Wrapping Cycles in Delaunay Complexes: Bridging Persistent Homology and Discrete Morse Theory

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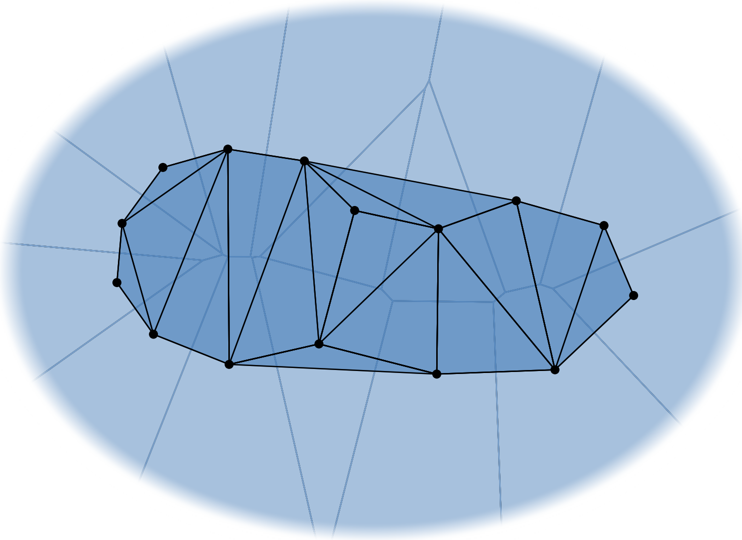
SoCG 2024, Athens

Joint work with Ulrich Bauer

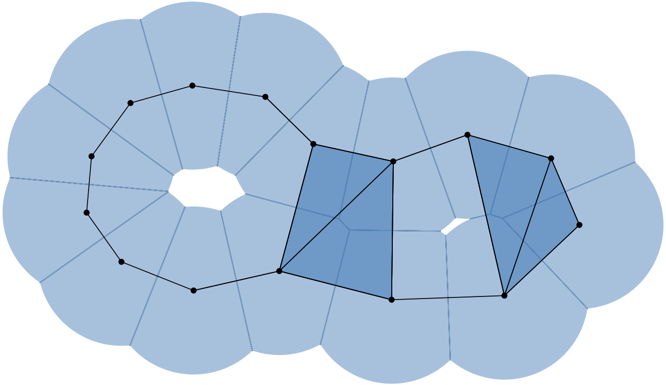
Delaunay complexes

Voronoi diagram and Delaunay triangulation



Delaunay complexes

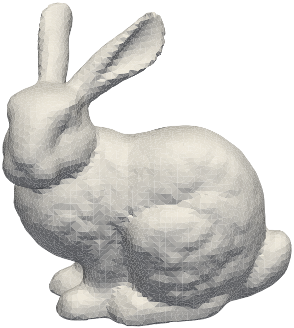
Definition. The *Delaunay complex* $\mathrm{Del}_r(X)$, or α-*shape*, of $X \subseteq \mathbb{R}^d$ is the nerve of the cover by closed Voronoi balls of radius r centered at points in X.



Wrap

- Originally introduced by Edelsbrunner (1995) as a subcomplex of the Delaunay triangulation for surface reconstruction, using flow lines associated to Euclidean distance functions
- Redeveloped using discrete Morse theory (Forman 1998) by Bauer & Edelsbrunner (2014/17)

Delaunay complex



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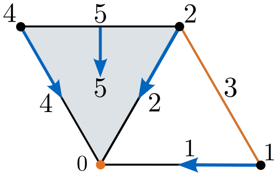
Discrete Morse theory

A *discrete Morse function* is a

- monotone function $f\colon K \to \mathbb{R}$ that
- partitions the complex into pairs and critical simplices, yielding the *discrete gradient V*



