Geometric Complexes and Applied Topology

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Graduate Seminar in Mathematics - LMU
May 12, 2023

Outline

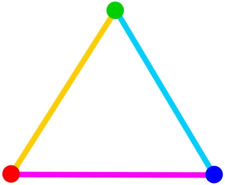
Nerve Theorems, Persistent Homology, and Homotopy Theory

Covid-19, Vietoris–Rips Complexes, and Gromov Hyperbolicity

The Alexandroff nerve (1928)

Definition. Let X be a topological space, and let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \{J \subseteq I \mid |J| < \infty \text{ and } U_J := \bigcap_{i \in J} U_i \neq \emptyset\}$$

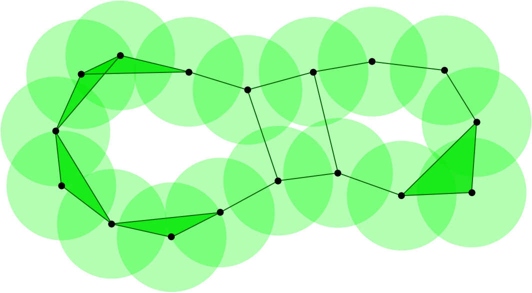


The Alexandroff nerve (1928)

Čech complex of a point cloud

Definition. The *Čech complex* of a subset $X \subseteq \mathbb{R}^d$ is the nerve of the cover by closed balls of radius r centered at points in X

$$\check{\mathrm{C}}\mathrm{ech}_r(X) = \mathrm{Nrv}((D_r(x))_{x \in X})$$



The nerve theorem

Theorem (Leray 1945, Borsuk 1948, Weil 1952, ...). Let $\mathcal{U}$ be a nice cover of X. Then $\mathrm{Nrv}(\mathcal{U})$ is equivalent to X.

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- finite, closed, convex cover
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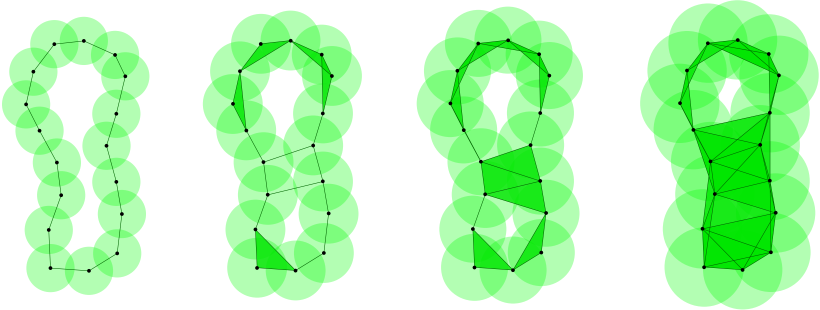
- Alexandroff 1928: Every compact metric space is the inverse limit of a sequence of nerves of "arbitrarily fine" closed covers.
- Čech 1932: Extends Alexandroff's ideas $\rightarrow$ Čech (co)homology

Functorial nerve theorem

Persistent homology



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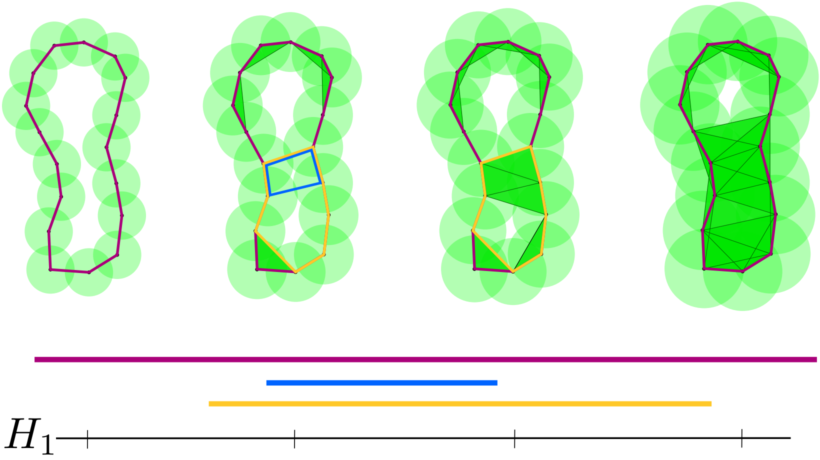

