Applied Topology: From Gromov Hyperbolicity to Algebraic Gradient Flows on Geometric Complexes

Fabian Roll (TUM)

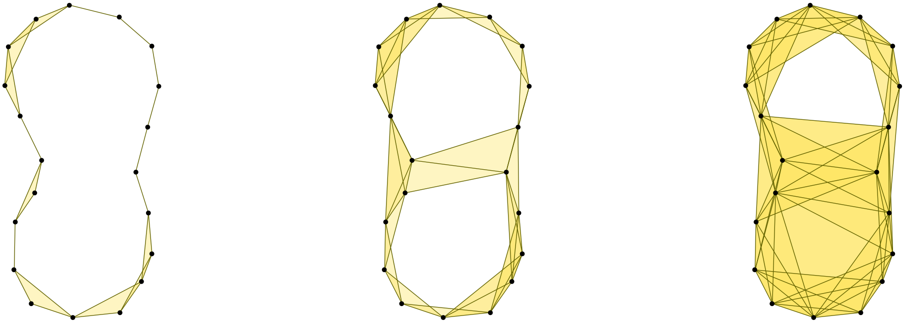
DGD Days 2023, Raitenhaslach

Project C04: Persistence and Stability of Geometric Complexes

Vietoris–Rips complexes

Definition. Let X be a metric space. The Vietoris-Rips complex at scale r is the simplicial complex

$$\mathrm{Rips}_r(X) = \{S \subseteq X \text{ finite} \mid S \neq \emptyset,\ \operatorname{diam} S \leq r\}.$$



Vietoris–Rips complexes

Applications

- For $r \to 0$: Used by Leopold Vietoris (1927) to extend homology theory to metric spaces.

Vietoris–Rips complexes

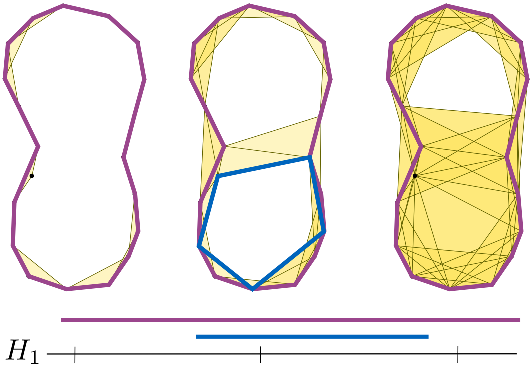
Applications

- For $r \to 0$: Used by Leopold Vietoris (1927) to extend homology theory to metric spaces.
- For $r \to \infty$: Used by Eliyahu Rips and Mikhael Gromov (1987) to study hyperbolic groups.

Vietoris–Rips complexes

Applications

- For $r \to 0$: Used by Leopold Vietoris (1927) to extend homology theory to metric spaces.
- For $r \to \infty$: Used by Eliyahu Rips and Mikhael Gromov (1987) to study hyperbolic groups.
- For all $r > 0$: Used in topological data analysis (nowadays) in the context of persistent homology:



Application of Vietoris–Rips persistent homology

COVID-19 genetic evolution data

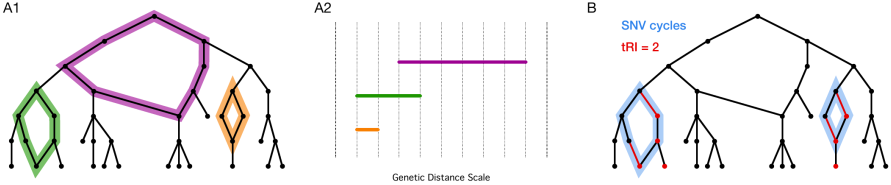


Figure 2. Topological data analysis quantifies convergent evolution. (A) Persistent homology detects reticulate events in viral evolution by means of a persistence barcode. Each bar in the barcode (A2) corresponds to a topological cycle in the reticulate phylogeny (A1). Bars at small genetic distance scales are expected to correspond mainly to homoplasies, while recombination events typically produce topological features at larger scales. (B) SNV cycles are topological cycles in the reticulate phylogeny for which adjacent sequences differ by single nucleotide variations (SNV) only. Under the assumption of single substitutions per site, any SNV in

