Discrete Morse Theory and Rips' Lemma for Trees

Fabian Roll (TUM)

DGD Days 2021
15.-17.09.21

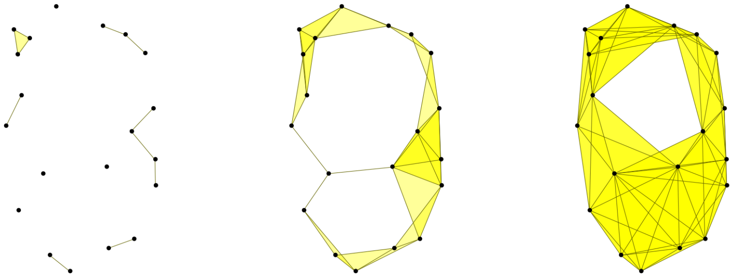
Project: C04 - Persistence and Stability of Geometric Complexes

Advisor: Prof. Dr. Ulrich Bauer

Background

Definition. X metric space. The Vietoris-Rips complex at scale r is the simplicial complex

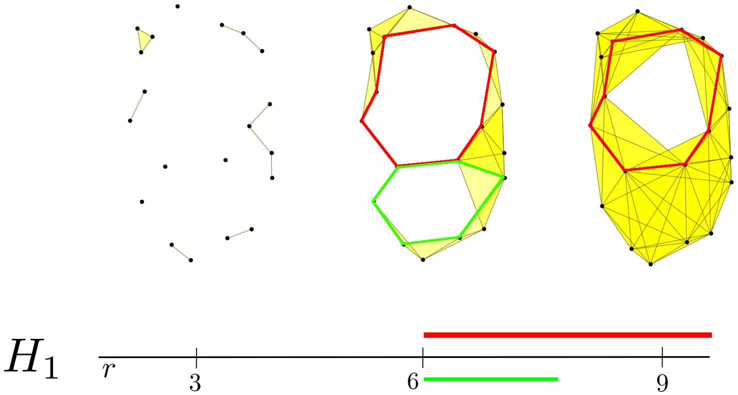
$$\mathrm{VR}_r(X) = \{S \subseteq X \text{ finite} \mid S \neq \emptyset,\ \operatorname{diam} S \leq r\}.$$



$r = 3, 6, 9$

Background

Persistent Homology



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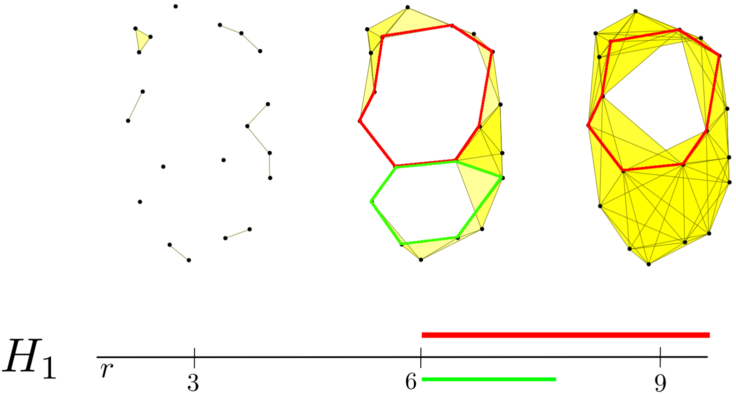

Software: U. Bauer "Ripser: efficient computation of Vietoris-Rips persistence barcodes"

Geometry and Combinatorics of Input influences Computation Time

5000 ordered points	time for H_0 & H_1
random in $\mathbb{R}^3$	1m 17s
random on S^2	5m 39s
graph (covid data)	12s

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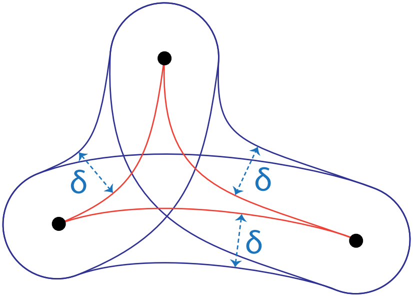
Gromov Hyperbolicity

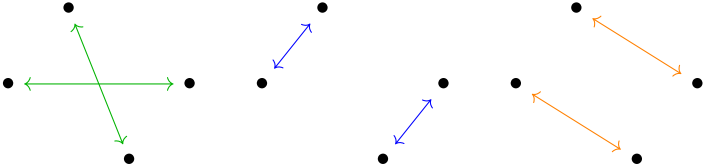
Definition. A length space is δ-hyperbolic if all triangles are δ-slim.



