V_k(\varphi) \geq -a, \quad \mathbb{P}\text{-a.s.}, \quad \forall k \geq 0$$

- A trading strategy is called **admissible** if it is a-admissible for some $a \geq 0$.

Remark:

- An admissible strategy can be interpreted as a strategy having some credit line which imposes a lower bound on the associated value process. So one may make debts, but only within clearly defined limits.

1.3 Properties of the market

Characterisation of financial markets via EMMs The description of a financial market model via EMMs can be summarized as follows:

- **Existence** of an EMM $\iff$ the market is **arbitrage-free** i.e. $\mathbb{P}_e(S) \neq \emptyset$ by the 1st FTAP
- **Uniqueness** of the EMM $\iff$ the market is **complete** i.e. $\#(\mathbb{P}_e(S)) = 1$ by the 2nd FTAP

1.3.1 Arbitrage

1st Fundamental Theorem of Asset Pricing (FTAP)

- Consider a financial market model in finite discrete time.
- Then S is arbitrage-free iff there exists an EMM for S, i.e.

$$\boxed{\text{(NA)} \iff \mathbb{P}_e(S) \neq \emptyset}$$

- In other words: If there exists a probability measure $\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$ s.t. S is a $\mathbb{Q}$-martingale, then S is arbitrage-free.

Arbitrage opportunity

- An arbitrage opportunity is an admissible self-financing strategy $\varphi = (0, \vartheta)$ with zero initial wealth, with $V_t(\varphi) \geq 0$, $\mathbb{P}$-a.s. and with $\mathbb{P}[V_T(\varphi) > 0] > 0$.
- The financial market $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, S^0, S)$ or shortly S is called **arbitrage-free** if there exist no arbitrage opportunities.
- The following statements are equivalent:
  - (i) S is arbitrage-free.
  - (ii) There exists no self-financing strategy $\varphi = (0, \vartheta)$ with zero initial wealth and satisfying $V_T(\varphi) \geq 0$, $\mathbb{P}$-a.s. and $\mathbb{P}[V_T(\varphi > 0] > 0$. In other words, S satisfies (NA').
  - (iii) For every (not necessarily admissible) self-financing strategy φ with $V_0(\varphi) = 0$, $\mathbb{P}$-a.s. and $V_T(\varphi) \geq 0$, $\mathbb{P}$-a.s., we have $V_T(\varphi) = 0$, $\mathbb{P}$-a.s.
  - (iv) For the space

    $$\mathcal{G}' := \{G_T(\varphi) | \nu \text{ is } \mathbb{R}^d\text{-valued and predictable}\}$$

    of all final wealths that one can generate from zero initial wealth through self-financing trading, we have

    $$\mathcal{G}' \cap \mathcal{L}^0_+(\mathcal{F}_T) = \{0\}$$

    where $\mathcal{L}^0_+(\mathcal{F}_T)$ denotes the space of all nonnegative $\mathcal{F}_T$-measurable RVs.

- *Interpretation:* Absence of arbitrage is a natural economic/financial requirement for a reasonable model of a financial market, since there cannot exist "money pumps" (at least not for long).

1.3.2 Completeness

2nd Fundamental Theorem of Asset Pricing (FTAP)

- Consider a financial market model in finite discrete time and assume that S is arbitrage-free, $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$.
- Then S is complete iff there is a unique EMM for S, i.e.

$$\boxed{\text{(NA)} + \text{completeness} \iff \#(\mathbb{P}_e(S)) = 1}$$

Completeness of the market

- A financial market model (in finite discrete time) is called complete if *every payoff $H \in \mathcal{L}^0_+(\mathcal{F}_T)$ is attainable.*
- Otherwise it is called *incomplete*.

Remark:

- If a financial market in discrete time is complete, then $\mathcal{F}_T$ is finite (i.e. completeness is quite restrictive).

1.4 Pricing of contingent claims H

Attainability

- A payoff $H \in \mathcal{L}^0_+(\mathcal{F}_T)$ is called attainable if there exists an admissible self-financing strategy $\varphi = (V_0, \vartheta)$ with $V_T(\varphi) = H$ $\mathbb{P}$-a.s.
- The stratety φ is then said to **replicate** H and is called a **replicating strategy** for H.

Valuation in complete markets (martingale pricing approach)

- Consider a financial market in finite discrete time and suppose that S is arbitrage-free and complete and $\mathcal{F}_0$ is trivial.
- Then for every payoff $H \in \mathcal{L}^0_+(\mathcal{F}_T)$, there is a unique price process $V^H = (V^H_k)_{k=0,1,\dots,T}$ which admits no arbitrage.
- V^H is given by:

$$\boxed{V^H_k = \mathbb{E}_{\mathbb{Q}}\left[H|\mathcal{F}_k\right] = V_k(V_0, \vartheta)}$$

  for $k = 0, 1, \dots, T$, for any EMM $\mathbb{Q}$ for S and for any replicating strategy $\varphi = (V_0, \vartheta)$ for H.

Characterization of attainable payoffs

- Consider a financial market in finite discrete time and suppose that S is arbitrage-free and $\mathcal{F}_0$ is trivial.
- For any payoff $H \in \mathcal{L}^0_+(\mathcal{F}_T)$, the following are equivalent:
  - (i) H is *attainable*.
  - (ii) $\sup_{\mathbb{Q} \in \mathbb{P}_e(S)} \mathbb{E}_{\mathbb{Q}}[H] < \infty$ is attained in some $\mathbb{Q}^* \in \mathbb{P}_e(S)$, i.e. the supremum is finite and a maximum.
    In other words, we have $\sup_{\mathbb{Q} \in \mathbb{P}_e(S)} \mathbb{E}_{\mathbb{Q}}[H] = \mathbb{E}_{\mathbb{Q}^*}[H] < \infty$ for some $\mathbb{Q}^* \in \mathbb{P}_e(S)$.
  - (iii) The mapping $\mathbb{P}_e(S) \to \mathbb{R}$, $\mathbb{Q} \mapsto \mathbb{E}_{\mathbb{Q}}[H]$ is constant, i.e. H has the same and finite expectation under all EMMs $\mathbb{Q}$ for S.
- Remark: Note that not all of these relationships necessarily hold for financial markets in infinite discrete time or continuous time.

Approach to valuing and hedging payoffs For a given payoff H in a financial market in finite discrete time (with $\mathcal{F}_0$ trivial):

(i) Check if S is arbitrage-free by finding at least one EMM $\mathbb{Q}$ for S.

(ii) Find all EMMs $\mathbb{Q}$ for S.

