Part 3 – TRANSMISSION LINE AND SIGNAL PROPAGATION (I)

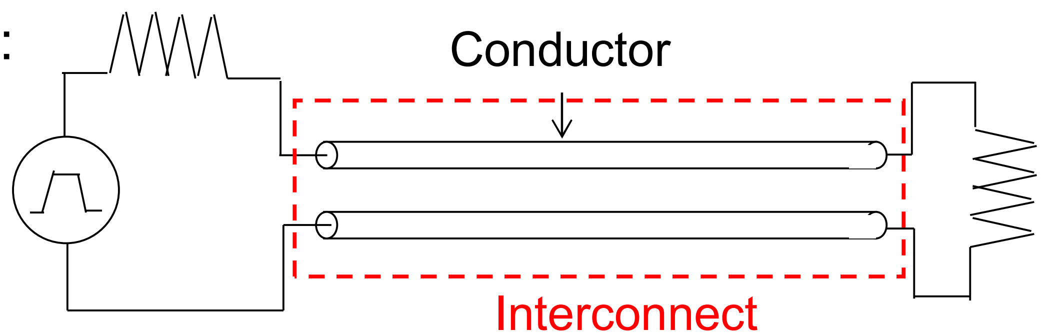
References

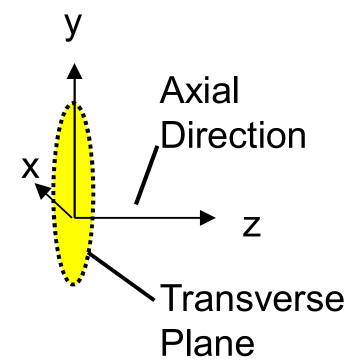
- [1] R.E. Collin, "Foundation for microwave engineering", 2nd edition, 1992, McGraw-Hill.
- [2] D.M. Pozar, "Microwave engineering", 2nd edition, 1998 John-Wiley & Sons (3^{rd} Edition, 2005 John-Wiley & Sons is also available).
- [3] S. Ramo, J.R. Whinnery, T.D. Van Duzer, "Field and waves in communication electronics" 3rd edition, 1994 John-Wiley & Sons.
- [4] http://pesona.mmu.edu.my/~wlkung/Master/mthesis.htm .
- [5] C. R. Paul, "Introduction to electromagnetic compatibility", John-Wiley & Sons, 1992. (2^{nd} edition 2006 available)
- [6] H. Johnson, M. Graham, "High-speed digital design – A handbook of black magic", Prentice-Hall, 1993.
- [7] H. Johnson, M. Graham, "High-speed signal propagation – Advanced black magic", Prentice-Hall, 2003.
- [8] T.S. Laverghetta, "Microwave materials and fabrication techniques", 3rd edition 2000, Artech House.
- [9] D. Brooks, "Signal integrity issues and printed circuit board design", 2003, Prentice Hall.

3.1 – Interconnection

Definition of Electrical Interconnect

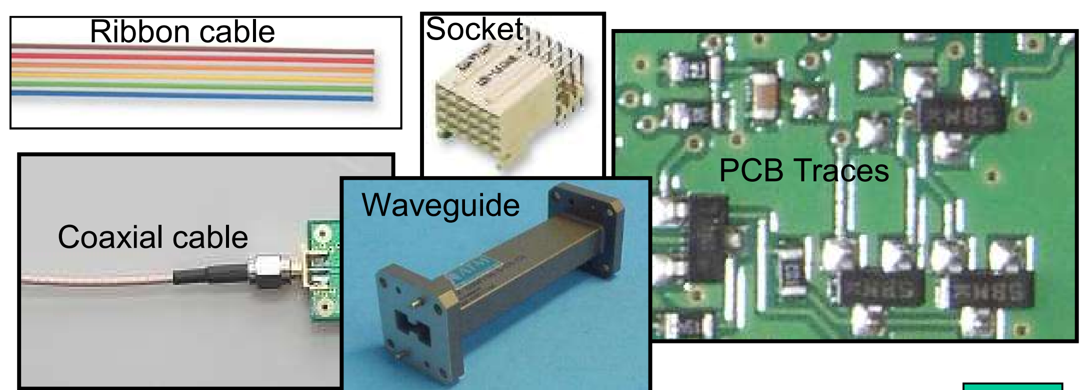
- Interconnect - metallic conductors that is used to transport electrical energy from one point of a circuit to another.
- Example:





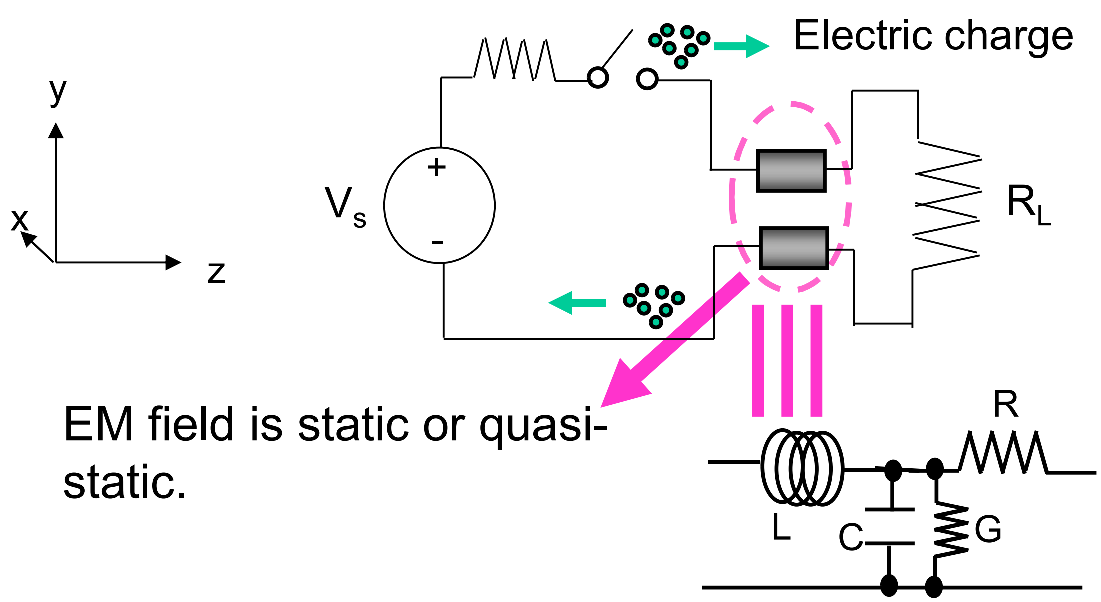
- Thus **cables**, **wires**, **conductive tracks** on printed circuit board (PCB), **sockets, packaging, metallic tubes** etc. are all examples of interconnect.

1. Usually contains <u>2 or more</u> conductors, to form a closed circuit.
2. Conductors assumed to be perfect electric conductors (PEC)



Short Interconnect – Lumped Circuit

- When the interconnect is <u>short</u>, a voltage will appear across R_L (as current flows through it) as soon as the switch is closed. The effect is instantaneous.
- Voltage and current are due to electric charges movement along the interconnect. Associated with the electric charges are **static** EM fields in the space surrounding the short interconnect.
- The short interconnect system can be modeled by lumped RLC circuit.



Static EM field changes uniformly, i.e. when field at one point increases, field at the other locations also increases.

$$\nabla \times \tilde{E} \cong 0$$

$$\nabla \times \tilde{B} \cong \mu \tilde{J}$$

$$\nabla \cdot \tilde{E} = \frac{\rho}{\varepsilon}$$

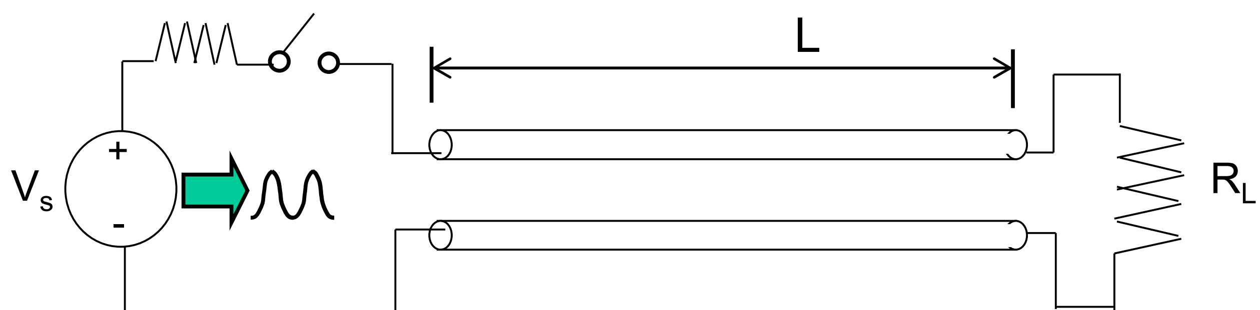
$$\nabla \cdot \tilde{B} = 0$$

Again we need to stress that typically the values of RLCG are very small, at low frequency their effect can be simply ignored.

Long Interconnect (1)

- If the interconnection is long, it takes some time for voltage and current to appear on the resistor R_L after the switch is closed.
- Electric charges move from V_s to the resistor R_L. As the charges move, there is an associated EM field which travels along with the charges.
- In effect there is a propagating EM field along the interconnect. The propagating EM field is called a wave and the interconnect is guiding the EM wave, this EM field is **dynamic**.

Long interconnect: When there is an appreciable delay between input and output

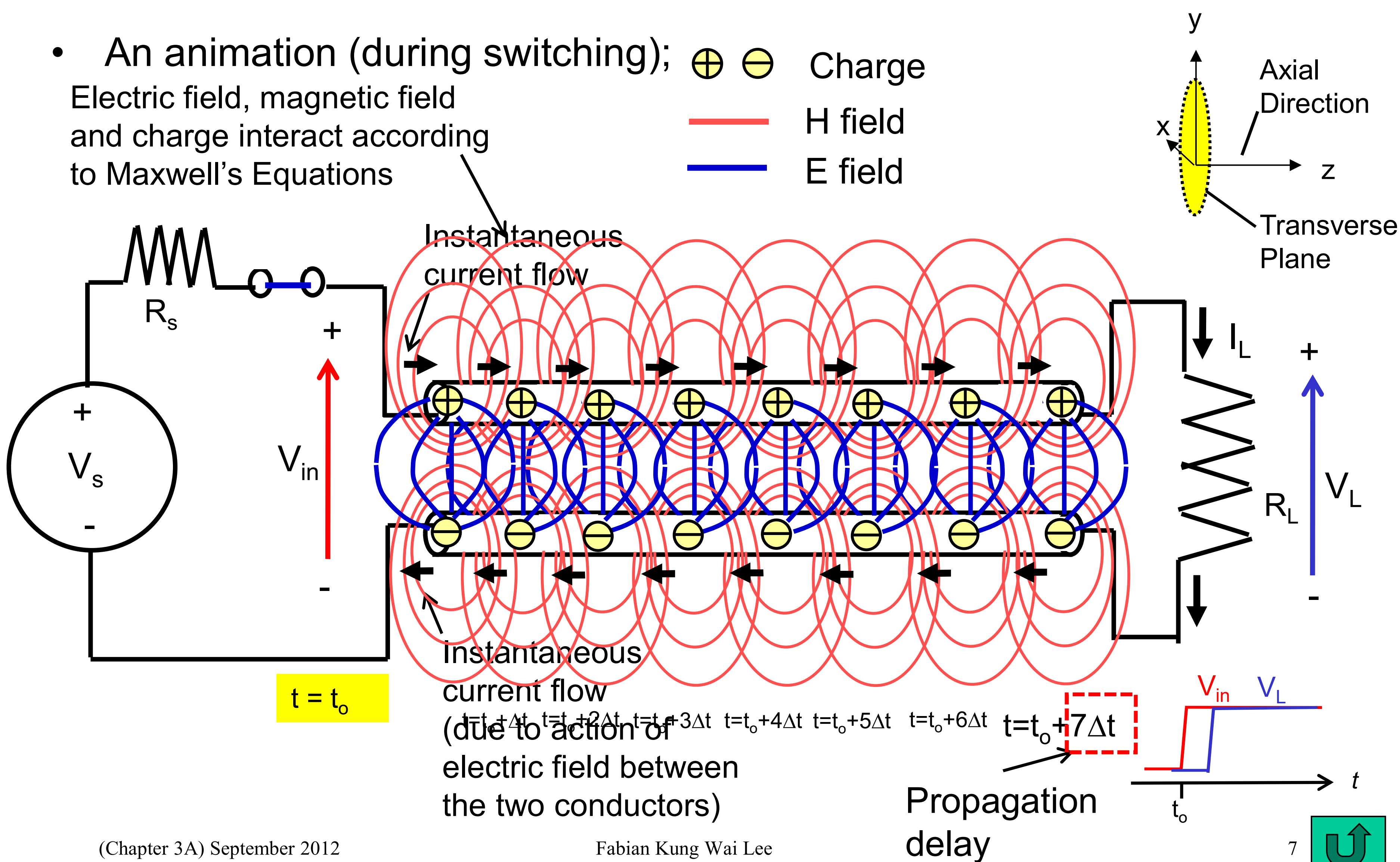


Without the metal conductors, the EM waves will disperse, i.e. radiated out into space.

Long Interconnect (2)

- An animation (during switching);

Electric field, magnetic field and charge interact according to Maxwell's Equations

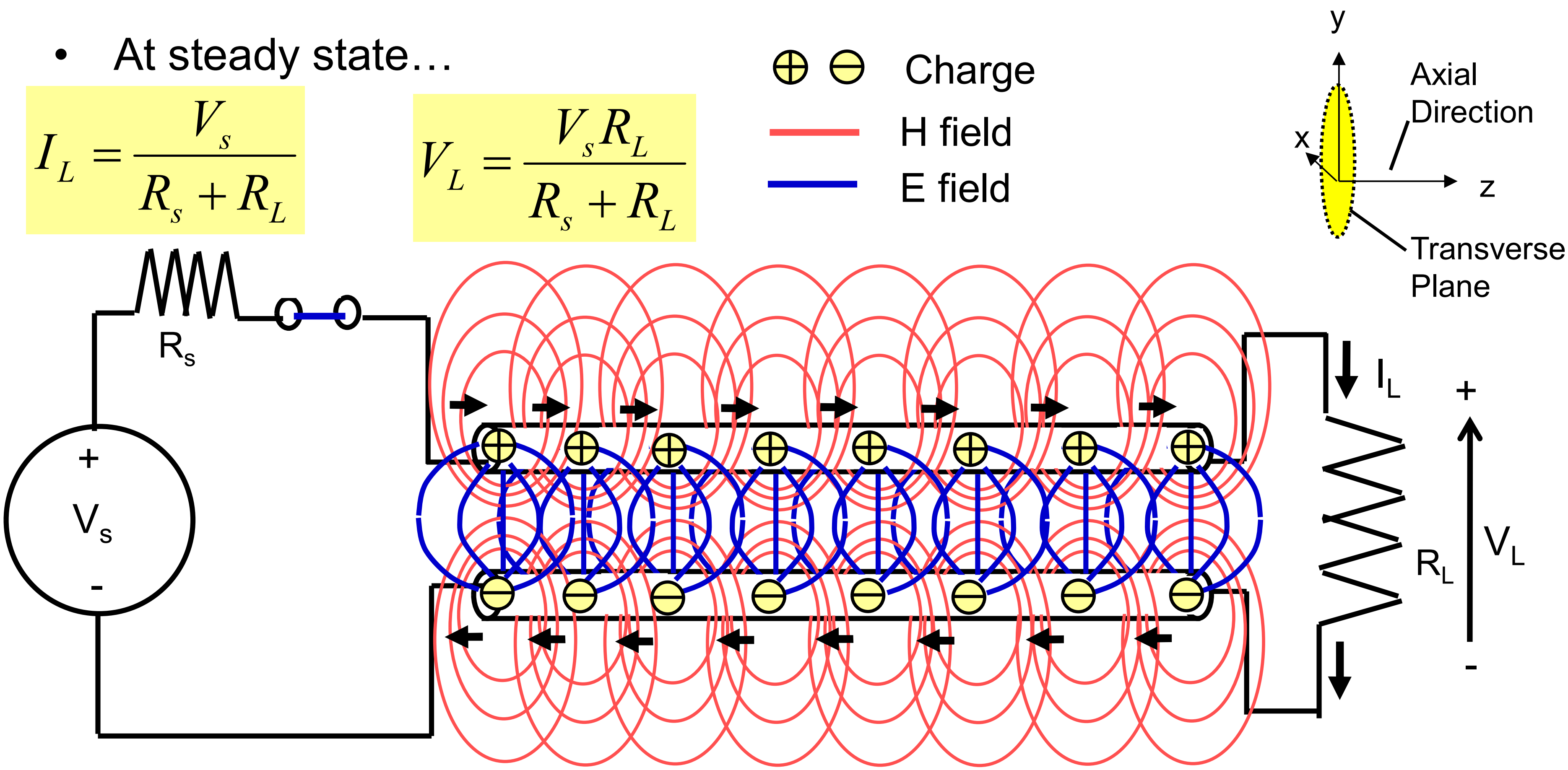


Long Interconnect (3)

- At steady state…

$$I_L = \frac{V_s}{R_s + R_L}$$

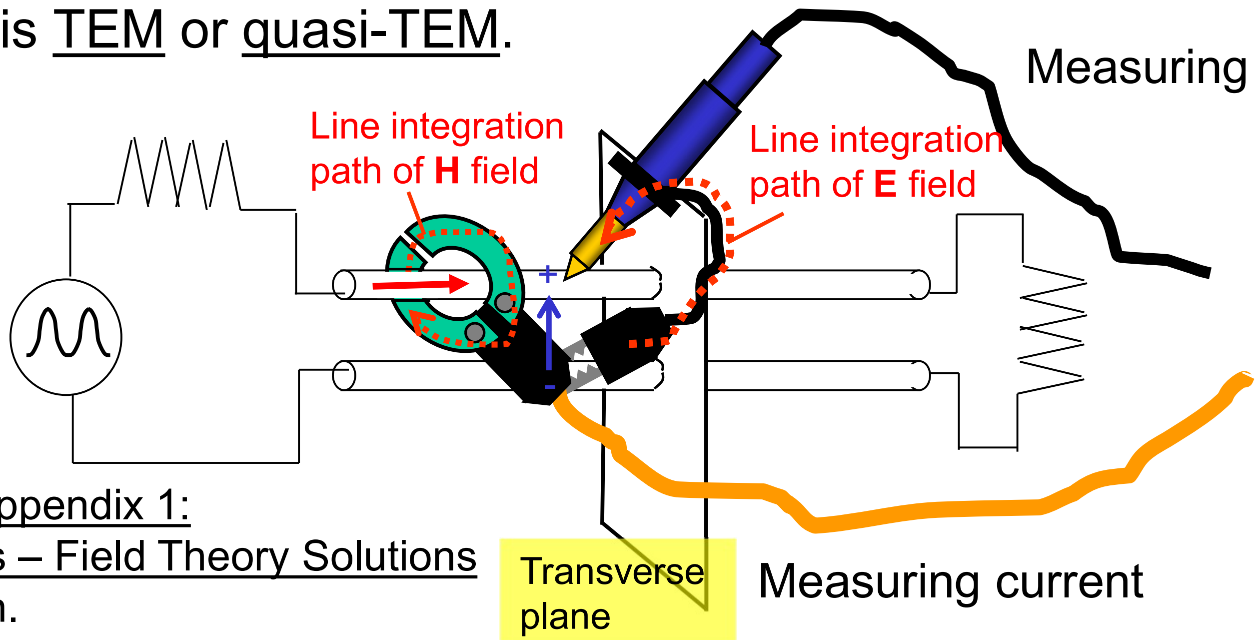
$$V_L = \frac{V_s R_L}{R_s + R_L}$$



Although we use positive charge here, we need to bear in mind that it is the free electrons that is moving in the metal, we can imagine the region with less free electrons is more positive than the region with more free electrons.

Voltage and Current on Interconnect

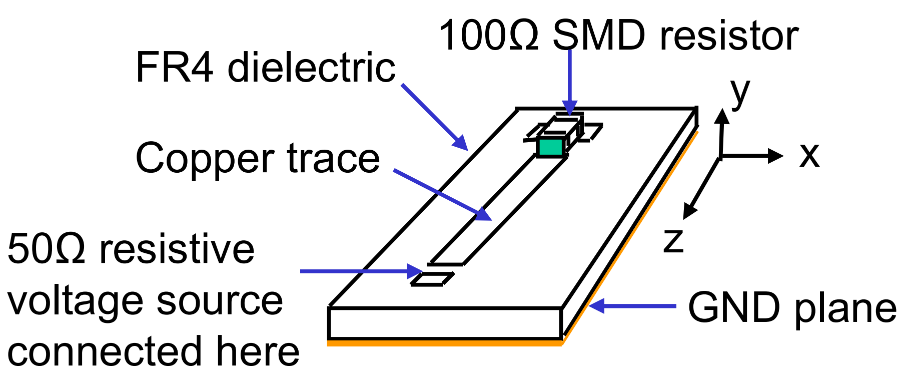
- V_t = Potential difference between two points on transverse plane and I_t = Rate of flow of electric charge across a conductor surface.
- V_t and I_t are related to the **E** and **H** fields phasors on the interconnect, and correspond to how we would measure them physically with probes.
- The V_t and I_t will be unique (e.g. do not depend on measurement setup, but only on the location) if and only if the EM field propagation mode in the interconnect is TEM or quasi-TEM.



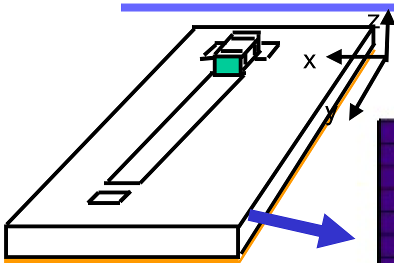
See discussion in Appendix 1: Advanced Concepts – Field Theory Solutions for more information.

Demonstration - Electromagnetic Field Propagation in Interconnect (1)

- The following example simulate the behavior of EM field in a simple interconnection system in time-domain.
- The system is a 3D model of a copper trace with a plane on the bottom.
- A numerical method, known as Finite-Difference Time-Domain (FDTD) is applied to Maxwell's Equations, to provide the approximate values of **E** and **H** fields at selected points on the model at every 1.0 picosecond interval. (Search on WWW or see http://pesona.mmu.edu.my/~wlkung/Phd/phdthesis.htm)
- Field values are displayed at an interval of 25.0 picoseconds.

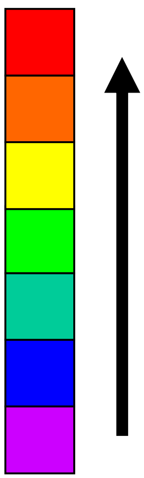


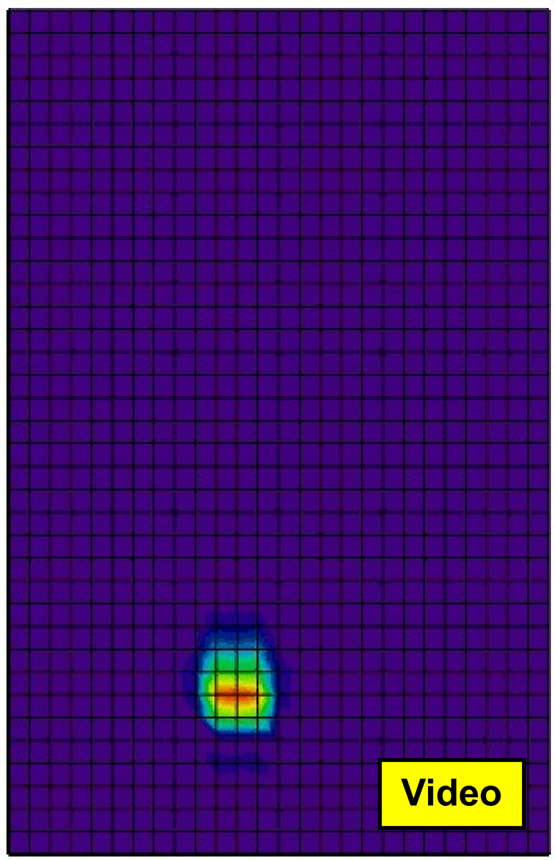
Demonstration - Electromagnetic Field Propagation in Interconnect (2)



Magnitude of E_z in dielectric

Intensity Scale





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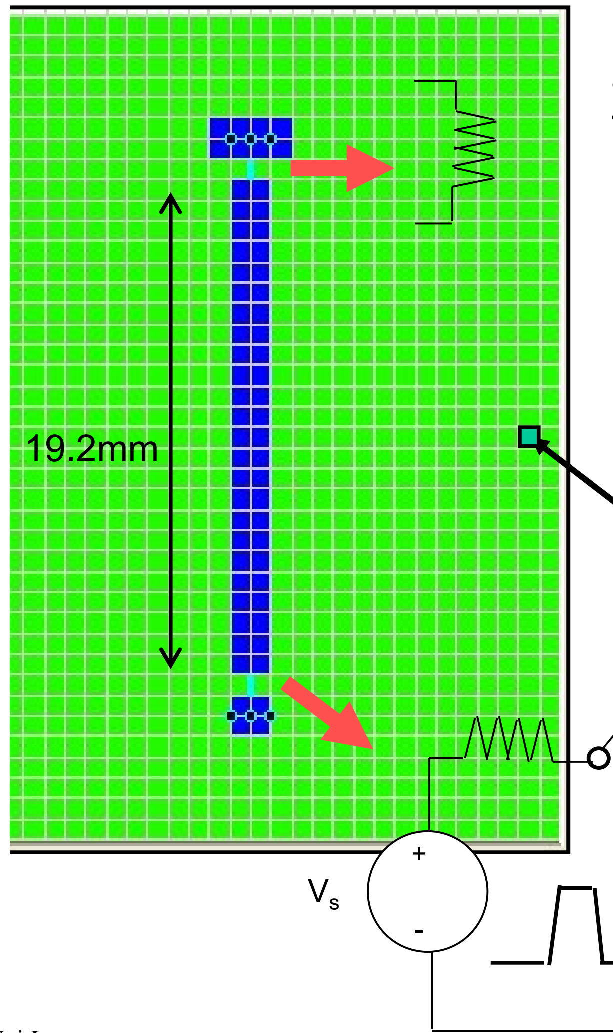
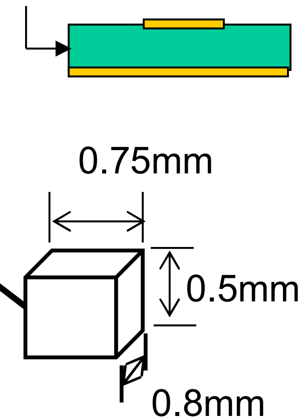
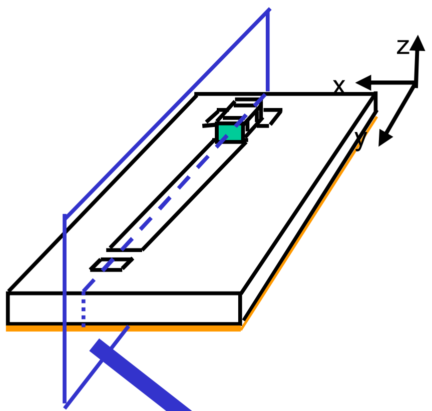
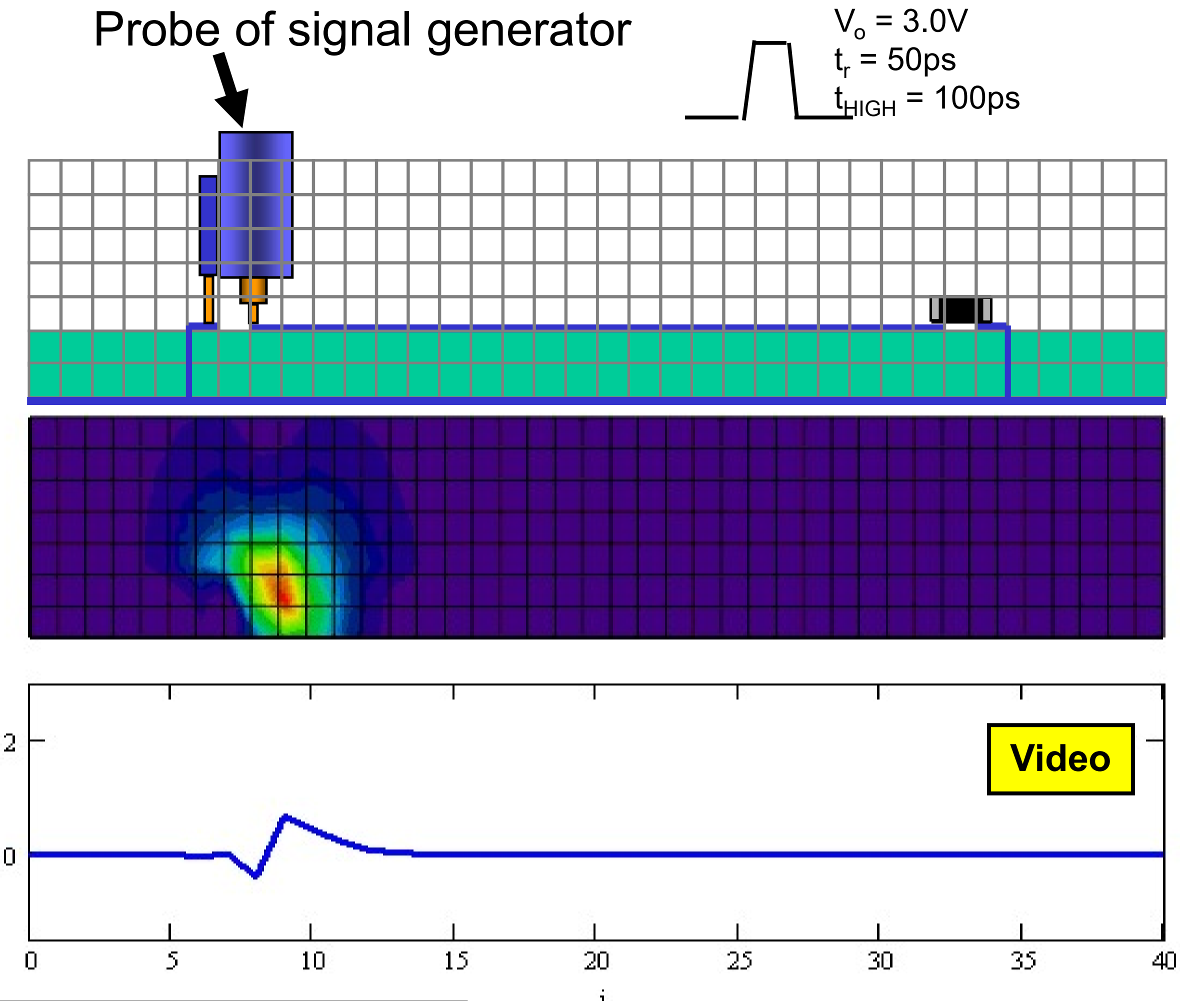


PCB dielectric: FR4, ε_r = 4.4, σ= 5. Thickness =1.0mm.



