ON A THEOREM OF LERAY* †

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The theorem of Leray under consideration states that under suitable conditions the homology of a space will be the same as the homology of the nerve complex of a covering. The original proof of Leray (as well as others in modified forms due to Borel, Cartan and Serre) is based on the couverture theory (or also the theory of sheaves and spectral sequences). We shall give here a proof in the pattern of treatment of Eilenberg-Steenrod. Though our proof works only for finite coverings, it has the advantage of being extendible easily to the case of fundamental groups, to which the known methods do not apply. Similar generalizations on homotopy groups and homotopy types are also studied.

§ 1. Canonical Mappings Associated with a Covering

Let $\mathscr{F}=\{F_\lambda\}$ be a finite closed covering of a normal space E and K its nerve complex with vertices a_λ corresponding to the sets F_λ of $\mathscr{F}$. Let $\mathscr{U}$ be a finite open covering $\{U_\lambda\}$ of E "similar" to $\mathscr{F}$ so that $F_\lambda \subset U_\lambda$, $F_{\lambda_1}\cdots F_{\lambda_r}=\emptyset$ if and only if $U_{\lambda_1}\cdots U_{\lambda_r}=\emptyset$. To each U_λ let f_λ be an arbitrary continuous function on E with $f_\lambda=0$ on $E-U_\lambda$ and $f_\lambda>0$ on U_λ which exists by Urysohn's Lemma. With respect to such a system $\{f_\lambda\}$ a *canonical mapping* φ of E into the space $|K|$ of K may then be defined by

$$\varphi(x)=\sum_\lambda \frac{f_\lambda(x)}{\sum_\mu f_\mu(x)}\cdot a_\lambda.$$

Let the mapping φ' of E into $|K|$ be another canonical mapping associated with a finite open covering $\mathscr{U}'=\{U'_\lambda\}$ and a system of functions f'_λ with similar properties. The finite open covering $\{U_\lambda \cap U'_\lambda\}$ is then again similar to $\mathscr{F}$ and with respect to it we may take

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a system of functions g_λ as before and denote by ψ the corresponding canonical mapping of E into $|K|$. Then

$$\varphi_t(x)=\sum_\lambda \frac{tg_\lambda(x)+(1-t)f_\lambda(x)}{t\cdot\sum_\mu g_\mu(x)+(1-t)\cdot\sum_\mu f_\mu(x)}\cdot a_\lambda,\quad 0\leqslant t\leqslant 1,$$

and

$$\varphi'_t(x)=\sum_\lambda \frac{t\cdot g_\lambda(x)+(1-t)f'_\lambda(x)}{t\cdot\sum_\mu g_\mu(x)+(1-t)\cdot\sum_\mu f'_\mu(x)}\cdot a_\lambda,\quad 0\leqslant t\leqslant 1$$

are evidently homotopies between φ, ψ and φ', ψ respectively. It follows that the homotopy class Φ of φ is independent of the covering $\mathscr{U}$ and the functions f_λ chosen, and there is induced a unique homomorphism

$$\Phi^*:\ H^*(K)\to H^*(E)$$

for any cohomology theory H^* defined on pairs of normal spaces in the sense of Eilenberg-Steenrod. We shall call Φ^* the *canonical homomorphism* associated with the covering $\mathscr{F}$.

Consider now a normal space E, a closed subspace E' of E, and a finite closed covering $\mathscr{F}$ of E. The non-empty intersections of sets in $\mathscr{F}$ with E' form then a finite closed covering $\mathscr{F}''$ of E' which will be called the *restriction* of $\mathscr{F}$ to E'. It may happen that a part of sets in $\mathscr{F}''$ forms already a closed covering, say $\mathscr{F}'$ of E'. Let K, K'' and K' be the nerve complexes of $\mathscr{F}, \mathscr{F}''$ and $\mathscr{F}'$ respectively. We shall identify the vertex of K', K'' corresponding to $F'_\lambda=F_\lambda\cdot E'\neq\emptyset$ of $\mathscr{F}'$ with the vertex a_λ of K corresponding to F_λ of $\mathscr{F}$ so that K' will be a subcomplex of K'' and K'' a subcomplex of K. Let the canonical homomorphisms associated with $\mathscr{F}$ and $\mathscr{F}'$ be respectively

$$\Phi^*:\ H^*(K)\to H^*(E),$$

and

$$\Phi'^*:\ H^*(K')\to H^*(E').$$

Then we have the following

Lemma 1. *Φ^* and Φ'^* are commutative with the coboundary homomorphisms δ^* and the injection homomorphisms induced by injections* i: E′⊂E *and* j: |K′|⊂|K|, *i.e., the following diagram is commutative:*

$$\begin{array}{ccccc}
H^*(K) & \xrightarrow{j^*} & H^*(K') & \xrightarrow{\delta^*} & H^*(K)\\
\Phi^*\big\downarrow & & \big\downarrow\Phi'^* & & \big\downarrow\Phi^*\\
H^*(E) & \xrightarrow{i^*} & H^*(E') & \xrightarrow{\delta^*} & H^*(E)
\end{array}$$

Proof. Let us take a finite open covering $\mathscr{U}=\{U_\lambda\}$ of E similar to the closed covering $\mathscr{F}$ of E. The restriction $\mathscr{U}''$ of $\mathscr{U}$ to E' is then also similar to $\mathscr{F}''$ of $\mathscr{F}$ to E', and the part of sets in $\mathscr{U}''$ corresponding to sets in $\mathscr{F}'$ forms also an open covering $\mathscr{U}'$ of E' similar to $\mathscr{F}'$. To each U_λ let us construct now a continuous function f_λ with $f_\lambda=0$ on $E-U_\lambda$ and $f_\lambda>0$ on U_λ as before. Let us define the canonical mapping $\varphi\colon E\to|K|$ with respect to the covering $\mathscr{U}$ and the system of functions f_λ, and the canonical mapping $\varphi'\colon E'\to|K'|$ (resp. $\varphi''\colon E'\to|K''|$) with respect to the covering $\mathscr{U}'$ (resp. $\mathscr{U}''$) and the system of functions f'_λ which are the restrictions of f_λ to $U_\lambda\cdot E'$. We have then for $x\in E'$,

$$\varphi(x)=\varphi''(x)=\Sigma'\frac{f_\lambda'(x)}{\Sigma' f_{\mu'}(x)+\Sigma'' f_{\mu''}(x)}\cdot a_\lambda'+$$

$$+\Sigma''\frac{f_\lambda''(x)}{\Sigma' f_{\mu'}(x)+\Sigma'' f_{\mu''}(x)}\cdot a_\lambda'',$$

$$\varphi'(x)=\Sigma'\frac{f_\lambda'(x)}{\Sigma' f_{\mu'}(x)}\cdot a_\lambda'.$$

