Summary: Numerical Analysis of Stochastic Ordinary Differential Equations (SODEs)

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1 Generation of random numbers

Pseudo random number generators Let $\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $U_n : \Omega \to \mathbb{R}, n \in \mathbb{N}$ be a sequence of $\mathbb{P}$-independent $\mathcal{U}_{(0,1)}$-distributed random variables. $\mathcal{U}_{(0,1)}$-pseudo random numbers are sequences of real number that are calculated by a deterministic algorithm and that have — in an appropriate sense — similar statistical properties as $(U_n)_{n\in\mathbb{N}}$.

1.1 Inversion method

Generalized inverse distribution function associated to a distribution function Let $F : \mathbb{R} \to [0,1]$ be a distribution function. Then we denote by $I_F : (0,1) \to \mathbb{R}$ the function with the property that $\forall y \in (0,1$ it hold that

$$I_F(y) = \inf\{x \in \mathbb{R} : F(x) \le y\} = \inf(F^{-1}([y,1]))$$

and we call I_F the generalized inverse distribution function associated to F.

Inversion method Let $F : \mathbb{R} \to [0,1]$ be a distribution function, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probabilit yspace, and let $U : \Omega \to \mathbb{R}$ be an $\mathcal{U}_{(0,1)}$-distributed RV with $U(\Omega) \subseteq (0,1)$. Then F is the distribution function of the $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mapping $I_F(U) = I_F \circ U : \Omega \to \mathbb{R}$, i.e. $\forall x \in \mathbb{R}$ it holds that

$$\mathbb{P}[I_F(U) \le x] = F(x)$$

1.2 Acceptance-rejection method

Result: Realization x of $X \sim \mathcal{U}_A$
Generate realization y of $Y \sim \mathcal{U}_B$
if $y \in A$ **then**
  $x = y$ (ACCEPT)
else
  Restart the algorithm (REJECT)

Result: Realization x of $X \sim X(P)_{\mathcal{B}(\mathbb{R}^d}$ (with unnormalized density f)
Generate realization y of $Y \sim Y(P)_{\mathcal{B}(\mathbb{R}^d}$ (with unnormalized density g)
Generate realization u of $U \sim \mathcal{U}_{(0,1)}$
if $g(y) \cdot u \le f(y)$ **(resp.** $u \le \frac{f(y)}{g(y)}$ **then**
  $x = y$ (ACCEPT)
else
  Restart the algorithm (REJECT)

1.3 Methods for the normal distribution

Box-Muller method Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $U_1, U_2 : \Omega \to \mathbb{R}$ be two independent $\mathcal{U}_{(0,1)}$-distributed RVs with the property that $U_1(\Omega) \subseteq (0,1)$ and $U_1(\Omega) \subseteq (0,1)$. Then the function $X = (X_1, X_2) : \Omega \to \mathbb{R}^2$ with the property that

$$X_1 = \sqrt{-2\log(U_1)}\cos(2\pi U_2)$$
$$X_2 = \sqrt{-2\log(U_1)}\sin(2\pi U_2)$$

is $\mathcal{N}_{0,\mathcal{I}_{\mathbb{R}^2}}$-distributed.

Result: Realization (x_1, x_2) of $(X_1, X_2) \sim \mathcal{N}_{0,\mathcal{I}_{\mathbb{R}^2}}$
Generate realization (u_1, u_2) of $(U_1, U_2) \sim \mathcal{U}_{(0,1)^2}$
$x_1 = \sqrt{-2\log(u_1)}\cos(2\pi u_2)$
$x_2 = \sqrt{-2\log(u_1)}\sin(2\pi u_2)$

Marsaglia polar method Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, let $U : \Omega \to \mathbb{R}^2$ be an $\mathcal{U}_{(x,y)\in\mathbb{R}^2 : x^2+y^2\in(0,1)}$-distributed RV with $U(\Omega) \subseteq \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \in (0,1)\}$, and let $X : \Omega \to \mathbb{R}^2$ be the function given by

$$X = \frac{U\sqrt{-2\log\left(\|U\|^2_{\mathbb{R}^2}\right)}}{\|U\|_{\mathbb{R}^2}}$$

Then X is $\mathcal{N}_{0,\mathcal{I}_{\mathbb{R}^2}}$-distributed.

