# Short-Time Fourier Analysis

In an effort to correct this deficiency, Dennis Gabor (1946) adapted the Fourier transform to analyze only a small section of the signal at a time -- a technique called *windowing* the signal. Gabor's adaptation, called the *Short-Time Fourier Transform* (STFT), maps a signal into a two-dimensional function of time and frequency.

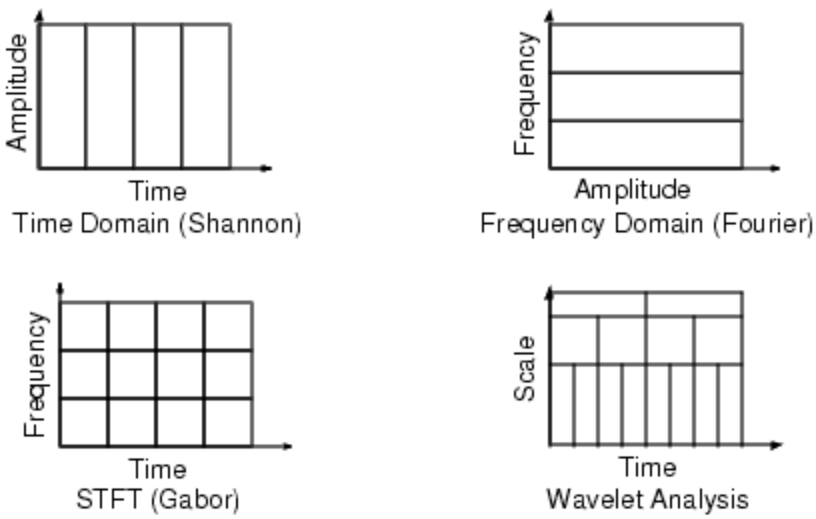

The STFT represents a sort of compromise between the time- and frequency-based views of a signal. It provides some information about both when and at what frequencies a signal event occurs. However, you can only obtain this information with limited precision, and that precision is determined by the size of the window.

While the STFT compromise between time and frequency information can be useful, the drawback is that once you choose a particular size for the time window, that window is the same for all frequencies. Many signals require a more flexible approach -- one where we can vary the window size to determine more accurately either time or frequency.

# Wavelet Analysis

Wavelet analysis represents the next logical step: a windowing technique with variable-sized regions. Wavelet analysis allows the use of long time intervals where we want more precise low-frequency information, and shorter regions where we want high-frequency information.


Here's what this looks like in contrast with the time-based, frequency-based, and STFT views of a signal:

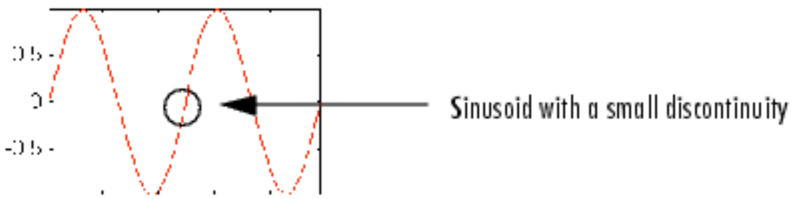



You may have noticed that wavelet analysis does not use a time-frequency region, but rather a time-*scale* region. For more information about the concept of scale and the link between scale and frequency, see [How to Connect Scale to Frequency?](#).

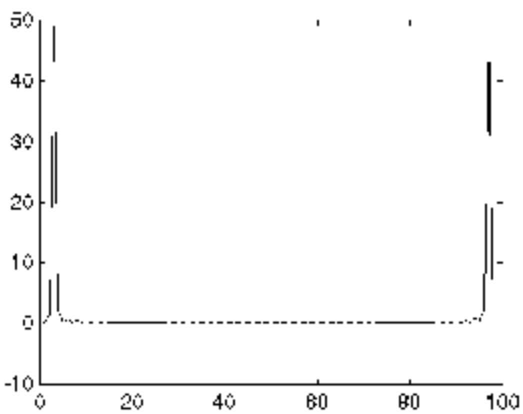
## What Can Wavelet Analysis Do?