In the summations λ', μ' run over indices λ for which $F_\lambda\cdot E'\in\mathscr{F}'$, and λ'', μ'' over indices λ for which $F_\lambda\cdot E'\in\mathscr{F}''$ but $\notin\mathscr{F}'$. Now $\sum' f_{\mu'}(x)>0$ for $x\in E'$ so that

$$\varphi_t(x)=\Sigma'\frac{f_\lambda'(x)}{\Sigma' f_{\mu'}(x)+(1-t)\Sigma'' f_{\mu''}(x)}\cdot a_\lambda'+$$

$$+(1-t)\cdot\Sigma''\frac{f_\lambda''(x)}{\Sigma' f_{\mu'}(x)+(1-t)\cdot\Sigma'' f_{\mu''}(x)}\cdot a_\lambda''$$

gives a homotopy of φ into φ', i.e., $\varphi i\simeq j\varphi'$. By extending the homotopy we get $\varphi\simeq\bar\varphi$ with $\bar\varphi i=j\varphi'/E'$ from which the commutativity of the diagram follows.

The preceding proof gives also the following

Lemma 2. *There exist mappings* φ *homotopic to a canonical mapping of* E *into* |K| *such that* $\varphi(\mathrm{E}')\subset|\mathrm{K}'|$. *Suppose that both* E *and* E′ *are arcwise connected. If we take any point* $\mathrm{O}\in\mathrm{E}'$ *as the reference point of* $\pi_1(\mathrm{E})$ *and* $\pi_1(\mathrm{E}')$, *and also* $\varphi(\mathrm{O})$ *as the reference point of* $\pi_1(|\mathrm{K}|)$ *and* $\pi_1(|\mathrm{K}'|)$, *then the diagram*

$$\begin{array}{ccc}\pi_1(E') & \xrightarrow{i_*} & \pi_1(E)\\ \varphi_*\big\downarrow & & \big\downarrow\varphi_*\\ \pi_1(|K'|) & \xrightarrow{j_*} & \pi_1(|K|)\end{array}$$

is commutative, in which φ_*, i_*, *and* j_* *are homomorphisms induced by* φ *and the injections* i: $\mathrm{E}'\subset\mathrm{E}$ *and* j: $|\mathrm{K}'|\subset|\mathrm{K}|$ *respectively.*

§ 2. Theorem of Leray

The original form of the theorem of Leray may be stated as follows:

Let E *be a bicompact Hausdorff space,* $\mathscr{F}=\{\mathrm{F}_\lambda\}$, *a finite closed covering of* E *with the following property: Each non-empty intersection* $\mathrm{F}_{\lambda_1}\cdots\mathrm{F}_{\lambda_s}\neq\emptyset$ *of sets in* $\mathscr{F}$ *has the same Alexander-Čech cohomology ring with respect to a coefficient ring* R *as that of a point. Then the nerve complex of* $\mathscr{F}$ *gives the same Alexander-Čech ring with respect to* R *as that of the space itself.*

Let us say that a cohomology theory defined on pairs of normal spaces verifying Eilenberg-Steenrod axioms possesses *strong excision property* if the following holds: For any continuous mapping φ of one pair of normal spaces (E_1, F_1) into a second pair (E_2, F_2) such that φ is a homeomorphism of E_1-F_1 onto E_2-F_2, the homomorphism $\varphi^*\colon H^*(E_2, F_2)\to H^*(E_1, F_1)$ induced is an isomorphism onto. The theorem of Leray will now be strengthened into the following form:

Theorem 1. *Let* H^* *be a cohomology theory defined on pairs of normal spaces which possess the strong excision property. Let* E *be a bicompact Hausdorff space with a finite closed covering* $\mathscr{F}$ *such that each non-empty intersection of sets in* $\mathscr{F}$ *has the same cohomology groups in dimensions* $\mathrm{r}-1$ *and* r *as those of a point, then the cohomology group in dimension* r *of* E *will be isomorphic to that of the nerve complex* K *of the covering* $\mathscr{F}$. *Moreover, the isomorphism between* $\mathrm{H}^r(|\mathrm{K}|)$ *and* $\mathrm{H}^r(\mathrm{E})$ *may be realized by any canonical mapping of* E *into* |K| *with respect to the covering* $\mathscr{F}$.

Proof. We shall prove the theorem by induction on the number of sets in $\mathscr{F}$. The theorem is trivial when there is only one set in $\mathscr{F}$. Suppose that it is true for $\leqq n$ sets and let us consider such a covering $\mathscr{F}$ consisting of $n+1$ sets $F_0, F_1,\cdots, F_n$. Denote the union of $F_1,\cdots, F_n$ by E_1, F_0 also by E_2 and $E_1\cdot E_2$ by E_0. As $E-E_1$ is identical with $E_2-E_1\cdot E_2$ and $E-E_2$ is identical with $E_1-E_1\cdot E_2$, the triple $(E; E_1, E_2)$ forms a proper triad owing to the strong excision property. Owing to the triviality of $H^*(E_2)$ the Mayer-Vietoris exact sequence of the triad reduces then to

$$\cdots\xrightarrow{j_1^*}H^{r-1}(E_1)\xrightarrow{j_0^*}H^{r-1}(E_0)\xrightarrow{\delta^*}H^r(E)\xrightarrow{j_1^*}H^r(E_1)\xrightarrow{j_0^*}H^r(E_0)\to\cdots$$

in which δ^* is the coboundary homomorphism and j_0^*, j_1^* are homomorphisms induced by inclusion maps. Denote now the restrictions of the covering $\mathscr{F}$ of E with F_0 removed to E_0 and E_1 by $\mathscr{F}_0$ and

