Neutron-proton system analysis

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21 de junio de 2025

1 System configuration

Deuteron ^{2}H is the second simplest stable nuclueus, just behind ^{1}H. It consists in two nucleons, one proton and one neutron. Diproton and dineutron nucleus do not exist in nature so the only bound system we can use to study the nuclear force is the deuteron. Systems with more than two nucleons include a complexitivity that does not allow the same precision to study the interaction between two nucleons. In the deuteron we have just the interaction between two nucleons therefore some properties we find here we can expand to the two-body interactions in any other nucleus.

Working inside the non-relativistic frame we use Schrödinguer equation to study this system:

$$i\hbar\partial_t\Psi(p,n,t) = H\Psi(p,n,t) = \left[V(p,n,t) + T_p + T_n\right]\Psi(p,n,t) \ , \tag{1.1}$$

where $V(p,n,t)$ and $\Phi(p,n,t)$ are implying that they can depend on any coordinate associted with the proton/neutron (spatial, spin, etc.) and time. T_p and T_n are the associated kinetic operators associated with each nucleon.

If you are interested in the stationary states the potential cannot depend on time and one solution for this equation is:

$$\Psi_\nu(p,n,t) = e^{\frac{-iE_\nu t}{\hbar}}\Phi_\nu(p,n) \ , \tag{1.2}$$

where Φ function doesn't depend on time and E_ν is a constant that represents the energy system in the ν state. Any expected value of this wave function is a constant. The most general solution is a linear combination of this kind of solution. That is:

$$\Psi(p,n,t) = \sum_\nu c_\nu e^{\frac{-iE_\nu t}{\hbar}}\Phi_\nu(p,n) \ . \tag{1.3}$$

The time-independent Schrödinguer equation to obtain the Φ_ν solutions is (from now on we obvious the subscript related to the state ν):

$$\left[-\frac{\hbar^2}{2m_p}\nabla_p^2 - \frac{\hbar^2}{2m_n}\nabla_n^2 + V(p,n)\right]\Phi(p,n) = E\Phi(p,n) \ . \tag{1.4}$$

For example, if the potential only depends on spatial coordinates we have:

$$\left[-\frac{\hbar^2}{2m_p}\nabla_p^2 - \frac{\hbar^2}{2m_n}\nabla_n^2 + V(\vec{r}_p,\vec{r}_n)\right]\Phi(\vec{r}_p,\vec{r}_n) = E\Phi(\vec{r}_p,\vec{r}_n) \ . \tag{1.5}$$

Obviously the only particles that generate any field in the system are the proton and the neutron so it is reasonable that the potencial only depends on a spatial vector that connects these nucleons $\vec{r} = \vec{r}_p - \vec{r}_n$. And we can rewrite our independent time Schrödinguer equation in terms of this new coordinate and the center of mass coordinate $\vec{R}$.

$$\left[-\frac{\hbar^2}{2(m_p + m_n)} \nabla_R^2 - \frac{\hbar^2}{2\mu} \nabla_r^2 + V(\vec{r}) \right] \Phi(\vec{r}, \vec{R}) = E\Phi(\vec{r}, \vec{R}) \ . \tag{1.6}$$

Due to this kind of potential the solutions are separable:

$$\Phi(\vec{r}, \vec{R}) = \phi_r(\vec{r})\phi_R(\vec{R}) \ . \tag{1.7}$$

And we have two independent equations, one for the center of mass and one for the relative movement between the particles. The one for the center-mass function is:

$$\left[-\frac{\hbar^2}{2(m_p + m_n)} \nabla_R^2 \right] \phi(\vec{R}) = E_R \phi(\vec{R}) \ . \tag{1.8}$$

And this is the equation for a free particle. This means that the center of mass moves freely, indeed this takes into account the traslation movement of the deuteron. We are not interested in it since this is the same for any free particle. The part that contains the information about the nuclear force is contained in the potential:

$$\left[-\frac{\hbar^2}{2\mu} \nabla_r^2 + V(\vec{r}) \right] \phi(\vec{r}) = E_r \phi_r(\vec{r}) \ . \tag{1.9}$$

where μ is the reduced mass:

$$\mu = \frac{m_p m_n}{m_p + m_n} \approx \frac{m_{nuc}}{2} \tag{1.10}$$

where we can expand the kinetic part as:

$$T = -\frac{\hbar^2}{2\mu} \nabla_r^2 = \frac{\hbar^2}{\mu} \left(-\frac{1}{r} \frac{d^2}{dr^2} r + \frac{l^2}{r^2} \right) \tag{1.11}$$

1.1 Central potential

In general the potential does depend on the three spatial coordinates, which in a spherical coordinate system that means $V(\vec{r}) = V(r, \varphi, \theta)$. In some cases, we have central potentials $V = V(r)$. This simplifies a lot the diferential equation. Indeed the solution is separable in radial part and angular part $\phi_r(\vec{r}) = R(r)Y(\varphi, \theta)$. The angular part is the same for all systems as the potential doesn't have any influence on it. The radial equation is:

$$\left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \epsilon - U(r) - \frac{l(l+1)}{r^2} \right] R_{El}(r) = 0 \ , \tag{1.12}$$

where $\epsilon = \frac{2\mu E_r}{\hbar^2}$ and $U(r) = \frac{2\mu V(r)}{\hbar^2}$. If we make the substitution $u(r) = R(r) \cdot r$ we have:

$$\left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} + V(r) \right] u(r) = E_r u(r) \ . \tag{1.13}$$

The normalization condition is:

$$\int u(r)dr = 1 \ . \tag{1.14}$$

And in order for the radial function to be finite at $r = 0$ and nullify at infinity it must follow:

$$u(r) = 0 \ \ / \ \ r = 0 \ , \ r \to \infty \ . \tag{1.15}$$

1.1.1 Shape of the potential

The potential generates an attractive force so it must be negative inside a certain radius, we called it R. For $r > R$ the experiments shows that the interaction disappears completely so it must be $V(r) = 0$. For $r < R$ we expect an increasing potential with incresing r, but keeping similar values in most of the zone. So it must have an exponentioal shape, that it could be even approximated by a constant. In summary, it must follow:

$$V(r) < 0 \quad r < R, \qquad V(r) \approx 0 \quad r > R, \qquad \frac{\partial V(r)}{\partial r} > 0 \quad r < R \ . \tag{1.16}$$

Assymptotically ($r \to \infty$) the radial equation becomes:

$$-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}u(r) = E_r u(r) \ . \tag{1.17}$$

1.1.2 Assuming l=0 state

Assuming the biding state of the deuteron is in a $l = 0$ state we can rewrite the radial equation as:

$$\frac{d^2u}{dr^2} + \kappa^2(r)u(r) = 0 \ , \tag{1.18}$$

where κ is the wave number defined as:

$$\kappa(r) = \pm\hbar^{-1}\left[2\mu\left(E - V(r)\right)\right]^{1/2} \ . \tag{1.19}$$

The potential increases from $r = 0$ to $r \to \infty$, and there is a special point $r = r_1$ in which for $r < r_1$ we have that $\kappa \in \mathbb{R}$ and for $r > r_1$ then $\kappa \in \mathbb{C}$. This r_1 can be considered as the radius R of the deuteron whose value is determined from experiment.

So for $r \to 0$ the solutions are:

$$u(r) = Asin(Kr) \ , \tag{1.20}$$

where K is an average of values of κ. And for $r \to \infty$ we have:

$$u(r) = Be^{-r/R_m} \ , \tag{1.21}$$

where $R_m = |\kappa|^{-1} = \hbar(2\mu|E|)^{-1/2} = 4.318fm$. We wonder about the form of the function in the inner zone. It must be continuous all over the domain and joins smoothly with the

exponential at $r = R$. The exponential goes to zero at infinity so it must have a smooth slope, so the exponential must join the sino function near a maximum of the sino. The sino function eq. [1.20] reaches its maximums at:

$$Kr = \frac{\pi}{2} n_{odd} \qquad n_{odd} = 1, 3, 5, ... \tag{1.22}$$

Therefore the range R must be a bit longer than $\frac{n_{odd}\pi}{2K}$ but pretty close to that value. The ground state must be related to $n_{odd} = 1$, in order for the wave function to be the somoothier as possible (which implies less kinetic energy). So we have:

$$R \approx \frac{\lambda}{4} = \frac{\pi}{2K} \ . \tag{1.23}$$

In order to help you visualize what we are doing here you can see figure [1], where the funtion joints at $r = R$ assuming that the sino takes it maximum at that position. This is not actually how the function should joint because as you can see the derivatives do not coincide at $r = R$, this is just an approximation. The maximum of the sino should be reached at value of r a bit smaller than R.

