An algebraic approach to a Kripkean theory of probability and truth

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February 21, 2019

Abstract

In her 2016 thesis, Catrin Campbell-Moore introduces a Kripkean theory of probability and truth which relies on the Strong Kleene evaluation scheme. In the context of plain theories of truth, Strong Kleene is probably the most important evaluation scheme. However, some authors also consider different schemes like Weak Kleene. With a different scheme, the resulting theory yields different results—both in the case of plain truth as well as for truth and probability. Therefore, one might want to have some flexibility regarding the evaluation scheme. In this text, we will develop an approach to a joint theory of probability and truth which can be instantiated with different evaluation schemes.

1 Introduction

In natural languages, we can formulate sentences like the following:

(1) Sentence (1) is false.

(2) The probability of sentence (2) is less than $\frac{1}{2}$.

Obviously, those sentences raise a number of questions. One seems to be what truth values one should assign to them. Regarding the first sentence—which is often called the liar sentence—this question leads to a paradox: If we assume that the sentence is true, then it should be the case that (1) is false, which is a contradiction. If we otherwise assume that (1) is false, then the truth condition for (1) is satisfied, so that we should be able to conclude that (1) is true, again a contradiction. That paradox, the liar paradox, motivated several theories of truth in the last century, where one suggestion by Saul Kripke was especially influential.

Very roughly, one might summarise the idea of a Kripkean theory as follows: As it both leads to a contradiction to assign to the liar sentence the truth value true (or $\mathbf{t}$), as it does with false (or $\mathbf{f}$), one might want to solve the issue by leaving the classical set of truth values $\{\mathbf{t}, \mathbf{f}\}$ and instead consider a richer set like $\{\mathbf{n}, \mathbf{t}, \mathbf{f}\}$. Then, the liar sentence can be assigned the truth value $\mathbf{n}$. One philosophical reading of this value would be that one has neither been told that the sentence at stake has the value true nor that it is false. Another reading would be that $\mathbf{n}$ means meaningless.

To turn those brief considerations into a comprehensive theory of truth there are many things that need to be done. One is the question of the evaluation scheme: In the classical case, the conjunction connective $\&$ receives the following semantic rule:

$$\mathrm{Val}(A\&B) = \mathbf{t} \quad \Leftrightarrow \quad \mathrm{Val}(A) = \mathbf{t} \quad \text{and} \quad \mathrm{Val}(B) = \mathbf{t}.$$

&	t	f
t	t	f
f	f	f

In the three-valued case however, this is not sufficient. We also have to specify how the function $\&$ operates on $\mathbf{n}$ as one of its arguments. Regarding the told interpretation, the following seems natural

&	t	f	n
t	t	f	n
f	f	f	f
n	n	f	n

For instance, if one has been told that not A and nothing about B, one might conclude that not $A\&B$, so $\mathbf{f}\&\mathbf{n} = \mathbf{f}$. We call one salient extension of such a rule to all logical connectives (and the false sentence $\perp$) the Strong Kleene Scheme (more about this later). So with the Strong Kleene scheme one would say that '(1) and $\perp$' is false. Regarding the "meaningless" interpretation, one would probably rather like to say that a complex sentences becomes meaningless when this is the case with one of its parts:

&	**t**	**f**	**n**
t	**t**	**f**	**n**
f	**f**	**f**	**n**
n	**n**	**n**	**n**

The according extension of this rule will be the Weak Kleene scheme. This assigns the value $\mathbf{n}$ to the sentence '(1) and $\perp$'. As a summary, we might say that there are different evaluation schemes with different philosophical benefits and different technical results e.g. for the truth value of sentences like '(1) and $\perp$'. (see [Gupta and Belnap, 1993, p. 40-44])

Now, one might not want to stop at this point of a plain theory of truth and also take probability sentences like (2) into account. This is what Catrin Campbell-Moore—beside other things—did in her thesis [Campbell-Moore, 2016]. In doing so, she decided to focus on the Strong Kleene evaluation scheme. Therefore, in her definition of the jump operator—a central technical part of a Kripkean theory of truth—the clauses are fixed in a way adapted to the Strong Kleene scheme. This choice is perfectly fine, since the Strong Kleene scheme is an important evaluation scheme. However, it raises the question whether a joint theory of truth and probability can be also formulated independently of that evaluation scheme.

In the following, we will develop a positive answer to this question. In order to so, we will start with an algebraic account of a Kripkean theory of truth which is independent of the evaluation scheme from a book by Gupta and Belnap. We will present the most important parts of this account in section 2. Afterwards, we are going to generalise it to the case of truth and probability. On one interpretation, this amounts to adding two layers of additional complexity: First, the semantic values are additionally relativised to possible world, and second, the theory yields the meaning of both a truth predicate as well as a probability predicate. For convenience, we will introduce those layers of additional complexity independently. In this vein, in section 3 we introduce a theory of independent truth predicates in their respective possible world. Afterwards, we will also introduce the probability predicate in section 4. Lastly, in section 5, we give a brief summary and mention some open issues in the outlook.[1]

[1]We will be brief with several proofs and give a detailed machine and human readable version in the appendix.

2 The base case of a plain theory of truth

In this section, we will repeat some results about a plain Kripkean theory of truth on which our construction relies. All this is a repetition of Chapter 2 of the book [Gupta and Belnap, 1993].

Syntax As usual, we start by introducing a syntax. Namely, a syntax will be generated by the following sets of symbols (and corresponding functions):

(1) A set of symbols for constants or names $\mathcal{C}$

(2) A set of symbols for functions $\mathcal{F}$, together with an arity function $\pi_F \colon \mathcal{F} \to \mathbb{N}$.

(3) A set of symbols for relations $\mathcal{R}$, together with an arity function $\pi_R \colon \mathcal{R} \to \mathbb{N}$.

(4) A set of variable symbols $\{v_1, v_2, \dots\}$.

In this paper, we will allow for different choices for the first three of those. The set of variables is assumed to be fixed once. For example, one could consider $v_i = i, i \in \mathbb{N}$.

