Expected Shortfall

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Structure

1. Basic Definitions and Properties of U-Statistics
2. von Mises Functional

Assumption

Our assumption throughout this presentation will be that

$$X_1, X_2, \ldots, X_n \text{ are i.i.d. on a space } (R, \mathcal{B}(\mathcal{R}))$$

with the distribution function F.

Definition: Parametric Functional

A **Parametric Functional** is a functional for which there exists an unbiased estimator, i.e. it can be written in the following way:

$$\theta(F) = \int \cdots \int \Phi(x_1, \ldots, x_m)\, F(dx_1) \ldots F(dx_m),$$

where $\Phi(x_1, \ldots, x_m)$ is a symmetric function of m variables called the kernel of the functional and m is called the degree.

Definition: U-Statistic

Let Φ be a symmetric kernel of a parametric functional. Then the **U-statistic** is defined as

$$U_n := \frac{1}{\binom{n}{m}} \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} \Phi(X_{i_1}, X_{i_2}, \ldots, X_{i_m}),$$

where $\binom{n}{m}$ is the binomial coefficient. U_n is an unbiased estimator of the parametric functional.

Example: Sample Mean

Consider the parametric functional for the mean:

$$\theta(F) = \mu(F) = \int x \, dF(x),$$

where $\mu(F)$ represents the expected value under the distribution F.
In this case, we choose the kernel $\Phi(x) = x$.
Then the U-statistic becomes

$$U_n = \frac{1}{n} \sum_{i=1}^{n} X_i = \overline{X},$$

where $\overline{X}$ is the sample mean.

Example: Squared Mean Functional

Consider the parametric functional for the squared mean:

$$\theta(F) = \mu^2(F) = \left(\int x \, dF(x) \right)^2 .$$

Here, the kernel is defined as

$$\Phi(x_1, x_2) = x_1 x_2 .$$

The corresponding U-statistic is given by

$$U_n = \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} X_i X_j .$$

Canonical Functions

Assume that $E|\Phi| < \infty$ and define

$$\Phi_c(x_1, \ldots, x_c) := E\left[\Phi(x_1, \ldots, x_c, X_{c+1}, \ldots, X_m)\right],$$

or equivalently,

$$\Phi_c(x_1, \ldots, x_c) = E\left(\Phi(X_1, \ldots, X_m) \mid X_1 = x_1, \ldots, X_c = x_c\right) \tag{1.1.4}$$

Let $\Phi_0 = \theta(F)$ and $\Phi_m = \Phi$.

Canonical Functions (2)

We set

$$\tilde{\Phi} := \Phi - \theta(P), \quad \tilde{\Phi}_c := \Phi_c - \theta(P),$$

and define the following functions:

$$g_1(x_1) := \tilde{\Phi}_1(x_1),$$

$$g_2(x_1, x_2) := \tilde{\Phi}_2(x_1, x_2) - g_1(x_1) - g_1(x_2),$$

$$g_3(x_1, x_2, x_3) := \tilde{\Phi}_3(x_1, x_2, x_3) - \sum_{i=1}^{3} g_1(x_i) - \sum_{1 \le i < j \le 3} g_2(x_i, x_j),$$

$$g_m(x_1, \dots, x_m) := \tilde{\Phi}_m(x_1, \dots, x_m) - \sum_{i=1}^{m} g_1(x_i)$$

$- \sum_{1 \le i < j \le m} g_2(x_i, x_j) - \dots$

$- \sum_{1 \le i_1 < \dots < i_{m-1} \le m} g_{m-1}(x_{i_1}, \dots, x_{i_{m-1}}).$

Canonical Functions (3)

Thus we get

$$\tilde{\Phi}_c(x_1, \ldots, x_c) = \sum_{d=1}^{c} \sum_{1 \le i_1 < \cdots < i_d \le c} g_d(x_{i_1}, \ldots, x_{i_d}), \qquad (1.1.7)$$

where g_d are symmetric functions and they possess the property of complete degeneracy:

$$Eg_1(X_1) = 0,$$

$$Eg_2(x_1, X_2) = 0,$$

$$\vdots$$

$$Eg_m(x_1, \ldots, X_m) = 0. \qquad (1.1.8)$$

Hoeffding Representation

Let us write the U-statistic U_n with a kernel Φ in the form

$$U_n - \theta(F) = \binom{n}{m}^{-1} S_n, \tag{1.1.16}$$

where

$$S_n := \sum_{1 \le i_1 < \cdots < i_m \le n} \left(\Phi(X_{i_1}, \ldots, X_{i_m}) - \theta(F) \right). \tag{1.1.17}$$