(iii) Compute $\mathbb{E}_{\mathbb{Q}}[H]$ for all EMMs $\mathbb{Q}$ for S and determine the supremum of $\mathbb{E}_{\mathbb{Q}}[H]$ over $\mathbb{Q}$.

(iv) If the supremum is finite and a maximum, i.e. attained in some $\mathbb{Q}^* \in \mathbb{P}_e(S)$, then H is attainble and its price process can be computed as $V^H_k = \mathbb{E}_{\mathbb{Q}}[H|\mathcal{F}_k]$, for any $\mathbb{Q} \in \mathbb{P}_(S)$.
If the supremum is not attained (or, equivalently for finite discrete time, there is a pair of EMMs $\mathbb{Q}_1, \mathbb{Q}_2$ with $\mathbb{E}_{\mathbb{Q}_1}[H] \neq \mathbb{E}_{\mathbb{Q}_2}[H]$), then H is not attainable.

Invariance of the risk-neutral pricing method under a change of numéraire

- The risk-neutral pricing method is invariant under a change of numéraire, i.e. all assets can be priced under a risk-neutral method independent of the chosen asset used for discounting.
- Denote with Q^{**} the EMM for $\hat{S}^0 := \frac{\tilde{S}^0}{\tilde{S}^1}$.
  Denote with Q^* the EMM for $S^1 = \frac{\tilde{S}^1}{\tilde{S}^0}$.
- Then it holds for a financial market $(\tilde{S}^0, \tilde{S}^1)$ and an undiscounted payoff $\tilde{H} \in L^0_+(\mathcal{F}_T)$ that:

$$\tilde{S}^0_k \mathbb{E}_{\mathbb{Q}^*}\left[\frac{\tilde{H}}{\tilde{S}^0_T}\middle|\mathcal{F}_k\right] = \tilde{S}^1_k \mathbb{E}_{\mathbb{Q}^{**}}\left[\frac{\tilde{H}}{\tilde{S}^1_T}\middle|\mathcal{F}_k\right]$$

EMMs in submarkets

- If a market $(S^0, S^1, \dots, S^k)$ is (NA), i.e. there exists an EMM $\mathbb{Q}$, then this EMM $\mathbb{Q}$ is also an EMM for all submarkets.
(e.g. for $(S^k, S^i), k \neq i$, for $(S^k, S^i, S^j), k \neq i \neq j$ etc.)
- If there exists a EMM $\mathbb{Q}^j$ for a submarket (S^0, S^j) which is not also an EMM for another submarket (S^0, S^k), $j \neq k$, then the whole market $(S^0, S^1, \dots, S^k)$ is not (NA), i.e. it admits arbitrage.

1.5 Multiplicative model

- Suppose that we start with the RVs $r_1, \dots, r_T$ and $Y_1, \dots, Y_T$.
- Define the **bank account/riskless asset** by:

$$\tilde{S}_k^0 := \prod_{j=1}^{k}(1+r_j), \qquad \frac{\tilde{S}_k^0}{\tilde{S}_{k-1}^0} = 1 + r_k, \qquad \tilde{S}_0^0 = 1$$

Remarks:

- $\tilde{S}_k^0$ is $\mathcal{F}_{k-1}$-measurable (i.e. predictable).
- r_k denotes the rate for $(k-1, k]$.

- Define the **stock/risky asset** by:

$$\tilde{S}_k^1 := S_0^1 \prod_{j=1}^{k} Y_j, \qquad \frac{\tilde{S}_k^1}{\tilde{S}_{k-1}^1} = Y_k, \qquad \tilde{S}_0^1 = \text{const.}, \quad \tilde{S}_0^1 \in \mathbb{R}$$

Remarks:

- $\tilde{S}_k^1$ is $\mathcal{F}_k$-measurable (i.e. adapted).
- Y_k denotes the growth factor for $(k-1, k]$.
- The rate of return R_k is given by $Y_k = 1 + R_k$.

1.5.1 Cox-Ross-Rubinstein (CRR) binomial model

Assumptions

- *Bank account/riskless asset:*
Suppose all the $r_k \in \mathbb{R}$ are constant with value $r > -1$.
This means that we have the same nonrandom interest rate over each period.
Then the bank account evolves as $\tilde{S}_k^0$ for $k = 0, 1, \dots, T$.
- *Stock/risky asset:*
Suppose that $Y_1, \dots Y_T \in \mathbb{R}$ are independent and only take two values, $1+u$ with probability p, and $1+d$ with probability $1-p$ (i.e. all Y_k are i.i.d.).
Then the stock prices at each step moves either up (by a factor $1+u$) or down (by a factor $1+d$).

Martingale property The discounted stock price $\frac{\tilde{S}^1}{\tilde{S}^0}$ is a $\mathbb{P}$-martingale iff $r = pu + (1-p)d$.

EMM

- In the binomial model, there exists a probability measure $\mathbb{Q} \approx \mathbb{P}$ s.t. $\frac{\tilde{S}^1}{\tilde{S}^0}$ is a $\mathbb{Q}$-martingale iff $u > r > d$.
- In that case, $\mathbb{Q}$ is unique (on $\mathcal{F}_T$) and characterised by the property that $Y_1, \dots, Y_T$ are i.i.d. under $\mathbb{Q}$ with parameter

$$q^* = \mathbb{Q}[Y_k = 1+u] = \frac{r-d}{u-d} \qquad (\nearrow \text{up})$$
$$1 - q^* = 1 - \mathbb{Q}[Y_k = 1+d] = \frac{u-r}{u-d} \qquad (\searrow \text{down})$$

Arbitrage and completeness The following statements are equivalent:

(i) $u > r > d$

(ii) $\exists$ a unique EMM $\mathbb{Q}^*$ for $\frac{\tilde{S}^1}{\tilde{S}^0}$ (on $\mathcal{F}_T$)

(iii) The market S is (NA) and complete.

