# Connecting Morse theory and persistent homology of geometric complexes

Fabian Roll (TUM)

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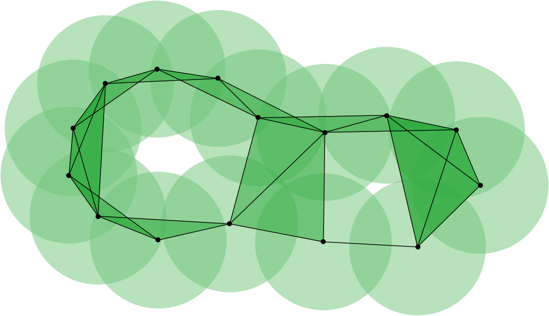
Minisymposium: Computational Topology and Geometry

Joint work with Ulrich Bauer

# Čech complexes

**Definition.** The *Čech complex* $\check{\mathrm{C}}\mathrm{ech}_r(X)$ of $X \subseteq \mathbb{R}^d$ is the simplicial complex

$$\check{\mathrm{C}}\mathrm{ech}_r(X) = \{J \subseteq X \mid |J| < \infty \text{ and } \bigcap_{y \in J} D_r(y) \neq \emptyset\}.$$



# Nerves

Definition (Alexandroff 1928). Let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of a space $X$. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \{J \subseteq I \mid |J| < \infty \text{ and } \bigcap_{i \in J} U_i \neq \emptyset\}$$

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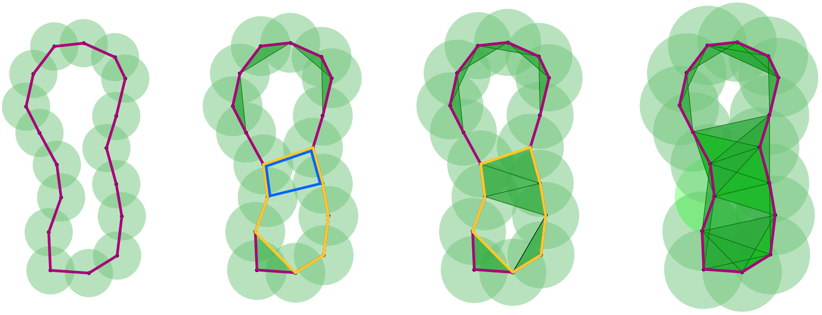
Here *nice* can mean different things:

- open, numerable cover, contractible intersections
- finite, closed, convex cover

U. Bauer, M. Kerber, F. Roll, and A. Rolle
A unified view on the functorial nerve theorem and its variations
Expositiones Mathematicae, 2023. doi:10.1016/j.exmath.2023.04.005

# Persistent homology



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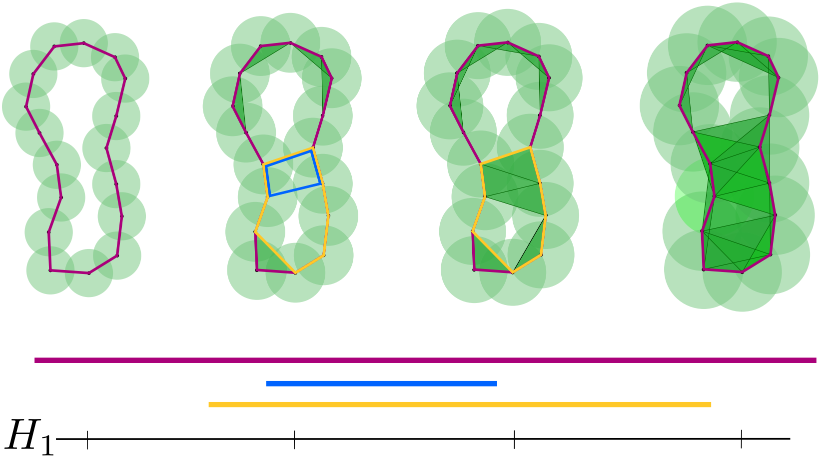