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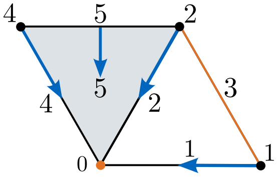
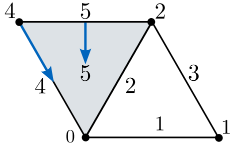


Discrete Morse functions - and their gradients - encode collapses:



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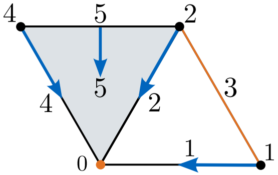

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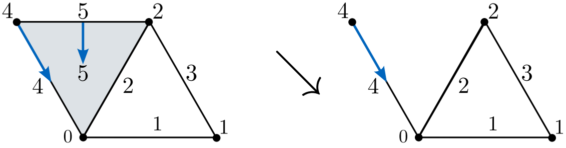

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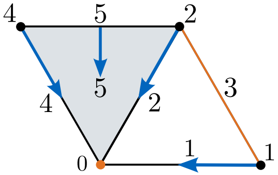

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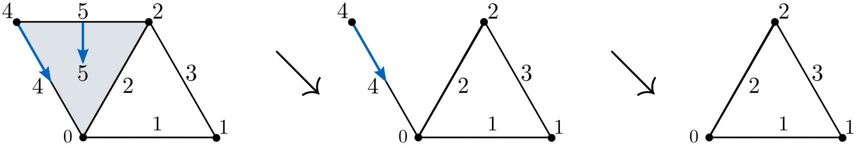
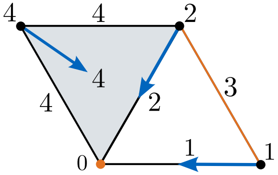


Generalized discrete Morse theory

Generalized gradients consist of intervals (in the face poset) instead of just facet pairs:



Morse Theory of Čech and Delaunay complexes

Proposition (Bauer, Edelsbrunner 2014). The Čech and Delaunay complexes are sublevel sets of generalized discrete Morse functions.

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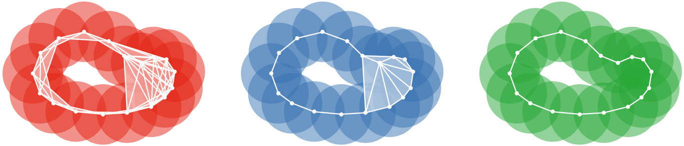
Čech, Delaunay, and Wrap complexes (at any scale r) are related by collapses encoded by a single discrete gradient field:

$$\check{\mathrm{C}}\mathrm{ech}_r(X) \searrow \mathrm{Del}_r(X) \searrow \mathrm{Wrap}_r(X).$$



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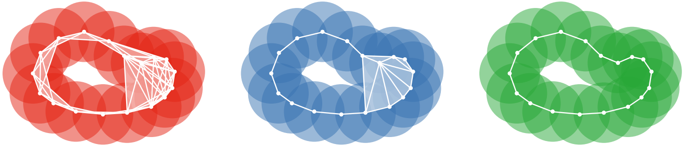
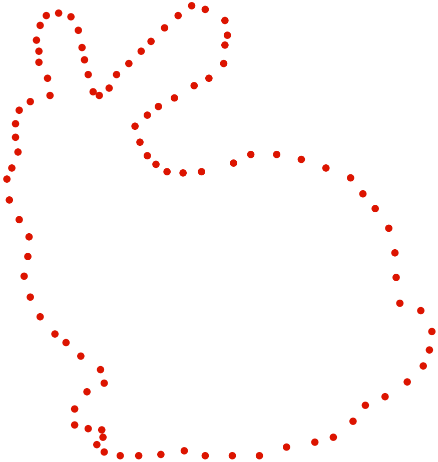


