Geometric complexes and their topological properties
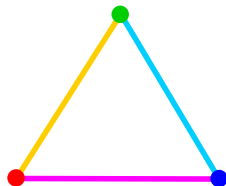
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TopMath–Talk
May 05, 2022

The Alexandroff nerve (1928)

Definition. Let X be a topological space, and let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \{J \subseteq I \mid |J| < \infty \text{ and } U_J := \bigcap_{i \in J} U_i \neq \emptyset\}$$
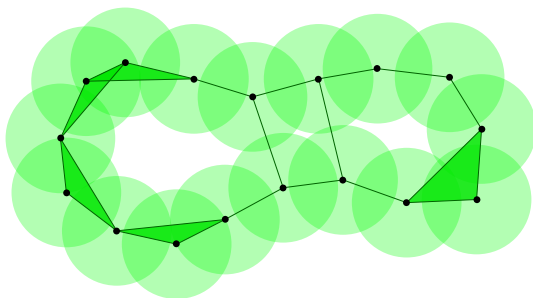
The Alexandroff nerve (1928)

Čech complex of a point cloud

Definition. The *Čech complex* of a subset $X \subseteq \mathbb{R}^d$ is the nerve of the cover by closed balls of radius r centered at points in X

$$\check{\mathrm{Cech}}_r(X) = \mathrm{Nrv}((D_r(X))_{x \in X})$$

The history of the nerve theorem

Theorem (Borsuk 1948, and many more). Let $\mathcal{U}$ be a nice cover of X. Then $\mathrm{Nrv}(\mathcal{U})$ is homotopy equivalent to X.

Here *nice* can mean different things:

- open, numerable cover, contractible intersections $\to$ many references
- finite, closed, convex cover $\to$ few references, mostly using outdated language and tools

Prior results?

- Alexandroff 1928: Every compact metric space is the inverse limit of a sequence of nerves of "arbitrarily fine" closed covers.
- Čech 1932: Extends Alexandroff's ideas $\to$ Čech (co)homology

Nerve theorem for closed convex sets

Theorem. If $X \subset \mathbb{R}^d$, and $\mathcal{A} = (C_i)_{i \in [n]}$ is a cover by closed convex subsets, then $\mathrm{Nrv}(\mathcal{A})$ is homotopy equivalent to X.

Proof strategy (R):

- Construct piecewise linear $\Gamma \colon \mathrm{Sd}\,\mathrm{Nrv}(\mathcal{A}) \to X$ with $\Gamma(\mathrm{bst}\,v_i) \subseteq C_i$.
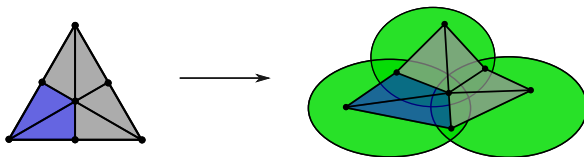
- Construct $\Phi \colon X \to \mathrm{Nrv}(\mathcal{A})$ using a partition of unity subordinate to an open thickening of the C_i with $\Phi(C_i) \subseteq \mathrm{bst}\,v_i$.
- Show that Φ is a homotopy inverse to Γ.

Nerve theorem for closed convex sets
Some proof details

- Take a collection of open sets $\mathcal{G} = (U_i \supseteq C_i)_{i \in [n]}$ satisfying $\mathrm{Nrv}(\mathcal{A}) = \mathrm{Nrv}(\mathcal{G}_\varepsilon)$.
- Take a partition of unity $(\psi_i)_{i \in [n]}$ on X subordinate to the cover $(U_i \cap X)_{i \in [n]}$ of X with $\psi_i|_{C_i} \equiv 1$.
- Define the map $\Phi \colon X \to \mathrm{Nrv}(\mathcal{A})$ by

$$\Phi \colon x \mapsto \sum_{i=0}^{n} \psi_i(x) \cdot v_i$$

- Then $\Gamma \circ \Phi(C_i) \subseteq C_i$ and $\Gamma \circ \Phi \simeq \mathrm{id}_X$ by a straight line homotopy.
- Then $\Phi \circ \Gamma(\mathrm{bst}\, v_i) \subseteq \mathrm{bst}\, v_i$ and $\Phi \circ \Gamma \simeq \mathrm{id}_{\mathrm{Sd}\,\mathrm{Nrv}(\mathcal{A})}$ by induction over the skeleton of $\mathrm{Sd}\,\mathrm{Nrv}(\mathcal{A})$; use that $(\mathrm{bst}\, v)_{v \in \mathrm{Vert}\,\mathrm{Nrv}(\mathcal{A})}$ is a good cover of $\mathrm{Sd}\,\mathrm{Nrv}(\mathcal{A})$.