$\mathscr{F}_1$ and their nerve complexes by K_0 and K_1 respectively, both considered as subcomplexes of the nerve complex K of the covering $\mathscr{F}$. Let the vertex in K corresponding to F_0 in $\mathscr{F}$ be a_0, and denote the join of a_0 and K_0 by K_2. Then K_0 is the intersection of K_1, K_2 while K is their union. As K_2 is homologically trivial, the Mayer-Vietoris exact sequence of the complex-triad $(K; K_1, K_2)$ reduces to

$$\cdots \xrightarrow{i_1^*} H^{r-1}(K_1) \xrightarrow{i_0^*} H^{r-1}(K_0) \xrightarrow{\delta^*} H^r(K) \xrightarrow{i_1^*} H^r(K_1) \xrightarrow{i_0^*} H^r(K_0) \rightarrow \cdots$$

in which δ^* is the coboundary homomorphism and i_0^*, i_1^* are homomorphisms induced by inclusion maps. Consider now the diagram below:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{r-1}(E_1) \arrow[r, "i_0^*"] & H^{r-1}(E_0) \arrow[r, "\delta^*"] & H^r(\mathrm{E}) \arrow[r, "i_1^*"] & H^r(E_1) \arrow[r, "i_0^*"] & H^r(E_0) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{r-1}(K_1) \arrow[r, "i_0^*"] \arrow[u, "\Phi_1^*"] & H^{r-1}(K_0) \arrow[r, "\delta^*"] \arrow[u, "\Phi_0^*"'] & H^r(K) \arrow[r, "i_1^*"] \arrow[u, "\Phi^*"'] & H^r(K_1) \arrow[r] \arrow[u, "\Phi_1^*"'] & H^r(K_0) \arrow[r] \arrow[u, "\Phi_0^*"'] & \cdots
\end{tikzcd}$$

in which $\Phi^*, \Phi_0^*, \Phi_1^*$ are the canonical homomorphisms associated with the corresponding coverings. Now the diagram is commutative by Lemma 1 in §1. As the numbers of sets in $\mathscr{F}_0$ and $\mathscr{F}_1$ are both $\leqslant n$, Φ_0^*, Φ_1^* are all isomorphisms by the induction hypothesis. By the "five lemma" Φ^* is then also an isomorphism.

Corollary. *Under the same hypothesis about* H^*, E *and* $\mathscr{F}$ *with the cohomology groups of each non-empty intersection of sets in* $\mathscr{F}$ *the same as those of a point in all dimensions,* $\mathrm{H}^*(|\mathrm{K}|)$ *will be ring isomorphic to* $\mathrm{H}^*(\mathrm{E})$ *when* H^* *admits a ring structure.*

Proof. This follows from the fact that $H^r(|K|) \approx H^r(E)$ are induced by a continuous mapping of E into $|K|$, namely, any canonical mapping associated with $\mathscr{F}$.

§ 3. Generalization to Fundamental Groups

Let E be an arcwise connected normal space and C_1, C_2 two arcwise connected open (resp. closed) subsets of E verifying the following conditions:

1°. The union of C_1 and C_2 is E.

2°. The intersection of C_1 and C_2 is the union of a finite number of disjoint open (resp. closed) sets B_j, $1 \leqslant j \leqslant m$, each of which is arcwise connected.

In the case that C_i, B_j are closed sets we shall make the further

Hypothesis (H). In E there are disjoint arcwise connected open sets U_j containing B_j such that each B_j is a deformation retract of U_j.

For such a triple $(E; C_1, C_2)$ let us take in each B_j a point O^j and let the reference points of the fundamental groups of B_j, C_1, C_2 and E be understood to be O_j, O_1, O_1 and O_1 respectively. For each $j \geqslant 1$ and $\leqslant m$ let us take a fixed path l_i^j in C_i from O_1 to O_j for which l_i^1 reduces to the point $O_1 (i=1, 2)$. The closed path $l_1^j (l_2^j)^{-1}$ represents then an element in $\pi_1(E)$ which will be denoted by β_j, $1 \leqslant j \leqslant m$, with $\beta_1 = 0$, $\beta_j \neq 0$ for $j > 1$. Furthermore, for each closed path l in B_j with end points at O_j, the correspondence $l \rightarrow l_i^j \cdot l \cdot (l_i^j)^{-1}$ induces a homomorphisms $\lambda_i^j: \pi_1(B_j) \rightarrow \pi_1(C_i)$, $i=1, 2$. Finally, the injections $C_i \subset E$ induces homomorphisms $\pi_1(C_i) \rightarrow \pi_1(E)$ which will be denoted by h_i^*, $i=1, 2$.

The following lemma is now a slightly modified form of a theorem due to Van Kampen[4] for which we omit the proof. (Cf. in this respect also a paper of P. Olum[6].)

Lemma. *Under the above hypothesis the structure of* $\pi_1(\mathrm{E})$ *is completely determined by* $\pi_1(\mathrm{C}_1), \pi_1(\mathrm{C}_2)$ *and the homomorphisms* λ_i^j. *More precisely, we have*

(i) *If* $\{\alpha_k^1\}$ *and* $\{\alpha_l^2\}$ *are systems of generators of* $\pi_1(\mathrm{C}_1)$ *and* $\pi_1(\mathrm{C}_2)$ *respectively, then* $\pi_1(\mathrm{E})$ *has a system of generators* $\{\mathrm{h}_1^*(\alpha_k^1), \mathrm{h}_2^*(\alpha_l^2), \beta_j\}$.

(ii) *If* $\mathrm{R}_s^1(\alpha_k^1) = 1$ *and* $\mathrm{R}_t^2(\alpha_l^2) = 1$ *are complete systems of relations for the above systems of generators of* $\pi_1(\mathrm{C}_1)$ *and* $\pi_1(\mathrm{C}_2)$ *respectively, then a complete system of relations for the above system of generators of* $\pi_1(\mathrm{E})$ *is given by*

$$R_s^1(h_1^*(\alpha_k^1)) = 1,$$
$$R_t^2(h_2^*(\alpha_l^2)) = 1,$$

and

$$\bar{\lambda}_1^j(r_{jt}) \cdot \beta_j = \beta_j \cdot \bar{\lambda}_2^j(r_{jt}), \qquad 1 \leqslant j \leqslant m,$$

where $\{\mathrm{r}_{jt}\}$ *are systems of generators of* $\pi_1(\mathrm{B}_j)$, $1 \leqslant \mathrm{j} \leqslant \mathrm{m}$, *and* $\bar{\lambda}_i^j = h_i^* \lambda_i^j$, $\mathrm{i} = 1, 2$.

