Gromov hyperbolicity, geodesic defect, and apparent pairs in Vietoris-Rips filtrations

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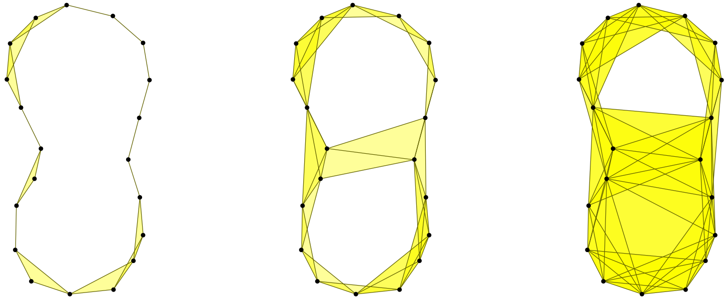
joint work with Ulrich Bauer

The Vietoris–Rips complex

Definition. Let X be a metric space. The Vietoris-Rips complex at scale r is the simplicial complex

$$\mathrm{Rips}_r(X) = \{S \subseteq X \text{ finite} \mid S \neq \emptyset,\ \operatorname{diam} S \leq r\}.$$



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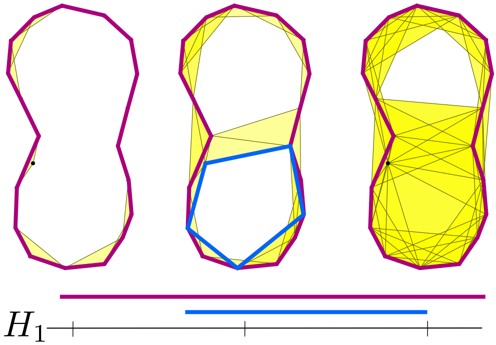
- In the limit $r \to \infty$: Used by Eliyahu Rips and Mikhael Gromov to study hyperbolic groups (1987).
- For all $r > 0$: Used in topological data analysis (nowadays).



Application of Vietoris–Rips persistent homology

COVID-19 genetic evolution data

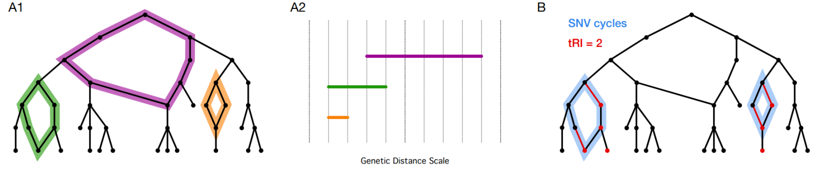


Figure 2. Topological data analysis quantifies convergent evolution. (A) Persistent homology detects reticulate events in viral evolution by means of a persistence barcode. Each bar in the barcode (A2) corresponds to a topological cycle in the reticulate phylogeny (A1). Bars at small genetic distance scales are expected to correspond mainly to homoplasies, while recombination events typically produce topological features at larger scales. (B) SNV cycles are topological cycles in the reticulate phylogeny for which adjacent sequences differ

M. Bleher, L. Hahn, J. A. Patino-Galindo, M. Carriere, U. Bauer, R. Rabadan, and A. Ott

Topological data analysis identifies emerging adaptive mutations in SARS-CoV-2

Preprint, arXiv:2106.07292, 2021

Application of Vietoris–Rips persistent homology

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covid data (9500 points)	Ripser's runtime
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Ripser: efficient computation of Vietoris–Rips persistence barcodes
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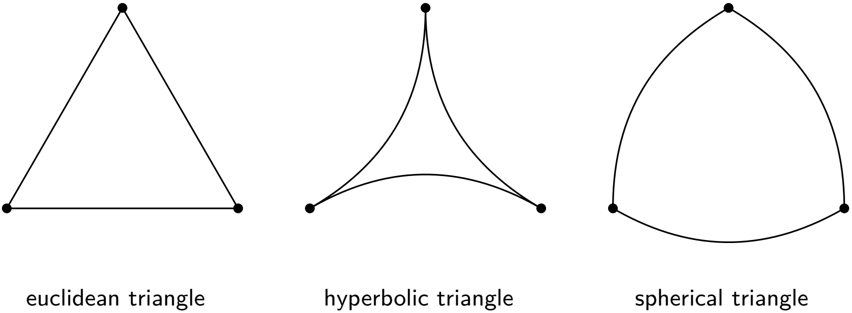
Rips contractibility lemma