Demonstration - Electromagnetic Field Propagation in Interconnect (3)





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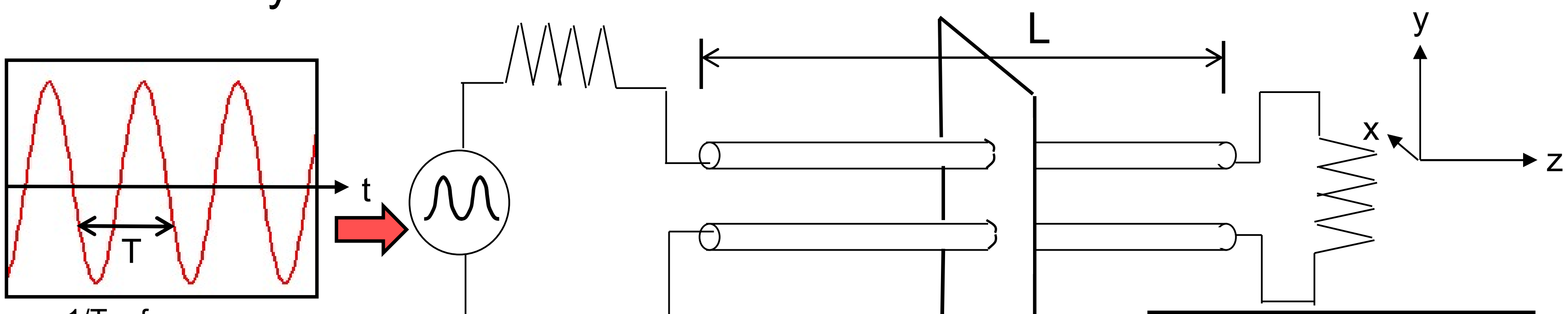
Interconnect Excited by Sinusoidal Source (1)

- If the source is sinusoidal, the corresponding EM field generated along the interconnect is also sinusoidal with respect to time and space.
- The sinusoidal EM fields on the interconnect are waves and can be described by Wave Functions.



1/T = frequency

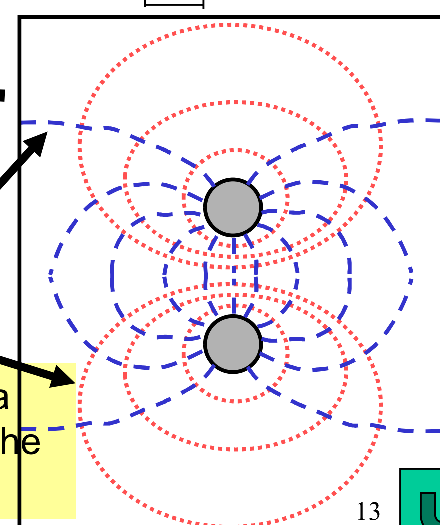
$$\vec{E}^{+} = e_x(x,y)e^{-j\beta z}\hat{x} + e_y(x,y)e^{-j\beta z}\hat{y} + e_z(x,y)e^{-j\beta z}\hat{z}$$
$$= \left(\vec{e}_t(x,y) + e_z(x,y)\hat{z}\right)e^{-j\beta z}$$

Propagating EM fields
(See Appendix 1)

$$\vec{H}^{+} = h_x(x,y)e^{-j\beta z}\hat{x} + h_y(x,y)e^{-j\beta z}\hat{y} + h_z(x,y)e^{-j\beta z}\hat{z}$$
$$= \left(\vec{h}_t(x,y) + h_z(x,y)\hat{z}\right)e^{-j\beta z}$$

Electric field (**E**)

Magnetic field (**H**)

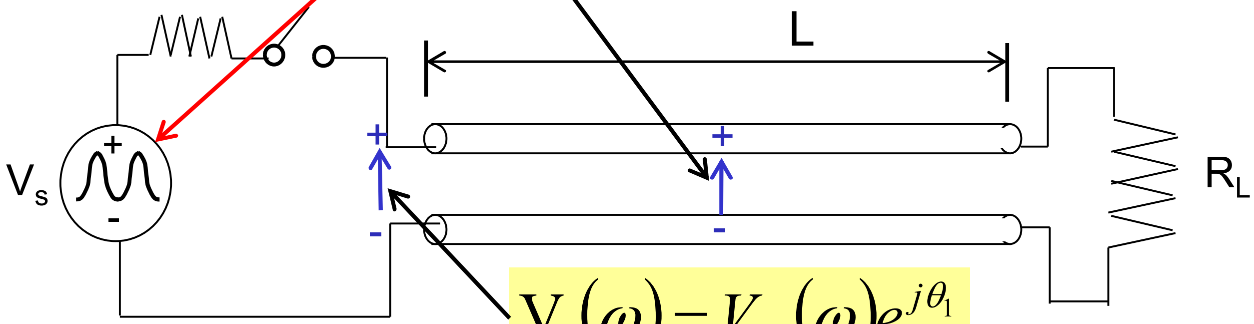


Snapshot of EM fields at a certain instant in time on the transverse plane

Interconnect Excited by Sinusoidal Source (2)

- Since any arbitrary waveforms can be decomposed into its sinusoidal components, let us consider V_s to be a sinusoidal source from now on.
- The sinusoidal EM field waves on the interconnect give rise to the corresponding voltage and current waves along the interconnect. We can represent the voltage (and current) phasors as shown.

$$V_t(\omega, z) = V_o^+(\omega)e^{-j\beta(\omega)z} + V_o^-(\omega)e^{j\beta(\omega)z}$$



$$V_1(\omega) = V_{o1}(\omega)e^{j\theta_1}$$

- These statements will be elaborated and proven mathematically later.

Concept of Wavelength

- An infinitely long trace being excited by a sinusoidal voltage source at 4GHz. The diagram below shows the instantaneous current density.

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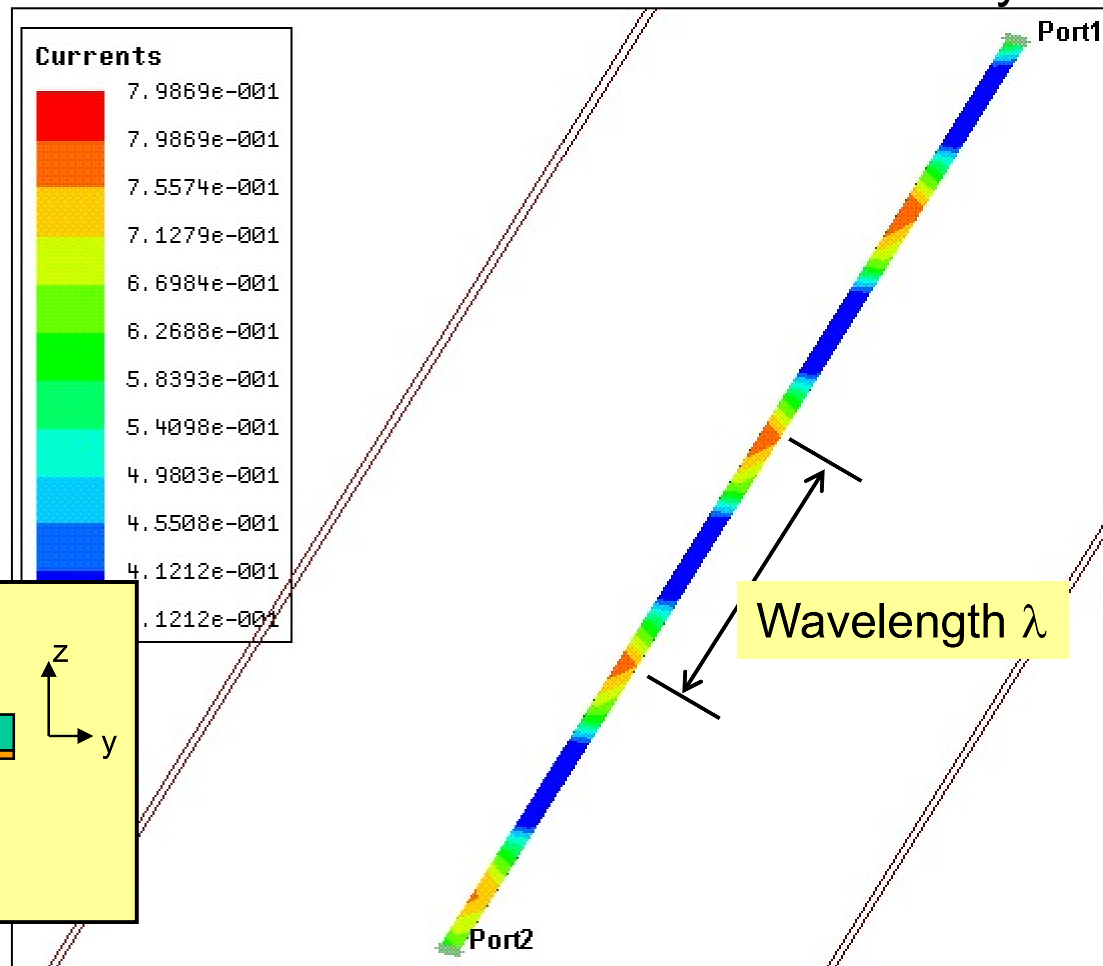
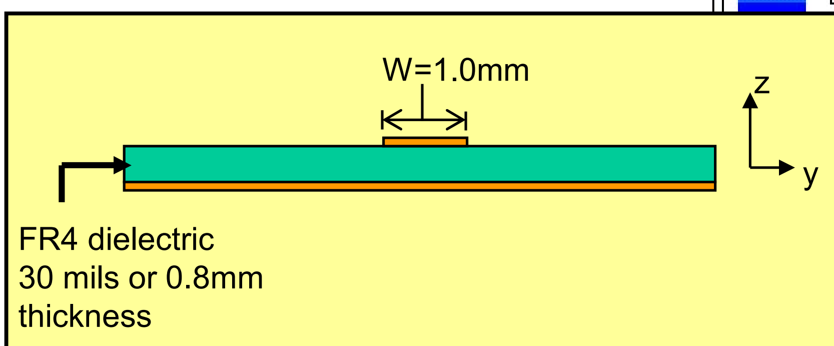
microstripline_4GHz_current.gif
microstripline_1GHz_current.gif

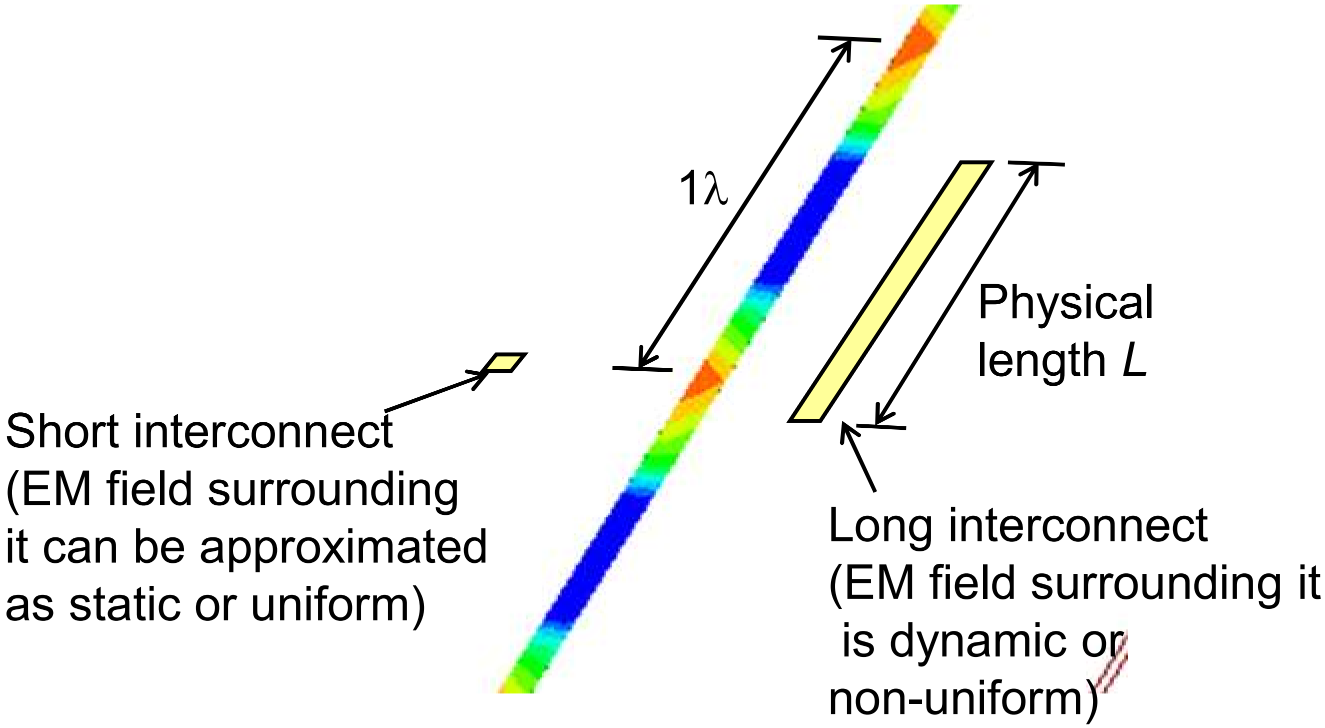


Side view of the trace model:



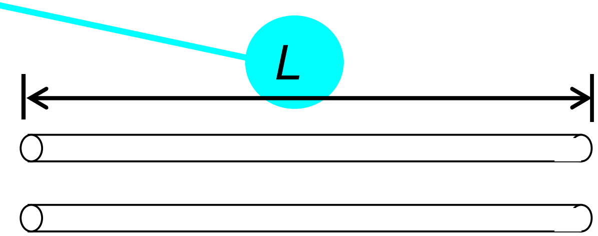
Physical Length Versus Wavelength

- If the physical length L is comparable to the wavelength λ, the EM fields along the interconnect is obviously dynamic, e.g. the field changes non-uniformly.
- Likewise it is clear that if L is small compare to λ, we can consider the EM fields along the interconnect to be static.



Long or Short Interconnect? Rule 1 -The Wavelength Rule-of-Thumb (Sinusoidal)

- We need a clear rule to determine whether an interconnect is long or short, i.e. delay between input and output is appreciable.
- Relative to wavelength for sinusoidal signals.
- Rule-of-Thumb: If $L < 0.05\lambda$, it is a short interconnect, otherwise it is considered a long interconnect. An example at the end of this section will illustrate this procedure clearly.



Interconnect

We call this the **5% rule**. Less conservative estimate will use 1/10=0.10 (the 10% Rule)

$$v_p = f\lambda \quad (1.1)$$

λ: wavelength

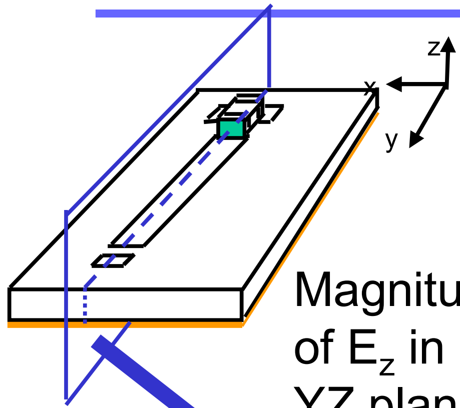
f: frequency

v_p: Phase velocity or propagation velocity

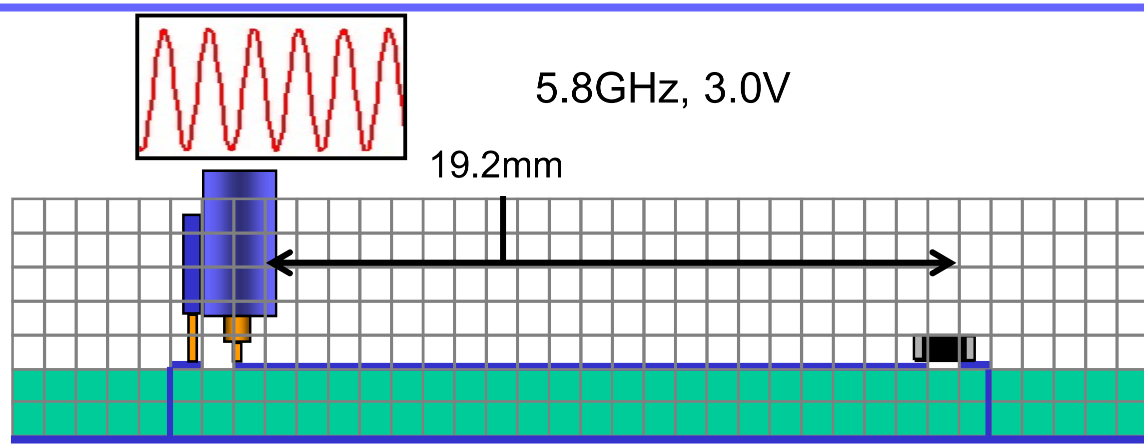
$f\uparrow \quad \lambda\downarrow$

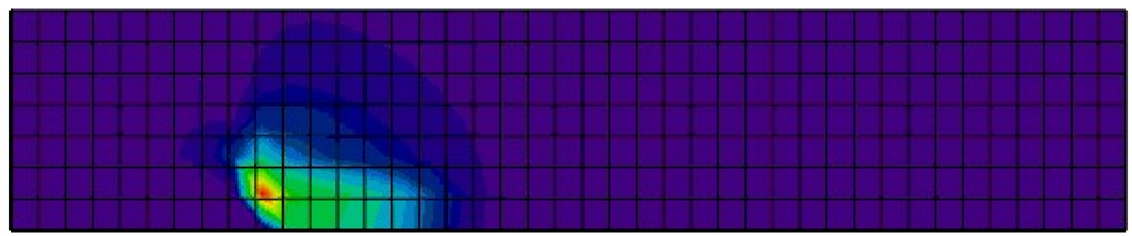
$f\downarrow \quad \lambda\uparrow$

Demonstration – Long Interconnect

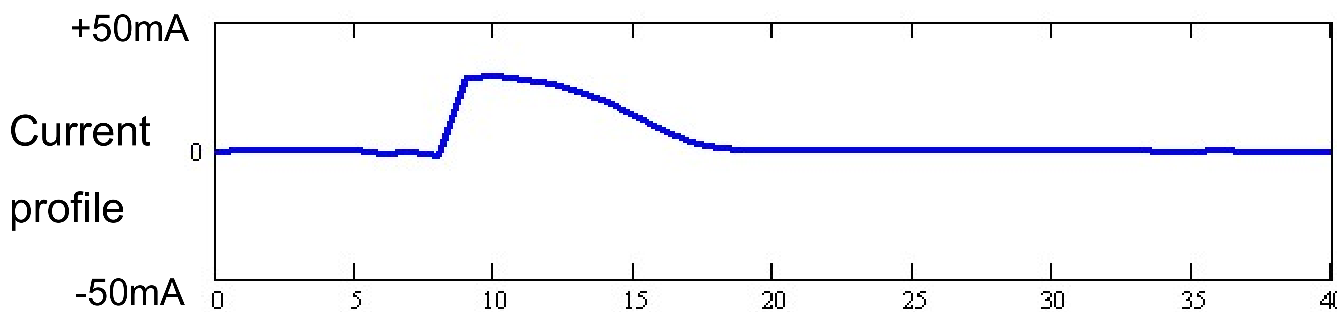


Magnitude of E_z in YZ plane





Each frame is displayed at 25psec interval



Video

Let us assume the EM wave travels at speed of light, C=2.998×10^8, then wavelength ≅ 52.0 mm

$$\lambda = \frac{C}{f}$$

19.2 mm is greater than 5% of 52 mm

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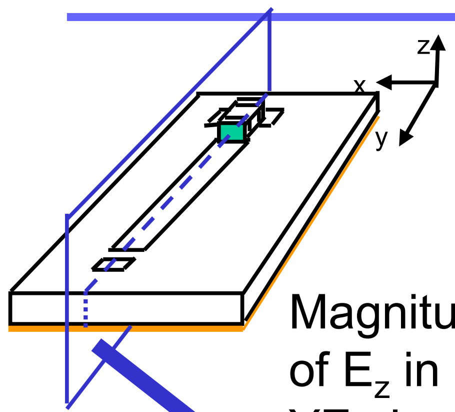
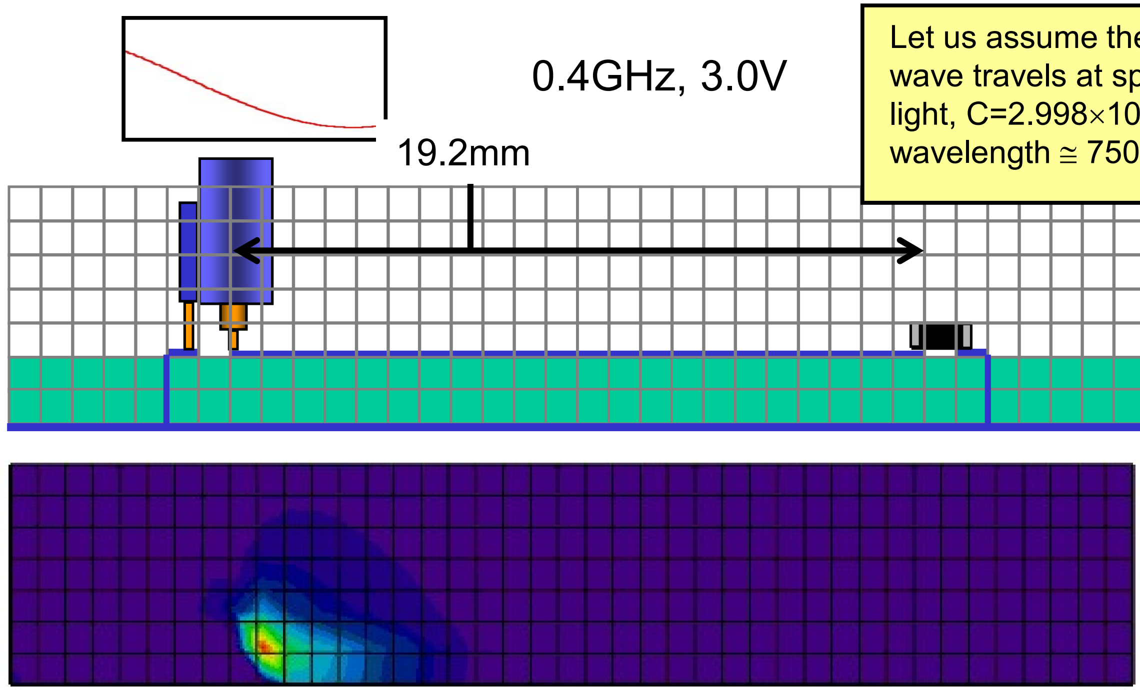
Demonstration – Short Interconnect



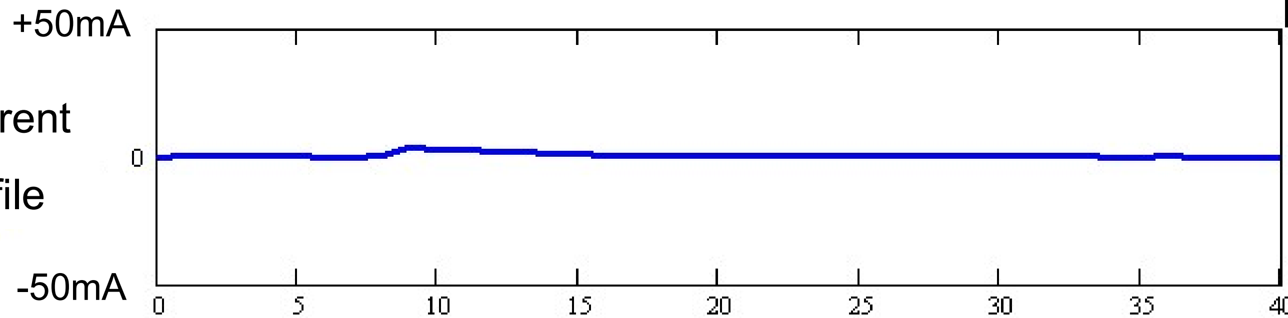
Magnitude of E_z in YZ plane



Let us assume the EM wave travels at speed of light, $C=2.998\times10^8$, then wavelength $\cong$ 750.0 mm

- At any instant in time the current profile is almost uniform along the axial direction.
- Interconnect can be considered lumped.

Video



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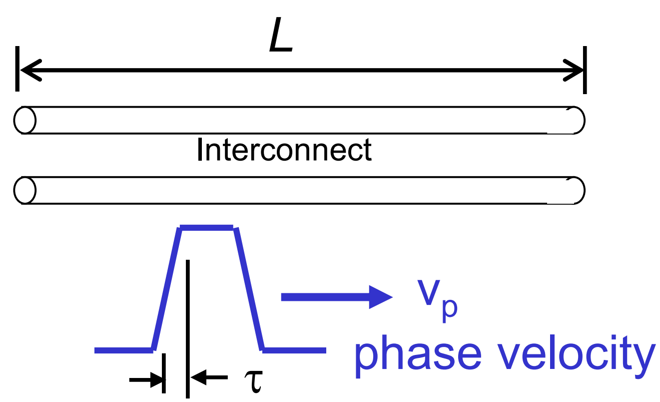
Long or Short Interconnect? Rule 2 -The Propagation Delay Rule (Pulse)

- The wavelength rule-of-thumb relates wavelength to physical length of the interconnect. The concept of wavelength is only valid for sinusoidal signals.
- In digital design we deal with non-sinusoidal signals and quantities such as rise/fall time (or edge rate) and propagation delay along the interconnect.
- In fact the wavelength rule-of-thumb can be modified to relate to rise/fall time and propagation delay in digital domain, as given here:

Propagation delay → T_{delay}

Rise/fall time → τ

$$\frac{T_{delay}}{\tau} \leq 0.1 \Rightarrow \text{Short interconnect} \qquad (1.2)$$



As long as we keep the propagation delay to be less than 10% of the rise/fall time, the interconnect can be considered short, i.e. it can be modeled as RLC network, which can be ignored if the effect is negligible

Example 1.1

- Assuming we are transmitting a digital pulse, of rise and fall time 100 ps along a copper trace of length 25.0 mm on a PCB. Make an estimate, whether the trace is a long or short interconnect. Consider the digital pulse to travel at speed of light of at 3.0×10^8 m/s.

$$T_{delay} = \frac{Length}{c} = \frac{0.025}{3.0\times10^8} = 83.33\,ps$$

$$\frac{T_{delay}}{\tau} = \frac{83.33}{100} \cong 0.83 > 0.1$$

Hence this copper trace can be considered a long interconnect.

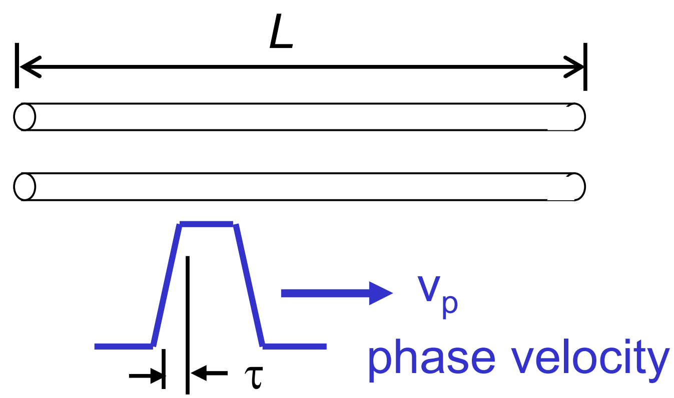
Proof of the Propagation Delay Rule-of-Thumb

Extra

- From $L \le 0.05\lambda$

$$\Rightarrow \frac{L}{v_p} \le 0.05\frac{\lambda}{v_p}$$

$$\Rightarrow T_{delay} \le 0.05\frac{1}{f}$$



- Here f is taken as the knee frequency, i.e. the highest frequency encountered. $T_{delay} \le 0.05\frac{1}{f} = 0.05(\pi\tau) = 0.1571\tau$

$$\Rightarrow \frac{T_{delay}}{\tau} \le 0.1571$$

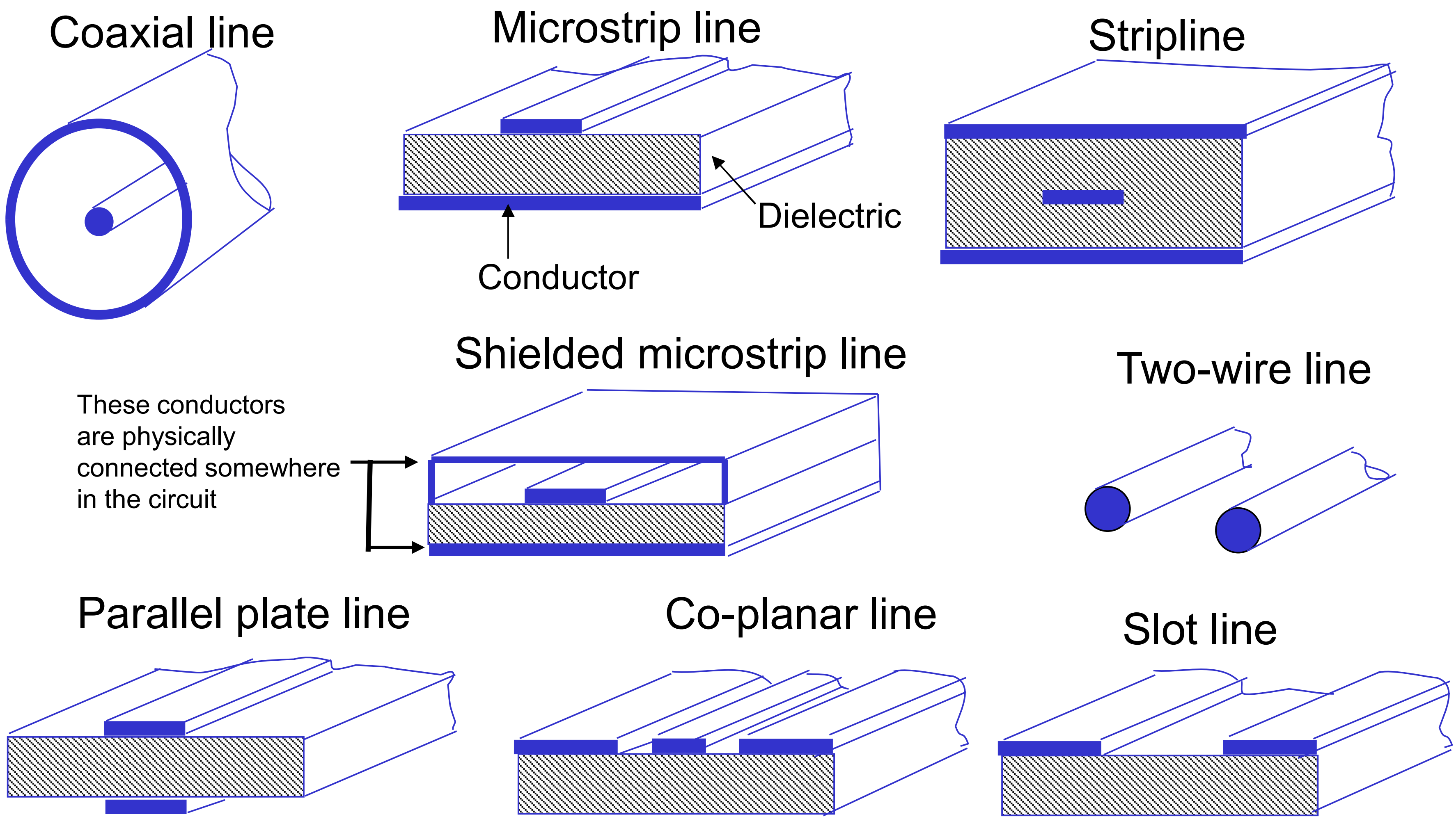
- We can just take the right-hand-side as 0.1, giving us the rule-of-thumb for short interconnect: $\frac{T_{delay}}{\tau} \le 0.1$

3.2 Transmission Line Concepts

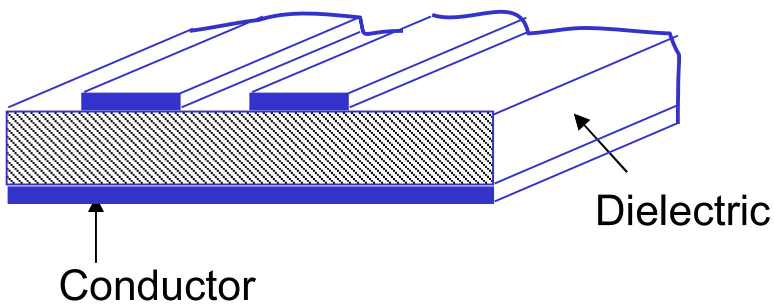
Definition of Transmission Line

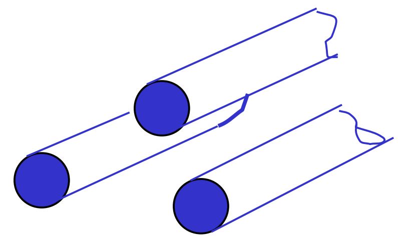
- A transmission line is a long interconnect with 2 conductors – the signal conductor and ground conductor for returning current.
- Multiconductor transmission line has more than 2 conductors, usually a few signal conductors and one ground conductor.
- Transmission lines are a subset of a broader class of devices, known as waveguide. Transmission line has at least 2 or more conductors, while waveguides refer collectively to any structures that can allow EM waves to propagate along the structure. This includes structures with only 1 conductor or no conductor at all.
- Widely known waveguides include the rectangular and circular waveguides for high power microwave system, and the optical fiber. Waveguide is used for system requiring (1) high power, (2) very low loss interconnect (3) high isolation between interconnects.
- Transmission line is more popular and is widely used in PCB. From now on we will be concentrating on transmission line, or **Tline** for short.