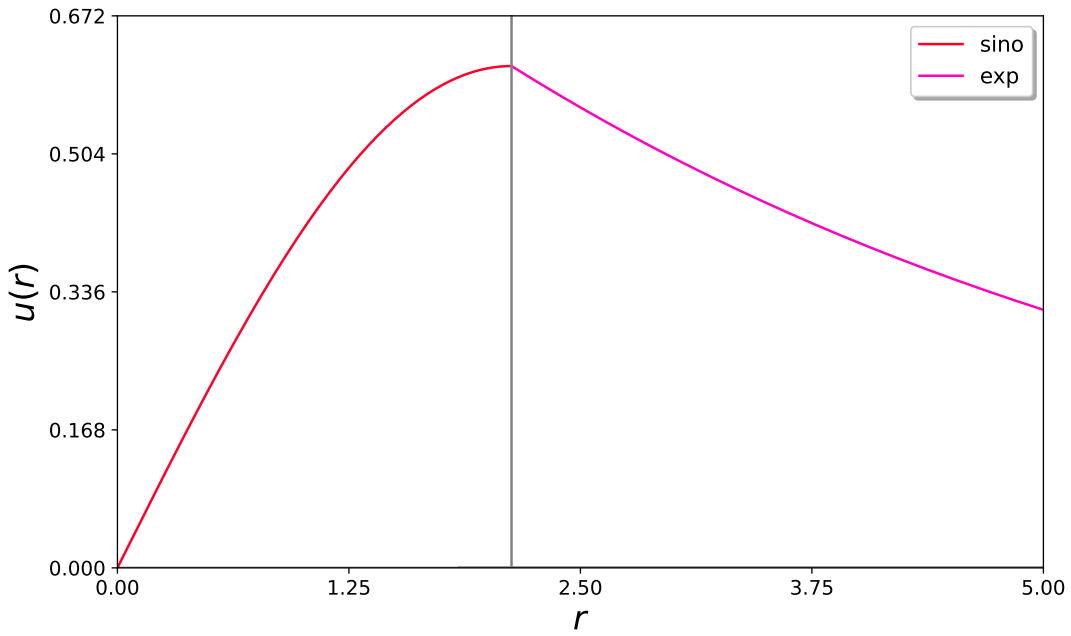

Figura 1: Joint of the functions at $r = R$ using the approximation in which the sino reaches its maximum at $r = R$.

We must take into account that the wavefunction keeps a notable height besides the point $r = R$, till $R_m > R$ this values are relevant. So we can consider the size of the deuteron quite bigger than R. Then the proton and neutron separates from each other more than the range of the force, therefore we can assume a weekly bound structure, meaning by that we assume E as being close to zero with respect the depth of the potential in the allowed zone. Then we can approximate K as:

$$K \approx \hbar^{-1} \left[2\mu|\hat{V}|\right]^{1/2} \quad , \tag{1.24}$$

where $\hat{V}$ is the average potential value in that zone and using eq.[1.23] we have a approximate value for this average potential:

$$|\hat{V}| \approx \frac{1}{2\mu}\left(\frac{\pi\hbar}{2R}\right)^2 = 22.6\ MeV \quad . \tag{1.25}$$

which leads to the result $K = 0.7382\ fm^{-1}$.

2 Using different potentials

2.1 Infinite square well

Let's start with a simple central potential. In this case we are treating the nuclear force as an atrractive interaction inside an sphere with radius a and outside of it there is a infinite potential so using this potential it's imposible for the nucleons to break the nucleus.

$$V(r) = \begin{cases} V_o & \text{if } r < R, \\ \infty & \text{if } r > R. \end{cases} \tag{2.1}$$

In this case we need to solve the following equation for $r < R$:

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \alpha^2 - \frac{l(l+1)}{r^2}\right] R_{El}(r) = 0 \quad , \tag{2.2}$$

where $\alpha = \sqrt{\epsilon - U_o}$. Remember that $\epsilon > U_{min} = U_o$ so α is always real. Because the potential gets an infinity value outside the nucleus the energy value has no meaning on its own, the important quantity is the difference with V_o. Outside the sphere the wavefunction is zero, inside the most general solution is:

$$R_{El}(r) = Aj_l(\alpha r) + By_l(\alpha r) \quad , \tag{2.3}$$

where j_l and y_l are the Bessel spherical functions of first and second kind respectevly. Because y_l diverges at $r = 0$ we find that $B = 0$ so the solution for the energy eigenvalues is the condition for $r = R$:

$$j_l(\alpha R) = 0 \quad . \tag{2.4}$$

Because of the centrifugal term we expect higher energies for higher values of angular momentum l. So let´s study only the first states, let's say until the first solutions till the first solution of $l = 4$. As we can see from fig. [2] until that solution, there are six solutions, 2 for $l = 0$, 2 for $l = 1$, 1 for $l = 2$, 1 for $l = 3$.

The solutions for αa are shown in table [1]:

If we call this values as X_{nl} the energy eigenvalues are:

$$\alpha = \frac{X_{nl}}{R} \tag{2.5}$$

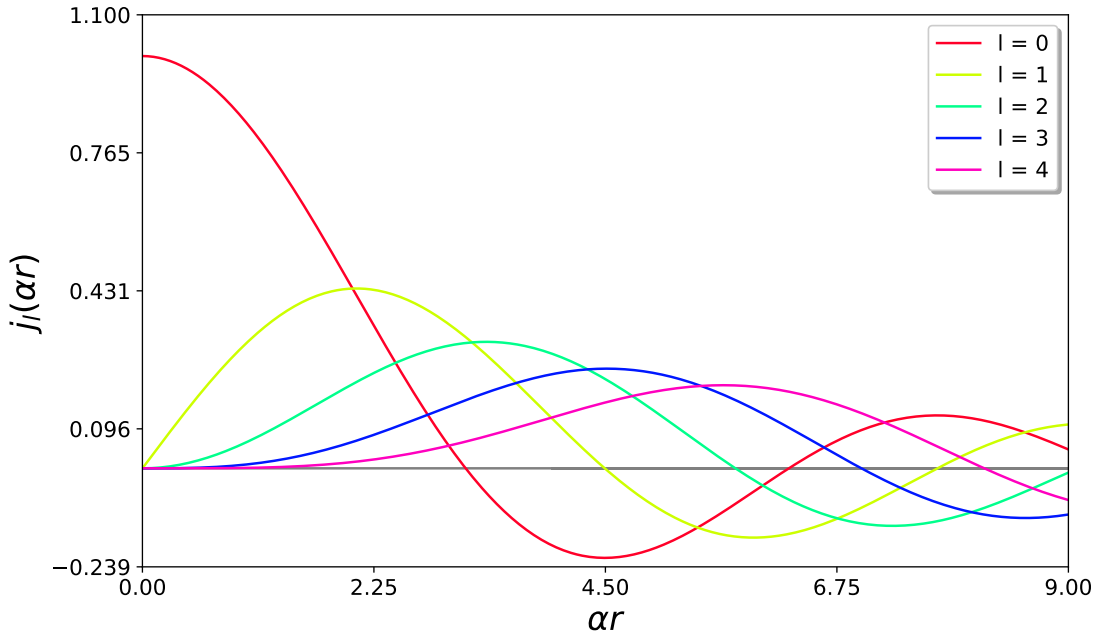

Figura 2: First spherical Bessel functions (from $l = 0$ to $l = 4$).

