A Unified View on the Functorial Nerve Theorem and its Variations

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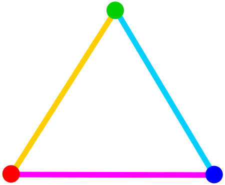
joint work with Ulrich Bauer, Michael Kerber, and Alexander Rolle

The Alexandrov nerve (1928)

Definition. Let X be a topological space, and let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \{J \subseteq I \mid |J| < \infty \text{ and } U_J := \bigcap_{i \in J} U_i \neq \emptyset\}$$

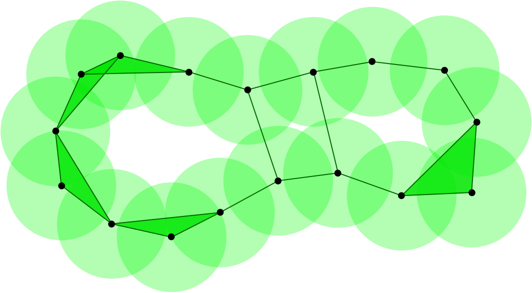


The Alexandrov nerve (1928)

Čech complex of a point cloud

Definition. The *Čech complex* of a subset $X \subseteq \mathbb{R}^d$ is the nerve of the cover by closed balls of radius r centered at points in X

$$\check{\mathrm{C}}\mathrm{ech}_r(X) = \mathrm{Nrv}((D_r(X))_{x \in X}))$$



The history of the nerve theorem

Theorem (Borsuk 1948, and many more). Let $\mathcal{U}$ be a nice cover of X. Then $\mathrm{Nrv}(\mathcal{U})$ is homotopy equivalent to X.

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- Čech 1932: Extends Alexandrov's ideas $\rightarrow$ Čech (co)homology

Nerve theorem for closed convex covers

Theorem. If $X \subset \mathbb{R}^d$, and $\mathcal{A} = (C_i)_{i \in [n]}$ is a cover by closed convex subsets, then $\mathrm{Nrv}(\mathcal{A})$ is homotopy equivalent to X.

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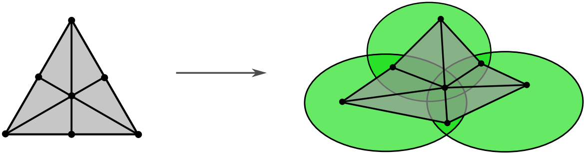
- Construct piecewise linear $\Gamma\colon \operatorname{Sd}\operatorname{Nrv}(\mathcal{A}) \to X$



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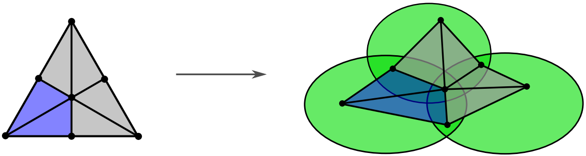

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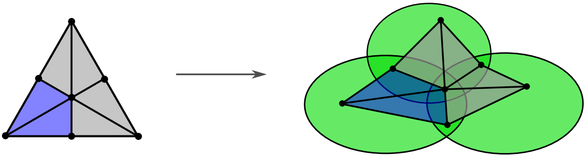

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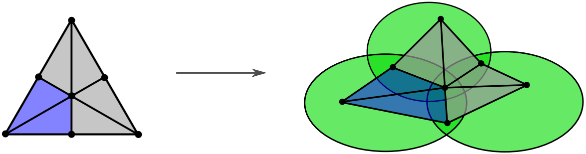

- Construct $\Phi\colon X \to \operatorname{Nrv}(\mathcal{A})$ using a partition of unity subordinate to an open thickening of the C_i with $\Phi(C_i) \subseteq \operatorname{bst} v_i$.
- Show that Φ is a homotopy inverse to Γ.