Theorem 2. *If* $\mathscr{U} = \{\mathrm{U}_\lambda\}$ *is a finite open covering of a normal connected space* E *such that each non-empty intersection* $\mathrm{U}_{\lambda_1} \cdots \mathrm{U}_{\lambda_r} \neq \emptyset$ *of sets in* $\mathscr{U}$ *is arcwise connected and simply connected, then* $\pi_1(\mathrm{E})$ *is isomorphic to* $\pi_1(|\mathrm{K}|)$ *where* K *is the nerve complex of* E. *Moreover, the isomorphism may be realized in the following manner. Consider any continuous mapping* φ *homotopic to a canonical mapping* $\Phi: \mathrm{E} \rightarrow |\mathrm{K}|$ *associated with the covering* $\mathscr{U}$ *and a system of continuous functions* f_λ *as in* §1. *Let* O *be any point in* E *and let* $\pi_1(\mathrm{E})$ *and* $\pi_1(|\mathrm{K}|)$ *be referred to* O *and* $\varphi(\mathrm{O})$ *respectively. Then* φ *induces such an isomorphism:*

$$\pi_1(E) \approx \pi_1(|K|).$$

Proof. The theorem is trivial when the number of sets in $\mathscr{U}$ is $n=1$. Suppose that the number of sets in $\mathscr{U}$ is $n>1$ and the theorem has been proved for $n-1$ or less. Let $C_1=U_1$, say, be a set in $\mathscr{U}$ such that $C_2=\sum_{i=2}^{n} U_i$ is arcwise connected. The intersection of C_1 and C_2 is then the union of a finite number of arcwise connected components B_j. The restriction $\mathscr{U}_j$ of $\mathscr{U}'=\{U_2,\cdots,U_n\}$ to B_j is an open covering of B_j verifying clearly the hypothesis of the theorem. Let a be the vertex in K corresponding to U_1 and L_j, K_2 the subcomplexes of K which are nerve complexes of $\mathscr{U}_j$ and the covering $\mathscr{U}'=\{U_2,\cdots,U_n\}$ of C_2. Then the join K_1 of a and the union of L_j may be considered as the nerve complex of the covering of $C_1=U_1$ which is the restriction of $\mathscr{U}$ to C_1. Remark that the subcomplexes L_j are connected and disjoint from each other since any non-empty intersection of sets in $\mathscr{U}$ is arcwise connected.

As in Lemma 2 of §1 we shall construct a mapping φ homotopic to a canonical mapping $\bar{\varphi}$ of E into $|K|$ such that $\varphi(B_j)\subset|L_j|$, and $\varphi(C_i)\subset|K_i|$. Take O_j in B_j and let us base the fundamental groups $\pi_1(B_j)$, $\pi_1(C_i)$ and $\pi_1(E)$ at the reference points O_j, O_1 and O_1 respectively. Similarly the fundamental groups $\pi_1(|L_j|)$, $\pi_1(|K_i|)$ and $\pi_1(|K|)$ will be based at $\varphi(O_j)$, $\varphi(O_1)$ and $\varphi(O_1)$ respectively. In C_i let us take paths l_j^i from O_1 to O_j with l_1^i reduced to point O_1 and define homomorphisms $\lambda_j^i\colon \pi_1(B_j)\to\pi_1(C_i)$ as before. The paths $\varphi(l_j^i)$ are then in $|K_i|$ running from $\varphi(O_1)$ to $\varphi(O_j)$ with $\varphi(l_1^i)$ reduced to $\varphi(O_1)$ which permits to define homomorphisms $\mu_j^i\colon \pi_1(|L_j|)\to\pi_1(|K_i|)$. Let $h_i^*\colon \pi_1(C_i)\to\pi_1(E)$ and $k_i^*\colon \pi_1(|K_i|)\to\pi_1(|K|)$ be homomorphisms induced by the injections $h_i\colon C_i\subset E$ and $k_i\colon |K_i|\subset|K|$ respectively, and let φ_* be the homomorphisms of corresponding fundamental groups induced by φ. The following diagram is now commutative, which follows from Lemma 2 of §1 for $i=2$ and is trivial for $i=1$ since $\pi_1(C_1)=\pi_1(|K_1|)=0$:

$$\begin{array}{ccccc}
\pi_1(B_j) & \xrightarrow{\lambda_j^i} & \pi_1(C_i) & \xrightarrow{h_i^*} & \pi_1(E) \\
\uparrow{\scriptstyle\varphi_*} & & \uparrow{\scriptstyle\varphi_*} & & \uparrow{\scriptstyle\varphi_*} \\
\pi_1(|L_j|) & \xrightarrow{\mu_j^i} & \pi_1(|K_i|) & \xrightarrow{k_i^*} & \pi_1(|K|).
\end{array}$$

By inductive hypothesis we have now

$$\varphi_*\colon\ \pi_1(|L_j|)\approx\pi_1(B_j),$$
$$\varphi_*\colon\ \pi_1(|K_2|)\approx\pi_1(C_2),$$

and trivially also

$$\pi_1(|K_1|)\approx\pi_1(C_1).$$

For the space $|K|$ of K we have $|K|=|K_1|\cup|K_2|$, $|K_1|\cdot|K_2|=\sum|L_j|$ and the triple $(|K|;|K_1|,|K_2|)$ verifies clearly the hypothesis (H). Applying the Lemma to the triples $(|K|;|K_1|,|K_2|)$ and $(E;C_1,C_2)$ and comparing, we get then $\pi_1(|K|)\approx\pi_1(E)$. The choice of base point O is clearly immaterial.

Remark. An alternative treatment without using Van Kampen's theorem, which works also in the case of higher homotopy groups, is given in the next section.

§4. Generalization to Homotopy Groups

As there is no analogue in the case of higher homotopy groups as the Mayer-Vietoris sequence in the case of homology groups or the theorem of Van Kampen in the case of fundamental groups, we require some modification of the techniques for the generalization of the theorem of Leray to higher homotopy groups. We start from the following simple case.