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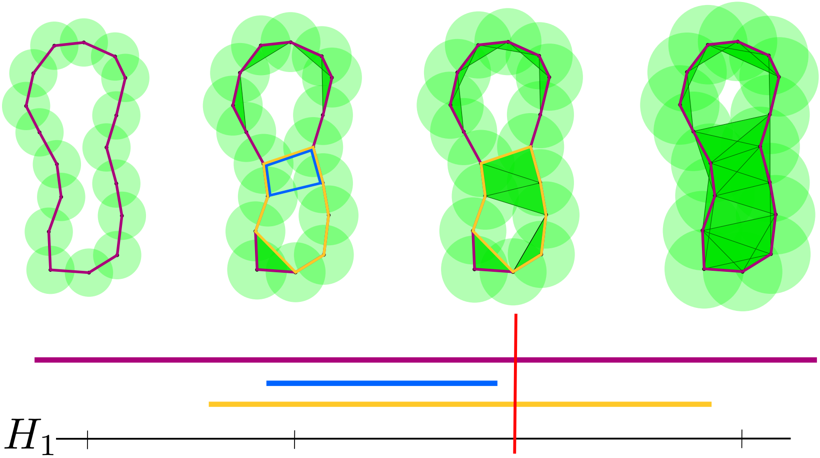

Functorial nerve theorem

Let $X \subseteq \mathbb{R}^d$, $\mathcal{U}_r = \{D_r(x)\}_{x \in X}$, and $X_r = \bigcup \mathcal{U}_r$. The nerve theorem guarantees for $r \leq l$ the homotopy equivalences

$$
\begin{tikzcd}
\mathrm{Nrv}(\mathcal{U}_r) & & \mathrm{Nrv}(\mathcal{U}_l) \\
X_r \arrow[u, "\simeq"] & & X_l \arrow[u, "\simeq"']
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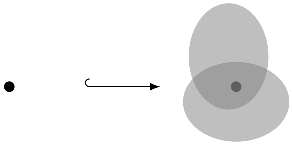
For an extensive treatment of functorial nerve theorems dealing with open and closed covers see

U. Bauer, M. Kerber, F. Roll, and A. Rolle
A Unified View on the Functorial Nerve Theorem and its Variations
Preprint, arXiv:2203.03571, 2022.

Functorial nerve theorem

Category of covered spaces



Functorial nerve theorem

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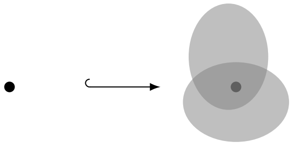

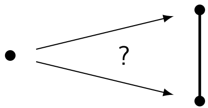

Definition. The *category of covered spaces* Cov has

- Obj: pairs of the form $(X, (U_i))$, with (U_i) a cover of X
- Mor: $(f, \varphi)\colon (X, (U_i)_{i \in I}) \to (Y, (V_\ell)_{\ell \in L})$, continuous map $f\colon X \to Y$ with $f(U_i) \subseteq V_{\varphi(i)}$.

Functorial nerve theorem

Category of covered spaces

Two functors

- Forgetting the cover: $\mathrm{Spc}\colon \mathsf{Cov} \to \mathsf{Top}$, $(X, \mathcal{U}) \mapsto X$
- The nerve: $\mathrm{Nrv}\colon \mathsf{Cov} \to \mathsf{Top}$, $(X, \mathcal{U}) \mapsto \mathrm{Nrv}(\mathcal{U})$

Functorial nerve theorem

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Remark. There are no natural transformations between Spc and Nrv!

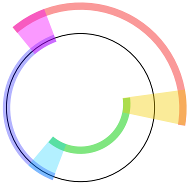
Functional nerve theorem

Blowup complex

Definition. For a finite cover $\mathcal{U} = (U_i)_{i \in [n]}$ of X, the *blowup complex* is

$$\operatorname{Blowup}(\mathcal{U}) = \bigcup_{J \in \operatorname{Nrv}(\mathcal{U})} U_J \times \Delta^{|J|-1} \subseteq X \times \operatorname{Nrv}(\mathcal{U}) \; ,$$

yielding a functor $\operatorname{Blowup}\colon \mathsf{Cov} \to \mathsf{Top}$.



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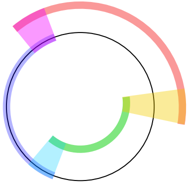


Remark. The blowup complex is (not naturally) homeomorphic to the *bar construction* of the nerve diagram $\to$ can exploit its good homotopy theoretic properties to prove nerve theorems

Functorial nerve theorem

Any $(f, \varphi)\colon (X, \mathcal{U}) \to (Y, \mathcal{V})$ induces a commuting diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & \mathrm{Blowup}(\mathcal{U}) \arrow[l, "\rho_S"'] \arrow[r, "\rho_N"] \arrow[d] & \mathrm{Nrv}(\mathcal{U}) \arrow[d, "\varphi_*"] \\
Y & \mathrm{Blowup}(\mathcal{V}) \arrow[l, "\rho_S"] \arrow[r, "\rho_N"'] & \mathrm{Nrv}(\mathcal{V})
\end{tikzcd}$$

Hence, there are natural transformations $\mathrm{Spc} \overset{\rho_S}{\Leftarrow} \mathrm{Blowup} \overset{\rho_N}{\Rightarrow} \mathrm{Nrv}$.