Wikipedia

Gromov Hyperbolicity

Definition. A metric space is δ-hyperbolic if for all four points



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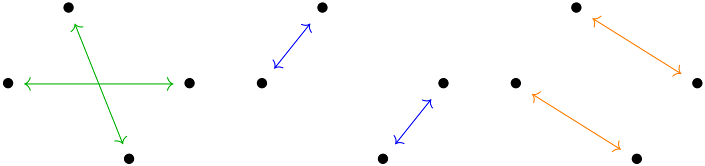

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we have $\delta \geq \ell - \ell$.

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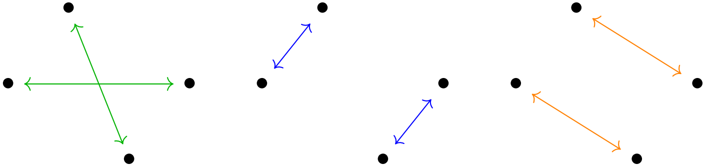

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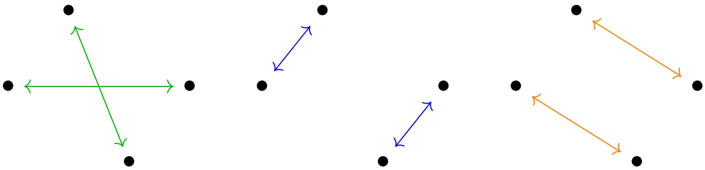

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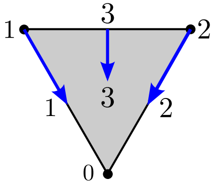
Lemma (Rips). X length space and δ-hyberbolic. Then, $\mathrm{VR}_r(X)$ is contractible for $r \geq 4\delta$.

Discrete Morse Theory



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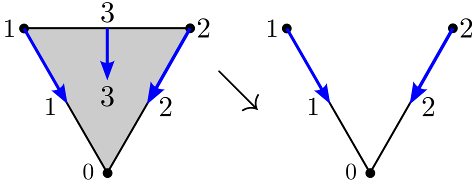

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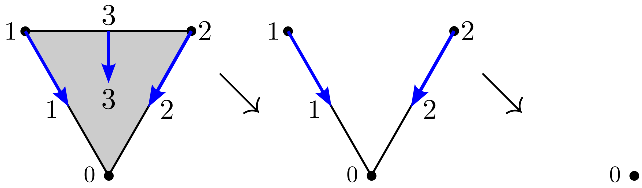

Apparent Pairs

Persistence computation: choose total order on K

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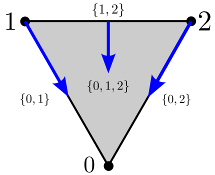

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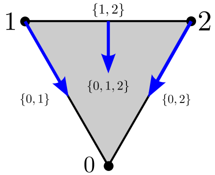

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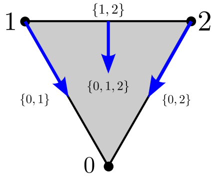

Lemma. The apparent pairs form a discrete gradient. $\rightarrow$ Ripser

Results

- X vertex set of a directed rooted tree
- sort simplices of $\mathsf{VR}_\infty(X)$ by diameter, then lexicographically

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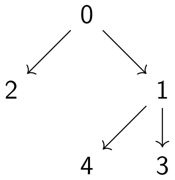

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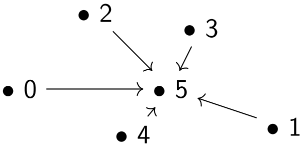
Remark. $r \geq r_c = 4\delta + r_c$ with $\delta = 0$ $\rightarrow$ similar to Rips' lemma

For general X sort points:



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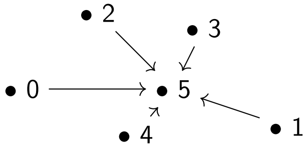

5000 ordered points	time for H_0 & H_1	non-apparent
random tree	7s	0
graph (covid data)	12s	4544
graph (covid data, reversed order)	6m 19s	19344
graph (covid data, random order)	2m 52s	13957

Future Work

- find analogous statements for general δ-hyperbolic spaces with sharp bounds
- use results/heuristics to find approximation algorithms for Vietoris-Rips persistent homology