Geometric complexes and their topological properties

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TopMath–Talk
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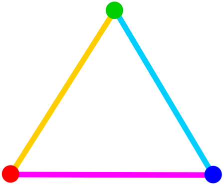
The Alexandroff nerve (1928)

Definition. Let X be a topological space, and let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \{J \subseteq I \mid |J| < \infty \text{ and } U_J := \bigcap_{i \in J} U_i \neq \emptyset\}$$

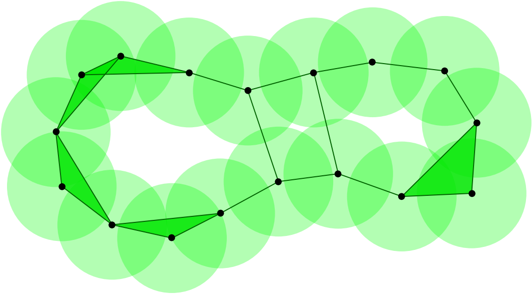


The Alexandroff nerve (1928)

Čech complex of a point cloud

Definition. The *Čech complex* of a subset $X \subseteq \mathbb{R}^d$ is the nerve of the cover by closed balls of radius r centered at points in X

$$\check{\mathrm{C}}\mathrm{ech}_r(X) = \mathrm{Nrv}((D_r(X))_{x \in X}))$$



The history of the nerve theorem

Theorem (Borsuk 1948, and many more). Let $\mathcal{U}$ be a nice cover of X. Then $\mathrm{Nrv}(\mathcal{U})$ is homotopy equivalent to X.

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- Čech 1932: Extends Alexandroff's ideas $\rightarrow$ Čech (co)homology

Nerve theorem for closed convex sets

Theorem. If $X \subset \mathbb{R}^d$, and $\mathcal{A} = (C_i)_{i \in [n]}$ is a cover by closed convex subsets, then $\mathrm{Nrv}(\mathcal{A})$ is homotopy equivalent to X.

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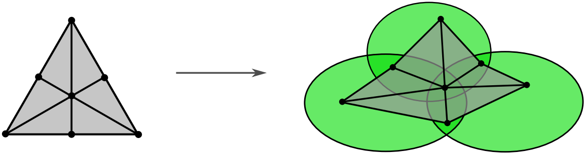
Proof strategy (R):

- Construct piecewise linear $\Gamma\colon\ \mathrm{Sd}\,\mathrm{Nrv}(\mathcal{A}) \to X$



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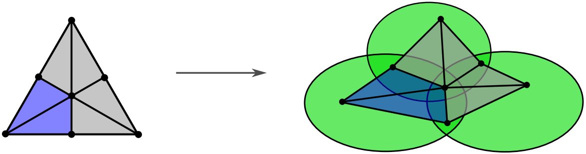

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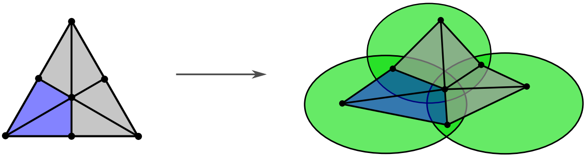

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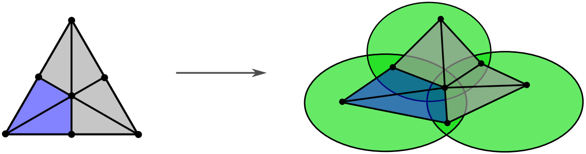

- Construct $\Phi\colon X \to \mathrm{Nrv}(\mathcal{A})$ using a partition of unity subordinate to an open thickening of the C_i with $\Phi(C_i) \subseteq \mathrm{bst}\, v_i$.
- Show that Φ is a homotopy inverse to Γ.

Nerve theorem for closed convex sets

Some proof details

- Take a collection of open sets $\mathcal{G} = (U_i \supseteq C_i)_{i \in [n]}$ satisfying $\mathrm{Nrv}(\mathcal{A}) = \mathrm{Nrv}(\mathcal{G}_\varepsilon)$.