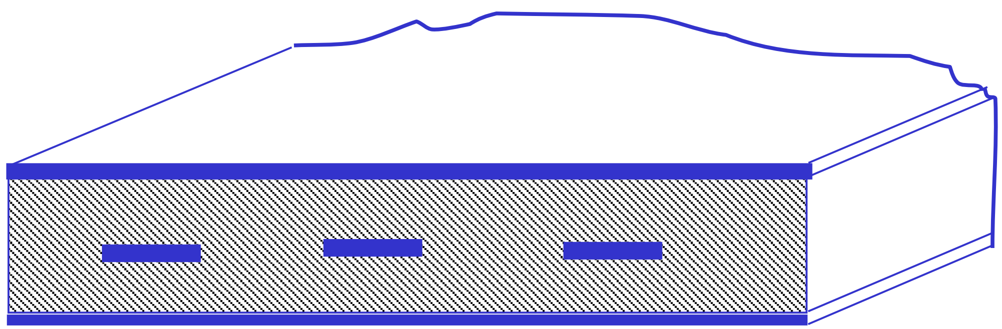
Typical Transmission Line Configurations

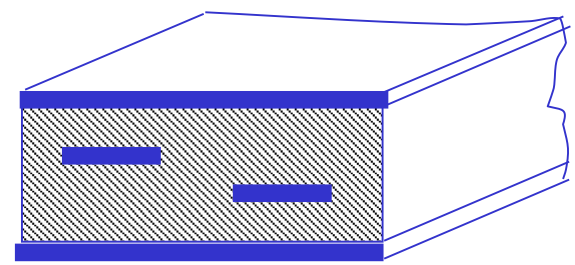


Some Multi-conductor Transmission Line Configurations



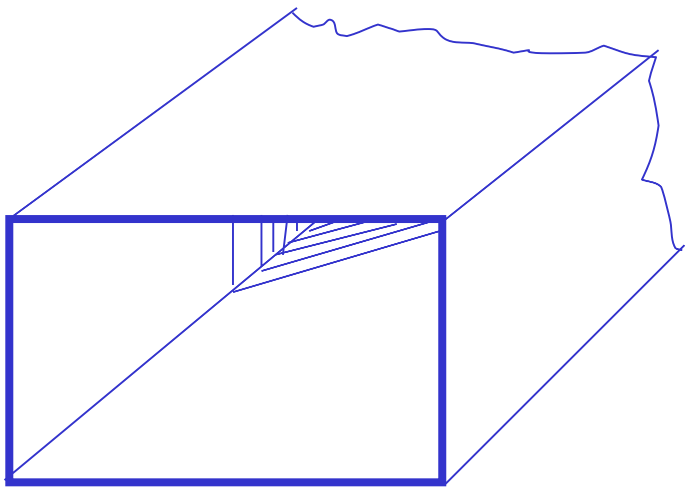




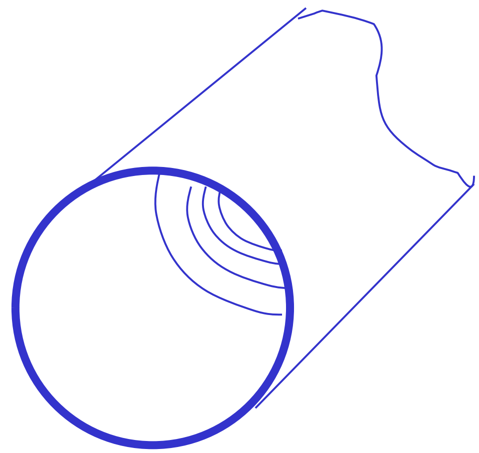


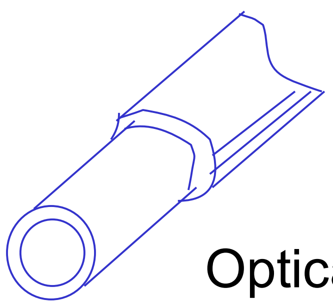
Typical Waveguide Configurations

Rectangular waveguide

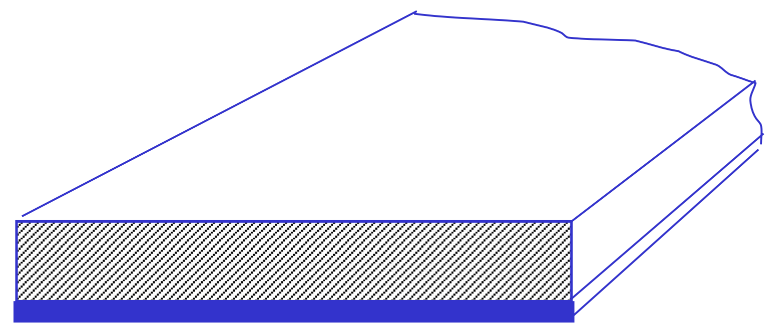


Circular waveguide



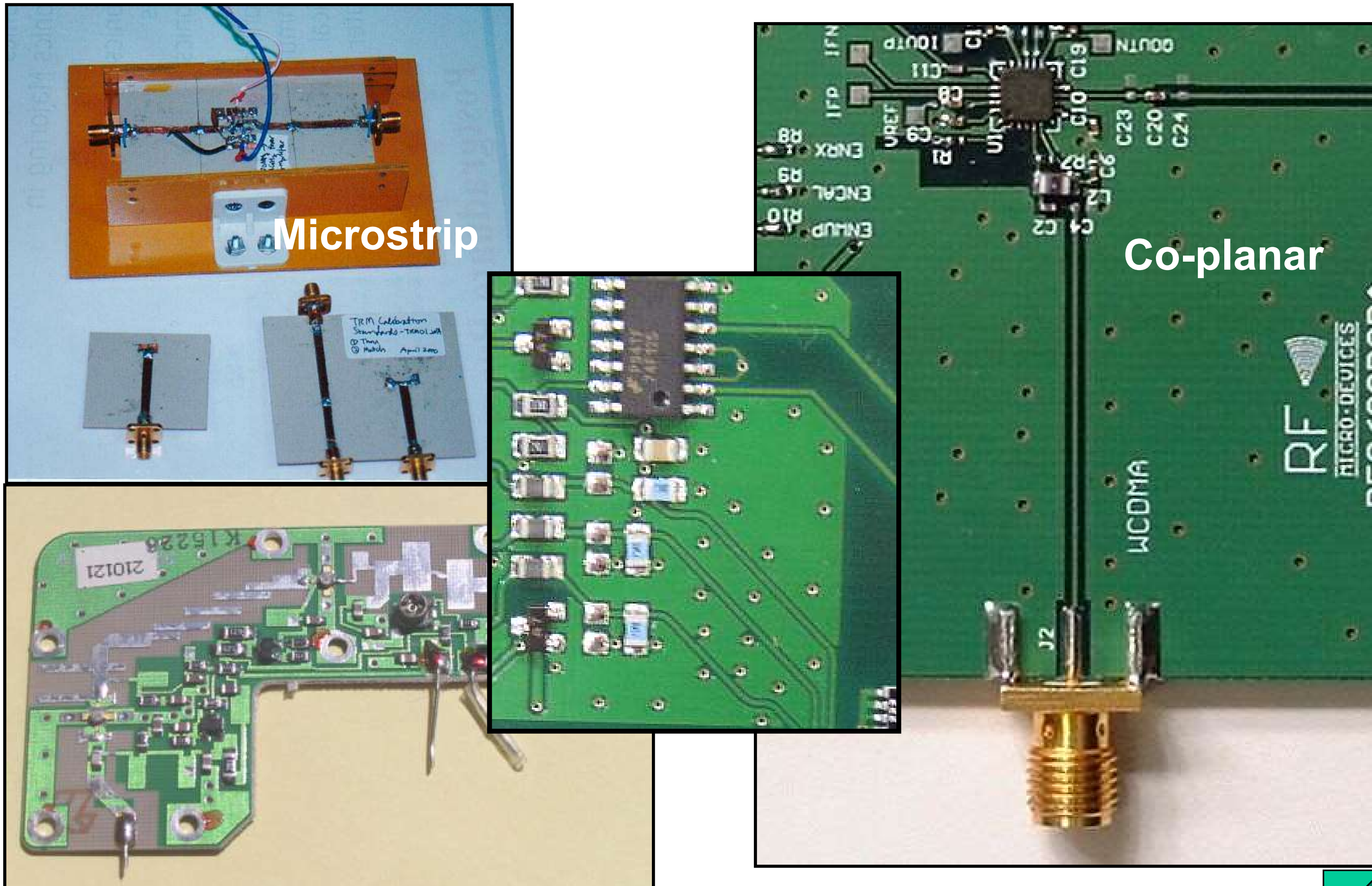


Optical Fiber



Dielectric waveguide

Examples of Microstrip and Co-planar Line



Summary of Short and Long Interconnections

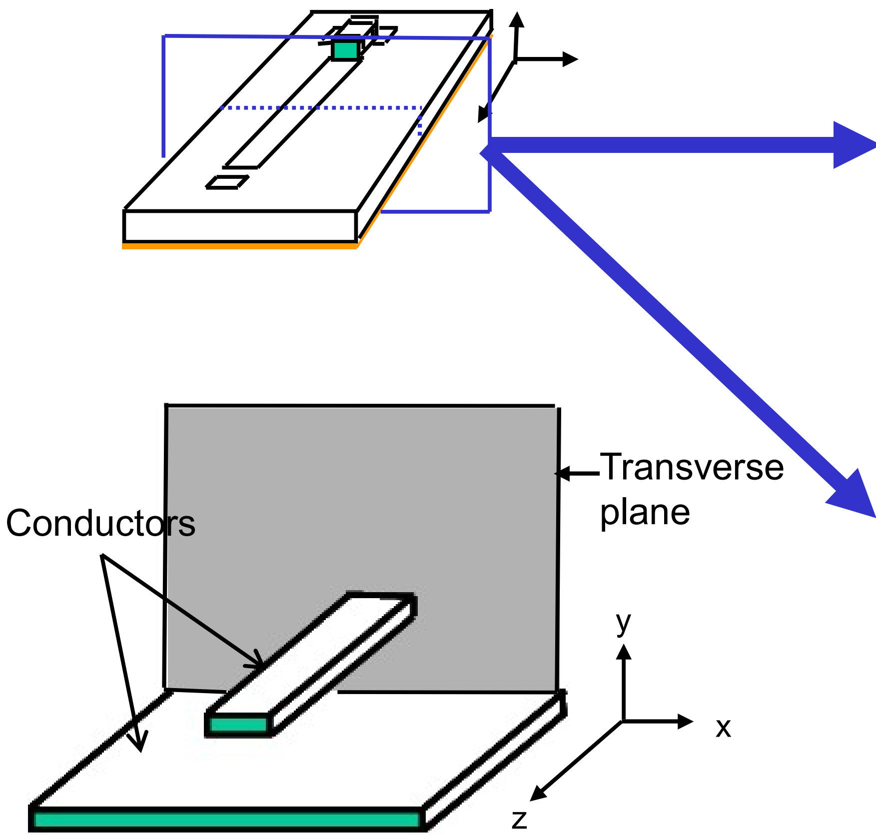
Short Interconnect	Long Interconnect
EM field in the system changes uniformly with time, static.	EM field in the system changes non-uniformly with time, dynamic.
Can be modeled as lumped RLCG network.	Cannot be modeled as lumped RLCG network, use distributed RLCG network if EM field is TEM or quasi-TEM.
Negligible propagation delay. $T_{delay} < 0.1T_{rise}$	Significant propagation delay. $T_{delay} > 0.1T_{rise}$
Physical length much shorter than shortest wavelength (5% rule).	Physical length long as compare to shortest wavelength (5% rule).

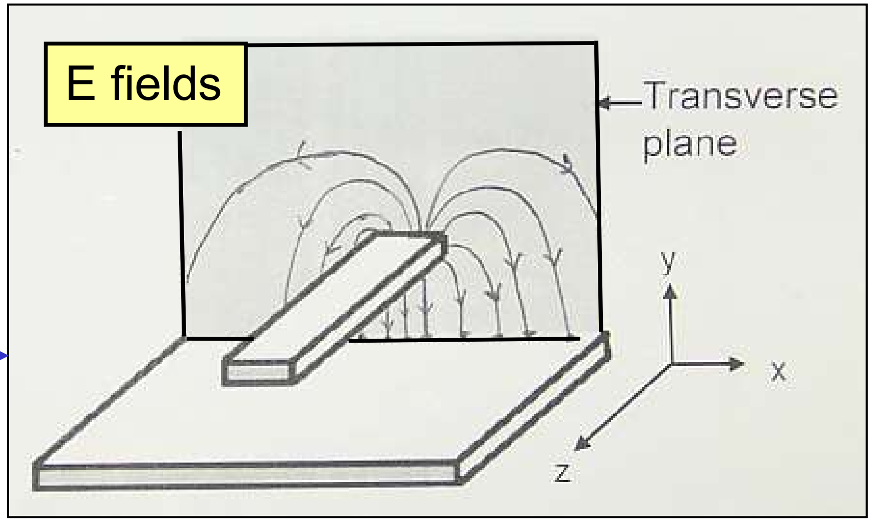
Propagation Modes

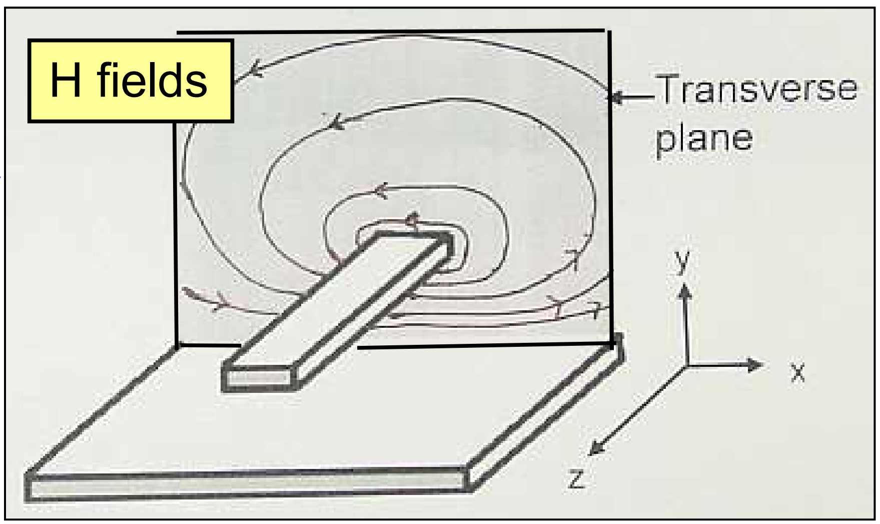
- The propagation of EM wave along a transmission line or waveguide can be described in terms of modes.
- Each mode is a pattern of electric and magnetic field distributions that is repeated along the transmission line at equal interval.
- A transmission line can support multiple modes.
- Each mode is distinguished from the others by the pattern of the E and H fields on the transverse plane (or cross section) of the transmission line.
- Before we begin, let us look at the concept of transverse and non-transverse fields.

Transverse E and H Field Patterns

Field patterns that lie in the Transverse Plane.

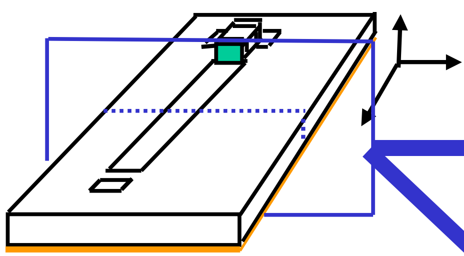




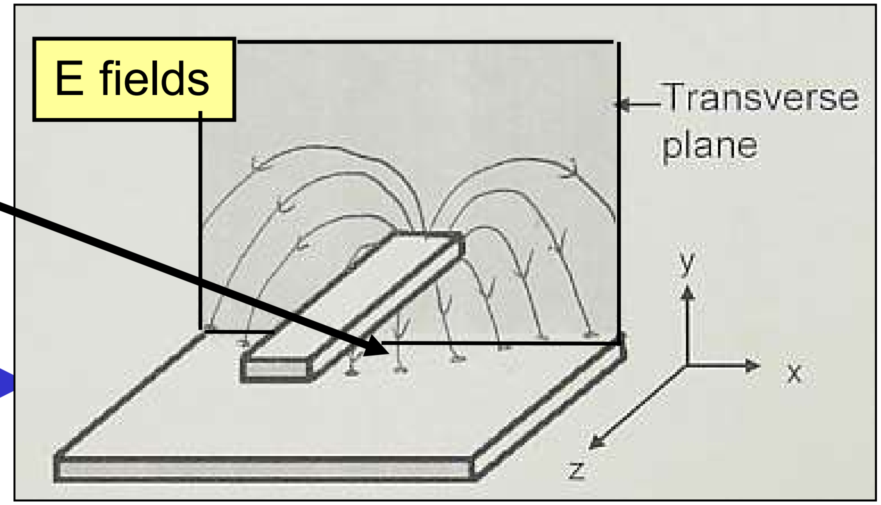


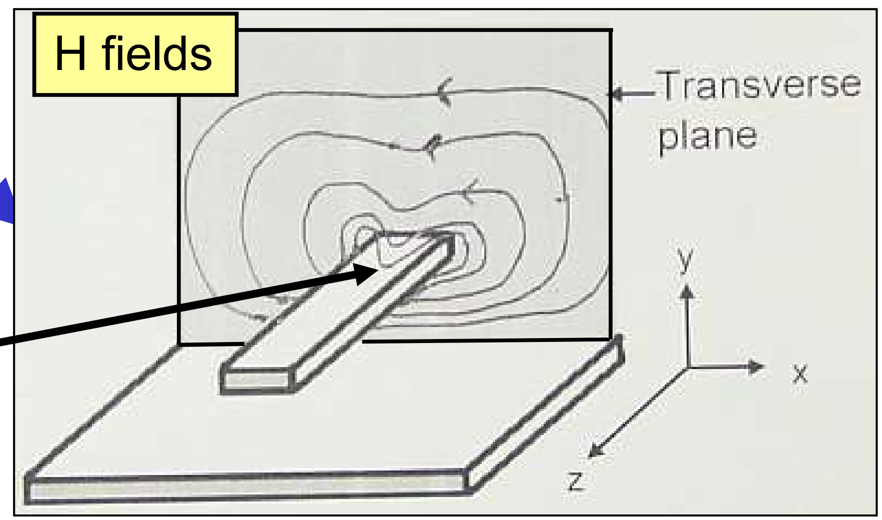
Non-transverse E and H Field Patterns

Field patterns that does not lie in the Transverse Plane.



Field contains z-component

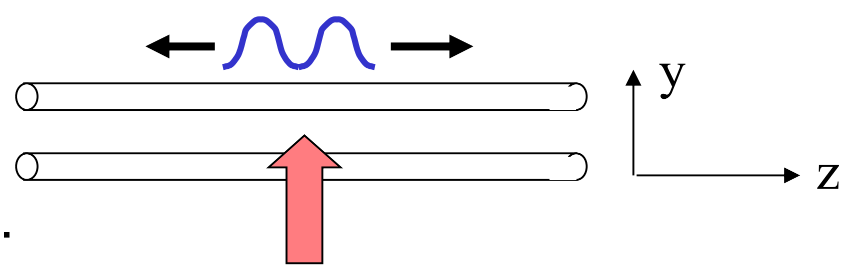




Field contains z-component

Propagation Modes for Transmission Line (1)

- Assuming the transmission line is parallel to z direction. The propagation of E and H fields along the line can be classified into 4 modes:
  - TE mode - where $E_z = 0$.
  - TM mode - where $H_z = 0$.
  - TEM mode - where E_z and H_z are 0.
  - Mix mode, any mixture of the above.

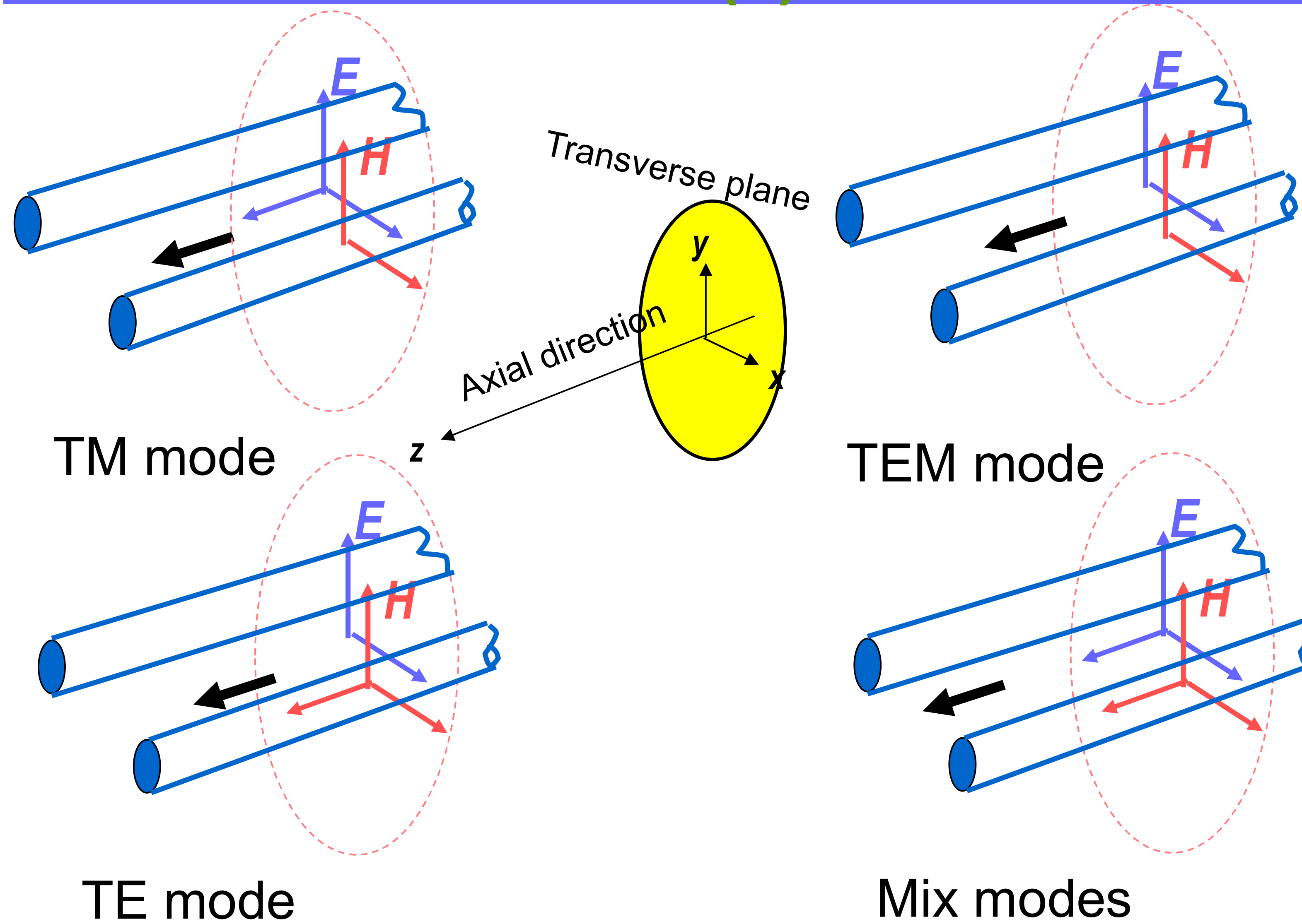


$$\vec{H}^{\pm} = \left(\vec{h}_t(x,y) + h_z(x,y)\hat{z}\right)e^{\mp j\beta z}$$

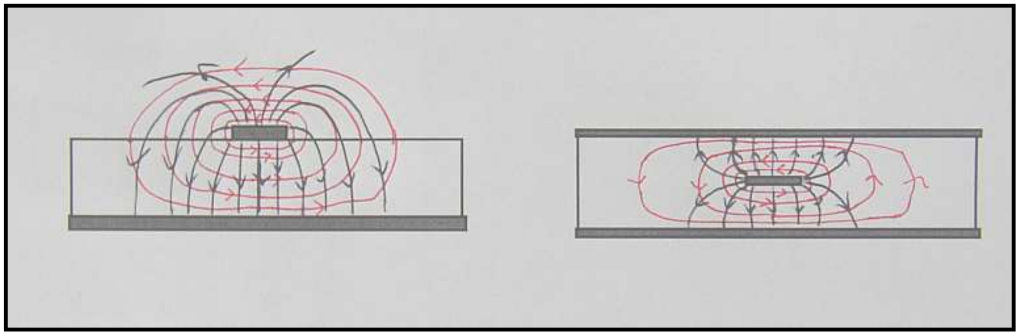
$$\vec{E}^{\pm} = \left(\vec{e}_t(x,y) + e_z(x,y)\hat{z}\right)e^{\mp j\beta z}$$

- A Tline can support a number of modes at any instance, however TE, TM or mix mode usually occur at very high frequency.
- There is another mode, known as quasi-TEM mode, which is supported by stripline structures with non-uniform dielectric. See discussion in Appendix 1.

