

Exam Notes for Analysis II

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Lookup Table

Differentiation and Integration

Differentiation Rules

$$y = \frac{u}{v} \Rightarrow y' = \frac{u'v - v'u}{v^2}$$

Integration Rules

$$\int_a^b f(x)g(x)dx = \left[F(x)g(x)\right]_a^b - \int_a^b F(x)g'(x)dx$$

Common Derivatives

Function	Derivative
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$

Common Antiderivatives

Function	Antiderivative
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\arcsin(x)$	$x \arcsin(x) + \sqrt{1-x^2} + C$
$\arccos(x)$	$x \arccos(x) - \sqrt{1-x^2} + C$
$\frac{1}{x}$	$\log(x) + C$

Trigonometric Functions

Trigonometric Values

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
\sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
\cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
\tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Taylor Series of Trigonometric Functions

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Euler Formulas

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

Trigonometric Equations

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\sin^3(x) = \frac{1}{4}(3\sin(x) - \sin(3x))$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

$$\cos^3(x) = \frac{1}{4}(3\cos(x) + \cos(3x))$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

$$\sin(3x) = 3\sin(x) - 4\sin^3(x)$$

$$\cos(3x) = 4\cos^3(x) - 3\cos(x)$$

Derivatives and Antiderivatives of Squared Trigonometric Functions

$$(\sin^2(x))' = 2\cos(x)\sin(x)$$

$$(\cos^2(x))' = -2\cos(x)\sin(x)$$

$$(\sin(x)\cos(x))' = \cos^2(x) - \sin^2(x)$$

$$\int \sin^2(x) dx = \frac{2x - \sin(2x)}{4} + C$$

$$\int \cos^2(x) dx = \frac{\sin(2x) + 2x}{4} + C$$

$$\int \sin(x)\cos(x) dx = \frac{\sin^2(x)}{2} + C = -\frac{\cos^2(x)}{2}$$

Other Formulas

Solution Formula for Quadratic Equations

To solve $ax^2 + bx + c = 0$:

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Ordinary Differential Equations

Differential equations are called *ordinary*, if the unknown function and its derivatives are evaluated at the same point.

Reducing an ODE to an Equation of Order 1

An *ODE* $y^{(k)} + a_{(k-1)}y^{(k-1)} + \dots + a_1y' + a_0y = b$ can be rewritten into a ODE of order 1. We first introduce $y_0 = y, y_1 = y' = y'_0, \dots, y_{k-1} = y^{(k-1)} = y'_{k-2}, y'_{k-1} = y^{(k)}$, furthermore let $Y = (y_0, \dots, y_{k-1})^T$. We now can write:

$$Y' = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{k-1} \end{pmatrix} Y + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{pmatrix}$$

Linear Differential Equations

Homogeneous and Inhomogeneous Linear Differential Equations

Inhomogeneous linear differential equations are of the form $y^{(k)} + a_{(k-1)}y^{(k-1)} + \dots + a_1y' + a_0y = b$. If $b = 0$, we call the equation *homogeneous*. Examples:

Equation	Ordinary	Linear	Linear Homogeneous
$f'(x) = f(x+1)$	No, because f and f' are evaluated at different points.	No, because not ordinary.	No, because not linear.
$y^2 = y'y''$	Yes	No, because y^2 is not linear.	No, because not linear.
$y'' + 2y = -e^{x^2}$	Yes	Yes	No, because $-e^{x^2}$ is in the equation.
$y^{(3)} + 6y' + y = 0$	Yes	Yes	Yes

If we can find any solution f_0 , then all the other solutions are of the form $f + f_0$.

Solving Linear Differential Equations of Order 1

A linear differential equation of order 1 has the form $y' + ay = b$. Because all solutions are of the form $f + f_0$, we first find f by solving the homogeneous equation $y' + ay = 0$, then find a solution f_0 of the inhomogeneous equation.

Solving the Homogeneous Equation

$$\begin{aligned}y' + a(x)y &= 0 \\y' &= -a(x)y \\ \frac{y'}{y} &= -a(x) \\ (\log(|y|))' &= -a(x) \\ \log(|y|) &= -\int a(x)dx \\ \log(|y|) &= -A(x) + c \\ y &= ze^{-A(x)}\end{aligned}$$

The so the solution is $f = ze^{-A(x)}$, for some z .