Result: Realization (x_1, x_2) of $(X_1, X_2) \sim \mathcal{N}_{0,\mathcal{I}_{\mathbb{R}^2}}$
repeat
  Generate realization (v_1, v_2) of $(V_1, V_2) \sim \mathcal{U}_{(0,1)^2}$
  $q = (2v_1 - 1)^2 + (2v_s - 1)$
until $q \in (0,1)$;
$w = \sqrt{-2\frac{\log(q)}{q}}$
$x_1 = (2v_1 - 1)w$
$x_2 = (2v_2 - 1)w$

The **acceptance probability** int eh acceptance-rejection algorithm in the Marsaglia polar method is $\frac{\pi}{4} \approx 0.78$. Thus on average the alogrithm in the Marsaglia polar method runs $\frac{4}{\pi} \approx 1.27$ times through the loop.

Normally distributed RVs with general mean and covariance matrix Let $\Omega, \mathcal{A}, \mathbb{P}$ be a probability space, let $d \in \mathbb{N}$ (dimension), $b \in \mathbb{R}^d$ (mean vector), $A \in \mathbb{R}^{d\times d}$ (covariance matrix), and let $X : \Omega \to \mathbb{R}^d$ be an $\mathcal{N}_{0,\mathcal{I}_{\mathbb{R}^d}}$-distributed RV. The the function

$$\Omega \ni \omega \to AX(\omega) + b \in \mathbb{R}^d$$

is $\mathcal{N}_{b,AA^T}$-distributed.

2 Sampling

2.1 Bias of an estimator

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $c \in \mathbb{R}$ be a real number, and let $X \in \mathcal{L}^1(\mathbb{P}; |\cdot|_{\mathbb{R}})$.

Bias $\mathrm{Bias}_{\mathbb{P},c}(X)$ is the $\mathbb{P}$-bias of X w.r.t. c, i.e.

$$\mathrm{Bias}_{\mathbb{P},c}(X) = \mathbb{E}_{\mathbb{P}}[X] - c$$

Unbiased If

$$\mathrm{Bias}_{\mathbb{P},c} = 0$$

then X is said to be $\mathbb{P}$-unbiased w.r.t. c.

Biased If

$$\mathrm{Bias}_{\mathbb{P},c} \ne 0$$

then X is said to be $\mathbb{P}$-biased w.r.t. c.

2.2 Approximations of the variance of a RV

Variance

$$\mathrm{Var}_{\mathbb{P}}[X_1] = \mathbb{E}_{\mathbb{P}}\left[(X_1 - \mathbb{E}_{\mathbb{P}}[X_1])^2\right]$$

Biased approximation of the variance The approximation of the variance $\mathrm{Var}_{\mathbb{P}}[X_1]$

$$\frac{1}{N}\sum_{n=1}^{N}\left(X_n - \frac{X_1 + \ldots + X_N}{N}\right)^2$$

is $\mathbb{P}$-biased w.r.t. $\mathrm{Var}_{\mathbb{P}}[X_1]$.

Unbiased approximation of the variance The approximation of the variance $\mathrm{Var}_{\mathbb{P}}[X_1]$

$$\frac{1}{N-1}\sum_{n=1}^{N}\left(X_n - \frac{X_1 + \ldots + X_N}{N}\right)^2$$

is $\mathbb{P}$-unbiased w.r.t. $\mathrm{Var}_{\mathbb{P}}[X_1]$.

Useful identity

$$\sum_{n=1}^{N}\left(x_n - \frac{x_1 + \ldots + x_N}{N}\right)^2 = \left(\sum_{n=1}^{N}(x_n)^2\right) - \frac{1}{N}\left(\sum_{n=1}^{N} x_n\right)^2$$

3 Deterministic numerical integration methods

Quadrature formula Let $d \in \mathbb{N}$, $A \in \mathcal{B}(\mathbb{R}^2)$, let I be a finite set, let $(x_i)_{i\in I} \subseteq A$, $(w_i)_{i\in I} \subseteq \mathbb{R}$, and let $Q : \mathcal{L}^1(B_A; |\cdot|_{\mathbb{R}}) \to \mathbb{R}$ be the function with the property that $\forall f \in \mathcal{L}^1(B_A; |\cdot|_{\mathbb{R}})$ it holds that

$$Q[f] = \sum_{i\in I} w_i f(x_i)$$

Then we call Q a quadrature formula (on A with quadrature nodes $(x_i)_{i\in I}$ and quadrature weights $(w_i)_{i\in I}$).

