Exam Notes for Analysis II

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Differentiation and Integration

Differentiation Rules

$$y = \frac{u}{v} \Rightarrow y' = \frac{u'v - v'u}{v^2}$$

Integration Rules

$$\int_a^b f(x)g(x)dx = \Big[F(x)g(x)\Big]_a^b - \int_a^b F(x)g'(x)dx$$

Common Derivatives

Function	Derivative
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$

Common Antiderivatives

Function	Antiderivative
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\frac{1}{x}$	$\log(\lvert x \rvert) + C$

Trigonometric Functions

Trigonometric Values

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Taylor Series of Trigonometric Functions

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Euler Formulas

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

Trigonometric Equations

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\sin^3(x) = \frac{1}{4}(3\sin(x) - \sin(3x))$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

$$\cos^3(x) = \frac{1}{4}(3\cos(x) + \cos(3x))$$

Ordinary Differential Equations

Differential equations are called *ordinary*, if the unknown function and its derivatives are evaluated at the same point.

Reducing an ODE to an Equation of Order 1

An *ODE* $y^{(k)} + a_{(k-1)} y^{(k-1)} + \cdots + a_1 y' + a_0 y = b$ can be rewritten into a ODE of order 1. We first introduce $y_0 = y, y_1 = y' = y_0', \ldots, y_{k-1} = y^{(k-1)} = y_{k-2}', y_{k-1}' = y^{(k)}$, furthermore let $Y = (y_0, \ldots, y_{k-1})^T$. We now can write:

$$Y' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{k-1} \end{pmatrix} Y + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{pmatrix}$$

Linear Differential Equations

Homogeneous and Inhomogeneous Linear Differential Equations

Inhomogeneous linear differential equations are of the form $y^{(k)} + a_{(k-1)}y^{(k-1)} + \cdots + a_1 y' + a_0 y = b$. If $b = 0$, we call the equation *homogeneous*. Examples:

Equation	Ordinary	Linear	Linear Homogeneous
$f'(x) = f(x+1)$	No, because f and f' are evaluated at different points.	No, because not ordinary.	No, because not linear.
$y^2 = y'y''$	Yes	No, because y^2 is not linear.	No, because not linear.
$y'' + 2y = -e^{x^2}$	Yes	Yes	No, because $-e^{x^2}$ is in the equation.
$y^{(3)} + 6y' + y = 0$	Yes	Yes	Yes

If we can find any solution f_0, then all the other solutions are of the form $f + f_0$.

Solving Linear Differential Equations of Order 1

A linear differential equation of order 1 has the form $y' + ay = b$. Because all solutions are of the form $f + f_0$, we first find f by solving the homogeneous equation $y' + ay = 0$, then find a solution f_0 of the inhomogeneous equation.

Solving the Homogeneous Equation

$$y' + a(x)y = 0$$
$$y' = -a(x)y$$
$$\frac{y'}{y} = -a(x)$$
$$(\log(|y|))' = -a(x)$$
$$\log(|y|) = -\int a(x)dx$$
$$\log(|y|) = -A(x) + c$$
$$y = ze^{-A(x)}$$

The so the solution is $f = ze^{-A(x)}$, for some z.

Solving the Inhomogeneous Equation

To solve the inhomogeneous equation, we use the method *variation of constants*, where we assume that z is a function $z(x)$. We insert the solution of the homogeneous equation for y.

$$
\begin{gathered}
y' + a(x)y = b(x) \\
(z(x)e^{-A(x)})' + a(x)(z(x)e^{-A(x)}) = b(x) \\
(z(x)'e^{-A(x)} + z(x)(-a(x))e^{-A(x)}) + a(x)(z(x)e^{-A(x)}) = b(x) \\
z(x)'e^{-A(x)} - a(x)z(x)e^{-A(x)} + a(x)z(x)e^{-A(x)} = b(x) \\
z(x)'e^{-A(x)} = b(x) \\
z(x)' = b(x)e^{A(x)} \\
z(x) = \int b(x)e^{A(x)}dx
\end{gathered}
$$

This gives us the result:

$$f_0 = \Big(\int b(x)e^{A(x)}dx\Big)e^{-A(x)}$$

The Final Result

The full solution space is now $f + f_0 = ze^{-A(x)} + (\int b(x)e^{A(x)}dx)e^{-A(x)} = e^{-A(x)}(z + \int b(x)e^{A(x)}dx)$ for some z. If we have some initial condition given for f, we can now simply insert it and solve for z.