Then, a term t is given (in Backus-Naur form) for $i \in \mathbb{N}, c \in \mathcal{C}, f \in \mathcal{F}$ as

$$t ::= v_i \mid c \mid f \underbrace{t\, t\, \dots\, t}_{\pi_F f \text{ times}}. \tag{1}$$

And for $r \in \mathcal{R}, i \in \mathbb{N}$, a formula ϕ is defined as

$$\phi ::= r \underbrace{t\, t\, \dots\, t}_{\pi_R r \text{ times}} \mid t = t \mid \perp \mid \phi \,\&\, \phi \mid \sim \phi \mid \forall v_i \phi. \tag{2}$$

Note that the set of terms depends on the sets $\mathcal{F}$ and $\mathcal{C}$, while the set of formula depends on $\mathcal{F}, \mathcal{C}$, and $\mathcal{R}$. Therefore we write $L^{(\mathcal{F},\mathcal{C},\mathcal{R})}$ for the latter. The subset of sentences (formula with no free variables) is denoted as $L^{(\mathcal{F},\mathcal{C},\mathcal{R})}_{sen}$.

Semantics The semantic value of those terms and sentences is defined in terms of an evaluation scheme and a model. Those two notions are defined as follows, where B is the set of truth values.

Definition (model) A model $M = (D, C_w, F_w, R_w)$ consists of a domain D, a constant valuation function $C_w \colon \mathcal{C} \to D$, a function valuation function $F_w \colon (\bigcup_{f \in \mathcal{F}} f \times D^{\pi_F f}) \to D$, and a relation valuation function $R_w \colon (\bigcup_{r \in \mathcal{R}} r \times D^{\pi_R r}) \to B$.

Definition (evaluation scheme) An evaluation scheme $S = (\mathbf{f}, \mathbf{t}, \mathbf{N}, \mathbf{A}, \mathbf{U})$ consists of a false truth value $\mathbf{f} \in B$, a true truth value $\mathbf{t} \in B$, a not function $\mathbf{N} \colon B \to B$, an and function $\mathbf{A} \colon B \times B \to B$, and an universalisation function satisfying $\mathbf{U}(D) \colon (D \to B) \to B$.

Then, we can define semantic value as follows:

Definition (semantic value) The semantic value relative to an evaluation scheme $S = (\mathbf{f}, \mathbf{t}, \mathbf{N}, \mathbf{A}, \mathbf{U})$, an assignment $s \colon \mathbb{N} \to D$, and a model $M = (D, C_w, F_w, R_w)$ is defined as

(1) For $c \in \mathcal{C}$, $\mathrm{Val}_{S,s,M}(c) = C_w(c)$.

(2) For $i \in \mathbb{N}$, $\mathrm{Val}_{S,s,M}(v_i) = s(i)$.

(3) For $f \in \mathcal{F}$ and terms $t_1, \ldots, t_{\pi_F f}$

$$\mathrm{Val}_{S,s,M}(f t_1 \ldots t_{\pi_F f}) = (F_w f)(\mathrm{Val}_{S,s,M}(t_1), \mathrm{Val}_{S,s,M}(t_2), \ldots, \mathrm{Val}_{S,s,M}(t_{\pi_F f})). \quad (3)$$

(4) $\mathrm{Val}_{S,s,M}(\bot) = \mathbf{f}$.

(5) For terms t_1 and t_2,

$$\mathrm{Val}_{S,s,M}(t_1 = t_2) = \begin{cases} \mathbf{t} \text{ if } \mathrm{Val}_{S,s,M}(t_1) = \mathrm{Val}_{S,s,M}(t_2) \\ \mathbf{f} \text{ else.} \end{cases} \quad (4)$$

(6) For $r \in \mathcal{R}$ and terms $t_1, \ldots, t_{\pi_R r}$

$$\mathrm{Val}_{S,s,M}(r t_1 \ldots t_{\pi_R r}) = (R_w r)(\mathrm{Val}_{S,s,M}(t_1), \mathrm{Val}_{S,s,M}(t_2), \ldots, \mathrm{Val}_{S,s,M}(t_{\pi_R r})). \quad (5)$$

(7) For a formula ϕ, $\mathrm{Val}_{S,s,M}(\sim \phi) = \mathbf{N}(\mathrm{Val}_{S,s,M})$.

(8) For formulas ϕ_1, ϕ_2, $\mathrm{Val}_{S,s,M}(\phi_1 \& \phi_2) = \mathbf{A}(\mathrm{Val}_{S,s,M}(\phi_1), \mathrm{Val}_{S,s,M}(\phi_2))$.

(9) For a formula ϕ and $i \in \mathbb{N}$,

$$\mathrm{Val}_{S,s,M}(\forall v_i \phi) = \mathbf{U}(D)(\lambda v. \mathrm{Val}_{S,\lambda k.s[i:=v],M}(\phi)), \quad (6)$$

where the assignment $s[i := v]$ should be v at i and s elsewhere.

To be concrete, we define the following evaluation schemes for the classical case with two truth values, $B = \{t_2, f_2\}$, the three-valued case $B = \{t_3, f_3, n_3\}$, and the four valued case $B = \{t_4, f_4, b_4, n_4\}$.

Definition (scheme τ) The scheme τ (the classical scheme) is defined as $\tau = (f_2, t_2, \mathbf{N}_\tau, \mathbf{A}_\tau, \mathbf{U}_\tau)$, where[2]

$$\begin{aligned} \mathbf{N}_\tau(b) = t_2 &\quad \text{iff } b = f_2 &(7)\\ \mathbf{A}_\tau(b_1, b_2) = t_2 &\quad \text{iff } b_1 = t_2 \text{ and } b_2 = t_2 &(8)\\ \mathbf{U}_\tau(D, f) = t_2 &\quad \text{iff for all } v \in D\ f(v) = t_2. &(9) \end{aligned}$$

Definition (scheme μ) The scheme μ (the Weak Kleene scheme) is defined as $\mu = (f_3, t_3, \mathbf{N}_\mu, \mathbf{A}_\mu, \mathbf{U}_\mu)$, where $\mathbf{N}_\mu$ and $\mathbf{A}_\mu$ are given as follows:

$\mathbf{N}_\mu$	
t_3	f_3
f_3	t_3
n_3	n_3

$\mathbf{A}_\mu$	t_3	f_3	n_3
t_3	t_3	f_3	n_3
f_3	f_3	f_3	n_3
n_3	n_3	n_3	n_3

The universalisation should generalise the and and therefore:

$$\mathbf{U}_\mu(D, f) = \begin{cases} t_3, & \text{if } f(v) = t_3 \text{ for all } v \in D,\\ n_3, & \text{if } f(v) = n_3 \text{ for some } v \in D,\\ f_3, & \text{else.} \end{cases} \tag{10}$$