Functorial nerve theorem

Persistent homology

H_1
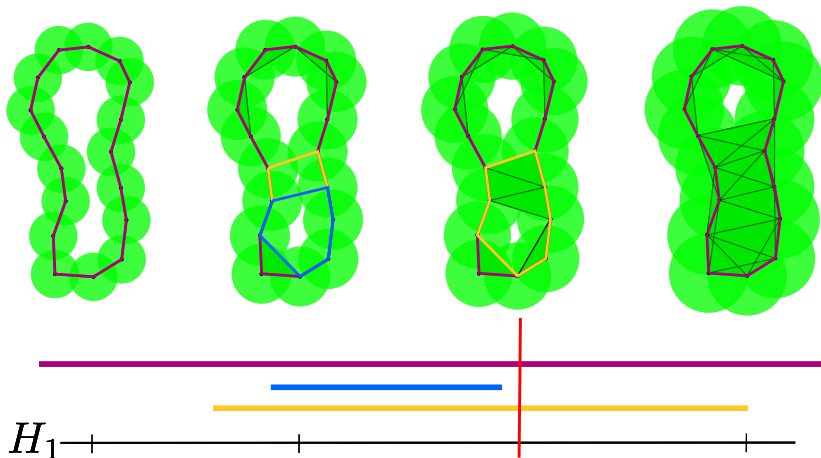
Functorial nerve theorem

Let $X \subseteq \mathbb{R}^d$, $\mathcal{U}_r = \{D_r(x)\}_{x \in X}$, and $X_r = \bigcup \mathcal{U}_r$. The nerve theorem guarantees for $r \leq l$ the homotopy equivalences

$$
\begin{array}{ccc}
\mathrm{Nrv}(\mathcal{U}_r) & \longrightarrow & \mathrm{Nrv}(\mathcal{U}_l) \\
\simeq \uparrow & \circlearrowleft? & \uparrow \simeq \\
X_r & \longrightarrow & X_l
\end{array}
$$

For an extensive treatment of functorial nerve theorems dealing with open and closed covers see
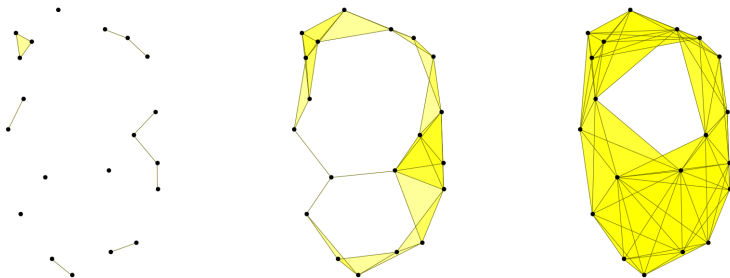
📄 U. Bauer, M. Kerber, F. Roll, and A. Rolle
A Unified View on the Functorial Nerve Theorem and its Variations
Preprint, arXiv:2203.03571, 2022.

The Vietoris–Rips complex (1927, 1987)

Definition. Let X be a metric space. The Vietoris-Rips complex at scale r is the simplicial complex

$$\mathrm{Rips}_r(X) = \{S \subseteq X \text{ finite} \mid S \neq \emptyset, \ \mathrm{diam}\, S \leq r\}.$$

The Vietoris–Rips complex (1927, 1987)

Applications

- In the limit $r \to 0$: Used by Leopold Vietoris to extend homology theory to metric spaces (1927).
- In the limit $r \to \infty$: Used by Eliyahu Rips and Mikhael Gromov to study hyperbolic groups (1987).
- For all $r > 0$: Used in topological data analysis (nowadays).

Theorem (Latschev, 2001). Let X be a closed Riemannian manifold. For small enough $r, \delta > 0$ and any metric space Y with $d_{GH}(X, Y) < \delta$:

$$\mathrm{Rips}_r(Y) \simeq X$$

The Vietoris–Rips complex (1927, 1987)

The circle S^1
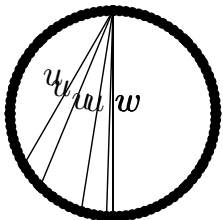
For $u \le r < w$:

$$|\operatorname{Rips}_r(X)| \simeq S^{2\ 3\ 9\ 49}$$

Theorem (Adamaszek, Adams 2015). For $l = 0, 1, \ldots$ there are homotopy equivalences

$$\operatorname{Rips}_r(S^1) \simeq \begin{cases} S^{2l+1} & \text{if } \frac{l}{2l+1} < r < \frac{l+1}{2l+3}, \\ \bigvee^{\mathfrak{c}} S^{2l} & \text{if } r = \frac{l}{2l+1}. \end{cases}$$

Application of Vietoris–Rips persistent homology
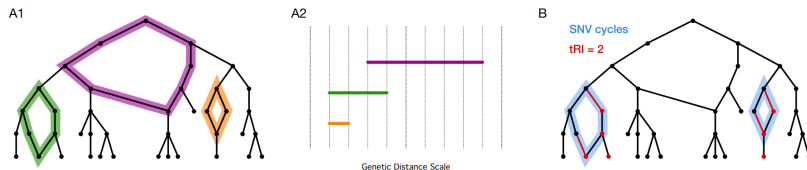
COVID-19 genetic evolution data

