A Unified View on the Functorial Nerve Theorem and its Variations

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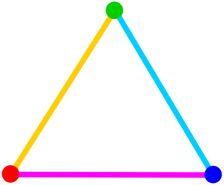
Seminar - Applied CATS
April 26, 2022

joint work with Ulrich Bauer, Michael Kerber, and Alexander Rolle

The Alexandrov nerve (1928)

Definition. Let X be a topological space, and let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \{J \subseteq I \mid |J| < \infty \text{ and } U_J := \bigcap_{i \in J} U_i \neq \emptyset\}$$

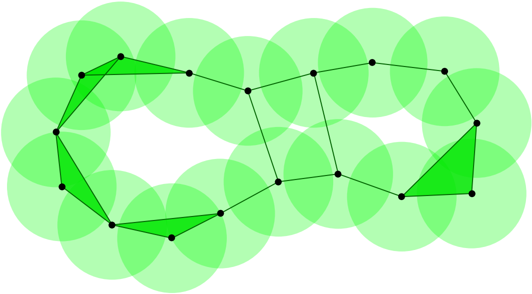


The Alexandrov nerve (1928)

Čech complex of a point cloud

Definition. The *Čech complex* of a subset $X \subseteq \mathbb{R}^d$ is the nerve of the cover by closed balls of radius r centered at points in X

$$\check{\mathrm{C}}\mathrm{ech}_r(X) = \mathrm{Nrv}((D_r(X))_{x \in X}))$$



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Theorem (Borsuk 1948, and many more). Let $\mathcal{U}$ be a nice cover of X. Then $\mathrm{Nrv}(\mathcal{U})$ is homotopy equivalent to X.

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- Čech 1932: Extends Alexandrov's ideas $\to$ Čech (co)homology

Applications to combinatorics

A puzzle from Gavin Wraith

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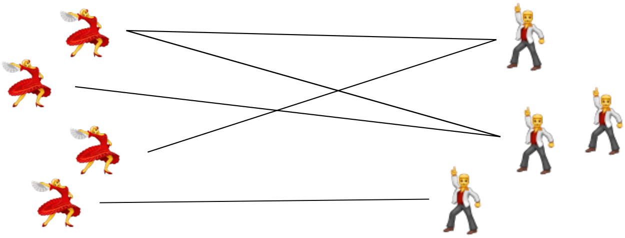

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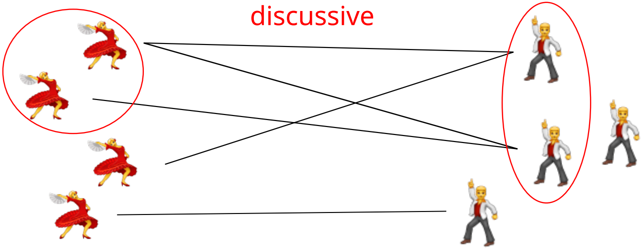

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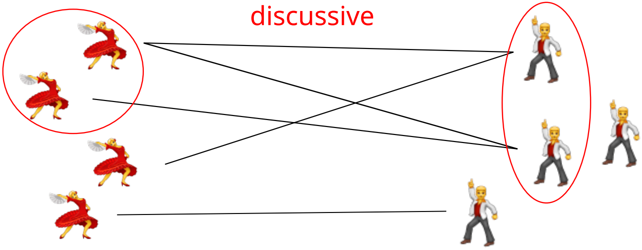

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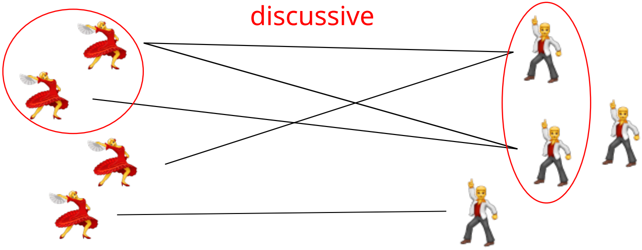

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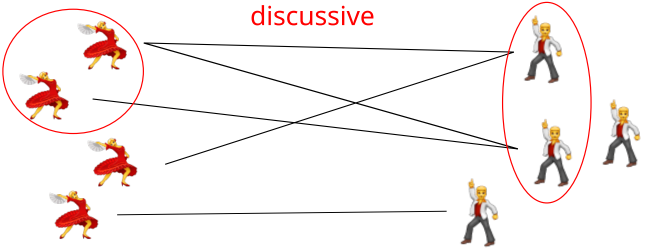

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Applications to combinatorics

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$$L = \{\sigma \subseteq A \mid \exists b \in B : \sigma \times \{b\} \subseteq S\}$$
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- Nerve theorem based proof of Rota's crosscut theorem
- Nerves appear in Lovász' proof (1978) of Kneser's conjecture (1955)
  $\rightarrow$ emergence of topological combinatorics

Nerve theorem for closed convex sets

Theorem. If $X \subset \mathbb{R}^d$, and $\mathcal{A} = (C_i)_{i \in [n]}$ is a cover by closed convex subsets, then $\mathrm{Nrv}(\mathcal{A})$ is homotopy equivalent to X.