Definition (scheme κ) The scheme κ (the Strong Kleene scheme) is defined as $\mu = (f_3, t_3, \mathbf{N}_\kappa, \mathbf{A}_\kappa, \mathbf{U}_\kappa)$, where $\mathbf{N}_\kappa$ and $\mathbf{A}_\kappa$ are given as follows:

$\mathbf{N}_\kappa$	
t_3	f_3
f_3	t_3
n_3	n_3

$\mathbf{A}_\kappa$	t_3	f_3	n_3
t_3	t_3	f_3	n_3
f_3	f_3	f_3	f_3
n_3	n_3	f_3	n_3

Again, the universalisation should generalise the and:

$$\mathbf{U}_\kappa(D, f) = \begin{cases} t_3, & \text{if } f(v) = t_3 \text{ for all } v \in D,\\ f_3, & \text{if } f(v) = f_3 \text{ for some } v \in D,\\ n_3, & \text{else.} \end{cases} \tag{11}$$

[2]We are not distinguishing between the curried and uncurried version of a function, so that we e.g. use $\mathbf{D}(D)(f)$ and $\mathbf{D}(D, f)$ interchangeable.

Definition (scheme ν) The scheme ν (the four valued scheme) is defined as $\nu = (f_4, t_4, \mathbf{N}_\nu, \mathbf{A}_\nu, \mathbf{U}_\nu)$, where $\mathbf{N}_\nu$ and $\mathbf{A}_\nu$ are given as follows:

$\mathbf{N}_\nu$	
t_4	f_4
f_4	t_4
n_4	n_4
b_4	b_4

$\mathbf{A}_\nu$	t_4	f_4	n_4	b_4
t_4	t_4	f_4	n_4	b_4
f_4	f_4	f_4	f_4	f_4
n_4	n_4	f_4	n_4	f_4
b_4	b_4	f_4	f_4	b_4

Again, the universalisation should generalise the and. We refer the reader to the appendix for an exact definition. Note that the four-valued scheme extends the Strong Kleene scheme and can be read alike in the told interpretation by regarding b_4 as "both" (one is been both told that the sentence at stake is true and false).

We can introduce partial orders on the sets of truth values. By partial order we mean a relation which is reflexive, antisymmetric, and transitive. Such relations can be depicted like in Fig. 1, which should be a definition of $\leq$ on the respective sets of truth values. It should be understood such that e.g. for $\{n_3, t_3, f_3\}$ it is the case that $a \leq a$ for $a \in \{n_3, t_3, f_3\}$ and $n_3 \leq b$ for $b \in \{t_3, f_3\}$. Apart from that $c \leq d$ should be false.[3]

These ordering can be extended to models as follows: Let $M_1 = (D_1, C_{w1}, F_{w1}, R_{w1})$ and $M_2 = (D_2, C_{w2}, F_{w2}, R_{w2})$ be models. Then

$$(D_1, C_{w1}, F_{w1}, R_{w1}) \leq (D_2, C_{w2}, F_{w2}, R_{w2}) \tag{12}$$
$$:\Leftrightarrow \quad D_1 = D_2 \quad \text{and} \quad C_{w1} = C_{w2} \quad \text{and} \quad F_{w1} = F_{w2} \quad \text{and} \tag{13}$$
$$\forall r \in \mathcal{R}, v_i \in D\, R_{w1}(r)(v_1, \ldots, v_{\pi_R r}) \leq R_{w2}(r)(v_1, \ldots, v_{\pi_R r}) \tag{14}$$

In terms of these relations, we can now state a monotony property of the non-classical schemes.

Lemma (Monotony in the schemes) Let M_1 and M_2 be two models with $M_1 \leq M_2$. Then the semantic value is monotone in the schemes $S \in \{\mu, \kappa, \nu\}$, namely

$$\mathrm{Val}_{S,s,M_1}(\phi) \leq \mathrm{Val}_{S,s,M_2}(\phi) \quad \forall \phi, s, S \in \{\mu, \kappa, \nu\}. \tag{15}$$

For the proof of this, one goes through all connectives in all schemes and proves the respective property. Then those results can be summarised in a proof by induction over the formula ϕ.

[3] Note that we depict the ordering through the height of the points and neglect to connect nodes with themselves.

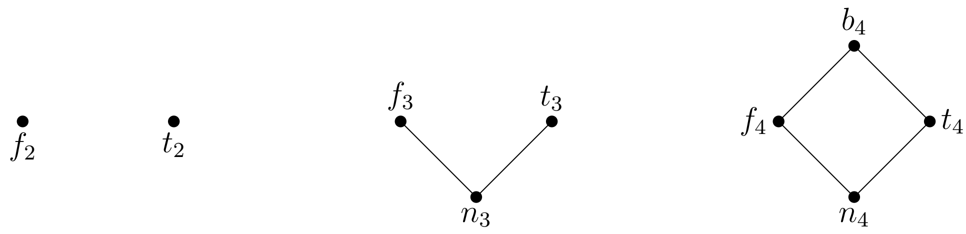

Figure 1: Orderings on the sets of truth values

In forthcoming constructions, we will need more structure than a mere partial order. Namely, the notion of ccpos (coherent complete partial orders) will be relevant:

Definition (Ccpo) A partial order $(M, \leq)$ is a ccpo, if each consistent set $X \subset M$ has a supremum in M. A $X \subset M$ is consistent if each $\{x, y\} \subset X$ has an upper bound in M.

One important implication of this definition is that a ccpo has a minimal element, since the empty set is consistent.

By a distinction of cases, one can show

Lemma (Examples of ccpos) $\{t_3, n_3, f_3\}$ and $\{t_4, b_4, f_4, n_4\}$ are ccpos, while $\{t_2, f_2\}$ is not.