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Some proof details

- Take a collection of open sets $\mathcal{G} = (U_i \supseteq C_i)_{i \in [n]}$ satisfying $\mathrm{Nrv}(\mathcal{A}) = \mathrm{Nrv}(\mathcal{G}_\varepsilon)$.

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- Then $\Phi \circ \Gamma(\operatorname{bst} v_i) \subseteq \operatorname{bst} v_i$ and $\Phi \circ \Gamma \simeq \mathrm{id}_{\mathrm{Sd}\,\mathrm{Nrv}(\mathcal{A})}$ by induction over the skeleton of $\mathrm{Sd}\,\mathrm{Nrv}(\mathcal{A})$;

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Nerve theorem for simplicial covers

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- Consider the product poset

$$Z = \{(\sigma, J) \mid \sigma \in K_J\} \subseteq \mathrm{Pos}(K) \times \mathrm{Pos}(\mathrm{Nrv}(\mathcal{A}))$$

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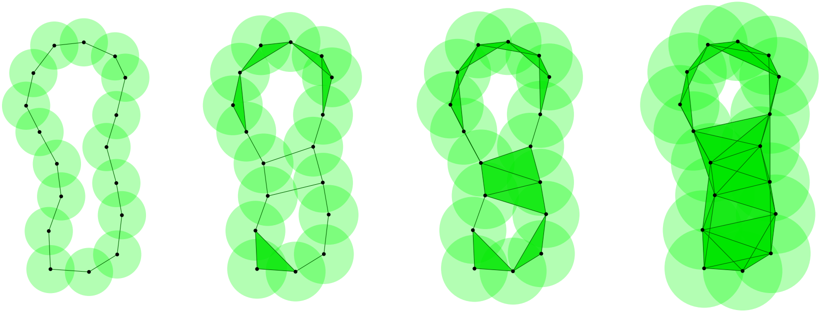
$$\begin{tikzcd} \mathrm{Pos}(K) \arrow[rr, bend right, "\text{locally finite cover}"'] & Z \arrow[l] \arrow[r] & \mathrm{Pos}(\mathrm{Nrv}(\mathcal{A}))^{\mathrm{op}} \end{tikzcd}$$

Functoriality

Persistent homology



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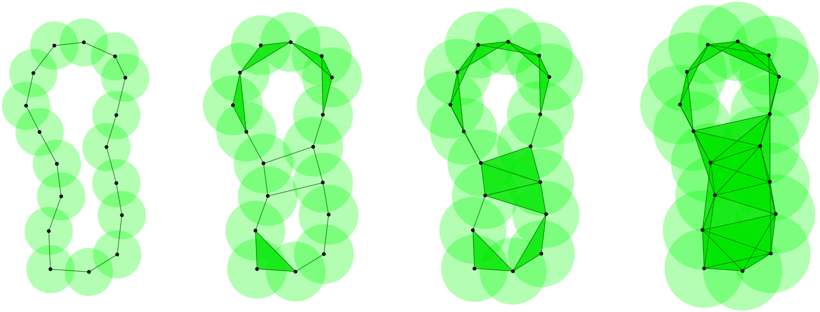

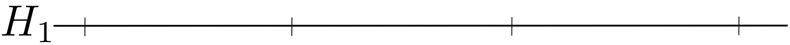

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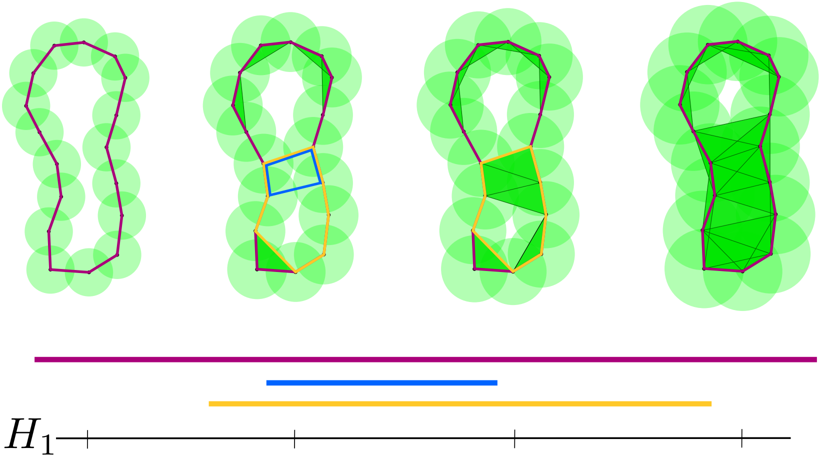