Lemma 1. *Let* $\{U_1, U_2\}$ *be an open covering of a space* E. *Suppose* U_1, U_2 *and* $U_1\cdot U_2$ *are all r–connected, i.e., arcwise connected with* $\pi_1=\pi_2=\cdots=\pi_r=0$. *Then* E *is also r–connected.*

Proof. Let f be a continuous mapping of an s–sphere S, $s\leq r$, in E. Triangulate S so that for each simplex σ of S the image $f(|\sigma|)$ is in one of the three open sets, U_1, U_2 or $U_1\cdot U_2$. Take a fixed point O in $U_1\cdot U_2$ and define a homotopy $h\colon S\times[0,1]\to E$ of $f\equiv h/S\times(0)$ to the constant map $h/S\times(1)=O$ as follows. For each vertex a of S, $h((a)\times[0,1])$ will be a path from $f(a)$ to O which lies wholly in $U_1\cdot U_2$ if $f(a)\in U_1\cdot U_2$, and lies wholly in U_1 (resp. U_2) if $f(a)\in U_1-U_1\cdot U_2$ (resp. $U_2-U_1\cdot U_2$). Suppose that h has been defined over $S^k\times[0,1]$, where S^k denotes the k–squelette of S, $k<s$, such that for each k–simplex σ^k of S, $h(|\sigma^k|\times[0,1])$ lies wholly in $U_1\cdot U_2$ if $f(|\sigma^k|)\subset U_1\cdot U_2$, or if not, lies wholly in U_1 or U_2. For any $(k+1)$–simplex σ^{k+1} of S, h has then been defined over the boundary of $\sigma^{k+1}\times[0,1]$ and the image lies wholly in $U_1\cdot U_2$ if $f(|\sigma^{k+1}|)\subset U_1\cdot U_2$, or if not, lies wholly in U_1 or U_2. As π_{k+1} of all these sets $=0$, h can be extended to $\sigma^{k+1}\times[0,1]$ with image lying wholly in $U_1\cdot U_2$, or U_1, or U_2. By induction we get thus the required homotopy $h\colon S\times[0,1]\to E$ which shows that $\pi_s(E)=0$.

Lemma 2. *Let* $\mathscr{U}=\{U_0, U_1,\cdots,U_n\}$ *be an open covering of a space* E *such that*

(i) $U_i\cdot U_0\neq\emptyset$, $i\neq0$;

(ii) *if* $U_{i_1}\cdot U_{i_j}\neq\emptyset$, $i,j=1,\cdots,s$, *then* $U_{i_1}\cdots U_{i_s}\neq\emptyset$ $(\lambda_i=0,1,\cdots,n)$; *and*

(iii) *any intersection* $U_{\lambda_1}\cdots U_{\lambda_s}(\lambda_i=0,1,\cdots,n)$, *if non-empty, is* r*-connected.*

Then E *is* r*-connected, and the same is also true for the nerve complex* K *of* $\mathscr{U}$.

Proof. When $n=1$, this follows from Lemma 1. Suppose our lemma is true for $n-1$ or less. Then $E_0=\sum_{i=0}^{n-1} U_i$ is r-connected. The space $E'=E_0\cdot U_n$ has an open covering $\mathscr{U}'$ consisting of non-empty sets among $U'_0,\cdots,U'_{n-1}$, where $U'_i=U_i\cdot U_n$. This covering has evidently properties analogous to (i), (ii), (iii) and hence by the induction hypothesis E' is r-connected. Applying Lemma 1 to the space E with the covering $\{E_0, U_n\}$, we see that E is r-connected.

For the nerve complex K let K' be the nerve complex of the covering $\mathscr{U}'$ which may be identified with a subcomplex of K. Similarly the nerve complex K_0 of the covering $\{U_0, U_1,\cdots,U_{n-1}\}$ of E_0 may be identified with a subcomplex of K which contains in turn K' as a subcomplex. Let a_n be the vertex of K corresponding to U_n. Then K is the union of the two subcomplexes K_0 and a_nK' whose intersection is the subcomplex K'. By the induction hypothesis both K' and K_0 are r-connected. This is also trivially true for the complex a_nK'. Now $|a_nK'|$ and $|K_0|$ have neighbourhoods V_n and V_0 in $|K|$ such that V_n, V_0 and $V_n\cdot V_0$ have respectively $|a_nK'|$, $|K_0|$, and $|K'|$ as deformation retracts. By Lemma 1 the space $V_0+V_n=|K|$ is then r-connected. This proves the second assertion of our lemma.

For any finite simplicial complex K let $Cl\sigma$ be the closed subcomplex of K determined by $\sigma\in K$, $St\,\sigma$ the open subcomplex consisting of simplexes having σ as a face and $\overline{St}\,\sigma$ the closed subcomplex consisting of all simplexes as well as their faces in $St\,\sigma$. For any closed subcomplex L of K the union of all $St\,\sigma$ with $\sigma\in L$ will be denoted by $St\,L$. For any open subcomplex L of K, $K-L$ is a closed subcomplex and the set $|K|-|K-L|$ will be denoted simply by $|L|$. We have then

Lemma 3. *For any closed subcomplex* L *of* K *let* f *be a continuous mapping of* $|L|$ *in* $|K|$ *such that for any* $\sigma\in L$ *and* $x\in|\sigma|$, *we have* $f(x)\in|St Cl\,\sigma|$. *Then* f *is homotopic to the identity mapping of* $|L|$.

Proof. For any $\sigma\in L$, the sets $|St\,\tau|$ with $\tau\prec\sigma$ form an open covering of $|St\,Cl\,\sigma|$ verifying evidently the conditions of Lemma 2 for any r ($|St\,\sigma|$ plays the role of U_0) so that $|St\,Cl\,\sigma|$ is r-connected for any r. Define now a homotopy h: $|L|\times[0,1]\to|K|$ between $f\equiv h/|L|\times(0)$ and $h/|L|\times(1)\equiv$ identity map of $|L|$ as follows. Suppose that h has been defined over $|L^k|$ with $h(|\tau|\times[0,1])\subset|St\,Cl\,\tau|$ for any $\tau\in L^k$. Consider any $(k+1)$-simplex $\sigma\in L$. Then h

has been defined over the boundary of $|\sigma|\times[0,1]$ with image $\subset|St\,Cl\,\sigma|$ and can thus be extended to $|\sigma|\times[0,1]$ with image $\subset|St\,Cl\,\sigma|$ owing to the $(k+1)$-connectedness of the latter set. By induction we arrive thus at the required homotopy h and therefore $f\simeq$ identity.