A very useful property of ccpos is the following

Lemma (Function space ccpos) Let $(X, \leq)$ be a ccpo and D some set. Then $\{f : D \to X\}$ is a ccpo with the ordering

$$f \leq g \quad :\Leftrightarrow \quad \forall d \in D \quad f(d) \leq g(d). \tag{16}$$

Furthermore, if we are given a ccpo $(X, \leq)$ and an operation on it, say $f : X \to X$, we define a notion of monotony:

Definition (Monotony of an operator) Let $(X, \leq)$ be a ccpo, $f : X \to X$. Then f is called monotone if for all $x, y \in X$ with $x \leq y$ it holds that $f(x) \leq f(y)$.

This is a property of interest because of the following

Theorem (Visser's fixed point theorem) Let $(X, \leq)$ be a ccpo, $f : X \to X$ monotone. Then the set of fixed points of f, $\{x \mid f(x) = x\}$, is a ccpo.

Especially, this means that there is a least fixed point. We will exploit this because fixed points of a certain jump operator yield interesting interpretations of a truth predicate. Before we can prove that, we collect several assumptions in the following

Definition (ground model) A model $M = (D, C_w, F_w, R_w)$ is a ground model regarding a symbol $G \in \mathcal{R}$ with $\pi_R(G) = 1$, a coding[4] $c\colon L^{(\mathcal{F},\mathcal{C},\mathcal{R})} \to D$ and an evaluation scheme $S = (\mathbf{f}, \mathbf{t}, \dots)$ iff $c|_{L_{sen}^{(\mathcal{F},\mathcal{C},\mathcal{R})}}$ is injective and

$$R_w(r)(t_1, \dots, t_{\pi_R r}) \in \{\mathbf{t}, \mathbf{f}\} \quad \text{for all } r \in \mathcal{R}\backslash\{G\}, \quad t_1, \dots, t_{\pi_R r} \text{ terms.} \tag{17}$$

The task is now to generate an interpretation for G based on a ground model. One could therefore also require that R_w is undefined on G as Gupta and Belnap do. For technical ease we allow R_w to be defined also on G, but we will plainly ignore its values there.

Regarding the interpretation for G, we now define

Definition (jump operator ρ_M) Let $M = (D, C_w, F_w, R_w)$ be a ground model regarding a $G \in \mathcal{R}$, a coding c and an evaluation scheme $S = (\mathbf{f}, \mathbf{t}, \mathbf{N}, \mathbf{A}, \mathbf{U})$. Then we define the jump operator $\rho_M\colon (D \to B) \to (D \to B)$ as

$$\rho_M(g)(v) = \begin{cases} \mathrm{Val}_{S,\hat{s},(D,C_w,F_w,R_w[G:=g])}(c^{-1}v) & \text{if } v \in c(L_{sen}^{(\mathcal{F},\mathcal{C},\mathcal{R})}), \\ \mathbf{f} & \text{otherwise.} \end{cases} \tag{18}$$

Here, $\hat{s}$ is some assignment, e.g. $\lambda n.c(\perp)$ (as we consider sentences, the semantic value is independent of it).

This jump operator is monotonous, as the following Lemma articulates:

Lemma (Monotony of ρ_M) Let $M = (D, C_w, F_w, R_w)$ be a ground model as in the definition of the jump operator with a scheme $S \in \{\mu, \kappa, \nu\}$. Furthermore, let $g_1, g_2\colon D \to B$ be two interpretations for the truth predicate G with $g_1 \leq g_2$. Then

$$\rho_M(g_1) \leq \rho_M(g_2). \tag{19}$$

[4]By coding we mean merely function of the specified type here.

Proof This is a consequence of the monotony in the non-classical schemes. Namely, the updated models in the definition of the jump operator satisfy

$$(D, C_w, F_w, R_w[G := g_1]) \leq (D, C_w, F_w, R_w[G := g_2]). \tag{20}$$

This implies monotony for sentences. For non-sentences, the jump operator is constant and the claim trivial.

This allows us to prove

Lemma (Fixed point property of ρ_M) For each ground model $M = (D, C_w, F_w, R_w)$ regarding a scheme $S \in \{\mu, \kappa, \nu\}$ there exists a $g\colon D \to B$ such that

$$\rho_M(g)(v) = g(v) \quad \forall v \in D. \tag{21}$$

Proof From the Lemma about function space ccpos, we know that $\{f\colon D \to B\}$ is a ccpo. ρ_M is a monotonous operation on that set as shown in the previous Lemma. From Visser's fixed point theorem we can conclude that the set of fixed points is a ccpo. Especially, this yields the existence of a least fixed point.

3 Independent truth predicates in their possible worlds

Let us now generalise the results from the previous section to a more general setting: We still fix a language L stemming from the sets $\mathcal{C}$, $\mathcal{F}$, $\mathcal{R}$. Further let S be a fixed evaluation scheme. Now we do not evaluate the truth of a sentences relative to one model, but we consider a set of possible worlds W, which come with respective models for each world but with a constant domain.[5]

Therefore we would like to generalise the notions of a model and a ground model into the notions of a W-model and a W-ground model as follows:

Definition (W-model) A W-model $\mathfrak{M} = (D, \mathfrak{C}_w, \mathfrak{F}_w, \mathfrak{R}_w)$ consists of a constant domain D and functions $\mathfrak{C}_w$, $\mathfrak{F}_w$, and $\mathfrak{R}_w$ defined on W such that $(D, \mathfrak{C}_w(w), \mathfrak{F}_w(w), \mathfrak{R}_w(w))$ is a model for each $w \in W$.

[5]One could also generalise this to an individual domain for each world. We opt for ease of presentation instead of flexibility here. Note that with a variable domain also a variable coding would be necessary later on.

Definition (ground W-model) A W-model $\mathfrak{M} = (D, \mathfrak{C}_w, \mathfrak{F}_w, \mathfrak{R}_w)$ is a ground W-model regarding a symbol $G \in \mathcal{R}$ with $\pi_R(G) = 1$, a coding $c\colon L^{(\mathcal{F},\mathcal{C},\mathcal{R})} \to D$ and an evaluation scheme $S = (\mathbf{f}, \mathbf{t}, \dots)$ iff $c|_{L_{sen}^{(\mathcal{F},\mathcal{C},\mathcal{R})}}$ is injective and

$$\mathfrak{R}_w(w)(r)(t_1, \dots, t_{\pi_R r}) \in \{\mathbf{t}, \mathbf{f}\} \quad \text{for all } r \in \mathcal{R}\backslash\{G\}, \quad t_1, \dots, t_{\pi_R r} \text{ terms}, \quad w \in W. \tag{22}$$

The jump operator is now defined again as an operator on interpretations of G, which in this context means $(W \to D \to B)$.

