Summary: Continuous Time Quantitative Finance

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1 Basic Black-Scholes model

1.1 Dynamics

$\mathbb{P}$-dynamics

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$
$$S_t = S_0 \cdot \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$$

where

r	risk-free rate	σ	volatility
μ	trend	B_t	$\mathbb{P}$-BM

$\mathbb{Q}$-dynamics

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t$$
$$S_t = S_0 \cdot \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$$

with W_t a $\mathbb{Q}$-BM, $\theta = \frac{\mu - r}{\sigma}$ the risk-premium and $\exists$ an EMM $\mathbb{Q}$ s.t.

$$\mathbb{Q}|_{\mathcal{F}_t} = \exp\left(-\theta B_t - \frac{1}{2}\theta 2t\right) \cdot \mathbb{P}|_{\mathcal{F}_t}$$

1.2 BS formula

BS formula for a European option

$$C_t = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$
$$P_t = Ke^{-r(T-t)}\Phi(-d_1) - S_t\Phi(-d_2)$$
$$d_{1,2} = \frac{\log\frac{S_t}{K} + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}$$

with $d_2 = d_1 - \sigma\sqrt{T-t}$.

BS PDE

$$\partial_t C + rs\partial_s C + \frac{1}{2}\sigma^2 s^2 \partial^2_{ss} - rC = 0 \qquad t \in (0,T)$$
$$C(s, t=T) = g(s)$$

$\forall s > 0$ with s the stock price in linear space, T the maturity and $g(s)$ the payoff function.

Self-financing portfolio

$$dV_t = \alpha_t dC_t + \beta_t dS_t$$

Hedging ratio

$$\frac{\beta_t}{\alpha_t} = -\partial_s C(S_t, t)$$

Martingale approach

$$\begin{aligned} C(S_0, T) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}(S_T - K)\mathbb{I}_{S_T \geq K}\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}S_T\mathbb{I}_{S_T \geq K}\right] - e^{-rT}K\mathbb{Q}\left[S_T \geq K\right] \end{aligned}$$

1.3 Greeks

Delta

$$\Delta = \frac{\partial C}{\partial s}$$
$$\Delta^C_E = \Phi(d_1) > 0$$
$$\Delta^P_E = -\Phi(-d_1) = \Phi(d_1) - 1 < 0$$

Gamma

$$\Gamma = \frac{\partial^2 C}{\partial s^2} \qquad \Gamma^C_E = \Gamma^P_E = \frac{\phi(d_1)}{s\sigma\sqrt{T-t}} > 0$$

Rho

$$\rho = \frac{\partial C}{\partial r}$$
$$\rho^C_E = K(T-t)e^{-r(T-t)}\Phi(d_2) > 0$$
$$\rho^P_E = -K(T-t)e^{-r(T-t)}\Phi(-d_2) < 0$$

Theta

$$\Theta = \frac{\partial C}{\partial t}$$
$$\Theta^C_E = -\frac{s\varphi(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi(d_2) < 0$$
$$\Theta^P_E = -\frac{s\varphi(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}\Phi(-d_2) > 0$$

Vega

$$\mathcal{V} = \frac{\partial C}{\partial \sigma} \qquad \mathcal{V}^C_E = \mathcal{V}^P_E = s\varphi(d_1)\sqrt{T-t} > 0$$

1.4 Mathematical tools

Girsanov's theorem The stochastic exponential or Doléans exponential is defined as:

$$\mathcal{E}(X)_t = \exp\left(X_t - \frac{1}{2}[X]_t\right)$$

Define the change of measure as:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}(X)_t$$

Then:

- it holds for the expectation of the RV ξ:

  $$\mathbb{E}_{\mathbb{Q}}[\xi] = \mathbb{E}_{\mathbb{P}}\left[\xi \cdot \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}\right]$$

- if W_t is a $\mathbb{P}$-BM, then a $\mathbb{Q}$-BM is defined as:

  $$W^{\mathbb{Q}}_t = W^{\mathbb{P}}_t - \left[X, W^{\mathbb{P}}\right]_t$$

  (e.g. if $X_t = \lambda W^{\mathbb{P}}_t$, then $W^{\mathbb{Q}}_t = W^{\mathbb{P}}_t - \lambda t$)

Laplace transform

- In general, the Laplace transform is defined as:

  $$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty e^{-st}f(t)dt$$

- In probability theory: X is a a RV with PDF f. Then:

  $$\mathcal{L}\{f\} = \mathbb{E}\left[e^{-sX}\right]$$

  Remark: Replace s with $-t$ IOT obtain the MGF (moment-generating function) of X, i.e. $\mathbb{E}\left[e^{tX}\right]$.

