

# Algorithms & Data Structures

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## Contents

### Preface

### References

### 1 Introduction

### 2 Basics

2.1 Incremental Algorithms	2
2.2 Loop Invariant	2
2.3 Correctness	2
2.4 Divide-and-Conquer Algorithms	2
2.5 Binary Search	2

### 3 Sorting

3.1 Insertion Sort	3
3.2 Merge Sort	3
3.3 Quick Sort (with Lomuto Partitioning)	3
3.4 Heap Sort	3
3.5 k-Smallest Element	4
3.6 Overview of Sorting Algorithms	4
3.7 Application	4

### 4 Data Structures

4.1 Stacks	5
4.2 Queues	5
4.3 Linked Lists	5
4.4 Dictionaries	5
4.5 Direct-Address Tables	5
4.6 Hash Tables	5
4.6.1 Chaining	6
4.6.2 Open Addressing	6
4.7 Binary Search Trees	6
4.7.1 Traversals	6
4.7.2 Basic Queries	6
4.7.3 Updates	7
4.7.4 Randomized Insertion	7
4.7.5 Rotations	7
4.7.6 Root Insertion	7
4.8 Red-Black Trees	7

### 5 Graphs

5.1 Representations of Graphs	8
5.2 Breadth-First Search	8
5.3 Depth-First Search	8
5.3.1 Topological Sort	9
5.3.2 Strongly Connected Components	9
5.4 Minimum Spanning Tree	9
5.5 Single-Source Shortest Paths	10

### 6 Design Techniques

6.1 Greedy Algorithms	10
6.2 Dynamic Programming	10

### 7 Complexity Theory

7.1 Decision Problems	11
7.2 Complexity Classes	11
7.3 Polynomial-Time Reductions	11
7.4 NP-Hardness and NP-Completeness	12

## Preface

This document is an unofficial student-made summary of the course Algorithms & Data Structures taught by Antonio Carzaniga in Spring 2025 at the Università della Svizzera italiana. It is based on the slides and other course materials, as well as [1]. The summary is not exhaustive and may contain errors. If you find any, please report them to [fabianlucasbosshard@gmail.com](mailto:fabianlucasbosshard@gmail.com) or open an issue at [https://github.com/fabianbosshard/USI\\_summaries](https://github.com/fabianbosshard/USI_summaries).

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## References

- [1] Thomas H. Cormen et al. Introduction to Algorithms. Fourth. MIT Press, 2022. ISBN: 9780262046305. URL: <https://mitpress.mit.edu/9780262046305/introduction-to-algorithms/>.

## 1 Introduction

1  $T(n)$  := number of **basic steps** needed to compute the result of a problem of size  $n$

1 **Example 1.** How many 1, 2-beats can one compose over a total of  $n$  beats?

1 Idea: the number of 1, 2-beats over  $n$  beats is the sum of the number of 1, 2-beats over  $n - 1$  beats and the number of 1, 2-beats over  $n - 2$  beats



### Algorithm 1 Pingala

```

1: function PINGALA( $n$ )                                ▷ Number of 1, 2-beats in  $n$  beats
2:   if  $n \leq 2$  then
3:     return  $n$ 
4:   return PINGALA( $n - 1$ ) + PINGALA( $n - 2$ )

```

$$T(1) = 1, \quad T(2) = 1, \quad T(n) = T(n-1) + T(n-2) + 2$$

$$T(n) \geq \underbrace{T(n-1) + T(n-2)}_{\geq T(n-2)} \geq 2T(n-2)$$

$$T(n) \geq 2T(n-2) \geq 2(2T(n-4)) = 2^2T(n-4) \geq \dots \geq 2^{n/2}T(0) = \sqrt{2}^n T(0)$$

This naïve recursive implementation has exponential running time,  $T(n) = \Omega(\sqrt{2}^n)$ . For this problem, we can do better if we avoid repeating the same computations over and over again (using Dynamic Programming, see 6.2). ◀

## 2 Basics

8 **Definition 1.** We define the following families of functions:

- $O(g(n)) := \{f(n) \mid \exists c > 0, n_0 \in \mathbb{N} \mid 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}$
- $\Omega(g(n)) := \{f(n) \mid \exists c > 0, n_0 \in \mathbb{N} \mid 0 \leq cg(n) \leq f(n) \text{ for all } n \geq n_0\}$
- $\Theta(g(n)) := \{f(n) \mid \exists c_1, c_2 > 0, n_0 \in \mathbb{N} \mid 0 \leq c_1g(n) \leq f(n) \leq c_2g(n) \forall n \geq n_0\}$
- $o(g(n)) := \{f(n) \mid \forall c > 0, \exists n_0 \in \mathbb{N} \mid 0 \leq f(n) < cg(n) \text{ for all } n \geq n_0\}$
- $\omega(g(n)) := \{f(n) \mid \forall c > 0, \exists n_0 \in \mathbb{N} \mid 0 \leq cg(n) < f(n) \text{ for all } n \geq n_0\}$  ◀

The notation ‘ $f(n) =$ ’ is used to denote that  $f(n)$  is an element of the set of functions on the right-hand side.

**Example 2.** Let  $\pi(n)$  be the number of primes less than or equal to  $n$ . Then

$$\pi(n) = \Theta\left(\frac{n}{\log n}\right). \quad \blacktriangleleft$$

The  $\Theta$ -notation,  $\Omega$ -notation, and  $O$ -notation can be viewed as the “asymptotic”  $=$ ,  $\geq$ , and  $\leq$  relations for functions. The  $o$ -notation and  $\omega$ -notation can be viewed as asymptotic  $<$  and  $>$ .

**Theorem 1.**  $f(n) = \Omega(g(n)) \wedge f(n) = O(g(n)) \Leftrightarrow f(n) = \Theta(g(n))$  ◀

The above theorem can be interpreted as saying

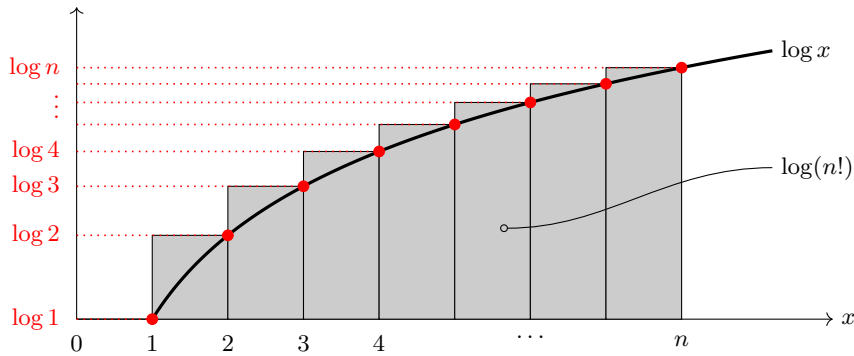
$$f \geq g \wedge f \leq g \Leftrightarrow f = g$$

**Example 3.**  $\log(n!) \in \Theta(n \log n)$ . We can rewrite  $\log(n!)$  as

$$\log(n!) = \log\left(\prod_{i=1}^n i\right) = \sum_{i=1}^n \log i \quad (1)$$

Clearly,  $\log(n!) \in O(n \log n)$ , since  $n \log n = \log(n^n) = \sum_{i=1}^n \log n \geq \sum_{i=1}^n \log i$ .

One way to understand why  $\log(n!) \in \Omega(n \log n)$  is to interpret (1) as a sum of areas, each rectangle has width 1 and height  $\log i$ :



We can see that the area of the gray rectangles is bounded from below by the area under the curve  $y = \log x$ , i.e.

$$\int_1^n \log x \, dx \leq \log(n!) \quad (2)$$

Using integration by parts, we can express the left-hand side of (2) as

$$\begin{aligned} \int_1^n \log x \, dx &= \int_1^n \underset{\downarrow}{1} \cdot \underset{\uparrow}{\log x} \, dx = \left[ x \log x - \int x \cdot \frac{1}{x} \, dx \right]_1^n = [x \log x - x]_1^n \\ &= n \log n - n + 1 \in \Omega(n \log n) \end{aligned}$$

Therefore,  $\log(n!) \in \Omega(n \log n) \wedge \log(n!) \in O(n \log n)$ . Using Theorem 1, we conclude that  $\log(n!) \in \Theta(n \log n)$ . ▶

Notation	Name	Example
$O(1)$	constant	Finding median of <b>sorted</b> array; computing $(-1)^n$ ; using a fixed-size lookup table.
$O(\alpha(n))$	inverse Ackermann	Amortized cost per operation in a disjoint-set data structure.
$O(\log^* n)$	iterated logarithmic	Distributed coloring of cycles (Cole-Vishkin algorithm).
$O(\log \log n)$	double logarithmic	Interpolation search on uniformly distributed data.
$O(\log n)$	logarithmic	Binary search; operations in balanced trees or a binomial heap.
$O(\log^c n)$ , $c > 1$	polylogarithmic	Matrix-chain ordering on a PRAM.
$O(n^c)$ , $0 < c < 1$	fractional power	Searching in a $k$ -d tree.
$O(\frac{n}{\log n})$		#Primes $\leq n$ (Example 1).
$O(n)$	linear	Scanning an unsorted list; ripple-carry addition of two $n$ -bit integers.
$O(n \log^* n)$		Seidel's polygon-triangulation algorithm.
$O(n \log n) = O(\log n!)$	linearithmic	Fastest comparison sorts; Fast Fourier transform.
$O(n^2)$	quadratic	Schoolbook multiplication; bubble/selection/insertion sort; worst-case quicksort; direct convolution.
$O(n^3)$	cubic	Naive $n \times n$ matrix multiplication; partial correlation.
$O(n^c)$ , $c > 1$	polynomial	TAG parsing; bipartite matching; determinant via LU.
$L_n[\alpha, c]$	sub-exponential	Factoring via number-field sieve.
$O(c^n)$ , $c > 1$	exponential	Exact TSP by DP; brute-force logical equivalence checking.
$O(n!)$	factorial	Brute-force TSP; enumerating permutations or partitions; determinant by Laplace expansion.

When  $f(n) = O(g(n))$ , we say that  $g(n)$  is an **upper bound** for  $f(n)$ , and that  $g(n)$  **dominates**  $f(n)$ .

When  $f(n) = \Omega(g(n))$ , we say that  $g(n)$  is a **lower bound** for  $f(n)$ .

When  $f(n) = \Theta(g(n))$ , we say that  $g(n)$  is a **tight bound** for  $f(n)$ .

We use the  $o$ -notation to denote an upper bound that is not asymptotically tight, and the  $\omega$ -notation to denote a lower bound that is not asymptotically tight. The following two implications hold:

$$f(n) = o(g(n)) \Rightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0 \quad f(n) = \omega(g(n)) \Rightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty$$

## 2.1 Incremental Algorithms

**Example 4 (Hand of cards).** Insertion sort (Algorithm 3) uses an algorithm design technique called **incremental** method: for each element  $A[i]$ , it inserts it into its proper place in the subarray  $A[1 : i]$ , having already sorted  $A[1 : i - 1]$ . This is reminiscent of how one might sort a hand of cards, where you pick up a card and insert it into the correct position in the already sorted hand.

At the start of each iteration of the for loop, the subarray  $A[1 : i - 1]$  consists of the elements originally in  $A[1 : i - 1]$ , but in sorted order. This is a loop invariant (Section 2.2). ▶

## 2.2 Loop Invariant

When using a **loop invariant**, 3 things need to be shown:

1. **Initialization:** It is true prior to the first iteration of the loop.
2. **Maintenance:** If it is true before an iteration of the loop, it remains true before the next iteration.
3. **Termination:** When the loop terminates, the invariant gives a useful property that helps show that the algorithm is correct.

A loop-invariant proof is a form of **mathematical induction**, where to prove that a property holds, you prove a base case and an inductive step. Here, showing that the invariant holds before the first iteration corresponds to the base case, and showing that the invariant holds from iteration to iteration corresponds to the inductive step. The third property is perhaps the most important one, since you are using the loop invariant to show correctness. Typically, you use the loop invariant along with the condition that caused the loop to terminate. Mathematical induction typically applies the inductive step infinitely, but in a loop invariant the “induction” stops when the loop terminates.

## 2.3 Correctness

You are given a problem  $P$  and an algorithm  $A$ .  $P$  formally defines a **correctness** condition. Assume, for simplicity, that  $A$  consists of one loop.

1. Formulate an invariant  $C$
2. **Initialization:** prove that  $C$  holds right before the first execution of the first instruction of the loop
3. **Management:** prove that if  $C$  holds right before the first instruction of the loop, then it holds also at the end of the loop
4. **Termination:** prove that the loop terminates, with some exit condition  $X$
5. Prove that  $X \wedge C \Rightarrow P$ , which means that  $A$  is correct

## 2.4 Divide-and-Conquer Algorithms

Many useful algorithms are **recursive**: they **recurse** (call themselves) one or more times to handle closely related subproblems. These algorithms typically follow the **divide-and-conquer** method: they break the problem into several subproblems that are similar to the original problem but smaller in size, solve the subproblems recursively, and then combine these solutions to create a solution to the original problem.