Theoretically, we can now compute all ODEs using the solution above and the reduction of ODEs to ODEs of Order 1 as seen previously. But for calcualtion by hand, this is mostly too complicated.

Solving Linear Differential Equations with Constant Coefficients

Linear ODEs with constant coefficients are of the form $y^{(k)} + a_{(k-1)}y^{(k-1)} + \cdots + a_1y' + a_0y = b(x)$.

Solving the Homogeneous Equation

We first solve the homogeneous equation $y^{(k)} + a_{(k-1)}y^{(k-1)} + \cdots + a_1y' + a_0y = 0$. Because we know that the coefficients are constants, we know that the solutions are of the form $f = e^{\alpha x}$. because of this, $fy^{(k)} + a_{(k-1)}f^{(k-1)} + \cdots + a_1f' + a_0f = e^{\alpha x}(\alpha^k + a_{k-1}\alpha^{k-1} + \cdots + a_1\alpha + a_0) = 0$.

Solving the Characteristic Polynomial The *characteristic polynomial* has the form $P(X) = X^k + a_{k-1}X^{k-1} + \cdots + a_1 X + a_0$. From the assumption above, we know that f is a solution of the ODE if and only if $P(\alpha) = 0$.

From the *fundamental theorem of Algebra*, we know that this polynomial of degree k has k complex roots $\alpha_1, \ldots, \alpha_k$ such that $P(X) = (X - \alpha_1) \cdots (X - \alpha_k)$.

Suppose that a root $\alpha = \beta + i\gamma$ is not in the real space, so that γ is non-zero. In this case, the conjugate $\bar{\alpha} = \beta - i\alpha$ is also a root of P. We can replace the two solutions $f_1 = e^{\alpha x}, f_2 = e^{\bar{\alpha} x}$ with the real-valued functions $\tilde{f_1} = e^{\beta x}\cos(\gamma x), f_2 = e^{\beta x}\sin(\gamma x)$.

Case 1: No Multiple Roots If $\alpha_i \neq \alpha_j$ for all $i \neq j$, the homogeneous solution is $f = z_1 f_1 + \cdots + z_k f_k = z_1 e^{\alpha_1 x} + \cdots + z_k e^{\alpha_k x}$, such that f spans the full space of solutions for the homogeneous ODE.

Case 2: Multiple Roots Suppose that α is a multiple root of order j. Then the j functions $f_{a,0} = e^{\alpha x}, f_{a,1} = x e^{\alpha x}, f_{a,j-1} = x^{j-1} e^{\alpha x}$ give a basis of the space of solutions.

Solving the Inhomogeneus Equation

Special Tricks to Avoid Variation of Constants

Variation of Constants Given the equation $y^{(k)} + a_{(k-1)} y^{(k-1)} + \cdots + a_1 y' + a_0 y = b(x)$. We search for a solution of the form $f = z_1(x) f_1(x) + \cdots + z_k(x) f_k(x)$, where z_i are now functions instead of variables, and the solutions to the homogeneous equation f_i are known. We know that:

$$\begin{pmatrix} f_1 & \cdots & f_k \\ \vdots & & \vdots \\ f_1^{(k-2)} & \cdots & f_k^{(k-2)} \end{pmatrix} \begin{pmatrix} z_1' \\ \vdots \\ z_k' \end{pmatrix} = 0$$

This gives us $k-1$ equations, plus the original $f = z_1 f_1 + \cdots + z_k f_k$, so in total k equations for k unknowns.

