Probabilities

Outline

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What are probabilities?

- Probability as frequency of outcome of events:
  - In this way of thinking about probabilities we look at the number of times a given event happens over a large number of tries.
  - It is difficult to define consistently however, without running into circular reasoning.
- Probability as degree of belief:
  - Subjective probability is associated with personal judgements about how likely something is to happen.
  - For example, 'I believe that team X will beat team Y, because teams Y's star player has an injury, while team X has been training really hard.' Such statements can be made even if teams X and Y have never played each other.
  - By requiring that two people arrive at the same conclusion if given the same assumptions and data, this definition of probability can be formalised into a mathematical system

equivalent to the other definitions.

- Probably derived from axioms:
  - Probability is a measure that satisfies a set of axioms derived from logic and set theory, such as the Kolmogorov or Cox axioms.
  - This sidesteps the frequentist vs Bayesian interpretation by sticking to purely mathematical concepts.

In this course we start out with this definition. In general we will follow the Bayesian degree-of-believe way of thinking about probability.

Notation and basic concepts

Set notation

- A set is a collection of elements, e.g.: $A = \{1, 2, 3\}$
- $e \in A$ means e is a member of the set A, e.g.: $1 \in \{1, 2, 3\}$
- A set can also be represented by a rule: $\{x | x \text{ follows a rule}\}$

  For example, the set E of even integers: $E = \{x | x = 2y,\ y \in \mathbb{Z}\}$

- Set inclusion ($\subseteq$). A is included in B (or is a subset of B) if all the elements of A are also elements of B.

  For example: $\{1, 2\} \subseteq \{1, 2, 3\}$

Set operators

Let $A = \{1, 3, 5\}$; $B = \{2, 3, 4\}$

- **Union** $\cup$ All elements of A and all elements of B $A \cup B = \{1, 2, 3, 4, 5\}$
- **Intersection** $\cap$: Elements that are in both A and B

  $A \cap B = \{3\}$

- **Difference** $\setminus$ Elements that are in A but not in B

  $A \setminus B = \{1, 5\}$

  $B \setminus A = \{2, 4\}$

- **Complement**:

  The complemnet of A in reference to Ω includes all elements in Ω that are not in A. For the die example $\Omega = \{1, 2, 3, 4, 5, 6\}$,

$A^c = \{2, 4, 6\}$ or

$A^c = \{\omega : \omega \in \Omega \text{ and } \omega \notin A\}$

- **Empty set** $\varnothing$

  The empty set, $\varnothing$, is the complement of the universal set:

  $\Omega^c = \varnothing$ and $\varnothing^c = \Omega$.

  This means, $A \cup \varnothing = A$ and $A \cap \varnothing = \varnothing$.

- **Power set**

  Collection of all possible sets of a given set

  $A = \{1, 3, 5\}$

  $\mathcal{P}(A) = \{\varnothing, \{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, \{1, 3, 5\}\}$

Outcomes, events, probability

Outcomes and sample space

The outcomes ω of an experiment are elements of the set of all possible outcomes, called the sample space Ω.

Consider the experiment of tossing a (fair) coin twice:

- $\omega = \mathrm{HH}$ ("two heads")
- $\omega \in \Omega = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$

Events and event space

An event F is a set of outcomes

- $F = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}\}$ ("at least one head")

Events are elements of the event space $\mathcal{F}$: the power set the sample space (the set of all possible outcomes)

Note that Ω and $\mathcal{F}$ are not the same. The sample space contains the basic outcomes and event space contains sets of outcomes.

Probability

The probability function P assigns a probability (a number between 0 and 1) to events

- $F = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}\}$
- $\Pr(F) = \frac{3}{4}$

Kolmogorov's axioms of probability

- The probability measure of events is a real number equal or larger than 1:

  $$0 \leq \Pr(A)$$

- The probability measure of the universal set is 1.

  $$\Pr(\Omega) = 1$$

- If the sets A_1, A_2, A_3 ... $\in \mathcal{F}$ are disjoint, then

  $$\Pr(A_1 \cup A_2 \cup \ldots) = \Pr(A_1) + \Pr(A_2) + \ldots$$

Consequences of the axioms of probability

- Numeric bound:

  $$0 \leq \Pr(A) \leq 1$$

- Monoticity:

  $A \subseteq B$ then $\Pr(A) \leq \Pr(B)$

- Complement rule:

  $$\Pr(A^c) = 1 - \Pr(A)$$

- Sum rule:

  $$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

Example, a single fair die:

- $\Omega = \{1, 2, 3, 4, 5, 6\}$
- $\Pr(F) = \frac{1}{6} \quad \forall F \in \Omega$
- Events $A = \{1, 3\}$ and $B = \{1, 2, 3, 4\}$

- Monoticity:

  $A \subseteq B$, $\Pr(A) = \frac{1}{3} \leq \Pr(B) = \frac{2}{3}$

- Complement rule:

  $$\Pr(A^c) = \Pr(\{2, 4, 5, 6\}) = \frac{2}{3} = 1 - \Pr(A)$$

- Sum rule:

  $$\Pr(A \cup B) = \Pr(\{1,2,3,4\}) = \frac{2}{3}$$

  $$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = \frac{1}{3} + \frac{2}{3} - \Pr(\{1,3\}) = \frac{2}{3}$$

Conditional probabilities and independence

Conditional probabilities

The conditional probability of event A happening, given that event B happened, is

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Instead of the Kolmogorov axioms, probability theory can also be defined in terms of conditional probabilities, using the Cox axioms.

Independence

If A is independent of B, $\Pr(A|B) = \Pr(A)$: the conditional probability of A given B does not depend on B. From this follows that

$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$

Law of total probability

Let $\{H_1, H_2, \dots\}$ be a countable collection of sets which is a partition of Ω, where

$H_i \cap H_j = \varnothing$ for $i \neq j$

$H_1 \cup H_2 \cup \dots = \Omega$

The probability of an event D can be calculated as

$$\Pr(D) = \Pr(D \cap H_1) + \Pr(D \cap H_2) + \dots$$

or in terms of conditional probabilities

$$\Pr(D) = \Pr(D|H_1)\Pr(H_1) + \Pr(D|H_2)\Pr(H_2) + \dots$$

Bayes' theorem

Applying the definition of the conditional probability twice we get Bayes' theorem:

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$$

Named after Thomas Bayes, British clergyman, 1702-1761

Example: a test for rare events

Let us assume there is a rare disease that affects 0.1% of the population. There is a test that can detect this disease. It has a detection efficiency of 99% and a probability of error (false-positive) of 2%.

What is the probability of having the disease when receiving a positive test?

$$\Pr(D|+) = \frac{\Pr(+|D)\Pr(D)}{\Pr(+)} = \frac{\Pr(+|D)\Pr(D)}{\Pr(+|D)\Pr(D)+\Pr(+|D^c)\Pr(D^c)}$$

- $\Pr(+|D) = 0.99$: probability of a postive test result, given the disease is present (detection efficiency of 99%)
- $\Pr(D) = 0.001$: the disease affects 0.1% of the population
- $\Pr(+|D^c) = 0.02$: probability of a postive test result, given the disease is not present (false-positive rate of 2%)

$$\Pr(D|+) = \frac{0.99 \cdot 0.001}{0.99 \cdot 0.001 + 0.02 \cdot 0.999} = 0.047$$

The disease is only present in 5% of the cases where the test is positive!

Exercise

Birthday problem

What is the probability of two people in this room sharing a birthday?

Find the probability of the event B_n, that two people out of n share a birthday?

Assume that birthdays are independent, equally likely to happend throughout the year, and the year having 365 days.

Monty Hall problem

In a game show the contestant is presented with three doors. Behind one door is a car and behind the two others are goats. The goal is to win the car by choosing the door with the car.

The contestant chooses a door (but the door remains closed for now). The host (who know behind which door the car is) then opens one of the two other doors and reveals a goat. The contestant can now choose whether to stick with their original choice of door or switch their choice to the other remaining closed door.

What is the best strategy here? What is the probability of winning the car when sticking with their original choice? What is the probability if they were to change their choice?

Find the probabilities $\Pr(C_1|H_3)$ and $\Pr(C_2|H_3)$ of the car being behind door 1 or 2, given that the host has opened door 3.

- Simulate the game to find these probabilities
- Find the probabilities by using the expression for conditional probabilities and Bayes' theorem

You are free to start with either task.