Definition (jump operator $\rho_{\mathfrak{M}}^W$) Let $\mathfrak{M} = (D, \mathfrak{C}_w, \mathfrak{F}_w, \mathfrak{R}_w)$ be a ground W-model regarding a $G \in \mathcal{R}$, a coding c and an evaluation scheme $S = (\mathbf{f}, \mathbf{t}, \mathbf{N}, \mathbf{A}, \mathbf{U})$. Then we define the jump operator $\rho_{\mathfrak{M}}^W\colon (W \to D \to B) \to (W \to D \to B)$ as

$$\rho_{\mathfrak{M}}^W(g)(w, v) = \begin{cases} \mathrm{Val}_{S,\hat{s},(D,\mathfrak{C}_w(w),\mathfrak{F}_w(w),\mathfrak{R}_w(w)[G:=g(w)])}(c^{-1}v) & \text{if } v \in c(L_{sen}^{(\mathcal{F},\mathcal{C},\mathcal{R})}), \\ \mathbf{f} & \text{otherwise.} \end{cases} \tag{23}$$

Lemma (Monotony of $\rho_{\mathfrak{M}}^W$) Let $\mathfrak{M} = (D, \mathfrak{C}_w, \mathfrak{F}_w, \mathfrak{R}_w)$ be a ground W-model as in the definition of the jump operator with a scheme $S \in \{\mu, \nu, \kappa\}$. Let $g_1, g_2\colon W \to D \to B$ with $g_1 \le g_2$. Then

$$\rho_{\mathfrak{M}}^W(g_1) \le \rho_{\mathfrak{M}}^W(g_2). \tag{24}$$

Proof Basically, we repeat the proof of the corresponding property of ρ_M in each possible world.

Again, that allows us to prove an important fixed point property:

Lemma (Fixed point property of $\rho_{\mathfrak{M}}^W$) For each ground W-model $\mathfrak{M} = (D, \mathfrak{C}_w, \mathfrak{F}_w, \mathfrak{R}_w)$ regarding a scheme $S \in \{\mu, \kappa, \nu\}$, there exists a function $g\colon (W \to D \to B)$ such that

$$\rho_{\mathfrak{M}}^W(g)(w, v) = g(w, v) \quad \forall w \in W, v \in D. \tag{25}$$

Proof By Currying, we observe that $\{f\colon W \to D \to B\}$ is essentially just $\{f\colon (W \times D) \to B\}$. Hence, we can again apply the Lemma about function space ccpos to show that $\{f\colon W \to D \to B\}$ is a ccpo.[6] Again, $\rho_{\mathfrak{M}}^W$ is a monotone operation on it and Visser's fixed point Theorem yields the existence of a fixed point.

[6] We could also apply the Lemma about function space ccpos twice.

4 A joint theory of truth and probability

Now, we keep the setting of the set of possible worlds W and introduce some interaction between them in terms of a probability predicate.

Definition (tp-model) A tp-model $\mathfrak{M} = (W, m, D, \mathfrak{C}_w, \mathfrak{F}_w, \mathfrak{R}_w)$ consists of a set of possible worlds W, a function $m\colon W \to \mathcal{P}(W) \to \mathbb{R}$ with

(1) $m(w)(W) = 1$ for all $w \in W$,

(2) $m(w)(A) \geq 0$ for all $A \subset W, w \in W$,

(3) for all $w \in W$, and $A, B \subset W$ with $A \cap B = \varnothing$, it holds that

$$m(w)(A \cup B) = m(w)(A) + m(w)(B), \tag{26}$$

and a W-model $(D, \mathfrak{C}_w, \mathfrak{F}_w, \mathfrak{R}_w)$.

Definition (ground tp-model) A tp-model $\mathfrak{M} = (W, m, D, \mathfrak{C}_w, \mathfrak{F}_w, \mathfrak{R}_w)$ is a ground tp-model regarding two symbols $G \in \mathcal{R}$ and $H \in \mathcal{R}$ with $\pi_R(G) = 1$ and $\pi_R(H) = 2$, a coding $c_1\colon L^{(\mathcal{F},\mathcal{C},\mathcal{R})} \to D$, a coding $c_2\colon \mathbb{Q} \to D$ and an evaluation scheme $S = (\mathbf{f}, \mathbf{t}, \dots)$ iff $c|_{L_{sen}^{(\mathcal{F},\mathcal{C},\mathcal{R})}}$ and c_2 are injective and

$$\mathfrak{R}_w(w)(r)(t_1, \dots, t_{\pi_R r}) \in \{\mathbf{t}, \mathbf{f}\} \quad \text{for all } r \in \mathcal{R} \backslash \{G, H\}, \quad t_1, \dots, t_{\pi_R r} \text{ terms}, \quad w \in W. \tag{27}$$

The pair of codings we require here is illustrated in Fig. 2 for convenience.

The jump operator is now defined again as an operator on interpretations of G and H, which means $(W \to D \to B) \times (W \to D \to D \to B)$.

Definition (jump operator $\rho_{\mathfrak{M}}^{tp}$) Let $\mathfrak{M} = (W, m, D, \mathfrak{C}_w, \mathfrak{F}_w, \mathfrak{R}_w)$ be a ground tp-model regarding G and H, c_1 and c_2, and an evaluation scheme $S = (\mathbf{f}, \mathbf{t}, \mathbf{N}, \mathbf{A}, \mathbf{U})$. Then we define the jump operator $\rho_{\mathfrak{M}}^{tp}\colon ((W \to D \to B) \times (W \to D \to D \to B)) \to ((W \to D \to B) \times (W \to D \to D \to B))$ as