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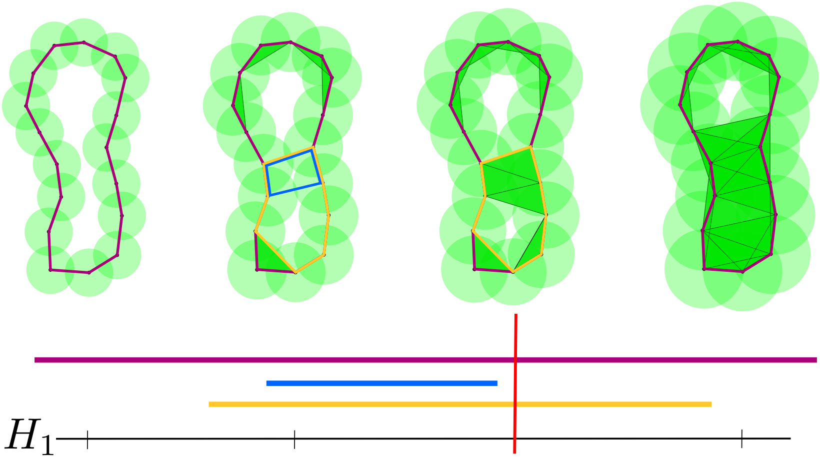

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Let $X \subseteq \mathbb{R}^d$, $\mathcal{U}_r = \{D_r(x)\}_{x \in X}$, and $X_r = \bigcup \mathcal{U}_r$. The nerve theorem guarantees for $r \leq l$ the homotopy equivalences

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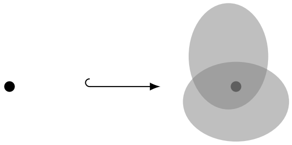
2. No "proper" functoriality $\rightarrow$ needed in some homotopy-theoretic approaches to TDA (e.g. Blumberg–Lesnick)

Functoriality

Category of covered spaces



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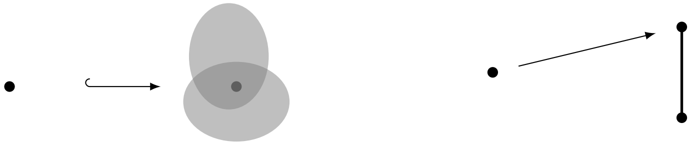

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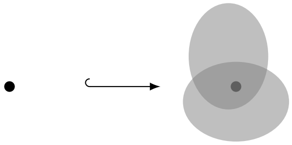

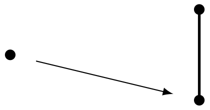

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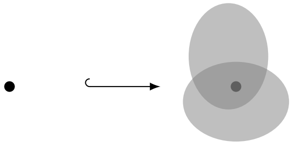

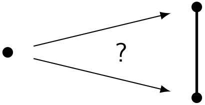

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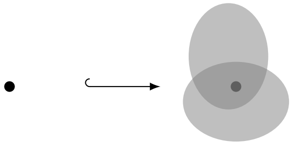

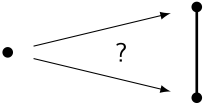

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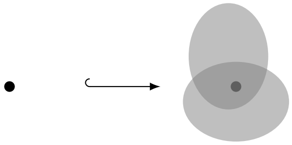