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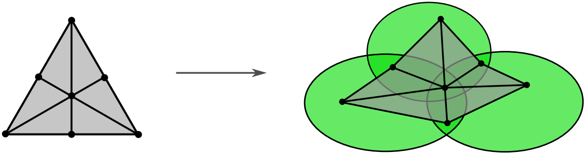

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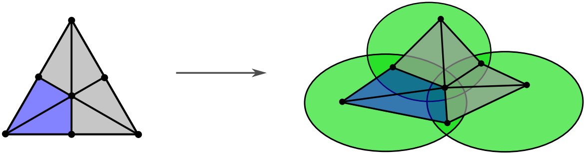

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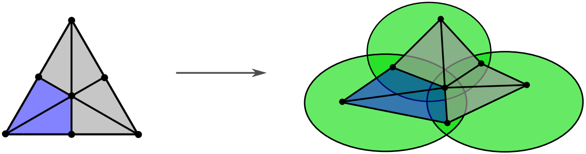

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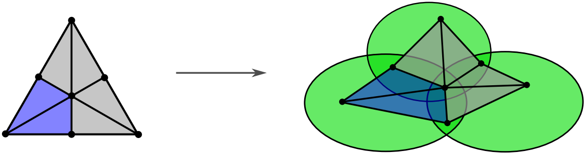

- Construct $\Phi\colon X \to \mathrm{Nrv}(\mathcal{A})$ using a partition of unity subordinate to an open thickening of the C_i with $\Phi(C_i) \subseteq \mathrm{bst}\, v_i$.
- Show that Φ is a homotopy inverse to Γ.

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Some proof details

- Take a collection of open sets $\mathcal{G} = (U_i \supseteq C_i)_{i \in [n]}$ satisfying $\mathrm{Nrv}(\mathcal{A}) = \mathrm{Nrv}(\mathcal{G}_\varepsilon)$.

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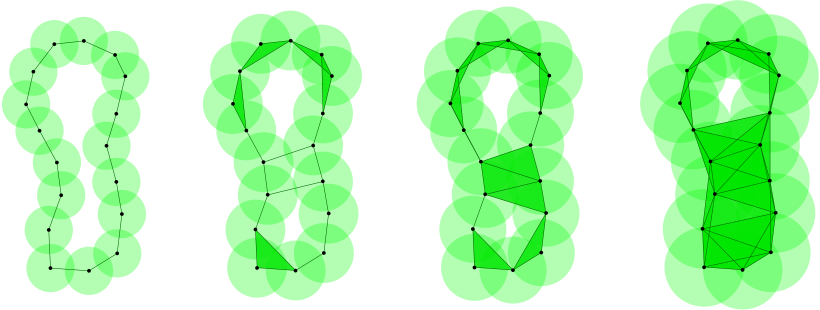
Functoriality

Persistent homology



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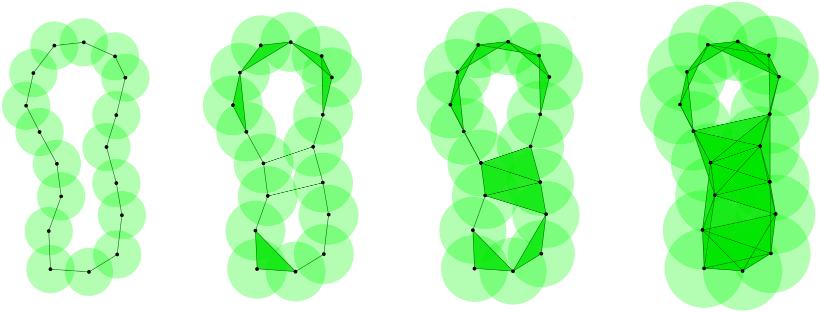