Leibnitz's rule for differentiation under the integral

$$\frac{d}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)dx = \frac{db(\alpha)}{d\alpha}f(b(\alpha),\alpha) - \frac{da(\alpha)}{d\alpha}f(a(\alpha),\alpha) + \int_{a(\alpha)}^{b(\alpha)}\frac{\partial f(x,\alpha)}{\partial\alpha}dx$$

2 American Currency Options

Garman-Kohlhagen model Under the risk-neutral probability $\mathbb{Q}$:

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW_t$$

with:

r	domestic risk-free rate	σ	currency volatility
δ	foreign risk-free rate	W_t	$\mathbb{Q}$-BM

American options

$$C_A(S_0, T) = \sup_{\tau \in \mathcal{T}(T)} \mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau}(S_\tau - K)^+\right]$$
$$P_A(S_0, T) = \sup_{\tau \in \mathcal{T}(T)} \mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau}(K - S_\tau)^+\right]$$

PDE (continuation region) In the continuation region, an American option satisfies the same PDE as a European option, i.e.

$$\frac{1}{2}\sigma^2 x^2 \partial^2_{xx} C_A(x,u) + (r-\delta)x\partial_x C_A(x,u) - rC_A(x,u) - \partial_u C_A(x,u) = 0$$

Impact of parameters on exercise boundaries In case of an American currency call:

- if the volatility σ increases: then the exercise boundary increases (i.e. we wait longer)
- if the domestic interest rate r increases: then the exercise boundary increases (i.e. we wait longer)
- if the foreign interest rate δ increases: then the exercise boundary decreases (i.e. we wait less)

2.1 Decompositions

American currency call price

$$\begin{aligned}C_A(S_t, T-t) &= C_E(S_t, T-t)\\ &+ \delta S_t \int_t^T e^{-\delta(s-t)}\Phi(d_1(S_t, b_c(T-s), s-t))ds\\ &- rK \int_t^T e^{-r(s-t)}\Phi(d_2(S_t, b_c(T-s), s-t))ds\end{aligned}$$

with

$$d_1(x,y,u) = \frac{\log\frac{x}{y} + \left(r-\delta+\frac{1}{2}\sigma^2\right)u}{\sigma\sqrt{u}}$$
$$d_2(x,y,u) = d_1(x,y,u) - \sigma\sqrt{u}$$

American currency put price

$$\begin{aligned}P_A(S_t, T-t) &= P_E(S_t, T-t)\\ &+ rK \int_t^T e^{-r(s-t)}\Phi(-d_2(S_t, b_p(T-s), s-t))ds\\ &- \delta S_t \int_t^T e^{-\delta(s-t)}\Phi(-d_1(S_t, b_p(T-s), s-t))ds\end{aligned}$$

with d_1, d_2 as defined above.

2.2 Perpetual American currency options

Now: $C_A(x) = C_A(x, +\infty)$ and $P_A(x) = P_A(x, +\infty)$

PDE approach

$$\frac{1}{2}\sigma^2 x^2 C''_A(x) + (r-\delta)xC'_A(x) - rC_A(x) = 0$$

with the following boundary conditions (continuity and smooth-pasting):

$$C_A(L^*) = L^* - K \qquad C'_A(L^*) = 1$$

Martingale approach

$$C_A(S_t) = \sup_\tau \mathbb{E}_\mathbb{Q}\left[e^{-r(\tau-t)}(S_\tau - K)\middle|\mathcal{F}_t\right]$$

By continuity of BM:

$$C_A(S_0) = \sup_L (L-K)\mathbb{E}_\mathbb{Q}\left[e^{-rT_L}\right]$$

Compute derivative $\partial_L(L-K)\mathbb{E}_\mathbb{Q}\left[e^{-rT_L}\right] = 0$ IOT get L^*.

Perpetual American call & put

- Continuation region ($x < L^*$)

  $$C_A(x) = (L^* - K)\left(\frac{x}{L_1^*}\right)^{\gamma_1}$$
  $$P_A(x) = (K - L^*)\left(\frac{x}{L_2^*}\right)^{\gamma_2}$$

  with

  $$L^*_{1/2} = \frac{\gamma_{1,2}}{\gamma_{1,2}-1}K \quad \geq K$$
  $$\gamma_{1,2} = \frac{-\nu \pm \sqrt{\nu^2 + 2r}}{\sigma}$$
  $$\nu = \frac{1}{\sigma}\left(r - \delta - \frac{1}{2}\sigma^2\right)$$

  i.e. $\gamma_{1,2}$ are the positive and negative root of:

  $$\frac{1}{2}\sigma^2\gamma^2 + \left(r-\delta-\frac{1}{2}\sigma^2\right)\gamma - r = 0$$

- Stopping region ($x \geq L^*$)

  $$C_A(x) = x - K$$
  $$P_A(x) = K - x$$

  i.e. the option price simply corresponds to its intrinsic value.