Put-Call parity

- Assuming $T = 1$, it holds that:

$$V_0^{C(K)} - V_0^{P(K)} = S_0^1 - \frac{K}{1+r}$$

Pricing binomial contingent claims H

- Assume time horizon $T = 1$, strike $K > 0$, and a payoff function $H(x, K) : \mathbb{R}_+ \to \mathbb{R}_+$ of a European style contingent claim with strike K.
- H may be a European call function $C(x, K) = (x-K)^+$
or a European put function $P(x, K) = (K-x)^+$.
- Then H can be replicated using a self-financing strategy $\varphi^{H(K)} = (V_0^{H(K)}, \vartheta^{H(K)})$ s.t.

$$V_1(\varphi^{H(K)} = \frac{H(\tilde{H}_1^1, K)}{1+r}, \qquad \mathbb{P} - a.s.$$

and $\varphi^{H(K)}$ is given by

$$V_0^{H(K)} = \frac{r-d}{u-d}\frac{H(S_0^1(1+u), K)}{1+r} + \frac{u-r}{u-d}\frac{H(S_0^1(1+d), K)}{1+r}$$
$$\vartheta_1^{H(K)} = \frac{H(1+u, K/S_0^1) - H(1+d, K/S_0^1)}{u-d}$$

- Note that this can also be expressed via the martingale pricing approach:

$$V_0^{H(K)} = \mathbb{E}_{\mathbb{Q}}\left[\frac{H(\tilde{S}_1^1, K)}{1+r}\right]$$

where

$$\mathbb{Q}\left[\tilde{S}_1^1 = S_1^0(1+u)\right] = q = \frac{r-d}{u-d}$$
$$\mathbb{Q}\left[\tilde{S}_1^1 = S_1^0(1+d)\right] = 1 - q = \frac{u-r}{u-d}$$

Binomial call pricing formula

$$\tilde{V}_k^{\tilde{H}} = \tilde{S}_k^1 \mathbb{Q}^{**}[W_{k,T} > x] - \tilde{K}\frac{\tilde{S}_k^0}{\tilde{S}_T^0}\mathbb{Q}^*[W_{k,T} > x]$$
$$x = \frac{\log \frac{\tilde{K}}{\tilde{S}_k^1} - (T-k)\log(1+d)}{\log \frac{1+u}{1+d}}$$

Remark:

- This is the discrete analogue of the Black-Scholes formula.

1.5.2 Multinomial model

EMM

- IOT construct an EMM for S^1, it needs to hold that:

$$\mathbb{E}_{\mathbb{Q}}[S_1^1] = S_0^1$$
$$\iff \mathbb{E}_{\mathbb{Q}}[Y_1] = 1 + r \iff \sum_{k=1}^{m} q_k(1+y_k) = 1 + r$$

with the further conditions:

$$\sum_{k=1}^{m} q_i = 1, \qquad q_1, \dots, q_m \in (0,1)$$

Arbitrage condition The following statements are equivalent:

(i) $y_1 < r < y_m$

(ii) $\exists$ an EMM $\mathbb{Q} \approx \mathbb{P}$ s.t. $\frac{\tilde{S}^1}{\tilde{S}^0}$ is a $\mathbb{Q}$-martingale.

(iii) The market S is (NA).

Completeness The multinomial model is

- **complete** whenever $m \leq 2$
  (i.e. there are *no* nodes that allow for more than two possible stock price evolutions)
- **incomplete** whenever $m > 2$
  (i.e. there is *at least one* node that allows for more than two possible stock price evolutions)

Inequality of the payoffs of Asian and European call options

- Consider a European call option $C_k^E = (\tilde{S}_k^1)^+$ and an Asian call option with

$$C_k^A = \left(\frac{1}{k} \sum_{j=1}^{k} \tilde{S}_j^1 - K \right)^+$$

- Then it holds for the $\mathbb{Q}$-expectation (risk-neutral) of the two payoffs that:

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{C_k^A}{\tilde{S}_k^0}\right] \leq \mathbb{E}_{\mathbb{Q}}\left[\frac{C_k^E}{\tilde{S}_k^0}\right]$$

- *Interpretation:* Since the volatility of an Asian style contingent claim is lower than the one of a European style contingent claim, the Asian option bears lower risk and thus yields also lower profit.

American options

- Consider an American option with maturity T and nonnegative adapted payoff process $U = (U_k)_{k=0,\ldots,T}$.
- Then the arbitrage-free price process $\bar{V} = (\bar{V}_k)_{k=0,\ldots,T}$ w.r.t. $\mathbb{Q}$ can be expressed as a backward recursive scheme such as:

$$\bar{V}_T = U_T$$
$$\bar{V}_k = \max(U_k, \mathbb{E}_{\mathbb{Q}}[\bar{V}_{k+1}|\mathcal{F}_k]), \qquad \text{for } k = 0, \ldots, T-1$$

2 Martingales

2.1 Martingales

Martingales

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\ldots,T}$.
- A (real-valued) stochastic process $X = (X_k)_{k=0,1,\ldots,T}$ is called a **martingale** (w.r.t. $\mathbb{P}$ and $\mathbb{F}$) if:
  (i) X is adapted to $\mathbb{F}$.
  (ii) X is $\mathbb{P}$-integrable in the sense that $X_k \in \mathcal{L}^1(\mathbb{P})$ for each k, i.e.

  $$\mathbb{E}_{\mathbb{P}}[|X|] < \infty$$

  (iii) X satisfies the martingale property:

  $$\mathbb{E}_{\mathbb{P}}[X_l|\mathcal{F}_k] = X_k \qquad \mathbb{P}\text{-a.s. for } k \leq l$$

- *Interpretation:* This means that the best prediction for the later value X_l given the information $\mathcal{F}_k$ is just the current value X_k. Hence the changes in a martingale cannot be predicted.
  In other words, a martingale describes a fair game in the sense that one cannot predict where it goes next.
- A **supermartingale** is defined the same but with the property

$$\mathbb{E}_{\mathbb{P}}[X_l|\mathcal{F}_l] \leq X_k$$

- A **submartingale** is defined the same but with the property

$$\mathbb{E}_{\mathbb{P}}[X_l|\mathcal{F}_l] \geq X_k$$

Equivalent Martingale Measure (EMM)

- $\mathbb{Q}$ is an EMM for the stochastic process $(S_k)_{k\geq 0}$ iff S is a martingale under $\mathbb{Q}$ and if $\mathbb{Q}$ is equivalent to $\mathbb{P}$.
- We denote by $\mathbb{P}_e(S)$ the set of all EMMs for S.
- The probability measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$ ($\mathbb{Q} \approx \mathbb{P}$) iff:
  (i) $\mathbb{Q}[A] > 0 \iff \mathbb{P}[A] > 0$
  (ii) $\mathbb{Q}[\Omega] = 1$
- A stochastic process $(S_k)_{k\geq 0}$ is a $\mathbb{Q}$-martingale iff:
  (i) S_k is adapted to the considered filtration.
  (ii) S_k is integrable: $\mathbb{E}_{\mathbb{Q}}[|S_k|] < \infty$.
  (iii) Martingale property: $\mathbb{E}_{\mathbb{Q}}[S_{k+1}|\mathcal{F}_k] = S_k$.