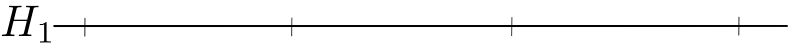

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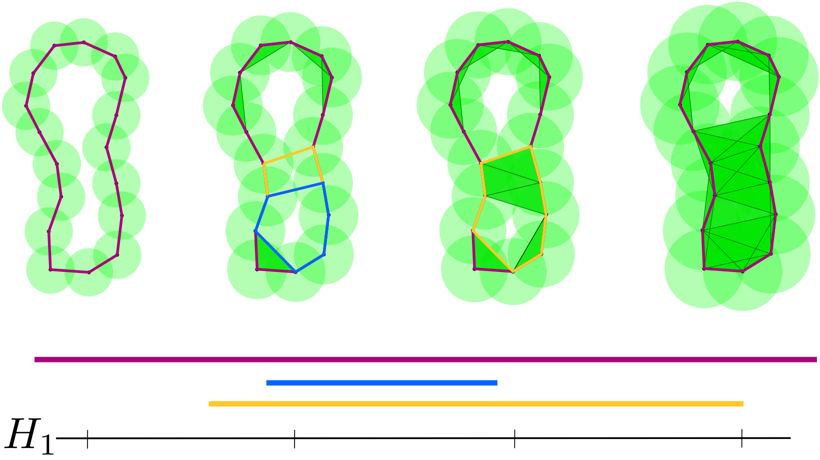

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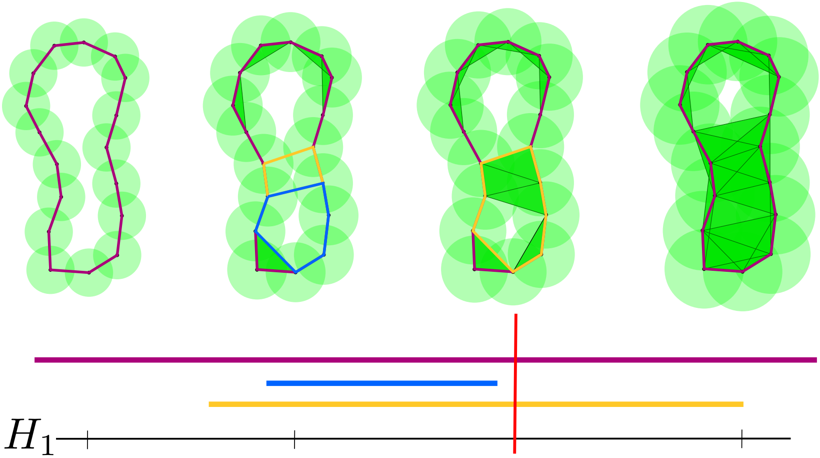

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Functoriality

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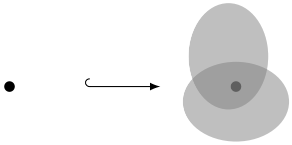

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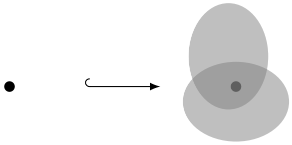

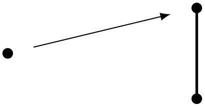

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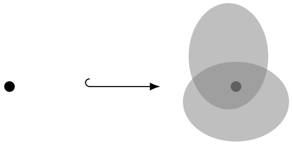
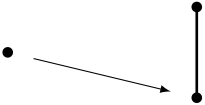
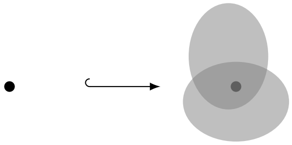
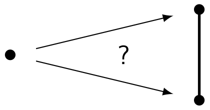




Functoriality

Category of covered spaces





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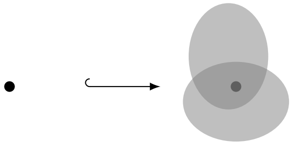

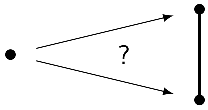

Definition. $(U_i)_{i\in I}$ a cover of X, and $(V_\ell)_{\ell\in L}$ a cover of Y. A *map of indexed covers* $\varphi\colon (U_i)_{i\in I} \to (V_\ell)_{\ell\in L}$ is formally a map $\varphi\colon I \to L$. A continuous map $f\colon X \to Y$ is *carried by* φ if $f(U_i) \subseteq V_{\varphi(i)}$.