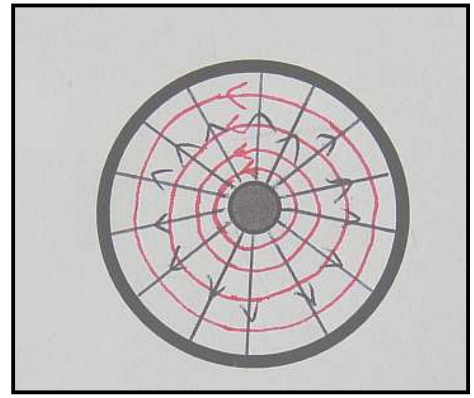
Propagation Modes for Transmission Line (2)



Examples of Field Patterns or Modes



TEM or quasi-TEM mode for microstrip and stripline

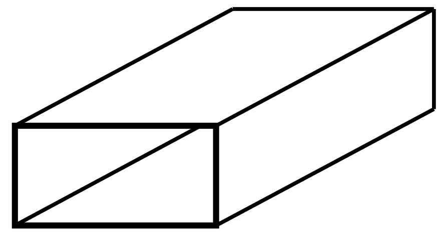


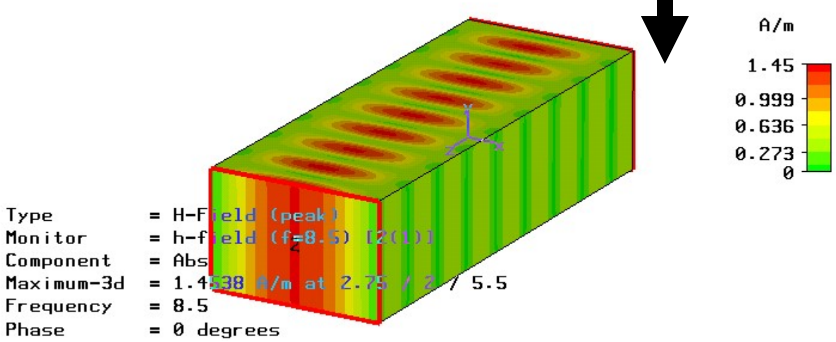
TEM field pattern for coaxial cable

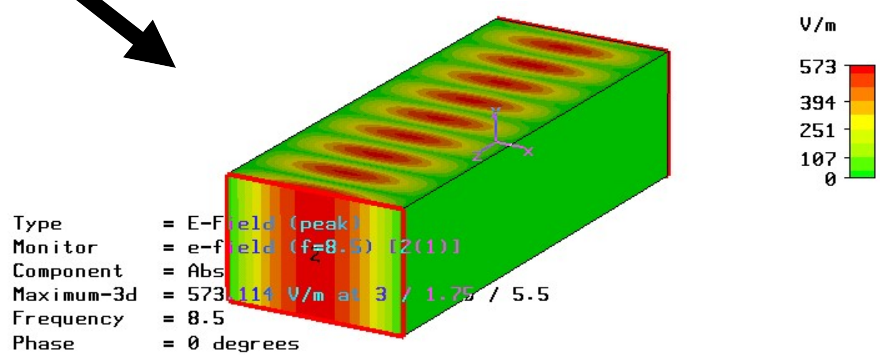
E field

H field

E and H field patterns for TE_{10} mode in rectangular waveguide, computed using CST Microwave Studio (Academic Edition)





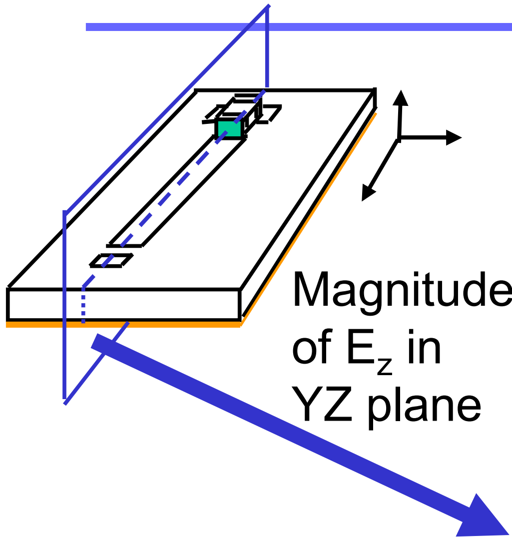


3.3 – Transmission Line Electrical Circuit Model and Signal Propagation

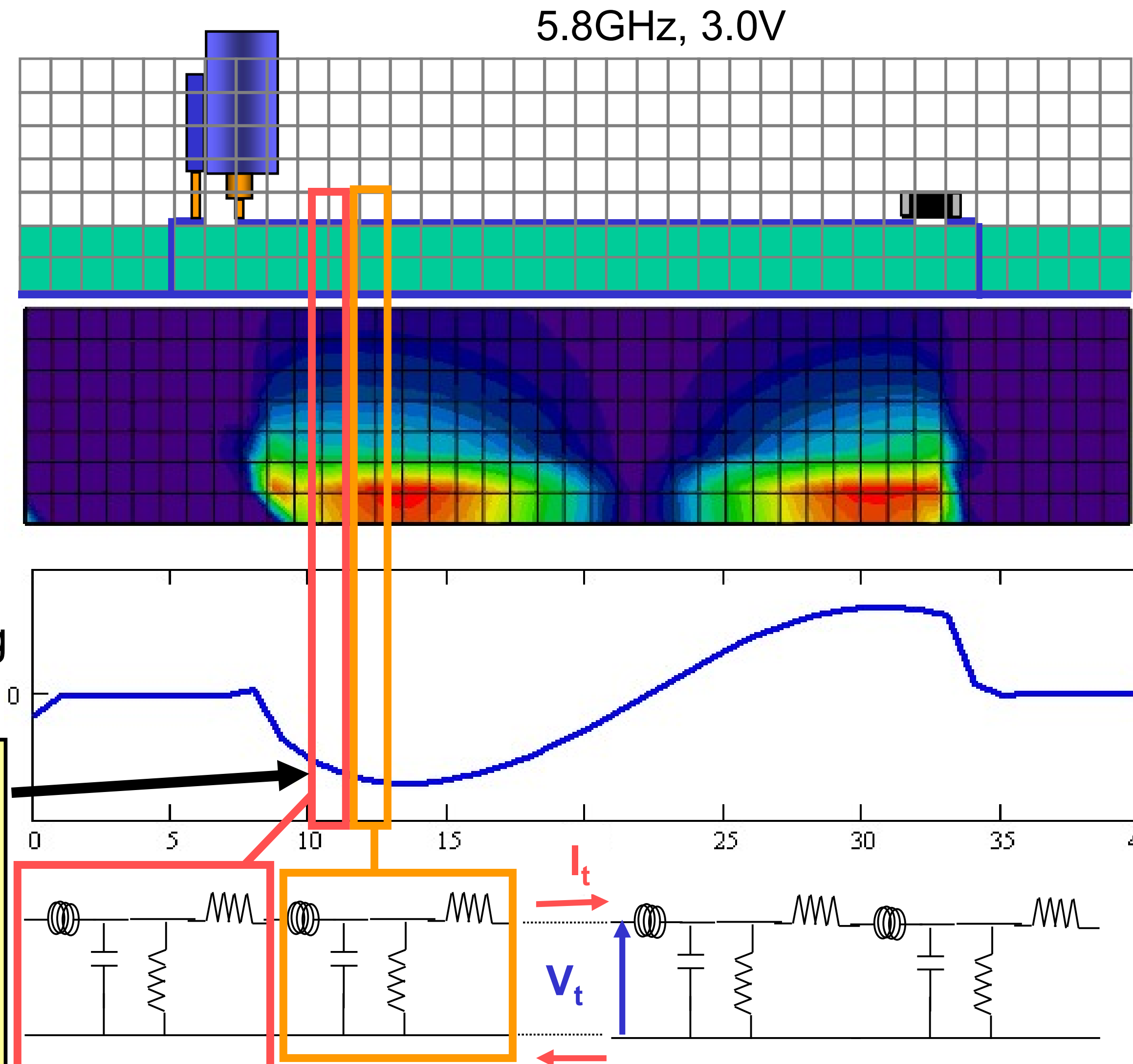
Modeling a Transmission Line With Circuit Elements (1)

- Since transmission line is a long interconnect, the field and current profile at any instant in time is not uniform along the line.
- It cannot be modeled by lumped RLC components.
- However if we divide the Tline into many short segments (each with length< 0.05λ), the field and current profile in each segment is almost uniform.
- Each of these short segments can be modeled as RLCG network.
- This assumption is true when the EM field propagation mode is TEM or quasi-TEM.
- **From now on we will assume all Tline under discussion support the dominant mode of TEM or quasi-TEM.**
- For transmission line, these associated R, L, C and G parameters are distributed, i.e. we use the per unit length values. The propagation of voltage and current on the transmission line can be described in terms of these distributed parameters.

Modeling a Transmission Line With Circuit Elements (2)



Magnitude of E_z in YZ plane



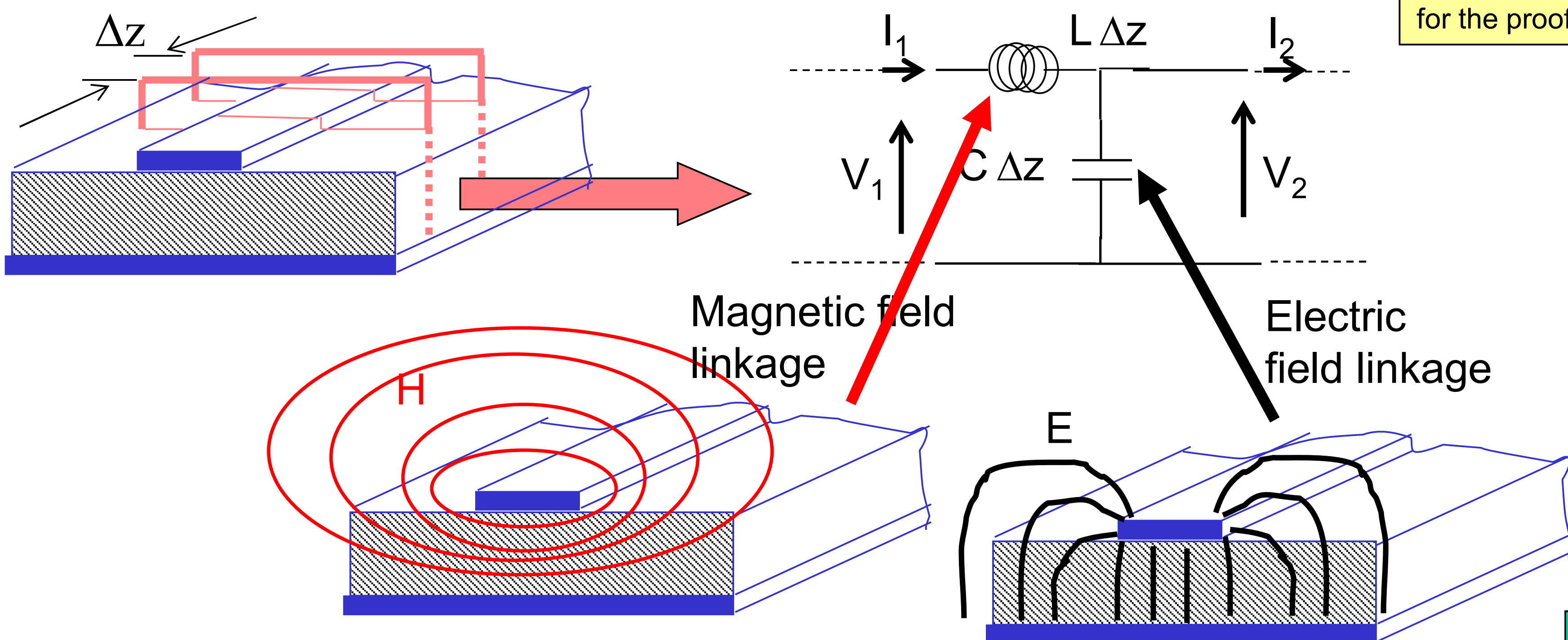
Current profile along conducting trace

Within each segment the current is more or less constant, in and out current is similar. Also the EM field can be considered static.

Distributed Parameters L and C

- The L and C elements in the electrical circuit model for Tline is due to magnetic flux linkage and electric field linkage between the conductors.
- It is more convenient to specify the inductance and capacitance relative to the segment length. Thus in the diagrams below L and C are the inductance and capacitance per 1 meter of the structure.

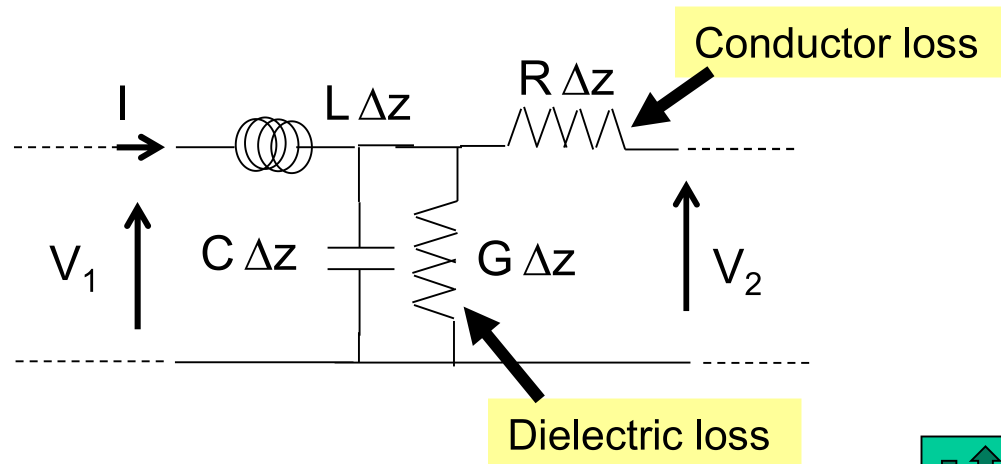
See Appendix II for the proofs.



Distributed Parameters R and G (1)

- Since the conductors has finite conductivity, a small amount of voltage drop occur across both conductors due to non-zero resistance (e.g. conductor loss)
- This is accounted for by the addition of series resistance into the model.
- The resistance R is frequency dependent, and increases with operating frequency due to a phenomenon known as skin effect and proximity effect.
- Generally skin effect implies that as the frequency increases, the current flowing within a conductor tends to 'crowd' to the surface, thereby increasing the effective resistance of the conductor.

Under lossy condition, R, L and G are usually function of frequency, hence the Tline is dispersive.

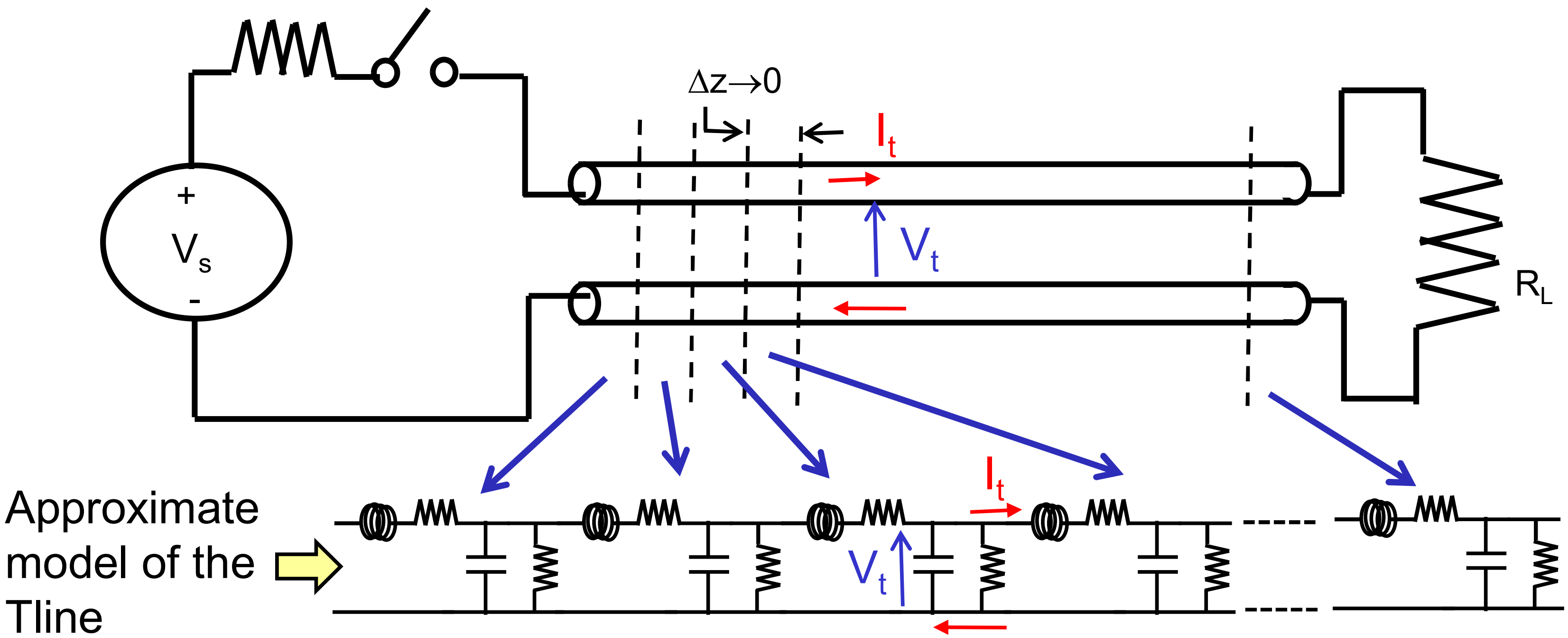


Distributed Parameters R and G (2)

- A non-ideal dielectric also suffers small amount of loss, in the form of dielectric leakage and polarization loss.
- When an AC electric field is imposed on the dielectric, the molecules within the dielectric tends to align with the **E** field. This action results in friction and heating of the dielectric (very small amount).
- Thus part of the incoming electromagnetic energy is converted into heat energy, and the outgoing EM field will be smaller.
- This effect can be modeled by the inclusion of a shunt conductance, G in the model. Again G is typically dependent of frequency.
- The equations for finding L, C, R, G under low loss condition are given in the following slide.

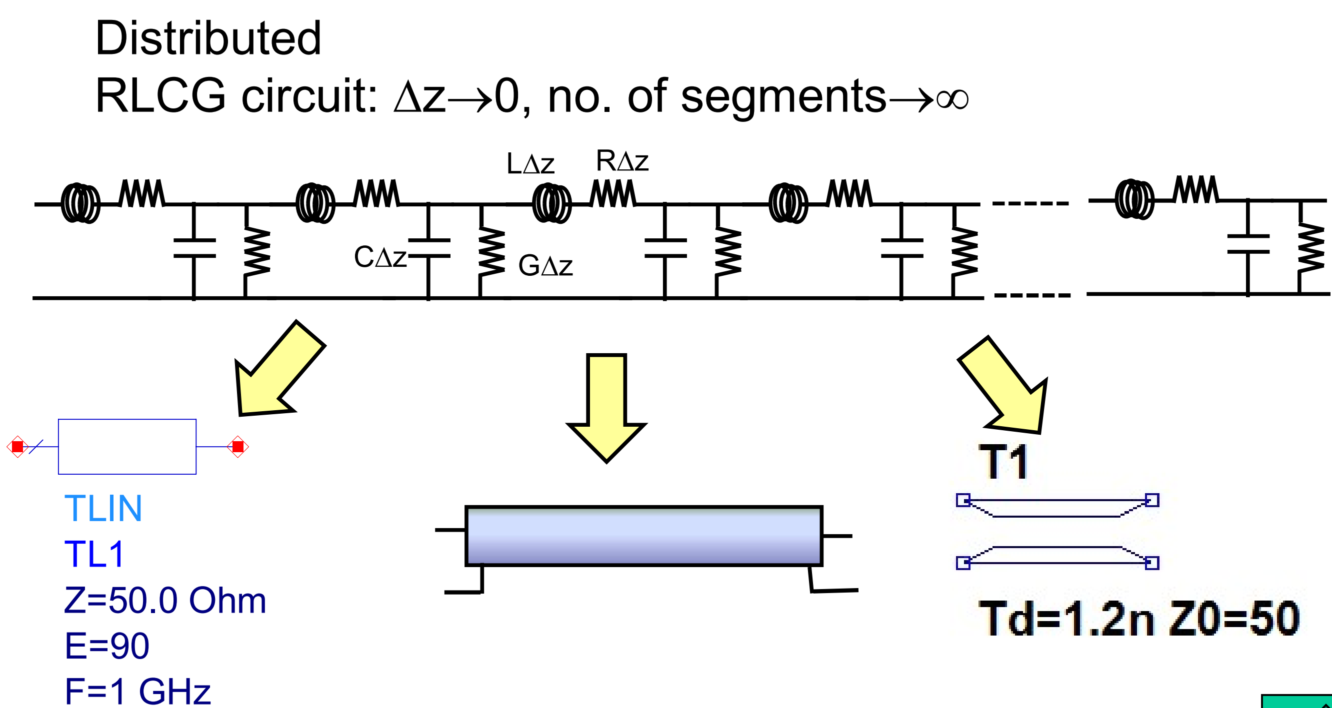
Distributed RLCG Model for Transmission Line (1)

- Thus a TEM or quasi-TEM mode transmission line can be considered as a cascade of many of these RLCG circuit sections. Working with circuit theory and circuit elements is much easier than working with **E** and **H** fields using Maxwell equations.



Distributed RLCG Model for Transmission Line (2)

- In order for this RLCG model for Tline to be valid from low to very high frequency, each segment length must approach zero, and the number of segment needed to accurately model the Tline becomes infinite.
- Under the above condition, the electrical circuit model for Tline is commonly known as Distributed RLCG Circuit Model.



Some special symbol used to represent distributed RLCG circuit in simulation software

Finding the Distributed RLCG Values

- The RLCG parameters can be obtained from the normal definition of inductance, resistance and capacitance. Or it can be obtained via energy consideration as follows:

$$L = \frac{\mu}{|I_t|^2} \iiint_V \left|\vec{H}_t\right|^2 dv \qquad C = \frac{\varepsilon'}{|V_t|^2} \iiint_V \left|\vec{E}_t\right|^2 dv$$

This indicates the volume enclosing the conductors

$$R = \frac{1}{\sigma_c \delta_s |I_t|^2} \int_{C_1+C_2} \left|\vec{H}_t\right|^2 dl \qquad G = \frac{\omega\varepsilon''}{|V_t|^2} \iiint_V \left|\vec{E}_t\right|^2 dv$$

(3.1a)

- See Section 3.9 of Collin [1].
- V is the volume surrounding the conductors with length of 1 meter along z axis.
- C_1 and C_2 are the paths surrounding the surface of conductor 1 and 2.

$$\varepsilon' = \varepsilon_r \varepsilon_o \qquad \varepsilon'' = \varepsilon_r \varepsilon_o \boxed{\tan\delta}$$

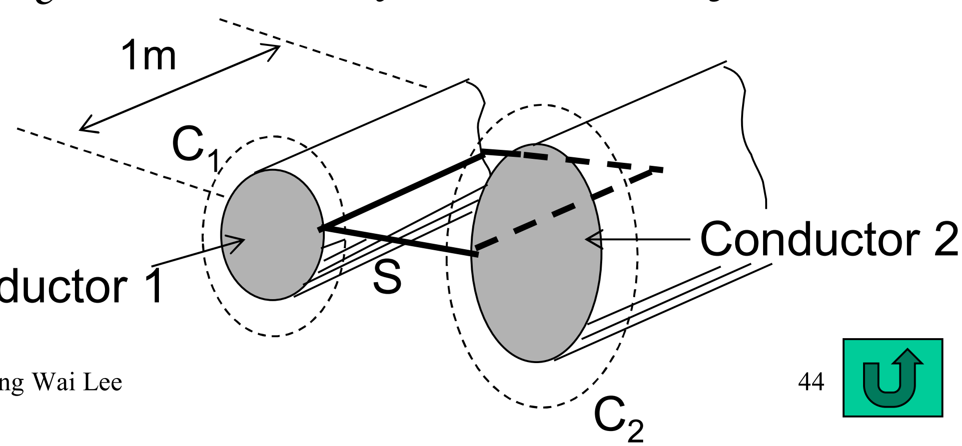
Loss tangent of the dielectric

$$\delta_s = \text{skin depth} = \sqrt{\frac{2}{\omega\sigma_c\mu}}$$

(3.1b)

$$\sigma_c = \text{conductivity of metalic object}$$

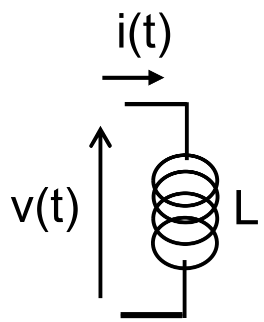
Skin depth and G depend on frequency



Note that conductor loss results in R, while dielectric loss results in G.

Example of Finding the L Parameters From Energy Consideration

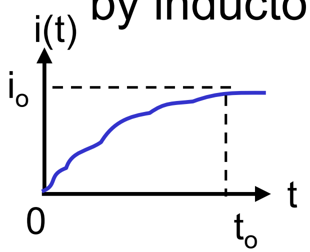
Extra



The instantaneous power absorbed by an inductor L is:

$$P_{ind}(t) = v(t)i(t)$$

Assuming i(t) increases from 0 at t = 0 to I_o at t = t_o, total energy stored by inductor is:

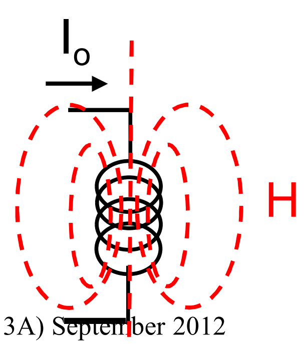


$$E_{ind} = \int_0^{t_o} P_{ind}(\tau)d\tau = \int_0^{t_o} v(\tau)i(\tau)d\tau = \int_0^{t_o} L\frac{di}{d\tau}(\tau)i(\tau)d\tau$$

$$\Rightarrow E_{ind} = \int_0^{I_o} Li \cdot di = \frac{1}{2}LI_o^2$$

This energy stored by the inductor is contained within the magnetic field created by the current (for instance, see D.J. Griffiths, "Introductory electrodynamics", Prentice Hall, 1999). Thus from EM theory the stored magnetic energy is:

$$E_H = \frac{\mu}{2}\iiint_V \left|\vec{H}\right| dxdydz$$



Both energy are the same, hence:

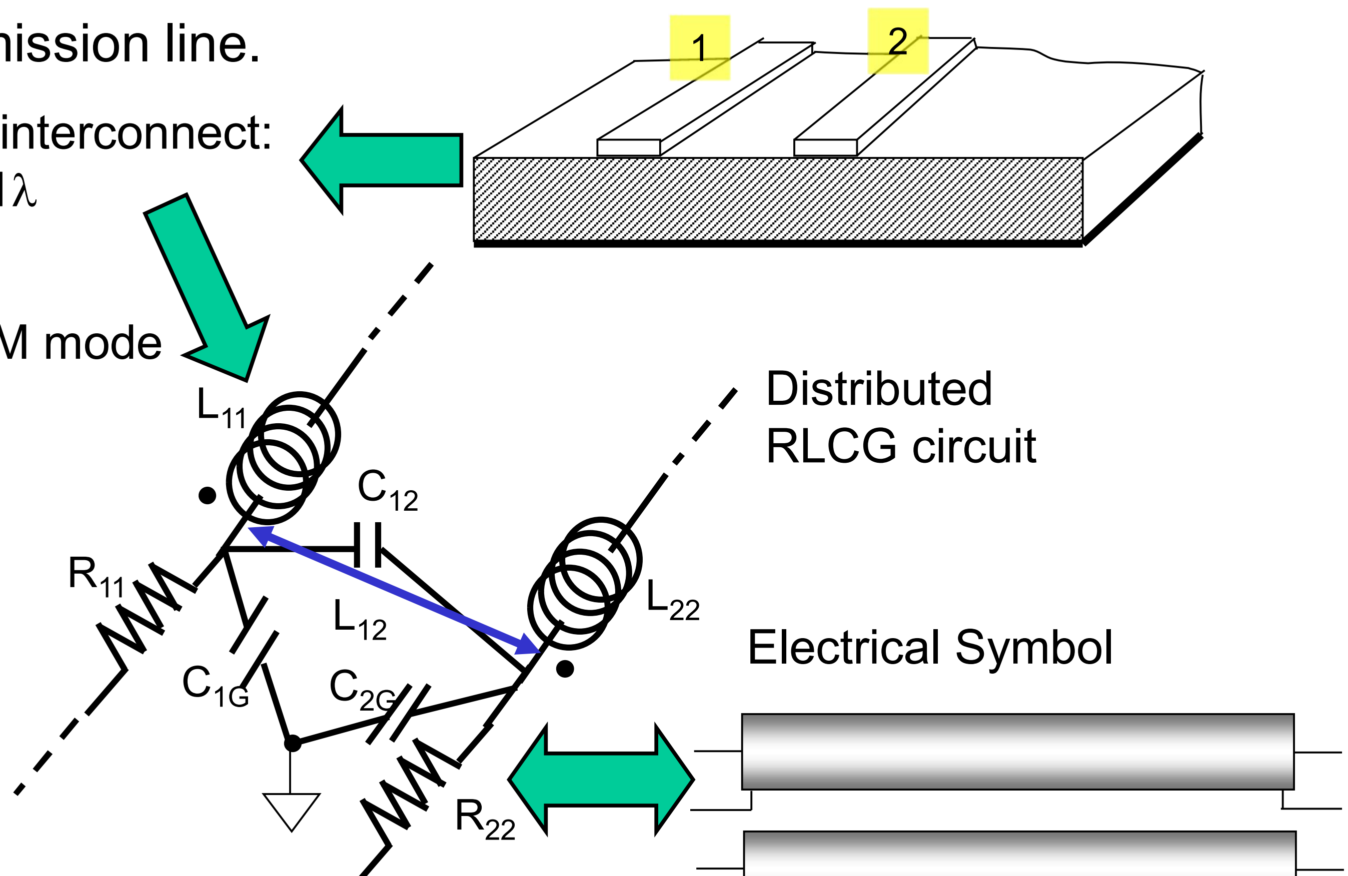
$$E_{ind} = E_H$$

$$\Rightarrow \frac{1}{2}LI_o^2 = \frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dxdydz$$

$$\Rightarrow L = \frac{\mu}{I_o^2}\iiint_V \left|\vec{H}\right|^2 dxdydz$$

Multi-Conductor Transmission Line Equivalent Circuit

- The distributed RLCG model can also be extended to the case of multi-conductor transmission line.



$$C_{1G} = C_{11} - C_{12}$$
$$C_{2G} = C_{22} - C_{12}$$

Parameters:
Per unit length
R, L, C, G matrices.
Usually as a function
Frequency.

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{bmatrix} \quad \begin{bmatrix} C_{11} & -C_{12} \\ -C_{12} & C_{22} \end{bmatrix}$$

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \quad \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

Voltage and Current Along Transmission Line – The Telegraphic Equations

- Much like the EM field in the physical model of the Tline is governed by Maxwell's Equations, we could show that the instantaneous transverse voltage V_t and current I_t on the distributed RLCG model are governed by a set of partial differential equations (PDE) called the Telegraphic Equations (See derivation in Appendix 2 - Advanced Concepts).
- For simplicity we will drop the subscript 't' from now.

In time-domain

$$\frac{\partial v}{\partial z} = -Ri - L\frac{\partial i}{\partial t}$$

$$\frac{\partial i}{\partial z} = -Gv - C\frac{\partial v}{\partial t} \qquad \textit{(3.2a)}$$

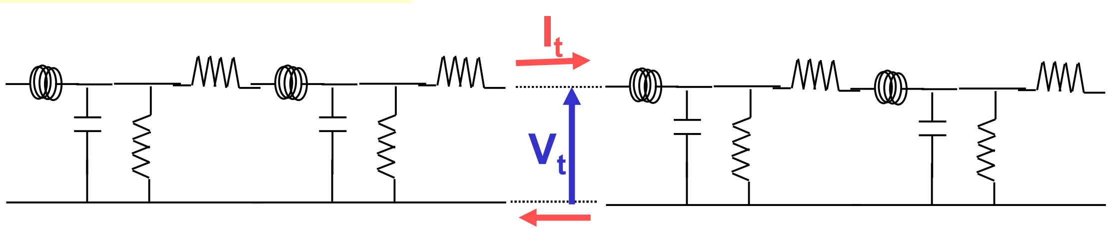
Fourier Transforms

Inverse Fourier Transforms

In time-harmonic form

$$\frac{\partial V}{\partial z} = -(R + j\omega L)I = -ZI$$

$$\frac{\partial I}{\partial z} = -(G + j\omega C)V = -YV \qquad \textit{(3.2b)}$$



Distributed RLCG circuit

Solutions of Telegraphic Equations (1)

- The expressions for V(z) and I(z) that satisfy the time-harmonic form of Telegraphic Equations (3.2b) are given as:

Wave travelling in +z direction

Wave travelling in -z direction

$$I(z) = I_o^+ e^{-\gamma z} + I_o^- e^{\gamma z} \quad \textbf{(3.3a)}$$

$$V(z) = \boxed{V_o^+ e^{-\gamma z}} + \boxed{V_o^- e^{\gamma z}} \quad \textbf{(3.3b)}$$

Attenuation factor

Phase factor

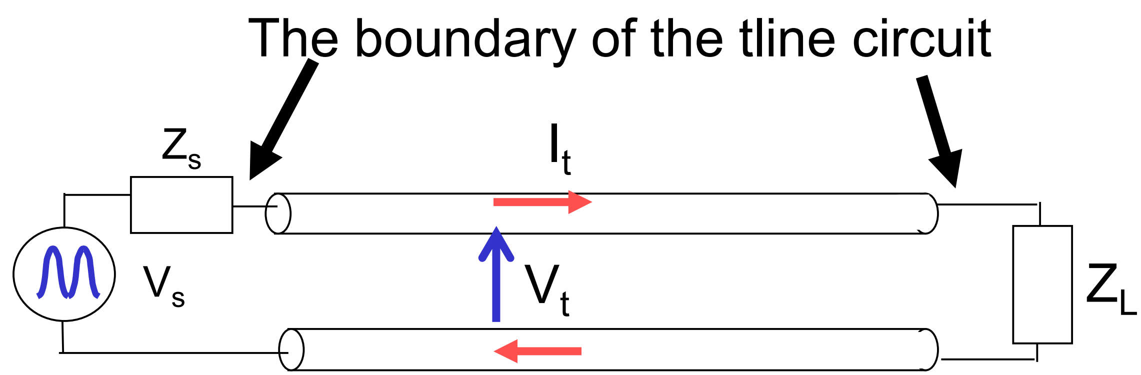
$$\gamma = \alpha(\omega) + j\beta(\omega) = \sqrt{(R + j\omega L)(G + j\omega C)} \quad \textbf{(3.3c)}$$

Propagation coefficient

- The expressions for current and voltage phasors in equations (3.3a) and (3.3b) describe propagating sinusoidal waves along the transmission line.