M. Bleher, L. Hahn, M. Neumann, J. A. Patino-Galindo, M. Carriere, U. Bauer, R. Rabadan, and A. Ott
Topological data analysis identifies emerging adaptive mutations in SARS-CoV-2
Preprint, arXiv:2106.07292, 2023

Application of Vietoris–Rips persistent homology

COVID-19 genetic evolution data

covid data (≈ 15000 points)	Ripser's runtime
ordered chronologically	1 day

U. Bauer
Ripser: efficient computation of Vietoris–Rips persistence barcodes
Journal of Applied and Computational Topology, doi:10.1007/s41468-021-00071-5, 2021

Application of Vietoris–Rips persistent homology

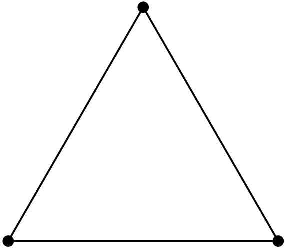
COVID-19 genetic evolution data

covid data (≈ 15000 points)	Ripser's runtime
ordered chronologically	1 day
ordered reversed chronologically	2 min

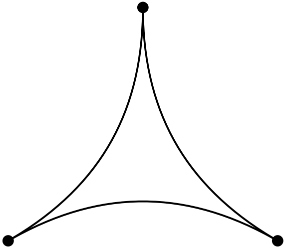
U. Bauer
Ripser: efficient computation of Vietoris–Rips persistence barcodes
Journal of Applied and Computational Topology, doi:10.1007/s41468-021-00071-5, 2021

Rips contractibility lemma

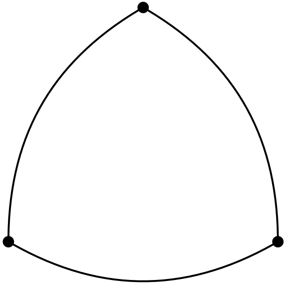
Gromov-hyperbolicity



euclidean triangle



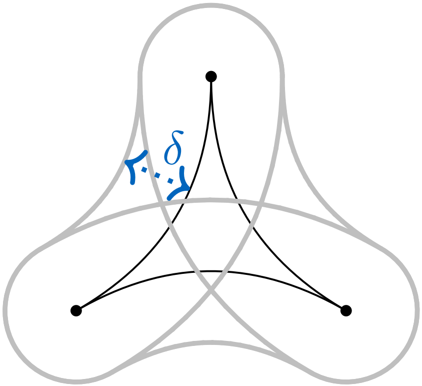
hyperbolic triangle



spherical triangle

Rips contractibility lemma

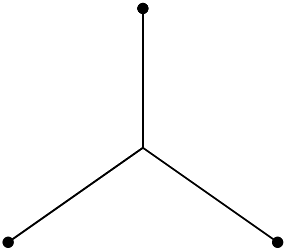
Gromov-hyperbolicity



hyperbolic triangle

Rips contractibility lemma

Gromov-hyperbolicity



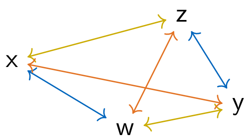
hyperbolic triangle

Rips contractibility lemma

Gromov-hyperbolicity

Definition (four-point condition). A metric space X is (Gromov) δ-*hyperbolic* if for all four points $w, x, y, z \in X$

$$d(x,w) + d(y,z) \leq \max\{d(x,y) + d(z,w), d(x,z) + d(y,w)\} + 2\delta$$



Example. finite metric spaces, trees are 0-hyperbolic, hyperbolic plane, ...

Rips contractibility lemma

Theorem (Rips, Gromov 1987). Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

Rips contractibility lemma

Theorem (Rips, Gromov 1987). Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

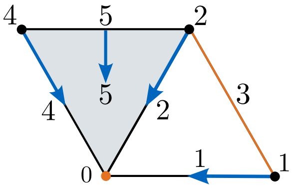
We address two questions:

1. What about non-geodesic spaces? Finite metric spaces?
2. Connections to Ripser?

Discrete Morse theory

A *discrete Morse function* is a

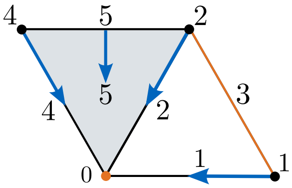
- monotone function $f\colon K \to \mathbb{R}$ that
- partitions the complex into pairs and critical simplices, yielding the *discrete gradient V*