3.1 Rectangle method

d-dimenstional left rectangle method Let $d \in \mathbb{N}$, $a, b \in \mathbb{R}$ with $a < b$. Then we denote by

$$R^n_{[a,b]^d} : \mathcal{L}^1(B_{[a,b]^d}; |\cdot|_{\mathbb{R}}) \to \mathbb{R}, n \in \mathbb{N}$$

the quadrature formulas with the property that $\forall n \in \mathbb{N}$, $f \in \mathcal{L}^1(B_{[a,b]^d}; |\cdot|_{\mathbb{R}})$ it holds that

$$R^n_{[a,b]^d} = \frac{(b-a)^d}{n^d} \cdot \sum_{i_i,\ldots,i_d \in \{0,1,\ldots,n-1\}} f\left(a + \frac{i_i}{n}(b-a), \ldots, \frac{i_d}{n}(b-a)\right)$$

and we call the sequence $R^n_{[a,b]^d}$, $n \in \mathbb{N}$, the d-dimensional left rectangle method.

Error estimate for the d-dimensional left rectangle method Let $d, n \in \mathbb{N}$, $\alpha \in (0,1]$, $a, b \in \mathbb{R}$ with $a < b$ and let $f \in \mathcal{L}^1(B_{[a,b]^d}; |\cdot|_{\mathbb{R}})$. Then

$$\begin{aligned}\left|R^n_{[a,b]^d}[f] - \int_{[a,b]^d} f(x)dx\right| &\le (b-a)^d w_f\left(\frac{(b-a)\sqrt{d}}{n}\right) \\ &\le \frac{(b-a)^{d+\alpha} d^{\frac{\alpha}{2}} \|f\|_{C^{0,\alpha}([a,b]^d,\mathbb{R})}}{n^\alpha}\end{aligned}$$

3.2 Trapezoidal rule

Trapezoidal method Let $a, b \in \mathbb{R}$ with $a < b$. Then we denote by $T^n_{[a,b]} : \mathcal{L}^1(B_{[a,b]}; |\cdot|_{\mathbb{R}}) \to \mathbb{R}$, $n \in \mathbb{N}$, the quadrature formulas with the property that $\forall n \in \mathbb{N}$, $f \in \mathcal{L}^1(B_{[a,b]}; |\cdot|_{\mathbb{R}})$ it holds that

$$\begin{aligned}T^n_{[a,b]}[f] &= \frac{b-a}{n}\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f\left(a + \frac{i}{n}(b-a)\right) \\ &= \frac{b-a}{n}\sum_{i=0}^{n-1}\frac{f\left(a + \frac{i}{n}(b-a)\right) + f\left(a + \frac{i+1}{n}(b-a)\right)}{2}\end{aligned}$$

and we call the sequence $T^n_{[a,b]}$, $n \in \mathbb{N}$ the 1-dimensional trapezoidal method.

Error estimate for the trapezoidal method Let $n \in \mathbb{N}$, $\alpha \in (0,1]$, $a, b, \in \mathbb{R}$ with $a < b$. Then $\forall f \in \mathcal{L}^1(B_{[a,b]}; |\cdot|_{\mathbb{R}})$ it holds that

$$\begin{aligned}\left|T^n_{[a,b]}[f] - \int_a^b f(x)dx\right| &\le (b-a)\cdot w_f\left(\frac{b-a}{2n}\right) \\ &\le \frac{(b-a)^{1+\alpha}\|f\|_{C^{0,\alpha}([a,b],\mathbb{R})}}{(2n)^\alpha}\end{aligned}$$

and $\forall f \in C^1([a,b], \mathbb{R})$ it holds that

$$\begin{aligned}\left|T^n_{[a,b]}[f] - \int_a^b f(x)dx\right| &\le \frac{(b-a)^2}{n}\cdot w_{f'}\left(\frac{b-a}{2n}\right) \\ &\le \frac{(b-a)^{2+\alpha}\|f''\|_{C^{0,\alpha}([a,b],\mathbb{R})}}{2^\alpha n^{1+\alpha}}\end{aligned}$$

4 Monte Carlo methods

Monte Carlo approximation of the expected value The Monte Carlo approximation of the expected value $\mathbb{E}_{\mathbb{P}}[X_1]$ is defined as

$$\frac{X_1 + \ldots + X_N}{N}$$

and is $\mathbb{P}$-unbiased w.r.t. $\mathbb{E}_{\mathbb{P}}$, i.e. $\forall N \in \mathbb{N}$ it holds that $\mathbb{E}_{\mathbb{P}}\left[\frac{1}{N}(X_1 + \ldots + X_N)\right] = \mathbb{E}_{\mathbb{P}}[X_1]$.