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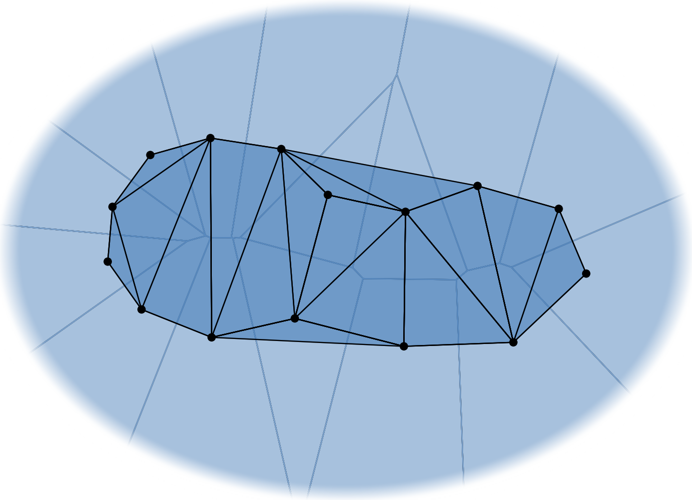
- determines the barcode through $\{[\operatorname{pivot} R_i, i) \mid R_i \neq 0\}$

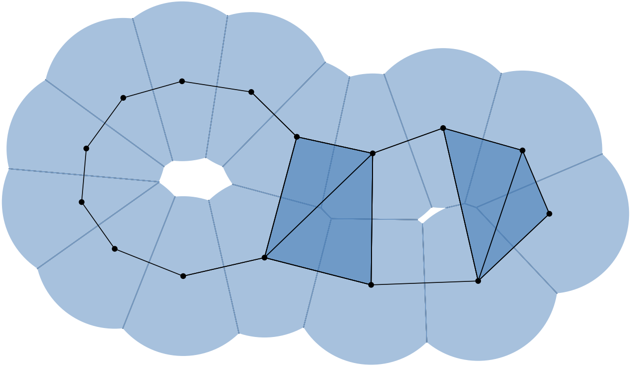
# Delaunay complexes

## Voronoi diagram and Delaunay triangulation

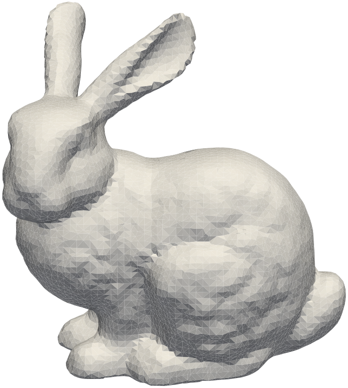


# Delaunay complexes

Definition. The *Delaunay complex* $\mathrm{Del}_r(X)$ of $X \subseteq \mathbb{R}^d$ is the nerve of the cover by closed Voronoi balls of radius $r$ centered at points in $X$.



Delaunay complex



Wrap complex

# Discrete Morse theory

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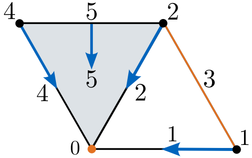
- monotone function $f\colon K \to \mathbb{R}$ that
- partitions the complex into pairs and critical simplices, yielding the *discrete gradient $V$*



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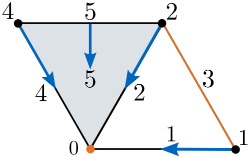


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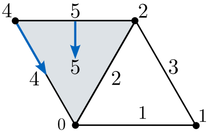

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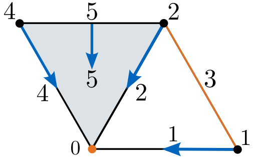


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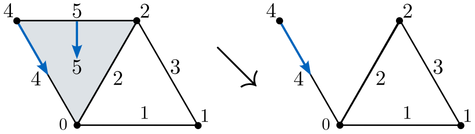

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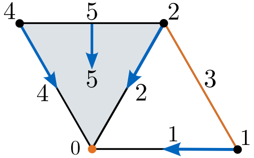


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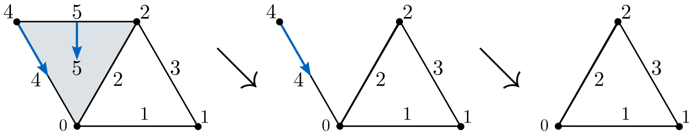
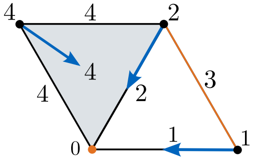


# Generalized discrete Morse theory

Generalized gradients consist of intervals (in the face poset) instead of just facet pairs:



# Morse Theory of Čech and Delaunay compelxes

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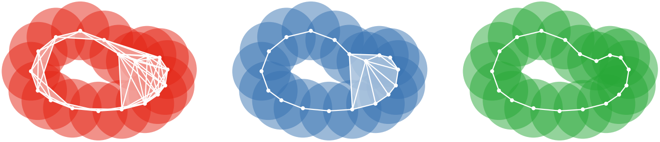
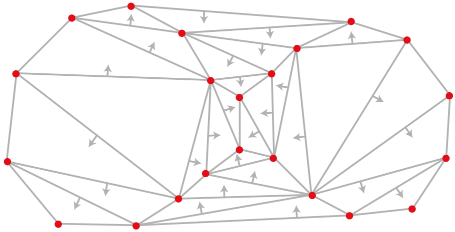
Theorem (Bauer, Edelsbrunner 2017).
Čech, Delaunay, and Wrap complexes (at any scale $r$) are related by collapses encoded by a single discrete gradient field:

$$\check{\mathrm{C}}\mathrm{ech}_r(X) \searrow \mathrm{Del}_r(X) \searrow \mathrm{Wrap}_r(X).$$



# Wrap complexes

Consider the gradient $V$ of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



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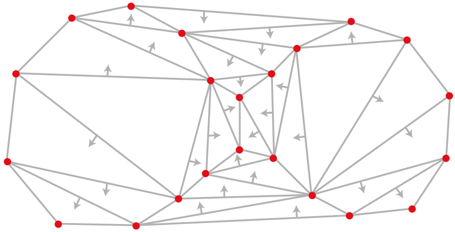

Definition (Edelsbrunner 1995; Bauer, Edelsbrunner 2017).
$\mathrm{Wrap}_r(X)$ is the *descending complex* of $V$ on $\mathrm{Del}_r(X)$: the smallest subcomplex of $\mathrm{Del}_r(X)$ that

- contains all critical simplices
- is a union of intervals of $V$.

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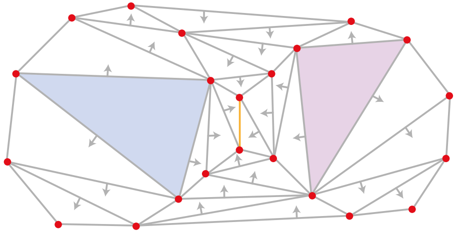

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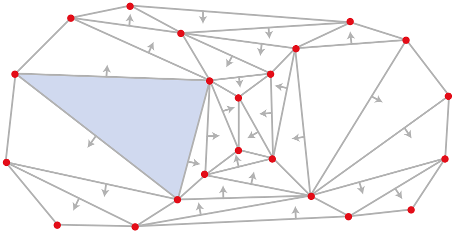

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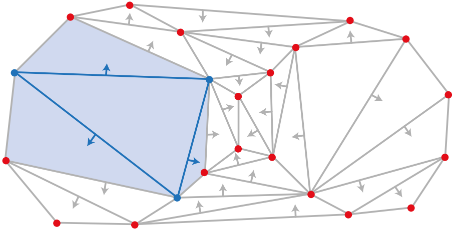

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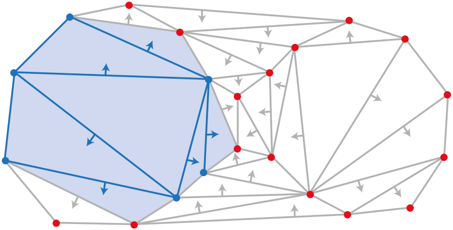

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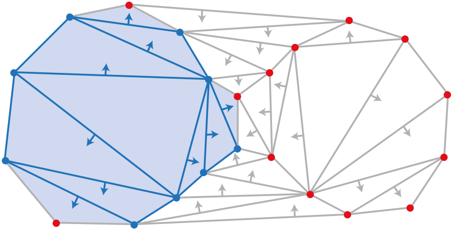

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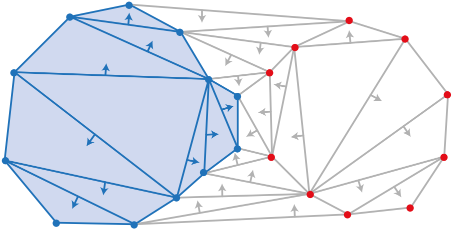