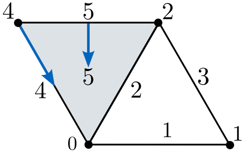
Discrete Morse theory

A *discrete Morse function* is a

- monotone function $f\colon K \to \mathbb{R}$ that
- partitions the complex into pairs and critical simplices, yielding the *discrete gradient V*



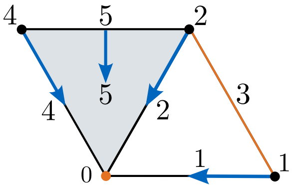
Discrete Morse functions - and their gradients - encode collapses:



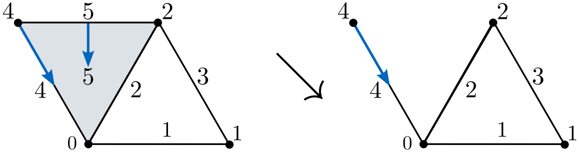
Discrete Morse theory

A *discrete Morse function* is a

- monotone function $f\colon K \to \mathbb{R}$ that
- partitions the complex into pairs and critical simplices, yielding the *discrete gradient* V



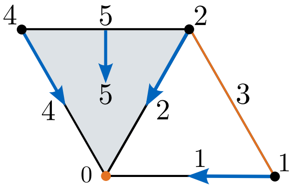
Discrete Morse functions - and their gradients - encode collapses:



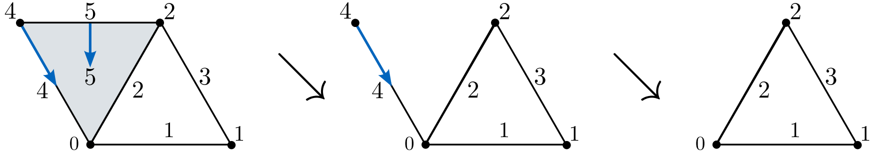
Discrete Morse theory

A *discrete Morse function* is a

- monotone function $f\colon K \to \mathbb{R}$ that
- partitions the complex into pairs and critical simplices, yielding the *discrete gradient V*

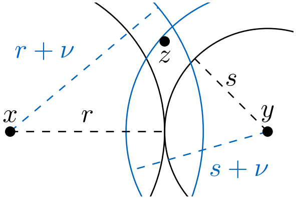


Discrete Morse functions - and their gradients - encode collapses:



The geodesic defect

Definition (Bonk, Schramm 2000). The metric space X is *ν-geodesic* if for all $x, y \in X$ and $r, s \geq 0$ with $r + s = d(x, y)$ there exists $z \in X$ with $d(x, z) \leq r + \nu$ and $d(y, z) \leq s + \nu$.



Generalized contractibility lemma

Theorem (Bauer, R. 2022). Let X be a δ-hyperbolic metric space. Then there exists a discrete gradient encoding the collapses

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}$$

for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect of X

Generalized contractibility lemma

Theorem (Bauer, R. 2022). Let X be a δ-hyperbolic metric space. Then there exists a discrete gradient encoding the collapses

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}$$

for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect of X

- Vietoris–Rips persistent homology of trees (0-hyperbolic) is concentrated in degree zero.

Generalized contractibility lemma

Theorem (Bauer, R. 2022). Let X be a δ-hyperbolic metric space. Then there exists a discrete gradient encoding the collapses

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}$$

for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect of X

- Vietoris–Rips persistent homology of trees (0-hyperbolic) is concentrated in degree zero.
- Connections to Ripser?

Generalized contractibility lemma

Theorem (Bauer, R. 2022). Let X be a δ-hyperbolic metric space. Then there exists a discrete gradient encoding the collapses

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}$$

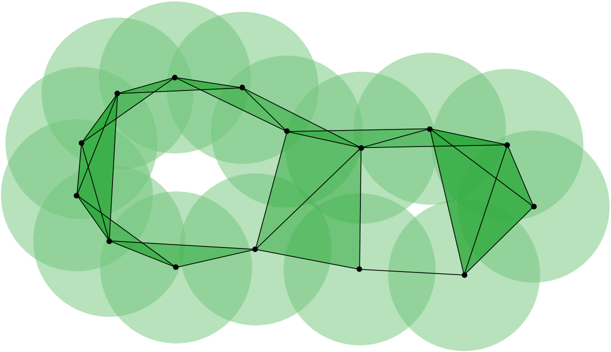
for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect of X

- Vietoris–Rips persistent homology of trees (0-hyperbolic) is concentrated in degree zero.
- Connections to Ripser?
- For a tree metric space, those are induced by apparent pairs.
    - Ripser computes the persistent homology without a single column operation.
    - Explains Ripser's outstanding performance on genetic distances.