(n,l)	X_{nl}
$(0,0)$	3.14157 ± 0.00006
$(0,1)$	4.49342 ± 0.00009
$(0,2)$	5.76346 ± 0.00006
$(1,0)$	6.28311 ± 0.00006
$(0,3)$	6.98793 ± 0.00009
$(1,1)$	7.72238 ± 0.00006
$(0,4)$	8.18283 ± 0.00006

Tabla 1: First roots for Bessel spherical fucntion of first kind

which means that:

$$E_{nl} - V_o = \frac{\hbar^2 X_{nl}^2}{2\mu R^2} \ . \tag{2.6}$$

If we assume the ground state is an eigenstate of L^2 with $L = 0$ and taking into account that the experimental energy is $2.23MeV$ or $E_{00} = -2.23MeV$ we have $V_o = -95.01MeV$.

2.2 Finite square well

In this case for $r > R$ we have:

$$[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \beta^2 - \frac{l(l+1)}{r^2}]R_{El}(r) = 0 \ , \tag{2.7}$$

(n,l)	$E_{nl} - V_o[MeV]$
$(0,0)$	92.78
$(0,1)$	189.82
$(0,2)$	312.29
$(1,0)$	371.14
$(0,3)$	459.08
$(1,1)$	560.65
$(0,4)$	629.50

Tabla 2: First roots for Bessel spherical fucntion of first kind

being $\beta = |\epsilon|^{1/2}$. We have the same solution for $r < R$ as in the infinite case. For $r > R$ we have:

$$R_{El}(r) = Cj_l(i\beta r) + Dy_l(i\beta r) \ . \tag{2.8}$$

In this case the square integrability condition (that is $R_{El}(r) = 0$ for r lím ínf) imposes $D = iC$. Therefore we have:

$$R_{El}(r) = Aj_l(\alpha r) \ \ for \ \ r < a \ , \tag{2.9}$$

and:

$$R_{El}(r) = Ch_l(i\beta r) \ \ for \ \ r > a \ , \tag{2.10}$$

and the continuity condition at $r = a$ provides the relation between the constants:

$$A = C\frac{h_l(i\beta a)}{j_l(\alpha a)} \ . \tag{2.11}$$

Finally the continuity of the derivative at $r = a$ gives us the equation for bound states.

$$\alpha\frac{j_{l+1}(\alpha a)}{j_l(\alpha a)} = i\beta\frac{h_{l+1}(i\beta a)}{h_l(i\beta a)} \ , \tag{2.12}$$

where h_l is the Hinkel Function defined as $h_l(z) = j_l(z) + iy_l(z)$. The left part of this equation is real (because $E > V_o$), then obviously the left part must be. Then if we define the following function $f_l(V_o)$ as:

$$f_l(V_o) = \alpha(V_o)\frac{j_{l+1}(\alpha(V_o)a)}{j_l(\alpha(V_o)a)} - i\beta\frac{h_{l+1}(i\beta a)}{h_l(i\beta a)} \ , \tag{2.13}$$

which must be real for all $V_o < E$ values. The strategy is to fix the values E and R and solve the previous equation numerically. You can see a plot of the function in fig. [3] for the range $V_o = -250 \ MeV$ to $V_o = E$. Here we can see two solutions, using the bisection method we can find these values and they are $V_o = -33.0081 \ MeV$ and $V_o = -214.4940 \ MeV$. Both could be valid solutions, we need some condition to eleminate one them. Experimentally we know there's only one bound state for deuteron. Fixing the V_o then the finite well function in eq.[2.13] can be expressed as a function of the energy:

$$f_l(E) = \alpha(E)\frac{j_{l+1}(\alpha(E)a)}{j_l(\alpha(E)a)} - i\beta(E)\frac{h_{l+1}(i\beta(E)a)}{h_l(i\beta(E)a)} \quad , \tag{2.14}$$

then we can find the solutions for energy from V_o to 0 which are the bound of the system. If the V_o value is valid, it needs to have only one bound state at the know bound state energy of the deuteron around. In fig[4] we can see the function $f(E)$ fixing $V_o - 33.0081\ MeV$ and in fig. [5] we can see the function $f(E)$ fixing $V_o = -214.4940\ MeV$, for the former case we only have the bound state energy but for the latter there is another solution. If this solution were valid, deuteron would have a bound state with a impossible huge energy that is not found in nature and makes no sense in theory neither.

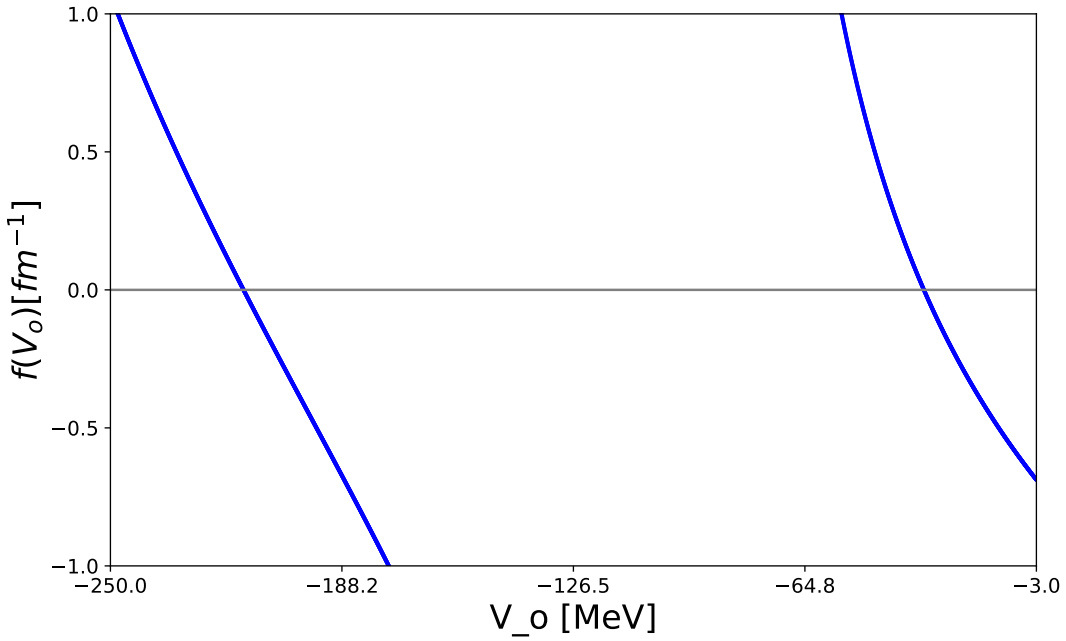

Figura 3: Function of the solutions for the finite well fixing the energy and the radius.

Actually there more solutions for the depth potential fixing the energy and radius as you can see in fig. [6], which is just a wider version of fig.[3]. But all the other solutions rather than $V_o = -33.0081$ have the same problem, they provide more than one bound states so they are not valid.