Monte Carlo approximation of the variance The Monte Carlo approximation of the variance $\mathrm{Var}_{\mathbb{P}}[X_1]$ is defined as

$$\frac{1}{N-1}\sum_{n=1}^{N}\left(X_n - \frac{X_1 + \ldots + X_N}{N}\right)^2$$

for $N \in \{2, 3, \ldots\}$ and is $\mathbb{P}$-unbiased w.r.t. $\mathrm{Var}_{\mathbb{P}}[X_1]$.

4.1 Confidence intervals for Monte Carlo methods

Definitions

- $\alpha \in [0, 1)$
- $c, \beta, \gamma \in [0, \infty)$
  with $\frac{1}{2\pi}\int_{-\beta}^{\gamma} e^{\frac{-x^2}{2}} dx \ge \alpha$
  and $\sqrt{\mathrm{Var}_{\mathbb{P}}(X_1)} \le c$
- $E_N = \frac{X_1 + \ldots + X_N}{N}$
- $V_N = \frac{1}{N-1}\sum_{n=1}^{N-1}(X_n - E_N)^2$

α-confidence intervals

$$\left[E_N \pm \frac{\sqrt{\mathrm{Var}_P[X_1]}}{\sqrt{(1-\alpha)N}}\right] \qquad \left[E_N \pm \frac{c}{\sqrt{(1-\alpha)N}}\right]$$

asymptotically valid α-confidence intervals

$$\left[E_N \pm \frac{\beta\sqrt{\mathrm{Var}_P[X_1]}}{\sqrt{N}}\right] \qquad \left[E_N \pm \frac{\beta c}{\sqrt{N}}\right]$$

asymptotically valid α-confidence intervals with variance approximation

$$\left[E_N \pm \frac{\sqrt{V_N}}{\sqrt{(1-\alpha)N}}\right] \qquad \left[E_N \pm \frac{\beta\sqrt{V_N}}{\sqrt{N}}\right]$$

Tails of the normal distribution $\forall x \in (0, \infty)$ it holds that

$$\mathcal{N}_{0,1}([x, \infty)) = \int_x^\infty \frac{e^{-\frac{1}{2}y^2}}{x\sqrt{2\pi}} < \frac{e^{-\frac{1}{2}x^2}}{x\sqrt{2\pi}}$$

$\forall \alpha \in [0, 1)$ it holds that

$$\frac{1}{\sqrt{2\pi}} \int_{\frac{-1}{\sqrt{1-\alpha}}}^{\frac{1}{\sqrt{1-\alpha}}} e^{-\frac{1}{2}x^2} dx > \alpha$$

4.2 Monte Carlo algorithms for numerical integration

Monte Carlo approximation I Real number to be approximated:

$$\int_A f(x)dx = \mathbb{E}[\lambda_{\mathbb{R}^d}(A) \cdot f(Y_1)]$$

Monte Carlo approximation:

$$\frac{\lambda_{\mathbb{R}^d}(A)}{N}(f(Y_1) + \ldots + f(Y_N))$$

Algorithm:

Result: Realization x of $X \sim \mathbb{P}_{\frac{\lambda_{\mathbb{R}^d}(A)}{N}(f(Y_1)+\ldots+f(Y_N))} \approx \int_A f(x)dx$
$s = 0$
for $n = 1 \to N$ **do**
  Generate realization y of $Y_n \sim \mathcal{U}_A$
  $s = s + f(u)$
end
$x = \frac{\lambda_{\mathbb{R}^d}(A)}{N} \cdot s$

Monte Carlo approximation II Real number to be approximated:

$$\int_A f(x)dx = \mathbb{E}_{\mathbb{P}}\left[\tilde{f}(U_1) \cdot \prod_{i=1}^d (b_i - a_i)\right]$$

Properties:

- $a_1, \ldots, a_d, b_1, \ldots, b_d \in \mathbb{R}$ and $a_1 \leq b_1, \ldots, a_d \leq b_d$
- $A \subseteq [a_1, b_1] \times \ldots \times [a_d, b_d]$
- function $\tilde{f} : [a_1, b_1] \times \ldots [a_d, b_d] \to \mathbb{R}$ with $\tilde{f}(x) = f(x) \quad \forall x \in A$ and $\tilde{f}(x) = 0$ else
- $U_n : \Omega \to [a_1, b_1] \times \ldots [a_d, b_d]$ are independent $\mathcal{U}_{[a_1,b_1]\times\ldots[a_d,b_d]}$-distributed RVs

Monte Carlo approximation:

$$\frac{\prod_{i=1}^d (b_i - a_i)}{N} \cdot \left(\tilde{f}(U_1) + \ldots + \tilde{f}(U_N)\right)$$

Algorithm:

Result: Realization x of $X \sim \mathbb{P}_{\frac{(b_1-a_1)\cdot\ldots\cdot(b_d-a_d)}{N}(\tilde{f}(Y_1)+\ldots+\tilde{f}(Y_N))} \approx \int_A f(x)dx$
$s = 0$
for $n = 1 \to N$ **do**
  Generate realization u of $U_n \sim \mathcal{U}_{[a_1,b_1]\times\ldots\times[a_d,b_d]}$
  if $u \in A$ **then**
    $s = s + f(u)$
  end
end
$x = \frac{(b_1-a_1)\cdot\ldots\cdot(b_d-a_d)}{N} \cdot s$

Confidence interval Properties:

- $\alpha \in (0, 1)$
- $c \in [0, \infty)$ s.t. $\left(\prod_{i=1}^d (b_i - a_i)\right) \sqrt{\mathrm{Var}_{\mathbb{P}}(\tilde{f}(U_1)} \leq c$

For the RVs $X_N^1, X_N^2 : \Omega \to \mathbb{R}, N \in \mathbb{N}$ defined by

$$X_N^1 = \frac{1}{N}\left(\prod_{i=1}^d (b_i - a_i)\right)\left(\tilde{f}(U_1) + \ldots + \tilde{f}(U_N)\right) - \frac{c}{\sqrt{(1-\alpha)N}}$$

$$X_N^2 = \frac{1}{N}\left(\prod_{i=1}^d (b_i - a_i)\right)\left(\tilde{f}(U_1) + \ldots + \tilde{f}(U_N)\right) + \frac{c}{\sqrt{(1-\alpha)N}}$$

it holds that

$$\mathbb{P}\left[\int_A f(x)dx \in [X_N^1, X_N^2]\right] \geq \alpha$$

Algorithm:

Result: Realization $(x_1, x_2$ of $X_N^1, X_N^2)$
$s = 0$
for $n = 1 \to N$ **do**
  Generate realization u of $U_n \sim \mathcal{U}_{[a_1,b_1]\times\ldots\times[a_d,b_d]}$
  if $u \in A$ **then**
    $s = s + f(u)$
  end
end
$x_1 = \frac{(b_1-a_1)\cdot\ldots\cdot(b_d-a_d)}{N} \cdot s - \frac{c}{\sqrt{(1-\alpha)N}}$
$x_1 = \frac{(b_1-a_1)\cdot\ldots\cdot(b_d-a_d)}{N} \cdot s + \frac{c}{\sqrt{(1-\alpha)N}}$

5 Stochastic Differential Equations (SDEs)

Geometric Brownian Motion

$$dX_t = \alpha X_t + \beta X_t dW_t, \qquad t \in [0, T], \qquad X_0 = \xi$$

$$X_t = \xi \cdot \exp\left(\left(\alpha - \frac{\beta^2}{2}\right)t + \beta W_t\right)$$

Since $\frac{d}{dt}\mathbb{E}[X_t] = \alpha\mathbb{E}[X_t]$ and $\mathbb{E}[X_0] = \xi$,

$$\mathbb{E}[X_t] = \xi e^{\alpha t}$$

Black-Scholes Model

$$\mu(x) = \begin{pmatrix} rx_1 \\ \alpha x_2 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ \beta x_2 \end{pmatrix}$$

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \end{pmatrix} = \begin{pmatrix} rX_t^1 dt \\ \alpha X_t^2 dt + \beta X_t^2 dW_t \end{pmatrix}, \qquad t \in [0, T], \qquad X_0 = \xi$$

$$\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} \xi^1 e^{rt} \\ \xi^2 \exp\left(\left(\alpha - \frac{\beta^2}{2}\right)t + \beta W_t\right) \end{pmatrix}$$