One major advantage afforded by wavelets is the ability to perform *local analysis* -- that is, to analyze a localized area of a larger signal.

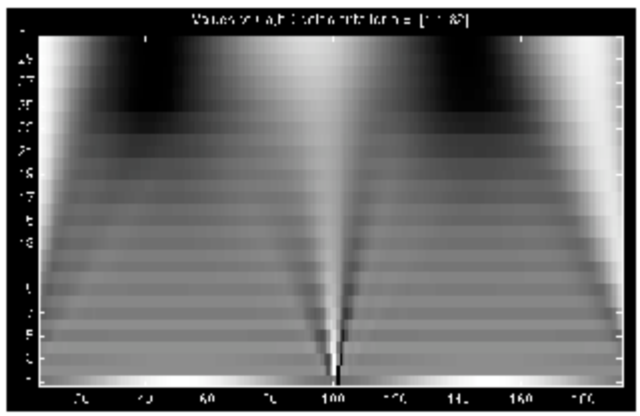
Consider a sinusoidal signal with a small discontinuity -- one so tiny as to be barely visible. Such a signal easily could be generated in the real world, perhaps by a power fluctuation or a noisy switch.




A plot of the Fourier coefficients (as provided by the `fft` command) of this signal shows nothing particularly interesting: a flat spectrum with two peaks representing a single frequency. However, a plot of wavelet coefficients clearly shows the exact location in time of the discontinuity.




Fourier Coefficients



Wavelet Coefficients

Wavelet analysis is capable of revealing aspects of data that other signal analysis techniques miss, aspects like trends, breakdown points, discontinuities in higher derivatives, and self-similarity. Furthermore, because it affords a different view of data than those presented by traditional techniques, wavelet analysis can often compress or de-noise a signal without appreciable degradation.

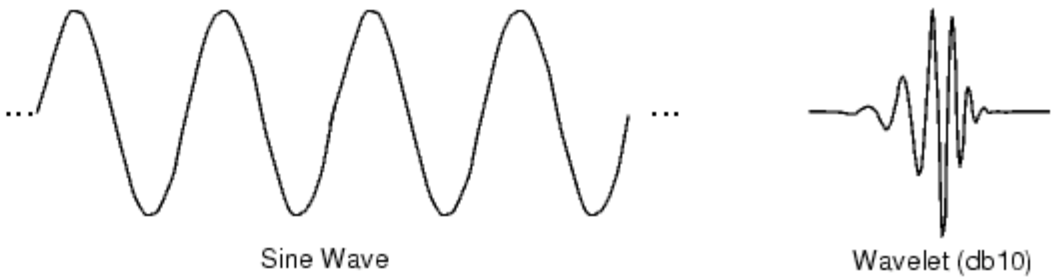
Indeed, in their brief history within the signal processing field, wavelets have already proven themselves to be an indispensable addition to the analyst's collection of tools and continue to enjoy a burgeoning popularity today.

# What Is Wavelet Analysis?

Now that we know some situations when wavelet analysis is useful, it is worthwhile asking "What is wavelet analysis?" and even more fundamentally, "What is a wavelet?"

A wavelet is a waveform of effectively limited duration that has an average value of zero.

Compare wavelets with sine waves, which are the basis of Fourier analysis. Sinusoids do not have limited duration -- they extend from minus to plus infinity. And where sinusoids are smooth and predictable, wavelets tend to be irregular and asymmetric.




Fourier analysis consists of breaking up a signal into sine waves of various frequencies. Similarly, wavelet analysis is the breaking up of a signal into shifted and scaled versions of the original (or *mother*) wavelet.

Just looking at pictures of wavelets and sine waves, you can see intuitively that signals with sharp changes might be better analyzed with an irregular wavelet than with a smooth sinusoid, just as some foods are better handled with a fork than a spoon.