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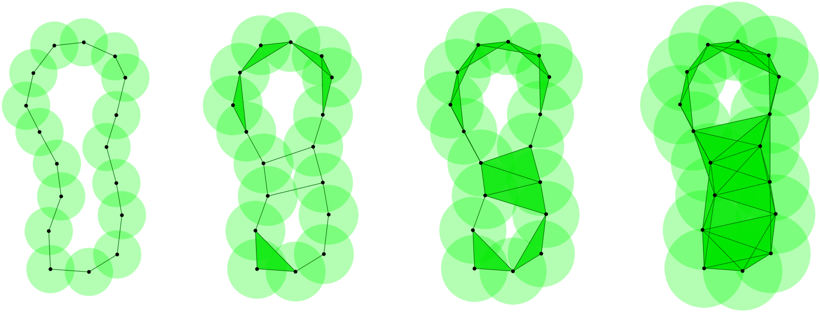
Functorial nerve theorem

Persistent homology



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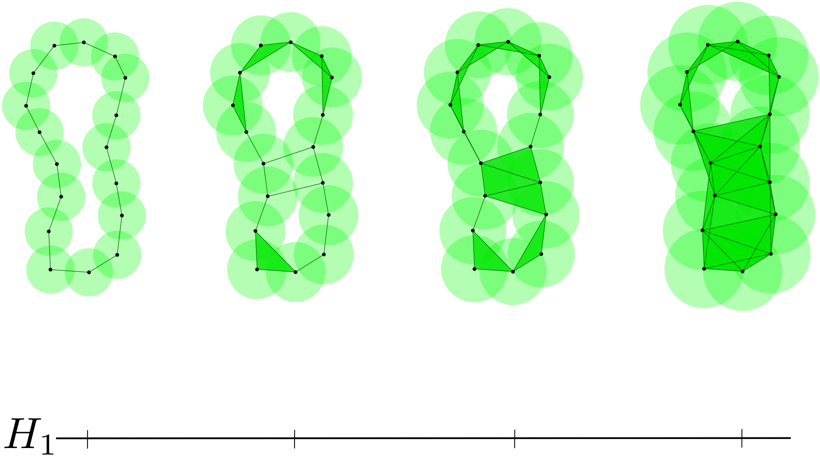

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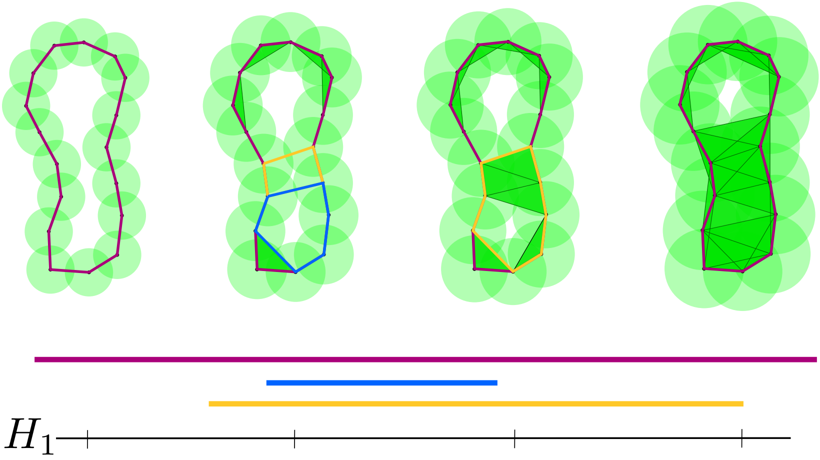

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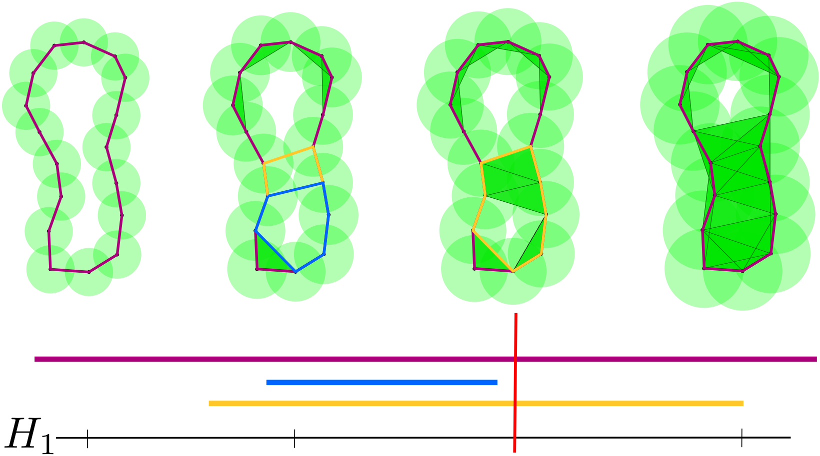

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Let $X \subseteq \mathbb{R}^d$, $\mathcal{U}_r = \{D_r(x)\}_{x \in X}$, and $X_r = \bigcup \mathcal{U}_r$. The nerve theorem guarantees for $r \leq l$ the homotopy equivalences

$$\begin{tikzcd} \mathrm{Nrv}(\mathcal{U}_r) & \mathrm{Nrv}(\mathcal{U}_l) \\ X_r \arrow[u, "\simeq"] & X_l \arrow[u, "\simeq"'] \end{tikzcd}$$