Hoeffding Representation (2)

For $c = 1, \dots, m$, we set

$$S_{nc} := \sum_{1 \le i_1 < \cdots < i_c \le n} g_c(X_{i_1}, \dots, X_{i_c}). \tag{1.1.18}$$

Inserting (1.1.18) with $c = m$ into (1.1.17) and using:

$$\tilde{\Phi}_c(x_1, \dots, x_c) = \sum_{d=1}^{c} \sum_{1 \le i_1 < \cdots < i_d \le c} g_d(x_{i_1}, \dots, x_{i_d}), \tag{1.1.7}$$

we obtain

$$S_n = S_{nm} + \sum_{c=1}^{m-1} \binom{n}{m}^{-1} \sum_{1 \le i_1 < \cdots < i_c \le n} g_c(X_{i_1}, \dots, X_{i_c}).$$

Hoeffding Representation (3)

For $c = 1$, the term on the right-hand side can be written as follows:

$$\sum_{1 \le i_1 < \cdots < i_m \le n} \sum_{s=1}^{m} g_1(X_s).$$

In this sum, each $g_1(X_s), 1 \le s \le n$, appears the same number of times. Since the sum contains $\binom{n}{m}$ terms, each term is encountered $n^{-1}\binom{n}{m}$ times.

Therefore, the sum $S_{n1} = \sum_{s=1}^{n} g_1(X_s)$ appears $\binom{n}{1}^{-1}\binom{n}{m}$ times.

Hoeffding Representation (4)

By repeating this argument for $c = 2, \ldots, m-1$, we obtain

$$S_n = \sum_{c=1}^{m} \binom{n}{c} \binom{n}{m}^{-1} S_{nc}.$$

This and (1.1.16) imply the following formula:

$$U_n - \theta = \sum_{c=1}^{m} \binom{n}{c} \binom{n}{m}^{-1} S_{nc}, \tag{1.1.19}$$

which is called the Hoeffding representation (or canonical decomposition) of a U-statistic.

Definition: Rank of a U-statistic

The **rank** of a U-statistic is defined as the smallest integer $r \geq 1$ for which the following conditions hold:

$$g_1 = \cdots = g_{r-1} = 0, \quad g_r \neq 0.$$

Then r is called the **rank** of the U-statistic.

- **Case** $r = 1$: The U-statistic is called **nondegenerate**.
- **Case** $r \geq 2$: The U-statistic is called **degenerate**, and r is referred to as the **order of degeneracy**.
- **Case** $r = m$: The U-statistic is called **complete degenerate**.

Rank of a U-statistic

We define

$$U_{nc} := \binom{n}{c}^{-1} \sum_{1 \leq i_1 < \cdots < i_c \leq n} g_c(X_{i_1}, \ldots, X_{i_c}), \qquad (1.1.21)$$

where g_c represents the degenerate kernel function.
Then the Hoeffding representation of a U-statistic can be written as:

$$U_n - \theta = \sum_{c=r}^{m} \binom{m}{c} U_{nc}, \qquad (1.1.22)$$

i.e., any U-statistic (1.1.3) is a linear combination of U-statistics (1.1.21), whose kernels have the property of complete degeneracy (1.1.8).

Martingale Structure of a U-statistic

Definition (Martingale): Let $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$ be the σ-algebra generated by $X_1, \ldots, X_n$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. A sequence $\{X_n\}$ is called a **martingale** with respect to a filtration $\{\mathcal{F}_n\}$ if it satisfies the following conditions:

- (M1) X_n is $\mathcal{F}_n$-measurable for all n.
- (M2) $\mathbb{E}[|X_n|] < \infty$ for all n.
- (M3) $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$ for all n.