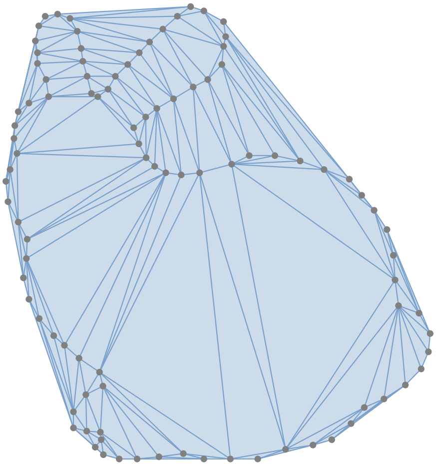
Definition. The Wrap complex $\mathrm{Wrap}_r(X)$ is the smallest subcomplex of $\mathrm{Del}_r(X)$ such that the Delaunay gradient induces a collapse $\mathrm{Del}_r(X) \searrow \mathrm{Wrap}_r(X)$.

Wrap complexes



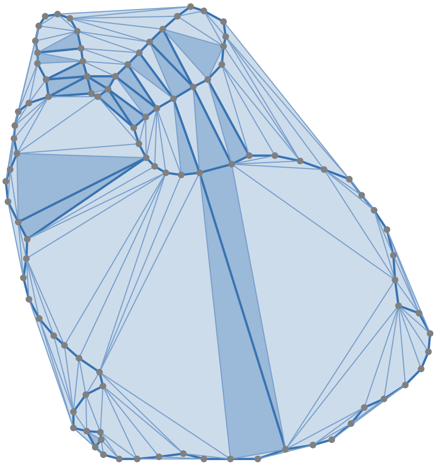
Point cloud

Wrap complexes



Delaunay triangulation

Wrap complexes



Critical simplices

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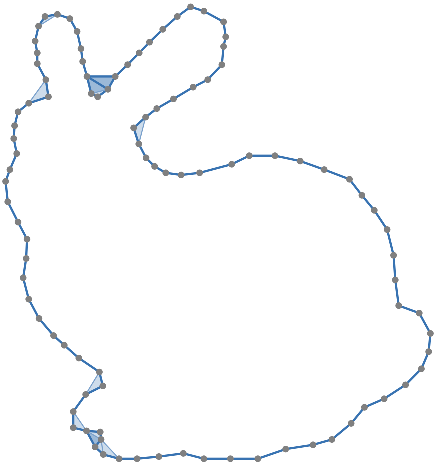

Delaunay complex

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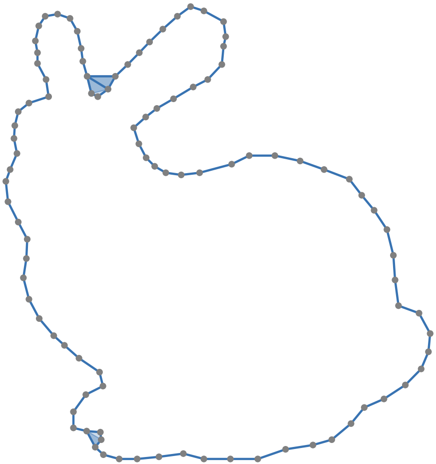

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- determines the barcode through $\{[\operatorname{pivot} R_i, i) \mid R_i \neq 0\}$

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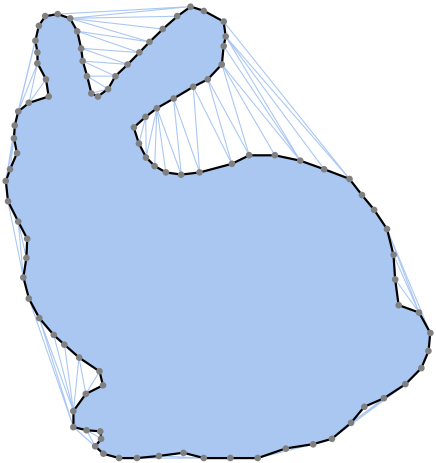