Solutions of Telegraphic Equations (2)

- V_o^+, V_o^-, I_o^+, I_o^- are unknown constants. When we study transmission line circuit, we will see how V_o^+, V_o^-, I_o^+, I_o^- can be determined from the 'boundary' of the Tline.
- For the rest of this discussions, exact values of these constants are not needed.



Signal Propagation on Transmission Line

- Considering sinusoidal sources, the expression for phasors V(z) and I(z) can be written in time domain as:

$$V_o^+ = \left|V_o^+\right| e^{j\phi_+} \qquad I_o^+ = \left|I_o^+\right| e^{j\theta_+}$$

$$v(z,t) = \left|V_o^+\right| \cos(\omega t - \beta z + \phi_+) e^{-\alpha z} + \left|V_o^-\right| \cos(\omega t + \beta z + \phi_-) e^{\alpha z} \quad \textbf{(3.4a)}$$

$$i(z,t) = \left|I_o^+\right| \cos(\omega t - \beta z + \theta_+) e^{-\alpha z} + \left|I_o^-\right| \cos(\omega t + \beta z + \theta_-) e^{\alpha z} \quad \textbf{(3.4b)}$$

- From the solution of the Telegraphic Equations, we can deduce a few properties of the equivalent voltage *v(z,t)* and current *i(z,t)* on a Tline structure.
  - *v(z,t)* and *i(z,t)* propagate, a signal will take finite time to travel from one location to another.
  - One can define an impedance, called the characteristic impedance of the line, it is the ratio of voltage wave over current wave.
  - That the traveling V_t and I_t experience dispersion and attenuation.
  - Other effects such as reflection to be discussed in later part.

Characteristic Impedance (Z_c)

- An important parameter in Tline is the ratio of voltage over current, called the Characteristic Impedance, Z_c.
- Since the voltage and current are waves, this ratio can be only be computed for voltage and current traveling in similar direction and at same location.

From Telegraphic Equations →

$$\frac{\partial V}{\partial z} = -ZI$$

$$\Rightarrow -\gamma V_o^+ e^{-\gamma z} = -Z I_o^+ e^{-\gamma z}$$

$$\Rightarrow V_o^+ = \frac{Z}{\gamma} I_o^+$$

$$Z_c = \frac{V_o^+ e^{-\gamma z}}{I_o^+ e^{-\gamma z}} = \frac{V_o^+}{I_o^+} = \frac{Z}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad \textbf{(3.5)}$$

Or

$$Z_c = \frac{V_o^-}{-I_o^-} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

← A function of frequency

Propagation Velocity (v_p)

- Compare the expression for v(z,t) of (3.4a) with a general expression for a traveling wave in positive and negative z direction:

$$V(z,t) = \left|V_o^+\right|\cos(\omega t - \beta z + \phi_+)e^{-\alpha z} + \left|V_o^-\right|\cos(\omega t + \beta z + \phi_-)e^{\alpha z}$$

Compare

$$f(\omega t - \beta z)$$

A general function describing propagating wave in +z direction

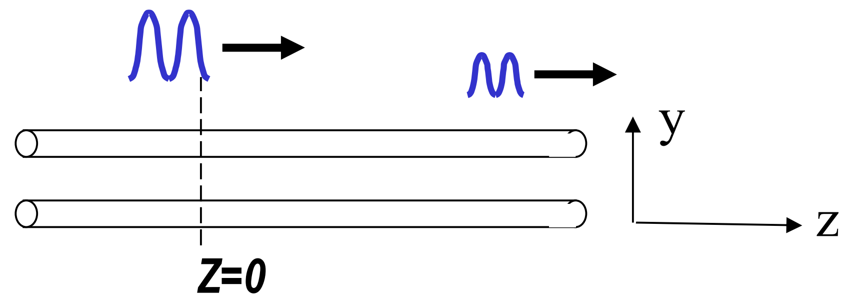
- And recognizing that both v(z,t) and i(z,t) are propagating waves, the phase velocity is given by:

$$v_p = \frac{\omega}{\beta} \qquad \textbf{(3.6)}$$

Attenuation (α)

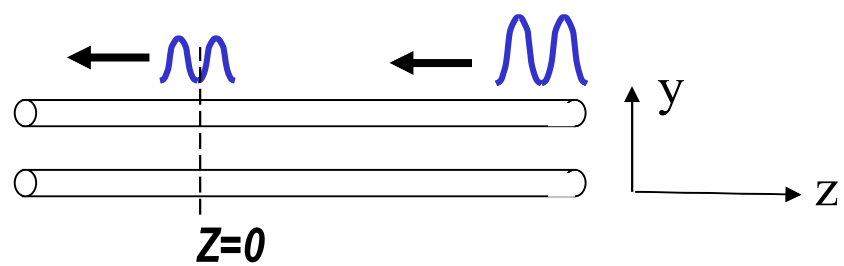
- The attenuation factor α decreases the amplitude of the voltage and current wave along the Tline.
- For +ve traveling wave:

$$\left|V_o^+\right|\cos\left(\omega t-\beta z+\phi_+\right)e^{-\alpha z}$$



- For -ve traveling wave:

$$\left|V_o^-\right|\cos\left(\omega t+\beta z+\phi_-\right)e^{\alpha z}$$



Dispersion (1)

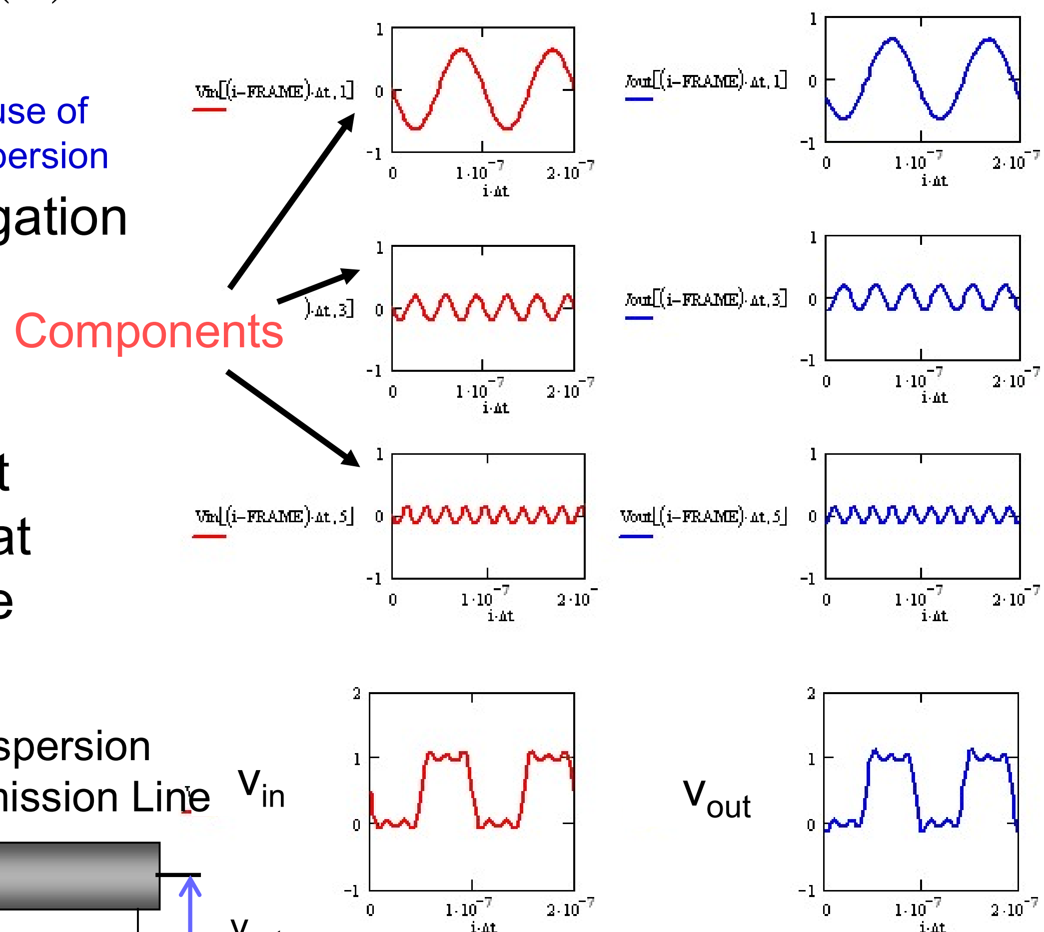
Since $\gamma = \gamma(\omega) = \alpha(\omega) + j\beta(\omega)$

$$v_p = \frac{\omega}{\beta(\omega)}$$

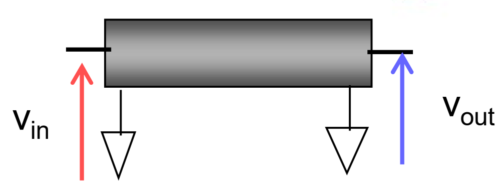
Cause of dispersion

Video

- We observe that the propagation velocity is a function of the wave's frequency.
- Different component of the signal propagates at different velocity (and also attenuate at different rate), resulting in the envelope of the signal being distorted at the output.



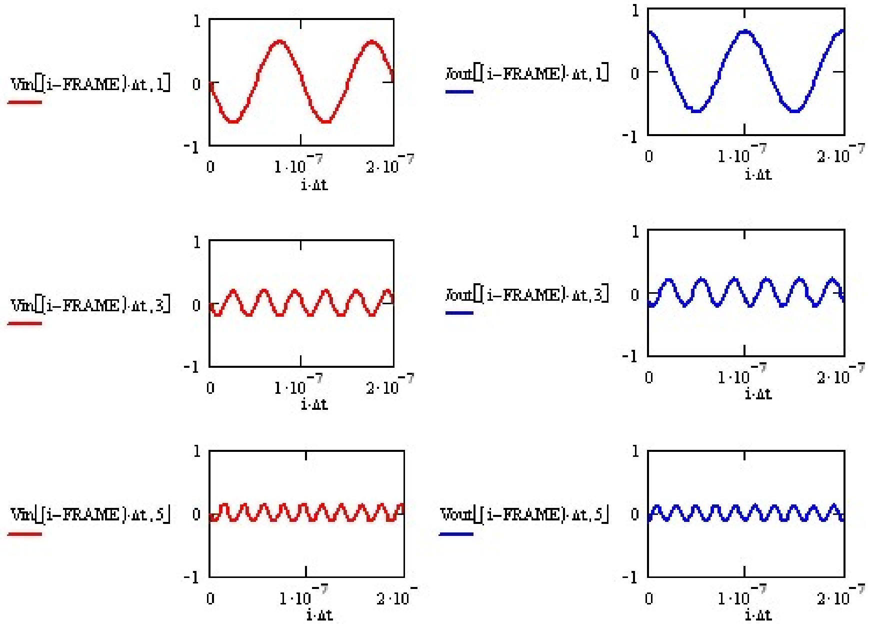
Low dispersion Transmission Line



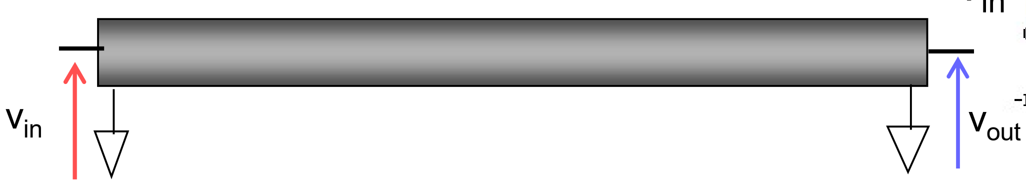
Dispersion (2)

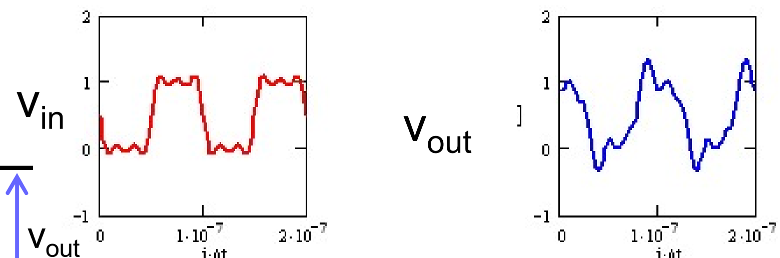
- Dispersion causes distortion of the signal propagating through a transmission line.
- This is particularly evident in a long line.

Video



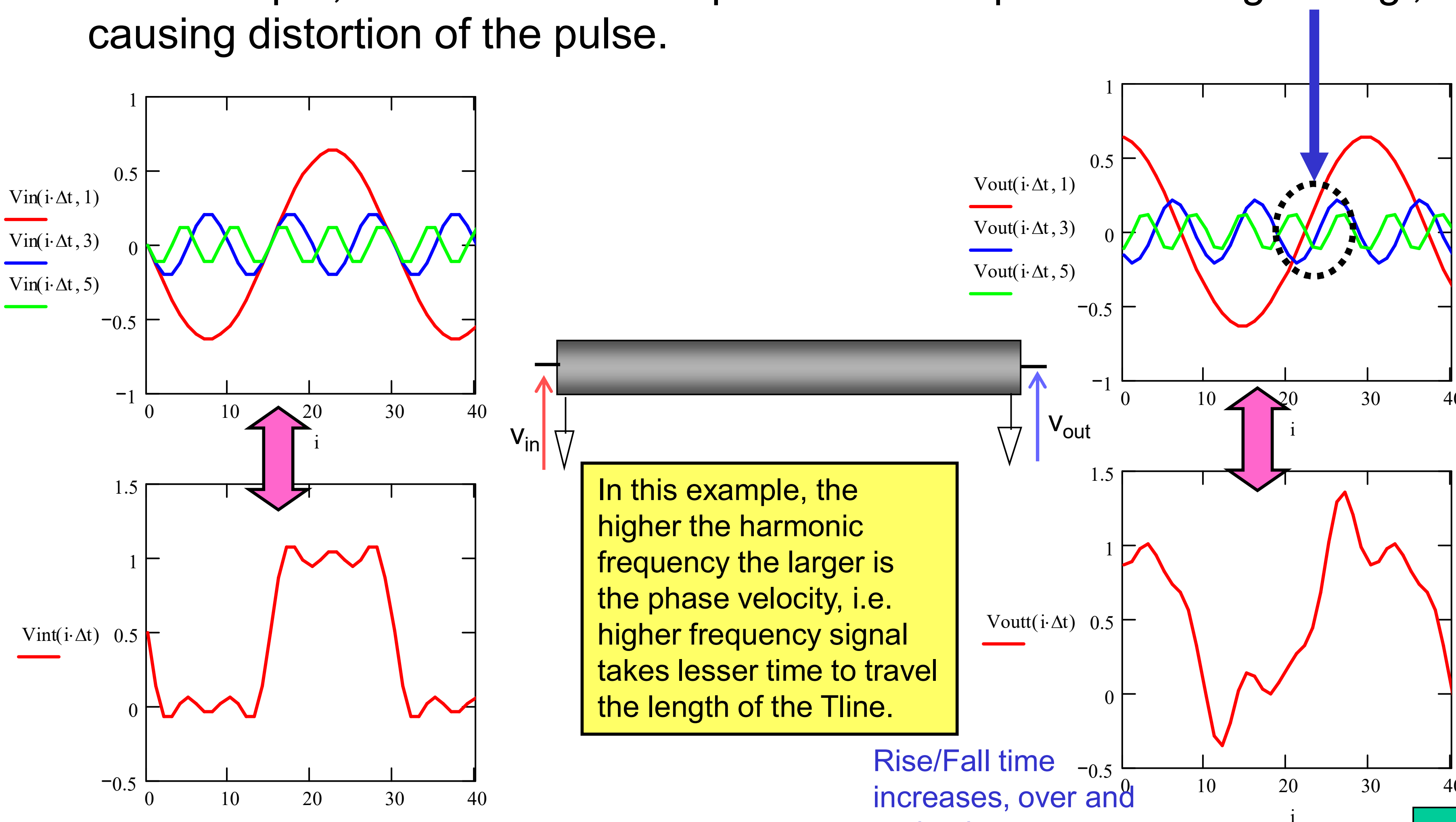
High dispersion Transmission Line



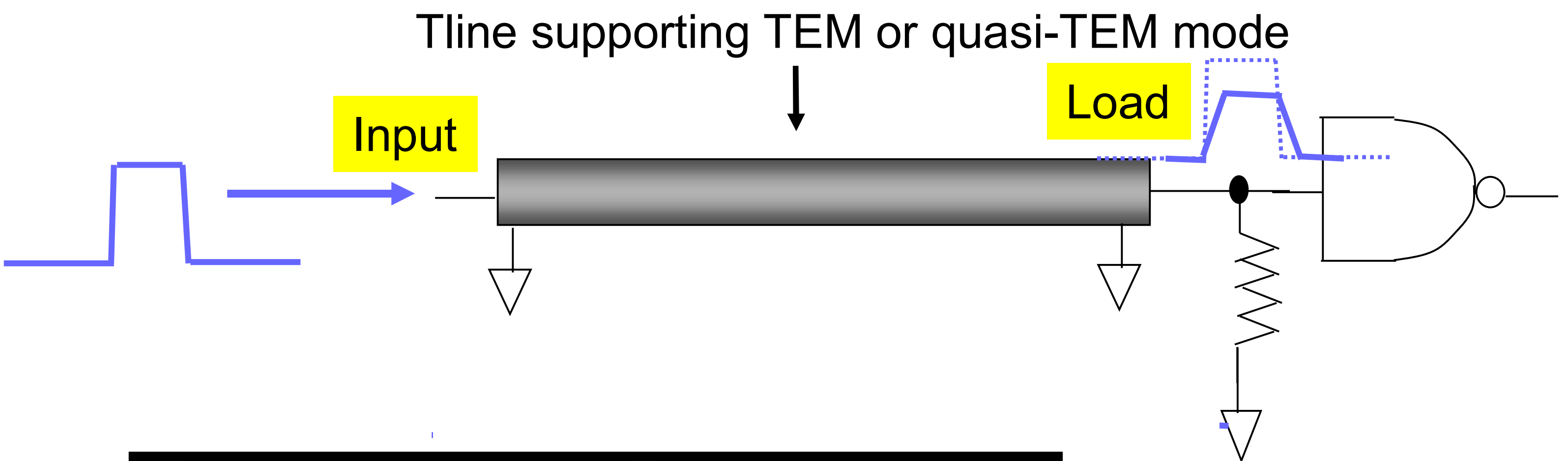


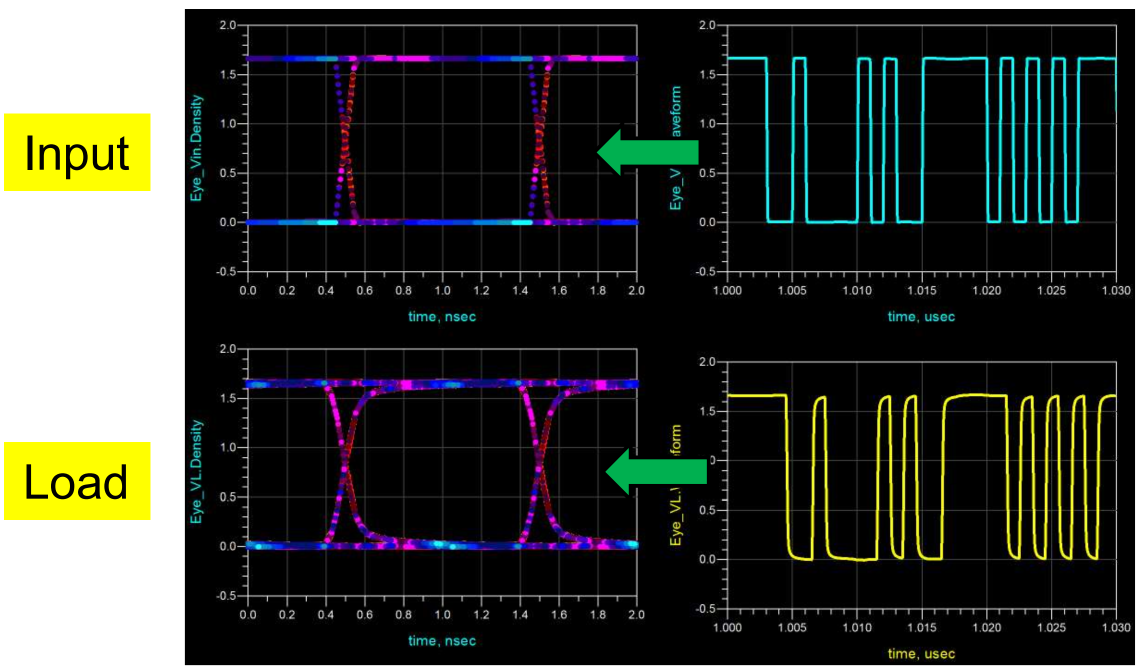
Dispersion (3)

- At the output, the sinusoidal components overlap at the wrong 'timing', causing distortion of the pulse.



The Implications to Pulse Traveling on a Long Transmission Line





Example of waveforms and eye-diagram of 1 Gbps data into a 250 mm long microstrip line (Z_c = 50Ω) on FR4 substrate.

The Lossless Transmission Line

- When the tline is lossless, R= 0 and G = 0.
- We have:

$$\gamma = j\beta = j\omega\sqrt{LC}$$

$$Z_c = \sqrt{\frac{L}{C}}$$

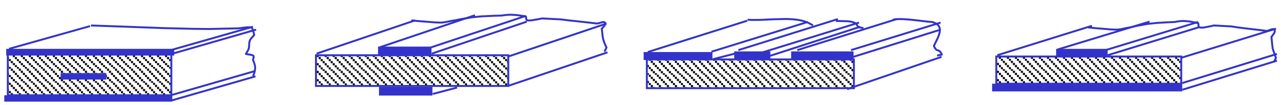
$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

- So the lossless transmission line has no attenuation, no dispersion and the characteristic impedance is real.
- Since lossless Tline is an ideal, in practical situation we try to reduce the loss to as small as possible, by using gold-plated conductor, and using good quality dielectric (low loss tangent).

3.4 – Designing Transmission Line On Printed Circuit Board (PCB) And Related Structures

Stripline Technology (1)

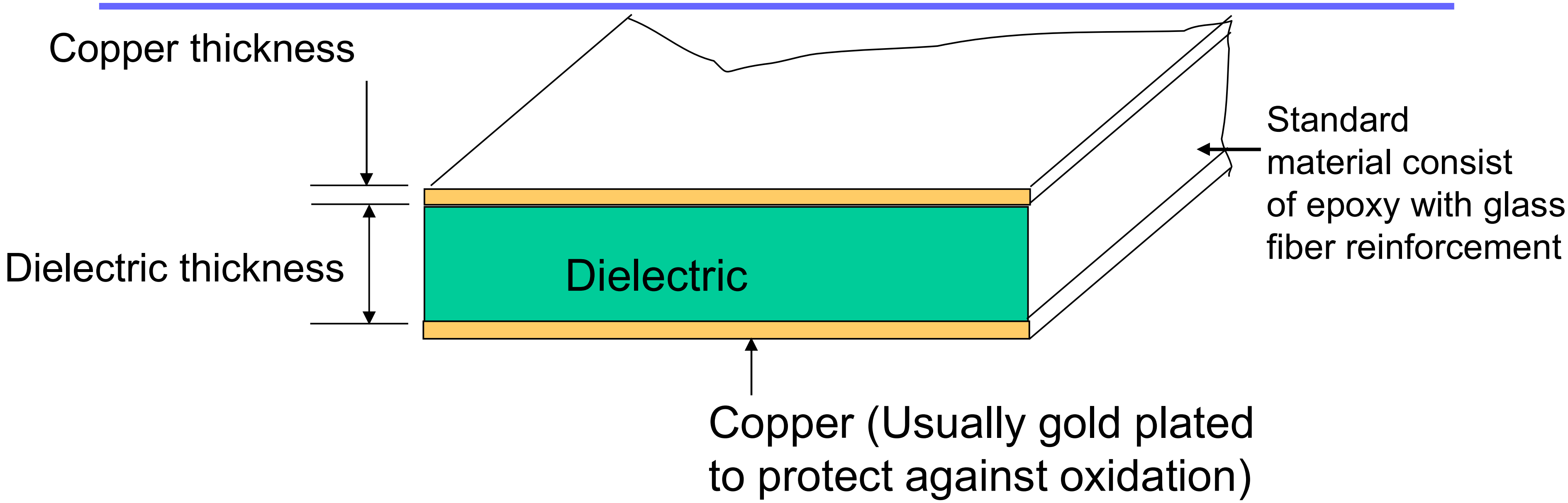
- Stripline is a planar-type Tline that lends itself well to microwave integrated circuit (MIC) and photolithograhpic fabrication.
- Stripline can be easily fabricated on a printed circuit board or semiconductor using various dielectric material such as epoxy resin, glass fiber such as FR4, polytetrafluoroethylene (PTFE) or commonly known as Teflon, Polyimide, aluminium oxide, titanium oxide and other ceramic materials or processes, for instance the low-temperature co-fired ceramic (LTCC).
- Three most common Tline configurations using stripline technology are microstrip line, stripline and co-planar stripline.



Stripline Technology (2)

- A variety of substrates, Thin and Thick-Film technologies can be employed.
- For more information on microstrip line circuit design, you can refer to T.C. Edwards, “Foundation for microstrip circuit design”, 2nd Edition 1992, John Wiley & Sons. (3rd and 4th editions are also available).
- For more information on stripline circuit design, you can consult H. Howe, “Stripline circuit design”, 1974, Artech House.
- For more information on microwave materials and fabrication techniques, you can refer to T.S. Laverghetta, “Microwave materials and fabrication techniques”, 3rd edition 2000, Artech House.

The Substrate or Laminate for Stripline Technology



- Typical dielectric thickness are 32mils (0.8mm), 62mils (1.56mm) for double sided board. For multi-layer board the thickness can be customized from 2 – 62 mils, in 1 mils step.
- Copper thickness is usually expressed in terms of the mass of copper spread over 1 square foot. Standard copper thickness are 0.5, 1.0, 1.5 and 2.0 oz/foot^2.

0.5 oz/foot^2 $\cong$ 0.7mils thick.

1.0 oz/foot2 $\cong$ 1.4mils thick.

2.0 oz /foot2 $\cong$ 2.8mils thick.

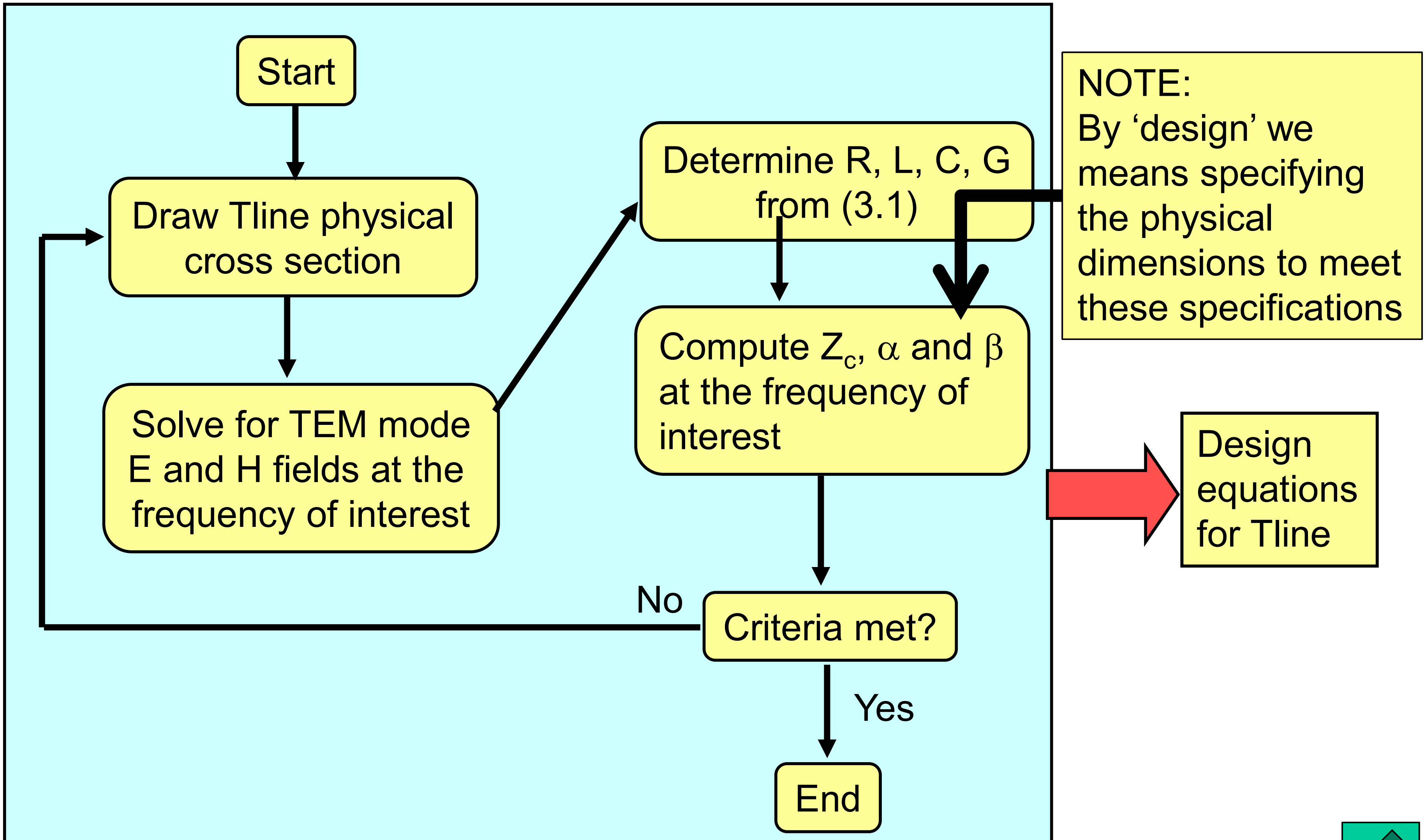
Factors Affecting Choices of Substrates

- Operating frequency.
- Electrical characteristics - e.g. nominal dielectric constant, anisotropy, loss tangent, dispersion of dielectric constant.
- Copper weight (affect low frequency resistance).
- T_g, the glass transition temperature.
- Cost.
- Tolerance.
- Manufacturing Technology - Thin or thick film technology.
- Thermal requirements - e.g. thermal conductivity, coefficient of thermal expansion (CTE) along x,y and z axis.
- Mechanical requirements - flatness, coefficient of thermal expansion, metal-film adhesion (peel strength), flame retardation, chemical and water resistance etc.