Put-Call symmetry

$$P_A(S_0, K, r, \delta) = C_A(K, S_0, \delta, r)$$

which comes from the fact that the right to sell a foreign currency corresponds to the right to buy the domestic one.

Perpetual exercise boundaries

$$b_c(K, r, \delta, T-t) \cdot b_p(K, \delta, r, T-t) = K^2$$

Laplace transforms of hitting times

- **Standard $\mathbb{Q}$-BM W_t**
  If T_y is the first hitting time of $y \in \mathbb{R}$ for a standard $\mathbb{Q}$-BM, i.e.

  $$T_y = \inf\{t \geq 0 : W_t = y\}$$

  then the Laplace transform of T_y is given as:

  $$\mathbb{E}_\mathbb{Q}\left[e^{-\frac{1}{2}\lambda^2 T_y}\right] = e^{-\lambda|y|}$$

- **Drifted BM $W_t + \nu t$**
  Let T_y be the first hitting time of $y \in \mathbb{R}$ for a drifted BM $W_t + \nu t$, i.e.

  $$T_y = \inf\{t \geq 0 : W_t + \nu t = y\}$$

  Then the corresponding measure $\mathbb{Q}^*$ is given according to Girsanov's theorem by:

  $$\left.\frac{\mathbb{Q}^*}{\mathbb{Q}}\right|_{\mathcal{F}_t} = e^{-\frac{1}{2}\nu^2 t - \nu W_t}, \qquad W_t^* = W_t + \nu t$$

  and the Laplace transform of T_y is given as:

  $$\mathbb{E}_\mathbb{Q}\left[e^{-\frac{1}{2}\lambda^2 T_y}\right] = e^{\nu y}e^{-|y|\sqrt{\nu^2+\lambda^2}}$$

Reflection principle

- If W_t a $\mathbb{Q}$-BM, τ_m the first passage time of W_t at the level m and if another level $\omega < m$ is considered, then:

  $$\mathbb{Q}\left[\tau_m \leq t, W_t \leq \omega\right] = \mathbb{Q}\left[W_t \geq 2m - \omega\right]$$

- If M_t is the running maximum of W_t, then:

  $$\mathbb{Q}\left[M_t \geq m, W_t \leq \omega\right] = \mathbb{Q}\left[W_t \geq 2m - \omega\right]$$

3 Stochastic Volatility

Time change

$$\Sigma_T = \int_0^T \sigma_u^2 du$$
$$\int_0^t \sigma_u dB_u = B^*_{\Sigma_t} = B^*_{\int_0^t \sigma_u^2 du}$$

where B_t is the original $\mathbb{Q}$-BM and B_t^* is the time-changed $\mathbb{Q}$-BM.
For B_t^*, it holds that:

$$B^*_{\Sigma_t} \sim \mathcal{N}(0, \Sigma_t), \qquad [B^*]_t = \int_0^t \sigma_u^2 du$$

General $\mathbb{P}$-dynamics for a currency

$$\frac{dS_t}{S_t} = \mu dt + \sigma_t d\tilde{B}_t$$
$$\frac{d\sigma_t}{\sigma_t} = f(\sigma_t)dt + \gamma d\tilde{W}_t$$

with $\tilde{W}_t, \tilde{B}_t$ two BM with correlation coefficient ρ.

General $\mathbb{Q}$-dynamics for a currency

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sigma_t dB_t$$
$$\frac{d\sigma_t}{\sigma_t} = \left(f(\sigma_t) - \Phi_t^\sigma\right)dt + \gamma dW_t$$

with W_t, B_t two BM with correlation coefficient ρ and Φ_t^σ the risk premium associated with volatility.

Under $\mathbb{Q}$, the underlying price S_T is then given by:

$$\begin{aligned}S_t &= S_0 e^{(r-\delta)t - \frac{1}{2}\int_0^t \sigma_u^2 du + \int_0^t dB_u}\\ &= S_0 e^{(r-\delta)t - \frac{1}{2}\Sigma_t^2 + B^*_{\Sigma_t}}\end{aligned}$$

3.1 Hull & White model

$\mathbb{Q}$-dynamics of the Hull & White model Volatility σ is assumed to follow a GBM.

$$\frac{dS_t}{S_t} = rdt + \sigma_t dW_t^{(1)} \qquad \frac{d\sigma_t}{\sigma_t} = kdt + \gamma dW_t$$

where $W_t^{(1)}, W_t$ two independent BM.