Without loss of generalization, let's consider the case $k = 2$. We have:

$$y'' + a_1 y' + a_0 y = b$$
$$f = z_1 f_1 + z_2 f_2$$
$$z_1' f_1 + z_2' f_2 = 0$$

By differentiation, we get:

$$f' = z_1' f_1 + z_2' f_2 + z_1 f_1' + z_2 f_2' = z_1 f_2' + z_2 f_2'$$

$$f'' = z_1' f_1' + z_2' f_2 0 + z_1 f_1'' + z_2 f_2''$$

We now insert:

$$y'' + a_1 y' + a_0 y = (z_1' f_1' + z_2' f_2 0 + z_1 f_1'' + z_2 f_2'') + a_1(z_1 f_2' + z_2 f_2') + a_0(z_1 f_1 + z_2 f_2)$$

$$y'' + a_1 y' + a_0 y = z_1(f_1'' + a_1 f_1' + a_0 f_1) + z_2(f_2'' + a_1 f_2' + a_0 f_2) + z_1' f_1' + z_2' f_2'$$

Because f_1 and f_2 solve the homogeneous equation, we get:

$$y'' + a_1 y' + a_0 y = z_1' f_1' + z_2' f_2' = b$$

Finally we get:

$$\begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \frac{1}{f_1 f_2' - f_2 f_1'} \begin{pmatrix} f_2' & -f_2 \\ -f_1' & f_1 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix}$$

Now, we just have to insert $f_0 = \int_0^x z_1(t) dt f_1 + \int_0^x z_2(t) dt f_2$.

The Final Result

The full solution space is again $f + f_0$. If we have some initial conditions given, we can insert them and solve for the unknowns.

Other Methods

Change of Variable

If we replace $f(x)$ with $h(x) = f(g(x))$ and we can find a result for $h(x)$, then $f(x) = h(g^{-1}(x))$.

Separation of Variable

If a differential equation of order 1 can be written as $(g(y))' = g(y)'y' = b$, this can be solved by writing $g(f(x)) = B(x)$ and then $f(x) = g^{-1}(B(x))$.

Differential Calculus in $\mathbb{R}^n$

Continuity in $\mathbb{R}^n$

Continuity of Functions

A sequence (x_k) *converges* to y as $k \to \infty$ if for all $\epsilon > 0$, there exists $N \geq 1$ such that for all $n \geq N$, we have $\|x_k - y\| \leq \epsilon$.

f is *continuous* at x_0 if for all $\epsilon > 0$, there exists $\delta > 0$ such that, if $\|x - x_0\| < \delta$, then $\|f(x) - f(x_0)\| < \epsilon$.

f is *continuous* on X, if it is continuous for all $x \in X$.

The composite of continuous functions is continuous.

Limit

f has the *limit* $\lim_{x \to x_0}(f(x))y$, if for every $\epsilon > 0$, there exists $\delta > 0$, such that for all $x \neq x_0$, $\|x - x_0\| < \delta$, we have $\|f(x) - y\| < \epsilon$.

We have $\lim_{x \to x_0}(f(x))y$ if and only if for every sequence (x_k) that converges to x, the sequence $(f(x_k))$ converges to y.

Bounded, Closed and Compact Sets

A subset $X \subset \mathbb{R}^n$ is *bounded* if the set of $\|x\|$ for $x \in X$ is bounded in $\mathbb{R}$.

A subset $X \subset \mathbb{R}^n$ is *closed* if for every sequence (x_k) in X that converges in $\mathbb{R}^n$ to some vector y, we have $y \in X$.

A subset X is *compact* if it is bounded and closed.

If f is continuous and Y is closed, then $f^{-1}(Y)$ is closed.

Partial Derivatives

Open Sets

A subset $X \subset R^n$ is *open* if, for any $x = (x_1, \ldots, x_n) \in X$, there exists $\delta > 0$ such that the set $\{y = (x_1, \ldots, y_n) \in R^n : |x_i - y_i| < \delta \text{ for all } i\}$ is contained in X.