$$\rho_{\mathfrak{M}}^{tp}(g, h) = (\rho_1, \rho_2) \text{ where} \tag{28}$$

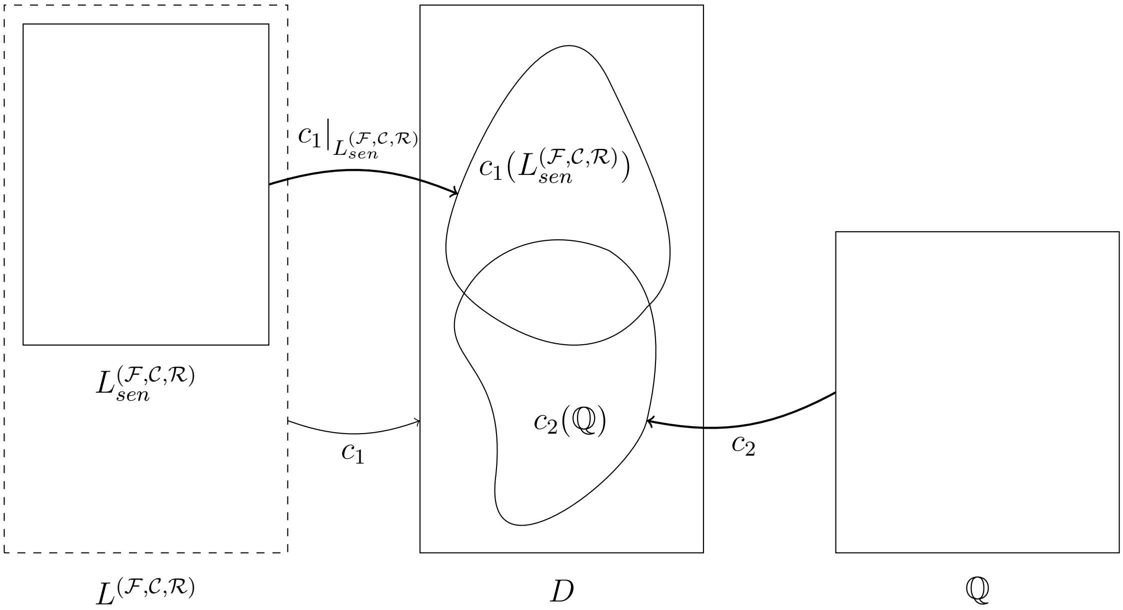

Figure 2: Illustration of the domains and ranges of the codings c_1 and c_2, and the domain of discourse D.

$$\rho_1 = \lambda w.\lambda v. \begin{cases} \mathrm{Val}_{S,\hat{s},(D,\mathfrak{C}_w(w),\mathfrak{F}_w(w),\mathfrak{R}_w(w)[G:=g(w)][H:=h(w)])}(c_1^{-1}v) & \text{if } v \in c_1(L_{sen}^{(\mathcal{F},\mathcal{C},\mathcal{R})}), \\ \mathbf{f} \quad \text{otherwise.} \end{cases} \tag{29}$$

$$\rho_2 = \lambda w.\lambda v_1.\lambda v_2. \begin{cases} \mathrm{pVal}_S(c_1^{-1}(v_1), c_2^{-1}(v_2)) & \text{if } v_1 \in c_1(L_{sen}^{(\mathcal{F},\mathcal{C},\mathcal{R})}) \text{ and } v_2 \in c_2(\mathbb{Q}), \\ \mathbf{f} \quad \text{otherwise.} \end{cases} \tag{30}$$

Now, we need to fix a probability evaluation function pVal. Like with the Val-operation we define it for each scheme $S \in \{\mu, \kappa, \nu\}$. Let us start with κ. To motivate the definition, we refer to the told interpretation of the truth values and the reading of m as a quantitative accessibility relation. So fix a world $w \in W$ in which we assume the agent to be, a sentence ϕ and a probability threshold q. The function $m(w)$ models how likely it is that the facts in w are as described in one of the worlds in the set of possible worlds. So if we had $m(w)\{w_1\} = \frac{1}{3}$, we know that the probability that ϕ (in w) has the truth value $\mathrm{Val}_{S,\hat{s},\mathfrak{C}_w(w_1),\mathfrak{F}_w(w_1),\mathfrak{R}_w(w_1)[G:=g(w_1)][H:=h(w_1)]}(\phi)$ is $\frac{1}{3}$. So the criterion for $\mathrm{pVal}_\kappa(\phi, q) = t_3$ should go as follows: We loop through all possible worlds and check whether ϕ is true there. Then we accumulate the probabilities of all the possible worlds where that is the case. If the result of this summation is $\geq q$, we say

$\mathrm{pVal}_\kappa(\phi, q) = t_3$. Therefore

$$\mathrm{pVal}_\kappa(\phi, q) = \begin{cases} t_3 & \text{if} \quad m(w)\{w_1 \in W \mid \mathrm{Val}^*(w_1) = t_3\} \geq q, \\ f_3 & \text{if} \quad m(w)\{w_1 \in W \mid \mathrm{Val}^*(w_1) = f_3\} > 1 - q, \\ n_3 & \text{else}, \end{cases} \tag{31}$$

where we use the abbreviation

$$\mathrm{Val}^*(w_1) = \mathrm{Val}_{S,\hat{s},\mathfrak{C}_w(w_1),\mathfrak{F}_w(w_1),\mathfrak{R}_w(w_1)[G:=g(w_1)][H:=h(w_1)]}(\phi). \tag{32}$$

Regarding ν we extend this scheme with the value both in its reading in the told interpretation:

$$\mathrm{pVal}_\nu(\phi, q) = \begin{cases} b_4 & \text{if} \quad m(w)\{w_1 \in W \mid \mathrm{Val}^*(w_1) \geq t_4\} \geq q \\ & \quad \text{and} \quad m(w)\{w_1 \in W \mid \mathrm{Val}^*(w_1) \geq f_4\} > 1 - q \\ t_4 & \text{if} \quad m(w)\{w_1 \in W \mid \mathrm{Val}^*(w_1) \geq t_4\} \geq q \\ & \quad \text{and not} \quad m(w)\{w_1 \in W \mid \mathrm{Val}^*(w_1) \geq f_4\} > 1 - q \\ f_4 & \text{if} \quad m(w)\{w_1 \in W \mid \mathrm{Val}^*(w_1) \geq f_4\} > 1 - q, \\ & \quad \text{and not} \quad m(w)\{w_1 \in W \mid \mathrm{Val}^*(w_1) \geq t_4\} \geq q \\ n_4 & \text{else}. \end{cases} \tag{33}$$

Note that $\geq$ in this context is the relation for $\{f_4, t_4, n_4, b_4\}$ depicted in Fig. 1. Then $a \geq f_4$ means that either $a = f_4$ or $a = b_4$, and accordingly for t_4.