Consider now an arbitrary finite covering $\mathscr{U}=\{U_\lambda\}$ of a space E with nerve complex K and vertices a_λ corresponding to U_λ. For any simplex $\sigma=(a_{\lambda_0}\cdots a_{\lambda_r})$ of K the sets $U_{\lambda_0}\cdots U_{\lambda_r}$ and $U_{\lambda_0}+\cdots+U_{\lambda_r}$ will be called the *support* and the *cover* of σ and will be denoted by Sup σ and Cov σ respectively. For any closed subcomplex L of K the union of all Cov σ with $\sigma\in L$ will then be denoted by Cov L. It is evident that if τ is a face of $\sigma\in K$: $\tau<\sigma$, then Sup $\tau\supset$ Sup σ while Cov $\tau\subset$ Cov σ. A continuous mapping f of a subcomplex L of K in E will be said to be *admissible* (*w.r.t.* $\mathscr{U}$) if for any $\sigma\in L$, $f(|\sigma|)\subset$ Cov σ. For any set $A\subset E$ let the totality of sets in $\mathscr{U}$ containing A, if there are any, be $U_{\lambda_0},\cdots,U_{\lambda_r}$, then the simplex $\sigma=(a_{\lambda_0}\cdots a_{\lambda_r})$ of K will be called the *carrier* of A and will be denoted by Car A. On the other hand the union of all sets U_λ in $\mathscr{U}$ for which $U_\lambda\cdot U_{\lambda_0}\cdots U_{\lambda_r}\neq\emptyset$ will be called the *expansion* of A and will be denoted by Exp A. It is clear that for any $A\subset B\subset E$ where B lies wholly in at least one of the sets of $\mathscr{U}$, we have Exp $A\subset$ Exp B, Car $B\prec$ Car A, $\overline{St}$ Car $A\subset\overline{St}$ Car B, and Cov $\sigma\subset$ Exp A for any $\sigma\in\overline{St}$ Car A. Let us call a covering $\mathscr{U}$ *star-disjoint* if for any finite number of sets $U_{\lambda_1},\cdots,U_{\lambda_s}$ of $\mathscr{U}$, $U_{\lambda_i}\cdot U_{\lambda_j}\neq\emptyset$, $i,j=1,\cdots,s$, implies always $U_{\lambda_1}\cdots U_{\lambda_s}\neq\emptyset$. For such a covering, $\overline{St}$ Car A is then easily seen to be the nerve complex of the covering of Exp A by the sets in $\mathscr{U}$ which have non-empty intersections with Sup Car A. Let us call $\mathscr{U}$ *r-connected* if any non-empty intersection of sets of $\mathscr{U}$ is r-connected. Then we have

Theorem 3. *If a normal connected space* E *has a finite open covering* $\mathscr{U}$ *which is star-disjoint and* r*-connected, then the space* E *has the same homotopy groups in the dimensions* $1,2,\cdots,r$ *as those of the nerve complex* K. *Moreover, the isomorphisms of homotopy groups may be realized by any canonical mapping* φ *of* E *in* $|K|$ *with respect to the covering* $\mathscr{U}$ *or any admissible mapping* ψ *of* $|K^{r+1}|$ *in* E *which necessarily exists.*

Proof. By Lemma 2 the set Cov σ for any $\sigma\in K$ is r-connected. Let us define now an admissible map ψ of $|K^{r+1}|$ in E as follows. For any vertex a_λ of K let us take any point $\bar{a}_\lambda$ in $U_\lambda=$ Cov (a_λ) as $\psi(a_\lambda)$. Suppose that ψ has been defined over $|K^k|$ with $k\leq r$ such that for each $\sigma\in K^k$, $\psi(|\sigma|)\subset$ Cov σ. Consider now any $(k+1)$-simplex σ of K. The map ψ has been defined over the boundary $\dot\sigma$ of σ with $\psi(|\dot\sigma|)\subset\sum_{\tau\in\dot\sigma}$ Cov $\tau\subset$ Cov σ which is r-connected. As $k\leq r$ the map ψ

can be extended to $|\sigma|$ with $\psi(|\sigma|)\subset \mathrm{Cov}\,\sigma$. This proves inductively the existence of ψ: $|K^{r+1}|\to E$ as asserted.

Next we shall prove that ψ and a canonical mapping φ with respect to the covering $\mathscr{U}$ induce both isomorphisms

$$\psi_*:\quad \pi_s(|K|)\approx \pi_s(E),$$
$$\varphi_*:\quad \pi_s(E)\approx \pi_s(|K|)$$