Assumptions:

$\delta = 0$ stock option on domestic market

$\rho = 0$ volatility follows a GBM and is uncorrelated with the stock price

$\Phi_\sigma = 0$ volatility has zero systematic risk

Thus, the drift of volatility is assumed to be constant, i.e. $f(\sigma) = k$.

Additional random variable: $V_T = \Sigma_T / T$.

Hull & White PDE Apply Itô's lemma to $C_E = f(S_t, \sigma_t)$ under $\mathbb{Q}$ and use the martingale property $\mathbb{E}_\mathbb{Q}[C_E] = rC_E$ IOT obtain:

$$\frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 C_E + \frac{1}{2}\gamma^2\sigma^2\partial_{\sigma\sigma}^2 C_E + rx\partial_x C_E + \sigma k \partial_\sigma C_E + \partial_t C_E - rC_E = 0$$

3.2 Scott model

$\mathbb{Q}$-dynamics of the Scott model The Scott model assumes that the logarithm of the volatility follows a Vasicek process, i.e. $f(\sigma) = \beta(a - \log\sigma) + \frac{1}{2}\gamma^2$. Then:

$$\frac{d\sigma_t}{\sigma_t} = \left(\beta(a - \log\sigma_t) + \frac{1}{2}\gamma^2\right)dt + \gamma d\tilde{W}_t$$
$$d\log\sigma_t = \beta(a - \log\sigma_t)dt + \gamma d\tilde{W}_t$$

Scott PDE There are two approaches on how to derive the Scott PDE:

(i) apply Itô's lemma to $C_E = f(S_t, \sigma_t)$ under $\mathbb{Q}$
use the martingale property $\mathbb{E}_\mathbb{Q}[C_E] = rC_E$
(as in the Hull & White model)

(ii) apply Itô's lemma to C_E under $\mathbb{P}$
use a continuous time version of the two factor APT (Arbitrage Pricing Theory model) with $\mathbb{E}[dC_E/C_E]$ and $\Phi_t^S = \mu + \delta - r$
equate $\mathbb{E}[\cdot]$

Then:

$$\frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 C_E + \frac{1}{2}\gamma^2\sigma^2\partial_{\sigma\sigma}^2 C_E + (r-\delta)x\partial_x C_E + \sigma(f(\sigma) - \Phi_t^\sigma)\partial_\sigma C_E + \partial_t C_E - rC_E = 0$$

4 Jump Models

4.1 Poisson processes

Standard Poisson process A *counting process*:

- *without explosion* (i.e. $T = \infty$)
- with *independent increments*
  i.e. for every $s, t \geq 0$ the RV $N_{t+s} - N_t$ is independent of $\mathcal{F}_\sqcup^N$
- with *stationary increments*
  i.e. for every $s, t \geq 0$ the RV $N_{t+s} - N_t$ has the same law as N_s

Important properties:

$$\mathbb{P}[N_t = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$
$$\mathbb{E}[N_t] = \lambda t \qquad \mathrm{Var}[N_t] = \lambda t$$
$$\lambda : \text{intensity of jumps}$$

Characteristic function:

$$\mathbb{E}\left[e^{iuN_t}\right] = e^{\lambda t(e^{iu}-1)}$$

Useful properties:

$$\mathbb{E}\left[e^{\alpha N_t}\right] = e^{\lambda t(e^\alpha - 1)} \qquad \mathbb{E}\left[x^{N_t}\right] = e^{\lambda t(x-1)}$$

Compensated Poisson process M_t The following expressions are $\mathbb{F}$-martingales:

$$M_t := N_t - \lambda t$$
$$M_t^2 - \lambda t = (N_t - \lambda t)^2 - \lambda t$$

Itô's formula

$$\begin{aligned} dX_t &= h_t dt + f_t dW_t + g_t dM_t \\ dF(t, X_t) &= \partial_t F(t, X_t)dt \\ &\quad + \partial_x F(t, X_{t-})(dX_t - g_t dN_t) \\ &\quad + \frac{1}{2}\partial_{xx}^2 F(t, X_t) d[X]_t \\ &\quad + (F(t, X_t) - F(t, X_{t-}))\, dN_t \\ F(t, X_t) &= F(0, X_0) + \int_0^t \partial_t F(s, X_s)ds \\ &\quad + \int_0^t \partial_x F(s, X_{s-})dX_s \\ &\quad + \frac{1}{2}\int_0^t \partial_{xx}^2 F(s, X_s) f_s^2 ds \\ &\quad + \int_0^t (F(s, X_s) - F(s, X_{s-}) \\ &\qquad - \partial_x F(s, X_{s-} g_s) dN_s \end{aligned}$$

Assumption: F is a $C^{1,2}$ function on $\mathbb{R}^+ \times \mathbb{R}$, i.e. $\partial_t F, \partial_x F, \partial_{xx}^2 F$ exist and are continuous.