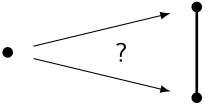

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Definition. The *category of covered spaces* Cov has

- Obj: pairs of the form $(X, (U_i))$, with (U_i) a cover of X
- Mor: $(f, \varphi)\colon (X, (U_i)) \to (Y, (V_\ell))$, continuous map $f\colon X \to Y$ carried by $\varphi\colon (U_i) \to (V_\ell)$

Functoriality

Category of covered spaces

Two functors

- Forgetting the cover: $\mathrm{Spc}\colon \mathsf{Cov} \to \mathsf{Top}$, $(X, \mathcal{U}) \mapsto X$
- The nerve: $\mathrm{Nrv}\colon \mathsf{Cov} \to \mathsf{Top}$, $(X, \mathcal{U}) \mapsto \mathrm{Nrv}(\mathcal{U})$

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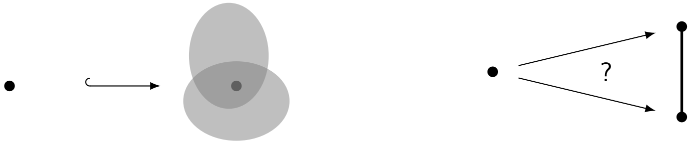
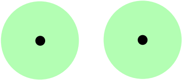
Remark. There are no natural transformations $\mathrm{Spc} \Rightarrow \mathrm{Nrv}$



and similarly no natural transformations $\mathrm{Nrv} \Rightarrow \mathrm{Spc}$.

Functoriality

Pointed covers



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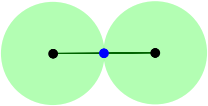

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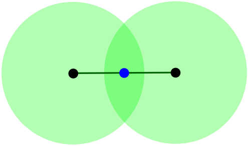

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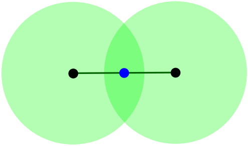

Definition. Only consider $X \subseteq \mathbb{R}^d$. The category $\mathsf{ClConv}_\bullet$ has

- Obj: $(X, \mathcal{A}_\bullet)$, $\mathcal{A}_\bullet$ a finite closed, convex, and *pointed cover* of X

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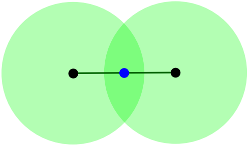

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Functoriality

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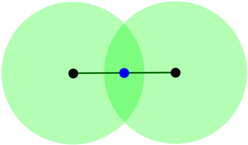

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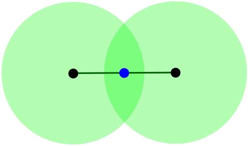

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Theorem. On ClConv$_\bullet$ there exists a pointwise homotopy equivalence

$$\operatorname{Sd}\operatorname{Nrv} \Rightarrow \operatorname{Spc}$$

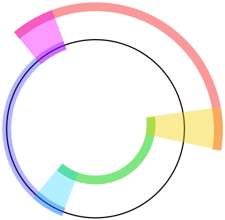
Functoriality

Blowup complex

Definition. For a finite cover $\mathcal{U} = (U_i)_{i \in [n]}$ of X, the *blowup complex* is

$$\mathrm{Blowup}(\mathcal{U}) = \bigcup_{J \in \mathrm{Nrv}(\mathcal{U})} U_J \times \Delta^{|J|-1} \subseteq X \times \Delta^n \,,$$

yielding a functor $\mathrm{Blowup} \colon \mathsf{Cov} \to \mathsf{Top}$.