Definition: Martingale-Difference

A sequence $\{X_n\}$ is called a **martingale-difference** with respect to a filtration $\{\mathcal{F}_n\}$ if it satisfies the following conditions:

- (M1) X_n is $\mathcal{F}_n$-measurable for all n.
- (M2) $\mathbb{E}[|X_n|] < \infty$ for all n.
- (M3) $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = 0$ for all n.

Connection between Martingales and Martingale-Differences

There is a close relationship between martingales and martingale-differences:

- If $\xi = \{\xi_n\}$ is a martingale, then defining

$$\eta_0 := \xi_0 \quad \text{and} \quad \eta_n := \xi_n - \xi_{n-1} \quad \text{for } n \geq 1$$

  yields a **martingale-difference sequence** $\eta = \{\eta_n\}$.
- Conversely, if $\eta = \{\eta_n\}$ is a martingale-difference sequence, then defining

$$\xi_n := \sum_{k=0}^{n} \eta_k$$

  gives a **martingale** $\xi = \{\xi_n\}$.

Definition: Reversed Martingale (Backwards Martingale)

Let $\{B_n\} = \{\sigma(U_n, U_{n+1}, \dots)\}$, $n = 1, 2, \dots$, sequence of σ-algebras. A sequence $\{\xi_n\}$ is called a **reversed martingale** (or **backwards martingale**) with respect to a sequence of σ-algebras $\{\mathcal{B}_n\}$ if it satisfies the following conditions:

- (M1) ξ_n is $\mathcal{B}_n$-measurable for all n.
- (M2) $\mathbb{E}[|\xi_n|] < \infty$ for all n.
- (M3) $\mathbb{E}[\xi_n \mid \mathcal{B}_{n+1}] = \xi_{n+1}$ for all n.

Lemma 1.1.1 (Hoeffding)

Suppose that $E|\Phi| < \infty$, and let $\mathcal{F}_k = \sigma\{X_1, \ldots, X_k\}$ be the σ-algebra generated by $X_1, \ldots, X_k$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
Then, for every $c = 1, \ldots, m$, we have:

$$\mathbb{E}(S_{nc} \mid \mathcal{F}_k) = S_{kc}, \quad c \leq k \leq n. \tag{1.1.25}$$

This implies that $(S_{nc}, \mathcal{F}_n)$ for $n \geq c$ forms a martingale.

Proof of Lemma 1.1.1

According to property (1.1.8), we have:

$$\mathbb{E}(g_c(X_{i_1}, \ldots, X_{i_c}) \mid X_1, \ldots, X_k) = 0$$

if at least one of the indices $i_1, \ldots, i_c$ does not belong to $\{1, \ldots, k\}$.
For example, if $i_1 \notin \{1, \ldots, k\}$, then:

$$\mathbb{E}(g_c(X_{i_1}, \ldots, X_{i_c}) \mid X_1, \ldots, X_k)$$

$$= \mathbb{E}\left(\mathbb{E}[g_c(X_{i_1}, \ldots, X_{i_c}) \mid X_1, \ldots, X_k, X_{i_2}, \ldots, X_{i_c}] \middle| X_1, \ldots, X_k\right)$$

$$= \mathbb{E}\left(\mathbb{E}[g_c(X_{i_1}, \ldots, X_{i_c}) \mid X_{i_2}, \ldots, X_{i_c}] \mid X_1, \ldots, X_k\right)$$

$$= \mathbb{E}[0 \mid X_1, \ldots, X_k] = 0.$$

Thus,

$$\mathbb{E}(S_{nc} \mid X_1, \ldots, X_k) = \sum_{1 \le i_1 < \cdots < i_c \le k} g_c(X_{i_1}, \ldots, X_{i_c}) = S_{kc}.$$

Lemma 1.1.3: Reversed Martingale Structure of U_n

Lemma 1.1.3: The sequence $\{U_n\}$ is a reversed martingale with respect to the sequence $\{B_n\} = \{\sigma(U_n, U_{n+1}, \dots)\}$, $n = 1, 2, \dots,$ of σ-algebras.

Proof: We want to show that U_n is a reversed martingale.