Outline

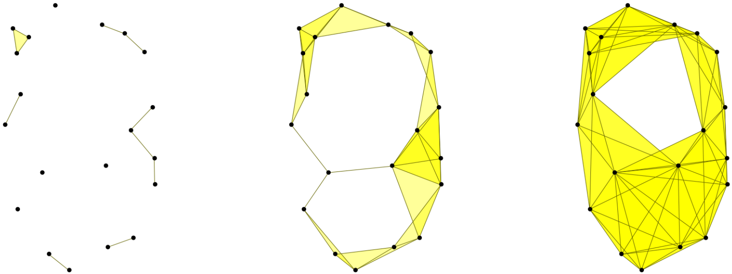
Nerve Theorems, Persistent Homology, and Homotopy Theory

Covid-19, Vietoris–Rips Complexes, and Gromov Hyperbolicity

The Vietoris–Rips complex (1927, 1987)

Definition. Let X be a metric space. The Vietoris-Rips complex at scale r is the simplicial complex

$$\mathrm{Rips}_r(X) = \{S \subseteq X \text{ finite} \mid S \neq \emptyset,\ \operatorname{diam} S \leq r\}.$$



The Vietoris–Rips complex (1927, 1987)

Applications

- In the limit $r \to 0$: Used by Leopold Vietoris to extend homology theory to metric spaces (1927).

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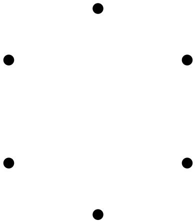
- For all $r > 0$: Used in topological data analysis (nowadays).

Theorem (Latschev, 2001). Let X be a closed Riemannian manifold. For small enough $r, \delta > 0$ and any metric space Y with $d_{GH}(X, Y) < \delta$:

$$\mathrm{Rips}_r(Y) \simeq X$$

The Vietoris–Rips complex (1927, 1987)

The circle S^1



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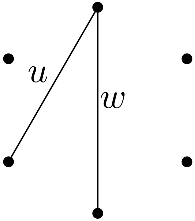

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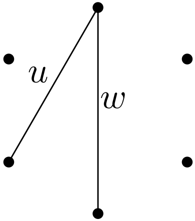
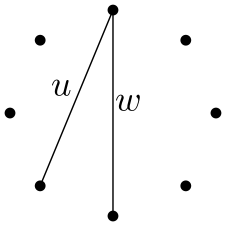


For $u \leq r < w$:

$$|\operatorname{Rips}_r(X)| \simeq S^2$$

The Vietoris–Rips complex (1927, 1987)

The circle S^1

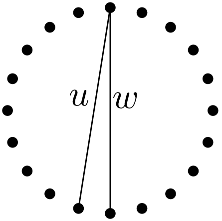


For $u \leq r < w$:

$$|\operatorname{Rips}_r(X)| \simeq S^3$$

The Vietoris–Rips complex (1927, 1987)

The circle S^1

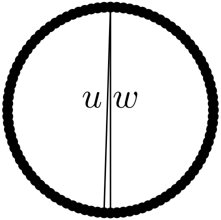


For $u \leq r < w$:

$$|\operatorname{Rips}_r(X)| \simeq S^9$$

The Vietoris–Rips complex (1927, 1987)

The circle S^1

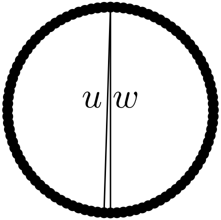


For $u \le r < w$:

$$|\mathrm{Rips}_r(X)| \simeq S^{49}$$

The Vietoris–Rips complex (1927, 1987)

The circle S^1



For $u \leq r < w$:

$$|\operatorname{Rips}_r(X)| \simeq S^{49}$$

Theorem (Adamaszek, Adams 2015). For $l = 0, 1, \ldots$ there are homotopy equivalences

$$\operatorname{Rips}_r(S^1) \simeq \begin{cases} S^{2l+1} & \text{if } \frac{l}{2l+1} < r < \frac{l+1}{2l+3}, \\ \bigvee^{\mathfrak{c}} S^{2l} & \text{if } r = \frac{l}{2l+1}. \end{cases}$$