Note that:

- In case of a jump at t: $X_t = X_{t-} + g_t$
  e.g. if $g_t = X_{t-}\phi$, then $X_t = (1+\phi)X_{t-}$
- $d[X]_t = f_t^2 dt$
- If $h_t = f_t = 0$ and $g_t = \phi$,
  then $S_t = S_0(1+\phi)^{N_t}$.

Example:

$$\frac{dS_t}{S_{t-}} = bdt + \sigma dW_t + \phi dM_t$$
$$d\log S_t = \left(b - \frac{1}{2}\sigma^2 - \lambda\phi\right)dt + \sigma dW_t + \log(1+\phi)dN_t$$
$$S_t = S_0 \underbrace{e^{bt}}_{\text{drift}} \underbrace{e^{\sigma W_t - \frac{1}{2}\sigma^2}}_{\text{martingale}} \underbrace{e^{\log(1+\phi)N_t - \lambda\phi t}}_{\text{martingale}}$$

Girsanov's theorem for Poisson processes Let L_t be the positive exponential martingale solution of:

$$dL_t = L_{t-}\phi dM_t, \qquad L_t = e^{\log(1+\phi)N_t - \lambda\phi t}$$

Let $\mathbb{Q}$ be the probability measure defined by:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t$$

Then it holds:

- under $\mathbb{P}$:
  - M_t is a $\mathbb{P}$-martingale
  - $(N_t)_{t\geq 0}$ is a $\mathbb{P}$-Poisson process of intensity λ
- under $\mathbb{Q}$:
  - $M_t^\phi = M_t - \phi\lambda t = N_t - (1+\phi)\lambda t$ is a $\mathbb{Q}$-martingale
  - $(N_t)_{t\geq 0}$ is a $\mathbb{Q}$-Poisson process of intensity $(1+\phi)\lambda$
- for the RV ξ that:

$$\mathbb{E}_\mathbb{Q}[\xi] = \mathbb{E}_\mathbb{P}\left[\xi \cdot \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}\right]$$

(i.e. as in the "normal" Girsanov theorem)

4.2 Lévy processes

Lévy process An $\mathbb{R}^d$-valued process X:

- s.t. $X_0 = 0$
- with *independent increments*
  i.e. for every $s, t \geq 0$ the RV $X_{t+s} - X_t$ is independent of $\mathcal{F}_t^X$
- with *stationary increments*
  i.e. for every $s, t \geq 0$ the RVs $X_{t+s} - X_t$ and X_s have the same law
- that is *continuous in probability*
  i.e. for fixed t, $\mathbb{P}[|X_t - X_u| > \epsilon] \to 0$ when $u \to t$ for every $\epsilon > 0$.

Lévy exponent If $\mathbb{E}[\exp(kX_1)] < \infty$ for any k, then the Lévy exponent ψ on $[0, \infty)$ of the Lévy process X is defined as:

$$\mathbb{E}[\exp(kX_1)] = \exp(\psi(k))$$

Note that the process $\left(e^{kX_t - t\psi(k)}\right)_{t\geq 0}$ is a martingale for any k s.t. $\psi(k) = \log\mathbb{E}\left[e^{kX_1}\right] < \infty$

Put-Call symmetries

- dynamics under the domestic risk-neutral probability $\mathbb{Q}$

$$\frac{dS_t}{S_{t-}} = (r-\delta)dt + \sigma dW_t + \phi dM_t$$
$$S_t = S_0 e^{(r-\delta)t} e^{\sigma W_t - \frac{1}{2}\sigma^2} e^{\log(1+\phi)N_t - \lambda\phi t}$$

- European put-call symmetry

$$P_E(x,K,r,\delta,\sigma,\phi,\lambda) = C_E\left(K,x,\delta,r,\sigma,\frac{-\phi}{1+\phi},\lambda(1+\phi)\right)$$