A set $X \subset R^n$ is open if and only if the complement $Y = \{x \in R^n : x \in X\}$ is closed.

If f is continuous and Y is open, then $f^{-1}(Y)$ is open.

Derivatives

$\frac{\partial f}{\partial x_i}(x) = \partial_{x_i} f(x) = \partial_i f(x)$ is the derivative of f in respect to the i-th variable.

Jacobi Matrix

For $f(x) = (f_1(x), \ldots, f_m(x))$, the Jacobi matrix is defined as $J_f(x) = (\partial_{x_j} f_i(x))_{1 \le i \le m, 1 \le j \le n}$. An example:

$$f(x,y) = \begin{pmatrix} \cos(x^2+y) \\ e^{\sin(\pi xy)} - 1 \\ y + \frac{1}{x^2+1} \end{pmatrix}, J_f(x,y) = \begin{pmatrix} -2x\sin(x^2+y) & -\sin(x^2+y) \\ \pi y \cos(\pi xy) e^{\sin(\pi xy)} & \pi x \cos(\pi xy) e^{\sin(\pi xy)} \\ \frac{-2x}{(1+x^2)^2} & 1 \end{pmatrix}$$

Gradient

$$\nabla f(x_0) = \begin{pmatrix} \partial_{x_1} f(x_0) \\ \cdots \\ \partial_{x_n} f(x_0) \end{pmatrix}$$

The Differential

f is *differentiable* at x_0 with the differential u if $\lim_{x \to x_0} \frac{1}{\|x - x_0\|}(f(x) - f(x_0) - u(x - x_0)) = 0$. We denote $df(x_0) = u$.

f is *differentiable* on X if f is differentiable for all $x \in X$.

If X is open and $f : X \to R^m$ a function that is differentiable on X, then f is continuous on X and admits derivatives with respect to each variable.

Directional Derivative

$w = \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$ is the directional derivative in the direction v, where $g(t) = f(x_0 + tv)$.

This means the directional derivative is $D_u f(a) = \frac{d}{dt} f(a + tu)$ with $t = 0$.

We can simply compute it using the gradient $D_u f(a) = \nabla f(a) \cdot u$.

Higher Derivatives

For higher derivatives, we have commutativity, $\partial_{x,y} f = \partial_{y,x} f$, $\partial_{x,y,z} f = \partial_{y,x,z} f = \partial_{z,x,y} f = \cdots$ and so on.

Hessian Matrix

The Hessian Matrix is defined as $H_f(x) = (\partial_{x_i, x_j} f)_{1 \le i,j \le n}$. An example:

$$f(x,y,z) = x^2 y - \cos(xz^3)$$
$$\partial_x f = 2xy + z^3 \sin(xz^3), \partial_y f = x^2, \partial_z f = 3xz^2 \sin(xz^3)$$

$$H_f(x,y,z) = \begin{pmatrix} 2y + z^6\sin(xz^3) & 2x & 3z^2\sin(xz^3) + xz^6\cos(xz^3) \\ 2x & 0 & 0 \\ 3z^2\sin(xz^3) + xz^6\cos(xz^3) & 0 & 6xz\sin(xz^3) + 9x^2z^6\cos(xz^3) \end{pmatrix}$$

Change of Variable

The derivative of $h = f \circ g$ is given by:

$$\partial_{y_1} h = \frac{\partial f}{\partial x_1}\frac{\partial g_1}{\partial y_1} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial g_n}{\partial y_1}$$

Or often written as:

$$\partial_{y_1} f = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_1} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_1}$$

Taylor Polynomials

Let $m! = m_1! \cdots m_n!$, $|m| = m_1 + \cdots + m_n$, $y^m = y_1^{m_1} \cdots y_n^{m_n}$:

$$T_k f(y; x_0) = \sum_{|m| \leq k} \frac{1}{m!} \partial_x^m f(x_0) y^m$$

$$T_2 f(y; x_0) = f(x_0) + \nabla f(x_0) y + \frac{1}{2} y^t H_f(x_0) y$$

Critical Points

The point x is a *critical point*, if $\nabla f(x) = 0$.

