Gromov hyperbolicity, geodesic defect, and apparent pairs in Vietoris-Rips filtrations

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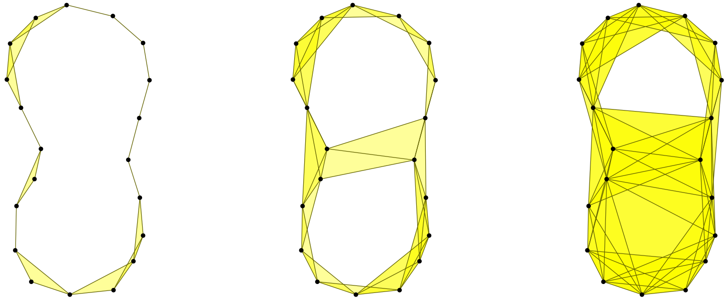
joint work with Ulrich Bauer

The Vietoris–Rips complex

Definition. Let X be a metric space. The Vietoris-Rips complex at scale r is the simplicial complex

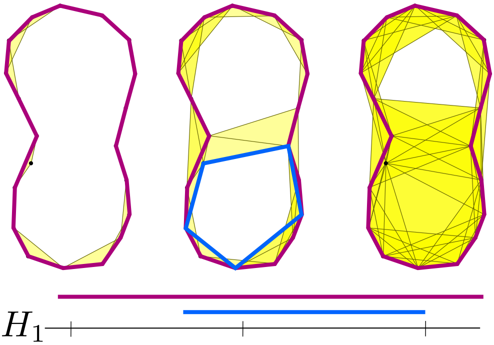
$$\mathrm{Rips}_r(X) = \{S \subseteq X \text{ finite} \mid S \neq \emptyset,\ \operatorname{diam} S \leq r\}.$$



The Vietoris–Rips complex

Applications

- In the limit $r \to 0$: Used by Leopold Vietoris to extend homology theory to metric spaces (1927).
- In the limit $r \to \infty$: Used by Eliyahu Rips and Mikhael Gromov to study hyperbolic groups (1987).
- For all $r > 0$: Used in topological data analysis (nowadays).



Application of Vietoris–Rips persistent homology

COVID-19 genetic evolution data

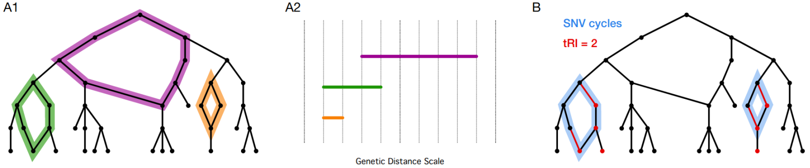


Figure 2. Topological data analysis quantifies convergent evolution. (A) Persistent homology detects reticulate events in viral evolution by means of a persistence barcode. Each bar in the barcode (A2) corresponds to a topological cycle in the reticulate phylogeny (A1). Bars at small genetic distance scales are expected to correspond mainly to homoplasies, while recombination events typically produce topological features at larger scales. (B) SNV cycles are topological cycles in the reticulate phylogeny for which adjacent sequences differ

M. Bleher, L. Hahn, J. A. Patino-Galindo, M. Carriere, U. Bauer, R. Rabadan, and A. Ott

Topological data analysis identifies emerging adaptive mutations in SARS-CoV-2

Preprint, arXiv:2106.07292, 2021

Application of Vietoris–Rips persistent homology

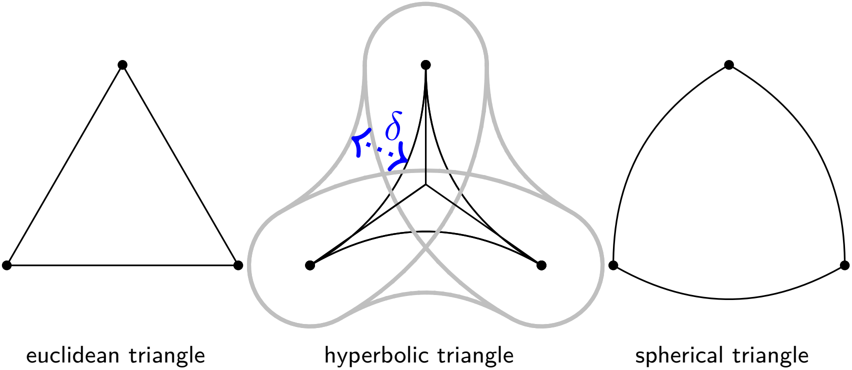
COVID-19 genetic evolution data

covid data (9500 points)	Ripser's runtime
ordered chronologically	49m 37s
ordered reversed chronologically	35s