- American put-call symmetry

$$P_A(x,K,r,\delta,\sigma,\phi,\lambda) = Kx\cdot C_A\left(\frac{1}{x},\frac{1}{K},\delta,r,\sigma,\frac{-\phi}{1+\phi},\lambda(1+\phi)\right)$$

- Symmetry for exercise boundaries

$$b_p(r,\delta,\phi,\lambda)\cdot b_c\left(\delta,r,\frac{-\phi}{1+\phi},\lambda(1+\phi)\right) = K^2$$

Hitting times

- dynamics of the underlying

$$S_t = S_0 e^{(b-\phi\lambda-\frac{1}{2}\sigma^2)t+\sigma W_t+\log(1+\phi)N_t} = S_0 e^{X_t}$$

- passage times

$$T_L(S) = \inf\{t\geq 0 : S_t \geq L\}$$
$$T_{\mathcal{L}}(X) = T_L(S) = \inf\{t\geq 0 : X_t \geq \mathcal{L}\}$$
$$\text{with } \mathcal{L} = \log\frac{L}{S_0}$$

- Laplace transform

$$\mathbb{E}\left[e^{-uT_{\mathcal{L}}}\right] = \begin{cases} e^{-\psi^{-1}(u)\mathcal{L}} & : \mathcal{L}>0 \\ 1 & : \text{otherwise}\end{cases}$$

with $e^{-\psi^{-1}(u)\mathcal{L}}$ the positive root of $\psi(k)=u$.

- Overshoot: if the jump size is strictly positive (i.e. $\phi>0$), there is a non-zero probability that $X_{T_{\mathcal{L}}}$ is strictly greater than $\mathcal{L}$, i.e.

$$\mathbb{P}[X_{T_{\mathcal{L}}} > \mathcal{L}] > 0$$

Then the overshoot is defined as

$$O_{\mathcal{L}} = X_{T_{\mathcal{L}}} - \mathcal{L}$$

If the the jump size is strictly negative (i.e. $-1<\phi<0$), then $O_{\mathcal{L}}=0$, i.e. X_t is continuous at the boundary.

Set of risk-neutral probability measures

- dynamics of the underlying

$$\frac{dS_t}{S_{t-}} = bdt + \sigma dW_t + \phi dM_t$$
$$S_t = S_0 e^{(b-\phi\lambda-\frac{1}{2}\sigma^2)t+\sigma W_t+\log(1+\phi)N_t}$$
$$R(t) = e^{-rt}$$

- martingale condition:
$d(RS)_t$ is a $\mathbb{P}^{\psi,\gamma}$-martingale, i.e.

$$\frac{d(RS)_t}{R_t S_t^-} = \sigma d\hat{W}_t + \phi d\hat{M}_t$$

- **set $\mathcal{Q}$ of EMMs** defined by:

$$\left.\frac{\mathbb{P}^{\psi,\gamma}}{\mathbb{P}}\right|_{\mathcal{F}_t} = L_t^{\psi,\gamma} = L_t^{\psi}(W)L_t^{\gamma}(M)$$

with

$$L_t^{\psi}(W) = e^{\psi W_t - \frac{1}{2}\psi^2 t} = \mathcal{E}(\psi W)_t$$
$$L_t^{\gamma}(M) = e^{\log(1+\gamma)N_t-\lambda\gamma t}$$

and the constraint

$$b - r + \sigma\psi + \lambda\psi\gamma = 0$$

- $\mathbb{P}^{\psi,\gamma}$-martingales:

$$\hat{W}_t = W_t - \psi t$$
$$\hat{M}_t = M_t - \lambda\gamma t = N_t - \lambda(1+\gamma)t$$

5 Real Options

5.1 Optimal entry

Setting

- a firm can invest at any time t the investment sum K_t to install a project which generates the sum of expected discounted future net cash-flows V_t
- the investment is irreversible
- both K_t and V_t are stochastic
- the maturity is infinite

Dynamics

- Historical probability $\mathbb{P}$:

$$\frac{dV_t}{V_t} = \alpha_1 dt + \sigma_1 dW_t$$
$$\frac{dK_t}{K_t} = \alpha_2 dt + \sigma_2 dB_t$$

where the two $\mathbb{P}$-BM W_t, B_t are correlated with ρ.

- Risk-neutral probability $\mathbb{Q}$:

$$\frac{d(V_t/K_t)}{V_t/K_t} = (\alpha_1-\alpha_2)dt + \Sigma dZ_t$$
$$\text{with } \Sigma = \sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}$$

where Z_t a $\mathbb{Q}$-BM.