It also makes sense that local features can be described better with wavelets that have local extent.

## Number of Dimensions

Thus far, we've discussed only one-dimensional data, which encompasses most ordinary signals. However, wavelet analysis can be applied to two-dimensional data (*images)* and, in principle, to higher dimensional data.

This toolbox uses only one- and two-dimensional analysis techniques.

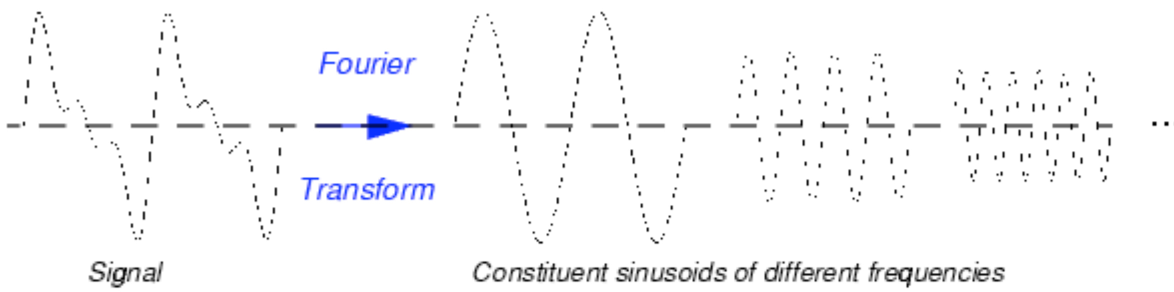
# The Continuous Wavelet Transform

Mathematically, the process of Fourier analysis is represented by the *Fourier transform*:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

which is the sum over all time of the signal *f(t)* multiplied by a complex exponential. (Recall that a complex exponential can be broken down into real and imaginary sinusoidal components.)

The results of the transform are the *Fourier coefficients* $F(\omega)$, which when multiplied by a sinusoid of frequency $\omega$ yield the constituent sinusoidal components of the original signal. Graphically, the process looks like

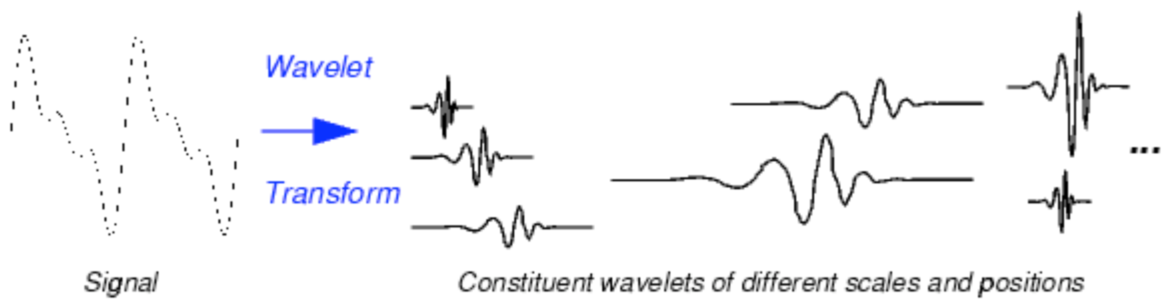



Similarly, the *continuous wavelet transform* (CWT) is defined as the sum over all time of the signal multiplied by scaled, shifted versions of the wavelet function $\psi$:

$$C(scale, position) = \int_{-\infty}^{\infty} f(t)\psi(scale, position, t)dt$$

The results of the CWT are many *wavelet coefficients* C, which are a function of scale and position.