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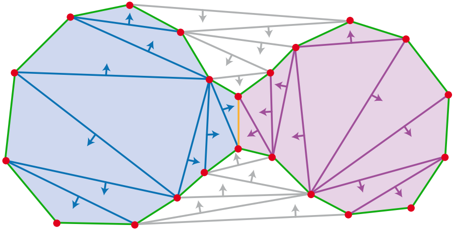

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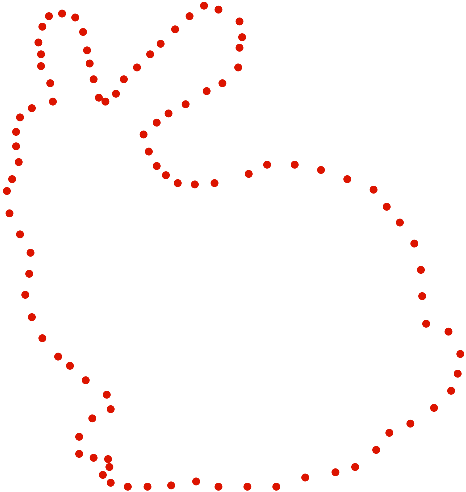

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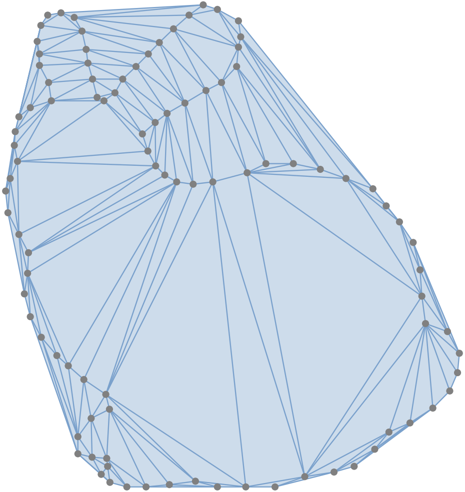

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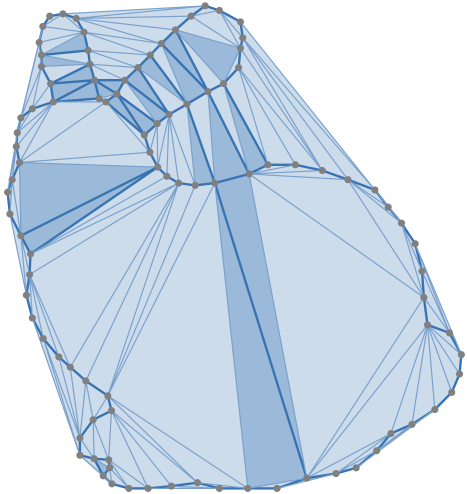

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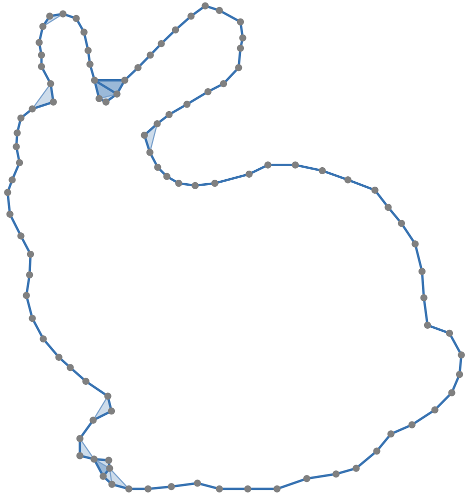

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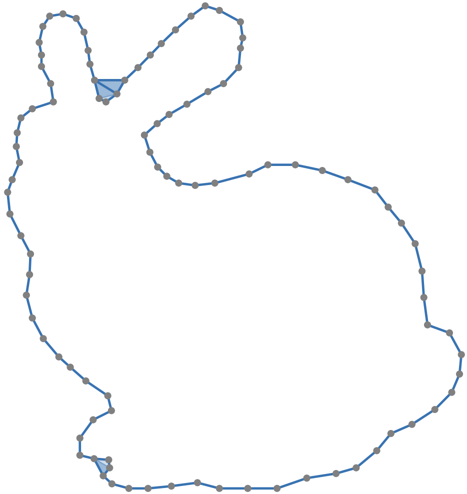