For the scheme μ one could (at least technically spoken) also use the evaluation function pVal_κ. However, we think that with the ”meaningless” reading in mind, one might want to consider another type of evaluation, namely one where a sentence is meaningless if it refers to at least one meaningless sentence. That motivates

$$\mathrm{pVal}_\mu(\phi, q) = \begin{cases} n_3 & \text{if} \quad m(w)\{w_1 \in W \mid \mathrm{Val}^*(w_1) = n_3\} > 0, \\ t_3 & \text{if} \quad m(w)\{w_1 \in W \mid \mathrm{Val}^*(w_1) = t_3\} \geq q \\ & \quad \text{and} \quad m(w)\{w_1 \in W \mid \mathrm{Val}^*(w_1) = n_3\} = 0, \\ f_3 & \text{if} \quad m(w)\{w_1 \in W \mid \mathrm{Val}^*(w_1) = f_3\} > 1 - q \\ & \quad \text{and} \quad m(w)\{w_1 \in W \mid \mathrm{Val}^*(w_1) = n_3\} = 0. \end{cases} \tag{34}$$

That finishes the definition of the jump operator. We will now show that the domains of these jump operators are ccpos and that the jump operators are monotonous. That will suffice to prove the fixed point property.

Lemma (Product ccpos) Let A and B be ccpos. Then $A \times B$ is also one, where the relevant ordering relation is

$$(a_1, b_1) \leq (a_2, b_2) \quad :\Leftrightarrow \quad a_1 \leq a_2 \quad \text{and} \quad b_1 \leq b_2. \tag{35}$$

Proof Let $X \subset A \times B$ be consistent. Introducing π_1 and π_2 as projections, namely $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$, we find that $\pi_1 X \subset A$ is consistent and $\pi_2 X \subset B$ also. From the assumption that A and B are ccpos, we obtain elements $b_A \in A$ and $b_B \in B$ which are the supremum of $\pi_1 X$ and $\pi_2 X$ respectively. Then, $(b_A, b_B) \in A \times B$ is the supremum of X.

Now we can come back to the monotony issue:

Lemma (Monotony of ρ_1) Let $\mathfrak{M} = (W, m, D, \mathfrak{C}_w, \mathfrak{F}_w, \mathfrak{R}_w)$ be a ground tp-model as in the definition of the jump operator, and $S \in \{\mu, \kappa, \nu\}$. Furthermore, let $g_1, g_2 \colon (W \to D \to B)$ and $h_1, h_2 \colon (W \to D \to D \to B)$ with $(g_1, h_1) \leq (g_2, h_2)$. Then

$$\pi_1(\rho^{tp}_{\mathfrak{M}}(g_1, h_1)) \leq \pi_1(\rho^{tp}_{\mathfrak{M}}(g_2, h_2)), \tag{36}$$

where π_1 should denote the first component of a tuple.

Proof In each possible world, we exploit the monotony in the schemes. Namely, from the assumption it follows that

$$\mathrm{Val}_{S,\hat{s},(D,\mathfrak{C}_w(w),\mathfrak{F}_w(w),\mathfrak{R}_w(w)[G:=g_1(w)][H:=h_1(w)])}\phi \tag{37}$$

$$\leq \mathrm{Val}_{S,\hat{s},(D,\mathfrak{C}_w(w),\mathfrak{F}_w(w),\mathfrak{R}_w(w)[G:=g_2(w)][H:=h_2(w)])}\phi, \tag{38}$$

which yields the result for the case of a sentence. For a non-sentence, it is again trivial.

Lemma (Monotony of ρ_2) Let $\mathfrak{M} = (W, m, D, \mathfrak{C}_w, \mathfrak{F}_w, \mathfrak{R}_w)$ be a ground tp-model as in the definition of the jump operator, and $S \in \{\mu, \kappa, \nu\}$. Furthermore, let $g_1, g_2 \colon (W \to D \to B)$ and $h_1, h_2 \colon (W \to D \to D \to B)$ with $(g_1, h_1) \leq (g_2, h_2)$. Then

$$\pi_2(\rho^{tp}_{\mathfrak{M}}(g_1, h_1)) \leq \pi_2(\rho^{tp}_{\mathfrak{M}}(g_2, h_2)), \tag{39}$$

where π_2 should denote the second component of a tuple.

Proof For the proof, we have to distinguish the cases of the respective schemes.

(κ) For κ, we have to show that $\mathrm{pVal}_\kappa(\phi, q)$ is monotone. The important thing to notice here is that as in the proof for the first component, we can exploit monotony in the scheme κ: If a sentence in a world w_1 is already true regarding (g_1, h_1), it has to be true also regarding (g_2, h_2). The same holds for falsity. Keeping that in mind, we can prove the result by a distinction of cases of the value of $\mathrm{pVal}_\kappa^{(g_1,h_1)}(\phi, q)$:

- If $\mathrm{pVal}_\kappa^{(g_1,h_1)}(\phi, q) = n_3$, the claim is trivial.
- If $\mathrm{pVal}_\kappa^{(g_1,h_1)}(\phi, q) = t_3$ we can show that $\mathrm{pVal}_\kappa^{(g_2,h_2)}(\phi, q) = t_3$: Namely, from the properties explained above we can conclude

  $$\{w_1 \in W \mid \mathrm{Val}^*_{(g_1,h_1)}(w_1) = t_3\} \subseteq \{w_1 \in W \mid \mathrm{Val}^*_{(g_2,h_2)}(w_1) = t_3\}. \tag{40}$$

  Since m is a probability measure, it assigns a higher value to the right hand side set, and as already the value for the left hand side set was greater or equal than the threshold q, that condition is also true regarding (g_2, h_2). Hence, $\mathrm{pVal}_\kappa^{(g_2,h_2)}(\phi, q) = t_3$.
- If $\mathrm{pVal}_\kappa^{(g_1,h_1)}(\phi, q) = f_3$, we do the same estimate with f_3 as we did for t_3 in the previous step. Furthermore we note that

  $$\{w_1 \in W \mid \mathrm{Val}^*_{(g_2,h_2)}(w_1) = t_3\} \cup \{w_1 \in W \mid \mathrm{Val}^*_{(g_2,h_2)}(w_1) = f_3\} \subseteq W \tag{41}$$

  and they are disjoint. Also, $m(w)(W) = 1$. Taking things together, we can conclude that the criterion for f_3 for (g_1, h_1) implies the criterion for (g_2, h_2). Hence, also $\mathrm{pVal}_\kappa^{(g_2,h_2)}(\phi, q) = f_3$.