Local martingale

- An adapted process $X = (X_k)_{k=0,1,\ldots,T}$ null at 0 (i.e. with $X_0 = 0$) is called a local martingale (w.r.t. $\mathbb{P}$ and $\mathbb{F}$) if there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ increasing to T s.t. for each $n \in \mathbb{N}$, the stopped process $X^{\tau_n} = (X_{k\wedge\tau_n})_{k=0,1,\ldots,T}$ is a $(\mathbb{P},\mathbb{F})$-martingale.
- We then call $(\tau_n)_{n\in\mathbb{N}}$ a **localising sequence**.
- For any martingale X and any stopping time τ, the stopped process X^τ is again a martingale.
  In particular, $\mathbb{E}_{\mathbb{P}}[X_{k\wedge\tau}] = \mathbb{E}_{\mathbb{P}}[X_0]$, $\forall k$.

2.2 Stopping times

Stopping time The stochastic process $S^\tau = (S_k^\tau)_{k=0,1,\ldots,T}$ defined by

$$S_k^\tau(\omega) := S_{k\wedge\tau}(\omega) := S_{k\wedge\tau(\omega)}(\omega)$$

is called the process S stopped at τ. It clearly behaves like S up to time τ and remains constant after time τ.

Stopping theorem

- Suppose that $M = (M_t)_{t\geq 0}$ is a $(\mathbb{P},\mathbb{F})$-martingale with RC trajectories, and σ, τ are $\mathbb{F}$-stopping times with $\sigma \leq \tau$.
- If either τ is bounded by some $T \in (0,\infty)$ or M is uniformly integrable, then M_τ, M_σ are both in $L^1(\mathbb{P})$ and

$$\mathbb{E}[M_\tau|\mathcal{F}_\sigma] = M_\sigma \qquad \mathbb{P}\text{-a.s.}$$

Cases of stopping times

- Define the stopping time τ_a for $a \in \mathbb{R}, a > 0$ as:

  $$\tau_a := \inf\{t \geq 0 | W_t > a\}$$

  Then it holds that:
  - $\tau_{a_1} \leq \tau_{a_2}, \mathbb{P}$-a.s. for $a_1 < a_2$.
  - $\mathbb{P}[\tau_a < \infty] = 1$.
  - $W_{\tau_a} = a, \mathbb{P}$-a.s.
  - $\mathbb{E}\left[W_{\tau_{a_2}} \middle| \mathcal{F}_{\tau_{a_1}}\right] \neq W_{\tau_{a_1}}, \mathbb{P}$-a.s.
    i.e. the stopping theorem fails for $\tau = \tau_{a_2}$ and $\sigma = \tau_{a_1}$.
  - $\lim_{n\to\infty} \tau_n(\omega) = \infty, \mathbb{P}$-a.s.
- Define the stopping time ρ_a for $a \in \mathbb{R}, a > 0$ as:

  $$\rho_a := \sup\{t \geq 0 | W_t > a\}$$

  Then it follows that $\rho_a = \infty$ with probability 1 under $\mathbb{P}$.

2.3 Density processes/Girsanov's theorem

Density in discrete time

- Assume $(\Omega, \mathcal{F})$ and a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\ldots,T}$ in finite discrete time.
  On $(\Omega, \mathcal{F})$, we have two probability measures $\mathbb{Q}$ and $\mathbb{P}$, and we assume $\mathbb{Q} \approx \mathbb{P}$.
- **Radon-Nykodin theorem:** There exists a density

  $$\frac{d\mathbb{Q}}{d\mathbb{P}} := \mathcal{D}$$

  This is a RV $\mathcal{D} > 0$, $\mathbb{P}$-a.s. s.t. for all $A \in \mathcal{F}_k$ and for all RVs $Y \geq 0$ it holds that:

  $$\mathbb{Q}[A] = \mathbb{E}_{\mathbb{P}}[\mathcal{D}\mathbb{I}_A], \qquad \mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}_{\mathbb{P}}[Y\mathcal{D}].$$

- This can also be written as

  $$\int_\Omega Y d\mathbb{Q} = \int_\Omega Y\mathcal{D} d\mathbb{P}$$

  This formula tells us how to compute $\mathbb{Q}$-expectations in terms of $\mathbb{P}$-expectations and vice-versa.

Density process in discrete time

- Assume the same setting as before.
- **Radon-Nykodin theorem:** The density process Z of $\mathbb{Q}$ w.r.t. $\mathbb{P}$, or also called the $\mathbb{P}$-martingale Z, is defined as

  $$Z_k := \mathbb{E}_{\mathbb{P}}[\mathcal{D}|\mathcal{F}_k] = \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathcal{F}_k\right] \qquad \text{for } k = 0, 1, \ldots, T$$

- Then for every $\mathcal{F}_k$-measurable RV $Y \geq 0$ or $Y \in \mathcal{L}^1(\mathbb{Q})$, it holds that

  $$\mathbb{E}_{\mathbb{Q}}[Y|\mathcal{F}_k] = \mathbb{E}_{\mathbb{P}}[YZ_k|\mathcal{F}_k]$$

  and for every $k \in \{0, 1, \ldots, T\}$ and any $A \in \mathcal{F}_k$, it holds that

  $$\mathbb{Q}[A] = \mathbb{E}_{\mathbb{P}}[Z_k\mathbb{I}_A]$$

- Properties:
  - Z_k is a RV and $Z_k > 0$, $\mathbb{P}$-a.s.
  - A process $N = (N_k)_{k=0,1,\ldots,T}$ which is adapted in $\mathbb{F}$ is a $\mathbb{Q}$-martingale iff the product ZN is a $\mathbb{P}$-martingale.
    (This tells us how martingale properties under $\mathbb{P}$ and $\mathbb{Q}$ are related to each other.)

- **Bayes formula:**
  If $j \leq k$ and U_k is $\mathcal{F}_k$-measurable and either ≥ 0 or in $\mathcal{L}^1(\mathbb{Q})$, then

  $$\mathbb{E}_{\mathbb{Q}}[U_k|\mathcal{F}_j] = \frac{1}{Z_j}\mathbb{E}_{\mathbb{P}}[Z_kU_k|\mathcal{F}_j] \qquad \mathbb{Q}\text{-a.s.}$$

  This tells us how conditional expectations under $\mathbb{Q}$ and $\mathbb{P}$ are related to each other.