Čech complexes

Definition. The *Čech complex* $\check{\mathrm{C}}\mathrm{ech}_r(X)$ of $X \subseteq \mathbb{R}^d$ is the simplicial complex

$$\check{\mathrm{C}}\mathrm{ech}_r(X) = \{J \subseteq X \mid |J| < \infty \text{ and } \bigcap_{y \in J} D_r(y) \neq \emptyset\}.$$



Nerves

Definition (Alexandroff 1928). Let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of a space X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \{J \subseteq I \mid |J| < \infty \text{ and } \bigcap_{i \in J} U_i \neq \emptyset\}$$

recording the intersection pattern of the cover.

Nerves

Definition (Alexandroff 1928). Let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of a space X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \{J \subseteq I \mid |J| < \infty \text{ and } \bigcap_{i \in J} U_i \neq \emptyset\}$$

recording the intersection pattern of the cover.

Theorem (Borsuk 1948, and many more). Let $\mathcal{U}$ be a nice cover of X. Then $\mathrm{Nrv}(\mathcal{U})$ is homotopy equivalent to X.

Nerves

Definition (Alexandroff 1928). Let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of a space X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \{J \subseteq I \mid |J| < \infty \text{ and } \bigcap_{i \in J} U_i \neq \emptyset\}$$

recording the intersection pattern of the cover.

Theorem (Borsuk 1948, and many more). Let $\mathcal{U}$ be a nice cover of X. Then $\mathrm{Nrv}(\mathcal{U})$ is homotopy equivalent to X.

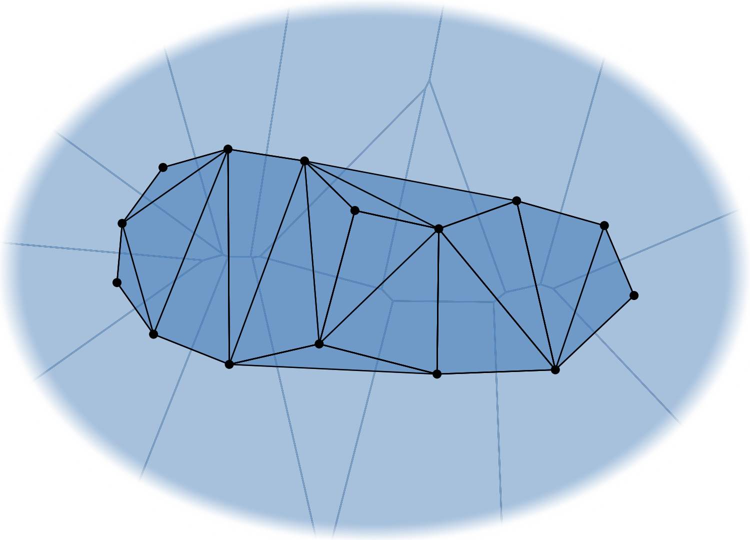
Here *nice* can mean different things:

- open, numerable cover, contractible intersections
- finite, closed, convex cover

U. Bauer, M. Kerber, F. Roll, and A. Rolle
A unified view on the functorial nerve theorem and its variations
Expositiones Mathematicae, 2023. doi:10.1016/j.exmath.2023.04.005