In the divide-and-conquer method, if the problem is small enough (the **base case**), you just solve it directly without recursing. Otherwise (the **recursive case**), you perform three characteristic steps:

1. **Divide** the problem into one or more subproblems that are smaller instances of the same problem.
2. **Conquer** the subproblems by solving them recursively.
3. **Combine** the subproblem solutions to form a solution to the original problem.

When an algorithm contains a recursive call, we can often describe its running time with a **recurrence relation**, which expresses the overall running time of a problem of size  $n$  in terms of the running time of the same algorithm on smaller inputs.

Let  $T(n)$  be the worst case running time on an input of size  $n$ . If the problem is small enough, say  $n \leq n_0$ , for some constant  $n_0$ , the straightforward solution takes constant time  $\Theta(1)$ . Suppose that the division of the problem yields  $a$  subproblems, each of size  $n/b$ .<sup>1</sup> If it takes  $D(n)$  time to divide the problem into subproblems and  $C(n)$  time to combine the solutions to the subproblems into the solution to the original problem, we get the recurrence

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq n_0 \\ D(n) + aT(n/b) + C(n) & \text{if } n > n_0 \end{cases} \quad (3)$$

**Theorem 2** (Master Theorem). Let  $a > 0$  and  $b > 1$  be constants, and let  $f(n)$  be a driving function that is defined and nonnegative on all sufficiently large reals. Define the recurrence  $T(n)$  on  $n \in \mathbb{N}$  by

$$T(n) = aT(n/b) + f(n)$$

where  $aT(n/b)$  actually means  $a'T(\lfloor n/b \rfloor) + a''T(\lceil n/b \rceil)$  for some constants  $a' \geq 0$  and  $a'' \geq 0$  satisfying  $a = a' + a''$ . Then the asymptotic behavior of  $T(n)$  can be characterized as follows:

1. If there exists a constant  $\epsilon > 0$  such that  $f(n) = O(n^{\log_b a - \epsilon})$ , then  $T(n) = \Theta(n^{\log_b a})$ .
2. If there exists a constant  $k \geq 0$  such that  $f(n) = \Theta(n^{\log_b a} \lg^k n)$ , then  $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$ .
3. If there exists a constant  $\epsilon > 0$  such that  $f(n) = \Omega(n^{\log_b a + \epsilon})$ , and if  $f(n)$  additionally satisfies the regularity condition  $af(n/b) \leq cf(n)$  for some constant  $c < 1$  and all sufficiently large  $n$ , then  $T(n) = \Theta(f(n))$ . ◁

## 2.5 Binary Search

Algorithm 2 is an efficient method for finding an element  $x$  in a sorted array  $A$ . By repeatedly halving the search interval, it reduces the problem size exponentially: at each step, it compares  $x$  to the middle element of the current interval and discards the half in which  $x$  cannot lie. This yields a worst-case running time of  $O(\log n)$ , a dramatic improvement over a linear search's  $O(n)$  behavior.

<sup>1</sup>For merge sort (Algorithm 4),  $a = b = 2$ , but there are other divide-and-conquer algorithms in which  $a \neq b$ .

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**Algorithm 2** Binary Search
 

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```

1: function BINARYSEARCH( $A, x$ )
2:    $l \leftarrow 1$                                  $\triangleright$  leftmost index
3:    $r \leftarrow \text{len}(A)$                          $\triangleright$  rightmost index
4:   while  $l \leq r$  do
5:      $m \leftarrow \lfloor (l + r)/2 \rfloor$                $\triangleright$  midpoint of  $A[l : r]$ 
6:     if  $A[m] < x$  then
7:        $l \leftarrow m + 1$                        $\triangleright$  search in right half
8:     else if  $A[m] > x$  then
9:        $r \leftarrow m - 1$                        $\triangleright$  search in left half
10:    else
11:      return  $m$                                  $\triangleright$  found  $x$  at index  $m$ 
12:  return 0                                     $\triangleright$  not found: 0 is not a valid index

```

---

### 3 Sorting

#### 3.1 Insertion Sort

Was already used to sort cards in Section 2.1, Example 4.

---

**Algorithm 3** Insertion Sort
 

---

```

1: function INSERTIONSORT( $A$ )
2:   for  $i = 2, \dots, \text{len}(A)$  do
3:      $j \leftarrow i$ 
4:     while  $j > 1 \wedge A[j-1] > A[j]$  do
5:       swap  $A[j]$  and  $A[j-1]$ 
6:        $j \leftarrow j - 1$ 

```

---

#### 3.2 Merge Sort

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**Algorithm 4** Merge Sort
 

---

```

1: function MERGESORT( $A$ )
2:   if  $\text{len}(A) \leq 1$  then                       $\triangleright$  base case (array is trivially sorted)
3:     return  $A$ 
4:    $m = \lfloor \text{len}(A)/2 \rfloor$                          $\triangleright$  midpoint of  $A$ 
5:    $A_L \leftarrow \text{MERGESORT}(A[1 : m])$            $\triangleright$  recursively sort  $A[1 : m]$ 
6:    $A_R \leftarrow \text{MERGESORT}(A[m+1 : |A|])$        $\triangleright$  recursively sort  $A[m+1 : |A|]$ 
7:   return MERGE( $A_L, A_R$ )                       $\triangleright$  merge the two sorted arrays

```

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**Algorithm 5** Merge
 

---

```

1: function MERGE( $A, B$ )
2:    $i, j \leftarrow 1$ 
3:    $C \leftarrow []$ 
4:   while  $i \leq \text{len}(A) \vee j \leq \text{len}(B)$  do
5:     if  $i \leq \text{len}(A) \wedge (j > \text{len}(B) \vee A[i] < B[j])$  then
6:       append  $A[i]$  to  $C$ 
7:        $i \leftarrow i + 1$ 
8:     else
9:       append  $B[j]$  to  $C$ 
10:     $j \leftarrow j + 1$ 
11:  return  $C$ 

```

---

We describe the running time of merge sort (Algorithm 4) as follows:

1. **Divide:** compute the middle of the array (Algorithm 4, Line 4), which takes constant time,  $D(n) = \Theta(1)$
2. **Conquer:** recursively solve two subproblems (Algorithm 4, Line 5, 6), each of size  $n/2$ , contributes  $2T(n/2)$
3. **Combine:** merge (Algorithm 5) the two sorted subarrays, which takes  $\Theta(n)$  time,  $C(n) = \Theta(n)$

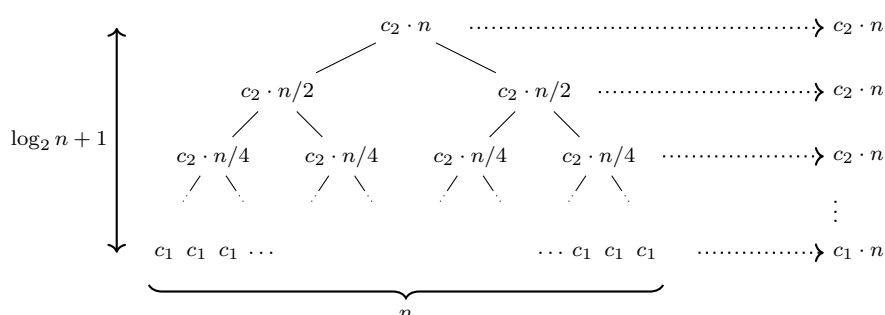
Using the so called ‘master theorem’:

$$T(n) = 2T(n/2) + \Theta(n) \xrightarrow{\text{master theorem}} T(n) = \Theta(n \log_2 n)$$

Intuitively we can also understand why that is the case without the master theorem. Assume for simplicity that  $n$  is an exact power of 2 and that the implicit base case is  $n = 1$ :

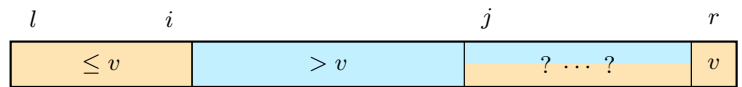
$$T(n) = \begin{cases} c_1 & \text{if } n = 1 \\ 2T(n/2) + c_2 n & \text{if } n > 1 \end{cases}$$

where  $c_1 > 0$  represents the time to solve the base case ( $n = 1$ ) and  $c_2 > 0$  is the time per element of the divide and combine steps.



$$\begin{aligned}
 T(n) &= 2T(n/2) + c_2 n = 2(2T((n/2)/2) + c_2(n/2)) + c_2 n \\
 &= 2^2 T(n/2^2) + 2c_2 n = 2^2(2T((n/2^2)/2) + c_2(n/2^2)) + 2c_2 n \\
 &= 2^3 T(n/2^3) + 3c_2 n = 2^3(2T((n/2^3)/2) + c_2(n/2^3)) + 3c_2 n \\
 &\vdots \\
 &= 2^{\log_2(n)} T(n/2^{\log_2(n)}) + \log_2(n) c_2 n \\
 &= nT(1) + \log_2(n) c_2 n \\
 &= c_1 n + c_2 n \log_2 n \\
 &= \Theta(n \log n)
 \end{aligned}$$

#### 3.3 Quick Sort (with Lomuto Partitioning)



Lomuto partitioning (Algorithm 7) divides an array  $A$  into two subarrays  $A[l : q - 1]$  and  $A[q + 1 : r]$  such that all elements in  $A[l : q - 1]$  are less than or equal to  $A[q]$  and all elements in  $A[q + 1 : r]$  are greater than  $A[q]$ .

---

**Algorithm 6** Quick Sort
 

---

```

1: function QUICKSORT( $A, l, r$ )
2:   if  $l < r$  then
3:      $q = \text{PARTITION}(A, l, r)$ 
4:     QUICKSORT( $A, l, q - 1$ )
5:     QUICKSORT( $A, q + 1, r$ )

```

---



---

**Algorithm 7** Lomuto Partitioning
 

---

```

1: function PARTITION( $A, l, r$ )
2:    $v = A[r]$                                  $\triangleright$  pick last element as pivot
3:    $i = l - 1$                                  $\triangleright$  highest index into the less-than-or-equal-partition
4:   for  $j = l, \dots, r$  do
5:     if  $A[j] \leq v$  then
6:        $i = i + 1$ 
7:       swap  $A[i]$  and  $A[j]$ 
8:   return  $i$                                  $\triangleright$  index of pivot

```

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#### 3.4 Heap Sort

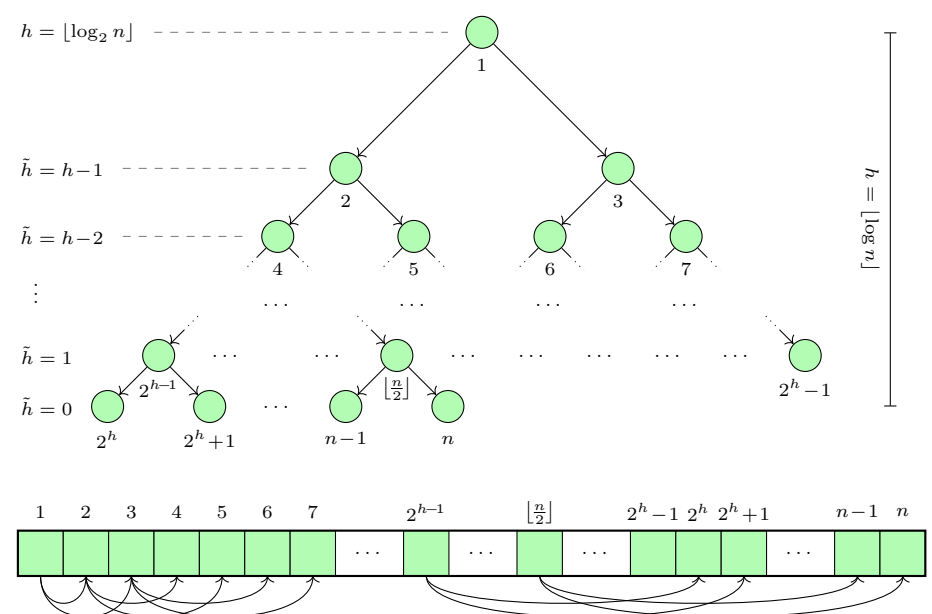
**Definition 2** (Heap). A **binary heap** is a nearly-complete binary tree stored in an array  $A[1 : n]$  satisfying the **max-heap property**:

$$\forall i > 1 : A[\text{Parent}(i)] \geq A[i]$$

The relationship between the indices of a binary heap is as follows:

$$\text{Parent}(i) = \lfloor i/2 \rfloor \quad \text{Left}(i) = 2i \quad \text{Right}(i) = 2i + 1 \quad \Leftarrow$$

Furthermore, an  $n$ -element heap has height  $h = \lfloor \log_2 n \rfloor$  and at most  $\lceil n/2^{\tilde{h}+1} \rceil$  nodes at any given height  $\tilde{h}$ , where  $\tilde{h}$  is defined as the longest path from the current node to a leaf node, measured in number of edges.



Algorithm 10 sorts an array  $A$  **in place** by first building a max-heap from the input array and then repeatedly extracting the maximum element (the root of the heap) and placing it at the end of the array. The complexities of Algorithms 8, 9, 10 are:

$$T_{\text{MAXHEAPIFY}}(n) = \Theta(\log n) \quad T_{\text{BUILDMAXHEAP}}(n) = \Theta(n)$$

$$T_{\text{HEAPSORT}}(n) = \Theta(n \log n)$$

The complexity of Algorithm 8 is determined by the height  $\tilde{h}$  of the node to be heapified, which is given by  $\tilde{h} = \lfloor \log i \rfloor$  for a node at index  $i$ .

Analyzing the complexity of Algorithm 9 is more involved. A simple upper bound on the running time is  $O(n \lg n)$ , since each call to MAXHEAPIFY costs  $O(\log n)$  and

**Algorithm 8** Max-Heapify

```

1: function MAXHEAPIFY( $A, i$ )
2:    $l \leftarrow \text{Left}(i)$ 
3:    $r \leftarrow \text{Right}(i)$ 
4:    $m \leftarrow i$  ▷ index of largest element among  $\{A[i], A[l], A[r]\}$ 
5:   if  $l \leq A.\text{heap-size} \wedge A[l] > A[m]$  then
6:      $m \leftarrow l$ 
7:   if  $r \leq A.\text{heap-size} \wedge A[r] > A[m]$  then
8:      $m \leftarrow r$ 
9:   if  $m \neq i$  then
10:    swap  $A[i]$  and  $A[m]$ 
11:    MAXHEAPIFY( $A, m$ )

```

**Algorithm 9** Build-Max-Heap

```

1: function BUILDMAXHEAP( $A$ )
2:    $A.\text{heap-size} \leftarrow |A|$ 
3:   for  $i = \lfloor |A|/2 \rfloor, \dots, 1$  do ▷ elements after  $\lfloor |A|/2 \rfloor$  are leaves
4:     MAXHEAPIFY( $A, i$ )

```

**Algorithm 10** Heap Sort

```

1: function HEAPSORT( $A$ )
2:   BUILDMAXHEAP( $A$ )
3:   for  $i = |A|, \dots, 2$  do
4:     swap  $A[1]$  and  $A[i]$ 
5:      $A.\text{heap-size} \leftarrow A.\text{heap-size} - 1$ 
6:     MAXHEAPIFY( $A, 1$ )

```

BUILDMAXHEAP makes  $O(n)$  such calls. This upper bound is correct but not asymptotically tight.

We can derive a tighter bound by recalling that the time for MAXHEAPIFY to run at a node  $i$  depends on the height  $\tilde{h}$  of that node in the tree, and that the height of most nodes is small.

Calling MAXHEAPIFY on a node of height  $\tilde{h}$  costs  $c\tilde{h}$ , and there are at most  $\lceil n/2^{\tilde{h}+1} \rceil$  nodes at that height. We can sum over all heights, starting from the leaves (with height 0) up to the root (with height  $\lfloor \log n \rfloor$ );

$$T(n) = \sum_{\tilde{h}=0}^{\lfloor \log n \rfloor} \left\lceil \frac{n}{2^{\tilde{h}+1}} \right\rceil c\tilde{h} \leq cn \sum_{\tilde{h}=0}^{\lfloor \log n \rfloor} \frac{\tilde{h}}{2^{\tilde{h}}} \leq cn \sum_{\tilde{h}=0}^{\infty} \frac{\tilde{h}}{2^{\tilde{h}}} \quad (4)$$

Recall the geometric series:

$$\sum_{\tilde{h}=0}^{\infty} x^{\tilde{h}} = \frac{1}{1-x}, \quad |x| < 1$$

Differentiating both sides with respect to  $x$  gives:

$$\frac{d}{dx} \left( \sum_{\tilde{h}=0}^{\infty} x^{\tilde{h}} \right) = \frac{d}{dx} \left( \frac{1}{1-x} \right) \implies \sum_{\tilde{h}=0}^{\infty} \tilde{h} x^{\tilde{h}-1} = \frac{1}{(1-x)^2}$$

Multiply through by  $x$  to shift the exponent back:

$$\sum_{\tilde{h}=0}^{\infty} \tilde{h} x^{\tilde{h}} = x \sum_{\tilde{h}=0}^{\infty} \tilde{h} x^{\tilde{h}-1} = \frac{x}{(1-x)^2}, \quad |x| < 1, \quad (5)$$

Setting  $x = \frac{1}{2}$  in (5) and substituting into (4) gives

$$T(n) \leq cn \sum_{\tilde{h}=0}^{\infty} \tilde{h} \left( \frac{1}{2} \right)^{\tilde{h}} = cn \cdot \frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2} = O(n) \quad (6)$$

Thus BUILDMAXHEAP builds a max-heap from an array in  $O(n)$  time.

For Algorithm 10, we have a similar situation, but different from BUILDMAXHEAP, where the majority of the calls to MAXHEAPIFY are done on nodes at the bottom of the heap (where the height is small), in HEAPSORT, the calls to MAXHEAPIFY are always done on the root of the heap, and the height is large for the majority of the calls.

After BUILDMAXHEAP has finished (line 2), the array  $A$  is a valid max-heap of size  $n$ . The HEAPSORT loop (lines 3-6) then performs exactly  $n-1$  iterations. At the  $k^{\text{th}}$  iteration the heap contains  $n-k+1$  elements, so the call to MAXHEAPIFY costs  $\Theta(\log(n-k+1))$ . The total time spent in the loop is therefore

$$\sum_{k=1}^{n-1} \Theta(\log(n-k+1)) = \Theta \left( \sum_{l=1}^{n-1} \log l \right) = \Theta(\log(n!)) \stackrel{\text{Ex. 3}}{=} \Theta(n \log n)$$

where the change of index  $l = n-k+1$  rewrites the sum in increasing order.

**3.5 k-Smallest Element**

Algorithm 11 is a randomized “divide-and-conquer” method for finding the  $k$ -th smallest element in an unsorted array in expected  $O(n)$  time. At each step it chooses a pivot uniformly at random, partitions the input into three subsets – those less than, equal to, and greater than the pivot – and then recurses only on the subset that must contain the desired element.

Partitioning around the pivot takes  $\Theta(n)$  time, and since the pivot is random the expected size of the recursive call is at most a constant fraction of  $n$ , yielding an overall expected running time of  $O(n)$ .

**Algorithm 11** Quick Select

```

1: function QUICKSELECT( $A, k$ ) ▷  $A$  is an unordered multiset (“bag”) of elements
2:    $v \leftarrow A[\text{randint}(1, |A|)]$  ▷ pick a random pivot
3:    $A_L, A_M, A_R \leftarrow \{\}_m$  ▷ three empty multisets
4:   for all  $a \in A$  do
5:     if  $a < v$  then
6:       add  $a$  to  $A_L$ 
7:     else if  $a = v$  then
8:       add  $a$  to  $A_M$ 
9:     else
10:      add  $a$  to  $A_R$ 
11:   if  $k \leq |A_L|$  then
12:     return QUICKSELECT( $A_L, k$ )
13:   else if  $k \leq |A_L| + |A_M|$  then
14:     return  $v$ 
15:   else
16:     return QUICKSELECT( $A_R, k - (|A_L| + |A_M|)$ )

```

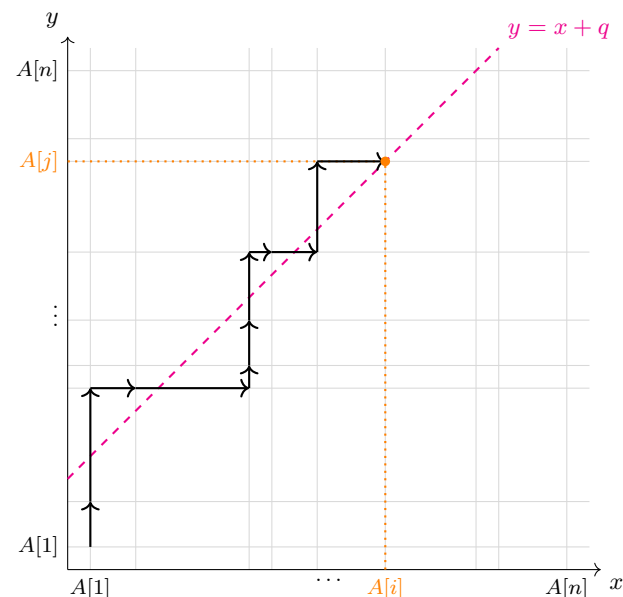
**3.6 Overview of Sorting Algorithms**

Algorithm	Time Complexity			in place?
	worst	average	best	
INSERTIONSORT	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n)$	✓
SELECTIONSORT	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	✓
MERGESORT	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$	✗
QUICKSORT	$\Theta(n^2)$	$\Theta(n \log n)$	$\Theta(n \log n)$	✓
HEAPSORT	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$	✓

**3.7 Application**

When an array is sorted, many operations and queries (e.g. the one in Example 5) can be performed much more efficiently than in an unsorted array.

**Example 5 (Caterpillar Method).** Checking if there a sorted array contains two elements  $A[i]$  and  $A[j]$  such that  $A[i] + q = A[j]$ , can be done in linear running time using the 🐛 method (sometimes also called “two-pointer” or “sliding-window” method).

**Algorithm 12** Checking for two elements with difference  $q$ 

```

1: function CATERPILLAR( $A, q$ )
2:    $l, r \leftarrow 1$ 
3:   while  $r \leq n$  do
4:     if  $A[r] < A[l] + q$  then
5:        $r \leftarrow r + 1$ 
6:     else if  $A[r] > A[l] + q$  then
7:        $l \leftarrow l + 1$ 
8:     else
9:       return true
10:  return false

```

Algorithm 12 maintains two indices  $l$  and  $r$ , both starting at the left end of  $A$  (Line 2). At each step, it compares  $A[r]$  and  $A[l]$  and checks if the difference  $A[r] - A[l]$  is less than, greater than, or equal to  $q$ :

- If  $A[r] - A[l] < q$ , move  $r$  one step to the right (to increase the difference).
- If  $A[r] - A[l] > q$ , move  $l$  one step to the right (to decrease the difference).
- If  $A[r] - A[l] = q$ , we found a valid pair and stop.



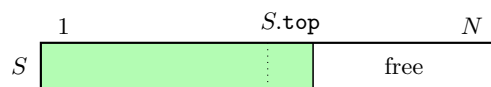
## 4 Data Structures

Operations on a dynamic set can be grouped into two categories, *queries*, which return information about the set, and *modifying operations*, which change the set:

- **SEARCH( $S, k$ )**  
*query* that, given a set  $S$  and a key value  $k$ , returns a pointer  $x$  to an element in  $S$  such that  $x.\text{key} = k$ , or NIL if no such element belongs to  $S$
- **INSERT( $S, x$ )**  
*modifying operation* that adds the element pointed to by  $x$  to the set  $S$  (we usually assume that any attributes in element  $x$  needed by the set implementation have already been initialized)
- **DELETE( $S, x$ )**  
*modifying operation* that, given a pointer  $x$  to an element in the set  $S$ , removes  $x$  from  $S$  (note that this operation takes a pointer to an element  $x$ , not a key value!)
- **MINIMUM( $S$ )** and **MAXIMUM( $S$ )**  
*queries* on a totally ordered set  $S$  that return a pointer to the element of  $S$  with the smallest (for MINIMUM) or largest (for MAXIMUM) key
- **SUCCESSOR( $S, x$ )**  
*query* that, given an element  $x$  whose key is from a totally ordered set  $S$ , returns a pointer to the next larger element in  $S$ , or NIL if  $x$  is the maximum element
- **PREDECESSOR( $S, x$ )**  
*query* that, given an element  $x$  whose key is from a totally ordered set  $S$ , returns a pointer to the next smaller element in  $S$ , or NIL if  $x$  is the minimum element

### 4.1 Stacks

**Definition 3** (Stack). A **stack** is a LIFO container that supports constant-time insertion and deletion at one end.  $\hookrightarrow$



Interface (Algorithm 13):

- **STACKEMPTY( $S$ )**: returns **true** iff the stack  $S$  contains no elements.
- **PUSH( $S, x$ )**: places element  $x$  on the top of the stack  $S$ .
- **POP( $S$ )**: removes and returns the top element of the stack  $S$ .

The stack has attributes  $S.\text{top}$ , indexing the most recently inserted element, and  $S.\text{length}$ , equaling the size  $N$  of the array.

#### Algorithm 13 Stack Operations (array-based)

```

1: function STACKEMPTY( $S$ )
2:   return ( $S.\text{top} = 0$ )
3: function PUSH( $S, x$ )
4:   if  $S.\text{top} = S.\text{length}$  then
5:     error "overflow"
6:    $S.\text{top} \leftarrow S.\text{top} + 1$ 
7:    $S[S.\text{top}] \leftarrow x$ 
8: function POP( $S$ )
9:   if STACKEMPTY( $S$ ) then
10:    error "underflow"
11:    $S.\text{top} \leftarrow S.\text{top} - 1$ 
12:   return  $S[S.\text{top} + 1]$ 

```

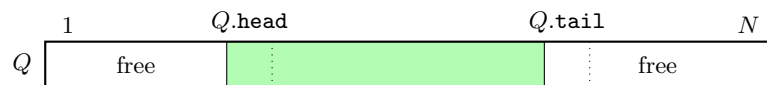
### 4.2 Queues

**Definition 4** (Queue). A **queue** is a FIFO container with constant-time insertion at the tail and deletion at the head.  $\hookrightarrow$

Interface (Algorithms 14 and 15):

- **ENQUEUE( $Q, x$ )**: insert  $x$  at the tail of  $Q$ .
- **DEQUEUE( $Q$ )**: remove and return the head element of  $Q$ .

The classic fixed-length **circular-array** implementation keeps two indices  $Q.\text{head}$  and  $Q.\text{tail}$  ( $Q.\text{head}$  points to the first element,  $Q.\text{tail}$  to the first free slot).



#### Algorithm 14 Enqueue (circular array)

```

1: function ENQUEUE( $Q, x$ )
2:   if  $Q.\text{queue-full}$  then
3:     error "overflow"
4:    $Q[Q.\text{tail}] \leftarrow x$ 
5:    $Q.\text{tail} \leftarrow (Q.\text{tail} \bmod Q.\text{length}) + 1$ 
6:    $Q.\text{queue-empty} \leftarrow \text{false}$ 
7:   if  $Q.\text{tail} = Q.\text{head}$  then
8:      $Q.\text{queue-full} \leftarrow \text{true}$ 

```

#### Algorithm 15 Dequeue (circular array)

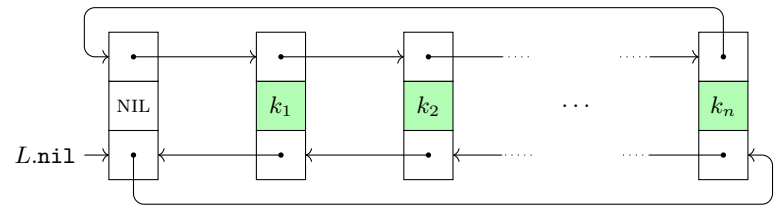
```

1: function DEQUEUE( $Q$ )
2:   if  $Q.\text{queue-empty}$  then
3:     error "underflow"
4:    $x \leftarrow Q[Q.\text{head}]$ 
5:    $Q.\text{head} \leftarrow (Q.\text{head} \bmod Q.\text{length}) + 1$ 
6:    $Q.\text{queue-full} \leftarrow \text{false}$ 
7:   if  $Q.\text{tail} = Q.\text{head}$  then
8:      $Q.\text{queue-empty} \leftarrow \text{true}$ 
9:   return  $x$ 

```

### 4.3 Linked Lists

**Definition 5** (Doubly-Linked List). A **doubly-linked list**  $L$  consists of nodes  $x$  that store a key  $x.\text{key}$  and two links  $x.\text{prev}$ ,  $x.\text{next}$ . A special sentinel node  $L.\text{nil}$  simplifies boundary cases because the list is empty iff  $L.\text{nil}.\text{next} = L.\text{nil}$ .  $\hookrightarrow$



#### Algorithm 16 Doubly-Linked List Operations (with sentinel)

```

1: function LISTINIT( $L$ )
2:    $L.\text{nil}.\text{prev} \leftarrow L.\text{nil}$ 
3:    $L.\text{nil}.\text{next} \leftarrow L.\text{nil}$ 
4: function LISTINSERT( $L, x$ ) ▷ insert at front
5:    $x.\text{next} \leftarrow L.\text{nil}.\text{next}$ 
6:    $L.\text{nil}.\text{next}.\text{prev} \leftarrow x$ 
7:    $L.\text{nil}.\text{next} \leftarrow x$ 
8:    $x.\text{prev} \leftarrow L.\text{nil}$ 
9: function LISTDELETE( $x$ )
10:   $x.\text{prev}.\text{next} \leftarrow x.\text{next}$ 
11:   $x.\text{next}.\text{prev} \leftarrow x.\text{prev}$ 
12: function LISTSEARCH( $L, k$ )
13:   $x \leftarrow L.\text{nil}.\text{next}$ 
14:  while  $x \neq L.\text{nil} \wedge x.\text{key} \neq k$  do
15:     $x \leftarrow x.\text{next}$ 
16:  return  $x$ 

```

Algorithm 16 shows the basic operations on a doubly-linked list. LISTINSERT and LISTDELETE take  $O(1)$  time, whereas LISTSEARCH takes  $\Theta(n)$  time in the worst case, with  $n$  the current length of  $L$ .

### 4.4 Dictionaries

**Definition 6** (Dictionary). A **dictionary** is an abstract data structure that represents a set of elements (or keys). It is a **dynamic set** that supports the following operations:

- **INSERT**: insert element into the set
- **DELETE**: delete element from the set
- **SEARCH**: test membership of an element in the set  $\hookrightarrow$

### 4.5 Direct-Address Tables

**Definition 7** (Direct-Address Table). A **direct-address table** implements a dictionary. Suppose the key universe is the set  $U = \{1, \dots, M\}$ . A **direct-address table** is an array  $T[1 : M]$  where slot  $T[k]$  stores a Boolean indicating membership of key  $k$  in the represented set.  $\hookrightarrow$

#### Algorithm 17 Direct-Address Table Operations

```

1: function DIRECTADDRESSINSERT( $T, k$ )
2:    $T[k] \leftarrow \text{true}$ 
3: function DIRECTADDRESSDELETE( $T, k$ )
4:    $T[k] \leftarrow \text{false}$ 
5: function DIRECTADDRESSSEARCH( $T, k$ )
6:   return  $T[k]$ 

```

All direct-address table operations (Algorithm 17) cost  $O(1)$  time, but the table occupies  $\Theta(|U|)$  space, which is prohibitive when the universe is large and the actual set is sparse. i.e., direct-address tables usually waste a lot of space.

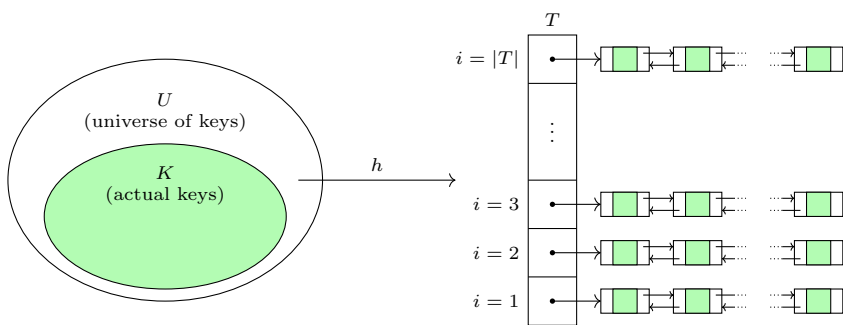
### 4.6 Hash Tables

To reduce the space overhead we use a smaller table  $T$  with  $|T| \ll |U|$  and map each key  $k \in U$  to a position in  $T$  using a **hash function**  $h : U \rightarrow \{1, \dots, |T|\}$ .

**Definition 8** (Load Factor). For a table of size  $|T|$  that currently stores  $n$  keys, the **load factor** is  $\alpha = \frac{n}{|T|}$ .  $\hookrightarrow$

#### 4.6.1 Chaining

Each slot  $T[i]$  stores a linked list of keys that hash to  $i$ .



#### Algorithm 18 Chained Hash Operations

```

1: function CHAINEDHASHINSERT( $T, k$ )
2:   LISTINSERT( $T[h(k)], k$ )
3: function CHAINEDHASHSEARCH( $T, k$ )
4:   return LISTSEARCH( $T[h(k)], k$ )
5: function CHAINEDHASHDELETE( $T, k$ )
6:    $x \leftarrow$  CHAINEDHASHSEARCH( $T, k$ )
7:   if  $x \neq \text{NIL}$  then
8:     LISTDELETE( $x$ )

```

We assume **uniform hashing** i.e.

$$P[h(k) = i] = \frac{1}{|T|} \quad \forall i \in \{1, \dots, |T|\}$$

So, given  $n$  distinct keys, the expected length  $n_i$  of the linked list at position  $i$  is

$$E[n_i] = \frac{n}{|T|} = \alpha$$

If we further assume  $h(k)$  can be computed in  $O(1)$ , the expected running time of CHAINEDHASHSEARCH is

$$\Theta(1 + \alpha)$$

#### 4.6.2 Open Addressing

Instead of using linked lists, keys are stored directly in the array. On a collision we **probe** other slots in  $T$  using a permutation  $h(k, 1), \dots, h(k, |T|)$ . So  $h(k, i)$  is a function of both  $k$  and  $i$ , where  $i$  is the probe number.

$h(k, \cdot)$  must be a **permutation** of  $\{1, \dots, |T|\}$ , i.e.,  $h(k, 1), \dots, h(k, |T|)$  must cover all slots in  $T$  exactly once.

We assume **independent uniform permutation hashing**: the probe sequence of each key is equally likely to be any of the  $|T|!$  permutations of  $\{1, \dots, |T|\}$ . Independent uniform permutation hashing generalizes the notion of independent uniform hashing introduced earlier to a hash function that produces not just a single slot number, but a whole probe sequence. True independent uniform permutation hashing is difficult to implement, however, and in practice suitable approximations (such as double hashing, Example 6) are used.

Neither double hashing nor its special case, linear probing, meets the assumption of independent uniform permutation hashing. Double hashing cannot generate more than  $|T|^2$  different probe sequences. Nonetheless, double hashing has a large number of possible probe sequences and seems to give good results. Linear probing is even more restricted, capable of generating only  $|T|$  different probe sequences.

**Example 6 (Double Hashing).** Double hashing offers one of the best methods available for open addressing because the permutations produced have many of the characteristics of randomly chosen permutations. Double hashing uses a hash function of the form

$$h(k, i) = (h_1(k) + i \cdot h_2(k)) \bmod |T| \quad (7)$$

where  $h_1$  and  $h_2$  are two different hash functions. The second hash function  $h_2(k)$  must be **relatively prime** to  $|T|$  (i.e.,  $\gcd(h_2(k), |T|) = 1$ ) to ensure that all slots in  $T$  are probed. A convenient way to achieve this is to let  $|T|$  be an exact power of 2 and to design  $h_2(k)$  so that it always produces an odd number. Another way is to let  $|T|$  be prime and to design  $h_2(k)$  so that it always returns a positive integer less than  $|T|$ . In (7) we can also set  $h_2(k) = 1$  for all  $k$ , in which case we get **linear probing**, a special case of double hashing.

#### Algorithm 19 Open-Address Hash Insert (generic probing)

```

1: function HASHINSERT( $T, k$ )
2:   for  $i = 1, \dots, |T|$  do
3:      $j \leftarrow h(k, i)$ 
4:     if  $T[j] = \text{NIL}$  then
5:        $T[j] \leftarrow k$ 
6:       return  $j$ 
7:   error "overflow"

```

To analyze the time complexity of Algorithm 19, we assume independent uniform permutation hashing for the hash function. We also assume that at least one slot is empty, i.e.,  $\alpha < 1$ . Because deleting from an open-address hash table does not really free up a slot, we assume as well that no deletions occur.

If we denote by  $X$  the number of probes performed until an empty slot is found, then on each probe we hit an occupied slot with probability  $\alpha = n/|T|$  and an empty slot with probability  $1 - \alpha$ . Hence

$$P[X = i] = \alpha^{i-1}(1 - \alpha), \quad i = 1, 2, \dots$$

so that  $X$  is geometrically distributed with success probability  $1 - \alpha$ . Hence

$$E[X] = \sum_{i=1}^{\infty} i \alpha^{i-1} (1 - \alpha) = \frac{1}{1 - \alpha}$$

Thus an insertion (or an unsuccessful search) requires on average  $\frac{1}{1-\alpha}$  probes; this grows rapidly as the load factor  $\alpha$  approaches 1.

#### 4.7 Binary Search Trees

**Definition 9 (Binary Search Tree).** A **binary search tree** implements a dynamic set. It stores keys from a totally ordered domain in nodes linked by **left** and **right** child pointers such that for every node  $x$

$$y \in \text{left-subtree}(x) \Leftrightarrow y.\text{key} \leq x.\text{key}, \quad z \in \text{right-subtree}(x) \Leftrightarrow z.\text{key} \geq x.\text{key}$$

**Example 7 (Lower Bound).** Let  $t$  be the root of a binary search tree that represents a set  $S$  of numbers. The size of the tree is  $|S| = n$ . The height of the tree is  $h$ . Algorithm 20 returns the node containing the least element  $y \in S$  such that  $x \leq y$ , or NIL if no such element exists.

#### Algorithm 20 Lower Bound for Binary Search Tree

```

1: function LOWERBOUND( $t, x$ )
2:   if  $t = \text{NIL}$  then ▷ base case
3:     return NIL
4:   if  $t.\text{key} < x$  then
5:     return LOWERBOUND( $t.\text{right}, x$ )
6:   else
7:      $y \leftarrow$  LOWERBOUND( $t.\text{left}, x$ )
8:     if  $y \neq \text{NIL}$  then
9:       return  $y$ 
10:    else
11:      return  $t$ 

```

The complexity is  $O(h)$ , where  $h$  is the height of the tree.

##### 4.7.1 Traversals

#### Algorithm 21 Inorder Tree Walk (recursive)

```

1: function TREEWALK( $x$ )
2:   if  $x \neq \text{NIL}$  then
3:     TREEWALK( $x.\text{left}$ )
4:     print  $x.\text{key}$ 
5:     TREEWALK( $x.\text{right}$ )

```

Algorithm 21 shows the INORDERTREEWALK algorithm. Three other variants can be obtained from Algorithm 21 by swapping the order of the recursive calls and the print statement (lines 3, 4, and 5). For PREORDERTREEWALK, we swap lines 3 and 4. For POSTORDERTREEWALK, we swap lines 4 and 5. For REVERSEORDERTREEWALK, we swap lines 3 and 5.

The general recurrence is

$$T(n) = T(n_L) + T(n - n_L - 1) + \Theta(1)$$

and all four variants run in  $\Theta(n)$  time. This can be proven using the **substitution method**. Can we do better? No, because the length of the output is  $\Theta(n)$ .

##### 4.7.2 Basic Queries

#### Algorithm 22 Searching a Binary Search Tree for a Key

```

1: function TREESearch( $x, k$ )
2:   if  $x = \text{NIL} \vee k = x.\text{key}$  then
3:     return  $x$ 
4:   if  $k < x.\text{key}$  then
5:     return TREESearch( $x.\text{left}, k$ )
6:   else
7:     return TREESearch( $x.\text{right}, k$ )

```

#### Algorithm 23 Minimum of a Binary Search Tree

```

1: function TREEMINIMUM( $x$ )
2:   while  $x.\text{left} \neq \text{NIL}$  do
3:      $x \leftarrow x.\text{left}$ 
4:   return  $x$ 

```

To find the maximum, replace all occurrences of **left** with **right** in Algorithm 23.

**Definition 10 (Successor).** The **successor** of a node  $x$  is the minimum of the right subtree of  $x$  if it exists, otherwise it is the lowest ancestor  $a$  of  $x$  such that  $x$  falls in the left subtree of  $a$ .

To find the predecessor, replace all occurrences of **right** with **left** and use TREEMAXIMUM instead of TREEMINIMUM in Algorithm 24.

**Algorithm 24** Successor of a Node in a Binary Search Tree

```

1: function TREESUCCESSOR( $x$ )
2:   if  $x.\text{right} \neq \text{NIL}$  then
3:     return TREEMINIMUM( $x.\text{right}$ )  $\triangleright$  leftmost node in right subtree
4:   while  $x.\text{parent} \neq \text{NIL} \wedge x.\text{parent}.\text{right} = x$  do
5:      $x \leftarrow x.\text{parent}$ 
6:   return  $x.\text{parent}$   $\triangleright$  lowest ancestor of  $x$  whose left child is an ancestor of  $x$ 

```

**4.7.3 Updates****Algorithm 25** Inserting a Node into a Binary Search Tree

```

1: function TREEINSERT( $T, z$ )
2:    $y, x \leftarrow \text{NIL}, T.\text{root}$ 
3:   while  $x \neq \text{NIL}$  do
4:      $y \leftarrow x$ 
5:     if  $z.\text{key} < x.\text{key}$  then
6:        $x \leftarrow x.\text{left}$ 
7:     else
8:        $x \leftarrow x.\text{right}$ 
9:    $z.\text{parent} \leftarrow y$ 
10:  if  $y = \text{NIL}$  then
11:     $T.\text{root} \leftarrow z$ 
12:  else if  $z.\text{key} < y.\text{key}$  then
13:     $y.\text{left} \leftarrow z$ 
14:  else
15:     $y.\text{right} \leftarrow z$ 

```

Deletion distinguishes three cases:

- $z$  is a leaf node, i.e. it has no children:
  - simply remove the node  $z$
- $z$  has one child:
  - remove  $z$
  - connect its child to its parent
- $z$  has two children:
  - find the successor  $y$  of  $z$  (since  $z$  has two children,  $y$  is guaranteed to be the minimum of the right subtree of  $z$  and thus have at most one child)
  - copy the key of  $y$  into  $z$
  - delete  $y$ , which is a leaf or has one right child, i.e. connect the child of  $y$  to the parent of  $y$

**Algorithm 26** Deleting a Node from a Binary Search Tree

```

1: function TREEDELETE( $T, z$ )
2:    $\triangleright z$  has two children: find successor, copy its key, then delete successor  $\triangleleft$ 
3:   if  $z.\text{left} \neq \text{NIL} \wedge z.\text{right} \neq \text{NIL}$  then
4:      $s \leftarrow \text{TREEMINIMUM}(z.\text{right})$ 
5:      $z.\text{key} \leftarrow s.\text{key}$   $\triangleright$  replace the key in  $z$  with the one from the successor
6:     return TREEDELETE( $T, s$ )
7:    $\triangleright z$  has at most one child: pick it (could be NIL)  $\triangleleft$ 
8:   if  $z.\text{left} \neq \text{NIL}$  then
9:      $c \leftarrow z.\text{left}$ 
10:  else
11:     $c \leftarrow z.\text{right}$ 
12:  if  $c \neq \text{NIL}$  then  $\triangleright$  if child exists, update its parent pointer
13:     $c.\text{parent} \leftarrow z.\text{parent}$ 
14:  if  $z.\text{parent} = \text{NIL}$  then  $\triangleright$  if  $z$  was the root, make child the new root
15:     $T.\text{root} \leftarrow c$ 
16:  else  $\triangleright$  otherwise, bypass  $z$  by connecting its child to its parent
17:    if  $z = z.\text{parent}.\text{left}$  then
18:       $z.\text{parent}.\text{left} \leftarrow c$ 
19:    else
20:       $z.\text{parent}.\text{right} \leftarrow c$ 
21:  return  $T$ 

```

Insertion, search and deletion operations have complexity  $\Theta(h)$  where  $h$  is the height of the tree. In the average case, the height is  $O(\log n)$  (i.e. with a random insertion order). In some particular cases, the height can be  $O(n)$  (i.e. with ordered sequence).

The problem is that the ‘worst case’ is not that uncommon. One way to avoid this is to instead of inserting  $A = [a_1, a_2, a_3, \dots, a_n]$  in order, insert a random permutation of  $A$ . The problem is that  $A$  is usually not known in advance. It is the application that calls the insertion procedure. But we can also obtain a random permutation of  $A$  by using a randomized insertion algorithm (see 4.7.4).

**4.7.4 Randomized Insertion**

In order to avoid the linear-height worst case one can insert into a tree using Algorithm 27. The idea behind the function TREERANDOMIZEDINSERT is as follows. We insert a new node  $z$  into the tree  $t$  as the new root of  $t$  with probability  $1/(t.\text{size} + 1)$ . If  $z$  is not inserted as the new root, we recursively insert it into the appropriate subtree of  $t$ . The additional attribute  $t.\text{size}$  represents the number of nodes in the subtree rooted at  $t$ .

With Algorithm 27 every insertion order is equally likely; the expected height of the tree is  $O(\log n)$  and all operations run in expected  $O(\log n)$  time.

**Algorithm 27** Randomized Insertion into a Binary Search Tree

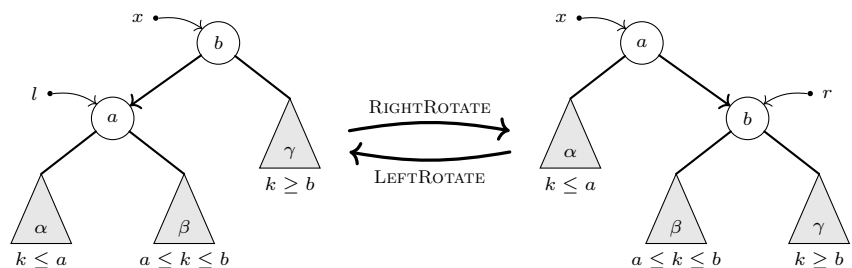
```

1: function TREERANDOMIZEDINSERT( $t, z$ )
2:   if  $t = \text{NIL}$  then
3:     return  $z$ 
4:    $r \leftarrow \text{RANDINT}(1, t.\text{size} + 1)$ 
5:   if  $r = 1$  then  $\triangleright P(r = 1) = P(z \text{ is inserted as the new root of } t) = \frac{1}{t.\text{size} + 1}$ 
6:      $z.\text{size} \leftarrow t.\text{size} + 1$ 
7:     return TREERANDOMIZEDINSERT( $t, z$ )  $\triangleright$  see 4.7.6
8:   if  $z.\text{key} < t.\text{key}$  then
9:      $t.\text{left} \leftarrow \text{TREERANDOMIZEDINSERT}(t.\text{left}, z)$ 
10:  else
11:     $t.\text{right} \leftarrow \text{TREERANDOMIZEDINSERT}(t.\text{right}, z)$ 
12:   $t.\text{size} \leftarrow t.\text{size} + 1$ 
13:  return  $t$ 

```

**4.7.5 Rotations**

**Definition 11** (Rotation). A **rotation** is a local restructuring of a BST that exchanges the relative position of a node  $x$  and one of its children while preserving the in-order sequence of keys. Rotations do **not** change the in-order ordering of the keys and they are the basic tool used by self-balancing trees.  $\triangleleft$

**Algorithm 28** Rotations in a Binary Search Tree

```

1: function RIGHTROTATE( $x$ )
2:    $l \leftarrow x.\text{left}$ 
3:    $x.\text{left} \leftarrow l.\text{right}$ 
4:    $l.\text{right} \leftarrow x$ 
5:   return  $l$ 
6: function LEFTROTATE( $x$ )
7:    $r \leftarrow x.\text{right}$ 
8:    $x.\text{right} \leftarrow r.\text{left}$ 
9:    $r.\text{left} \leftarrow x$ 
10:  return  $r$ 

```

Rotations are very useful operations. For example, we can use it to perform

**4.7.6 Root Insertion****Algorithm 29** Root Insertion into a Binary Search Tree

```

1: function TREEROOTINSERT( $x, z$ )
2:   if  $x = \text{NIL}$  then
3:     return  $z$ 
4:   if  $z.\text{key} < x.\text{key}$  then
5:      $x.\text{left} \leftarrow \text{TREEROOTINSERT}(x.\text{left}, z)$ 
6:     return RIGHTROTATE( $x$ )
7:   else
8:      $x.\text{right} \leftarrow \text{TREEROOTINSERT}(x.\text{right}, z)$ 
9:     return LEFTROTATE( $x$ )

```

General strategies to deal with complexity in the worst case:

- **Randomization**: can make the scenario  $h = n$  highly unlikely
- **Amortized Maintenance**: relatively expensive but ‘amortized’ operations
- **Self-Balancing**: e.g. Red-Black trees (see 4.8)

**4.8 Red-Black Trees**

A **red-black tree** is a binary search tree with one extra bit of storage per node: its **color**, which can be either RED or BLACK. By constraining the node colors on any simple path from the root to a leaf, red-black trees ensure that no such path is more than twice as long as any other, so that the tree is approximately balanced.

**Definition 12** (Red-Black Tree). A red-black tree is a binary search tree that satisfies the following **red-black properties**:

1. Every node is either red or black.
2. The root is black.
3. Every leaf (NIL) is black.
4. If a node is red, then both its children are black.
5. For each node, all simple paths from the node to descendant leaves contain the same number of black nodes.  $\triangleleft$



**Definition 13** (Black-Height). The number of black nodes on any simple path from, but not including, a node  $x$  down to, and including, a leaf is called the **black-height** of the node  $x$ :

$$\text{bh}(x) := \# \text{ black nodes on path from (excluding) } x \text{ to (including) leafs}$$

By property 5, the notion of black-height is well defined, since all descending paths have the same number of black nodes.  $\neq$

**Lemma 3** (Height of a Red-Black Tree). The height  $h(x)$  of a red-black tree with  $n = \text{size}(x)$  internal nodes is at most  $2\log(n+1)$ .  $\leq$

**Proof.** First, we prove (by induction) that the subtree rooted at any node  $x$  contains at least

$$2^{\text{bh}(x)} - 1 \quad (8)$$

internal nodes.

- (i) base case:  $x$  is a leaf, so  $\text{size}(x) = 0$  and  $\text{bh}(x) = 0$ , fulfilling (8)  $\checkmark$
- (ii) induction step: consider  $x, y_1, y_2$  such that  $x = y_1.\text{parent} = y_2.\text{parent}$

$$\text{size}(x) = \text{size}(y_1) + \text{size}(y_2) + 1 \geq (2^{\text{bh}(y_1)} - 1) + (2^{\text{bh}(y_2)} - 1) + 1$$

By rule 5, both children must have the same black-height. Let  $\text{bh}(y) = \text{bh}(y_1) = \text{bh}(y_2)$ . Then

$$\text{size}(x) \geq 2 \cdot (2^{\text{bh}(y)} - 1) + 1 = 2^{\text{bh}(y)+1} - 1$$

The black-height of a child  $y$  differs at most by one from the black-height of the parent  $x$ , i.e.  $\text{bh}(y) \in \{\text{bh}(x), \text{bh}(x) - 1\}$ . In both cases, we have  $\text{size}(x) \geq 2^{\text{bh}(x)} - 1$ , fulfilling (8)  $\checkmark$

By property 4, the black-height of a node  $x$  is at least half the height of the tree, i.e.  $\text{bh}(x) \geq \frac{h(x)}{2}$ . Therefore,

$$n = \text{size}(x) \geq 2^{\text{bh}(x)} - 1 \geq 2^{\frac{h(x)}{2}} - 1$$

which can be rearranged to

$$h(x) \leq 2\log(n+1) \quad \square$$

## 5 Graphs

### 5.1 Representations of Graphs

There are two standard ways to represent a graph: as a collection of adjacency lists or as an adjacency matrix. Because the adjacency-list representation provides a compact way to represent **sparse** graphs (those for which  $|E| \ll |V|^2$ ), it is usually the method of choice. The adjacency-matrix representation might be preferred when the graph is **dense** (i.e.,  $|E| \approx |V|^2$ ), or you need to be able to tell quickly whether there is an edge connecting two given vertices.

The space required for the adjacency-list representation is  $\Theta(|V| + |E|)$ , and finding each edge in the graph also takes  $\Theta(|V| + |E|)$  time, since each of the  $|V|$  adjacency lists must be examined.

The space required for the adjacency-matrix representation is  $\Theta(|V|^2)$ , and finding each edge in the graph takes  $\Theta(|V|^2)$  time, since the entire adjacency matrix must be examined. For an undirected graph, the adjacency matrix is symmetric.

The complexity for different operations is summarized in the following table.

Operation	Representation	
	Adjacency List	Adjacency Matrix
accessing vertex $u$	$O(1)$ optimal	$O(1)$ optimal
iteration through $V$	$\Theta( V )$ optimal	$\Theta( V )$ optimal
iteration through $E$	$\Theta( V  +  E )$ okay (not optimal)	$\Theta( V ^2)$ possibly very bad
checking $(u, v) \in E$	$O( V )$ bad	$O(1)$ optimal
space complexity	$\Theta( V  +  E )$ optimal	$\Theta( V ^2)$ possibly very bad

### 5.2 Breadth-First Search

Breadth-First Search is one of the simplest but also a fundamental algorithm, as it is the archetype of many important algorithms.

We enqueue a vertex only if it is WHITE, and we immediately color it GRAY; thus, we enqueue every vertex at most once. So the (dequeue) while loop at Line 7 executes  $O(|V|)$  times. For each vertex  $u$ , the inner loop at Line 9 executes  $\Theta(|\Gamma(u)|)$  times, for a total of  $O(|E|)$  steps. Thus, the total running time is  $O(|V| + |E|)$ .

The minimum number of edges in any path from  $s$  to  $v$  is called the shortest-path distance from  $s$  to  $v$  and is denoted by  $\delta(s, v)$ . If there is no path from  $s$  to  $v$ , then  $\delta(s, v) = \infty$ . We define the predecessor subgraph as  $G_\pi = (V_\pi, E_\pi)$  where

$$V_\pi = \{v \in V \mid v.\pi \neq \text{NIL}\} \cup \{s\}$$

and

$$E_\pi = \{(v.\pi, v) \mid v \in V_\pi \setminus \{s\}\}$$

The predecessor subgraph  $G_\pi$  is a **breadth-first tree** if  $V_\pi$  consists of all vertices reachable from  $s$  and, for all  $v \in V_\pi$ , the subgraph  $G_\pi$  contains a unique simple path from  $s$  to  $v$  that is also a shortest path from  $s$  to  $v$  in  $G$ .

#### Algorithm 30 Breadth-First Search

```

1: function BFS( $G, s$ ) ▷  $s$  is the source
2:   for all  $u \in V \setminus \{s\}$  do
3:      $u.\text{color}, u.\delta, u.\pi \leftarrow \text{WHITE}, \infty, \text{NIL}$ 
4:    $s.\text{color}, s.\delta, s.\pi \leftarrow \text{GRAY}, 0, \text{NIL}$ 
5:    $Q \leftarrow \emptyset$  ▷ initialize empty queue for vertices to visit
6:   ENQUEUE( $Q, s$ )
7:   while  $Q \neq \emptyset$  do
8:      $u \leftarrow \text{DEQUEUE}(Q)$ 
9:     for all  $v \in \Gamma(u)$  do
10:      if  $v.\text{color} = \text{WHITE}$  then
11:         $v.\text{color} \leftarrow \text{GRAY}$ 
12:         $v.\delta \leftarrow u.\delta + 1$ 
13:         $v.\pi \leftarrow u$ 
14:      ENQUEUE( $Q, v$ )
15:    $u.\text{color} \leftarrow \text{BLACK}$ 

```

Breadth-First Search constructs  $\pi$  so that the predecessor subgraph  $G_\pi = (V_\pi, E_\pi)$  is a breadth-first tree rooted at  $s$ . Upon termination,  $v.\delta = \delta(s, v)$  for all  $v \in V$ . For any vertex  $v$  reachable from  $s$ , the simple path in the breadth-first tree  $G_\pi$  from  $s$  to  $v$  corresponds to a shortest path (that is, a path containing the smallest number of edges) from  $s$  to  $v$  in the original graph  $G$ .

Assuming that Breadth-First Search has computed a breadth-first tree  $G_\pi$ , we can print the shortest path from  $s$  to  $v$  using (the recursive) Algorithm 31.

#### Algorithm 31 Print Shortest Path

```

1: function PRINTPATH( $G, s, v$ )
2:   if  $v = s$  then
3:     PRINT( $s$ )
4:   else if  $v.\pi = \text{NIL}$  then
5:     PRINT("No path exists")
6:   else
7:     PRINTPATH( $G, s, v.\pi$ )
8:     PRINT( $v$ )

```

### 5.3 Depth-First Search

As its name implies, Depth-First Search searches “deeper” in the graph whenever possible. Depth-First Search explores edges out of the most recently discovered vertex  $v$  that still has unexplored edges leaving it. Once all of  $v$ ’s edges have been explored, the search **backtracks** to explore edges leaving the vertex from which  $v$  was discovered. This process continues until all vertices that are reachable from the original source vertex have been discovered. If any undiscovered vertices remain, then Depth-First Search selects one of them as a new source, repeating the search from that source. This process is repeated until every vertex has been discovered.<sup>2</sup>

#### Algorithm 32 Depth-First Search

```

1: function DFS( $G$ )
2:   for all  $u \in V$  do
3:      $u.\text{color}, u.\pi \leftarrow \text{WHITE}, \text{NIL}$ 
4:    $T \leftarrow 0$  ▷ global time variable, for timestamps
5:   for all  $u \in V$  do
6:     if  $u.\text{color} = \text{WHITE}$  then
7:       DFS-VISIT( $u$ ) ▷ new tree in forest
8:   function DFS-VISIT( $u$ )
9:      $u.\text{color} \leftarrow \text{GRAY}$  ▷  $u$  has just been discovered
10:     $T \leftarrow T + 1$ 
11:     $u.d \leftarrow T$ 
12:    for all  $v \in \Gamma(u)$  do ▷ explore each edge leaving  $u$ 
13:      if  $v.\text{color} = \text{WHITE}$  then
14:         $v.\pi \leftarrow u$ 
15:      DFS-VISIT( $v$ ) ▷ recursively visit  $v$  if it undiscovered
16:     $u.\text{color} \leftarrow \text{BLACK}$  ▷ blacken  $u$ ; it is finished
17:     $T \leftarrow T + 1$ 
18:     $u.f \leftarrow T$ 

```

Since Depth-First Search always explores all vertices, we define the predecessor subgraph as  $G_\pi = (V, E_\pi)$ , where

$$E_\pi = \{(v.\pi, v) \mid v \in V \wedge v.\pi \neq \text{NIL}\}$$

It is a **depth-first forest**, comprising several **depth-first trees**.

Each vertex is initially white, is grayed when it is discovered in the search (Line 9), and is blackened when it is finished, that is, when its neighborhood has been examined completely (Line 16).

Each vertex  $u$  has two timestamps: the first,  $u.d$ , records when  $u$  is first discovered (and grayed), and the second,  $u.f$ , records when the search finished examining  $u$ ’s neighborhood (and blackens  $u$ ). For each vertex  $u \in V$ , we have

$$u.d, u.f \in \{1, \dots, 2|V|\} \quad \text{and} \quad u.d < u.f$$

since there is one discovery and one finishing event for each of the  $|V|$  vertices.  $u$  is WHITE before  $u.d$ , GRAY between  $u.d$  and  $u.f$ , and BLACK after  $u.f$ .

<sup>2</sup>Although Breadth-First Search could proceed from multiple sources and Depth-First Search could be limited to one source source, the approach here reflects how they are typically used. Breadth-First Search usually serves to find shortest-path distances and the associated predecessor subgraph from a given source, while Depth-First Search is often a subroutine in another algorithm.



Upon every call of  $\text{DFS-VISIT}(u)$  in Line 7, a new depth-first tree is created, rooted at  $u$ . In each call  $\text{DFS-VISIT}(u)$ ,  $u$  is initially **WHITE**. Lines 12–15 examine each vertex  $v$  adjacent to  $u$  and recursively visit  $v$  if it is **WHITE**.

The result depends on the order in which Line 5 examines the vertices and upon the order in which Line 12 visits the neighbors of  $u$ . Usually, these different visitation orders tend not to cause problems, because many applications can use any of those.

We call  $\text{DFS-VISIT}(u)$  exactly once (either in Line 7 or recursively in Line 15) for each vertex  $u$ , because we call it only if  $u.\text{color} = \text{WHITE}$ , but then we immediately set  $u.\text{color} = \text{GRAY}$  in Line 9. The loop in Lines 12–15 executes  $\Theta(|\Gamma(u)|)$  times. So, the total running time is  $\Theta(|V| + |E|)$ .

Properties of the Depth-First Forest  $G_\pi$ :

- $v$  is a descendant of  $u \iff v$  is discovered during the time in which  $u$  is gray
- discovery and finish times have **parenthesis structure**: if in  $\text{DFS-VISIT}$  we printed ‘(u’ when we discovered  $u$  and ‘u)’ when we finished  $u$ , then the printed expression would be well formed in the sense that the parentheses are properly nested
- for any two vertices  $u_1$  and  $u_2$ , exactly one of the following conditions holds
  - $[u_1.d, u_1.f] \cap [u_2.d, u_2.f] = \emptyset \iff$  neither is a descendant of the other
  - $[u_1.d, u_1.f] \subset [u_2.d, u_2.f] \iff u_1$  is a descendant of  $u_2$
  - $[u_1.d, u_1.f] \supset [u_2.d, u_2.f] \iff u_2$  is a descendant of  $u_1$
- $v$  is a descendant of  $u \iff$  at the time  $u.d$  that the search discovers  $u$ , there is a path from  $u$  to  $v$  consisting entirely of white vertices

Classification of Edges in  $E$ :

- **tree edges** are edges  $(u, v) \in E_\pi$  in the depth-first forest  $G_\pi$ .  $(u, v)$  is a tree edge if  $v$  was first discovered by exploring the edge  $(u, v)$  in Line 12.
- **back edges** are edges  $(u, v) \notin E_\pi$  connecting  $u$  to an ancestor  $v$  in the depth-first forest  $G_\pi$ .
- **forward edges** are edges  $(u, v) \notin E_\pi$  connecting  $u$  to a descendant  $v$  in the depth-first forest  $G_\pi$ .
- **cross edges** are edges  $(u, v) \notin E_\pi$  are all other edges.

**Lemma 4.** A directed graph is acyclic if and only if a Depth-First Search yields no back edges.  $\triangleleft$

**Theorem 5.** In an undirected graph, a Depth-First Search yields only tree edges and back edges.  $\triangleleft$

When during an Depth-First Search an edge  $(u, v)$  is first explored, the color of  $v$  yields information about the edge:

- if  $v$  is **WHITE**, then  $(u, v)$  is a tree edge
- if  $v$  is **GRAY**, then  $(u, v)$  is a back edge
- if  $v$  is **BLACK**, then  $(u, v)$  is a
  - forward edge if  $u.d < v.d$
  - cross edge if  $u.d > v.d$

Observe that the gray vertices always form a linear chain of descendants corresponding to the stack of active  $\text{DFS-VISIT}$  invocations.

### 5.3.1 Topological Sort

A **topological sort** of a directed acyclic graph (“dag”)  $G = (V, E)$  is a linear ordering of the vertices such that if  $G$  contains an edge  $(u, v)$ , then  $u$  appears before  $v$  in the ordering.

**Algorithm 33** Topological Sort

```

1: function TOPOLOGICALSORT( $G$ )
2:   call  $\text{DFS}(G)$  to compute the finish time  $v.f$  for each  $v \in V$ 
3:   as each vertex  $v$  is finished, insert it onto the front of a linked list  $L$ 
4:   return  $L$ 

```

### 5.3.2 Strongly Connected Components

**Definition 14** (Strongly Connected Component). A **strongly connected component** of a directed graph  $G = (V, E)$  is a maximal set of vertices  $C \subseteq V$  such that for every pair of vertices  $u, v \in C$ , both  $u \rightsquigarrow v$  and  $v \rightsquigarrow u$ , that is,  $u$  and  $v$  are reachable from each other.  $\triangleleft$

Algorithm 34 uses the transpose of  $G$ ,  $G^T = (V, E^T)$ , where  $E^T = \{(u, v) : (v, u) \in E\}$ . That is,  $E^T$  consists of the edges of  $G$  with their directions reversed. Given an adjacency-list representation of  $G$ , the time to create  $G^T$  is  $\Theta(V + E)$ . The graphs  $G$  and  $G^T$  have exactly the same strongly connected components:  $u$  and  $v$  are reachable from each other in  $G$  if and only if they are reachable from each other in  $G^T$ .

The linear-time (i.e.,  $\Theta(V + E)$ -time) Algorithm 34 computes the strongly connected components of a directed graph  $G = (V, E)$  using two depth-first searches, one on  $G$  and one on  $G^T$ .

**Definition 15** (Component Graph). Suppose that  $G$  has strongly connected components  $C_1, \dots, C_k$ . The vertex set  $V^{\text{SCC}}$  is  $\{v_1, \dots, v_k\}$ , and it contains one vertex  $v_i$  for each strongly connected component  $C_i$  of  $G$ . There is an edge  $(v_i, v_j) \in E^{\text{SCC}}$  if  $G$  contains a directed edge  $(x, y)$  for some  $x \in C_i$  and some  $y \in C_j$ .

Looked at another way, if we contract all edges whose incident vertices are within the same strongly connected component of  $G$  so that only a single vertex remains, the resulting graph is  $G^{\text{SCC}}$ .  $\triangleleft$

**Algorithm 34** Strongly Connected Components

```

1: function STRONGLYCONNECTEDCOMPONENTS( $G$ )
2:   call  $\text{DFS}(G)$  to compute the finish time  $v.f$  for each  $v \in V$ 
3:   create the transpose graph  $G^T$ 
4:   call  $\text{DFS}(G^T)$ , but in Line 5, consider the vertices in order of decreasing  $u.f$ 
5:   return the depth-first trees of  $G_\pi$  formed in Line 4

```

The Strongly Connected Components, defined in Definition 14 and the Component Graph  $G^{\text{SCC}} = (V^{\text{SCC}}, E^{\text{SCC}})$ , defined in Definition 15 have the following properties:

If  $C_i$  and  $C_j$  are two distinct Strongly Connected Components in a directed graph  $G = (V, E), \dots$

1. ...and  $u_i, v_j \in C_i$  and  $u_j, v_j \in C_j$ , and  $G$  contains a path  $u_i \rightsquigarrow u_j$ , then  $G$  cannot also contain a path  $v_j \rightsquigarrow v_i$ , i.e. the Component Graph  $G^{\text{SCC}}$  is acyclic.
2. ...and there is an edge  $(u, v) \in E$  with  $u \in C_j$  and  $v \in C_i$ , then  $C_j.f > C_i.f$ .
3. ...and suppose that  $C_i.f > C_j.f$ , then  $E^T$  contains no edge  $(v, u)$  such that  $u \in C_j$  and  $v \in C_i$ .

Property 1 implies that the Component Graph  $G^{\text{SCC}}$  can be topologically sorted (see Section 5.3.1). And in fact, Algorithm 34 visits the vertices of the Component Graph in topologically sorted order, by considering vertices in the second depthfirst search (Line 4) in decreasing order of the finish times that were computed in the first depth-first search (Line 2).

In Properties 2 and 3, the finish time of a Strongly Connected Component,  $C.f$ , refers to the maximal finish time of any vertex in  $C$  as computed by Line 2 of Algorithm 34, i.e.,

$$C.f := \max_{v \in C} (v.f)$$

Property 3 provides the key idea behind Algorithm 34: When the Depth-First Search of  $G^T$  in Line 4 of Algorithm 34 visits any Strongly Connected Component, any edges out of that component must be to components that the search has already visited. Each depth-first tree produced by Line 4, therefore, corresponds to exactly one Strongly Connected Component of  $G$ .

Another way to see this is to consider the component graph  $(G^T)^{\text{SCC}}$  of  $G^T$ . If we map each strongly connected component visited in the second depth-first search to a vertex of  $(G^T)^{\text{SCC}}$ , the second depth-first search visits vertices of  $(G^T)^{\text{SCC}}$  in the reverse of a topologically sorted order.

## 5.4 Minimum Spanning Tree

Given a graph  $G = (V, E)$  with weight function  $w : E \rightarrow \mathbb{R}$  we want to find an acyclic subset  $T \subseteq E$  such that  $T$  touches all vertices in  $V$  and

$$w(T) = \sum_{e \in T} w(e)$$

that is, the total weight of the tree  $T$  is minimal. This  $T$  is called a **minimum spanning tree** (MST) of  $G$ .

Conceptually, Kruskal’s algorithm (1956) is similar to the Strongly Connected Components algorithm, while Prim’s algorithm (1957) resembles Dijkstra’s shortest-paths algorithm. Both run in  $O(|E| \log |V|)$  time.

The general idea is to build  $T$  by adding one edge  $e \in E$  at a time, such that  $e$  is the lightest edge that does not create a cycle. Both Kruskal and Prim are greedy algorithms (see 6.1), that is, in each step they make the choice that seems best at the moment. Even though such a strategy does not generally yield an optimal solution, for the MST problem one can prove that certain greedy strategies yield an optimal solution.

**Algorithm 35** Generic Minimum Spanning Tree

```

1: function GENERIC-MST( $G, w$ )
2:    $A \leftarrow \emptyset$  ▷ invariant:  $A$  is part of a minimum spanning tree
3:   while  $A$  is not a spanning tree do
4:     find a safe edge  $e = (u, v)$  ▷ a safe edge maintains the invariant
5:      $A \leftarrow A \cup \{e\}$ 

```

The strategy is given in Algorithm 35. The invariant used is that  $A$  is a subset of a minimum spanning tree and a **safe edge** is an edge that maintains this invariant.

A **cut** of a graph  $G = (V, E)$  is a partition of  $V$  into two disjoint sets  $S$  and  $V \setminus S$ . An edge  $e \in E$  **crosses** the cut if one of its endpoints is in  $S$  and the other is in  $V \setminus S$ . A cut **respects** a set of edges  $A$  if no edges in  $A$  cross the cut.

**Theorem 6.** Let  $A \subseteq E$  be included in a minimum spanning tree and  $(S, V \setminus S)$  be a cut that respects  $A$ . Then a minimum-weight edge  $e$  crossing the cut  $(S, V \setminus S)$  is a safe edge for  $A$ .  $\triangleleft$

Theorem 6 provides a rule for recognizing safe edges and is the key to why Algorithm 35 works. At any point in the execution, the graph  $G_A = (V, A)$  is a forest and each of the connected components of  $G_A$  is a tree. Any safe edge  $e$  for  $A$  connects two distinct components of  $G_A$ , since  $A \cup \{e\}$  must be acyclic.

The loop in Lines 3–5 of Algorithm 35 executes  $|V| - 1$  times, because it finds one of the  $|V| - 1$  edges of a minimum spanning tree in each iteration. Initially, when  $A = \emptyset$ , there are  $|V|$  trees in  $G_A$  and each iteration reduces that number by one, until the forest contains only a single tree, upon which the method terminates.

Kruskal and Prim both use the following corollary of Theorem 6.

**Corollary 7.** Let  $A \subseteq E$  be included in a minimum spanning tree and let  $C = (V_C, E_C)$  be a connected component (tree) of  $G_A = (V, A)$ . Then a minimum-weight edge  $e$  connecting  $C$  to some other component in  $G_A$  is a safe edge for  $A$ .  $\triangleleft$

Kruskal and Prim use a different, specific rule to determine a safe edge in Line 4 of Algorithm 35. In Kruskal’s algorithm,  $A$  is a forest whose vertices are all those of the given graph and the safe edge added is always a lowest-weight edge that connects two distinct components. In Prim’s algorithm,  $A$  is a single tree and the safe edge added is always a lowest-weight edge connecting the tree to a vertex not in the tree.

For Kruskal, we need the **disjoint-set data structure**:

- $\text{MAKESET}(x)$  creates a set containing only the element  $x$
- $\text{FIND}(x)$  returns the representative of the set containing  $x$
- $\text{UNION}(x, y)$  joins the sets containing  $x$  and  $y$  into a single set

---

**Algorithm 36** Kruskal

---

```

1: function KRUSKAL( $G, w : E \rightarrow \mathbb{R}$ )
2:    $A \leftarrow \emptyset$ 
3:   for all  $v \in V$  do
4:      $\text{MAKESET}(v)$ 
5:   sort  $E$  in non-decreasing order by  $w$ 
6:   for all  $(u, v) \in E$  in non-decreasing  $w$ -order do
7:     if  $\text{FIND}(u) \neq \text{FIND}(v)$  then
8:        $A \leftarrow A \cup \{(u, v)\}$ 
9:        $\text{UNION}(u, v)$ 

```

---



---

**Algorithm 37** Prim

---

```

1: function PRIM( $G, w : E \rightarrow \mathbb{R}, r$ )
2:    $T \leftarrow (\emptyset, \emptyset)$   $\triangleright$  minimal spanning tree
3:   for all  $v \in V(G)$  do
4:      $v.\text{weight} \leftarrow \infty$   $\triangleright$  best known cost of connecting  $v$  to  $T$ 
5:      $v.\pi \leftarrow \text{NIL}$   $\triangleright u \in T$  such that  $(u, v)$  is the least-cost edge connecting  $v$  to  $T$ 
6:    $r.\text{weight} \leftarrow 0$   $\triangleright r$  is the root of the spanning tree
7:   while  $V(T) \neq V(G)$  do  $\triangleright$  while  $T$  does not span all vertices
8:     find  $u \notin V(T)$  such that  $u.\text{weight}$  is minimal
9:      $T \leftarrow T \cup \{u\}$   $\triangleright$  add  $u$  to the spanning tree  $T$ 
10:    for all  $v \in \Gamma(u) \setminus V(T)$  do
11:      if  $w(u, v) < v.\text{weight}$  then
12:         $v.\text{weight} \leftarrow w(u, v)$ 
13:         $v.\pi \leftarrow u$ 

```

---

## 5.5 Single-Source Shortest Paths

Dijkstra’s algorithm is a greedy, label-setting method for nonnegative edge weights. At each step it finalizes the closest unreachable vertex and relaxes its outgoing edges, yielding a time complexity of  $O((|V| + |E|) \log |V|)$  with a binary-heap implementation.

---

**Algorithm 38** Dijkstra

---

```

1: function DIJKSTRA( $G, w : E \rightarrow \mathbb{R}_{\geq 0}, s$ )
2:    $N \leftarrow \emptyset$   $\triangleright$  nodes of  $G$  whose least-cost path from  $s$  is definitely known
3:   for all  $v \in V(G)$  do
4:      $v.\delta \leftarrow \infty$   $\triangleright$  best known cost from  $s$  to  $v$ 
5:      $v.\pi \leftarrow \text{NIL}$   $\triangleright$  node preceding  $v$  on the least-cost path from  $s$ 
6:    $s.\delta \leftarrow 0$   $\triangleright s$  is the source
7:   while  $N \neq V(G)$  do  $\triangleright$  while we do not know the least-cost path to all nodes
8:     find  $u \notin N$  such that  $u.\delta$  is minimal
9:      $N \leftarrow N \cup \{u\}$   $\triangleright$  add  $u$  to the nodes  $N$ 
10:    for all  $v \in \Gamma(u) \setminus N$  do
11:      if  $u.\delta + w(u, v) < v.\delta$  then
12:         $v.\delta \leftarrow u.\delta + w(u, v)$ 
13:         $v.\pi \leftarrow u$ 

```

---

Bellman-Ford algorithm is a dynamic-programming, label-correcting method that supports arbitrary (i.e. including negative) weights and detects negative-weight cycles. It relaxes all edges in up to  $|V| - 1$  passes for  $O(|V| \times |E|)$  time and uses one extra pass to check for cycles.

The Bellman-Ford equation:

$$\delta(s, v) = \min_{(u, v) \in E} (\delta(s, u) + w(u, v)) \quad (9)$$

---

**Algorithm 39** Bellman-Ford

---

```

1: function BELLMANFORD( $G, w : E \rightarrow \mathbb{R}, s$ )
2:   for all  $v \in V(G)$  do
3:      $v.\delta \leftarrow \infty$ 
4:      $v.\pi \leftarrow \text{NIL}$ 
5:    $s.\delta \leftarrow 0$ 
6:   for  $i = 1$  to  $|V(G)| - 1$  do
7:     for all  $(u, v) \in E(G)$  do
8:       if  $u.\delta + w(u, v) < v.\delta$  then
9:          $v.\delta \leftarrow u.\delta + w(u, v)$   $\triangleright$  applying Equation (9)
10:         $v.\pi \leftarrow u$ 
11:   for all  $(u, v) \in E(G)$  do
12:     if  $u.\delta + w(u, v) < v.\delta$  then
13:       return false  $\triangleright$  negative-weight cycle detected
14:   return true

```

---

## 6 Design Techniques

### 6.1 Greedy Algorithms

Greedy algorithms construct a solution by always choosing the option that looks best at the moment. That is, they make a locally optimal choice in the hope that this choice leads to a globally optimal solution.

At every step, we consider only what is best best in the current problem, not considering the results of the subproblems.

The key ingredients of a problem for a greedy strategy to work:

1. **Greedy-Choice**: One can always arrive at a globally optimal solution by making a locally optimal choice.
2. **Optimal Substructure**: An optimal solution to the problem contains within it optimal solutions to subproblems.

It is natural to prove property 2 by induction: if the solution to the subproblem is optimal, then combining the greedy choice with that solution yields an optimal solution.

The proof pattern when designing a greedy algorithm is:

- (i) **Cast** the problem as one where we make a **greedy choice** and are left with a **subproblem**.
- (ii) **Prove** that (at least) one optimal solution (not necessarily always the same one) contains the greedy choice (Property 1).
- (iii) **Prove** that the remaining subproblem is such that combining the greedy choice with the optimal solution of the subproblem gives an optimal solution to the original problem.

**Remark 1** (Designing Greedy Algorithms).

- Inventing a greedy algorithm is easy (easy to come up with greedy choices)
- Proving it optimal may be hard (requires deep understanding of the structure of the problem)  $\blacktriangleleft$

**Remark 2** (Greedy vs. Dynamic Programming). Greedy algorithms have many similarities to dynamic programming. In particular, problems for which dynamic programming works also need to have optimal substructure (Property 2).

One major difference is that instead of first finding optimal solutions to subproblems and then making an informed choice, greedy algorithms first make a greedy choice (the choice that looks best at the time) and then solve **one** resulting subproblem, without bothering to solve all possible related smaller subproblems.

Dynamic programming is more general since it does not need the greedy-choice property (Property 1). It usually looks at several subproblems and “dynamically” chooses one of them to obtain a global solution.  $\blacktriangleleft$

### 6.2 Dynamic Programming

Dynamic programming is a method for solving (typically optimization or counting, sometimes also probability) problems by decomposing them into (slightly smaller) subproblems that share subsubproblems. To make such a recursive approach efficient, the optimal solutions to those subproblems are stored in an array/table of appropriate size and dimension, so that each subsubproblem is solved just once. Instead of solving the problem top-down (recursively, with memoization) it is also possible (and often slightly more efficient due to the overhead of recursion) to solve the problem bottom-up (iteratively filling the table, starting with the base cases).

Not all problems can be solved with dynamic programming. They need to have certain properties. The key ingredients of a problem for a dynamic programming method to work are:

1. **Optimal Substructure**: An optimal solution to the problem contains within it optimal solutions to subproblems.
2. **Overlapping Subproblems**: The subproblems share subsubproblems. A recursive solution therefore revisits the same subsubproblem repeatedly.

**Example 8** (Unweighted shortest/longest path). Given  $G = (V, E)$ ,

- find the length of the shortest path from  $u$  to  $v$ 
  - decompose  $u \rightsquigarrow v$  into  $u \rightsquigarrow w \rightsquigarrow v$
  - easy to prove that, if  $u \rightsquigarrow w \rightsquigarrow v$  is minimal, then  $w \rightsquigarrow v$  is also minimal
  - this is the **optimal substructure property**!
- find the length of the longest simple (i.e., no cycles) path from  $u$  to  $v$ 
  - we can also decompose  $u \rightsquigarrow v$  into  $u \rightsquigarrow w \rightsquigarrow v$
  - however, it is not the case that, if  $u \rightsquigarrow w \rightsquigarrow v$  is maximal, then  $w \rightsquigarrow v$  is also maximal, as can be shown by a counterexample  $\blacktriangleleft$

Typical steps to solve a problem with dynamic programming:

1. Identify a problem as a dynamic programming problem: It should satisfy the above properties.

2. State expression<sup>3</sup>: Choose a set of state variables so that each valid combination represents a distinct subproblem. A good rule of thumb is to try to keep it as simple as possible.
3. Formulate state and transition relationship: Derive a recurrence (transition) relation that expresses the cost/value of a state in terms of smaller states, i.e., derive the solution to the main problem from (one of) the solutions to the  $O(1), O(n), O(n^2), \dots$  subproblems.
4. Identify the base case(s): Subproblem(s) that can't be divided into any smaller subproblems and don't depend on any other subproblems for a solution.
5. Apply tabulation (bottom-up) or memoization (top-down).

**Remark 3** (Dynamic Programming vs. Divide and Conquer). Both decompose a problem into subproblems. But in divide-and-conquer the subproblems are disjoint, that is, they do not share subsubproblems. The subproblems in divide-and-conquer are also typically much smaller than the original problem, while in dynamic programming the subproblems are typically only slightly smaller than the original problem (e.g. reducing  $L(j)$  to  $L(j-1)$ )

**Example 9** (Pingala). Using memoization, we can make the recursive algorithm from Example 1 much more efficient.

**Algorithm 40** Pingala, top-down with memoization

```

1: memo ← [NIL] × n                                ▷ memoization array of size n
2: function PINGALA(n)                               ▷ Number of 1, 2-beats in n beats
3:   if n ≤ 2 then
4:     return n
5:   if memo[n] = NIL then
6:     memo[n] ← PINGALA(n-1) + PINGALA(n-2)
7:   return memo[n]
```

Algorithm 40 runs in  $O(n)$  time.

## 7 Complexity Theory

**Definition 16** (Polynomial-Time Algorithm). A **polynomial-time algorithm** is one whose worst-case running time  $T(n)$ , on an input of size  $n$  bits, is  $O(n^k)$  for some constant  $k$ .

you have  $n$  objects:

all pairs	polynomial:	$\Theta(n^2)$
all triples	polynomial:	$\Theta(n^3)$
all $k$ -tuples for a fixed $k$	polynomial:	$\Theta(n^k)$
all subsets	super-polynomial:	$\Theta(2^n)$
all permutations	super-polynomial:	$\Theta(n!)$

you have a graph over  $n$  vertexes:

all edges	polynomial:	$\Theta(n^2)$
all trees	super-polynomial:	$\Theta(n^{n-2})$
all complete tours	super-polynomial:	$\Theta(n!)$
all cuts	super-polynomial:	$\Theta(2^n)$

### 7.1 Decision Problems

**Definition 17** (Problem). A **problem**  $Q$  is a binary relation between a set  $I$  of **instances** and a set  $S$  of **solutions**.

**Definition 18** (Concrete Problem). A **concrete problem**  $Q$  is a problem where  $I$  and  $S$  are the set of binary strings  $\{0, 1\}^*$ .

For all practical purposes, instances and solutions can be encoded as binary strings (i.e. mapped into  $\{0, 1\}^*$ ).

**Definition 19** (Decision Problem). A **decision problem**  $Q$  is a problem where the set of solutions is  $S = \{0, 1\}$ .

**Example 10** (Primality Testing). Input: the binary representation of an integer  $n$ . Output: **true** if  $n$  is prime, else **false**.

We focus on decision problems only.

- An optimization problem is **at least as hard** as its corresponding decision problem
  - having a solution to the optimization gives an immediate solution to the decision problem
- An optimization problem is **not much harder** than the corresponding decision problem
  - having a solution to the decision problem does not give an immediate solution to the optimization problem
  - but we can typically use the decision problem as a subroutine in some kind of (binary) search to solve the corresponding optimization problem

**Example 11** (Shortest Path).

<sup>3</sup>Dynamic Programming problems are all about the state and its transition. This is the most basic step which must be done very carefully because the state transition depends on the choice of state definition you make. A state can be defined as the set of parameters that can uniquely identify a certain position or standing in the given problem. This set of parameters should be as small as possible to reduce state space.

- as an optimization problem:

$$\underbrace{G = (V, E), v_{\text{source}}, v_{\text{destination}}}_{\text{instance}} \longrightarrow \underbrace{(v_{\text{source}}, \dots, v_{\text{destination}})}_{\text{solution}}$$

- Input: a graph  $G$ , a source vertex, and a destination vertex
- Output: a sequence of vertexes

- as a decision problem:

$$\underbrace{G = (V, E), v_{\text{start}}, v_{\text{end}}, \delta}_{\text{instance}} \longrightarrow \underbrace{1/0}_{\text{solution}}$$

- Input: a graph  $G$ , a start vertex, an end vertex, and a path length ( $\delta$ )
- Output: 1 if there is a path of (at most) the given length, else 0

### 7.2 Complexity Classes

**Definition 20** (Class P). A concrete decision problem  $Q$  is **polynomial-time solvable** if there is a polynomial-time algorithm  $A$  that solves it. The complexity class P is the set of all concrete decision problems that are polynomial-time solvable:

$$Q \in P \iff \exists A \text{ that solves } Q \text{ in polynomial time (worst case)}$$

**Example 12.** Primality Testing (2002, by Agrawal and Bachelor students Kayal and Saxena), Context-free Language Parsing, Shortest-Path (decision variant).

**Definition 21** (Class NP). A concrete decision problem  $Q$  is **polynomial-time verifiable** if there is a polynomial-time algorithm  $A$  such that for every instance  $x \in I$  that has a **true** solution ( $Q(x) = 1$ ), there is a certificate  $y$  of polynomial size  $|y| = O(|x|^c)$  for some constant  $c$  such that  $A(x, y) = 1$ .  $A(x, y)$  verifies in polynomial time that  $y$  proves that  $Q(x) = 1$ . The complexity class NP is the set of all concrete decision problems that are polynomial-time verifiable:

$$Q \in NP \iff \exists A \text{ that verifies } Q \text{ in polynomial time (worst case)}$$

**Example 13.** Longest-Path (decision variant), Vertex Cover, Satisfiability, Knapsack (decision variant).

**Theorem 8.**  $P \subseteq NP$

**Proof.** Given a polynomial-time solver for a problem, construct a verifier that ignores the certificate and simply runs the solver.  $\square$

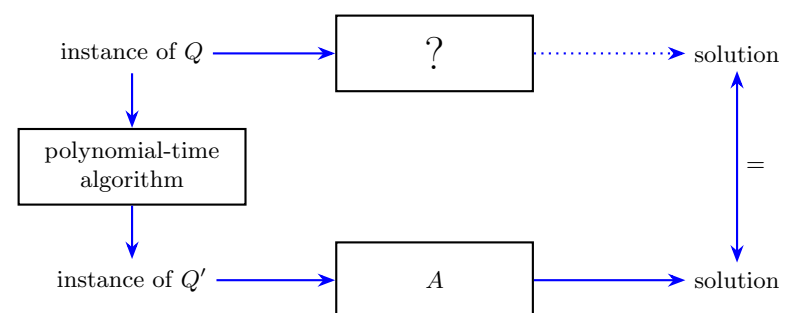
polynomial-time solvable  $\implies$  polynomial-time verifiable

polynomial-time verifiable  $\stackrel{?}{\implies}$  polynomial-time solvable

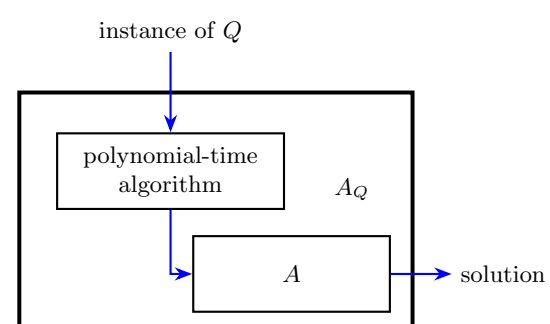
**Conjecture 9** (P vs. NP).  $P = NP$ ? The prevailing opinion is  $P \neq NP$ .

### 7.3 Polynomial-Time Reductions

**Definition 22** (Polynomial-Time Reduction). A problem  $Q$  is **polynomial-time reducible** to another problem  $Q'$  if there is a polynomial-time reduction from  $Q$  to  $Q'$ . That means there is a polynomial-time algorithm that transforms every instance  $q$  of  $Q$  into an instance  $q'$  of  $Q'$  such that the solution to  $q$  is **true** if and only if the solution to  $q'$  is **true**.



**Example 14** (2-CNF-SAT). When every clause of  $\varphi$  has at most two literals, satisfiability can be tested via an implication graph and two runs of Depth-First Search. A formula is no satisfiable if and only if  $x_i \rightsquigarrow \neg x_i \rightsquigarrow x_i$  for some  $i$ . Therefore  $2\text{-CNF-SAT} \in P$ .





## 7.4 NP-Hardness and NP-Completeness

**Definition 23** (NP-Hard). A problem  $Q'$  is NP-hard if all problems  $Q \in \text{NP}$  are polynomial-time reducible to  $Q'$ .  $\triangleleft$

**Definition 24** (NP-Complete). A problem  $Q'$  is NP-complete if  $Q' \in \text{NP}$  and  $Q'$  is NP-hard.  $\triangleleft$

**Example 15** (Circuit Satisfiability (SAT)). was the first problem proven to be NP-hard and, since  $\text{SAT} \in \text{NP}$ , it is also NP-complete. Many problems were then proved to be NP-complete through polynomial reductions from SAT, for example Vertex Cover.  $\triangleleft$

**Theorem 10.** If  $Q'$  is NP-hard and polynomial-time reducible to  $Q''$ , then  $Q''$  is NP-hard.  $\triangleleft$

**Theorem 11.** If  $Q'$  is NP-hard and polynomial-time solvable, then  $\text{P} = \text{NP}$ . Most people believe there is no such  $Q'$ .  $\triangleleft$