Real option

- Supremum

$$C_{\mathrm{RO}} = \sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathbb{P}}\left[e^{-r\tau}(V_\tau - K_\tau)\right] = \sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathbb{P}}\left[e^{-r\tau}K_\tau\left(\frac{V_\tau}{K_\tau}-1\right)\right] = \sup_{\tau\in\mathcal{T}} K_0\mathbb{E}_{\mathbb{Q}}\left[e^{-(r-\alpha_2)\tau}(V_\tau - K_\tau)\right]$$

- Since this problem is in principle a perpetual American call option:

$$C_{\mathrm{RO}}(V_0,K_0) = K_0(L^*-1)\left(\frac{V_0/K_0}{L^*}\right)^{\epsilon}$$
$$L^* = \frac{\epsilon}{\epsilon-1}$$

with

$$\epsilon = \sqrt{\left(\frac{\alpha_1-\alpha_2}{\Sigma^2}-\frac{1}{2}\right)^2 + \frac{2(r-\alpha_2)}{\Sigma^2}} - \left(\frac{\alpha_1-\alpha_2}{\Sigma^2}-\frac{1}{2}\right)$$

Interpretation In the neoclassical framework, it is optimal to invest if expected discounted earnings are higher than expected discounted costs, i.e. if $X_t>1$.
This framework, however, takes also the risk appropriately into account and thus the optimal time to invest is at T_{L^*} with $L^*>1$. In other words, this frameworks generally suggests a certain delay compared to the neoclassical framework IOT account for the risk taken.

5.2 Optimal entry and optimal exit

Setting

- no competition
- decision to *invest* and irreversible decision to *disinvest*
- decision to *invest* (with entry cost K_i) and irreversible decision to *disinvest* (with exit cost K_d)
- corresponds to embedded perpetual American options, i.e. first an American call (investment) and an American put (disinvestment)

Parameters

K_i	investment cost
K_d	disinvestment cost
K	sum of all future discounted costs
$\alpha<r$	drift strictly smaller than risk-free rate

Supremum

$$VF(S_0) = C_A(S_0) + P_A(S_0) = \sup_{L_i,L_d} \phi(L_i)\mathbb{E}\left[e^{-rT_{L_i}}\right] + \psi(L_d)\mathbb{E}\left[e^{-rT_{L_d}}\right]$$

with

$$\phi(L_i) = \frac{L_i}{r-\alpha} - K - K_i$$
$$\psi(L_d) = K - \frac{L_d}{r-\alpha} - K_d$$

Laplace transforms

$$\mathbb{E}\left[e^{-rT_i}\right] = \left(\frac{S_0}{L_i}\right)^{\gamma_1}$$
$$\mathbb{E}\left[e^{-r(T_d-T_i)}\right] = \left(\frac{L_i}{L_d}\right)^{\gamma_2}$$

with:

$$\gamma_{1,2} = \frac{-\theta\pm\sqrt{2r+\theta^2}}{\sigma}, \quad \theta = \frac{\alpha-\frac{1}{2}\sigma^2}{\sigma}$$

Interpretation The possibility to disinvest gives the firm incentives to invest earlier than in the irreversible investment case.

6 Systemic Risk

Introduction

- notional value of all derivatives (globally) corresponds to app. 12 times global GDP
- if derivatives were only used for hedging, notional value would amount to app. 30-40 % of global GDP
- conflict of interest for rating agencies: companies pay for their ratings ...

CDS (Credit Default Swaps)

- only half of the CDS in the USA actually covered risks, the other half served only speculative purposes
- thus, banks can actually benefit more from financial distress of their client companies
- **ISDA** (International Swaps & Derivatives Association): makes finally the decision whether a credit event takes place or not (which is relevant for CDS), but: members of the voting board of ISDA are the major banks who also purchased the CDS

Food speculation

- **Impact of food speculation on prices:**
  - Paradox: usually, the futures prices converge to the spot price at maturity
  - however: on the food market it is vice-versa, i.e. the spot prices converge to the futures prices
- demand for commodities was (is) dominated by speculators (between 65–80 %)

High-Frequency Trading (HFT)

- the economy does not work in milliseconds, but takes days, weeks, years to adapt ...
- **Front trading:**
  - buying a stock just before an investor wants to buy it IOT sell the stock to the same investor for a slightly higher price
  - this is a kind of a tax on investors who do not have access to HFT
  - of course this is illegal (kind of insider trading)
- counter-regulation: e.g. micro-tax on electronic payments