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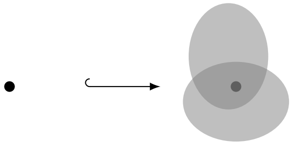

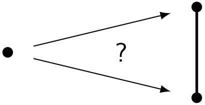

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Definition. The *category of covered spaces* Cov has

- Obj: pairs of the form $(X, (U_i))$, with (U_i) a cover of X
- Mor: $(f, \varphi)\colon (X, (U_i)) \to (Y, (V_\ell))$, continuous map $f\colon X \to Y$ carried by $\varphi\colon (U_i) \to (V_\ell)$

Functoriality

Category of covered spaces

Two functors

- Forgetting the cover: $\mathrm{Spc}\colon \mathsf{Cov} \to \mathsf{Top}$, $(X, \mathcal{U}) \mapsto X$
- The nerve: $\mathrm{Nrv}\colon \mathsf{Cov} \to \mathsf{Top}$, $(X, \mathcal{U}) \mapsto \mathrm{Nrv}(\mathcal{U})$

Functoriality

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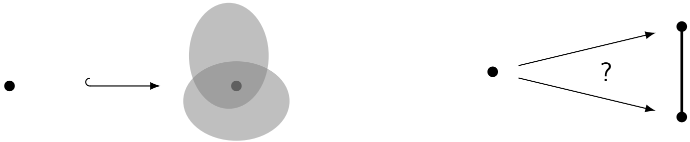
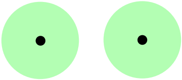
Remark. There are no natural transformations $\mathrm{Spc} \Rightarrow \mathrm{Nrv}$



and similarly no natural transformations $\mathrm{Nrv} \Rightarrow \mathrm{Spc}$.

Functoriality

Pointed covers



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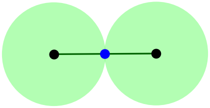

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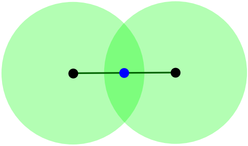

Functoriality

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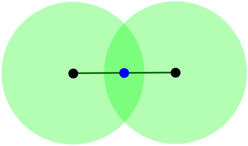
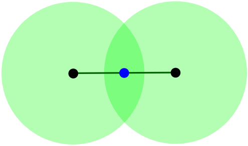


Definition. Only consider $X \subseteq \mathbb{R}^d$. The category $\mathsf{ClConv}_\bullet$ has

- Obj: $(X, \mathcal{A}_\bullet)$, $\mathcal{A}_\bullet$ a finite closed, convex, and *pointed cover* of X

Functoriality

Pointed covers



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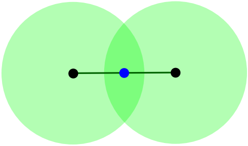
- Mor: $(f, \varphi)\colon (X, \mathcal{A}_\bullet) \to (Y, \mathcal{B}_\bullet)$, $f\colon X \to Y$ carried by $\varphi\colon \mathcal{A} \to \mathcal{B}$:
  - f preserves the basepoints
  - f is affine linear on each cover element

Functoriality

Pointed covers



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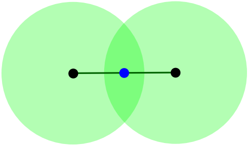
The map $\Gamma\colon \operatorname{Sd}\operatorname{Nrv}(\mathcal{A}) \xrightarrow{\simeq} X$ is natural w.r.t morphisms in ClConv$_\bullet$.

Functoriality

Pointed covers



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Theorem. On $\mathsf{ClConv}_\bullet$ there exists a pointwise homotopy equivalence

$$\operatorname{Sd}\operatorname{Nrv} \Rightarrow \operatorname{Spc}$$

Functoriality

Blowup complex

Definition. For a finite cover $\mathcal{U} = (U_i)_{i \in [n]}$ of X, the *blowup complex* is

$$\mathrm{Blowup}(\mathcal{U}) = \bigcup_{J \in \mathrm{Nrv}(\mathcal{U})} U_J \times \Delta^{|J|-1} \subseteq X \times \Delta^n \,,$$

yielding a functor $\mathrm{Blowup}\colon \mathsf{Cov} \to \mathsf{Top}$.

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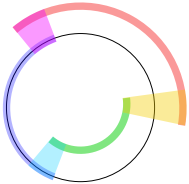

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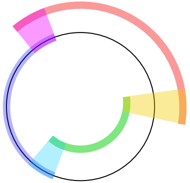

Remark. The blowup complex is (not naturally) homeomorphic to the *bar construction* of the nerve diagram $\to$ can exploit its good homotopy theoretic properties to prove nerve theorems

Functoriality

Any $(f, \varphi)\colon (X, \mathcal{U}) \to (Y, \mathcal{V})$ induces a commuting diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & \mathrm{Blowup}(\mathcal{U}) \arrow[l, "\rho_S"'] \arrow[r, "\rho_N"] \arrow[d] & \mathrm{Nrv}(\mathcal{U}) \arrow[d, "\varphi_*"] \\ Y & \mathrm{Blowup}(\mathcal{V}) \arrow[l, "\rho_S"] \arrow[r, "\rho_N"'] & \mathrm{Nrv}(\mathcal{V}) \end{tikzcd}$$

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Hence, there are natural transformations $\mathrm{Spc} \overset{\rho_S}{\Leftarrow} \mathrm{Blowup} \overset{\rho_N}{\Rightarrow} \mathrm{Nrv}$.