Definition (reversed martingale): A sequence $\{\xi_n\}$ is a reversed martingale if it satisfies the following conditions:

- **(M1)** ξ_n is B_n-measurable.
- **(M2)** $\mathbb{E}[|\xi_n|] < \infty$ for all n.
- **(M3)** $\mathbb{E}[\xi_n \mid B_{n+1}] = \xi_{n+1}$ for all n.

We already know that (M1) and (M2) are satisfied, so it remains to prove (M3) for reversed martingales.

Proof of Lemma 1.1.3 (continued)

(M1) is satisfied because the σ-algebras are chosen such that U_n is B_n-measurable. Any permutation of the variables $X_1, X_2, \ldots, X_n$ does not change the sequence $U_n, U_{n+1}, \ldots$, since a U-statistic is a symmetric function (the kernel Φ is symmetric). This fact, combined with the independence of X_i, $i = 1, \ldots, n$, for any $(i_1, \ldots, i_m)$ such that $1 \leq i_1 < \ldots < i_m \leq n$, leads to the relation:

$$\mathbb{E}\left(\Phi(X_{i_1}, \ldots, X_{i_m}) \mid B_n\right) = \mathbb{E}\left(\Phi(X_1, \ldots, X_m) \mid B_n\right). \tag{1}$$

Carrying out the summation over all indices $1 \leq i_1 < \ldots < i_m \leq n$ on both sides of (1), we obtain:

$$\mathbb{E}(U_n \mid B_n) = \mathbb{E}(\Phi(X_1, \ldots, X_m) \mid B_n).$$

Proof of Lemma 1.1.3 (continued)

Since $\mathbb{E}(U_n \mid B_n) = U_n$, because U_n is B_n-measurable, we have:

$$U_n = \mathbb{E}(\Phi(X_1, \ldots, X_m) \mid B_n), \qquad (2)$$

for all $n = m, m+1, \ldots$. Now, let us take the conditional expectation with respect to the σ-algebra B_{n+1} in (2). Then:

$$\mathbb{E}(U_n \mid B_{n+1}) = \mathbb{E}\left(\mathbb{E}(\Phi(X_1, \ldots, X_m) \mid B_n) \mid B_{n+1}\right).$$

Using the property of conditional expectation (smaller σ-algebra wins) and the inclusion $B_n \supseteq B_{n+1}$, we obtain:

$$\mathbb{E}(U_n \mid B_{n+1}) = \mathbb{E}(\Phi(X_1, \ldots, X_m) \mid B_{n+1}) = U_{n+1}.$$

This shows that U_n is a reversed martingale.
q.e.d.

Dispersion of U-Statistics

Denote

$$\eta_c := \mathbb{E}\left[\tilde{\Phi}_c^2(X_1, \ldots, X_c)\right], \quad c = 0, 1, \ldots, m,$$

By virtue of the symmetry of $\tilde{\Phi}$ and the independence of $X_1, \ldots, X_n$, we have:

$$\mathbb{E}\left(\tilde{\Phi}(X_{i_1}, \ldots, X_{i_m})\tilde{\Phi}(X_{j_1}, \ldots, X_{j_m})\right) = \eta_c,$$

where c represents the number of identical elements in the two sets $\{X_{i_1}, \ldots, X_{i_m}\}$ and $\{X_{j_1}, \ldots, X_{j_m}\}$. By definition, the dispersion of a U-statistic U_n is given by:

$$\sigma^2(U_n) := \mathbb{E}\left[(U_n - \theta)^2\right].$$

Expanding this, we obtain:

$$\sigma^2(U_n) = \binom{n}{m}^{-1} \sum_{c=0}^{m} \sum_{(c)} \mathbb{E}\left(\tilde{\Phi}(X_{i_1}, \ldots, X_{i_m})\tilde{\Phi}(X_{j_1}, \ldots, X_{j_m})\right).$$

Dispersion of U-Statistics (continued)

We get:

$$\sigma^2(U_n) = \binom{n}{m}^{-1} \sum_{c=1}^{m} \binom{m}{c} \binom{n-m}{m-c} \eta_c, \tag{1.1.34}$$

where $\eta_c = \mathbb{E}[\tilde{\Phi}_c^2(X_1, \ldots, X_c)]$ for $c = 1, \ldots, m$. Let

$$\delta_c := \mathbb{E}\left[g_c^2(X_1, \ldots, X_c)\right], \quad c = 1, \ldots, m.$$

We aim to represent $\sigma^2(U_n)$ in terms of the variables δ_c by using expression (1.1.9).