Reduction process

Algebraic gradient flows and persistent homology

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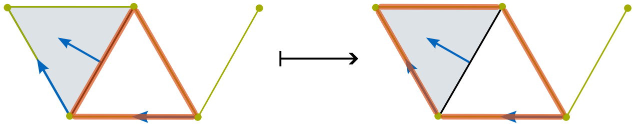

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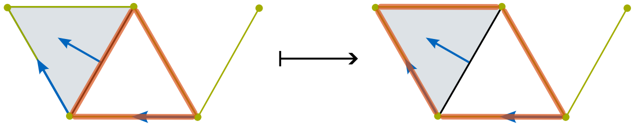

- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to based chain complexes (*algebraic Morse theory*)
  - the basis elements take the role of the simplices in discrete Morse theory
  - all other notions translate straightforwardly

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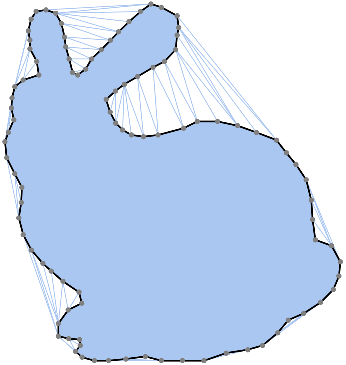
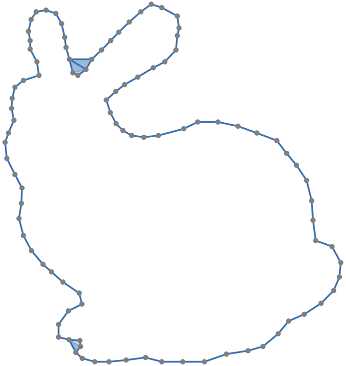
- exhaustive Matrix reduction corresponds to gradient flow
- the lexicographically minimal cycles are invariant under the algebraic gradient flow
- connects to generalized discrete Morse theory, and hence to the Wrap complex, through gradient refinements (by apparent pairs)

Minimal cycles and Wrap complexes

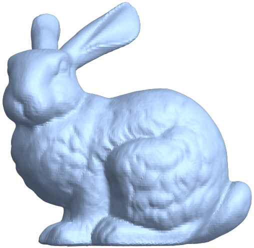
Theorem (Bauer, R). Let $X \subset \mathbb{R}^d$ be a finite subset in general position and let $r \in \mathbb{R}$. Then the lexicographically minimal cycles of $\mathrm{Del}_r(X)$, with respect to the Delaunay-lexicographic order on the simplices, are supported on $\mathrm{Wrap}_r(X)$.

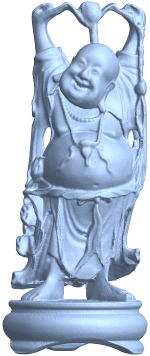


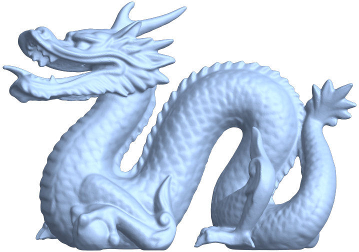


Point cloud reconstruction with most persistent features

The lexicographically minimal cycle, with respect to the Delaunay-lexicographic order on the simplices, corresponding to the interval in the persistence barcode of the Delaunay filtration with the largest death/birth ratio:







```
$ docker build -o output github.com/fabian-roll/wrappingcycles
```

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- Provide a tight link between persistent homology and discrete Morse theory
  - such that the corresponding algebraic gradient flow can be viewed as a variant of the reduction algorithm for computing persistent homology
- Establish a strong connection between Morse-theoretic and homological approaches to shape reconstruction
  - lexicographically minimal cycles of $\mathrm{Del}_r(X)$ are supported on the Wrap complex $\mathrm{Wrap}_r(X)$

Funding: DFG – TRR109 *Discretization in Geometry and Dynamics*