Exam Notes for Analysis II

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Lookup Table

Common Derivatives

Function	Derivative
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$

Common Antiderivatives

Function	Antiderivative
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\frac{1}{x}$	$\log(\lvert x\rvert) + C$

Ordinary Differential Equations

Differential equations are called *ordinary*, if the unknown function and its derivatives are evaluated at the same point.

Reducing an ODE to an Equation of Order 1

An *ODE* $y^{(k)}+a_{(k-1)}y^{(k-1)}+\cdots+a_1y'+a_0y = b$ can be rewritten into a ODE of order 1. We first introduce $y_0 = y, y_1 = y' = y_0', \ldots, y_{k-1} = y^{(k-1)} = y_{k-2}', y_{k-1}' = y^{(k)}$, furthermore let $Y = (y_0, \ldots, y_{k-1})^T$. We now can write:

$$Y' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{k-1} \end{pmatrix} Y + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{pmatrix}$$

Linear Differential Equations

Homogeneous and Inhomogeneous Linear Differential Equations

Inhomogeneous linear differential equations are of the form $y^{(k)} + a_{(k-1)}y^{(k-1)} + \cdots + a_1y' + a_0y = b$. If $b = 0$, we call the equation *homogeneous*. Examples:

Equation	Ordinary	Linear	Linear Homogeneous
$f'(x) = f(x+1)$	No, because f and f' are evaluated at different points.	No, because not ordinary.	No, because not linear.
$y^2 = y'y''$	Yes	No, because y^2 is not linear.	No, because not linear.
$y'' + 2y = -e^{x^2}$	Yes	Yes	No, because $-e^{x^2}$ is in the equation.
$y^{(3)} + 6y' + y = 0$	Yes	Yes	Yes

If we can find any solution f_0, then all the other solutions are of the form $f + f_0$.

Solving Linear Differential Equations of Order 1

A linear differential equation of order 1 has the form $y' + ay = b$. Because all solutions are of the form $f + f_0$, we first find f by solving the homogeneous equation $y' + ay = 0$, then find a solution f_0 of the inhomogeneous equation.

Solving the Homogeneous Equation

$$y' + a(x)y = 0$$

$$y' = -a(x)y$$

$$\frac{y'}{y} = -a(x)$$

$$(\log(|y|))' = -a(x)$$

$$\log(|y|) = -\int a(x)dx$$

$$\log(|y|) = -A(x) + c$$

$$y = ze^{-A(x)}$$

The so the solution is $f = ze^{-A(x)}$, for some z.

Solving the Inhomogeneous Equation

To solve the inhomogeneous equation, we use the method *variation of constants*, where we assume that z is a function $z(x)$. We insert the solution of the homogeneous equation for y.

$$y' + a(x)y = b(x)$$
$$(z(x)e^{-A(x)})' + a(x)(z(x)e^{-A(x)}) = b(x)$$
$$(z(x)'e^{-A(x)} + z(x)(-a(x))e^{-A(x)}) + a(x)(z(x)e^{-A(x)}) = b(x)$$
$$z(x)'e^{-A(x)} - a(x)z(x)e^{-A(x)} + a(x)z(x)e^{-A(x)} = b(x)$$
$$z(x)'e^{-A(x)} = b(x)$$
$$z(x)' = b(x)e^{A(x)}$$
$$z(x) = \int b(x)e^{A(x)}dx$$

This gives us the result $f_0 = (\int b(x)e^{A(x)}dx)e^{-A(x)}$.

The Final Result

The full solution space is now $f + f_0 = ze^{-A(x)} + (\int b(x)e^{A(x)}dx)e^{-A(x)} = e^{-A(x)}(z + \int b(x)e^{A(x)}dx)$ for some z. If we have some initial condition given for f, we can now simply insert it and solve for z.

Theoretically, we can now compute all ODEs using the solution above and the reduction of ODEs to ODEs of Order 1 as seen previously. But for calcualtion by hand, this is mostly too complicated.

Solving Linear Differential Equations with Constant Coefficients

Linear ODEs with constant coefficients are of the form $y^{(k)} + a_{(k-1)}y^{(k-1)} + \dots + a_1y' + a_0y = b(x)$.

Solving the Homogeneous Equation

We first solve the homogeneous equation $y^{(k)} + a_{(k-1)}y^{(k-1)} + \dots + a_1y' + a_0y = 0$. Because we know that the coefficients are constants, we know that the solutions are of the form $f = e^{\alpha x}$. because of this, $fy^{(k)} + a_{(k-1)}f^{(k-1)} + \dots + a_1f' + a_0f = e^{\alpha x}(\alpha^k + a_{k-1}\alpha^{k-1} + \dots + a_1\alpha + a_0) = 0$.

Solving the Characteristic Polynomial The *characteristic polynomial* has the form $P(X) = X^k + a_{k-1}X^{k-1} + \cdots + a_1 X + a_0$. From the assumption above, we know that f is a solution of the ODE if and only if $P(\alpha) = 0$.

From the *fundamental theorem of Algebra*, we know that this polynomial of degree k has k complex roots $\alpha_1, \ldots, \alpha_k$ such that $P(X) = (X - \alpha_1) \cdots (X - \alpha_k)$.

Suppose that a root $\alpha = \beta + i\gamma$ is not in the real space, so that γ is non-zero. In this case, the conjugate $\bar{\alpha} = \beta - i\alpha$ is also a root of P. We can replace the two solutions $f_1 = e^{\alpha x}, f_2 = e^{\bar{\alpha} x}$ with the real-valued functions $\tilde{f_1} = e^{\beta x} \cos(\gamma x), f_2 = e^{\beta x} \sin(\gamma x)$.

Case 1: No Multiple Roots If $\alpha_i \neq \alpha_j$ for all $i \neq j$, the homogeneous solution is $f = z_1 e^{\alpha_1 x} + \cdots + z_k e^{\alpha_k x}$, such that f spans the full space of solutions for the homogeneous ODE.

Case 2: Multiple Roots This special case is left out for the moment.

Solving the Inhomogeneus Equation

Special Tricks to Avoid Variation of Constants

Variation of Constants Given the equation $y^{(k)} + a_{(k-1)} y^{(k-1)} + \cdots + a_1 y' + a_0 y = b(x)$. We search for a solution of the form $f = z_1(x) f_1(x) + \cdots + z_k(x) f_k(x)$, where z_i are now functions instead of variables, and the solutions to the homogeneous equation f_i are known. We know that:

$$\begin{pmatrix} f_1 & \cdots & f_k \\ \vdots & & \vdots \\ f_1^{(k-2)} & \cdots & f_k^{(k-2)} \end{pmatrix} \begin{pmatrix} z_1' \\ \vdots \\ z_k' \end{pmatrix} = 0$$

This gives us $k-1$ equations, plus the original $f = z_1 f_1 + \cdots + z_k f_k$, so in total k equations for k unknowns.

Without loss of generalization, let's consider the case $k = 2$. We have:

$$y'' + a_1 y' + a_0 y$$
$$f = z_1 f_1 + z_2 f_2$$
$$z_1' f_1 + z_2' f_2 = 0$$

By differentiation, we get:

$$f' = z_1' f_1 + z_2' f_2 + z_1 f_1' + z_2 f_2' = z_1 f_2' + z_2 f_2'$$
$$f'' = z_1' f_1' + z_2' f_2 0 + z_1 f_1'' + z_2 f_2''$$

We now insert:

$$y'' + a_1 y' + a_0 y = (z_1' f_1' + z_2' f_2 0 + z_1 f_1'' + z_2 f_2'') + a_1(z_1 f_2' + z_2 f_2') + a_0(z_1 f_1 + z_2 f_2)$$

$$y'' + a_1 y' + a_0 y = z_1(f_1'' + a_1 f_1' + a_0 f_1) + z_2(f_2'' + a_1 f_2' + a_0 f_2) + z_1' f_1' + z_2' f_2'$$

Because f_1 and f_2 solve the homogeneous equation, we get:

$$y'' + a_1 y' + a_0 y = b = z_1' f_1' + z_2' f_2'$$

Finally we get:

$$\begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \frac{1}{f_1 f_2' - f_2 f_1'} \begin{pmatrix} f_2' & -f_2 \\ -f_1' & f_1 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix}$$

The Final Result

The full solution space is again $f + f_0$ for some z. If we have some initial conditions given, we can insert them and solve for the unknowns.

Other Methods

Differential Calculus in $\mathbb{R}^n$

Continuity in $\mathbb{R}^n$

Continuity of Functions

f is *continuous* at x_0 if for all $\epsilon > 0$, there exists $\delta > 0$ such that, if $\|x - x_0\| < \delta$, then $\|f(x) - f(x_0)\| < \epsilon$.

f is *continuous* on X, if it is continuous for all $x \in X$.

Limit

f has the *limit* $\lim_{x \to x_0}(f(x))y$, if for every $\epsilon > 0$, there exists $\delta > 0$, such that for all $x \neq x_0$, $\|x - x_0\| < \delta$, we have $\|f(x) - y\| < \epsilon$.

We have $\lim_{x \to x_0}(f(x))y$ if and only if for every sequence (x_k) that converges to x, the sequence $(f(x_k))$ converges to y.

Bounded, Closed and Compact Sets

A subset $X \subset \mathbb{R}^n$ is *bounded* if the set of $\|x\|$ for $x \in X$ is bounded in $\mathbb{R}$.

A subset $X \subset \mathbb{R}^n$ is *closed* if for every sequence (x_k) in X that converges in $\mathbb{R}^n$ to some vector y, we have $y \in X$.

A subset X is *compact* if it is bounded and closed in X.

Partial Derivatives

Jacobi Matrix

For $f(x) = (f_1(x), \ldots, f_m(x))$, the Jacobi matrix is defined as $J_f(x) = (\partial_{x_j} f_i(x))_{1 \leq i \leq m, 1 \leq j \leq n}$

Gradient

$$\nabla f(x_0) = \begin{pmatrix} \partial_{x_1} f(x_0) \\ \cdots \\ \partial_{x_n} f(x_0) \end{pmatrix}$$

Hessian Matrix

$H_f(x) = (\partial_{x_i, x_j} f)_{1 \leq i,j \leq n}$

The Differential

f is *differentiable* at x_0 with the differential u if $\lim_{x \to x_0} \frac{1}{\|x - x_0\|}(f(x) - f(x_0) - u(x - x_0)) = 0$. We denote $df(x_0) = u$.

f is *differentiable* on X if f is differentiable for all $x \in X$.

Directional Derivative

$w = \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$ is the directional derivative in the direction v, where $g(t) = f(x_0 + tv)$.

This means the directional derivative is $D_u f(a) = \frac{d}{dt} f(a + tu)$ with $t = 0$.

We can simply compute it using the gradient $D_u f(a) = \nabla f(a) \cdot u$.

Higher Derivatives

Change of Variable

Taylor Polynomials

Critical Points

Lagrange Multipliers

Directional Derivative

Integration in $\mathbb{R}^n$