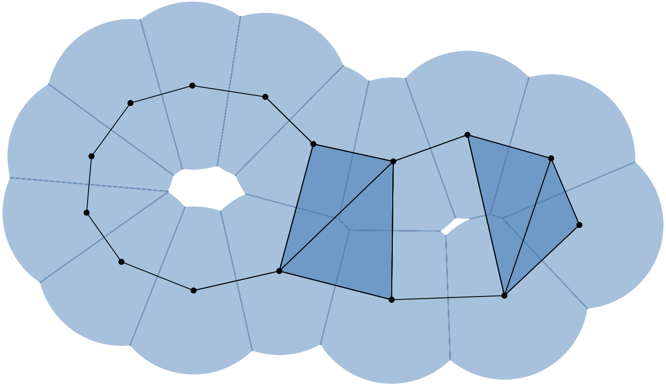
Delaunay complexes

Voronoi diagram and Delaunay triangulation

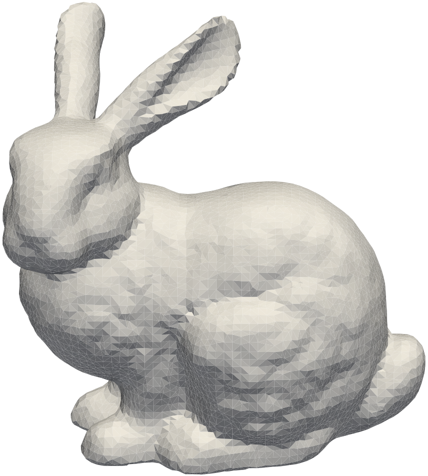


Delaunay complexes

Definition. The *Delaunay complex* $\mathrm{Del}_r(X)$ of $X \subseteq \mathbb{R}^d$ is the nerve of the cover by closed Voronoi balls of radius r centered at points in X.



Delaunay complex



Wrap complex

Morse Theory of Čech and Delaunay compelxes

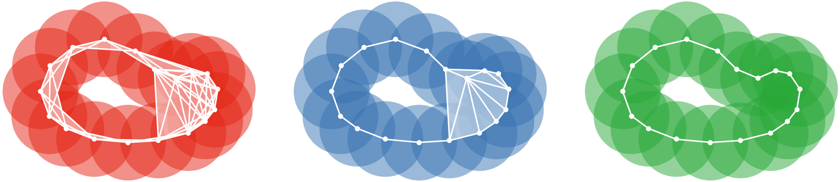
Proposition (Bauer, Edelsbrunner 2014). The Čech and Delaunay complexes are sublevel sets of (generalized) discrete Morse functions.

Morse Theory of Čech and Delaunay compelxes

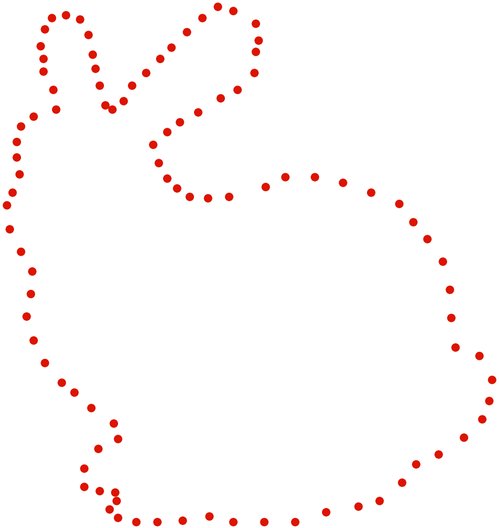
Proposition (Bauer, Edelsbrunner 2014). The Čech and Delaunay complexes are sublevel sets of (generalized) discrete Morse functions.

Theorem (Bauer, Edelsbrunner 2017).
Čech, Delaunay, and Wrap complexes (at any scale r) are related by collapses encoded by a single discrete gradient field:

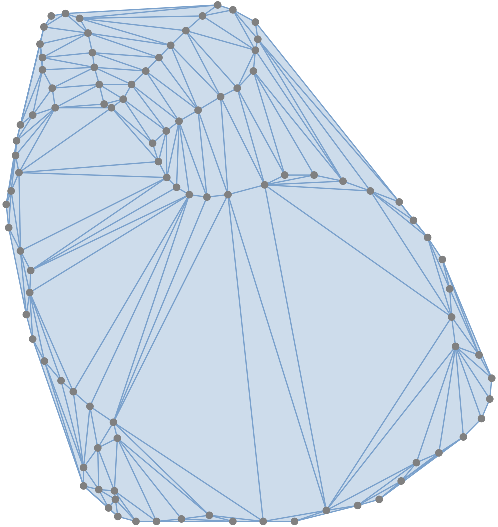
$$\check{\mathrm{C}}\mathrm{ech}_r(X) \searrow \mathrm{Del}_r(X) \searrow \mathrm{Wrap}_r(X).$$