Figure 2. Topological data analysis quantifies convergent evolution. (A) Persistent homology detects reticulate events in viral evolution by means of a persistence barcode. Each bar in the barcode (A2) corresponds to a topological cycle in the reticulate phylogeny (A1). Bars at small genetic distance scales are expected to correspond mainly to homoplasies, while recombination events typically produce topological features at larger scales. (B) SNV cycles are topological cycles in the reticulate phylogeny for which adjacent sequences differ

M. Bleher, L. Hahn, J. A. Patino-Galindo, M. Carriere, U. Bauer, R. Rabadan, and A. Ott

Topological data analysis identifies emerging adaptive mutations in SARS-CoV-2

Preprint, arXiv:2106.07292, 2021

Application of Vietoris–Rips persistent homology

COVID-19 genetic evolution data

5000 ordered points	time for H_0 & H_1
random in $\mathbb{R}^3$	1m 17s
random on S^2	5m 39s
graph (covid data)	12s
graph (covid data, reversed order)	6m 19s
graph (covid data, random order)	2m 52s

Rips contractibility lemma

Gromov-hyperbolicity

Definition. A metric space X is (Gromov) δ-*hyperbolic* if for all four points $w, x, y, z \in X$

$$d(x,w) + d(y,z) \leq \max\{d(x,y) + d(z,w), d(x,z) + d(y,w)\} + 2\delta$$
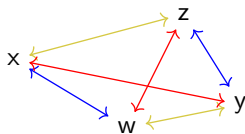
Example. finite metric space, trees are 0-hyperbolic, hyperbolic plane, ...

Rips contractibility lemma

Theorem (Rips, Gromov 1987). Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

Any hyperbolic group G

- is finitely generated and finitely presented.
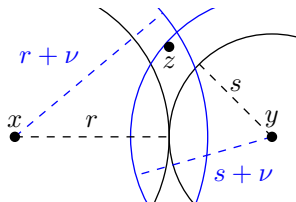- admits an Eilenberg–MacLane space $K(G, 1)$ with finitely many cells in each dimension.

We address two questions:

1. What about non-geodesic spaces? Finite metric spaces?
2. Connection to Ripser?

Generalized contractibility lemma

The geodesic defect

Definition (Bonk, Schramm 2000). The metric space X is ν-*geodesic* if for all $x, y \in X$ and $r, s \geq 0$ with $r + s = d(x, y)$ there exists $z \in X$:

Generalized contractibility lemma

Theorem (Bauer, R 2021). Let X be a finite δ-hyperbolic metric space. Then there exists a discrete gradient encoding the collapses

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}$$

for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect of X
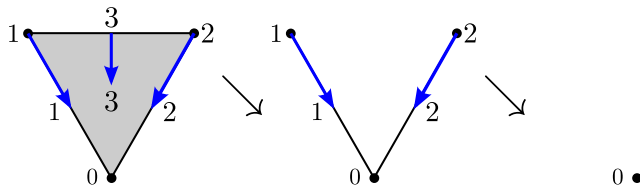
U. Bauer, F. Roll

Gromov hyperbolicity, geodesic defect, and apparent pairs in Vietoris-Rips filtrations

Accepted to SoCG 2022, Extended version on arXiv:2112.06781, 2022

Discrete Morse theory
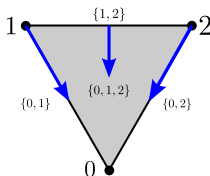
Discrete Morse theory

Apparent Pairs

Ripser uses the following construction for a computational shortcut:

Definition. In a simplexwise filtration $(K_i = \{\sigma_1, \ldots, \sigma_i\})_i$, a pair of simplices (σ, τ) is an apparent pair if

- σ latest proper face of τ, and
- τ is the earliest proper coface of σ.

Lemma. The apparent pairs form a discrete gradient.

Collapsing Rips complexes of trees

Let X be the path length metric for a weighted tree $T = (V, E)$.

- Choose a root and extend the tree order to a total order.

Theorem (Bauer, R). The apparent pairs gradient induces, for every $u > t > 0$ such that no edge $e \in E$ has length $l(e) \in (t, u]$, the collapses

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow T_t,$$

where T_t is the graph on V with the edges in E of length at most t.

- Explains Ripser's outstanding performance on genetic distances.

Future work

- Approximate nerve theorems
- Discuss Latschev's result on the reconstruction of Riemannian manifolds and its functoriality by using *Vietoris–Rips good covers*
- Extend our collapsing result for trees to more general graphs.
- Further investigate spaces that are not tree-like with an eye towards the apparent pairs gradient.