2.2.1 Normalization constants

In this case the normalization condition expressed in eq. [4.8] is:

$$C\left[\left|\frac{h_l(i\beta a)}{j_l(\alpha a)}\right|^2 \int_0^a |j_l(\alpha r)|^2 r^2 dr + \int_a^\infty |h_l(i\beta r)|^2 r^2 dr\right] = 1 \tag{2.15}$$

from this relation we can get C and therefore A too. For $l = 0$ we obtain:

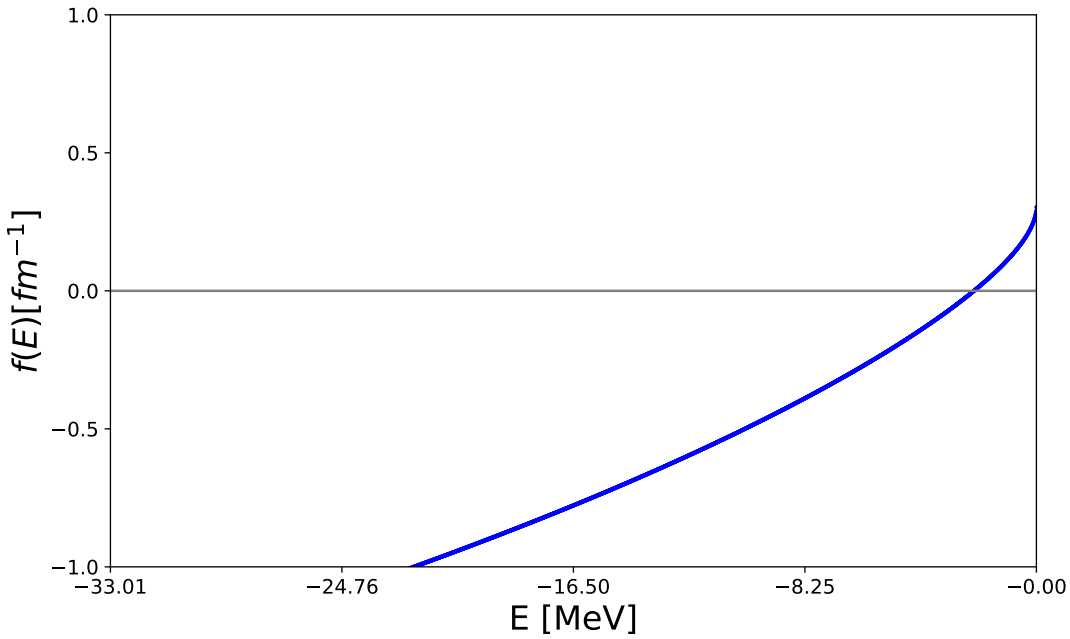

Figura 4: Function of the solutions for the finite well fixing the depth potential at $V_o = -33.0081\ MeV$ and the radius.

$$C = 0.04160 \qquad A = -0.09788 \tag{2.16}$$

with this values the radial eigenfunction is negative for all its domain, to have it positive we can multiply it by -1, which is like including a phase which has no physical meaning. This function can be observed in fig. [7].

And we can observe the radial density in fig. [8]. This gives us a visualization of the distance that the neutron and the proton keep themseleve apart.

Finally we could check the function $u(r)$ in fig.[9] and compare it to the approximation we showed in fig. [1].

2.3 Magnetic dipole moment

The magnetic dipole moment definition is:

$$\mu = \langle JJ|\mu_z|JJ\rangle \tag{2.17}$$

Using Landé formula we have:

$$\langle JJ|\mu_z|JJ\rangle = \frac{1}{J(J+1)}\langle JJ|(\vec{J}\vec{\mu})|JJ\rangle\langle JJ|J_z|JJ\rangle = \frac{1}{(J+1)}\langle JJ|(\vec{J}\vec{\mu})|JJ\rangle \tag{2.18}$$

The $\vec{m}u$ magnetic dipole moment operator is:

$$\vec{\mu} = g_p\vec{s}_p + g_n\vec{s}_n + g_{lp}\vec{l}_p + g_{ln}\vec{l}_n \tag{2.19}$$

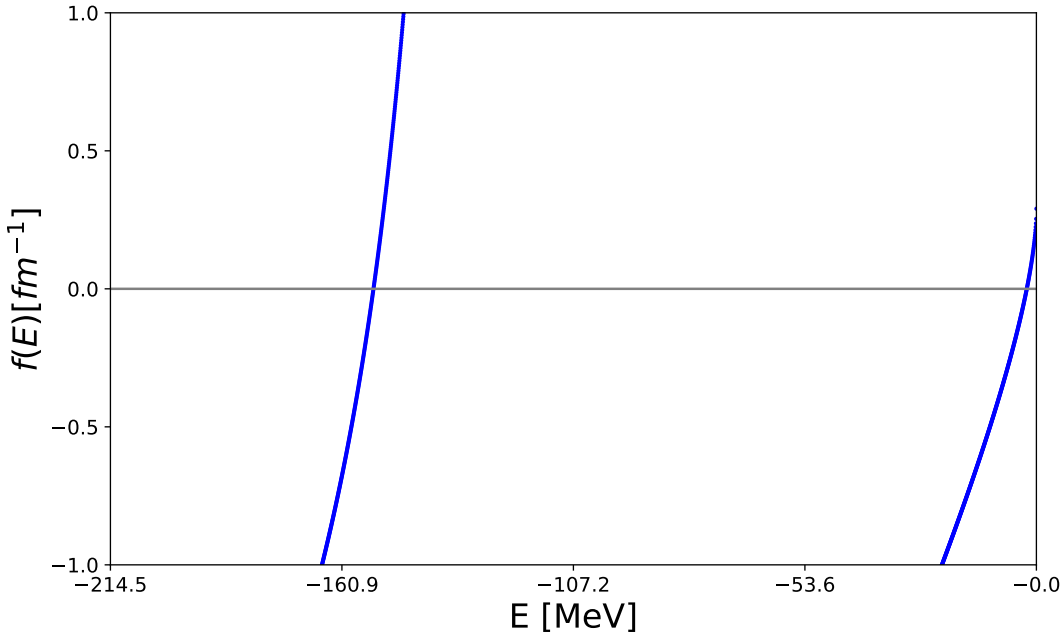

Figura 5: Function of the solutions for the finite well fixing the depth potential at $V_o = -214.4940\ MeV$ and the radius.

because of the neutron not having charge we have $g_{ln} = 0$, while for the proton we have $g_{lp} = 1$. And because the similar masses of protons and neutrons we have that $\vec{l}_p = \frac{\vec{L}}{2}$. Then we can write:

$$\vec{\mu} = g_p \vec{s}_p + g_n \vec{s}_n + \frac{1}{2}\vec{L} \tag{2.20}$$

using $\vec{S} = \vec{s}_n + \vec{s}_p$ we can write:

$$\vec{\mu} = \frac{1}{2}\left[(g_p + g_s)\vec{S}(g_p - g_s)(\vec{s}_p - \vec{s}_n) + \vec{L}\right] \tag{2.21}$$

then:

$$\vec{\mu}\vec{J} = \vec{\mu}() \tag{2.22}$$

3 Tensor force

Deuteron can not be in the a $L = 0$ state because it has non-zero quadrupole moment and if the potencial were central the ground state would be $L = 0$ so the conclusion is that it needs to be a combination of different L-state with a pair value (in order to have a symmetric spatial part). The simplest state it is a combination of $L = 0$-state and $L = 2$-state.

So the potential is not central anymore and with requirements of different symmetries it has a term with the following form:

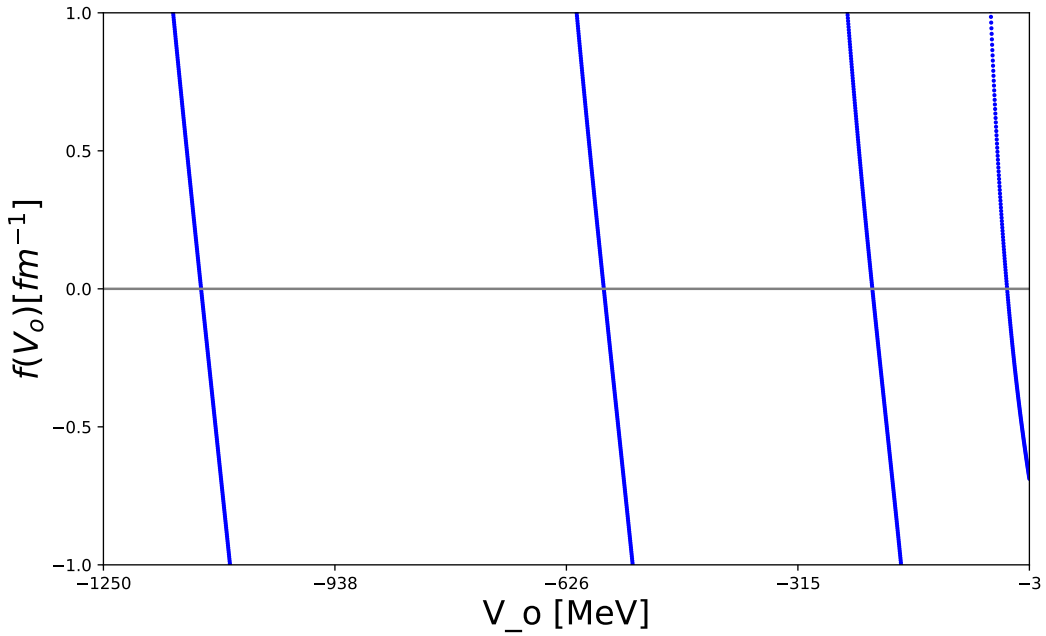

Figura 6: Function of the solutions for the finite well fixing the energy and the radius.