Solving the Inhomogeneous Equation

To solve the inhomogeneous equation, we use the method *variation of constants*, where we assume that z is a function $z(x)$. We insert the solution of the homogeneous equation for y .

$$\begin{aligned}y' + a(x)y &= b(x) \\ (z(x)e^{-A(x)})' + a(x)(z(x)e^{-A(x)}) &= b(x) \\ (z(x)'e^{-A(x)} + z(x)(-a(x))e^{-A(x)} + a(x)(z(x)e^{-A(x)})) &= b(x) \\ z(x)'e^{-A(x)} - a(x)z(x)e^{-A(x)} + a(x)z(x)e^{-A(x)} &= b(x) \\ z(x)'e^{-A(x)} &= b(x) \\ z(x)' &= b(x)e^{A(x)} \\ z(x) &= \int b(x)e^{A(x)}dx\end{aligned}$$

This gives us the result:

$$f_0 = \left(\int b(x)e^{A(x)}dx \right) e^{-A(x)}$$

The Final Result

The full solution space is now $f + f_0 = ze^{-A(x)} + (\int b(x)e^{A(x)}dx)e^{-A(x)} = e^{-A(x)}(z + \int b(x)e^{A(x)}dx)$ for some z . If we have some initial condition given for f , we can now simply insert it and solve for z .

Theoretically, we can now compute all ODEs using the solution above and the reduction of ODEs to ODEs of Order 1 as seen previously. But for calculation by hand, this is mostly too complicated.

Solving Linear Differential Equations with Constant Coefficients

Linear ODEs with constant coefficients are of the form $y^{(k)} + a_{(k-1)}y^{(k-1)} + \dots + a_1y' + a_0y = b(x)$.

Solving the Homogeneous Equation

We first solve the homogeneous equation $y^{(k)} + a_{(k-1)}y^{(k-1)} + \dots + a_1y' + a_0y = 0$. Because we know that the coefficients are constants, we know that the solutions are of the form $f = e^{\alpha x}$. because of this, $fy^{(k)} + a_{(k-1)}f^{(k-1)} + \dots + a_1f' + a_0f = e^{\alpha x}(\alpha^k + a_{k-1}\alpha^{k-1} + \dots + a_1\alpha + a_0) = 0$.

Solving the Characteristic Polynomial The *characteristic polynomial* has the form $P(X) = X^k + a_{k-1}X^{k-1} + \dots + a_1X + a_0$. From the assumption above, we know that f is a solution of the ODE if and only if $P(\alpha) = 0$.

From the *fundamental theorem of Algebra*, we know that this polynomial of degree k has k complex roots $\alpha_1, \dots, \alpha_k$ such that $P(X) = (X - \alpha_1) \dots (X - \alpha_k)$.

Suppose that a root $\alpha = \beta + i\gamma$ is not in the real space, so that γ is non-zero. In this case, the conjugate $\bar{\alpha} = \beta - i\alpha$ is also a root of P . We can replace the two solutions $f_1 = e^{\alpha x}, f_2 = e^{\bar{\alpha}x}$ with the real-valued functions $\tilde{f}_1 = e^{\beta x} \cos(\gamma x), f_2 = e^{\beta x} \sin(\gamma x)$.

Case 1: No Multiple Roots If $\alpha_i \neq \alpha_j$ for all $i \neq j$, the homogeneous solution is $f = z_1f_1 + \dots + z_kf_k = z_1e^{\alpha_1x} + \dots + z_ke^{\alpha_kx}$, such that f spans the full space of solutions for the homogeneous ODE.

Case 2: Multiple Roots Suppose that α is a multiple root of order j . Then the j functions $f_{a,0} = e^{\alpha x}, f_{a,1} = xe^{\alpha x}, f_{a,j-1} = x^{j-1}e^{\alpha x}$ give a basis of the space of solutions.

Solving the Inhomogeneous Equation

Special Tricks to Avoid Variation of Constants

1. If $b(x) = x^d e^{\beta x}$ for some integer $d \geq 0$ and some number β which is not a root of P , then we look for a solution of the form $f(x) = Q(x)e^{\beta x}$ where Q is a polynomial of degree d .
2. If $b(x) = x^d \cos(\beta x)$ or $b(x) = x^d \sin(\beta x)$ for some integer $d \geq 0$ and some number β which is not a root of P , then we look for a solution of the form $f(x) = Q_1(x) \cos(\beta x) + Q_2(x) \sin(\beta x)$ where Q_1 and Q_2 are polynomials of degree d .
3. If 1. or 2. apply with β is a root of P , then we look for an analogue solution but with Q being a polynomial of degree $d + 1$.
4. If $\beta = 0$ and 0 is a root of P , we look for a Q of degree $d + j$, where j is the multiplicity of 0 as a root of P .

Variation of Constants Given the equation $y^{(k)} + a_{(k-1)}y^{(k-1)} + \dots + a_1y' + a_0y = b(x)$. We search for a solution of the form $f = z_1(x)f_1(x) + \dots + z_k(x)f_k(x)$, where z_i are now functions instead of variables, and the solutions to the homogeneous equation f_i are known. We know that:

$$\begin{pmatrix} f_1 & \cdots & f_k \\ \vdots & & \vdots \\ f_1^{(k-2)} & \cdots & f_k^{(k-2)} \end{pmatrix} \begin{pmatrix} z_1' \\ \vdots \\ z_k' \end{pmatrix} = 0$$

This gives us $k - 1$ equations, plus the original $f = z_1f_1 + \dots + z_kf_k$, so in total k equations for k unknowns.

Without loss of generalization, let's consider the case $k = 2$. We have:

$$\begin{aligned} y'' + a_1y' + a_0y &= b \\ f &= z_1f_1 + z_2f_2 \\ z_1'f_1 + z_2'f_2 &= 0 \end{aligned}$$

By differentiation, we get:

$$\begin{aligned} f' &= z_1'f_1 + z_2'f_2 + z_1f_1' + z_2f_2' = z_1f_2' + z_2f_2' \\ f'' &= z_1'f_1' + z_2'f_2' + z_1f_1'' + z_2f_2'' \end{aligned}$$

We now insert:

$$\begin{aligned} y'' + a_1y' + a_0y &= (z_1'f_1' + z_2'f_2' + z_1f_1'' + z_2f_2'') + a_1(z_1f_2' + z_2f_2') + a_0(z_1f_1 + z_2f_2) \\ y'' + a_1y' + a_0y &= z_1(f_1'' + a_1f_1' + a_0f_1) + z_2(f_2'' + a_1f_2' + a_0f_2) + z_1'f_1' + z_2'f_2' \end{aligned}$$

Because f_1 and f_2 solve the homogeneous equation, we get:

$$y'' + a_1y' + a_0y = z'_1f'_1 + z'_2f'_2 = b$$

Finally we get:

$$\begin{pmatrix} f_1 & f_2 \\ f'_1 & f'_2 \end{pmatrix} \begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$\begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix} = \frac{1}{f_1f'_2 - f_2f'_1} \begin{pmatrix} f'_2 & -f_2 \\ -f'_1 & f_1 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix}$$

Now, we just have to insert $f_0 = \int_0^x z_1(t)dt f_1 + \int_0^x z_2(t)dt f_2$.

The Final Result

The full solution space is again $f + f_0$. If we have some initial conditions given, we can insert them and solve for the unknowns.

Other Methods

Change of Variable

If we replace $f(x)$ with $h(x) = f(g(x))$ and we can find a result for $h(x)$, then $f(x) = h(g^{-1}(x))$.

Separation of Variable

If a differential equation of order 1 can be written as $(g(y))' = g(y)'y' = b$, this can be solved by writing $g(f(x)) = B(x)$ and then $f(x) = g^{-1}(B(x))$.

Differential Calculus in \mathbb{R}^n

Continuity in \mathbb{R}^n

Continuity of Functions

A sequence (x_k) *converges* to y as $k \rightarrow \infty$ if for all $\epsilon > 0$, there exists $N \geq 1$ such that for all $n \geq N$, we have $\|x_k - y\| \leq \epsilon$.

f is *continuous* at x_0 if for all $\epsilon > 0$, there exists $\delta > 0$ such that, if $\|x - x_0\| < \delta$, then $\|f(x) - f(x_0)\| < \epsilon$.

f is *continuous* on X , if it is continuous for all $x \in X$.

The composite of continuous functions is continuous.

Limit

f has the *limit* $\lim_{x \rightarrow x_0} (f(x))y$, if for every $\epsilon > 0$, there exists $\delta > 0$, such that for all $x \neq x_0$, $\|x - x_0\| < \delta$, we have $\|f(x) - y\| < \epsilon$.

We have $\lim_{x \rightarrow x_0} (f(x))y$ if and only if for every sequence (x_k) that converges to x , the sequence $(f(x_k))$ converges to y .

Bounded, Closed and Compact Sets

A subset $X \subset \mathbb{R}^n$ is *bounded* if the set of $\|x\|$ for $x \in X$ is bounded in \mathbb{R} .

A subset $X \subset \mathbb{R}^n$ is *closed* if for every sequence (x_k) in X that converges in \mathbb{R}^n to some vector y , we have $y \in X$.

A subset X is *compact* if it is bounded and closed.

If f is continuous and Y is closed, then $f^{-1}(Y)$ is closed.

Partial Derivatives

Open Sets

A subset $X \subset \mathbb{R}^n$ is *open* if, for any $x = (x_1, \dots, x_n) \in X$, there exists $\delta > 0$ such that the set $\{y = (x_1, \dots, y_n) \in \mathbb{R}^n : |x_i - y_i| < \delta \text{ for all } i\}$ is contained in X .

A set $X \subset \mathbb{R}^n$ is open if and only if the complement $Y = \{x \in \mathbb{R}^n : x \notin X\}$ is closed.

If f is continuous and Y is open, then $f^{-1}(Y)$ is open.

Derivatives

$\frac{\partial f}{\partial x_i}(x) = \partial_{x_i} f(x) = \partial_i f(x)$ is the derivative of f in respect to the i -th variable.

Jacobi Matrix

For $f(x) = (f_1(x), \dots, f_m(x))$, the Jacobi matrix is defined as $J_f(x) = (\partial_{x_j} f_i(x))_{1 \leq i \leq m, 1 \leq j \leq n}$. An example:

$$f(x, y) = \begin{pmatrix} \cos(x^2 + y) \\ e^{\sin(\pi xy)} - 1 \\ y + \frac{1}{x^2 + 1} \end{pmatrix}, J_f(x, y) = \begin{pmatrix} -2x \sin(x^2 + y) & -\sin(x^2 + y) \\ \pi y \cos(\pi xy) e^{\sin(\pi xy)} & \pi x \cos(\pi xy) e^{\sin(\pi xy)} \\ \frac{-2x}{(1+x^2)^2} & 1 \end{pmatrix}$$

Gradient

$$\nabla f(x_0) = \begin{pmatrix} \partial_{x_1} f(x_0) \\ \vdots \\ \partial_{x_n} f(x_0) \end{pmatrix}$$

The Differential

f is *differentiable* at x_0 with the differential u if $\lim_{x \rightarrow x_0} \frac{1}{\|x - x_0\|} (f(x) - f(x_0) - u(x - x_0)) = 0$. We denote $df(x_0) = u$.

f is *differentiable* on X if f is differentiable for all $x \in X$.

If X is open and $f : X \rightarrow \mathbb{R}^m$ a function that is differentiable on X , then f is continuous on X and admits derivatives with respect to each variable.

Directional Derivative

$w = \lim_{t \rightarrow 0} \frac{f(x_0 + tv) - f(x_0)}{t}$ is the directional derivative in the direction v , where $g(t) = f(x_0 + tv)$.

This means the directional derivative is $D_u f(a) = \frac{d}{dt} f(a + tu)$ with $t = 0$.

We can simply compute it using the gradient $D_u f(a) = \nabla f(a) \cdot u$.

Higher Derivatives

For higher derivatives, we have commutativity, $\partial_{x,y} f = \partial_{y,x} f$, $\partial_{x,y,z} f = \partial_{y,x,z} f = \partial_{z,x,y} f = \dots$ and so on.

Hessian Matrix

The Hessian Matrix is defined as $H_f(x) = (\partial_{x_i, x_j} f)_{1 \leq i, j \leq n}$. An example:

$$\begin{aligned} f(x, y, z) &= x^2 y - \cos(xz^3) \\ \partial_x f &= 2xy + z^3 \sin(xz^3), \partial_y f = x^2, \partial_z f = 3xz^2 \sin(xz^3) \\ H_f(x, y, z) &= \begin{pmatrix} 2y + z^6 \sin(xz^3) & 2x & 3z^2 \sin(xz^3) + xz^6 \cos(xz^3) \\ 2x & 0 & 0 \\ 3z^2 \sin(xz^3) + xz^6 \cos(xz^3) & 0 & 6xz \sin(xz^3) + 9x^2 z^6 \cos(xz^3) \end{pmatrix} \end{aligned}$$

Change of Variable

The derivative of $h = f \circ g$ is given by:

$$\partial_{y_1} h = \frac{\partial f}{\partial x_1} \frac{\partial g_1}{\partial y_1} + \dots + \frac{\partial f}{\partial x_n} \frac{\partial g_n}{\partial y_1}$$

Or often written as:

$$\partial_{y_1} f = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial y_1} + \dots + \frac{\partial f}{\partial x_n} \frac{\partial x_n}{\partial y_1}$$

Taylor Polynomials

Let $m! = m_1! \cdots m_n!$, $|m| = m_1 + \cdots + m_n$, $y^m = y_1^{m_1} \cdots y_n^{m_n}$ ($y = x - x_0$ for approximations of f at point x):

$$T_k f(y; x_0) = \sum_{|m| \leq k} \frac{1}{m!} \partial_x^m f(x_0) y^m$$

$$T_2 f(y; x_0) = f(x_0) + \nabla f(x_0) y + \frac{1}{2} y^t H_f(x_0) y$$

Critical Points

The point x is a *critical point*, if $\nabla f(x) = 0$.

Let p and q be the number of positive and negative eigenvalues of $H_f(x)$:

1. If $p = n$, f has a local minimum at x .
2. If $q = n$, f has a local maximum at x .
3. If $pq \neq 0$, f has a saddle point at x .

Or in other words:

1. $\det(H_f(x)) > 0$ and $f_{xx}(x) > 0$, then x is a local minimum of f .
2. $\det(H_f(x)) > 0$ and $f_{xx}(x) < 0$, then x is a local maximum of f .
3. $\det(H_f(x)) < 0$, then x is a saddle point of f .
4. $\det(H_f(x)) = 0$, then the test is inconclusive.

Integration in \mathbb{R}^n

Line Integrals

The *line integral* of f , where f is often called a *vector field*, along $\gamma: [a, b] \rightarrow \mathbb{R}^n$ is denoted:

$$\int_{\gamma} f(s) ds = \int_a^b f(\gamma(t)) \gamma'(t) dt$$

Conservative Vector Fields

A vector field f is *conservative*, if for any x_1, x_2 , the line integral from x_1 to x_2 is independent from the choice of γ . Equivalently, f is conservative if and only if the line integral over γ is zero if $\gamma(a) = \gamma(b)$.

If f is conservative, then there exists a C^1 function g such that $f = \nabla g$.

Further, if f is conservative, then we have $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ for all $i \neq j$.

A subset $X \subset \mathbb{R}^n$ is *star-shaped* around x_0 if there exists an $x_0 \in X$ such that, for all $x \in X$, the line segment joining x_0 to x is contained in X .

If $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ for all $i \neq j$ on a star-shaped open subset of \mathbb{R}^n , then f is conservative.

The *potential* $\omega(x, y)$ is the scalar field whose gradient is the given vector field $f = \nabla \omega(x, y)$.

Curl

$$\text{curl}(f) = \nabla \times f = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix} = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{pmatrix}$$

f is conservative if and only if $\text{curl}(f) = 0$.

The Riemann Integral in \mathbb{R}^n

Improper Integrals

$$\lim_{x \rightarrow \infty} \int_{[a, x] \times I} f(x, y) dx dy = \lim_{x \rightarrow \infty} \int_a^x \left(\int_I f(x, y) dy \right) dx = \lim_{x \rightarrow \infty} \int_I \left(\int_a^x f(x, y) dx \right) dy$$

The Change of Variable Formula

Let $\varphi : X \rightarrow Y$, f a continuous function. We have:

$$\int_X f(\varphi(x)) |\det(J_\varphi(x))| dx = \int_Y f(y) dy$$

From this follows:

$$\text{Vol}(AX) = |\det(A)| \text{Vol}(X)$$

We have:

$$\gamma = \begin{pmatrix} r \cos(\varphi) \\ r \sin(\varphi) \end{pmatrix}, |\det(J_\gamma)| = r$$

$$\gamma = \begin{pmatrix} r \cos(\theta) \sin(\varphi) \\ r \sin(\theta) \sin(\varphi) \\ r \cos(\varphi) \end{pmatrix}, |\det(J_\gamma)| = r^2 \sin(\varphi)$$

Geometric Applications of Integrals

Center of Mass

Let $\bar{x} = (\bar{x}_n, \dots, \bar{x}_n)$ be the *center of mass* of X .

$$\bar{x}_i = \frac{1}{\text{Vol}(X)} \int_X x_i dx$$

Surface Area

The *surface area* of the set $\Gamma = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in [a, b] \times [c, d], z = f(x, y)\}$ where $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is given by:

$$\int_a^b \int_c^d \sqrt{1 + (\partial_x f(x, y))^2 + (\partial_y f(x, y))^2} dx dy$$

The Green Formula

$$\int_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx dy = \sum_{i=1}^k \int_{\gamma_i} f \cdot ds = \sum_{i=1}^k \int_{a_i}^{b_i} f(\gamma_i(t)) \cdot \gamma'_i(t) dt$$

If we want to compute the area, $\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) = 1$ and thus $f = (0, x)$, $f = (0, -y)$, and infinitely others.

From this, we get:

$$\text{Vol}(X) = \sum_{i=1}^k \int_{a_i}^{b_i} \gamma_{i,1}(t) \gamma'_{i,2}(t) dt$$