Dispersion of U-Statistics (continued)

The dispersion of the U-statistic U_n can be expressed in terms of δ_c as follows:

$$\sigma^2(U_n) = \sum_{c=1}^{m} \binom{m}{c}^2 \binom{n}{c}^{-1} \delta_c. \tag{1.1.36}$$

From expressions (1.1.7) and (1.1.35), we have:

$$\eta_c = \delta_c + \binom{c}{1}\delta_{c-1} + \cdots + \binom{c}{c-1}\delta_1, \tag{1.1.37}$$

for $c = 1, \ldots, m$.

Dispersion of U-Statistics (continued)

By induction, we find from (1.1.37) that:

$$\delta_c = \eta_c - \binom{c}{1}\eta_{c-1} + \binom{c}{2}\eta_{c-2} - \cdots + (-1)^{c-1}\binom{c}{c-1}\eta_1, \quad (1.1.38)$$

for all $c = 1, \ldots, m$. Relations (1.1.37) and (1.1.38) show that if r is the rank of a U-statistic, then:
If $\eta_0 = \cdots = \eta_{r-1} = 0$ and $\eta_r \neq 0$,
this is equivalent to the condition $\delta_c = 0$ for $c < r$.

Lemma 1.1.4 (Hoeffding)

Let r be the rank of a U-statistic defined by (1.1.20). We define

$$\gamma_r(m,n) = \binom{n}{m}^{-1} \sum_{c=r}^{m} \binom{m}{c}\binom{n-m}{m-c}\eta_c.$$

The following relations hold:

- For $1 \leq c < d \leq m$,

$$\eta_c \leq \frac{c}{d}\eta_d. \tag{1.1.39}$$

- The variance $\sigma^2(U_n)$ satisfies

$$\frac{m^2}{n}\eta_1 \leq \sigma^2(U_n) \leq \frac{m}{n}\eta_m. \tag{1.1.40}$$

- The variable $n\sigma^2(U_n)$ is decreasing with respect to n:

$$(n+1)\sigma^2(U_{n+1}) \leq n\sigma^2(U_n). \tag{1.1.41}$$

- Furthermore,

$$\frac{m}{d}\gamma_r(m,n)\eta_d \leq \sigma^2(U_n) \leq \gamma_r(m,n)\eta_m. \tag{1.1.42}$$

Proof of Inequality (1.1.39)

We start with the expression:

$$\eta_c = \delta_c + \binom{c}{1}\delta_{c-1} + \cdots + \binom{c}{c-1}\delta_1, \qquad (1.1.37)$$

for $c = 1, \ldots, m$.
For $c \leq d$, using (1.1.37), we can write:

$$\begin{aligned} c\eta_d - d\eta_c &= c\sum_{i=1}^{d}\binom{d}{i}\delta_i - d\sum_{i=1}^{c}\binom{c}{i}\delta_i \\ &= \sum_{i=1}^{c}\left[c\binom{d}{i} - d\binom{c}{i}\right]\delta_i + c\sum_{i=c+1}^{d}\binom{d}{i}\delta_i. \end{aligned}$$

This implies inequality (1.1.39), since $c\binom{d}{i} - d\binom{c}{i} \geq 0$ for $1 \leq i \leq c \leq d$.

Definition: Projection of a U-Statistic

Assume $\mathbb{E}|\Phi| < \infty$. The projection of a U-statistic is defined as

$$\hat{U}_n := \sum_{i=1}^{n} \mathbb{E}(U_n \mid X_i) - (n-1)\theta, \tag{1.1.45}$$

where θ is the parameter being estimated. In terms of Φ_1, we have:

$$\hat{U}_n - \theta = \frac{m}{n} \sum_{j=1}^{n} \tilde{\Phi}_1(X_j). \tag{1.1.46}$$

In the general case where $\eta_0 = \cdots = \eta_{r-1} = 0 < \eta_r$, we get:

$$\hat{U}_n - \theta = \sum_{1 \le i_1 < \cdots < i_r \le n} \mathbb{E}(U_n \mid X_{i_1}, \ldots, X_{i_r}) - \binom{n}{r}\theta. \tag{1.1.47}$$