Comparison between Various Transmission Lines

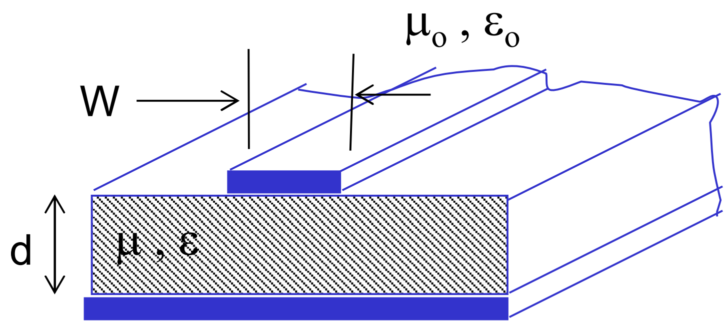
Microstrip line	Stripline	Co-planar line
Suffers from dispersion and non-TEM modes	Pure TEM mode	Suffers from dispersion and non-TEM modes
Easy to fabricate	Difficult to fabricate	Fairly difficult to fabricate
High density trace	Mid density trace	Low density trace
Fair for coupled line structures	Good for coupled line structures	Not suitable for coupled line structures
Need through holes to connect to ground	Need through holes to connect to ground	No through hole required to connect to ground

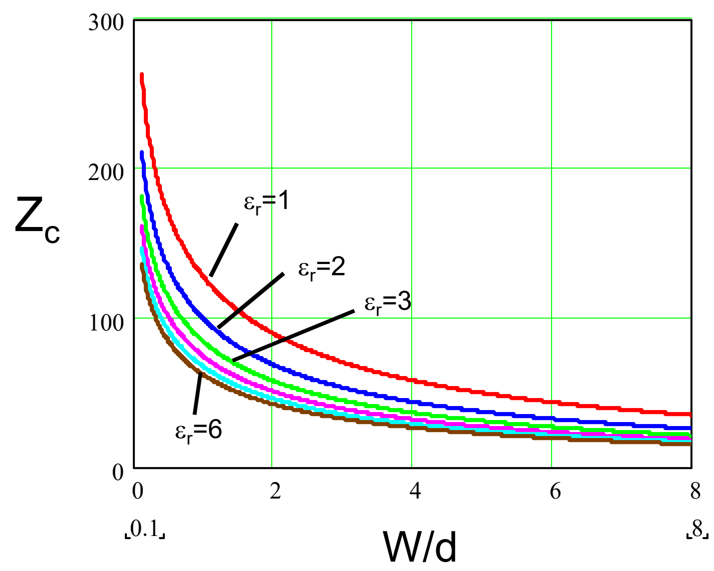
Typical Iterative Flow for Transmission Line Design



Design Equations

- By varying the physical dimensions and using the flow of the previous slide, one can obtain a collection of results (Z_c, α, β) .
- These results can be plotted as points on a graph.
- Curve-fitting techniques can then be used to derive equations that match the results with the physical parameters of the Tline.

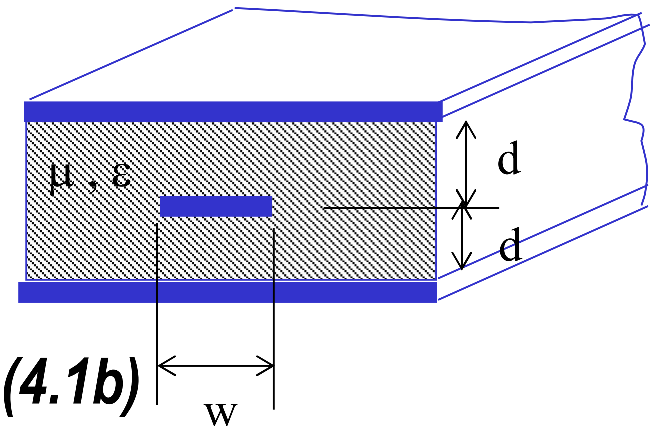




Design Equations for Stripline

Stripline (see reference [3], Chapter 8):

$$Z_c \cong \frac{Z_o}{4} \cdot \frac{K(k)}{K\left(\sqrt{1-k^2}\right)} \qquad \textit{(4.1a)}$$



$$K(x) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-x^2\sin^2\phi}} \qquad k = \left[\cosh\left(\frac{\pi w}{4d}\right)\right]^{-1} \qquad \textit{(4.1b)}$$

Complete elliptic integral of the 2nd kind

$$Z_o = \sqrt{\frac{\mu}{\varepsilon}}$$

$$v_p = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \textit{(4.1c)}$$

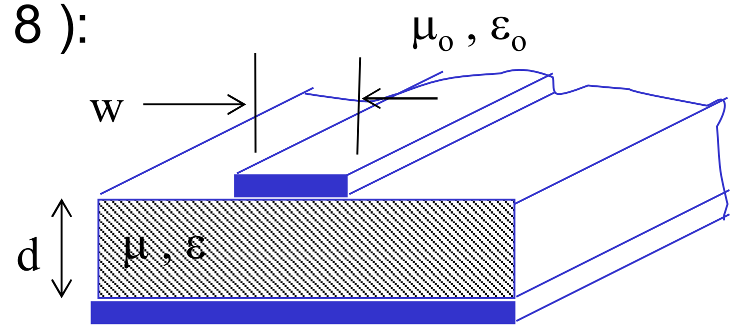
Assumptions:
- TEM or quasi-TEM approximation and very low loss condition.
- Also conductor thickness for the trace is assumed to be zero.

Design Equations for Microstrip Line

Microstrip Line (see reference [3], Chapter 8):

Effective dielectric constant (See Appendix 3)

$$\varepsilon_{eff} = 1 + \frac{\varepsilon_r - 1}{2}\left[1 + \frac{1}{\sqrt{1 + \frac{10d}{w}}}\right] \quad \textbf{(4.2a)}$$



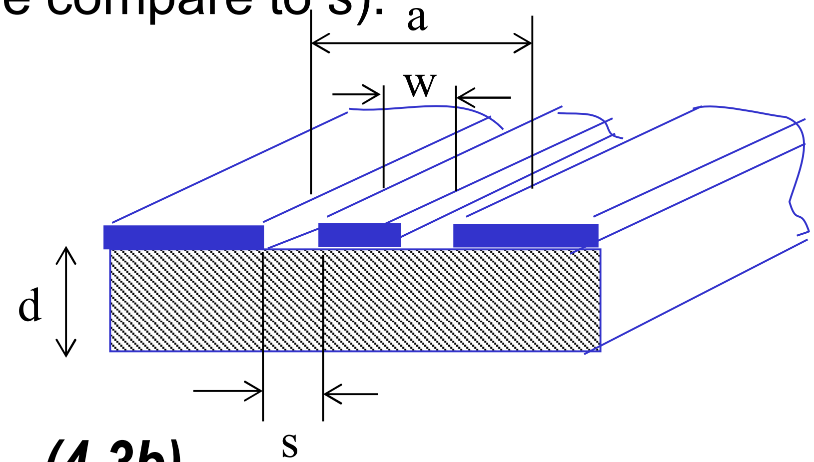
$$Z_c = \frac{377}{\sqrt{\varepsilon_{eff}}}\left[\frac{w}{d} + 1.98\left(\frac{w}{d}\right)^{0.172}\right]^{-1} \quad \textbf{(4.2b)}$$

$$v_p = \frac{1}{\sqrt{\mu \varepsilon_{eff} \varepsilon_o}} \quad \textbf{(4.2c)}$$

Assumptions:
- TEM or quasi-TEM approximation and very low loss condition.
- Also conductor thickness for the trace is assumed to be zero.

Design Equations for Co-planar Line

Co-planar Line (see [3], Assume *d* is large compare to *s*):



$$\varepsilon_{eff} = \frac{\varepsilon_r + 1}{2} \qquad \textbf{(4.3a)}$$

$$Z_c \cong \frac{Z_o}{\pi\sqrt{\varepsilon_{eff}}} \ln\left(2\sqrt{\frac{a}{w}}\right) \quad \text{for } 0 < \frac{w}{a} < 0.173$$

$$Z_c \cong \frac{\pi Z_o}{4\sqrt{\varepsilon_{eff}}} \left[\ln\left(2\frac{1+\sqrt{w/a}}{1-\sqrt{w/a}}\right)\right]^{-1} \quad \text{for } 0.173 < \frac{w}{a} < 1 \qquad \textbf{(4.3b)}$$

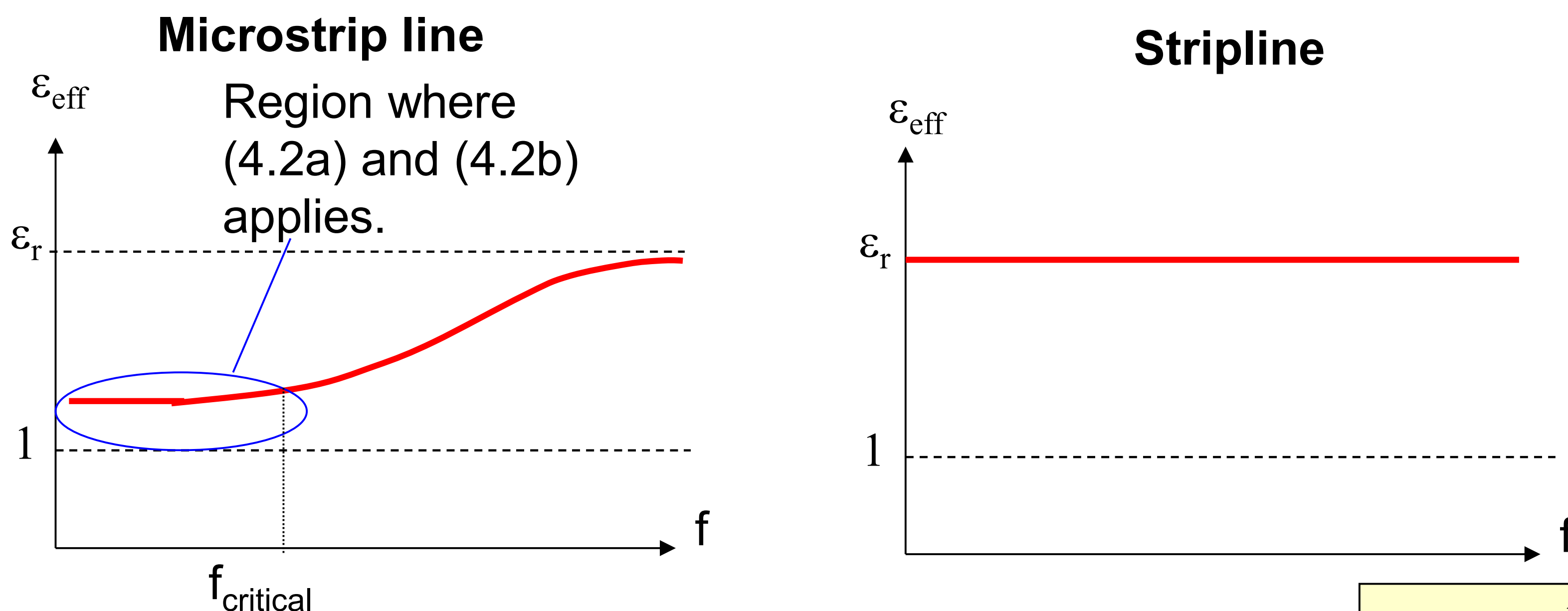
$$Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} \qquad v_p = \frac{1}{\sqrt{\mu\varepsilon_{eff}\varepsilon_o}} \qquad \textbf{(4.3c)}$$

Assumptions:
- TEM or quasi-TEM approximation and very low loss condition.
- Also conductor thickness for the trace is assumed to be zero.
- *d* > 4*s*

Dispersive Property of Microstrip and Co-planar Lines (1)

- The actual propagation mode for microstrip and co-planar lines are a combination TM and TE modes. Both modes are dispersive. The phase velocity of the EM wave is dependent on the frequency (see references [1] and [2], or discussion in Appendix 1).
- This change in phase velocity is reflected by effective dielectric constant that changes with frequency.
- At low frequency ($f < f_{critical}$), when the propagation modes for microstrip and co-planar lines approaches quasi-TEM, the phase velocity is almost constant.
- Appendix 1 shows a simple method to estimate $f_{critical}$ for microstrip and co-planar lines.
- Thus $f_{critical}$ is usually taken as the upper frequency limit for microstrip and co-planar lines. Typical value is 5 – 100 GHz depending on dielectric thickness.
- Stripline does not experience this effect as theoretically it can support pure TEM mode.

Dispersive Property of Microstrip and Co-planar Line (2)



Limit for quasi-TEM approximation, see Example A2 and Example 4.1 on how to estimate $f_{critical}$

$$v_p = \frac{1}{\sqrt{\mu \varepsilon_{eff} \varepsilon_o}}$$

For microstrip line, the EM field is partly in the air and dielectric. Hence the effective dielectric ε_{eff} constant is between those of air and dielectric.

Note:
Beyond $f_{critical}$ the concept of characteristic impedance becomes meaningless.

Example 4.1 - Microstrip Line Design

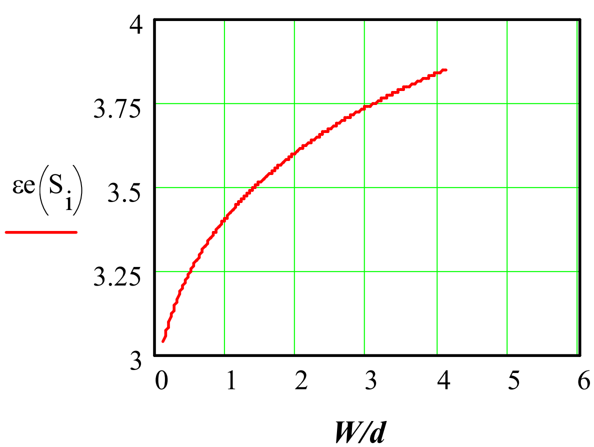
- In this example we will use the design equations formula to design some transmission line. This can be done on a spreadsheet program.
- (a) Design a 50Ω microstrip line on a dielectric substrate with two copper layers, given that d = 0.51 mm (20mils) and dielectric constant = 4.6. Assume low loss.

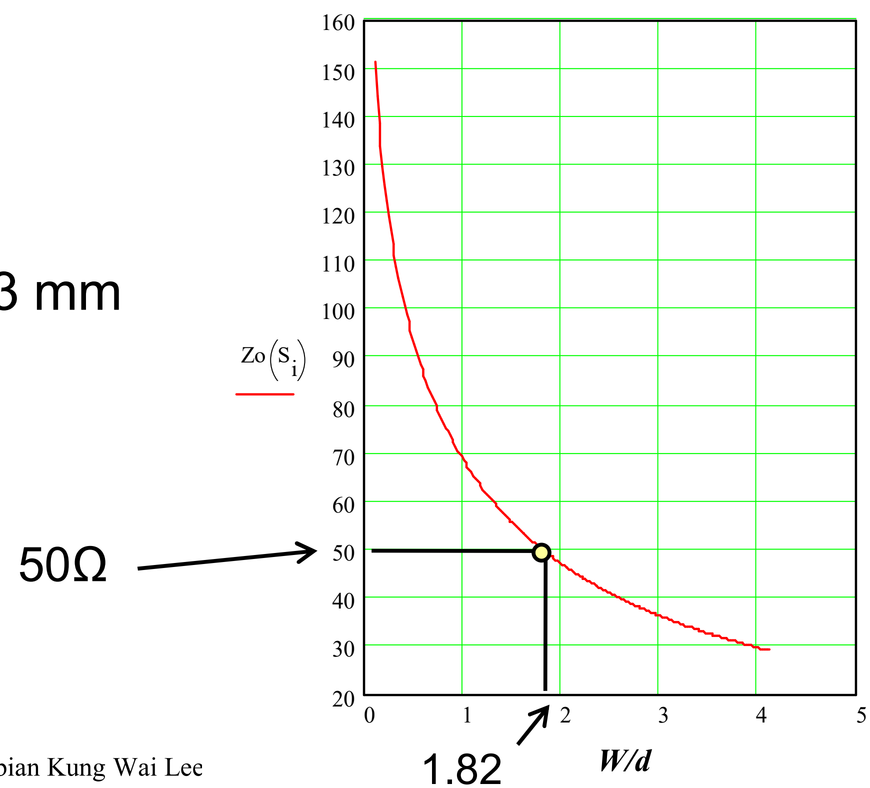
•Steps...

•Plot out Z_c versus (w/d).

• From the curve, we see that w/d = 1.82 for 50Ω.

• Thus w = 1.8 x 0.51 mm = 0.93 mm



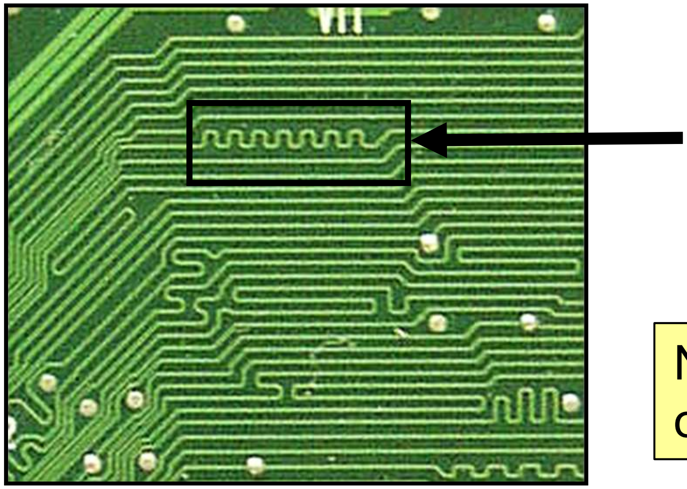


Example 4.1 Cont...

- (b) If the length of the Tline is 6.5 cm, find the propagation delay.
- (c) Using the *l* < 0.05λ rule, find the frequency range where the microstrip line can be represented by lumped RLCG circuit (e.g. the line becomes short interconnect).

From ε_{eff} versus w/d, we see that ε_{eff} = 3.51 at w/d = 1.82. Therefore:

$$v_p = \frac{1}{\sqrt{\varepsilon_o \mu_o \cdot 3.51}} = 1.601 \times 10^8 ms^{-1} \Rightarrow T_{delay} = \frac{0.065}{v_p} \cong \underline{406 ps}$$



An example of Microstrip delay line on PCB using 'serpentine' trace.

Note that this delay only depends on w/d ratio, not on d.

Example 4.1 Cont...

- **(**c) Using the *l* < 0.05λ rule, find the frequency range where the microstrip line can be represented by lumped RLCG circuit (e.g. the line becomes short interconnect).

To be represented as lumped, the wavelength λ must be > 20 x *Length:*

$$\lambda > 20 \cdot l = 1.30\,\mathrm{m}$$

$$\Rightarrow \frac{v_p}{f} > 1.30$$

$$\Rightarrow f < \frac{v_p}{1.30} = 123.2\,\mathrm{MHz}$$

$$f_{lumped} = 123.2\,\mathrm{MHz}$$

* Using the result from Part 2:
For digital pulse, this corresponds to a minimum rise/fall time of:

$$\frac{1}{\pi\tau} = f_{lumped} < 123.2 \times 10^6$$

$$\Rightarrow \tau > \frac{1}{123.2 \times 10^6 \times \pi} = 2.59\,\mathrm{nsec}$$

Example 4.1 Cont...

- (d) When the low loss microstrip line is considered short, derived its equivalent LC network.

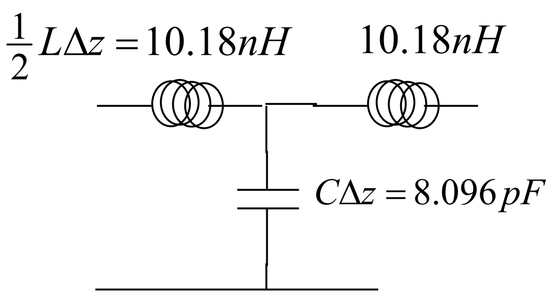
$$Z_c v_p = \sqrt{\frac{L}{C}} \times \frac{1}{\sqrt{LC}} = \frac{1}{C}$$

$$L = {Z_c}^2 C = 313.2\text{nH/m}$$

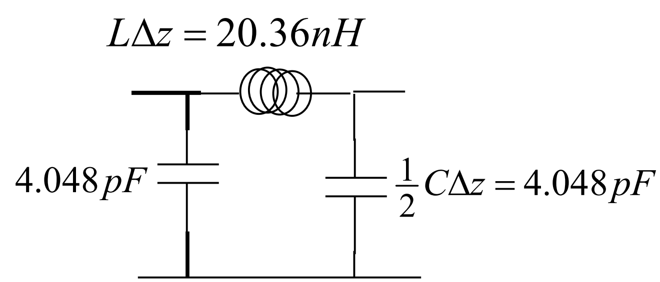
$$C = \frac{1}{Z_c v_p} = \frac{1}{50 \times 1.601 \times 10^8} = 124.6\text{pF/m}$$

The low-frequency LC equivalent for this interconnect: $C \cdot 0.065 = 8.096\,pF$

$$L \cdot 0.065 = 20.36\,nH$$



or



Example 4.1 Cont...

- (e) Finally estimate $f_{critical}$, the limit where quasi-TEM approximation begins to break down.

Again using the criteria that wavelength > 20*d* for quasi-TEM mode to propagate:

$$\lambda = \frac{v_p}{f_{critical}} > 20d = 0.01 \Rightarrow f_{critical} < \frac{v_p}{0.01} = 15.5GHz$$

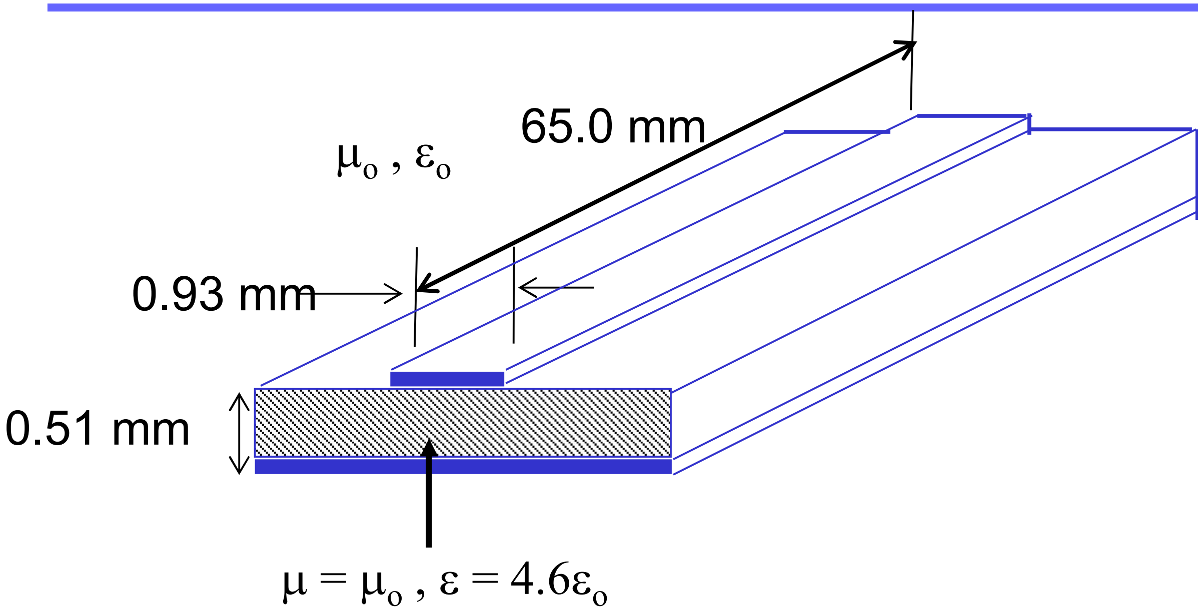
We see that to increase $f_{critical}$, smaller thickness d should be used.

* Using the result from Part 2:
For digital pulse, this corresponds
To a minimum rise/fall time of:

$$\frac{1}{\pi\tau} = f_{dist} < 15.5\times10^9$$

$$\Rightarrow \tau > \frac{1}{15.5\times10^9\times\pi} \cong 0.02\,\text{nsec} = 20\text{psec}$$

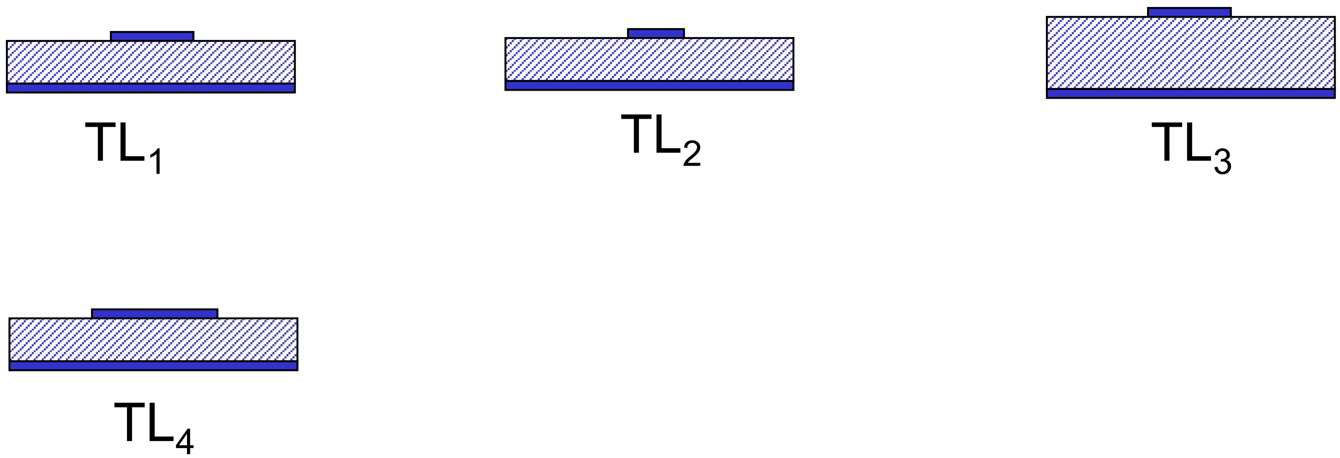
Example 4.1 Cont...



Z_c = 50Ω
v_p = 1.601x10^8 m/s
t_{delay} = 406psec
Maximum usable frequency = $f_{critical}$ = 15.5 GHz. (We can interpret this as the bandwidth of this microstrip line)
Or t_{rise} (t_{fall}) = 0.02nsec
Short interconnect limit = 123.2 MHz or t_{rise} (t_{fall}) = 2.59nsec

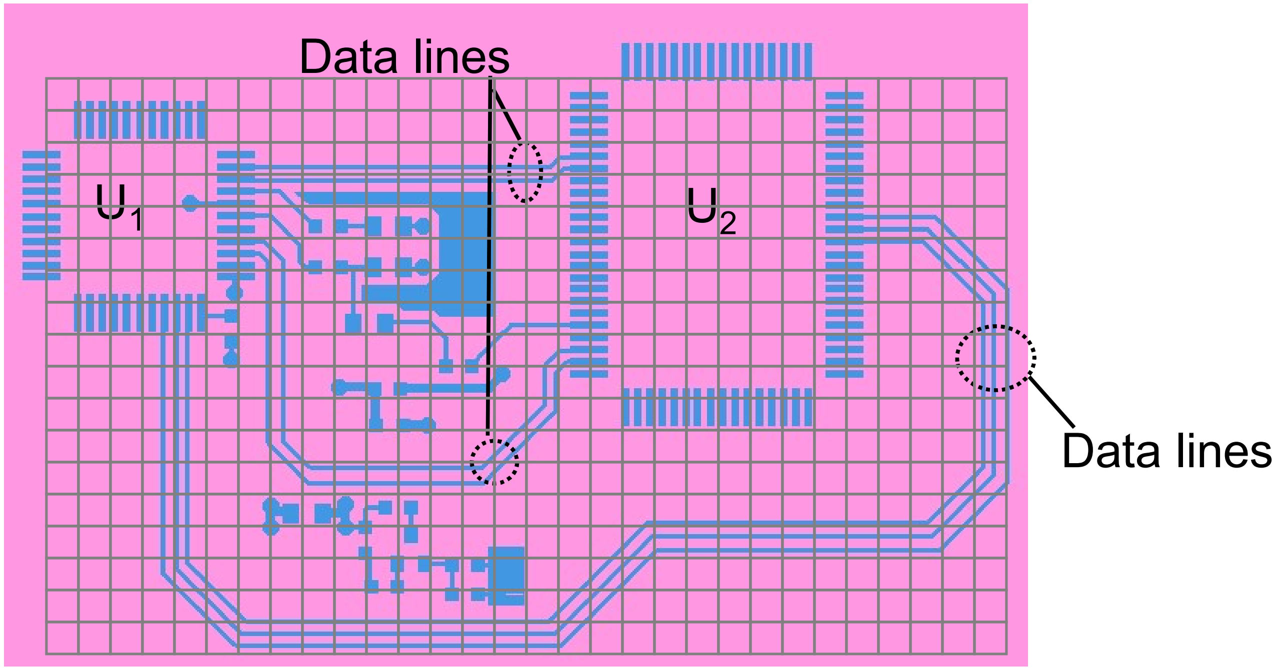
Example 4.2 – Estimating the Effect of Trace Width and Dielectric Thickness on Z_C

- Consider the following microstrip line cross sections, assuming lossless Tline, make a comparison of the characteristic impedance of each line.



Example 4.3 – Linking What We Have Learnt in Part 1, 2 and 3

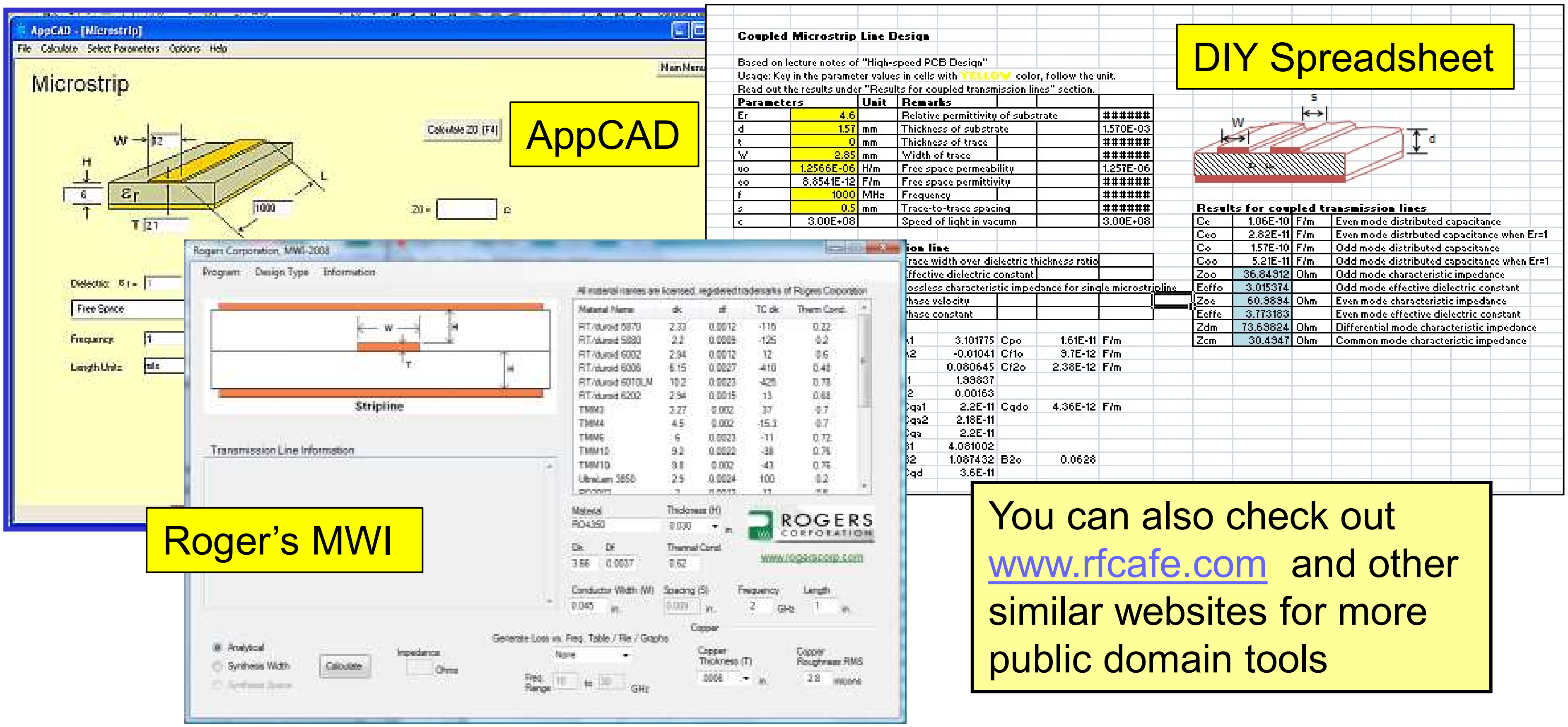
- Suppose we have the top layer of a PCB layout as shown below. All the traces that link Integrated circuit U_1 and U_2 are data lines, with width of 10.0 mils. There is an internal ground layer underneath the top layer. The dielectric of the PCB is FR4, separation between trace and ground plane is 12 mils. Determine which traces are considered as transmission line and which are not. The data rate is 166 Mbits/sec with rise/fall time of 0.5 nsec. Each grid is 1.0 mm x 1.0 mm.