Stochastic Ginzburg-Landau equation

$$\mu(x) = \alpha x - \delta x^3 \qquad \sigma(x) = \beta x - \bar{\beta}$$

$$dX_t = (\alpha X_t - \delta X_t^3)dt + (\beta X_t + \bar{\beta})dW_t, \quad t \in [0, T], X_0 = \xi$$

Stochastic Verhulst equation

$$\mu(x) = (\eta + \frac{c^2}{2})x - \lambda x^2 \qquad \sigma(x) = cx$$

$$dX_t = \left(\left(\eta + \frac{c^2}{2}X_t - \lambda X_t^2\right)\right)dt + cX_t dW_t$$

$$t \in [0, T], \qquad X_0 = \xi$$

Stochastic predator-prey model

$$\mu\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1(\alpha - \beta x_2) \\ x_2(\gamma x_1 - \delta) \end{pmatrix} \qquad \sigma\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 x_1 & 0 \\ 0 & c_2 x_2 \end{pmatrix}$$

$$dX_t = \begin{pmatrix} X_t^1(\alpha - \beta X_t^2) \\ X_t^2(\gamma X_t^1 - \delta \end{pmatrix} dt + \begin{pmatrix} c_1 X_t^1 & 0 \\ 0 & c_2 X_t^2 \end{pmatrix} dW_t$$

$$t \in [0, T], \qquad X_0 = \xi$$

Deterministic case: $c_1 = c_2 = 0$.

Cox-Ingersoll-Ross process

$$dX_t = (\delta - \alpha X_t)dt + \beta\sqrt{X_t}dW_t, \qquad t \in [0,T], X_0 = \xi$$

Simplified Ait-Sahalia interest rate model

$$dX_t = (\delta + \gamma X_t - \alpha X_t^2)dt + \beta X_t^b dW_t, \qquad t \in [0,T], X_0 = \xi$$

Volatility process in the Lewis stochastic volatility model

$$dX_t = (\gamma X_t - \alpha X_t^2)dt + \beta X_t^{\frac{3}{2}} dW_t, \qquad t \in [0,T], X_0 = \xi$$

6 Strong Approximations for SDEs

6.1 Convergence

6.2 Euler-Maruyama scheme

SDE

$$X_t = \xi + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s$$

For $t - t_0$ "sufficiently small" it holds $\mathbb{P} - a.s.$ that

$$X_t \approx X_{t_0} + \mu(X_{t_0}(t - t_0) + \sigma(X_{t_0}(W_t - W_{t_0})$$

Euler-Maruyama scheme

$$Y_{n+1} = Y_n + \mu(Y_n)\frac{T}{N} + \sigma(Y_n)\left(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\right)$$

Linearly interpolated Euler-Maruyama approximation

$$Y_t = Y_{\frac{nT}{N}} + \left(\frac{tN}{T} - n\right)\left(\mu\left(Y_{\frac{nT}{N}}\right)\frac{T}{N} + \sigma\left(Y_{\frac{nT}{N}}\right)\left(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\right)\right)$$

7 Distributions

7.1 Discrete distributions

Bernoulli distribution Let $p \in [0,1]$ be a real number and let $F : \mathbb{R} \to [0,1]$ be the distribution function of the Bernoulli distribution with paramet $p \in [0,1]$, i.e. assume that $\forall x \in \mathbb{R}$ it holds that

$$\begin{aligned} F(x) &= \mathrm{Ber}_p(-\infty, x]) \\ &= (1-p)\delta_0((-\infty,x]) + p\delta_1((-\infty,x]) \\ &= \begin{cases} 0 & : x < 0 \\ 1-p & : 0 \le x < 1 \\ 1 & : x \ge 1 \end{cases} \end{aligned}$$

Then the generalized inverse distribution function $I_F : (0,1) \to \mathbb{R}$ associated to F satisfies that $\forall y \in (0,1)$ it holds that

$$\begin{aligned} I_F(y) &= \inf\{x \in [x,\infty) : F(x) \ge y\} \\ &= \begin{cases} 0 & : 0 < y \le 1-p \\ 1 & : 1-p < y < 1 \end{cases} \end{aligned}$$

Result: Realization x of $X \sim \mathrm{Ber}_p$
Generate realization u of $U \sim \mathcal{U}_{(0,1)}$
if $u \le 1-p$ **then**
  $x = 0$
else
  $x = 1$

Result: Realization x of $X \sim \mathrm{Ber}_p$
Generate realization u of $U \sim \mathcal{U}_{(0,1)}$
if $u < p$ **then**
  $x = 1$
else
  $x = 0$

Binomial distribution Let $n \in \mathbb{N}$ and $p \in (0,1)$ be real numbers and assume that X is $\mathrm{Bin}_{n,p}$-distributed. The it holds $\forall k \in \{0,1,\dots,n\}$ that

$$p_k = \binom{n}{k}p^k(1-p)^{n-k}$$

and it holds $\forall k \in \{n+1, n+2, \dots\}$ that $p_k = 0$. Furthermore it holds that

$$p_{k+1} = \frac{p}{1-p}\frac{n-k}{k+1}p_k$$

Result: Realization x of $X \sim \mathrm{Bin}_{n,p}$
Generate realization u of $U \sim \mathcal{U}_{(0,1)}$
$k = 0$
$r = \frac{p}{1-p}$
$q = (1-p)^n$
$F = q$
while $u > F$ **do**
  $q = r \cdot q \cdot \frac{n-k}{k+1}$
  $F = F + q$
  $k = k + 1$
end
$x = k$

Geometric distribution Let $p \in (0,1)$ be a real number and assume that X is Geom_p-distributed. Then it holds $\forall n \in \mathbb{N}_0$ that

$$p_n = p(1-p)^n$$

This implies $\forall n \in \{-1,0,1,2,\dots\}$ that

$$F(n) = 1 - (1-p)^{n+1}$$

This shows $\forall u \in (0,1)$ that

$$I_F(u) = \left\lceil \frac{\log(1-u)}{\log(1-p)} \right\rceil - 1$$

Hence

$$X = \left\lfloor \frac{\log(U)}{\log(1-p)} \right\rfloor$$

Poisson distribution Let $\lambda \in (0,\infty)$ be a real number and assume that X is Poi_λ-distributed. Then it holds $\forall x \in \mathbb{N}_0$ that $p_n = \frac{\lambda^n}{n!e^\lambda}$. Hence, we obtain $\forall n \in \mathbb{N}_0$ that

$$p_{n+1} = \frac{\lambda^{n+1}}{(n+1)!e^\lambda} = \frac{\lambda}{n+1}p_n$$

Result: Realization x of $X \sim \mathrm{Poi}_\lambda$
Generate realization u of $U \sim \mathcal{U}_{(0,1)}$
$n = 0$
$p = e^{-\lambda}$
$F = p$
while $u > F$ **do**
  $p = \frac{p \cdot \lambda}{n+1}$
  $F = F + p$
  $n = n + 1$
end
$x = n$

7.2 Continuous distributions

Exponential distribution Let $\lambda \in (0,\infty)$ be a real number and assume that X is $\exp_\lambda$-distributed. Then it holds $\forall x \in \mathbb{R}$ and $\forall y \in (0,1)$, respectively, that

$$F(x) = \exp_\lambda((-\infty,x]) = \begin{cases} 0 & : x < 0 \\ 1 - e^{-\lambda x} & : x \ge 0 \end{cases}$$

$$I_F(y) = \frac{-\log(1-y)}{\lambda}$$

hence

$$X = \frac{-\log(U)}{\lambda}$$

Cauchy distribution Let $\mu \in \mathbb{R}$ and $\lambda \in (0, \infty)$ be real numbers and assume that X is $\mathrm{Cau}_{\mu,\lambda}$-distributed. Then it holds $\forall x \in \mathbb{R}$ and $\forall y \in (0, 1)$, respectively, that

$$F(x) = \mathrm{Cau}_{\mu,\lambda}((-\infty, x]) = \frac{\arctan\left(\frac{x-\mu}{\lambda}\right)}{\pi} + \frac{1}{2}$$
$$I_F(y) = \lambda \tan\left(\pi\left(y - \frac{1}{2}\right)\right) + \mu$$

Hence

$$X = \lambda \tan\left(\pi\left(U - \frac{1}{2}\right)\right) + \mu$$

Laplace distribution

8 MATLAB commands

rand	text
rand(1,4)	text