$$V_{12} = V_T(r) S_{12}(\vec{r}, \vec{\sigma_1}, \vec{\sigma_2}) \ , \tag{3.1}$$

with S_12 being:

$$S_{12} = 3(\vec{\sigma_1} \cdot \vec{r})(\vec{\sigma_2} \cdot \vec{r}) - \vec{\sigma_1} \cdot \vec{\sigma_2} \ . \tag{3.2}$$

And we will use a base of L-states in order to diagonalize the hamiltonian and get the ground state energy. Assuming the ground state is a combination of $L = 0$-state and $L = 2$-state we only need to calculate three terms.

$$H_{11} = \langle L = 0 \ S = 1 | V_{12} | L = 0 \ S = 1 \rangle \tag{3.3}$$

$$H_{22} = \langle L = 2 \ S = 1 | V_{12} | L = 2 \ S = 1 \rangle \tag{3.4}$$

$$H_{12} = \langle L = 2 \ S = 1 | V_{12} | L = 0 \ S = 1 \rangle \tag{3.5}$$

The part of $\vec{\sigma_1} \cdot \vec{\sigma_2}$ only depends on spin coordinates.

First of all let's be precised in the definitions we are gonna use. From the angular momentum operators (s_x, s_y, s_z) the Pauli matrices are $\sigma_k = 2s_k$. And the definition of stair angular momentum operators that is used:

$$s_+ = s_x + i s_y \qquad s_- = s_x - i s_y \tag{3.6}$$

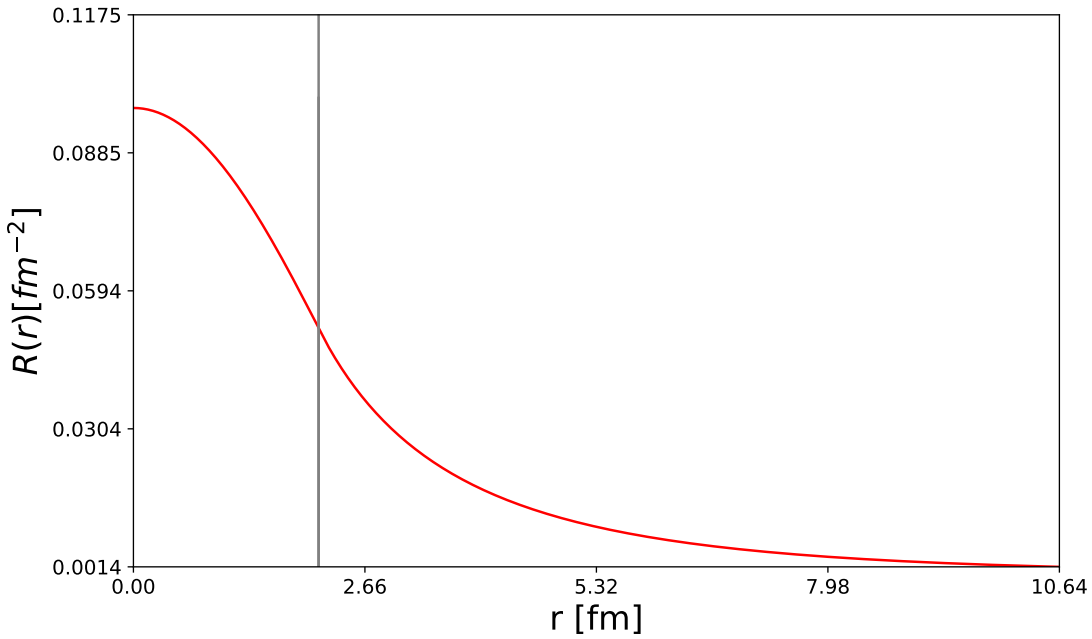

Figura 7: Radial eigenfunction for $l = 0$.

And the stairs Pauli operators are defined in the following way:

$$\sigma_+ = -\frac{1}{\sqrt{2}}(\sigma_x + i\sigma_y) \qquad \sigma_- = \frac{1}{\sqrt{2}}(\sigma_x - i\sigma_y) \tag{3.7}$$

$$\sigma_x = \frac{1}{\sqrt{2}}(\sigma_- \sigma_+) \qquad \sigma_y = \frac{i}{\sqrt{2}}(\sigma_+ + \sigma_-) \tag{3.8}$$

And we have: $\vec{\sigma_1} \cdot \vec{\sigma_2}|S = 1\rangle = |S = 1\rangle$

Developing the first term of the tensor force we have:

$$\frac{3}{r^2}(\sigma_1 \cdot r)(\sigma_2 \cdot r) = \frac{3}{2r^2}[\sigma_-(1)\sigma_-(2)x^2 - \sigma_-(1)\sigma_(2)x^2 + 2i\sigma_-(1)\sigma_-(2)xy + \sigma_-(1)\sigma_z(2)xz - \sigma_+(1)\sigma_-(2)x^2 + \sigma_+(\tag{3.9}$$

We have to make the expected value of this operator on $|S = 1\rangle$. We choose $|++\rangle$. So the only part of this operator that has non-zero value on the expected value is $\frac{1}{2}\sigma_z(1)\sigma_z(2)z^2$.

$$\langle ++|\frac{3}{r^2}(\sigma_1 \cdot r)(\sigma_2 \cdot r)|++\rangle = \frac{3z^2}{2r^2} \tag{3.10}$$

So we have:

$$H_{11} = \langle L = 0|V_T(r)\frac{3z^2}{2r^2}|L = 0\rangle \tag{3.11}$$

$$H_{22} = \langle L = 2|V_T(r)\frac{3z^2}{2r^2}|L = 2\rangle \tag{3.12}$$

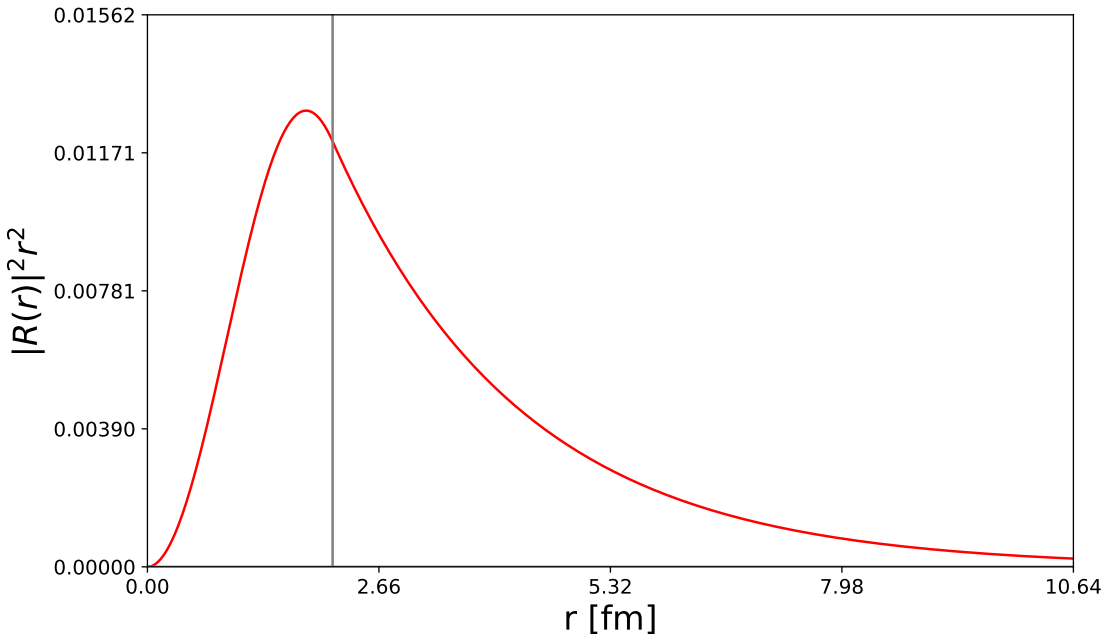

Figura 8: Radial density for $l = 0$.