Steps:
- From the data rate find the bandwidth of the digital signal.
- Compute the f_{lumped} for each relevant trace.
- Trace with $f_{flump} < f_{BW}$ are transmission lines.

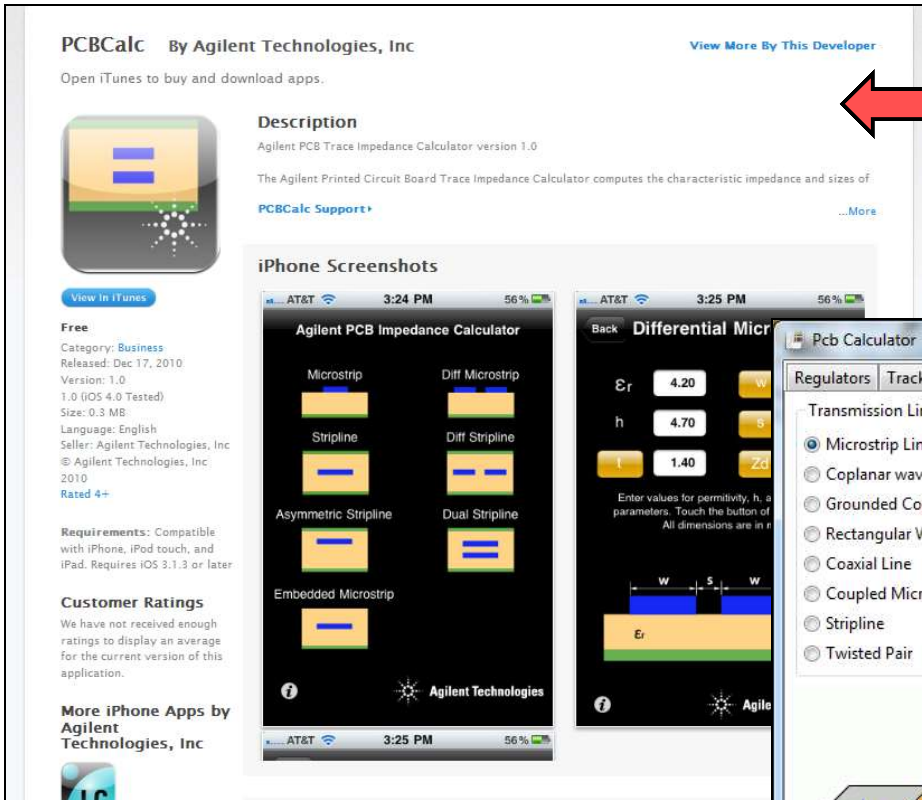
Free Tools for Stripline Design (1)

- Here are some examples of software which encapsulate the design equations for transmission line into a nice graphical user interface.

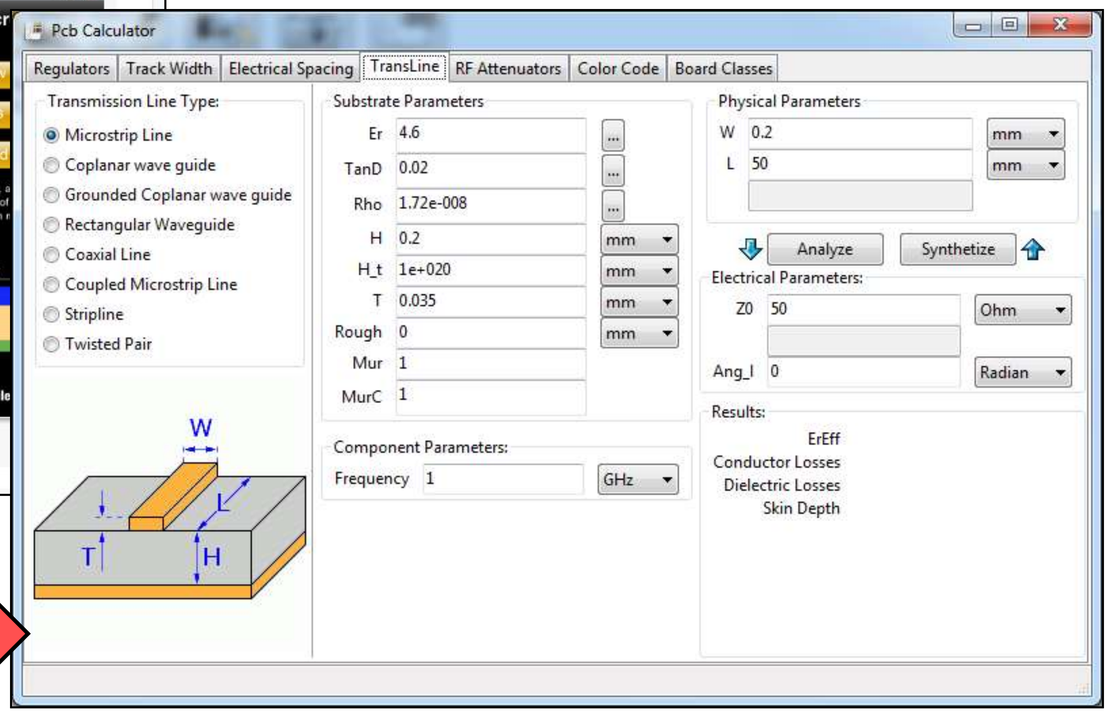


You can also check out www.rfcafe.com and other similar websites for more public domain tools

Free Tools for Stripline Design (2)



Agilent's PCB Trace Calculator (Apple iPad, iPhone)



KiCAD PCB Calculator for Windows™ PC

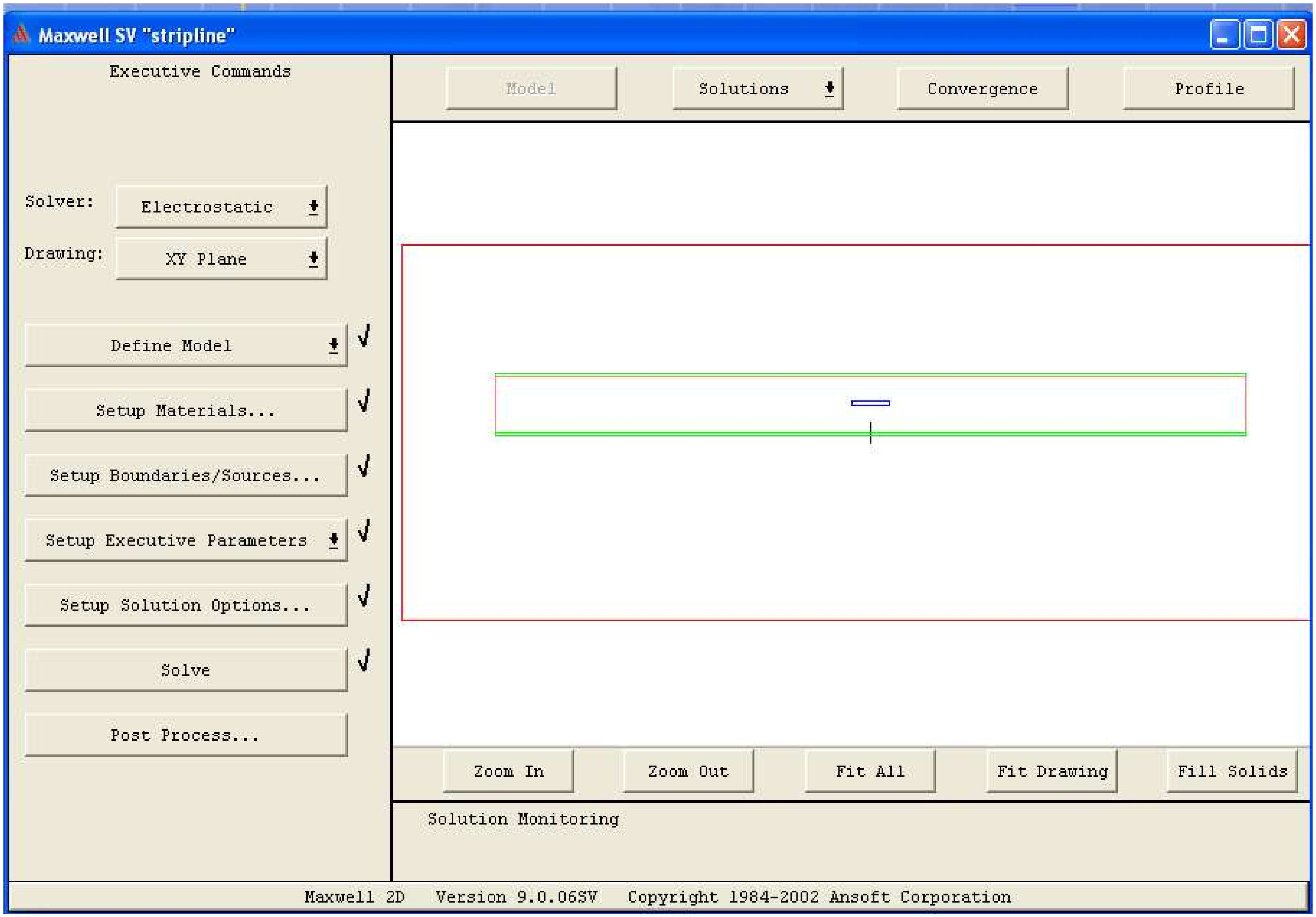
Example 4.4 - Stripline Design Example Using 2D EM Field Solver Program

Extra

- Here we demonstrate the use of a program called Maxwell SV Ver 9.0 (2004), a free version from Ansoft Corp., to design a stripline.
- Today Ansoft has become Ansys, unfortunately the free student version is no longer available. But you can still find the old versions such as this for download on some sites.
- The software uses finite element method (FEM) to compute the twp-dimensional (2D) static E and H field of an array of metallic objects.
- It is assumed that the stripline is lossless.
- Two projects are created, one is the Electrostatic problem for calculation of static electric field and distributed capacitance, the other is Magnetostatic problem, for calculation of static magnetic field and distributed inductance.
- Characteristic impedance of the stripline can then be computed from the distributed capacitance and inductance.

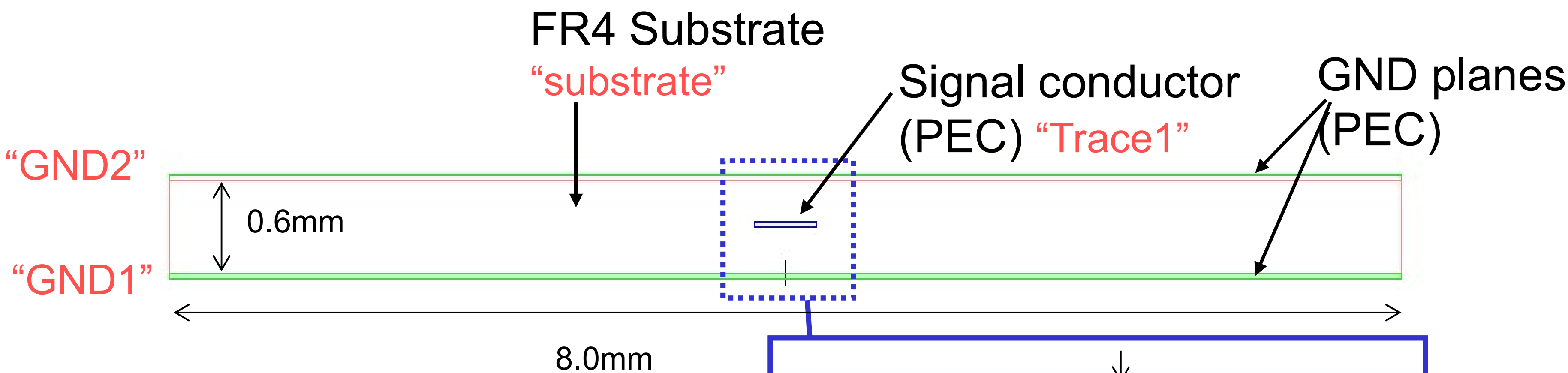
Extra

Example 4.4 - Screen Shot

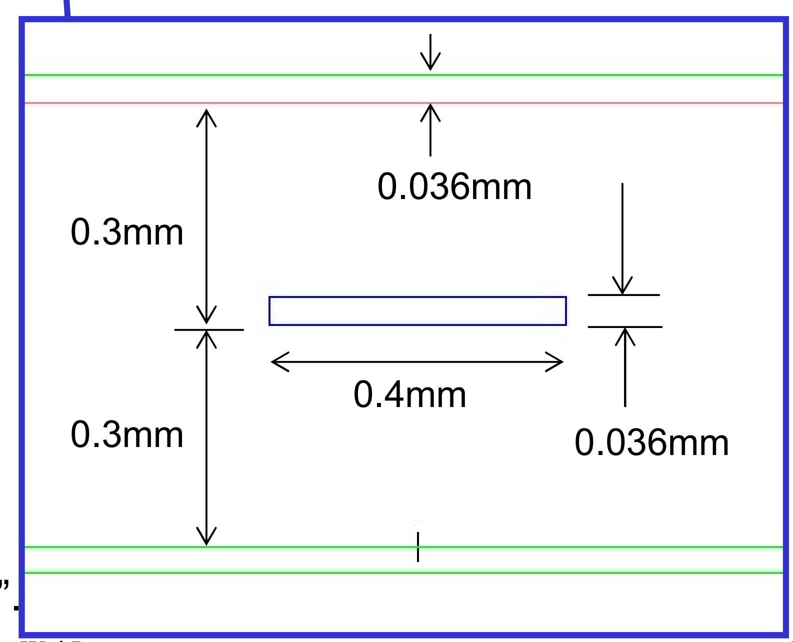


Example 4.4 - Stripline Cross Section

- Draw the cross section of the model and assign material.



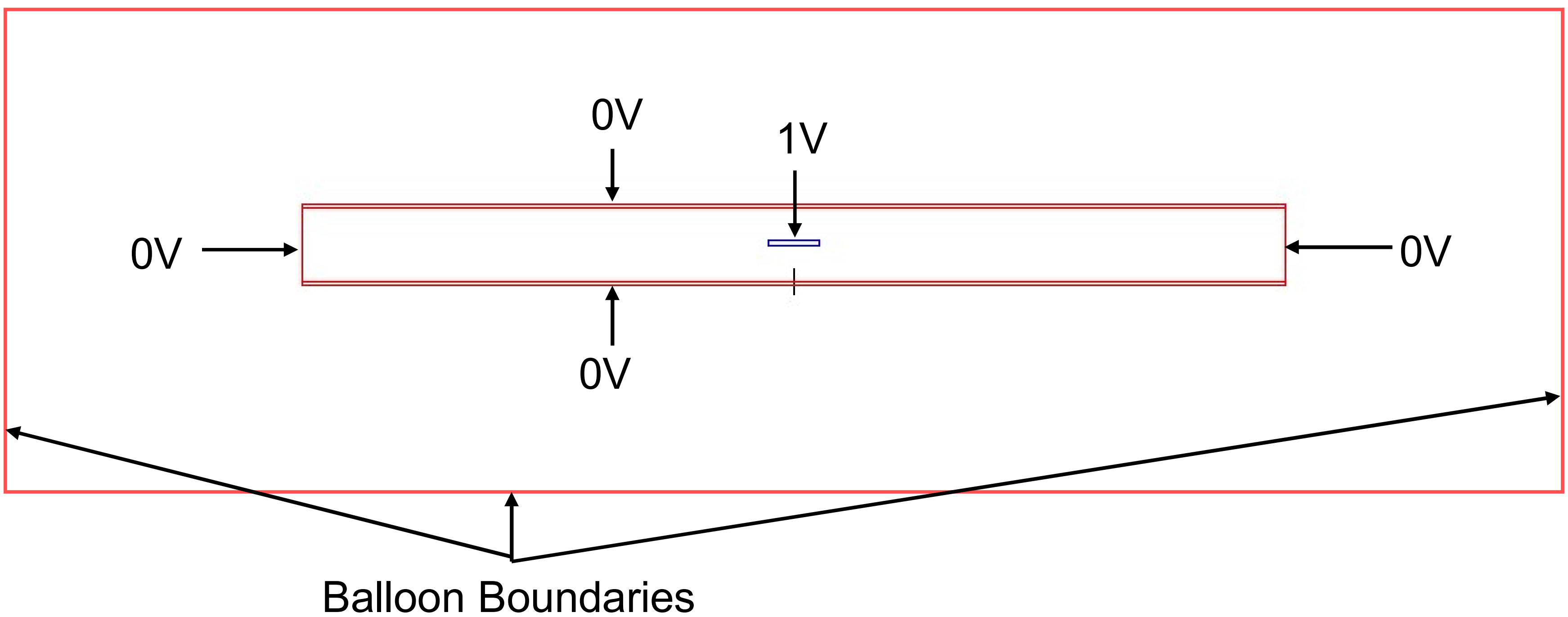
- Set drawing units to micron.
- Set drawing size to 10000um for x and 4000um for y.
- Set grid to dU = dV = 100um.
- Draw model, use direct entry mode for conducting structures like GND planes and signal.
- Name signal conductor “Trace1” and GND planes “GND1” and “GND2”.
- Name FR4 as “substrate”, then group both “GND1” and “GND2” as one object “GND”.



Example 4.4 - Electrostatics: Setup Boundary Conditions

Extra

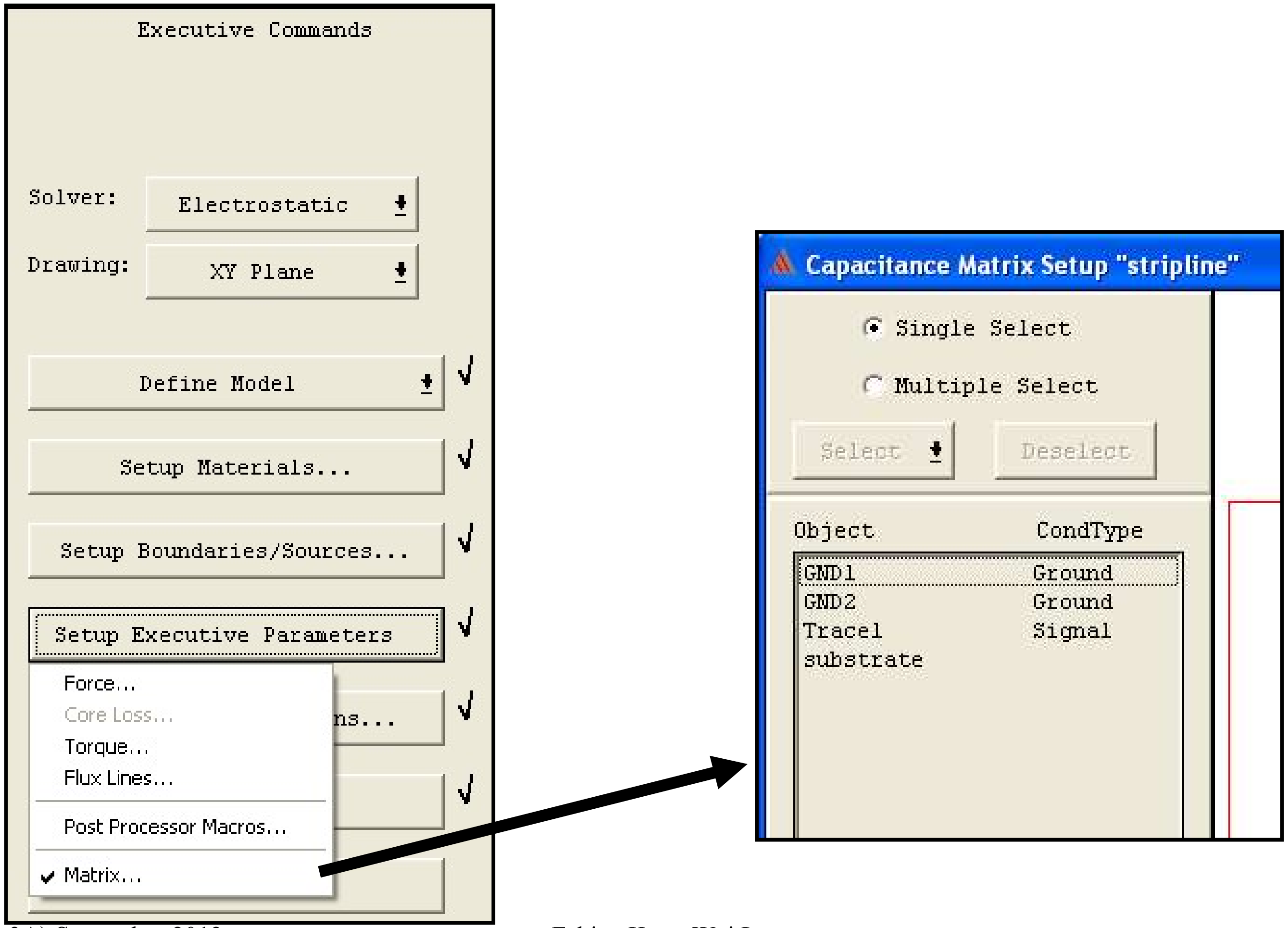
- Set the boundary conditions.
- All boundary are Dirichlet type, i.e. voltages are specified.
- Let the edges of the model domain remain as Balloon Boundary.



Extra

Example 4.4 - Electrostatics: Setup Executive Parameters

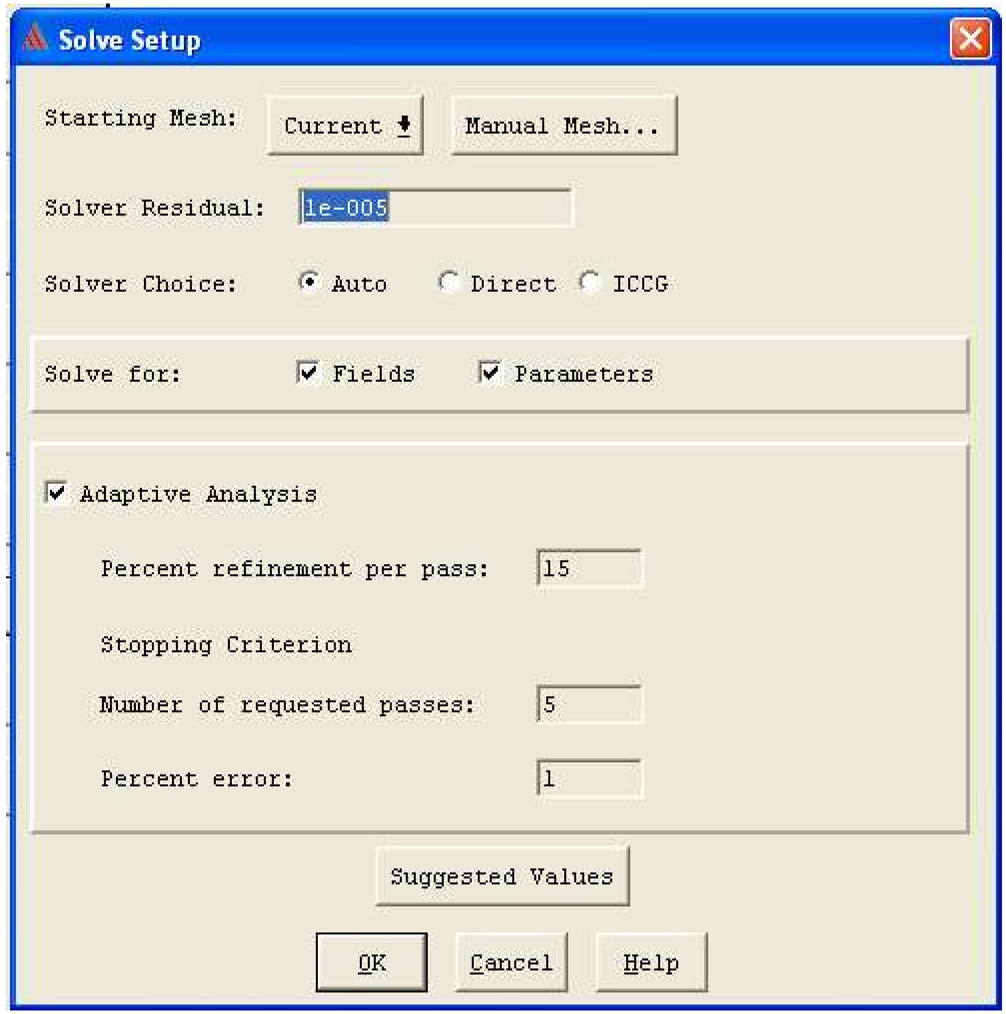
- Under ‘Setup Executive Parameters’ tab, select ‘Matrix…’ and proceed to perform the capacitance matrix setup as shown.



Example 4.4 - Electrostatics: Setup Solver and Solve for Scalar Potential ϕ

Extra

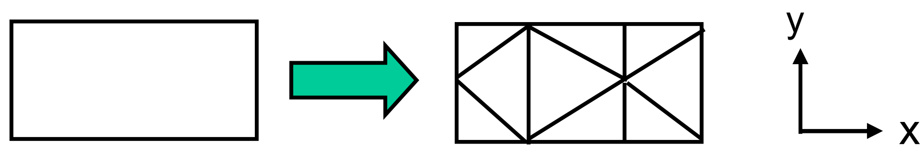
- Setup the solver and solve for the approximate potential solution. Use the 'Suggested Values' if you are not sure.



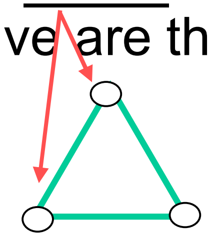
Example 4.4 - Finite Element Method (1)

Extra

- In finite-element method (FEM) an object is thought to consist of many smaller elements, usually triangle for 2D object and tetrahedron for 3D object.



- FEM is used to solve for the approximate scalar potential V (or ϕ) for electrostatic problem and vector potential A for magnetostatic problem at the vertex of each triangle. The partial differential equations (PDE) to solve are the Poisson's equations (in a lossless uniform region):



$$\nabla^2 V = \frac{\rho}{\varepsilon} \qquad \nabla^2 \vec{A} = \mu \vec{J}$$

Note: Actually a more general form of the PDEs for V and **A**, assuming non-uniform dielectric with loss (e.g. σ>0) are used by Ansoft software:

$$\nabla \times \left(\tfrac{1}{\mu} \nabla \times \vec{A}\right) = -j\omega\sigma\vec{A} + \vec{J} \qquad \nabla \cdot \left[(\sigma + j\omega\varepsilon)\nabla V\right] = 0$$

- Potential value inside the triangle can be estimated via interpolation.
- For 2D problem the PDE can be written as:

$$\nabla_t^2 V_t = \frac{\rho}{\varepsilon} \qquad \nabla_t = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}$$

$$V_t = V_t(x, y) \quad \rho = \rho(x, y)$$

$$\nabla_t^2 A_z = \mu \vec{J}_z$$

$$A_z = A_z(x, y) \quad J_z = J_z(x, y)$$

Example 4.4 - Finite Element Method (2)

Extra

- 2D quasi-static E field can then be obtained by:

$$\vec{E}_t(x,y) = -\nabla_t V_t(x,y)$$

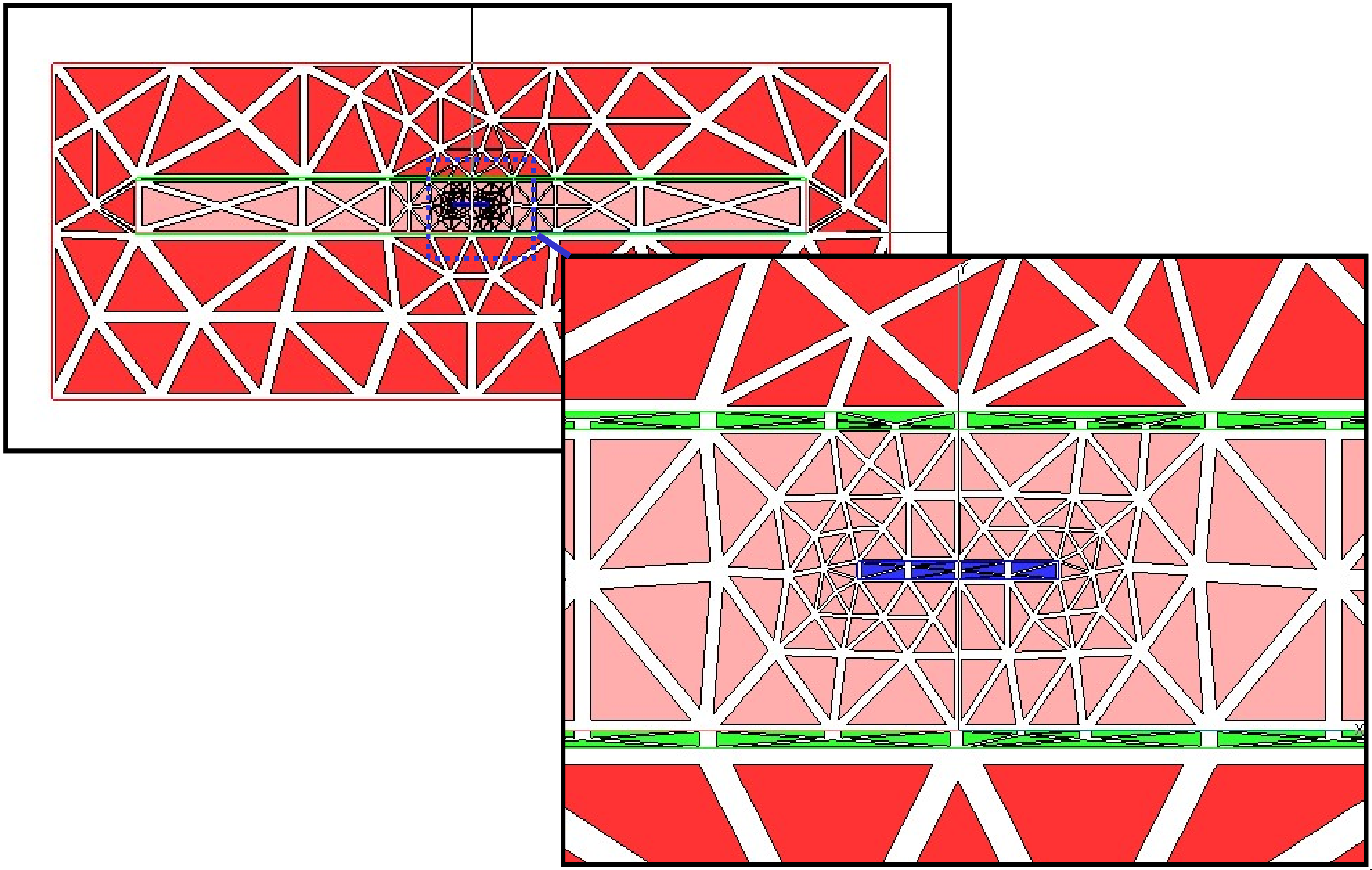
- Similarly magnetic flux intensity H can be obtained from:

$$\vec{H}_t(x,y) = -\frac{1}{\mu}\nabla_t \times \left[A_z(x,y)\hat{z}\right]$$

- For more information, refer to
  - T. Itoh (editor), “Numerical techniques for microwave and milimeter-wave passive structures”, John-Wiley & Sons, 1989.
  - P. P. Silvester, R. L. Ferrari, “Finite elements for electrical engineers”, Cambridge University Press, 1990.
  - Other newer books.