in dimensions $s\leqslant r$, which are moreover inverse of each other. (The reference points are, say, a vertex of K and its image by ψ.) The proof will be accomplished by showing that $\varphi_*\psi_*=$ identity and ψ_* is onto in the dimensions $s\leqslant r$. To see that $\varphi_*\psi_*=$ identity, let us consider any point x of a k-simplex $\sigma=(a_{\lambda_0}\cdots a_{\lambda_k})$ of K, $k\leqslant r+1$. Then $\psi(x)\in U_{\lambda_0}+\cdots+U_{\lambda_k}=\mathrm{Cov}\,\sigma$. Let $U_{\lambda i_1},\cdots,U_{\lambda i_a}(a\geqslant 1)$ and eventually $U_{\mu_1},\cdots,U_{\mu_b}$ be the totality of sets in $\mathscr{U}$ each of which contains $\psi(x)$. Then $\varphi\psi(x)$ is in the simplex $a_{\lambda i_1}\cdots a_{\lambda i_a}, a_{\mu_1}\cdots a_{\mu_b}$ but not on the face $a_{\mu_1}\cdots a_{\mu_b}$ so that $\varphi\psi(x)\in|St\,Cl\,\sigma|$. By Lemma 3 we have therefore $\varphi\psi/K^{r+1}\simeq$ identity so that $\varphi_*\psi_*=$ identity in the dimensions $s\leqslant r$, as asserted. To see that ψ_* is onto in dimensions $s\leqslant r$, let us consider any continuous mapping f of an s-sphere S in E where $s\leqslant r$. Triangulate S so that for each simplex ξ of S, $f(|\xi|)$ lies wholly in one of the open sets U_λ. We shall define a homotopy h: $S\times[0,1]\to E$ of $f\equiv h/S\times(0)$ to a map $g\equiv h/S\times(1)$ with $g=\psi g_0$, g_0: $S\to|K'|$ as follows. For any vertex b of S let U_λ be one of the sets in $\mathscr{U}$ containing $f(b)$, then $h((b)\times[0,1])$ will be a path lying wholly in U_λ which joins $f(b)$ to $\psi(a_\lambda)$. We set also $g_0(b)=a_\lambda$. [For the reference point b of S, a_λ will be taken to be the reference point of $\pi_s(|K|)$ and $h(b\times[0,1])$ will be the path reduced to the point $f(b)=\psi(a_\lambda)$.] Suppose that h over $S^k\times[0,1]$ and g_0 over S^k have been defined, verifying the following properties ($k\subset s$): for any k-simplex ξ of S we have (i) $h(|\xi|\times[0,1])\subset \mathrm{Exp}\,f(|\xi|)$, and (ii) $g_0(|\xi|)\subset|[\overline{St}\,\mathrm{Car}\,f(|\xi|)]^{r+1}|$. Consider now any $(k+1)$-simplex η of S. Then $g_0(|\dot\eta|)$ has been defined and $\subset\sum_{\zeta\in\dot\eta}|[\overline{St}\,\mathrm{Car}\,f(|\zeta|)]^{r+1}|\subset|[\overline{St}\,\mathrm{Car}\,f(|\eta|)]^{r+1}|$. Now $\overline{St}\,\mathrm{Car}\,f(|\eta|)$ is the nerve complex of the covering of $\mathrm{Exp}\,f(|\eta|)$ by sets in $\mathscr{U}$ having non-empty intersections with $\mathrm{Sup}\,\mathrm{Car}\,f(|\eta|)$. By Lemma 2 the complex $\overline{St}\,\mathrm{Car}\,f(|\eta|)$ is r-connected so that $g_0/|\dot\eta|$ may be extended to g_0: $|\eta|\to|[\overline{St}\,\mathrm{Car}\,f(|\eta|)]^{r+1}|$. It follows that $\psi g_0(|\eta|)\subset\psi(|[\overline{St}\,\mathrm{Car}\,f(|\eta|)]^{r+1}|)\subset \mathrm{Cov}\,[\overline{St}\,\mathrm{Car}\,f(|\eta|)]^{r+1}\subset \mathrm{Exp}\,f(|\eta|)$. Define $h/|\eta|\times(1)$ as $\psi g_0/|\eta|$, then h is defined on the boundary of $|\eta|\times[0,1]$ with its image contained in $\mathrm{Exp}\,f(|\eta|)$. It follows again by Lemma 2 that $\mathrm{Exp}\,f(|\eta|)$ is r-connected. Therefore h may be extended to $|\eta|\times[0,1]$ with its image $\subset \mathrm{Exp}\,f(|\eta|)$. Thus by induction we reach finally a map $h(S\times[0,1])$ which gives a

homotopy of f to a map ψg_0 where g_0: $S\to|K^{r+1}|$. This proves that ψ_* is onto in the dimensions $s\leqslant r$ as asserted, q.e.d.

Remark. For the homotopy groups there are no analogues to Theorem 1 concerning homology groups. In fact, let E be the 2-sphere and U_1, U_2 two slightly enlarged open hemispheres with $U_1\cdot U_2$ a zone about the equator. Then $U_1, U_2, U_1\cdot U_2$ all have π_{r-1} and $\pi_r=0$, when $r>2$. However $\pi_r(E)$ is known to be non-zero for $r=3, 4$, etc.

§5. Generalization to Homotopy Types

A connected compactum which is an ANR is called by J. H. C. Whitehead a CR-space. As proved by Whitehead, a mapping f of a CR-space X in a CR-space Y is a homotopy equivalence if f induces an isomorphism onto $\pi_n(X)\approx\pi_n(Y)$ in each dimension $n\geqslant 1$. (Cf. [5]). Combining with the preceding theorem, we get

Theorem 4. *If $\mathscr{U}$ is a finite star-disjoint open covering of a CR-space* E *such that each intersection of any number of sets in $\mathscr{U}$, if non-empty, is contractible in itself, then the space* E *has the same homotopy type as the nerve complex* K *of $\mathscr{U}$. Moreover, a homotopy equivalence may be realized by any canonical mapping φ:* E$\to$|K| *with respect to the covering $\mathscr{U}$ and any admissible mapping of* |K| *in* E, *which necessarily exists, is a homotopy inverse of φ.*

In what follows we shall give a modification of Theorem 4 using closed coverings instead of open coverings. For this purpose let us recall first that a regular separable space E is said to be an *absolute retract* if for any regular separable spaces A, B where A is a closed subset of B, any continuous mapping of A in E may be extended to B. The following theorem is due to Aronszajn and Borsuk (Cf. [1]):

Lemma 1. *If the regular separable space* E *is the union of two closed subsets* F_1 *and* F_2 *such that* F_1, F_2 *and* $F_1\cdot F_2$ *are all absolute retracts, then* E *is itself an absolute retract.*

The next Lemma is an analogue of Lemma 2 of §4.

Lemma 2. *Let* $\mathscr{F}=\{F_0, F_1,\cdots,F_n\}$ *be a closed covering of a regular separable space* E *such that*

(i) $F_i\cdot F_0\neq\emptyset$, $i\neq 0$;

(ii) *if* $F_{\lambda_i}\cdot F_{\lambda_j}\neq\emptyset$, i,j $=,\cdots 1, r$, *then* $F_{\lambda_1}\cdots F_{\lambda_r}\neq\emptyset$, ($\lambda_i=0,1,\cdots$, n); *and*

(iii) *any intersection* $F_{\lambda_1}\cdots F_{\lambda_k}$ *of sets in* $\mathscr{F}$, *if non-empty, is an absolute retract.*

Then E *is itself an abstract retract.*

Proof. Using induction with respect to *n*.