Density process in continuous time

- Suppose we have $\mathbb{P}$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$.
  Fix $T \in (0, \infty)$ and assume only that $\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$.
- Then the density process Z of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ on $[0, T]$ is defined as

  $$Z_t := \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}|_{\mathcal{F}_T}}{d\mathbb{P}|_{\mathcal{F}_T}}\middle|\mathcal{F}_t\right] \qquad \text{for } 0 \leq t \leq T$$

- **Bayes formula:**
  For $s \leq t \leq T$ and every U_t which is $\mathcal{F}_t$-measurable and either ≥ 0 or in $L^1(\mathbb{Q})$, it holds that

  $$\mathbb{E}_{\mathbb{Q}}[U_t|\mathcal{F}_s] = \frac{1}{Z_s}\mathbb{E}_{\mathbb{P}}[Z_tU_t|\mathcal{F}_s] \qquad \mathbb{Q}\text{-a.s.}$$

- It also holds that an adapted process $Y = (Y_t)_{0\leq t\leq T}$ is a (local) $\mathbb{Q}$-martingale iff the product ZY is a (local) $\mathbb{P}$-martingale.

Girsanov's theorem

- Suppose that $\mathbb{Q} \underset{\text{loc}}{\approx} \mathbb{P}$ with density process Z.
- If M is a local $\mathbb{P}$-martingale null at 0, then

  $$\tilde{M} := M - \int \frac{1}{Z} d[Z, M]$$

  is a local $\mathbb{Q}$-martingale null at 0.
- In particular, every $\mathbb{P}$-semimartingale is also a $\mathbb{Q}$-semimartingale (and vice-versa, by symmetry).

Girsanov (continuous version)

- Suppose that $\mathbb{Q} \underset{\text{loc}}{\approx} \mathbb{P}$ with continuous density process Z.
  Write $Z = Z_0\mathcal{E}(L)$.
- If M is a local $\mathbb{P}$-martingale null at 0, then

  $$\tilde{M} := M - [L, M] = M - \langle L, M\rangle$$

  is a local $\mathbb{Q}$-martingale null at 0.
- Moreover, if W is a $\mathbb{P}$-BM, then $\tilde{W}$ is a $\mathbb{Q}$-BM.
- In particular, if $L = \int \nu dW$ for some $\nu \in \mathcal{L}^2_{\text{loc}}(W)$, then $\tilde{W} = W - \langle \int \nu dW, W\rangle = W - \int \nu_s ds$ so that the $\mathbb{P}$-BM $W = \tilde{W} + \int \nu_s ds$ becomes under $\mathbb{Q}$ a BM with (instantaneous) drift ν.

3 Stochastic Integration and Calculus

3.1 Brownian motion & Poisson processes

Brownian motion

- A Brownian motion w.r.t. $\mathbb{P}$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a real-valued stochastic process $W = (W_t)_{t\geq 0}$ which satisfies the following properties:

  (BM0) *null at zero*
  W is adapted to $\mathbb{F}$ and null at 0 (i.e. $W_0 \equiv 0$, $\mathbb{P}$-a.s.).

  (BM1) *independent and stationary increments*
  For $s \leq t$, the increment $W_t - W_s$ is independent (under $\mathbb{P}$) of $\mathcal{F}_s$ and satisfies under $\mathbb{P}$: $W_t - W_s \sim \mathcal{N}(0, t - s)$).

  (BM2) *continuous sample paths*
  W has continuous trajectories, i.e. for $\mathbb{P}$-a.a. $\omega \in \Omega$, the function $t \to W_t(\omega)$ on $[0, \infty)$ is continuous.

Remarks:

- Brownian motion in $\mathbb{R}^m$ is simply an adapted $\mathbb{R}^m$-valued stochastic process null at 0 and with the increment $W_t - W_s$ having the normal distribution $\mathcal{N}(0, (t-s)\mathbb{I}_{m\times m})$, where $\mathbb{I}_{m\times m}$ denotes the identity matrix.
- For $\mathbb{P}$-a.a. $\omega \in \Omega$, the function $t \mapsto W_t(\omega)$ from $[0, \infty)$ to $\mathbb{R}$ is continuous, but *nowhere differentiable*.

Transformations of BM The following stochastic processes are BMs:

- $W^1 := -W$
- Restarting at a fixed time T:

  $$W^2_t := W_{T+t} - W_T$$

  for $t \geq 0$ and for any $T \in (0, \infty)$.
- Rescaling in space and time:

  $$W^3_t := cW_{\frac{t}{c^2}}$$

  for $t \geq 0$ and for any $c \in \mathbb{R}$, $c \neq 0$.
- Time-reversal on $[0, T]$:

  $$W^4_t := W_{T-t} - W_T$$

  for $0 \leq t \leq T$ and for any $T \in (0, \infty)$.
- Inversion of small and large times:

  $$W^5_t := \begin{cases} tW_{\frac{1}{t}} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

  for $t \geq 0$.

- $W_t^6 := (W_t)^2 - t = 2\int_0^t W_s dW_s, \qquad t \geq 0$
- $W_t^7 := \exp\left(\alpha W_t - \frac{1}{2}\alpha^2 t\right)$ for $t \geq 0$ and for any $\alpha \in \mathbb{R}$.

Laws on BM

- Law of large numbers:

$$\lim_{t\to\infty} \frac{W_t}{t} = 0, \qquad \mathbb{P}\text{-a.s.}$$

i.e. BM grows asymptotically less than linearly (as $t \to \infty$).

- Global law of the iterated logarithm (LIL):
With $\psi_{\text{glob}}(t) := \sqrt{2t\log(\log t)}$, it holds for $t \geq 0$ that:

$$\lim_{t\to\infty} \sup \frac{W_t}{\psi_{\text{glob}}(t)} = +1$$
$$\lim_{t\to\infty} \inf \frac{W_t}{\psi_{\text{glob}}(t)} = -1$$

i.e. for $\mathbb{P}$-a.a. ω, the function $t \to W_t(\omega)$ for $t \to \infty$ oscillates precisely between $t \to \pm\psi_{\text{glob}}(t)$.

- Local law of the iterated logarithm (LIL):
With $\psi_{\text{loc}}(h) := \sqrt{2h\log(\log \frac{1}{h})}$, it holds for $t \geq 0$ that:

$$\lim_{h\searrow 0} \sup \frac{W_{t+h} - W_t}{\psi_{\text{loc}}(h)} = +1$$
$$\lim_{h\searrow 0} \inf \frac{W_{t+h} - W_t}{\psi_{\text{loc}}(h)} = -1$$

i.e. for $\mathbb{P}$-a.a. ω, to the right of t, the trajectory $u \to W_u(\omega)$ around the level $W_t(\omega)$ oscillates precisely between $h \to \pm\psi_{\text{loc}}(h)$.