(ν) For ν, we note that monotony in the schemes yields also that $\mathrm{Val}^*_{(g_1,h_1)} \geq t_4$ implies that $\mathrm{Val}^*_{(g_2,h_2)} \geq t_4$. And also, this property holds for f_4 and we have $\mathrm{Val}^*_{(g_1,h_1)} = b_4 \Rightarrow \mathrm{Val}^*_{(g_2,h_2)} = b_4$. Then we have to make the same distinction of cases for the value $\mathrm{pVal}_\kappa^{(g_1,h_1)}(\phi, q)$ and put the parts together similarly as above.

(μ) For the scheme μ, the new complication is that $m(w)\{w_1 \in W \mid \mathrm{Val}^*_{(g_1,h_1)}(w_1) = n_3\} = 0$ needs to imply that $m(w)\{w_1 \in W \mid \mathrm{Val}^*_{(g_2,h_2)}(w_1) = n_3\} = 0$. But monotony in μ

yields that

$$\{w_1 \in W \mid \mathrm{Val}^*_{(g_2,h_2)}(w_1) = n_3\} \subset \{w_1 \in W \mid \mathrm{Val}^*_{(g_1,h_1)}(w_1) = n_3\}. \tag{42}$$

Hence, the fact that m is a probability measure is sufficient to show the desired implication. Apart from that, we make the same distinction of cases and also the implications for the other components are similar.

Now we can collect those two results in the following Corollary:

Corollary (Monotony of $\rho^{tp}_{\mathfrak{M}}$) Let $\mathfrak{M} = (W, m, D, \mathfrak{C}_w, \mathfrak{F}_w, \mathfrak{R}_w)$ be a ground tp-model as in the definition of the jump operator, and $S \in \{\mu, \kappa, \nu\}$. Furthermore, let $g_1, g_2 \colon (W \to D \to B)$ and $h_1, h_2 \colon (W \to D \to D \to B)$ with $(g_1, h_1) \le (g_2, h_2)$. Then

$$\rho^{tp}_{\mathfrak{M}}(g_1, h_1) \le \rho^{tp}_{\mathfrak{M}}(g_2, h_2). \tag{43}$$

Taking things together, we arrive at

Lemma (Fixed point property of $\rho^{tp}_{\mathfrak{M}}$) Let $\mathfrak{M} = (W, m, D, \mathfrak{C}_w, \mathfrak{F}_w, \mathfrak{R}_w)$ be a ground tp-model as in the definition of the jump operator, and $S \in \{\mu, \kappa, \nu\}$. Then there exist functions $g \colon W \to D \to B$ and $h \colon W \to D \to D \to B$ such that

$$\rho^{tp}_{\mathfrak{M}}(g, h) = (g, h). \tag{44}$$

Proof From the Lemma about function space ccpos, we can respectively conclude that $\{f \colon W \to D \to B\}$ and $\{f \colon W \to D \to D \to B\}$ are ccpos. Hence by the Lemma about product ccpos their Cartesian product is also a ccpo. The jump operator is monotone on it and therefore Visser's fixed point theorem yields the existence of a least fixed point.

5 Summary and outlook

Let us summarise the most important developments and results of this text. We began by briefly introducing the idea of a Kripkean theory of truth and the role different evaluation schemes play there. Namely, the Weak Kleene scheme can be motivated by means of the ”meaningless” reading of n_3, while the Strong Kleene scheme represents intuitions which arise in the context of the ”told” interpretation. In the literature, there are not only Kripkean theories of truth but also theories of truth and probability. The latter are especially studied for the Strong Kleene scheme. That motivated the general account of this text which can be instantiated with different schemes.

To develop this account, we followed this general strategy: First, we repeated results of a suiting account for a plain theory of truth from a book by Gupta and Belnap. Afterwards, we generalised it in two steps: We considered the case of several truth predicates in their respective possible worlds and then introduced also the probability predicate. Roughly, the proof techniques known from the case of a plain theory of truth could be exploited to arrive at the fixed point property results also in the other cases. An exception of this is to some extend the new part of the jump operator which evaluate probability claims. There, we suggested several options for the different schemes and proved monotony and the fixed point property of the corresponding jump operators.

Those results raise several interesting open question we leave for further research:

Equivalence to Campbell-Moore's account One question would be whether our account is equivalent to the one suggested by Catrin Campbell-Moore in [Campbell-Moore, 2016]. We conjecture that there is a negative answer to this question if equivalence is read in a pedantic form. Namely, in bullet point 5 of Definition 3.2.3 the truth value true seems to be assigned to probability statements where the first place is filled with a non-sentence and the second place is filled with a rational number less or equal zero. We assigned those sentences the value false. We are not sure whether we misunderstood something in Campbell-Moore's account by that interpretation or how the equivalence question more broadly construed should be answered. One reason why we leave this question unanswered is that in Campbell-Moore's account a specific type of coding is "imported" from a proof of Gödel's incompleteness theorems and we are not optimistic that several functions which are appearing in the Definition 3.2.3 and elsewhere are formalisable with moderate effort.

Truth-and-Probability predicates The natural next step in the generalisation of Gupta and Belnap's account to the case at hand would be to generalise the notion of the T-predicate to that of a TP-predicate (Truth and Probability) and show that the fixed points of the jump operator give rise to such interpretations of the predicates G and H. That would probably look like Prop. 3.2.11 in [Campbell-Moore, 2016].

Further desiderata to a joint theory of truth and probability and the new schemes By showing that all our considered schemes lead to jump operators which have the fixed point property we showed that they satisfy some kind of minimal desideratum to a theory of truth and probability one might have. However, there are further such desiderata, as for example the one mentioned in the previous paragraph. But there are far more. For example, Campbell-Moore investigates how her theory fares in terms of several such in [Campbell-Moore, 2016] from page 65 on. So the interesting question would now be how the different schemes of this account behave in terms of those desiderata.

We admit that those questions are probably even more interesting than those we answered in this text. However, we regard the definitions and lemmata provided as an important groundwork for the task of finding an answer to the more advanced further questions.

References

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