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X & & \mathrm{Nrv}(\mathcal{A}) \arrow[ll, "\simeq", "\Gamma"']
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Unified nerve theorem

Theorem. Let X be a topological space and $\mathcal{A} = (A_i)_{i \in I}$ a cover of X.

1. Consider the natural map $\rho_S \colon \mathrm{Blowup}(\mathcal{A}) \to X$.

2. Consider the natural map $\rho_N \colon \mathrm{Blowup}(\mathcal{A}) \to \mathrm{Nrv}(\mathcal{A})$.

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   b) If for all $J \in \mathrm{Nrv}(\mathcal{A})$ the space A_J is compactly generated and $\mathcal{A}$ is homologically good with respect to a coefficient ring R, then ρ_N is an R-homology isomorphism.

Unified nerve theorem

Counterexamples

1. Paracompactness is crucial to guarantee a homotopy equivalence

Unified nerve theorem

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Unified nerve theorem

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- $\mathcal{A} = (L_{<p})_{p \in \omega_1}$ is a good open cover and it follows from 1.a) and 2.a) that the nerve $\operatorname{Nrv} \mathcal{A}$ is weakly contractible and hence contractible by Whitehead's theorem

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- Thus, L and $\operatorname{Nrv} \mathcal{A}$ are not homotopy equivalent

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2. For closed good covers some "finiteness" is needed:

- Consider the cover of S^1 by its points
- As the nerve $\operatorname{Nrv}\mathcal{A}$ is a disjoint union of points, it is not homotopy equivalent to S^1
- All conditions in 1.b) and 2.a) are satisfied except the locally finiteness assumption

Unified nerve theorem

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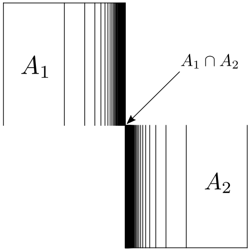
3. The “latching assumption” is not a proof artefact:

Unified nerve theorem

Counterexamples

3. The “latching assumption” is not a proof artefact:

- Consider the *double comb space* C and denote the two combs by A_1 and A_2



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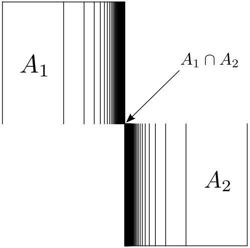

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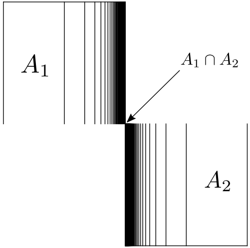

- The nerve $\operatorname{Nrv} \mathcal{A}$ is contractible, but C is not
- The pairs $(A_1, A_1 \cap A_2)$ and $(A_2, A_1 \cap A_2)$ do not satisfy the homotopy extension property

Unified nerve theorem

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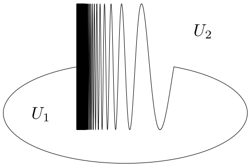
Unified nerve theorem

Counterexamples

3. The “latching assumption” is not a proof artefact; even if we only care about the homologies:

- Consider the *Warsaw circle* $W \subseteq S^2$ that separates the sphere into two connected components U_1 and U_2



Unified nerve theorem

Counterexamples

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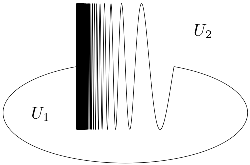

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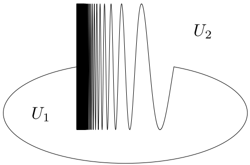

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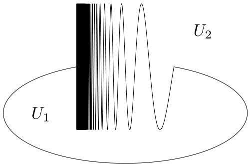

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Future work

Funding: DFG (SFB/TRR 109 *Discretization in Geometry and Dynamics*)

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- Approximate nerve theorems
- Discuss Latschev's result on the reconstruction of Riemannian manifolds and its functoriality by using *Vietoris–Rips good covers*

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Selected references

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