Application of Vietoris–Rips persistent homology

COVID-19 genetic evolution data

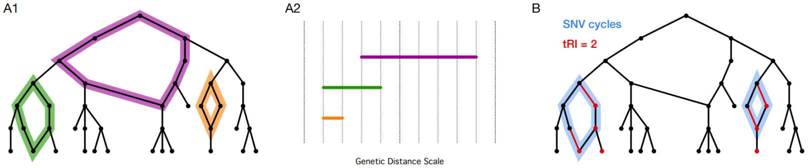


Figure 2. Topological data analysis quantifies convergent evolution. (A) Persistent homology detects reticulate events in viral evolution by means of a persistence barcode. Each bar in the barcode (A2) corresponds to a topological cycle in the reticulate phylogeny (A1). Bars at small genetic distance scales are expected to correspond mainly to homoplasies, while recombination events typically produce topological features at larger scales. (B) SNV cycles are topological cycles in the reticulate phylogeny for which adjacent sequences differ

M. Bleher, L. Hahn, J. A. Patino-Galindo, M. Carriere, U. Bauer, R. Rabadan, and A. Ott

Topological data analysis identifies emerging adaptive mutations in SARS-CoV-2

Preprint, arXiv:2106.07292, 2021

Application of Vietoris–Rips persistent homology

COVID-19 genetic evolution data

covid data	Ripser's runtime
ordered chronologically	full day
ordered reversed chronologically	2 minutes

Circle Limit III, M. C. Escher

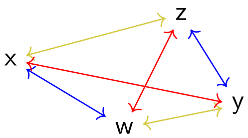
U. Bauer
Ripser: efficient computation of Vietoris–Rips persistence barcodes
Journal of Applied and Computational Topology,
DOI:10.1007/s41468-021-00071-5, 2021

Rips contractibility lemma

Gromov-hyperbolicity

Definition. A metric space X is (Gromov) δ-*hyperbolic* if for all four points $w, x, y, z \in X$

$$d(x,w) + d(y,z) \leq \max\{d(x,y) + d(z,w), d(x,z) + d(y,w)\} + 2\delta$$



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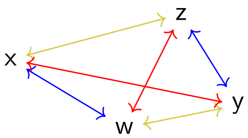

Example. finite metric space, trees are 0-hyperbolic, hyperbolic plane, ...

Rips contractibility lemma

Theorem (Rips, Gromov 1987). Let X be a δ-hyperbolic geodesic metric space. Then $\operatorname{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

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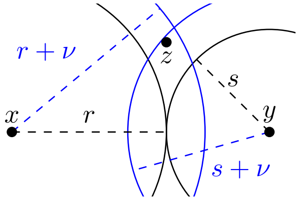
We address two questions:

1. What about non-geodesic spaces? Finite metric spaces?
2. Connection to Ripser?

Generalized contractibility lemma

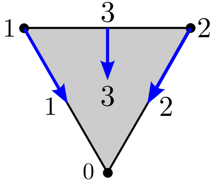
The geodesic defect

Definition (Bonk, Schramm 2000). The metric space X is *ν-geodesic* if for all $x, y \in X$ and $r, s \geq 0$ with $r + s = d(x, y)$ there exists $z \in X$:



Generalized contractibility lemma

Discrete Morse theory



Generalized contractibility lemma

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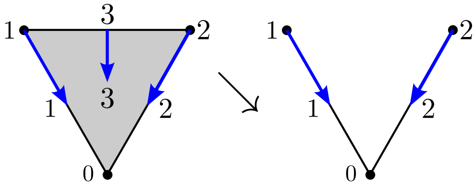

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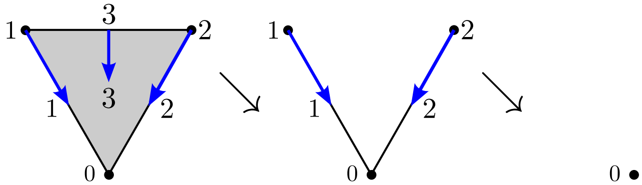

Generalized contractibility lemma

Theorem (Bauer, R 2021). Let X be a finite δ-hyperbolic metric space. Then there exists a discrete gradient encoding the collapses

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}$$

for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect of X

U. Bauer, F. Roll
Gromov hyperbolicity, geodesic defect, and apparent pairs in Vietoris-Rips filtrations
SoCG 2022, extended version: arXiv:2112.06781

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- Can relate this result to Ripser's outstanding performance on genetic distances by considering the *apparent pairs gradient*.

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