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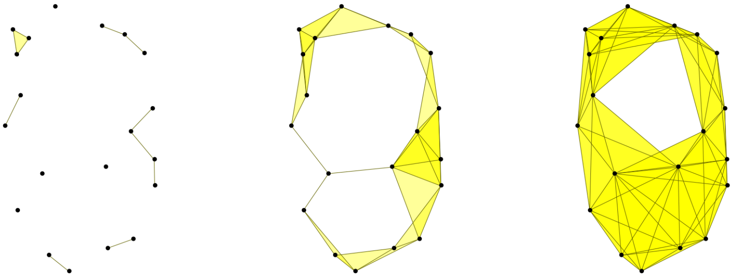
For an extensive treatment of functorial nerve theorems dealing with open and closed covers see

U. Bauer, M. Kerber, F. Roll, and A. Rolle
A Unified View on the Functorial Nerve Theorem and its Variations
Preprint, arXiv:2203.03571, 2022.

The Vietoris–Rips complex (1927, 1987)

Definition. Let X be a metric space. The Vietoris-Rips complex at scale r is the simplicial complex

$$\mathrm{Rips}_r(X) = \{S \subseteq X \text{ finite} \mid S \neq \emptyset,\ \operatorname{diam} S \leq r\}.$$



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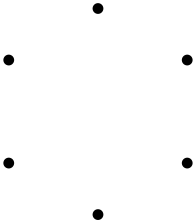
Theorem (Latschev, 2001). Let X be a closed Riemannian manifold. For small enough $r, \delta > 0$ and any metric space Y with $d_{GH}(X, Y) < \delta$:

$$\mathrm{Rips}_r(Y) \simeq X$$

The Vietoris–Rips complex (1927, 1987)

The circle S^1



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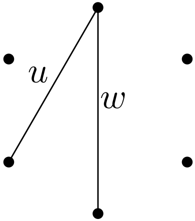

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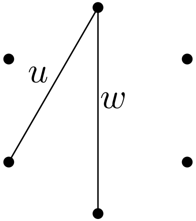

For $u \leq r < w$:

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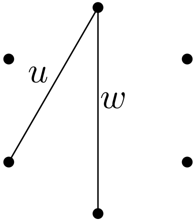

For $u \leq r < w$:

$$|\operatorname{Rips}_r(X)| \simeq S^2$$

The Vietoris–Rips complex (1927, 1987)

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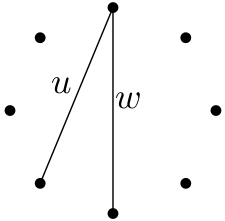

For $u \le r < w$:

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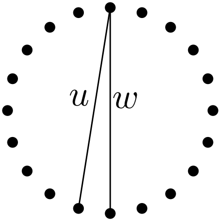
The Vietoris–Rips complex (1927, 1987)

The circle S^1



For $u \leq r < w$:

$$|\operatorname{Rips}_r(X)| \simeq S^{9}$$

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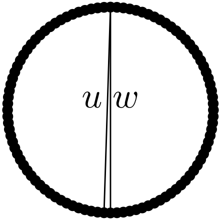

For $u \leq r < w$:

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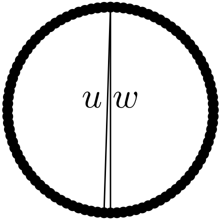

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Theorem (Adamaszek, Adams 2015). For $l = 0, 1, \ldots$ there are homotopy equivalences

$$\operatorname{Rips}_r(S^1) \simeq \begin{cases} S^{2l+1} & \text{if } \frac{l}{2l+1} < r < \frac{l+1}{2l+3}, \\ \bigvee^{\mathfrak{c}} S^{2l} & \text{if } r = \frac{l}{2l+1}. \end{cases}$$

Application of Vietoris–Rips persistent homology

COVID-19 genetic evolution data

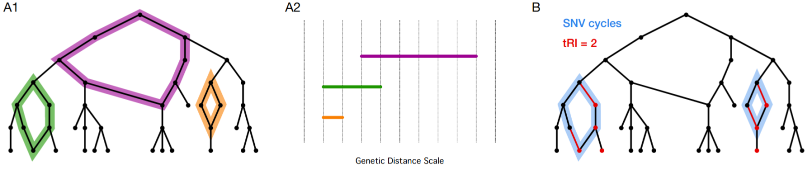

Figure 2. Topological data analysis quantifies convergent evolution. (A) Persistent homology detects reticulate events in viral evolution by means of a persistence barcode. Each bar in the barcode (A2) corresponds to a topological cycle in the reticulate phylogeny (A1). Bars at small genetic distance scales are expected to correspond mainly to homoplasies, while recombination events typically produce topological features at larger scales. (B) SNV cycles are topological cycles in the reticulate phylogeny for which adjacent sequences differ