$$H_{12} = \langle L = 2|V_T(r)\frac{3z^2}{2r^2}|L = 0\rangle \tag{3.13}$$

For the L-states I will be using eigenstates of the finite square well, that is:

$$\langle\vec{r}|L = 0\rangle = Aj_0(\alpha r) \qquad \langle\vec{r}|L = 2\rangle = Bj_2(\alpha r) \tag{3.14}$$

for $0 <= r <= R$. And:

$$\langle\vec{r}|L = 0\rangle = A'h_0(i\beta r) \qquad \langle\vec{r}|L = 2\rangle = B'h_2(i\beta r)\ , \tag{3.15}$$

for $R < r$.

3.1 Total potential

The total potential that follows all the symmetry requirements is:

$$V(\vec{r},\sigma) = V_d(r) + V_\sigma(r)(\vec{\sigma_1}\cdot\vec{\sigma_2}) + V_T(r)S_{12}(\vec{r},\sigma) \tag{3.16}$$

The two first terms are central, we need the second part to include a central potential which interacts with the spin of the particles.

3.2 Wavefunction of the ground state

From experiment we know that $J = 1$ and we know the internal spin is in a triplet state $S = 1$. Therefore the possible values for the orbital angular momentum are $L = 0, 1, 2$. But $L = 1$ implies an antisymmetric spatial function. The total wavefunction is:

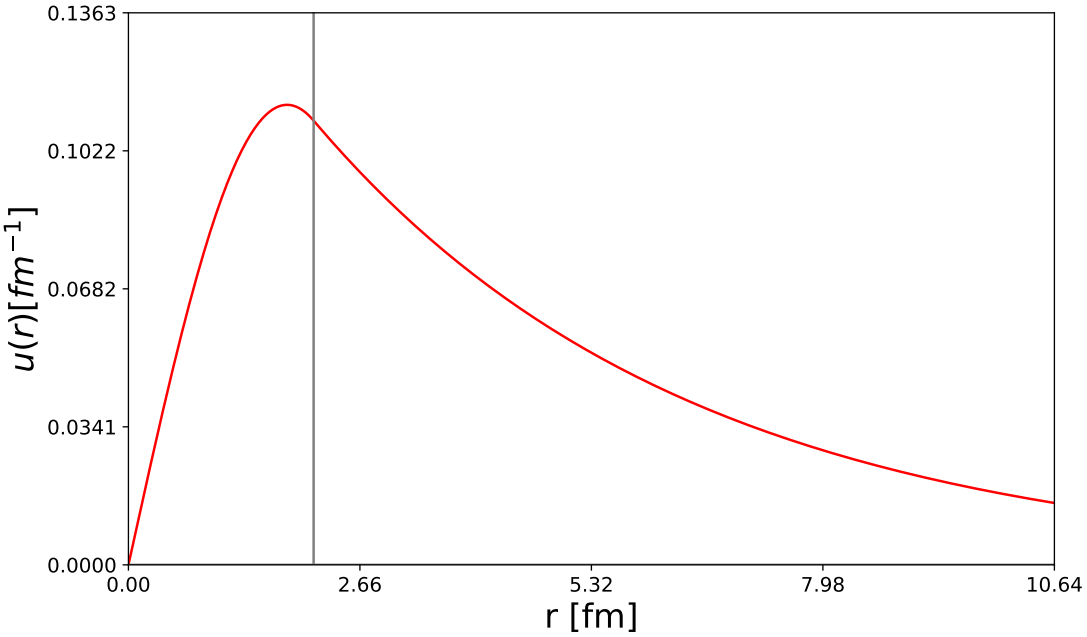

Figura 9: Radial function $u(r)$ for $l = 0$.

$$\Psi_M^{J=1S=1\pi=+}(\vec{r},\sigma) = \psi_{L=0}(\vec{r},\sigma) + \psi_{L=2}(\vec{r},\sigma)$$
$$= \frac{u(r)}{r} Y_M^{J=1L=0S=1}(\theta,\varphi,\sigma) + \frac{w(r)}{r} Y_M^{J=1L=2S=1}(\theta,\varphi,\sigma) \quad (3.17)$$

where $u(r)$ and $w(r)$ are the radial parts to be found out from the radial equation. Note that $u(r)$ and $w(r)$ are not the eigenfunctions for the more simple radial equation in eq.[1.13], they are not normalized and their squares are actually the density probability radial function. What we exactly know in eq. [3.17] are the generalized spherical harmonics:

$$Y_M^{J=1L=0S=1}(\theta,\varphi,\sigma) = Y_{M_L=0}^{L=0}(\theta,\varphi)\chi_{S=1M}(\sigma) = \sqrt{\frac{1}{4\pi}}\chi_{S=1M}(\sigma) \quad (3.18)$$

$$Y_M^{J=1L=2S=1}(\theta,\varphi,\sigma) = \sum_{M_L+M_S=M} \langle 21M_LM_S|121M\rangle Y_{M_LM_S}^{J=1L=2S=1}(\theta,\varphi,\sigma) \quad (3.19)$$

The probability of finding the deuteron in the $L = 0$ state is:

$$p_S = \int_0^\infty u(r)r^2dr \quad (3.20)$$

And in the $L = 2$ state:

$$p_D = \int_0^\infty w(r)r^2dr \quad (3.21)$$

And obviously:

$$\langle \Psi | \Psi \rangle = p_S + p_D = 1 \tag{3.22}$$

In order to get the radial functions $u(r)$ and $w(r)$ we insert the total wavefunction in the total time independent equation in eq. [1.12]. We use the following results:

$$\vec{\sigma_1} \cdot \vec{\sigma_2} \chi_{1M_S} = 1 \tag{3.23}$$

$$S_{12} Y_M^{J=1L=0S=1} = \sqrt{8} Y_M^{J=1L=2S=1} \tag{3.24}$$

$$S_{12} Y_M^{J=1L=2S=1} = \sqrt{8} Y_M^{J=1L=0S=1} - 2 Y_M^{J=1L=2S=1} \tag{3.25}$$

And we get the following equation:

$$\begin{aligned} &\left[-\frac{\hbar^2}{\mu} \frac{d^2 u}{dr^2} + V_c(r) u(r) \right] Y^{101} + \sqrt{8} V_T(r) u(r) Y^{121} + \\ &\left[-\frac{\hbar^2}{\mu} \left(\frac{d^2 w}{dr^2} - \frac{6w}{r^2} \right) + (V_c(r) - 2V_T(r)) w(r) \right] Y^{121} + \sqrt{8} V_T(r) w(r) Y^{101} = \\ &E \left(u(r) Y^{101} + w(r) Y^{121} \right) \end{aligned} \tag{3.26}$$

Then we have two related equations, one being the term that multiplies Y^{101} and the other being the therm that multiplies Y^{121}:

$$-\frac{\hbar^2}{\mu} \frac{d^2 u}{dr^2} + V_c(r) u(r) - E u(r) = -\sqrt{8} V_T(r) w(r) \tag{3.27}$$

$$-\frac{\hbar^2}{\mu} \left(\frac{d^2 w}{dr^2} - \frac{6w}{r^2} \right) + [V_c(r) - 2V_T(r)] w(r) - E w(r) = -\sqrt{8} V_T(r) u(r) \tag{3.28}$$