Fiscal arbitrage

- e.g. Goldman Sachs created a structured product (basket) IOT allow Greece to convert debt in USD into EUR at an arbitrary higher exchange rate
  result: high commission for Goldman Sach and a mean for the Greek government to hide debt

Insider trading

- criteria:
  (i) surprisingly high volume (and open interest)
  (ii) impressive profits within a short period (e.g. 200–500 %)
  (iii) transactions without hedging
- insider trading for stocks: can be justified since private information is revealed
- insider trading for options: cannot be justified since no information is actually revealed (e.g. open interest is not considered in the BS formula)

7 Environmental Finance

Environmental Finance

- part of environmental economics, exploits financial instruments IOT deal with ecological issues
  e.g. land use planning, natural resource preservation, urban growth issues
- focuses on financial and quantitative issues, proposed by environmental economists
- quantitative analysis of the impact of market-based environmental policies
  e.g. European Emission Trading Scheme

Global warming

- **perspectives on how to tackle global warming:**
  - economics: introduce taxes
  - finance: introduce a market
- **ecological footprint:** measures how much land and water area a human population requires to produce the resources it consumes and to absorb its wastes, using prevailing technology
  e.g. in 2012: humanity used 1.8 planets (i.e. 2 yrs of regeneration)
- **CO_2 concentration in the atmosphere:**
  - since industrialisation, CO_2 concentration has been continuously increasing
  - today's concentration is 400ppm, while the safe upper limit might rather be 350 ppm
- **CO_2 emissions**
  - if all CO_2 emissions were globally stopped, it would take 1,000 yrs to decrease global temperature by 1 K
  - biggest carbon emissioning states: USA and China
  - average CO_2 emissions per capita, e.g. CHE 4.6 t (in 2011)
  - goal: 1 t p.c. and p.a.
- **temperature increase:**
  - current projection if no actions are taken: +4.5 K (catastrophic outcome ...)
  - reconstruction of temeprature curves: data from ice layers, different models provide different reconstructions of temperature curves
- **consequences of global warming:**
- more extreme events (e.g. insurance industry)
- adaption costs (e.g. in developping countries IOT mitigate effects)

Kyoto protocol

- Greenhouse gas emissions (GHG) of developed countries to be reduced by 5.2 % from 1990 to 2012 (Europe: 8 %)
  However, in 2011, global emissions were 42 % higher than in 1990
- **market-based mechanisms:** emission permits
- **project-based mechanisms:** certificates of emission reduction

EU ETS

- started in 2005 and covers 5 main sectors: pulp & paper, steel, cement, energy production, ore & mining
- each relevant company received a pre-specified amount of permits (emission allowances)
- **price development:**
  - start with a high price, decline after 2005, crashed by 2010 (partly also due to the financial crisis)
  - Brussels had no clue about CO_2-emissions, thus companies were polled, but they initially reported way too much
  - Stiglitz: CO_2 price per t will increase to 100 EUR at some point
    (40 EUR per t is required as an incentive for sustainable development)
- **issues:**
  - allocation criteria: grandfathering vs. auctioning
  - windfall profits, e.g. energy sector can pass opportunity costs of emission permits on the end-consumer
  - duration of the scheme: the longer, the better?
  - relevant sectors to include, e.g. what about aviation, transportation, households?
- **emission allowance:**
  - a limited, transferable right to emit one ton of an offending gas or carbon dioxide equivalent (CO_2e)
  - goal of the provisions: cost-efficiency
- **marginal cost theory:**
  - price of a permit equal to the marginal cost of abatement, i.e.

    $$S_t = MC_t$$

    i.e. if $S_t > MC_t$, sell permit & adapt clean technology
    i.e. if $S_t < MC_t$, buy permit & adapt clean technology
  - companies should reduce their expected discounted costs:

    $$\min_{X_0} \left\{ P_0 X_0 + (1+\eta)^{-1} \mathbb{E}\left[g(Q) \cdot (P_1 + P_2)\right] \right\}$$

- but: marginal cost theory inconsistent with reality
- "special" stochastic process S_t: on $[0, T)$ a BM, but at T either 0 or P ...
- unknowns: X_1 (emissions of company 1), X_2 (emissions of company 2), S_T (price of CO_2-certificates at T)

intentionally left blank

Notation

φ	PDF of the standard normal distribution
Φ	CDF of the standard normal distribution

Abbreviations

BM	Brownian motion
CDF	cumulative distribution function
e.g.	exempli gratia
EMM	equivalent martingale measure
i.e.	id est
IOT	in order to
p.a.	per annum
p.c.	per capita
PDE	partial differential equation
PDF	probability density function
RV	random variable
s.t.	such that

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