M. Bleher, L. Hahn, J. A. Patino-Galindo, M. Carriere, U. Bauer, R. Rabadan, and A. Ott

Topological data analysis identifies emerging adaptive mutations in SARS-CoV-2

Preprint, arXiv:2106.07292, 2021

Application of Vietoris–Rips persistent homology

COVID-19 genetic evolution data

5000 ordered points	time for H_0 & H_1
random in $\mathbb{R}^3$	1m 17s
random on S^2	5m 39s
graph (covid data)	12s

U. Bauer
Ripser: efficient computation of Vietoris–Rips persistence barcodes
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graph (covid data, reversed order)	6m 19s
graph (covid data, random order)	2m 52s

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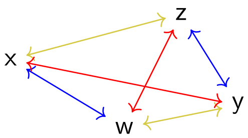
Rips contractibility lemma

Gromov-hyperbolicity

Definition. A metric space X is (Gromov) δ-*hyperbolic* if for all four points $w, x, y, z \in X$

$$d(x,w) + d(y,z) \leq \max\{d(x,y) + d(z,w), d(x,z) + d(y,w)\} + 2\delta$$



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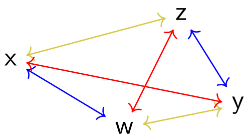

Example. finite metric space, trees are 0-hyperbolic, hyperbolic plane, ...

Rips contractibility lemma

Theorem (Rips, Gromov 1987). Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

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Any hyperbolic group G

- is finitely generated and finitely presented.
- admits an Eilenberg–MacLane space $K(G,1)$ with finitely many cells in each dimension.

Rips contractibility lemma

Theorem (Rips, Gromov 1987). Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

Any hyperbolic group G

- is finitely generated and finitely presented.
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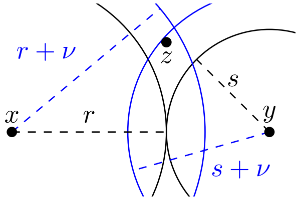
We address two questions:

1. What about non-geodesic spaces? Finite metric spaces?
2. Connection to Ripser?

Generalized contractibility lemma

The geodesic defect

Definition (Bonk, Schramm 2000). The metric space X is ν-*geodesic* if for all $x, y \in X$ and $r, s \geq 0$ with $r + s = d(x, y)$ there exists $z \in X$:



Generalized contractibility lemma

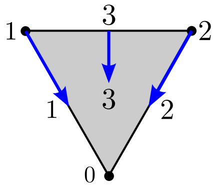
Theorem (Bauer, R 2021). Let X be a finite δ-hyperbolic metric space. Then there exists a discrete gradient encoding the collapses

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}$$

for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect of X

U. Bauer, F. Roll
Gromov hyperbolicity, geodesic defect, and apparent pairs in Vietoris-Rips filtrations
Accepted to SoCG 2022, Extended version on arXiv:2112.06781, 2022

Discrete Morse theory



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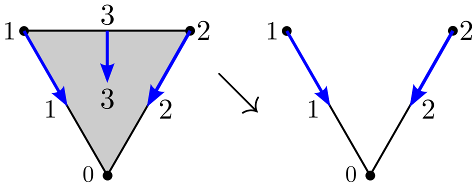

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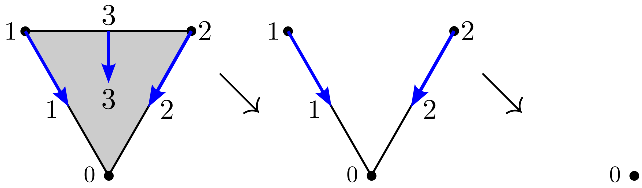

Discrete Morse theory

Apparent Pairs

Ripser uses the following construction for a computational shortcut:

Definition. In a simplexwise filtration $(K_i = \{\sigma_1, \ldots, \sigma_i\})_i$, a pair of simplices (σ, τ) is an apparent pair if

- σ latest proper face of τ, and
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Discrete Morse theory

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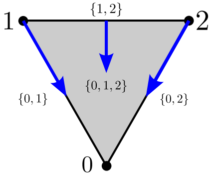
Lemma. The apparent pairs form a discrete gradient.



Collapsing Rips complexes of trees

Let X be the path length metric for a weighted tree $T = (V, E)$.

- Choose a root and extend the tree order to a total order.

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- Explains Ripser's outstanding performance on genetic distances.

Future work

Funding: DFG (SFB/TRR 109 *Discretization in Geometry and Dynamics*)

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- Further investigate spaces that are not tree-like with an eye towards the apparent pairs gradient.

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