In total there are four free parameters whose values are to be adjusted: b_c range of the central force, b_T range of the tensor force, V_{oc} depth of the central potential and V_{oT} depth of the tensor force. Being established that, let's evaluate the solutions outside the range of the forces:

$$-\frac{\hbar^2}{\mu} \frac{d^2}{dr^2} u(r) = E u(r) \tag{3.29}$$

$$-\frac{\hbar^2}{\mu} \left(\frac{d^2}{dr^2} - \frac{6}{r^2} \right) w(r) = E w(r) \tag{3.30}$$

The first equation has as solutions the decreasing exponential for bound states:

$$u(r) = N e^{-\frac{r}{R_m}} \tag{3.31}$$

and:

$$w(r) = N' e^{-\frac{r}{R_m}} \left[3 \left(\frac{R_m}{r} \right)^2 + 3 \left(\frac{R_m}{r} \right) + 1 \right] \tag{3.32}$$

Now we ask ourselves what is the minimum value of r for what these solutions become valid because now we have two different ranges in comparision to the simpler case. From an analysis we don't include here we know that in the first equation [3.27] the terms $V_c(r)u(r)$ and $-\sqrt{8}V_T(r)w(r)$ are comparable so the value is the maximum between b_c and b_T. For the second differential equation [3.28] the term $V_c(r)w(r)$ turns out to be small compared to the $-\frac{6w}{r^2}$ term, so it is b_T the one which determines the validity of this solution on the r-axis. Using a Yukawa potential, good values were found for the ranges, being them:

$$b_c = 2.47fm \qquad b_T = 3.68fm \tag{3.33}$$

Therefore, being $b_T > b_c$, the ranges for both out of the region force solutions $u(r)$[3.31] and $w(r)$[3.32] are b_T.

Let's suppose for the region $r < b_T$ we have approximately usual shapes of radial functions. u as S-state and w as D-state in a finite square well potential:

$$u(r) = Arj_0(\alpha r) = A\frac{sin(\alpha r)}{\alpha} \tag{3.34}$$

$$w(r) = Brj_2(\alpha r) = B\left[\left(\frac{3}{\alpha^3 r^2} - 1\right)sin(\alpha r) - \frac{3}{\alpha^2 r}cos(\alpha r)\right] \tag{3.35}$$

4 Scattering at low energies

Only in S-states scattering is considerable. We work in the center-mass system which means $E = \frac{E_{beam}}{2}$, where E_{beam} is the energy beam of neutrons in the laboratory system and $\theta_{CM} = 2\theta_{lab}$. The range of energy considered here is $1eV < E < 10MeV$. We need it to be $E > 1eV$ ir oder for the molecular bind of the proton to be considered neligable. (It also needs to be $E > 0.1eV$ to consider neutronic capture neligable).

We are concerned again about the relative motion of the system. In this case for positive energies. We imposse the wavefunction that describes the system to represent an incident beam moving across the z-axe and a beam formed by the scattering process when $r \to \infty$. The amount of beam in each direction is account in the quantity $f(\theta)$ called scattering amplitude:

$$\psi(\vec{r}) \to e^{ikz} + f(\theta)\frac{e^{ikr}}{r} \quad . \tag{4.1}$$

It can not depend on the azimutal angle because the way we chose the coordinate system and that the particles are not polarized in any direction, where k is defined as:

$$k = \left(\frac{2\mu E}{\hbar^2}\right)^{1/2} \quad . \tag{4.2}$$

The differential cross section for elastic scattering is:

$$d\sigma = |f(\theta)|^2 d\Omega \quad . \tag{4.3}$$

The wave function is assumed to be a S-state so it can be expressed as $\psi(\vec{r}) = \phi(r)Y_{00}$. We are intereseted in the radial part as it must be an eigenfunction of the radial equation. We have:

$$\phi(r) = Y_{00} \int \psi(\vec{r}) d\Omega \ . \tag{4.4}$$

For the asymtotic region $r \to \infty$ we have:

$$\phi(r) = i\pi^{1/2}(kr)^{-1}\left[e^{-ikr} - e^{ikr}\right] + (4\pi)^{1/2}\hat{f}\frac{e^{ikr}}{r} \quad , r \to \infty \ . \tag{4.5}$$

Of course $u(r) = r\phi(r)$ must follow the radial equation (eq. [1.13]). E is positive so our radial function has the following form:

$$u(r) = Csin(kr + \delta) \ . \tag{4.6}$$

We decide to normalize the function fixing its shape at infinity:

$$u(r) \to v(r) = \frac{sin(kr + \delta)}{sin(\delta)} \tag{4.7}$$

that means the normalization is:

$$v(0) = 1 \tag{4.8}$$

It is important to notice that this normalization does not have the same meaning as if it were a bound state where you always need to impose $u(0) = 0$. These are not bound states and they are indeed not normalizable. Here we say normalization to the process of fixing a value for C.

So the radial part in the assymtotic region is:

$$\phi(r) \to \frac{A}{sin\delta}\left[e^{i\delta}\frac{e^{ikr}}{2ir} - e^{-i\delta}\frac{e^{-ikr}}{2ir}\right] \ , \tag{4.9}$$

where we include the coefficient A just because it is suitable. It is not neccesary but it is clearer this way.

Comparing the part multiplying the e^{-ikr} factor in eq.[4.5] and eq.[4.9] we have:

$$A = \frac{2\pi^{1/2}}{k}e^{i\delta}sin(\delta) \ . \tag{4.10}$$

Putting this in eq.[4.9] we get:

$$\phi(r) = \frac{2\pi^{1/2}}{k}e^{i\delta}sin(\delta)\frac{u(r)}{r} \tag{4.11}$$

Comparing with [4.5] (the part multiplying the e^{ikr}) we get:

$$\hat{f} = k^{-1}\left(\frac{e^{2i\delta} - 1}{2i}\right) = k^{-1}e^{i\delta}sin(\delta) \tag{4.12}$$

Therefore the elastic differential cross section has the following form:

$$d\sigma = \frac{sen^2(\delta)}{k^2}d\Omega \tag{4.13}$$

The phase shift δ is determined by the regular condition $u(0) = 0$, so we are interested in the behaviour of the wave function near $r = 0$. Now let's call $u_0(r)$ to the radial function at zero enegies $E = 0$, being this not real zero energy because in that situation we would need to consider chemical bindings of the target, we will consider it as zero energies but not considering the chemical bindings.

Beyond the force region at zero energies we have the following radial equation:

$$\frac{d^2u_0}{dr^2} = 0 \ . \tag{4.14}$$

And the solutions are:

$$u_0(r) = D(r - a) \tag{4.15}$$

where a is called the scattering length, which is just the point where this function intersects with the r axis.

Impossing the same normalization as in eq.[4.8], that's it $u(0) = 1$ we have:

$$u_0(r) \to v_0(r) = 1 - \frac{r}{a} \tag{4.16}$$

In the limit $k \to 0$ we have $v_0(r) = 1 + rkcot(\delta)$, so comparing we have:

$$tan(\delta) = -ka(k \to 0) \tag{4.17}$$

So we obtaing the cross section at zero energy:

$$\sigma = 4\pi a^2(k \to 0) \tag{4.18}$$

The measurement of the cross section at "zero.^{en}ergy determines the absolute value of the scattering length, but not its sign. Actually the sign of a has a deep meaning because one only can have a bound state with a being positive. We can see from the graph in fig. [10] that only the function with positive a could join the decreasing exponential form of a bound state. Later this conclusion will be asserted from the equations.