Multiplying each coefficient by the appropriately scaled and shifted wavelet yields the constituent wavelets of the original signal:

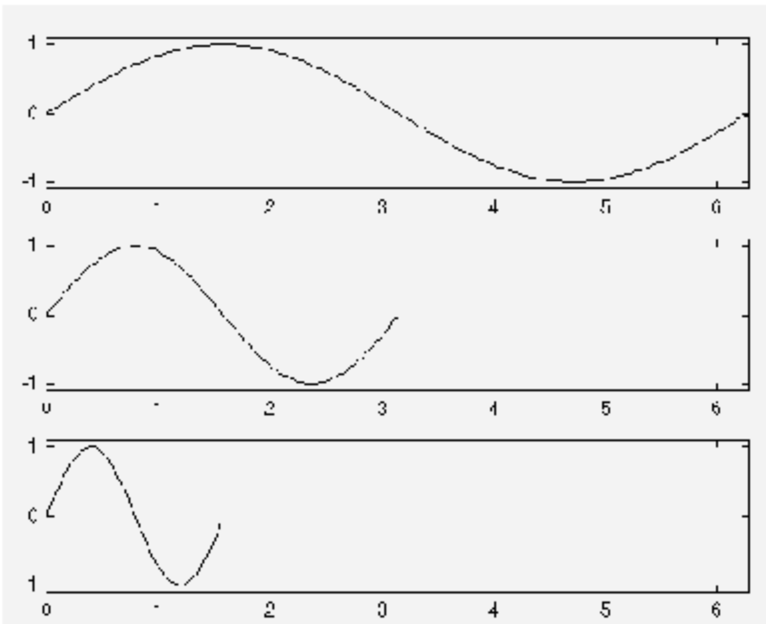


## Scaling

We've already alluded to the fact that wavelet analysis produces a time-scale view of a signal, and now we're talking about scaling and shifting wavelets. What exactly do we mean by *scale* in this context?

Scaling a wavelet simply means stretching (or compressing) it.

To go beyond colloquial descriptions such as "stretching," we introduce the *scale factor*, often denoted by the letter $a$. If we're talking about sinusoids, for example, the effect of the scale factor is very easy to see:

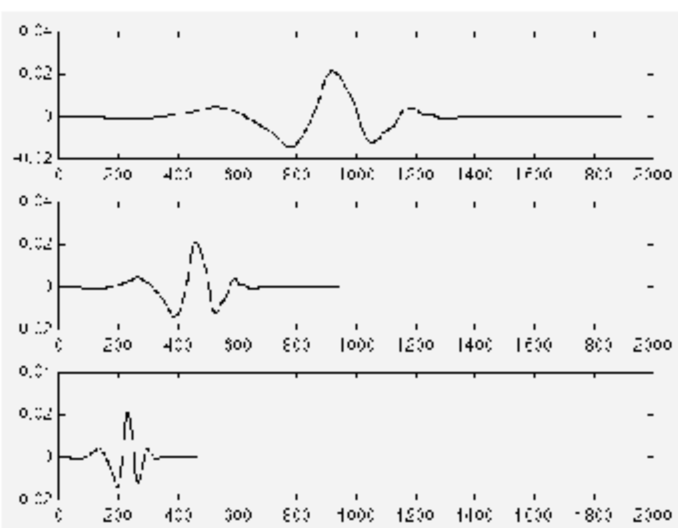


$$f(t) = \sin(t)\,; \quad a = 1$$

$$f(t) = \sin(2t)\,; \quad a = \frac{1}{2}$$

$$f(t) = \sin(4t)\,; \quad a = \frac{1}{4}$$

The scale factor works exactly the same with wavelets. The smaller the scale factor, the more "compressed" the wavelet.



$$f(t) = \psi(t) \quad ; \quad a = 1$$

$$f(t) = \psi(2t)\,; \quad a = \frac{1}{2}$$

$$f(t) = \psi(4t)\,; \quad a = \frac{1}{4}$$

It is clear from the diagrams that, for a sinusoid $\sin(\omega t)$, the scale factor $a$ is related (inversely) to the radian frequency $\omega$. Similarly, with wavelet analysis, the scale is related to the frequency of the signal. We'll return to this topic later.

## Shifting

Shifting a wavelet simply means delaying (or hastening) its onset. Mathematically, delaying a function $f(t)$ by *k* is represented by $f(t-k)$: