Projection

A projection is a transformation of a vector onto a subspace.[1]

There are different types of projections, but the one that's relevant for us here is *orthogonal projection*, which projects a vector onto the nearest point on a subspace, where "nearest" usually refers to the Euclidean distance.[2] Why this type of projection is called "orthogonal projection" will be come clear below.

Projecting from 2-D onto 1-D

Projecting a vector in two-dimensional space onto a line that goes through the origin is a nice way to build an understanding for what a projection does.[3]

Say we want to orthogonally project the vector b onto a line defined by another vector, a, and we will call the resulting projection p. Hence, p is the point on the line defined by a that is nearest to the (tip of) the vector b.

We can think of the line as being generated by scaling vector a with a scalar x, so that choosing a suitable x allows us to reach any point on the line. Finding p then boils down to finding the value of x that gets us to that point of the line that is closest to b. We can thus write $p = ax$.

[todo: insert figure here]

Let's start by finding p. We can find it in different ways.

Using calculus:

Minimising the distance between the (tip of) the vector, b, and the projection, p, is akin to solving the following problem:

[1] A subspace is a subset of a vector space that is itself a vector space in which any possible linear combination of two vectors in the space is also in the space. For instance, a 2-dimensional plane is a subspace of $\mathbb{R}^3$ if it contains all possible linear combinations of any 2-dimensional vectors. For this to be the case, the plane has to go through the origin – the point (0, 0, 0) – to contain linear combinations with the zero scalar. Similarly, a line that goes through the origin is also a valid subspace, since any linear combination of two vectors that lie on the line will also lie on the line.

[2] The Euclidean distance between two points x and $\bar{x}$ in $\mathbb{R}^N$ is defined as $\sqrt{\sum_{i=1}^{N} (\bar{x_i} - x_i)^2}$.

[3] A line through the origin is a subspace of a two-dimensional vector because it is 1-dimensional (and thus a subset of the 2-dimensional vector) and because all possible linear combinations of vectors on the line will also lie on the line (the line needs to pass through the origin for this latter statement to be true, see the footnote on subspaces).

$$
\begin{aligned}
argmin_x \sqrt{\sum_{i=1}^{2} (b_i - p_i)^2} &= argmin_x \sum_{i=1}^{2} (b_i - p_i)^2 \\
&= argmin_x \sum_{i=1}^{2} (b_i - xa_i)^2 \\
&= argmin_x (b - xa)'(b - xa),
\end{aligned}
$$

Calculating the derivative with respect to x to zero we get:

$$
\begin{aligned}
\frac{d}{dx}(b - xa)'(b - xa) &= (-a)'(b - xa) + (b - xa)'(-a) \\
&= -a'b + xa'a - a'b + xa'a \\
&= -2a'b + 2xa'a = 0
\end{aligned}
$$

Solving for x we get:

$$
\begin{aligned}
-2a'b + 2xa'a &= 0 \\
xa'a &= a'b \\
x &= (a'a)^{-1}a'b
\end{aligned}
$$

Hence, given that $p = ax$, we have:

$$
p = ax = \underbrace{a(a'a)^{-1}a'}_{P_a} b,
$$

where P_a is the projection matrix.

Let's reflect for a moment what this all means. In general, pre-multiplying a vector by a matrix transforms the vector in a particular way. When we perform orthogonal projection, we pre-multiply a vector by a matrix that transforms the vector into that point on a subspace that it closest to the original vector. In our case here, pre-multiplying our initial vector b by the projection matrix P_a transforms b into that point on a that is closest to b, which we call p. Given that we define "nearest" using the Euclidean distance, it makes sense that the projection matrix would emerge out of the solution to the minimisation problem of finding the point on the subspace that minimises the Euclidean distance to the original vector.

Using basic geometry:

We could also find p using our understanding of basic geometry. Looking at the figure above, it is intuitively obvious that the shortest path between the tip of b and the projection p onto