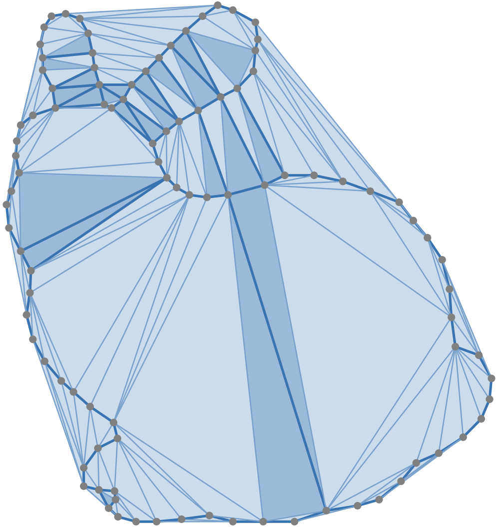
Wrap complexes and exhaustively reduced cycles



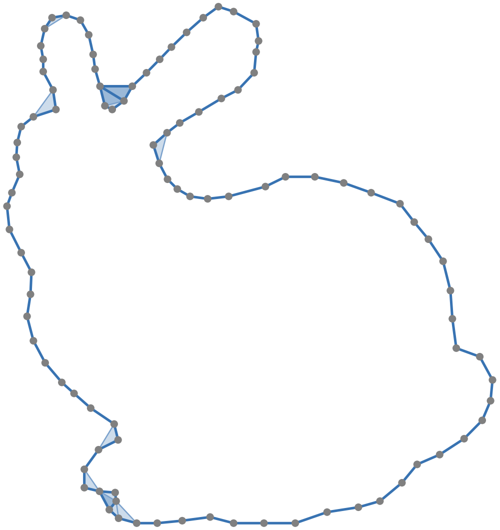
Wrap complexes and exhaustively reduced cycles



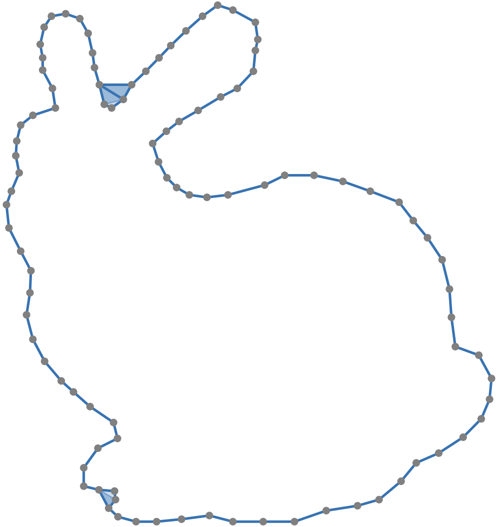
Wrap complexes and exhaustively reduced cycles



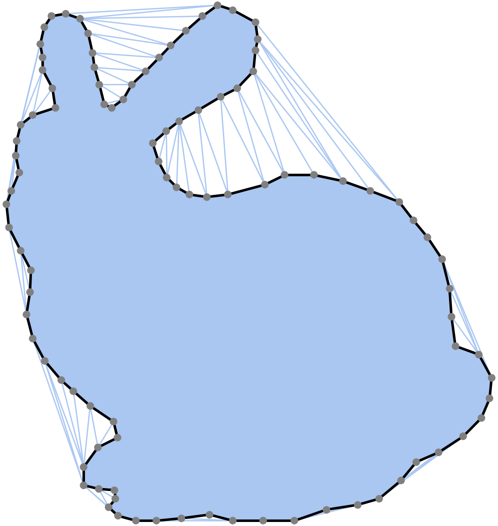
Wrap complexes and exhaustively reduced cycles



Wrap complexes and exhaustively reduced cycles



Wrap complexes and exhaustively reduced cycles



Algebraic gradient flows and persistent homology

Loose ends in the literature:

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a discrete gradient

Algebraic gradient flows and persistent homology

Loose ends in the literature:

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a discrete gradient
- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to based chain complexes (*algebraic Morse theory*)

Algebraic gradient flows and persistent homology

Loose ends in the literature:

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a discrete gradient
- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to based chain complexes (*algebraic Morse theory*)
- Cohen-Steiner–Lieutier–Vuillamy (2022) consider shape reconstruction from point clouds using lexicographically minimal cycles