Theorem 4'. *If $\mathscr{F}=\{F_\lambda\}$ is a finite star-disjoint closed covering of a CR-space E such that any intersection of sets in $\mathscr{F}$, if non-empty, is an absolute retract, then the space E has the same homotopy type as the nerve complex K of $\mathscr{F}$. Moreover, a homotopy equivalence may be realized by any canonical mapping φ: E$\to$|K| with respect to the covering $\mathscr{F}$ and any admissible mapping of |K| in E, which necessarily exists, is a homotopy inverse of φ.*

Proof. As each absolute retract is contractible in itself, it follows that admissible mappings $\psi: |K| \to E$ with $\psi(|\sigma|) \subset \mathrm{Cov}\,\sigma$ exist as in the proof of Theorem 3.

Let φ be any canonical mapping of E in $|K|$ with respect to the closed covering $\mathscr{F}$. The proof of the theorem will be accomplished by showing that $\varphi\psi \simeq$ identity and $\psi\varphi \simeq$ identity.

To see the first statement, let us remark that for any point $a \in |\sigma|$, $\sigma=(a_{i_0}\cdots a_{i_r}) \in K$, we have $\psi(a)=\bar{a} \in \mathrm{Cov}\,\sigma$ so that $\varphi(\bar{a})=\varphi\psi(a) \in |St\, Cl\,\sigma|$. By Lemma 3 of §4 we have therefore $\varphi\psi \simeq$ identity.

To see the second, let us remark that for any point $\bar{a}$ of E, $\psi\varphi(\bar{a}) \in \mathrm{Exp}\,(\bar{a})$, which is an absolute retract by Lemma 2 (any set in $\mathscr{F}$ containing $\bar{a}$ will play the role of F_0 in Lemma 2). Let us construct now a homotopy $h: E\times[0,1]\to E$ with $h/E\times(0)\equiv\psi\varphi$, $h/E\times(1)\equiv$ identity map of E, and $h/((\bar{a})\times[0,1])\subset \mathrm{Exp}\,(\bar{a})$ for any $\bar{a}\in E$. Suppose that h has been constructed for all $(\bar{a})\times[0,1]$, $\bar{a}\in \mathrm{Sup}\,\tau$, $\tau \in K$, dim $\tau > k$ and satisfies the above conditions. Consider now any k-simplex σ of K. Let σ_j, $j=1,\cdots,s$, be the totality of $(k+1)$-simplexes of K having σ as a face. Then $\mathrm{Sup}\,\sigma \supset \sum \mathrm{Sup}\,\sigma_j$ and h has been defined over $\sum \mathrm{Sup}\,\sigma_j\times[0,1]+\mathrm{Sup}\,\sigma\times(0)+\mathrm{Sup}\,\sigma\times(1) \subset \mathrm{Sup}\,\sigma\times[0,1]$ with its image $\subset \mathrm{Exp}\,\mathrm{Sup}\,\sigma$ which is an absolute retract by Lemma 2. We can therefore extend h to $\mathrm{Sup}\,\sigma\times[0,1]$ with its image $\subset \mathrm{Exp}\,\mathrm{Sup}\,\sigma$. For any point $\bar{a}\in \mathrm{Sup}\,\sigma$ but not $\in \sum \mathrm{Sup}\,\sigma_j$ we have then Car $\bar{a}=\sigma$ so that $h((\bar{a})\times[0,1]) \subset \mathrm{Exp}\,\mathrm{Sup}\,\sigma = \mathrm{Exp}\,(\bar{a})$. By induction we get therefore the required homotopy h and $\psi\varphi \simeq$ identity as asserted.

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ORDER STATISTICS WITH INVARIANT MEAN AND VARIANCE

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§1. Introduction

One of the most important and fundamental problems of mathematical statistics has been to define a satisfactory test for "goodness of fit". Such tests have been developed to study the agreement between a set of actual observations and some hypothetical probability distribution. The tests also allow us to explore whether a random phenomenon observed in the real world may be described by some underlying statistical law.

The form of statistical problem treated in this article is as follows. Given a random variable ξ with an unknown distribution function. Let

$$x_1, x_2, \cdots, x_n \tag{1}$$

be the outcome of n independent observations of the random variable ξ (i.e., a simple random sample). Let $F(x)$ be some completely specified distribution function. Given a set of sample values (1), we want to investigate whether the variable ξ does really follow the given distribution function $F(x)$. The classical question has been, "What method should we use to investigate this problem?"

In order to answer the proposed question, we first set up the hypothesis H: "the variable ξ follows the given distribution function $F(x)$." We would like to see if this hypothesis is correct or not. Secondly, we choose a suitable statistic

$$\alpha_n = \alpha[x_1, x_2, \cdots, x_n; F(x)]$$

to measure the discrepancy of sample (1) and values from the hypothetical distribution function $F(x)$. In general, α_n is nonnegative. If the hypothesis H is true, the random fluctuation in α_n will force us to consider the sampling distribution function of α_n, say $G_{\alpha_n}(y)$. If the hypothesis H is not true, α_n will usually tend to have a larger value than if H is true. Thus the following approach seems to be reasonable: Let $G_{\alpha_n}[z(\epsilon)] = 1-\epsilon$, i.e., if H is true $P\{\alpha_n \geq z(\epsilon)\} = \epsilon$ where ϵ is a predetermined number (e.g. = .01 or .05). According to the given sample values (1) and the given distribution function $F(x)$, if $\alpha_n \geq z(\epsilon)$, we will reject the hypothesis H at the significant level ϵ; if $\alpha_n < z(\epsilon)$ we say that the hypothesis H will not be rejected as the result of testing α_n. The significance level ϵ is the probability of rejecting H when H is true.

We will discuss the sampling distribution of α_n when H is true (i.e., $F(x)$ is the distribution of random variable ξ). In many cases, although it is difficult to calculate the sampling distribution $G_{\alpha_n}(y)$, the limiting distribution of $G_{\alpha_n}(y)$, say $G_\alpha(y)$ can be readily found. If the error of approximation is considered to be small, we may use $G_\alpha(y)$ in place of the distribution $G_{\alpha_n}(y)$. In what follows we shall treat separately the formation of test statistics for discrete and continuous cases.

When ξ is a discrete random variable, we have $P\{\xi = a_j\} = p_j$, $p_j > 0$, $\sum_j p_j = 1$. If the number of

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