Let p and q be the number of positive and negative eigenvalues of $H_f(x)$:

1. If $p = n$, f has a local minimum at x.
2. If $q = n$, f has a local maximum at x.
3. If $pq \neq 0$, f has a saddle point at x.

Or in other words:

1. $\det(H_f(x)) > 0$ and $f_{xx}(x) > 0$, then x is a local minimum of f.
2. $\det(H_f(x)) > 0$ and $f_{xx}(x) < 0$, then x is a local maximum of f.
3. $\det(H_f(x)) < 0$, then x is a saddle point of f.
4. $\det(H_f(x)) = 0$, then the test is inconclusive.

Integration in $\mathbb{R}^n$

Line Integrals

The *line integeral* of f, where f is often called a *vector field*, along $\gamma : [a, b] \to \mathbb{R}^n$ is denoted:

$$\int_\gamma f(s)ds = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Conservative Vector Fields

A vector field f is *conservative*, if for any x_1, x_2, the line integral from x_1 to x_2 is independent from the choice of γ. Equivalently, f is conservative if and only if the line integral over γ is zero if $\gamma(a) = \gamma(b)$.

If f is conservative, then there exists a C^1 function g such that $f = \nabla g$.

Further, if f is conservative, then we have $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ for all $i \neq j$.

A subset $X \subset \mathbb{R}^n$ is *star-shaped* around x_0 if there exists an $x_0 \in X$ such that, for all $x \in X$, the line segment joining x_0 to x is contained in X.

If $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ for all $i \neq j$ on a star-shaped open subset of $\mathbb{R}^n$, then f is conservative.

The *potential* $\omega(x, y)$ is the scalar field whose gradient is the given vector field $f = \nabla\omega(x, y)$.

Curl

$$\operatorname{curl}(f) = \nabla \times f = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix} = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{pmatrix}$$

f is conservative if and only if $\operatorname{curl}(f) = 0$.

The Riemann Integral in $\mathbb{R}^n$

Improper Integrals

$$\lim_{x\to\infty} \int_{[a,x]\times I} f(x, y)dxdy = \lim_{x\to\infty} \int_a^x \Big(\int_I f(x, y)dy\Big)dx = \lim_{x\to\infty} \int_I \Big(\int_a^x f(x, y)dx\Big)dy$$

The Change of Variable Formula

Let $\varphi : X \to Y$, f a continuous function. We have:

$$\int_X f(\varphi(x))|\det(J_\varphi(x))|dx = \int_Y f(y)dy$$

From this follows:

$$\mathrm{Vol}(AX) = |\det(A)|\mathrm{Vol}(X)$$

Geometric Applications of Integrals

Center of Mass

Let $\bar{x} = (\bar{x}_n, \dots, \bar{x}_n)$ be the *center of mass* of X.

$$\bar{x}_i = \frac{1}{\mathrm{Vol}(X)} \int_X x_i dx$$

Surface Area

The *surface area* of the set $\Gamma = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in [a, b] \times [c, d], z = f(x, y)\}$ where $f : [a, b] \times [c, d] \to \mathbb{R}$ is given by:

$$\int_a^b \int_c^d \sqrt{1 + (\partial_x f(x, y))^2 + (\partial_y f(x, y))^2} dxdy$$

The Green Formula

$$\int_X \Big(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\Big) dxdy = \sum_{i=1}^{k} \int_{\gamma_i} f \cdot ds$$

If we want to compute the area, $\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) = 1$ and thus $f = (0, x)$.

From this, we get:

$$\mathrm{Vol}(X) = \sum_{i=1}^{k} \int_{a_i}^{b_i} \gamma_{i,1}(t) \gamma'_{i,2}(t) dt$$