Algebraic gradient flows and persistent homology

Loose ends in the literature:

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a discrete gradient
- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to based chain complexes (*algebraic Morse theory*)
- Cohen-Steiner–Lieutier–Vuillamy (2022) consider shape reconstruction from point clouds using lexicographically minimal cycles

We interpret persistent homology in terms of algebraic Morse theory:

- persistence pairs correspond to algebraic gradient pairs

Algebraic gradient flows and persistent homology

Loose ends in the literature:

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a discrete gradient
- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to based chain complexes (*algebraic Morse theory*)
- Cohen-Steiner–Lieutier–Vuillamy (2022) consider shape reconstruction from point clouds using lexicographically minimal cycles

We interpret persistent homology in terms of algebraic Morse theory:

- persistence pairs correspond to algebraic gradient pairs
- exhaustive Matrix reduction corresponds to gradient flow

Algebraic gradient flows and persistent homology

Loose ends in the literature:

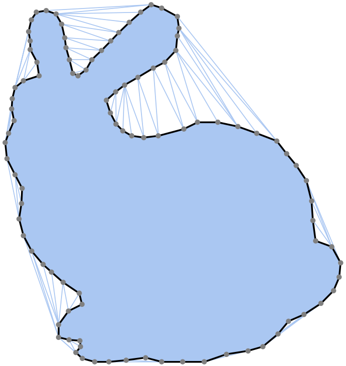
- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a discrete gradient
- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to based chain complexes (*algebraic Morse theory*)
- Cohen-Steiner–Lieutier–Vuillamy (2022) consider shape reconstruction from point clouds using lexicographically minimal cycles

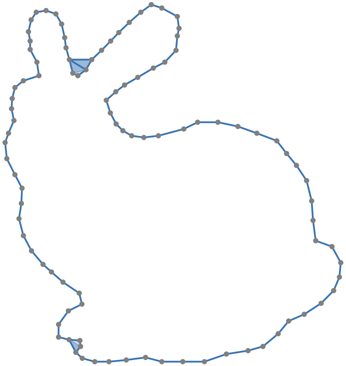
We interpret persistent homology in terms of algebraic Morse theory:

- persistence pairs correspond to algebraic gradient pairs
- exhaustive Matrix reduction corresponds to gradient flow
- the resuling representative cycles are lexicographically minimal

Minimal cycles and Wrap complexes

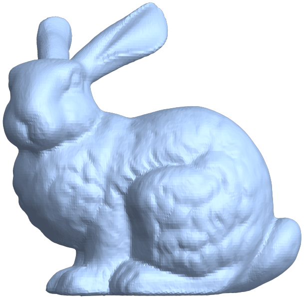
Theorem (Bauer, R. 2022). Let $X \subset \mathbb{R}^d$ be a finite subset in general position and let $r \in \mathbb{R}$. Then the lexicographically minimal cycles of $\mathrm{Del}_r(X)$ are supported on $\mathrm{Wrap}_r(X)$.

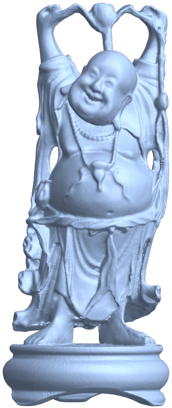


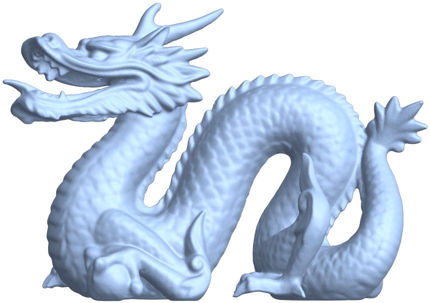


Point cloud reconstruction with most persistent features

The lexicographically minimal cycle corresponding to the interval in the persistence barcode of the Delaunay function with the largest death/birth ratio:







Summary

- Saw a generalization of Rips contractibility lemma beyond geodesic spaces
- Learned that the geoemtry and combinatorics of the input influences the persistent homology computation time
- Saw that persistent homology relates to algebraic gradient flows
- Learned that lexicographically minimal cycles of $\mathrm{Del}_r(X)$ are supported on $\mathrm{Wrap}_r(X)$