From the wave equations of the functions we have used u, u_0, v and v_0 we have the following equations:

$$\frac{d}{dr}(uu_0' - u_0u') = k^2uu_0 \tag{4.19}$$

and:

$$\frac{d}{dr}(vv_0' - v_0v') = k^2vv_0 \tag{4.20}$$

Substracting and integrating from zero to infinity we have:

$$\int_0^\infty \frac{d}{dr}\left(uu_0' - u_0u' - vv_0' + v_0v'\right) dr = k^2 \int_0^\infty (uu_0 - vv_0) \tag{4.21}$$

All the functions must nullify at infinity and we use $u_0(0) = u(0) = 0$, $v_0(0) = v(0) = 1$, $v'(0) = kcot(\delta)$ and $v_0'(0) = -a^{-1}$. And we get:

$$kcot(\delta) = -a^{-1} + k^2 \int_0^\infty ((vv_0 - uu_0)\, dr \tag{4.22}$$

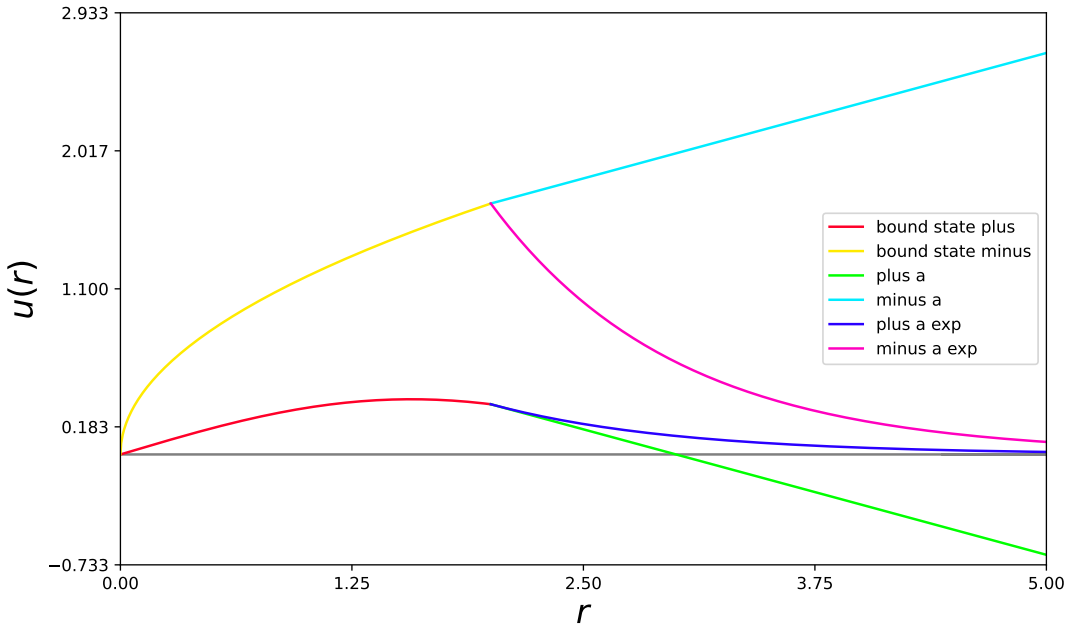

Figura 10: Joint of the functions for $a = +4$ and $a = -4$ with the exponential of a bound state.

We put our attention in the integral. For large values of r it nullifies because v is the asymptotic variation of u, the integrand adquires considerable values in the region where the potential is greater, that's it, inside the range of the nuclear force. Because the behaviour of the wave functions v and u depends much more on the potential than the energy in that region we can approximate the equation as:

$$kcot(\delta) = -a^{-1} + \frac{1}{2} r_0 k^2 \tag{4.23}$$

where r_0 is the so called effective range:

$$r_0 = 2 \int_0^\infty \left(v_0^2 - u_0^2\right) dr \tag{4.24}$$

which is a function of the potential, it does not depend on the energy.

We can follow a similar derivation of the former equations using the bound state. In that case we have that $k^2 = -R_m^2$, the asymptotic form of the function now changes to $v(r) = e^{\frac{-r}{R_m}}$ and what changes is $v'(0) = {R_m}^{-1}$. Therefore we have:

$${R_m}^{-1} = a^{-1} + \frac{1}{2} r_0 {R_m}^{-2} \tag{4.25}$$

This equation establishes that a must be positive for a bound state. Take into account that a is approximately the equivalent to R_m for zero energy and the term with r_0 is a correction. You can see that in the definition of r_0. Therefor if a is negative, R_m is negative which can not be by definition.

Now we can derive an approximate expression for the cross section at any energy:

$$\sigma = \frac{4\pi}{\left(k^2 + R_m^{-2}\right)\left[1 - \left(\frac{r_0}{R_m}\right) + \frac{1}{4}r_0^2\left(k^2 + R_m^{-2}\right)\right]} \tag{4.26}$$

where k represents the energy given by eq.[4.2] and R_m is given by eq.[1.21]. So the adjustable parameter is r_0. We can use any energy to determined this coefficiente. For $k \to 0$ we have:

$$\sigma_0 = 4\pi R_m^2 \left[1 - \frac{r_0}{2R_m}\right]^{-2} \tag{4.27}$$

The maximum possible value for r_0 is R_m. So the values for σ_0 goes from $2.33 \cdot 10^{-24} cm^2$ to $9.32 \cdot 10^{-24} cm^2$. While the experimental value is around $20.36 \cdot 10^{-24} cm^2$.

The reason for this disagreenment dwells in the assumption that nuclear force is not dependent on the spins of the nucleons. But if they are, that means if the neutron and proton froms and state of $J = 0$ we can not use the wave functions we use and the deuteron data which is all asumed to be in a $L = 0$ and $J = 1$ state. In the situation of the experiment nucleons are not polarized so their spins are oriented at random. Just because there are three possible states for the triplet state and one for the single state we have that:

$$\sigma = \frac{3}{4}\sigma_t + \frac{1}{4}\sigma_s \tag{4.28}$$

So using our calculations for $(\sigma_t)_0$ and the exprimental value of σ_0 we can find that $74.6 \cdot 10^{-24} > (\sigma_s)_0 > 53.6 \cdot 10^{-24} cm^2$

We should use radial functions with $l = 1$ and develop what we did in this section, this is not to be done here but it's known that all the development is valid except for eq.[4.25] and eq.[4.26], in which we used values from the deuteron states.

The signs of a establishes wether the wave function can describe a bound state. If it is positive the form leads for $E < 0$ can be a bound state, if it is negative that's impossible.

We use two different forms for the cross sections of the single state and triplet state:

$$\sigma_s = \frac{4\pi}{\left(a_s^{-1} - \frac{1}{2}r_{0s}k^2\right)^2 + k^2} \tag{4.29}$$

which is valid in general, not only for the single state, but for the triplet state we prefer to take advantage of the known value of R_m:

$$\sigma_t = \frac{4\pi}{\left(k^2 + R_m^{-2}\right)\left[1 - \left(\frac{r_{0t}}{R_m}\right) + \frac{1}{4}r_{0t}^2\left(k^2 + R_m^{-2}\right)\right]} \tag{4.30}$$

a_t and r_0 are related by eq.[4.25] (equation which is only valid for triplet state). Using eq.[4.18] we have a relation for a_s and a_t:

$$\sigma_0 = 4\pi(a_s^2 + a_t^2) \tag{4.31}$$

There then remain two independently variable parameters in the cross section at higher energies. We shall take these parameters to be the two effective ranges, r_{0t} and r_{0s}

4.1 Independence on nuclear charge

What we can infer from the development we have done is that the interaction depends on the spin coupling of the two particles and that's how the cross section are different for a interaction between the same particles in different states of total spin.

Still one could think that because not having a neutron-neutron bound system that means the nuclear force has a huge dependence on the nuclear charge. But we can see it is precisely the opposite. Let's think about the total wavefunction of system of two particles:

$$\Psi = \psi_l(\vec{r})\chi_S \tag{4.32}$$

In the special case of a system of two particles we have that for odd l the spatial function is antisymmetric and for even l it is symmetric. Assuming ground state with $l = 0$ for the system neutron-neutron in order to follow Pauli's principle then we need to have a $S = 0$ spin state. We can not have $S = 1$ as in the deuteron. And we have that for the system neutron-proton with $S = 0$ we don't have a bound state, therefore that's actually the difference. It is the coupling in spin which really affects the interaction between two nucleons rather than the particular pair nucleons we are dealing with.

Of course in the case of a system proton-proton we need to take into account the electric repulsion. And in order to really know the independence on nuclear charge, neutron-neutron and proton-proton systems need to be studied.