Poisson processes

- A Poisson process $N = (N_t)_{t\geq 0}$ with parameter $\lambda \in \mathbb{R}, \lambda > 0$ and w.r.t. $(\mathbb{P}, \mathbb{F})$ is a real-valued stochastic process satisfying the following properties:

(PP0) *null at zero*
N is adapted to $\mathbb{F}$ and null at 0 (i.e. $N_0 \equiv 0, \mathbb{P}$-a.s.).

(PP1) *independent and stationary increments*
For $0 \leq s < t$, the increment $N_t - N_s$ is independent (under $\mathbb{P}$) of $\mathcal{F}_s$ and follows (under $\mathbb{P}$) the Poisson distribution with parameter $\lambda(t-s)$, i.e. $N_t - N_s \sim \mathrm{Poi}(\lambda(t-s))$, i.e

$$\mathbb{P}[N_t = k] = \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}$$

(PP2) *counting process*
N is a counting process with jumps of size 1, i.e. for $\mathbb{P}$-a.a. ω, the function $t \mapsto N_t(\omega)$ is RCLL, piecewise constant and $\mathbb{N}_0$-valued, and increases by jumps of size 1.

- Important properties of Poisson processes: if $X \sim \mathrm{Poi}(\lambda)$, then:

$$\mathbb{E}[X] = \lambda, \qquad \mathrm{Var}[X] = \lambda$$

The quadratic variation of a Poisson process equals itself, i.e.

$$[N]_t = N_t$$

- Examples of Poisson processes: the following Poisson processes are $(\mathbb{P}, \mathbb{F})$-martingales:
  - **Compensated Poisson process:**

  $$\tilde{N}_t = N_t - \lambda t, \qquad t \geq 0$$

  - **Geometric Poisson process:**

  $$S_t = \exp\left(N_t \log(1+\sigma) - \lambda\sigma t\right), \qquad t \geq 0$$

  where $\sigma \in \mathbb{R}, \sigma > -1$.
  - Two cases of squared compensated Poisson processes:

  $$\left(\tilde{N}_t\right)^2 - N_t, \qquad \left(\tilde{N}_t\right)^2 - \lambda t, \qquad t \geq 0$$

  It follows that $[\tilde{N}]_t = N_t$.

3.2 Stochastic integration

Optional quadratic variation/square bracket process

- For any local martingale $M = (M_t)_{t\geq 0}$ null at 0, there exists a unique adapted increasing RCLL process $[M] = ([M]_t)_{t\geq 0}$ and having the property that $M^2 - [M]$ is also a local martingale.
- This process can be obtained as the quadratic variation of M in the following sense.
There exists a sequence $(\Pi_n)_{n\in\mathbb{N}}$ of partitions $[0,)$ with $|\pi| \to 0$ as $n \to \infty$ s.t.

$$\mathbb{P}\left[[M]_t(\omega) = \lim_{n\to\infty} \sum_{t_i \in \Pi_n} \left(M_{t_i\wedge t}(\omega) - M_{t_{i-1}\wedge t}(\omega)\right)^2 \forall t \geq 0\right] = 1$$

We call $[M]$ the optional quadratic variation or square bracket process of M.

- If M satisfies $\sup_{0\leq s\leq t} |M_s| \in \mathcal{L}^2$ for each $t \geq 0$ (and hence is in particular a martingale), then $[M]$ is integrable (i.e. $[M]_t \in \mathcal{L}^1$ for every $t \geq 0$) and $M^2 - [M]$ is a martingale.

(Optional) covariation process

- For two local martingales M, N null at 0, we define the (optional) covariation process $[M, N]$ by polarisation, i.e.

$$[M, N] := \frac{1}{4}\left([M+N] - [M-N]\right)$$

- The operation $[\cdot, \cdot]$ is bilinear.

Set of all bounded elementary processes

- We denote by $b\mathcal{E}$ the set of all bounded elementary process of the form

$$H = \sum_{i=0}^{n-1} h_i \mathbb{I}_{(t_i, t_{i+1}]}$$

with $n \in \mathbb{N}$, $0 \leq t_0 < t_1 < \ldots < t_n < \infty$ and each h_i a bounded (real-valued) $\mathcal{F}_{t_i}$-measurable RV.

Stochastic integral

- For any stochastic process $X = (X_t)_{t\geq 0}$, the stochastic integral $\int H dX$ of $H \in b\mathcal{E}$ is defined as

$$\int_0^t H_s dX_s := H \cdot X_t := \sum_{i=0}^{n-1} h_i \left(X_{t_{i+1}\wedge t} - X_{t_i\wedge t}\right) \quad \text{for } t \geq 0$$

- If X and H are both $\mathbb{R}^d$-valued, the integral is still real-valued, and we simply replace products by scalar products everywhere.
- The following fundamental identity applies to the quadratic variation of stochastic integrals:

$$\left[\int_0^t H_s dX_s\right] = \int_0^t H_s^2 d[X]_s$$

Isometry property

- Suppose M is a square-integrable martingale (i.e. $M_t \in \mathcal{L}^2$ for all $t \geq 0$).
- For every $H \in b\mathcal{E}$, the stochastic integral process $H \cdot M = \int H dM$ is then also a square-integrable martingale, and we have the isometry property

$$\mathbb{E}\left[(H \cdot M_\infty)^2\right] = \mathbb{E}\left[\int_0^\infty H_s^2 d[M]_s\right]$$

Properties

- (Local) Martingale properties
- Linearity
- Associativity
- Behaviour under stopping
- Quadratic variation and covaration
- Jumps

3.3 Stochastic calculus

Itô's formula I

- Suppose $X = (X_t)_{t\geq 0}$ is a continuous real-valued semimartingale and $f : \mathbb{R} \to \mathbb{R}$ is in C^2 (i.e. f is twice continuously differentiable).
- Then $f(X) = (f(X_t))_{t\geq 0}$ is again a continuous (real-valued) semimartingale, and we explicitly have $\mathbb{P}$-a.s. $\forall t \geq 0$:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle X\rangle_s$$
$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X\rangle_t$$

Remarks:

- The dX-integral is a stochastic integral. It is well-defined since X is a semimartingale and $f'(X)$ is adapted and continuous, hence predictable and locally bounded.
  The $d\langle X\rangle$-integral is a classical Lebesgue-Stieltjes integral since $\langle X\rangle$ has increasing trajectories. It is also well-defined since $f''(X)$ is also predictable and locally bounded.
- In comparison to the classical chain rule, the $d\langle X\rangle$-integral is an extra second-order term coming from the quadratic variation of X. Hence Itô's formula can be viewed as an extension of the chain rule.
- The important message of this formula is that when one is dealing with stochastic models, a simple linear approximation is not good enough, since one also has to account for the second-order behaviour of X.
- $\langle X\rangle_t = \langle M\rangle_t$

Itô's formula II

- Suppose $X = (X_t)_{t\geq 0}$ is a general $\mathbb{R}^d$-valued semimartingale and $f : \mathbb{R}^d \to \mathbb{R}$ is in C^2.
- Then $f(X) = (f(X_t))_{t\geq 0}$ is again a (real-valued) semimartingale, $\mathbb{P}$-a.s. and $\forall t \geq 0$.
- If X has continuous trajectories:

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x^i}(X_s)dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)d\langle X^i, X^j\rangle_s$$

- If a stochastic process $X_t = f(t, W_t)$ with $f : \mathbb{R}^2 \to \mathbb{R}$ in $C^{1,2}$ (i.e. once continuously differentiable in time t and twice continuously differentiable in W_t), then:

$$\boxed{X_t = X_0 + \underbrace{\int_0^t \frac{\partial f}{\partial w}(W_s, s)dW_s}_{\text{local } (\mathbb{P},\mathbb{F}) \text{ martingale}} + \int_0^t \left(\frac{\partial f}{\partial t}(W_s, s) + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}(W_s, s)\right) ds}$$

  Note that X is a (continuous) local $(\mathbb{P},\mathbb{F})$-martingale iff

$$\int_0^t \left(\frac{\partial f}{\partial t}(W_s, s) + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}(W_s, s)\right) ds = 0, \qquad \forall t \geq 0$$

- **Itô's formula with jumps**
  If $d = 1$, X real-valued but not necessarily continuous, then it holds that:

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s + \frac{1}{2}\int_0^t f''(X_{s-})d[X]_s + \sum_{0<s\leq t}\Big(f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s - \frac{1}{2}f''(X_{s-})(\Delta X_s)^2\Big)$$

- If $f : \mathbb{R} \to \mathbb{R}$ is in C^2, $\alpha, \beta \in \mathbb{R}$ and the semimartingale $X = (X_t)_{t\geq 0}$ is given by $X_t = \alpha t + \beta N_t$, then:

$$f(X_t) = f(X_0) + \alpha\int_0^t f'(X_{s-})ds + \sum_{0<s\leq t}(f(X_s) - f(X_{s-}))$$

Stochastic exponential

- For a general real-valued semimartingale X null at 0, the stochastic exponential of X is defined as the unique solution Z of the SDE

$$dZ_t = Z_{t-}dX_t, \qquad Z_0 = 1$$

  and it follows that the unique solution to this SDE is:

$$Z_t := \mathcal{E}(X) = 1 + \int_0^t Z_{s-}dX_s \qquad \forall t \geq 0$$

$$\boxed{\mathcal{E}(X)_t = \exp\left(X_t - \frac{1}{2}[X]_t\right)}$$

- **Yor's formula:**

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y])$$

Itô process

- An Itô process is a stochastic process of the form

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s, \qquad \forall t \geq 0$$

  where W is some Brownian motion and μ and σ are predictable processes.
- More generally, X, μ, W could be vector-valued and σ could be matrix-valued.
- For any C^2 function f, the process $f(X)$ is again an Itô process, and Itô's formula gives

$$f(X_t) = f(X_0) + \int_0^t \left(f'(X_s)\mu_s + \frac{1}{2}f''(X_s)\sigma_s^2\right) ds + \int_0^t f'(X_s)\sigma_s dW_s$$

Itô's representation theorem

- Suppose that $W = (W_t)_{t\geq 0}$ is a $\mathbb{R}^m$-valued BM.
- Then every RV $H \in \mathcal{L}^1(\mathcal{F}_\infty^W, P)$ has a unique representation as

$$H = \mathbb{E}[H] + \int_0^\infty \psi_s dW_s, \qquad \text{a.s.}$$

  for an $\mathbb{R}^m$-valued integrand $\psi \in \mathcal{L}^2_{\text{loc}}(W)$.
- ψ has the additional property that $\int \psi dW$ is a $(P, \mathbb{F}^W)$-martingale on $[0, \infty]$ (and is thus uniformly integrable).

Itô product formula

- Define the stochastic process $Z = XY$, where X and Y are two continuous real-valued semimartingales.
- Then Z can be written as the sum of stochastic integrals:

$$Z_t - Z_0 = \int_0^t Y_s dX_s + \int_0^t X_s dY_s + \int_0^t d[X, Y]_s$$

General properties/results

- Any continuous, adapted process H is also predictable and locally bounded.
  It furthermore holds for any predictable, locally bounded process H that $H \in L^2_{\text{loc}}(W)$.
- Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary continuous convex function. Then the process $(f(W_t))_{t\geq 0}$ is integrable and is a $(\mathbb{P}, \mathbb{F})$-submartingale.
- Given a $(\mathbb{P}, \mathbb{F})$-martingale $(M_t)_{t\geq 0}$ and a measurable function $g : \mathbb{R}_+ \to \mathbb{R}$, the process

  $$(M_t + g(t))_{t\geq 0}$$

  is a:
  - $(\mathbb{P}, \mathbb{F})$-supermartingale iff g is decreasing;
  - $(\mathbb{P}, \mathbb{F})$-submartingale iff g is increasing.
- A continuous local martingale of finite variation is identically constant (and hence vanishes if it is null at 0).
- For a function $f : \mathbb{R} \to \mathbb{R}$ in C^1, the stochastic integral $\int_0^\cdot f'(W_s)dW_s$ is a continuous local martingale.
  Furthermore, for $f \in C^2$ it holds that $f(W)$ is a continuous local martingale iff $\int_0^\cdot f''(W_s)ds = 0$.
- If a predictable process $H = (H_t)_{t\geq 0}$ satisfies

  $$\mathbb{E}\left[H_s^2 ds\right] < \infty, \qquad \forall T \geq 0$$

  then $\int H dW_s$ is a square-integrable martingale.
- If $f : \mathbb{R} \to \mathbb{R}$ is bounded and continuous, then the stochastic integral $\int f(W)dW$ is a square-integrable martingale.
- If a process $H = (H_t)_{t\geq 0}$ is predictable and the map $s \mapsto \mathbb{E}[H_s^2]$ is continuous, then the stochastic integral $\int H dW$ is a square-integrable martingale.
- If $f : \mathbb{R} \to \mathbb{R}$ is polynomial, then the stochastic integral $\int f(W)dW$ is a square-integrable martingale.