Projection of a U-Statistic (continued)

Applying the Hoeffding representation (1.1.19) and the martingale relation (1.1.25),

$$\mathbb{E}(S_{nc} \mid \mathcal{F}_k) = S_{kc}, \quad c \leq k \leq n,$$

we can rewrite the general definition as:

$$\hat{U}_n - \theta = \binom{m}{r} \binom{n}{r}^{-1} S_{nr}, \tag{1.1.48}$$

since $S_{n0} = \cdots = S_{n,r-1} = 0$.
For $r = 1$, we recover (1.1.46).

The Hoeffding Formula

Let $k = \lfloor \frac{n}{m} \rfloor$ be the integral part of $\frac{n}{m}$. We set

$$h(x_1, \dots, x_n) := k^{-1}(\Phi(x_1, \dots, x_m) + \Phi(x_{m+1}, \dots, x_{2m}) + \dots$$

$$+ \Phi(x_{km-m+1}, \dots, x_{km})).$$

Then, by using the symmetry of the kernel Φ, we get:

$$k \sum_{\sigma \in S_n} h(x_{\sigma(i_1)}, \dots, x_{\sigma(i_n)}) = km!(n-m)! \sum_{c} \Phi(x_{i_1}, \dots, x_{i_m}).$$

The Hoeffding Formula (continued)

Or, using (1.1.3):

$$\sum_{\sigma \in S_n} h(X_{\sigma(i_1)}, \ldots, X_{\sigma(i_n)}) = m!(n-m)! \binom{n}{m} U_n.$$

Thus,

$$U_n = \frac{1}{n!} \sum_{\sigma \in S_n} h(X_{\sigma(i_1)}, \ldots, X_{\sigma(i_n)}), \qquad (1.1.50)$$

i.e., U_n is represented as an average of $n!$ terms, each being, in turn, an average of k independent identically distributed random variables.

Definition: Von Mises' Functionals (or von Mises' Statistics)

The von Mises' statistic V_n is defined as:

$$V_n := n^{-m} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \Phi(X_{i_1}, \ldots, X_{i_m}). \qquad (1.3.1)$$

With the help of the Dirac measure, we define the empirical measure P_n by the formula:

$$P_n(A) := \frac{1}{n} \sum_{j=1}^{n} \delta_{X_j}(A), \quad A \in \mathcal{X}. \qquad (1.3.2)$$

Von Mises' Functionals (continued)

Then, we can write (1.3.1) in the integral form:

$$V_n = \int \cdots \int \Phi(x_1, \ldots, x_m)\, dP_n(x_1) \cdots dP_n(x_m). \tag{1.3.3}$$

Consequently, $V_n = \theta(P_n)$, where the functional $\theta(F)$ is defined by (1.1.1).
Though P_n is an unbiased estimate of F, the functional V_n is **not** an unbiased estimate of $\theta(F)$.

Canonical Decomposition

We set

$$V_{nc} := \int \cdots \int \Phi_c(x_1, \ldots, x_c) \prod_{j=1}^{c} d(P_n(x_j) - P(x_j)), \qquad (1.3.6)$$

for $c = 1, \ldots, m$.
In the notation of (1.1.13):

$$g_c(x_1, \ldots, x_c) = \int \cdots \int \Phi_c(y_1, \ldots, y_c) \prod_{s=1}^{c} d(\delta_{x_s}(y_s) - P(y_s)).$$

Canonical Decomposition (continued)

This equality takes the form:

$$V_{nc} = \frac{1}{n^c} \sum_{i_1=1}^{n} \cdots \sum_{i_c=1}^{n} g_c(X_{i_1}, \ldots, X_{i_c}). \tag{1.3.7}$$

For every $c = 1, \ldots, m$, V_{nc} is a von Mises' functional with the kernel g_c possessing the property of complete degeneracy.
Using (1.1.11), we obtain

$$\theta(P_n) - \theta(P) = \sum_{c=1}^{m} \binom{m}{c} V_{nc}. \tag{1.3.8}$$

Here, we also take into account that $V_n = \theta(P_n)$.