U. Bauer
Ripser: efficient computation of Vietoris–Rips persistence barcodes
Journal of Applied and Computational Topology, doi:10.1007/s41468-021-00071-5, 2021

Rips contractibility lemma

Gromov-hyperbolicity

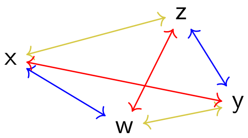


Rips contractibility lemma

Gromov-hyperbolicity

Definition. A metric space X is (Gromov) δ-*hyperbolic* if for all four points $w, x, y, z \in X$

$$d(x,w) + d(y,z) \leq \max\{d(x,y) + d(z,w), d(x,z) + d(y,w)\} + 2\delta$$



Example. finite metric spaces, trees are 0-hyperbolic, hyperbolic plane, ...

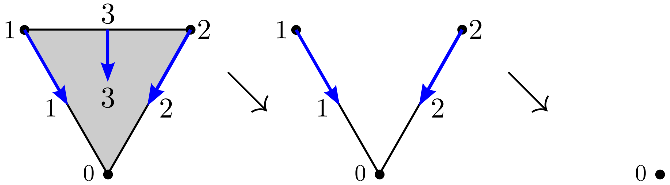
Rips contractibility lemma

Theorem (Rips, Gromov 1987). Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

We address two questions:

1. What about non-geodesic spaces? Finite metric spaces?
2. Connections to Ripser?

Discrete Morse theory

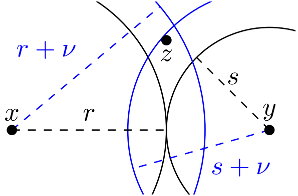


- Discrete Morse function $K \to \mathbb{R}$ with discrete gradient.
- They induce collapses that preserve the homotopy type.

Generalized contractibility lemma

The geodesic defect

Definition (Bonk, Schramm 2000). The metric space X is *ν-geodesic* if for all $x, y \in X$ and $r, s \geq 0$ with $r + s = d(x, y)$ there exists $z \in X$ with $d(x, z) \leq r + \nu$ and $d(y, z) \leq s + \nu$.



Generalized contractibility lemma

Theorem (Bauer, R). Let X be a finite δ-hyperbolic metric space. Then there exists a discrete gradient encoding the collapses

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}$$

for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect of X

- Vietoris–Rips persistent homology of trees (0-hyperbolic) is concentrated in degree zero.
- Connections to Ripser?

Discrete Morse theory

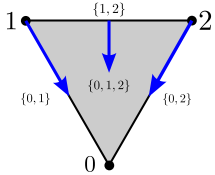
Apparent pairs

Ripser uses the following construction for a computational shortcut:

Definition. In a simplexwise filtration $(K_i = \{\sigma_1, \ldots, \sigma_i\})_i$, a pair of simplices (σ, τ) is an apparent pair if

- σ latest proper face of τ, and
- τ is the earliest proper coface of σ.

Lemma. The apparent pairs form a discrete gradient.



Collapsing Rips complexes of trees

Let X be the path length metric space for a weighted tree $T = (V, E)$.

- order points away from an arbitrarily chosen root
- T_t subforest on V with all edges in E of length at most t

Theorem (Bauer, R). If X is ordered in a compatible way, the apparent pairs gradient induces a sequence of collapses

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow T_t$$

for every $u > t > 0$ such that no edge $e \in E$ has length $l(e) \in (t, u]$.

- Ripser computes the persistent homology of X without a single column operation.
- Explains Ripser's outstanding performance on genetic distances.

Conclusion

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- Identified a subclass of metric spaces for which the persistent homology computation is very efficient.
- Extended the Contractibility Lemma to finite metric spaces and made it filtration compatible.

Collapsing Rips complexes of trees

Some proof details

If X is generic

- $\mathrm{diam}\colon \mathrm{Cl}(V) \to \mathbb{R}$ is a (generalized) discrete Morse function.
- for *any* total order on V the apparent pairs gradient for the lexicographically refined Vietoris–Rips filtration induces the collapses.

If X is arbitrary

- for a *compatible* total order on V a symbolic perturbation scheme on the edges is induced, establishing the generic situation.