4 Black-Scholes Formula

4.1 Black-Scholes (BS) model

BS model (undiscounted, historical measure $\mathbb{P}$)

$$\tilde{S}_t^0 = e^{rt} \qquad \tilde{S}_t^1 = \tilde{S}_0^1 \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right)$$

$$\frac{d\tilde{S}_t^0}{\tilde{S}_t^0} = r dt \qquad \frac{d\tilde{S}_t^1}{\tilde{S}_t^1} = \mu dt + \sigma dW_t$$

BS model (discounted, historical measure $\mathbb{P}$)

$$S_t^0 = 1 \qquad S_t^1 = S_0^1 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right) t\right)$$

$$\frac{dS_t^1}{S_t^1} = (\mu - r)dt + \sigma dW_t$$

BS model (discounted, risk-neutral measure $\mathbb{Q}$)

$$dS_t^1 = S_t^1 \sigma \left(dW_t + \frac{\mu - r}{\sigma} dt\right) = S_t^1 \sigma dW_t^*$$

$$S_t^1 = S_0^1 + \int_0^t S_u^1 \sigma dW_u^* = S_0^1 \exp\left(\sigma W_t^* - \frac{1}{2}\sigma^2 t\right)$$

where

$$W_t^* := W_t + \frac{\mu - r}{\sigma} t = W_t + \int_0^t \lambda ds$$

Market price of risk

- The market price of risk or infinitemsimal **Sharpe ratio** of S^1 is defined as

  $$\lambda^* = \frac{\mu - r}{\sigma}$$

4.2 Black-Scholes PDE

Black-Scholes PDE

$$0 = \frac{\partial \tilde{v}}{\partial t} + r\tilde{x}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \frac{1}{2}\sigma^2 \tilde{x}^2 \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} - r\tilde{v}, \qquad \tilde{v}(T, \cdot) = \tilde{h}(\cdot)$$

4.3 Black-Scholes formula for option pricing

Martingale pricing approach

- The discounted arbitrage-free value at time t of any discounted payoff $H \in L^1_+(\mathcal{F}_T, \mathbb{Q}^*)$, $H_T = H(\tilde{S}_T^0, \tilde{S}_T^1)$, is given by

  $$\boxed{V_t^* = \mathbb{E}_{\mathbb{Q}}\left[H | \mathcal{F}_t\right] := \vartheta(t, S_t^1)}$$

- Then the discounted payoff H can be hedged via the replicating strategy (V_0, ϑ) s.t.

  $$V_0 + \int_0^T \vartheta_u dS_u^1 = H(\tilde{S}_T^0, \tilde{S}_T^1)$$

  Using Itô's representation theorem the replicating strategy can be expressed as

  $$\boxed{\begin{aligned} V_t^* &= \vartheta(t, S_t^1) = V_0 + \int_0^t \vartheta_s dS_s^1 + \underbrace{\textit{cont. FV process}}_{\text{"usually" vanishes}} \\ V_0 &= \vartheta(0, S_0^1), \qquad \vartheta_t = \frac{\partial \vartheta}{\partial x}(t, S_t^1) \end{aligned}}$$

Black-Scholes formula for a European call option

$$\tilde{V}_t^{\tilde{H}} = \tilde{v}(t, \tilde{S}_t^1) = \tilde{S}_t^1 \Phi(d_1) - \tilde{K} e^{-r(T-t)} \Phi(d_2)$$

$$d_{1,2} = \frac{\log\left(\frac{\tilde{S}_t^1}{\tilde{K}}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}$$

$$\Phi(y) = Q^*[Y \leq y] = \int_{-\infty}^y \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

Remarks:

- Φ denotes the CDF of the standard normal distribution $\mathcal{N}(0, 1)$.
- Note that the drift μ of the stock does not appear here. This is analogous to the result that the probability p of an up move in the CRR binomial model does not appear in the binomial option pricing formula.

Replicating strategy for a European call option

$$\nu_t^H = \frac{\partial \tilde{v}}{\partial \tilde{x}}(t, \tilde{S}_t^1) = \Phi(d_1)$$

Greeks The derivatives of the option price w.r.t. the various parameters, i.e. the sensitivities of the option price w.r.t. to the parameters, are called Greeks.

5 Appendix

Markov's inequality For X any nonnegative integrable RV and $a \in \mathbb{R}, a > 0$:

$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$$

Chebyshev's inequality For X an integrable RV with finite expected value $\mu \in \mathbb{R}$ and finite non-zero variance $\sigma^2, \sigma \in \mathbb{R}$ and for any real number $k > 0$:

$$\mathbb{P}[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2}$$

Jensen's inequality For X a RV and f a convex function, it holds that

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$$

Sets/families

- L^0_+: familiy of all nonnegative RVs

Correlation and Independence

- Let X and Y be two RVs.
- Then X, Y are **uncorrelated** iff

  $$\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

- Then X, Y are **independent** iff

  $$\mathbb{P}[X = x, Y = y] = \mathbb{P}[X = x] \cdot \mathbb{P}[Y = y]$$

  Note that independence of X, Y implies that X, Y are uncorrelated (but not vice-versa!).

Independence of equations

- If there is e.g. a system of equations such as

  $$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \\ a_3x + b_3y = c_3 \end{cases}$$

  then this system admits a solution iff

  $$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = 0$$

Fubini's lemma

$$\mathbb{E}\left[\int_0^T H_s^2 ds\right] = \int_0^T \mathbb{E}\left[H_s^2\right] ds$$

intentionally left blank

Abbreviations

a.a.	almost all
a.s.	almost surely
BM	Brownian motion
CDF	cumulative distribution function
iff	if and only if
IOT	in order to
PDE	partial differential equation
PDF	probability density function
RCLL	right-continuous with left limits
RV	random variable
SDE	stochastic differential equation
s.t.	such that
w.r.t.	with respect to

Disclaimer

- This summary is work in progress, i.e. neither completeness nor correctness of the content are guaranteed by the author.
- This summary may be extended or modified at the discretion of the readers.
- Source: Lecture Mathematical Foundations for Finance, autumn semester 2015/16, ETHZ (lecture notes, script and exercises).
- The layout of this summary is mainly based on the summaries of several courses of the BSc ETH ME from Jonas LIECHTI.