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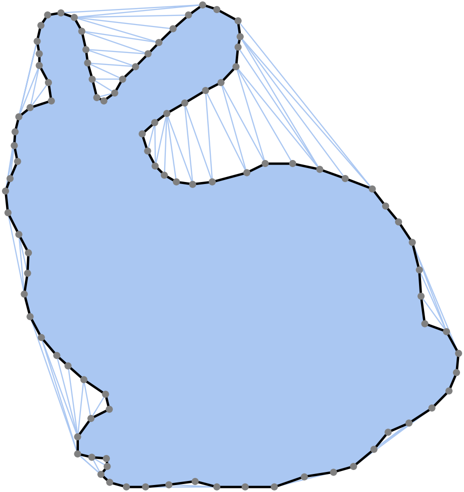

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# Lexicographic minimality

Given an total order $\sigma_1 < \cdots < \sigma_m$ of the simplices of $K$

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$$c_1 < c_2 :\Leftrightarrow \max\{(c_1 \setminus c_2) \cup (c_2 \setminus c_1)\} \in c_2$$

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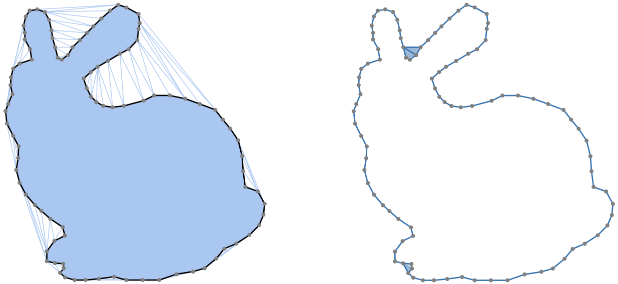
- then by the lexicographic order induced by the total vertex order.

# Minimal cycles and Wrap complexes

Theorem (Bauer, R 2022). Let $X \subset \mathbb{R}^d$ be a finite subset in general position and let $r \in \mathbb{R}$. Then the lexicographically minimal cycles of $\mathrm{Del}_r(X)$ are supported on $\mathrm{Wrap}_r(X)$.



# Making discrete Morse theory algebraic

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- The basis elements $\Sigma_*$ take the role of the simplices in discrete Morse theory.
- All further definitions of discrete Morse theory apply verbatim.

# Algebraic gradients from persistent homology

Let $R = D \cdot V$ be the reduced boundary matrix
(for the Delaunay filtration, breaking ties lexicographically).

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- If $(\sigma, \tau)$ is an *apparent pair*, then $(\mathrm{pivot}\, R_\tau, V_\tau) = (\sigma, \tau)$.
- Thus the apparent pairs gradient is a subset of the persistence gradient.
- The apparent pairs of persistence zero refine the gradient of the Delaunay radius function.

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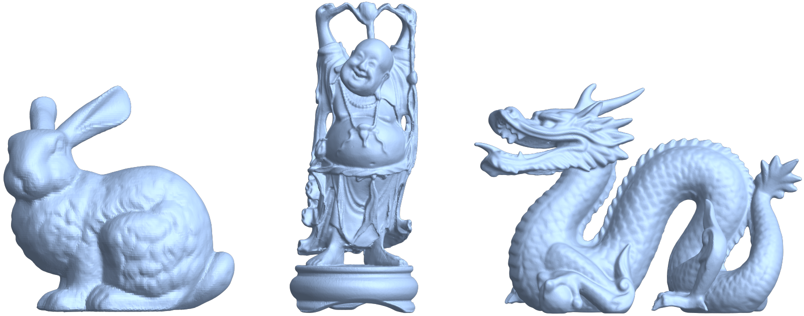
- Smaller gradients have more flow-invariant cycles.
  - Thus, an invariant cycle for the algebraic flow (by the persistence gradient) is also invariant for the geometric flow (by apparent pairs of persistence zero).
- Coarser gradients have larger descending complexes.
  - Thus, the minimal cycles are also supported on the Wrap complex.

# Point cloud reconstruction with most persistent features

The lexicographically minimal cycle corresponding to the interval in the persistence barcode of the Delaunay function with the largest death/birth ratio:



# Summary

- Got reminded about discrete Morse theory and the Wrap complex
- Saw how persistent homology relates to algebraic gradient flows
- Learned that lexicographically minimal cycles of $\mathrm{Del}_r(X)$ are supported on $\mathrm{Wrap}_r(X)$

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