Functoriality

Bar construction

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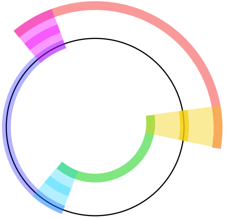

$\rightarrow$ can exploit its good homotopical properties to prove nerve theorems

Functoriality

Any $(f, \varphi)\colon (X, \mathcal{U}) \to (Y, \mathcal{V})$ induces a commuting diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & \mathrm{Blowup}(\mathcal{U}) \arrow[l, "\rho_S"'] \arrow[r, "\rho_N"] \arrow[d] & \mathrm{Nrv}(\mathcal{U}) \arrow[d, "\varphi_*"] \\
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Hence, there are natural transformations $\mathrm{Spc} \overset{\rho_S}{\Leftarrow} \mathrm{Blowup} \overset{\rho_N}{\Rightarrow} \mathrm{Nrv}$.

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Functoriality

Simplicial covers

Definition. For a cover $\mathcal{A} = (K_i \subseteq K)_{i \in I}$ by subcomplexes, consider the subposet

$$\mathrm{PoBar}(\mathcal{A}) = \{(\sigma, J) \mid J \subseteq I \text{ finite, } \sigma \in K_J\}$$

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Remark. There exists a commuting diagram

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Unified nerve theorem

Theorem. Let X be a topological space and $\mathcal{A} = (A_i)_{i \in I}$ a cover of X.

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Unified nerve theorem

Counterexamples

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Unified nerve theorem

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Unified nerve theorem

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- Thus, L and $\operatorname{Nrv} \mathcal{A}$ are not homotopy equivalent

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2. For closed good covers some “finiteness” is needed:

- Consider the cover of S^1 by its points
- As the nerve $\operatorname{Nrv}\mathcal{A}$ is a disjoint union of points, it is not homotopy equivalent to S^1
- All conditions in 1.b) and 2.a) are satisfied except the locally finiteness assumption

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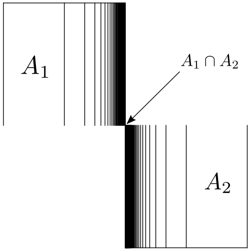

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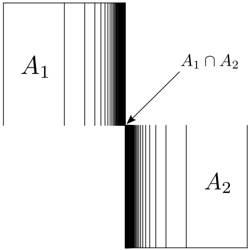

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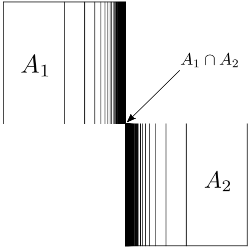

- The nerve $\operatorname{Nrv} \mathcal{A}$ is contractible, but C is not
- The pairs $(A_1, A_1 \cap A_2)$ and $(A_2, A_1 \cap A_2)$ do not satisfy the homotopy extension property

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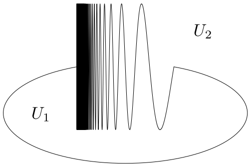
- Consider the *Warsaw circle* $W \subseteq S^2$ that separates the sphere into two connected components U_1 and U_2



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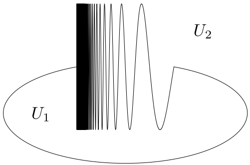

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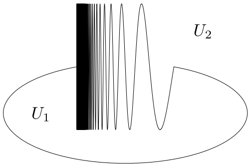

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- $A_1 \cap A_2 = W$ is acyclic and hence $\mathcal{A} = \{A_1, A_2\}$ is a homologically good closed cover of S^2

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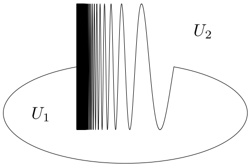

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- $A_1 \cap A_2 = W$ is acyclic and hence $\mathcal{A} = \{A_1, A_2\}$ is a homologically good closed cover of S^2
- S^2 and $\operatorname{Nrv} \mathcal{A}$ do not have isomorphic homology groups

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Funding: DFG (SFB/TRR 109 *Discretization in Geometry and Dynamics*)

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- Discuss approximate nerve theorems.
- Discuss Latschev's result on the reconstruction of Riemannian manifolds and its functoriality by using *Vietoris–Rips good covers*.

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