Gromov-hyperbolicity



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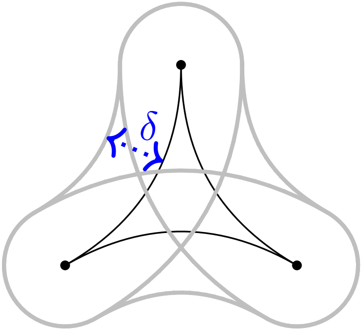

hyperbolic triangle

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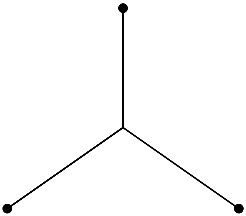

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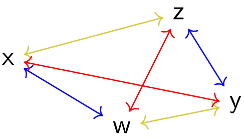
Definition. A metric space X is (Gromov) δ-*hyperbolic* if for all four points $w, x, y, z \in X$

$$d(x,w) + d(y,z) \leq \max\{d(x,y) + d(z,w), d(x,z) + d(y,w)\} + 2\delta$$



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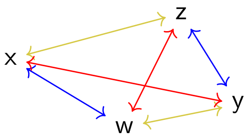

Example. finite metric spaces, trees are 0-hyperbolic, hyperbolic plane, ...

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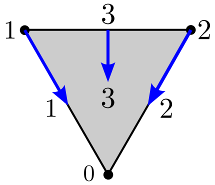
We address two questions:

1. What about non-geodesic spaces? Finite metric spaces?
2. Connections to Ripser?

Discrete Morse theory



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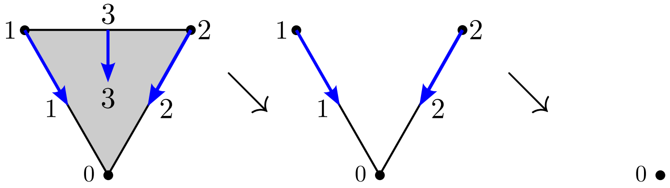
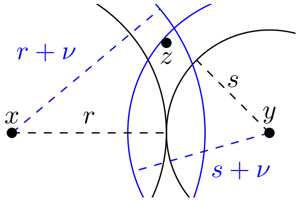


- Discrete Morse function $K \to \mathbb{R}$ with discrete gradient.
- They induce collapses that preserve the homotopy type.

Generalized contractibility lemma

The geodesic defect

Definition (Bonk, Schramm 2000). The metric space X is *ν-geodesic* if for all $x, y \in X$ and $r, s \geq 0$ with $r + s = d(x, y)$ there exists $z \in X$ with $d(x, z) \leq r + \nu$ and $d(y, z) \leq s + \nu$.



Generalized contractibility lemma

Theorem (Bauer, R). Let X be a finite δ-hyperbolic metric space. Then there exists a discrete gradient encoding the collapses

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}$$

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- Connections to Ripser?

Discrete Morse theory

Apparent pairs

Ripser uses the following construction for a computational shortcut:

Definition. In a simplexwise filtration $(K_i = \{\sigma_1, \ldots, \sigma_i\})_i$, a pair of simplices (σ, τ) is an apparent pair if

- σ latest proper face of τ, and
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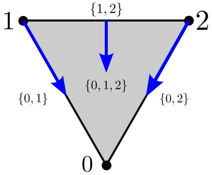
Lemma. The apparent pairs form a discrete gradient.



Collapsing Rips complexes of trees

Let X be the path length metric space for a weighted tree $T = (V, E)$.

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for every $u > t > 0$ such that no edge $e \in E$ has length $l(e) \in (t, u]$.

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- Ripser computes the persistent homology of X without a single column operation.
- Explains Ripser's outstanding performance on genetic distances.

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- Identified a subclass of metric spaces for which the persistent homology computation is very efficient.
- Extended the Contractibility Lemma to finite metric spaces and made it filtration compatible.

Collapsing Rips complexes of trees

Some proof details

If X is generic

- $\mathrm{diam}\colon \mathrm{Cl}(V) \to \mathbb{R}$ is a (generalized) discrete Morse function.
- for *any* total order on V the apparent pairs gradient for the lexicographically refined Vietoris–Rips filtration induces the collapses.

If X is arbitrary

- for a *compatible* total order on V a symbolic perturbation scheme on the edges is induced, establishing the generic situation.