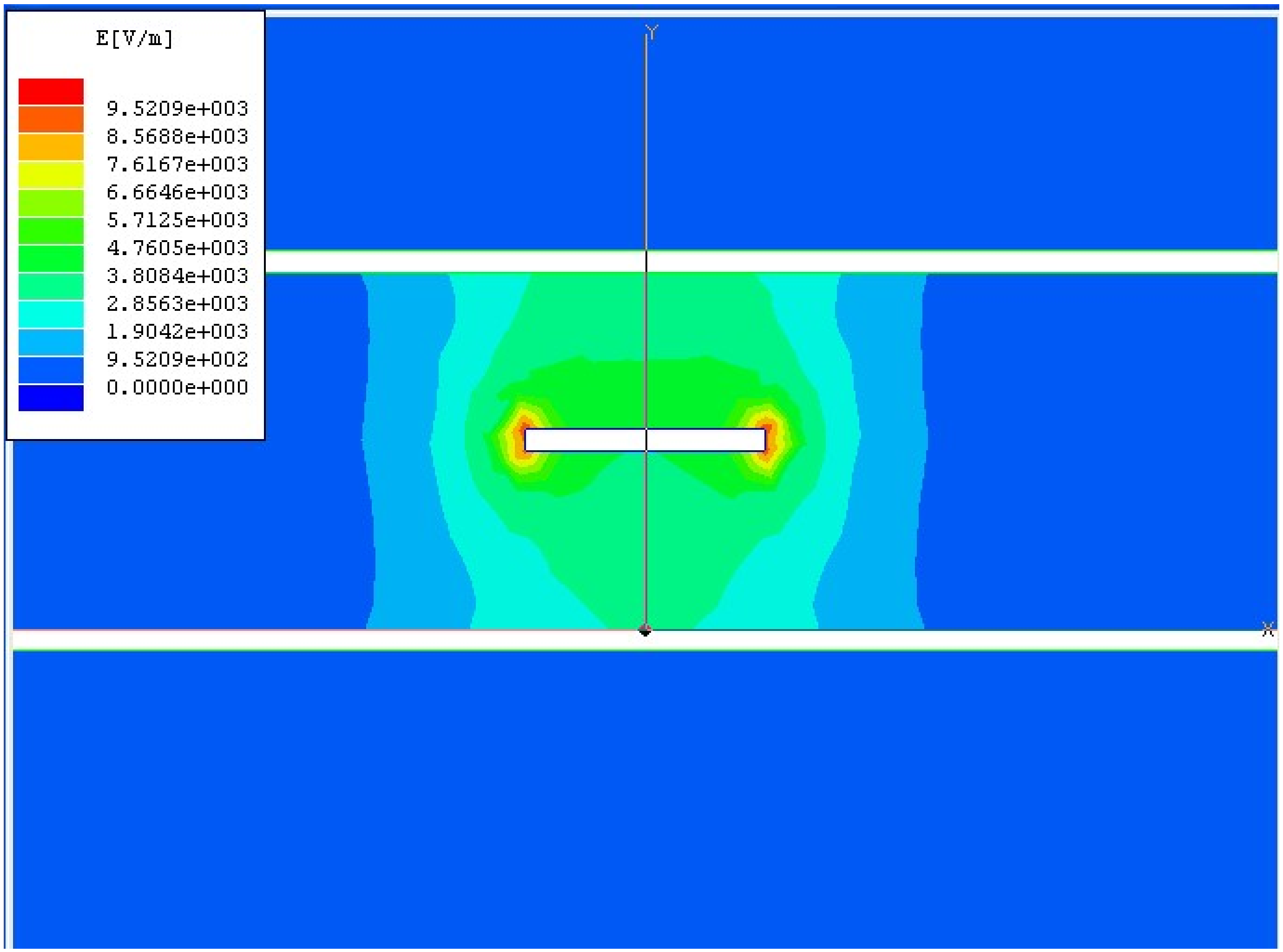
Example 4.4 - Electrostatics: The Triangular Mesh

Extra



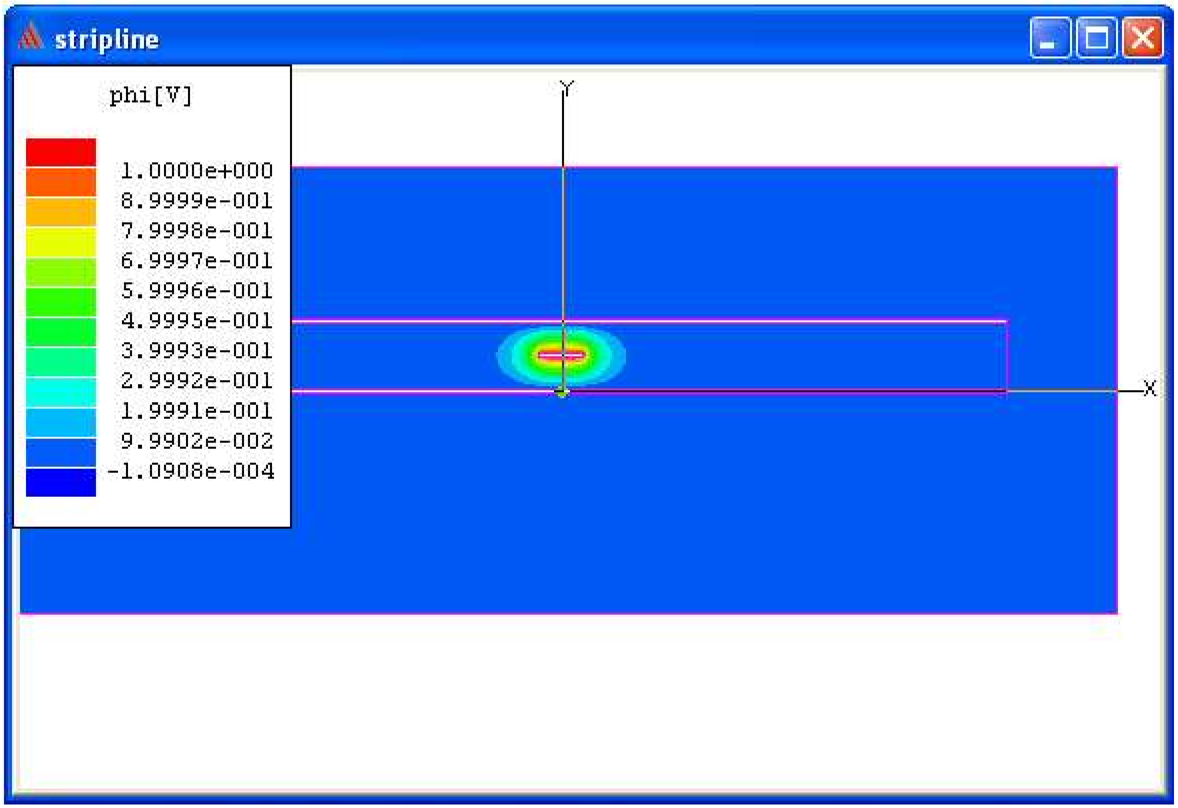
Example 4.4 - Electrostatics: Plot of E field Magnitude

Extra



Example 4.4 - Electrostatics: Plot of Voltage Contour

Extra



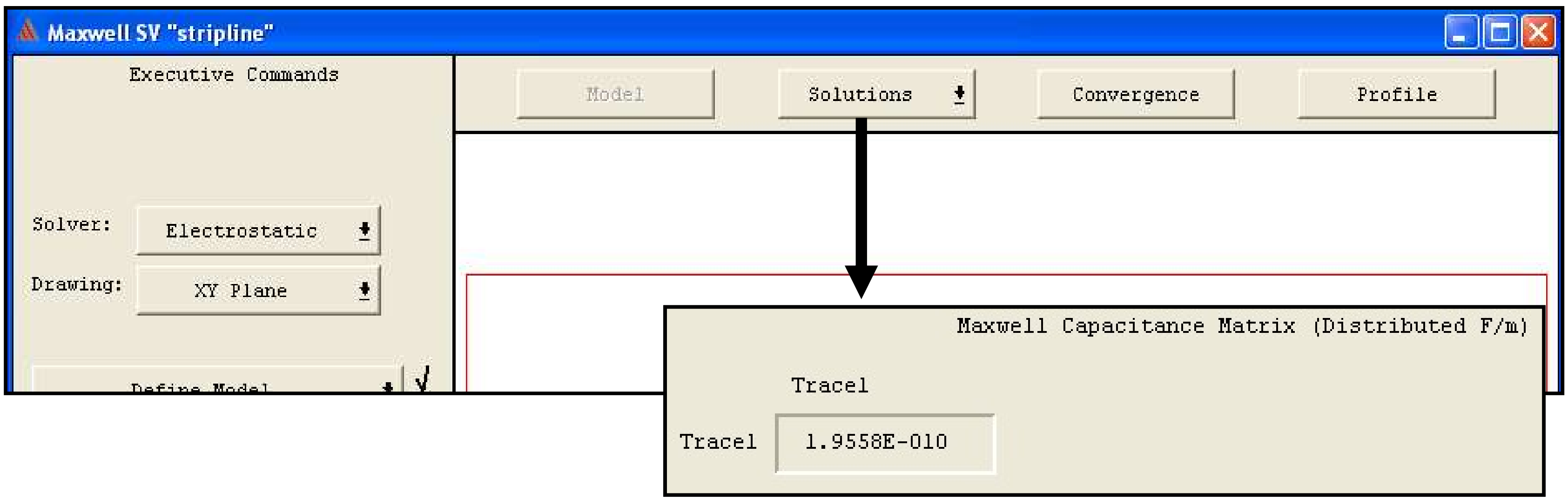
Extra

Example 4.4 - Electrostatics: Capacitance

- The estimated distributed capacitance is then computed using:

$$C = \frac{\varepsilon'}{\left|V_t\right|^2} \iiint_V \left|\vec{E}_t\right|^2 dv$$

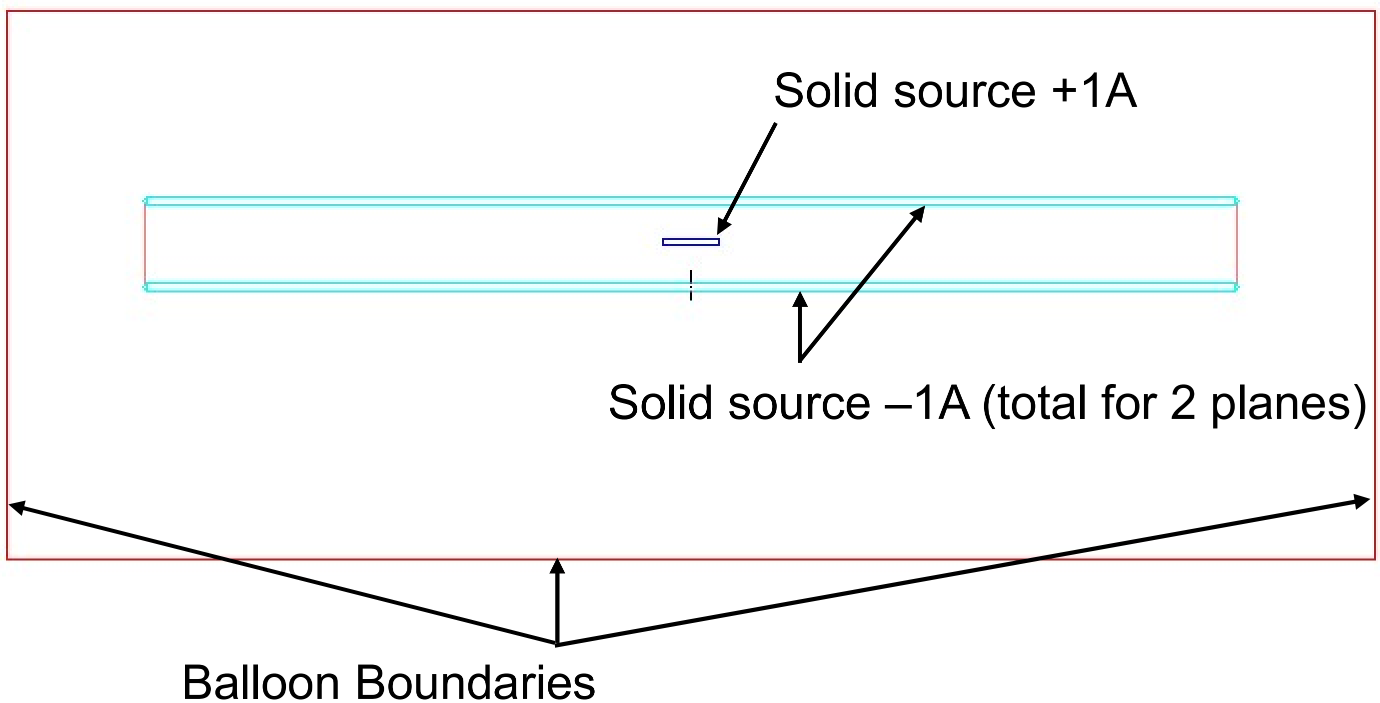
- Approximation to the integration using summation is performed by the software. The result is shown below:



- $C \approx 1.9958 \times 10^{-10}$ F/m or 199.58 pF/m.

Example 4.4 - Magnetostatics: Setup Boundary Conditions

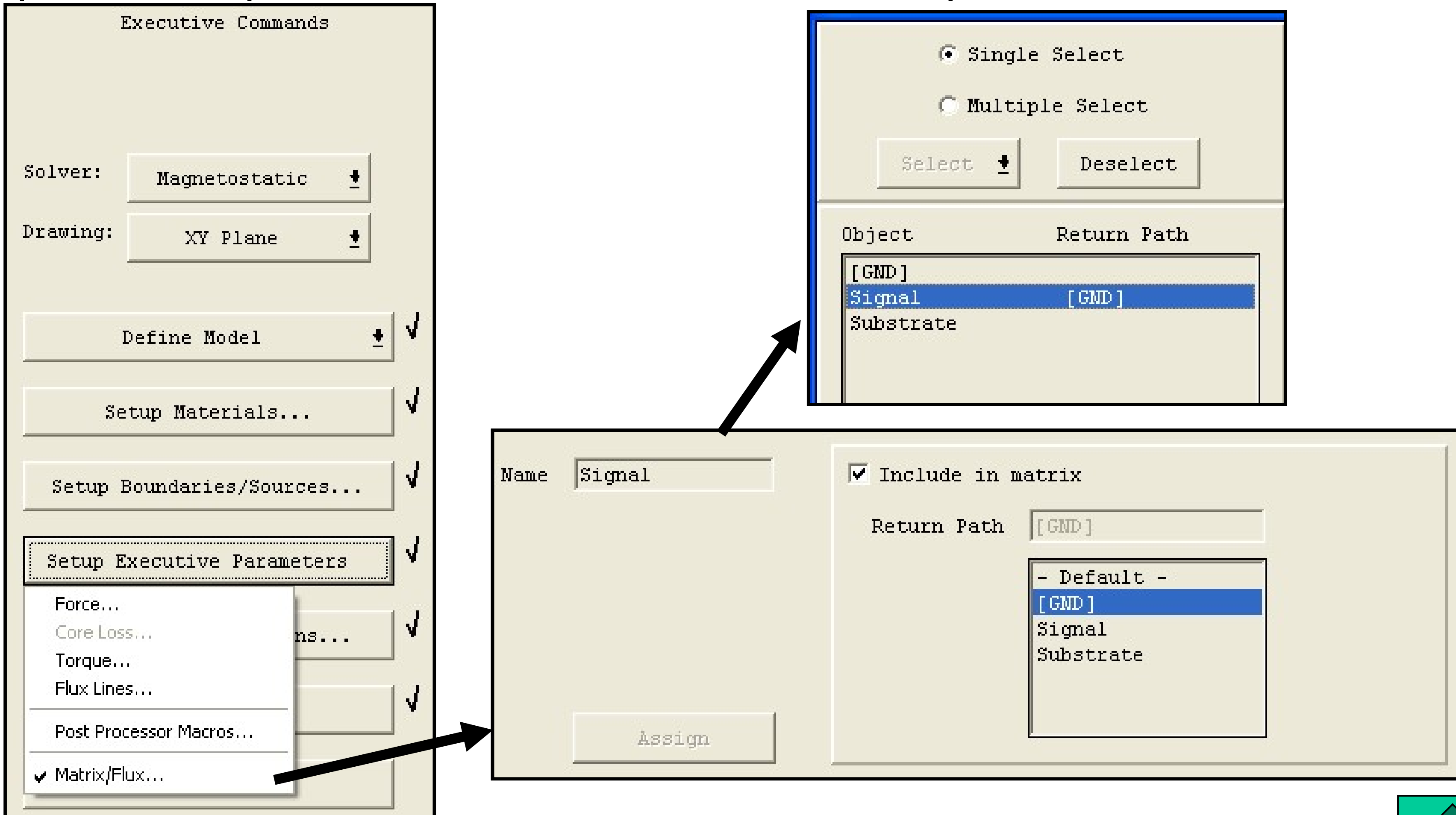
Extra



Example 4.4 - Magnetostatics: Setup Executive Parameters

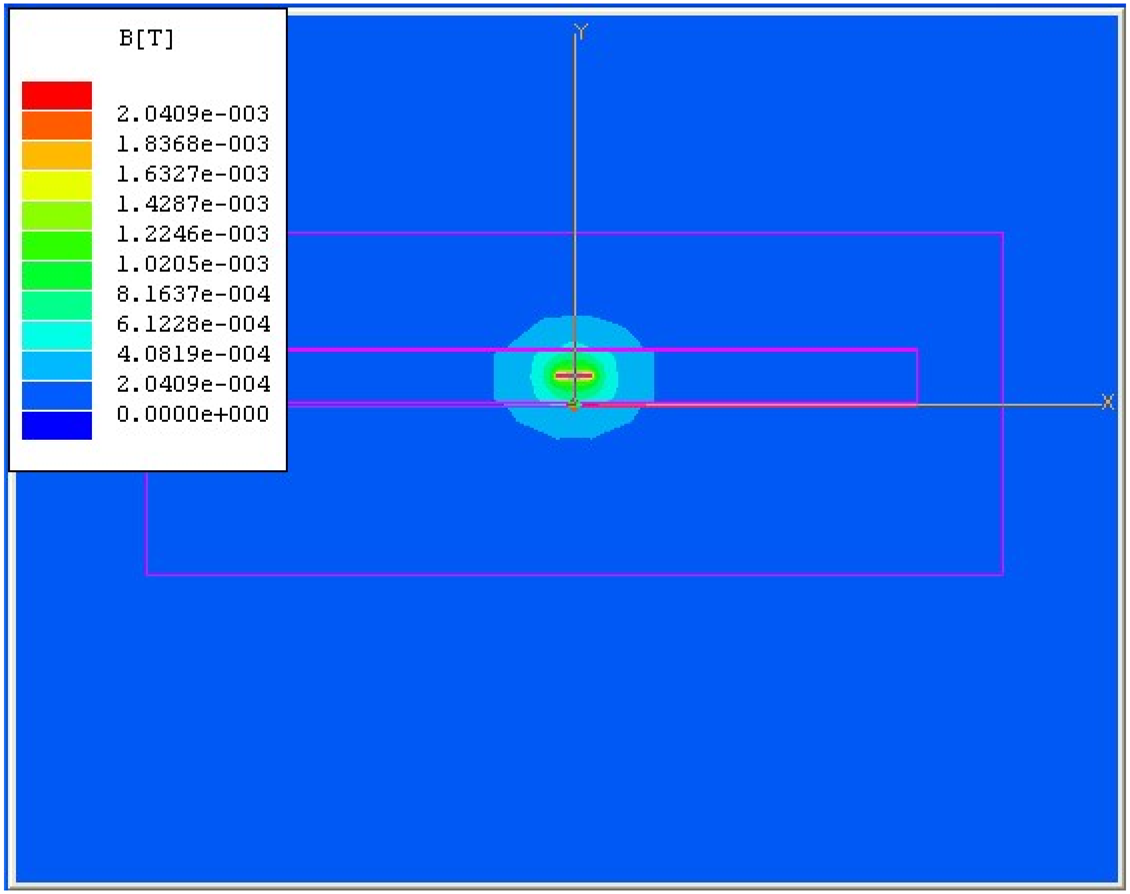
Extra

- Under 'Setup Executive Parameters' tab, select 'Matrix/Flux…' and proceed to perform the inductance matrix setup as shown.



Example 4.4 - Magnetostatics: Plot of B field Magnitude

Extra



Extra

Example 4.4 - Magnetostatics: Inductance

- The estimated distributed capacitance is then computed using:

$$L = \frac{\mu}{\left|I_t\right|^2} \iiint_V \left|\vec{H}_t\right|^2 dv$$

Inductance Matrix (Distributed H/m)

	Signal
Signal	2.5093E-007

- L ≈ 2.5093×10^{-7} H/m or 250.93 nH/m.

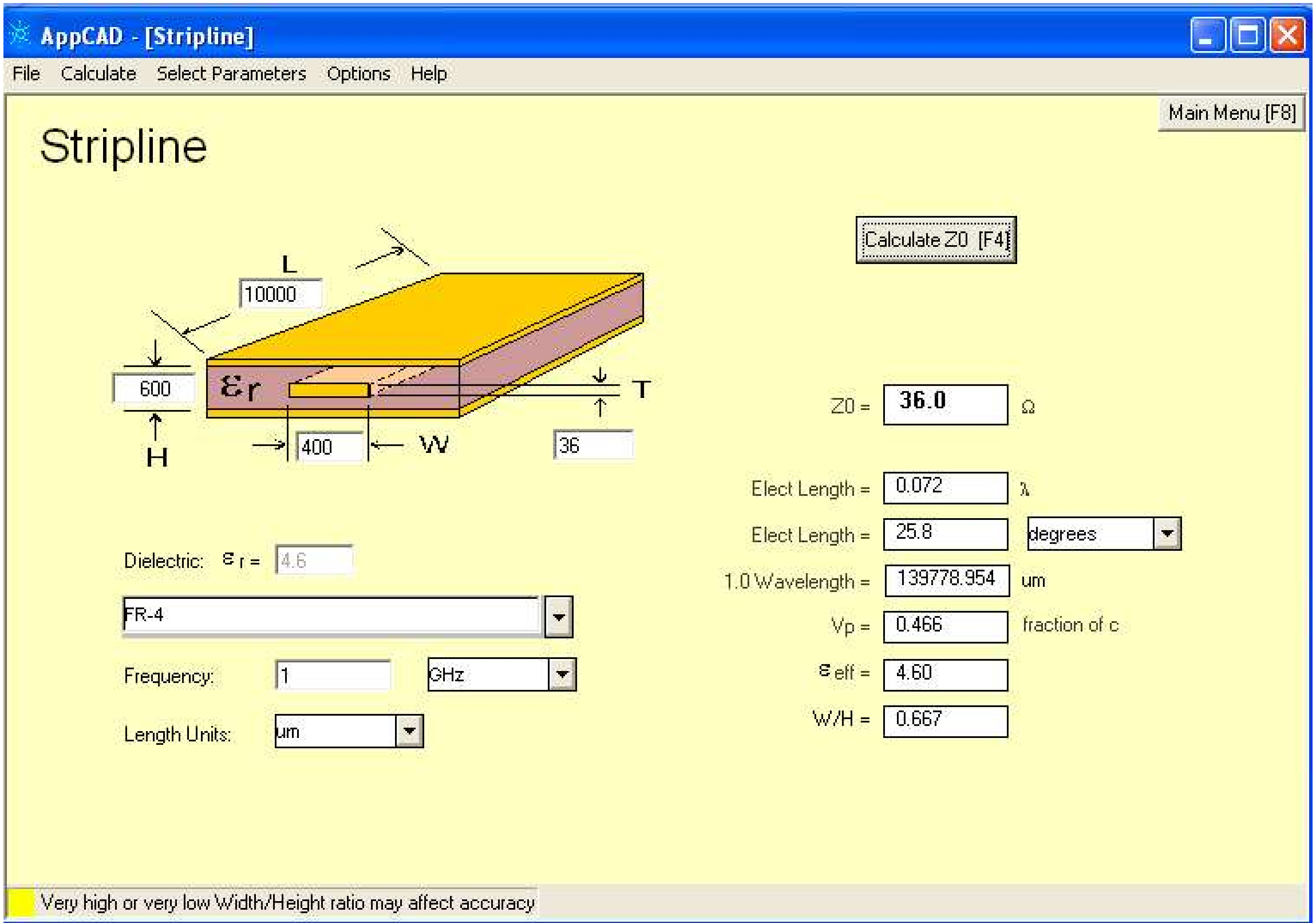
- Finally from the per unit length L and C information, we can work out the lossless characteristic impedance and phase velocity of this transmission line as follows:

$$Z_c = \sqrt{\frac{L}{C}} = \sqrt{\frac{2.5093\times 10^{-7}}{1.9558\times 10^{-10}}} = 35.819\Omega$$

$$v_p = \frac{1}{\sqrt{LC}} = 1.427\times 10^{8}\, ms^{-1}$$

Example 4.5 – Tline Design Using Agilent's AppCAD V3.02

Extra



Example 4.5 – Derivation of LC Parameters from AppCAD V3.02 Results

Extra

$$Z_c = 36.0$$

$$v_p = 0.466 v_{p(vacumn)}$$

$$v_{p(vacumn)} = 2.998 \times 10^8$$

$$C = \frac{1}{Z_c v_p} = 1.988 \times 10^{-10} F/m$$

$$L = Z_c^{\,2} C = 2.577 \times 10^{-7} H/m$$

- As we can see, both the results using EM field solver and using closed-form solution (AppCAD) are very close.
- Bear in mind that this is only approximate solution, as skin-effect loss and dielectric loss are ignored.
- It is however accurate design equations are, most still do not predict the effect of solder masks, silkscreen, tolerance and other structure on the transmission line parameters.

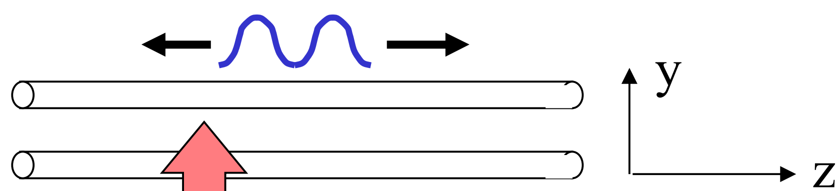
Key Learnings for Part 3A

- Concepts of transmission line, under what condition a trace/interconnect becomes a transmission line (Tline).
- Propagation modes for Tline – TEM and non-TEM modes.
- Electrical circuit model for Tline, and the implication to voltage and currents.
- Transmission line effect – non-uniform EM fields, propagation, dispersion, attenuation, characteristic impedance.
- Theory for Tline design – EM field solution method.
- Practical Tline design using design equations.

Appendix 1
Advanced Concepts – Field Theory Solutions, Propagation Modes

Field Theory Solution

- The nature of E and H fields in the space between conductors can be studied by solving the Maxwell's Equations or Wave Equations (which can be derived from Maxwell's Equations) (See [1], [2], [3]). Assuming the condition of long interconnection, the solutions of E and H fields are propagating fields or waves.
- We assume time-harmonic EM fields with $e^{j\omega t}$ dependence and wave propagation along positive and negative z-axis.



Maxwell Equations

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = \vec{J} + j\omega\varepsilon\vec{E}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$$

$$\nabla \cdot \vec{H} = 0$$

+

Boundary conditions

For instance tangential E field component on PEC must be zero, continuity of E and H field components across different dielectric material, etc.

+

Wave Equations

$$\nabla^2\vec{E} + k_o^2\vec{E} = 0$$

$$\nabla^2\vec{H} + k_o^2\vec{H} = 0 \qquad \textbf{(A.1)}$$

$$k_o = \omega\sqrt{\varepsilon\mu}$$

Extra: Deriving the Hemholtz Wave Equations From Maxwell Equations

Extra

Performing curl operation on Faraday's Law $\nabla \times \vec{E} = -j\omega\mu\vec{H}$:

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E} = -j\omega\mu\left(\nabla \times \vec{H}\right)$$

$$\Rightarrow \nabla^2 \vec{E} + \omega^2 \mu\varepsilon\vec{E} = \boxed{j\omega\mu\vec{J} + \nabla\left(\frac{\rho}{\varepsilon}\right)}$$

These are the sources for the E field

Note: use the well-known vector calculus identity

$$\nabla \times \nabla \times \vec{A} = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

In free space no electric charge and current:

$$\nabla^2 \vec{E} + \omega^2 \mu\varepsilon\vec{E} = 0$$

Similar procedure can be used to obtain

$$\nabla^2 \vec{H} + \omega^2 \mu\varepsilon\vec{H} = -\nabla \times \vec{J}$$

Or in free space

$$\nabla^2 \vec{H} + \omega^2 \mu\varepsilon\vec{H} = 0$$

Note: This derivation is valid for time-harmonic case under linear medium only. See more advanced text for general wave equation. For Example:

C. A. Balanis, "Advanced engineering Electromagnetics", John-Wiley, 1989.

Obtaining the Expressions for E and H (1)

Extra

- Assuming an ordinary differential equation (ODE) system as shown:

ODE → $$\frac{d^2y}{dx^2} + k^2 y = 0, \quad y = f(x), \; x \in [0, b]$$ ← The Domain

$$y(0) = C_1 \text{ and } y(b) = C_2$$ ← Boundary conditions

- To obtain a solution to the above system (a solution means a function that when substituted into the ODE, will cause left and right hand side to be equal), many approaches can be used (for instance see *E. Kreyszig, "Advance engineering mathematics", 1998, John Wiley*).
- One popular approach is the Trial and Error/substitution method, where we guess a functional form for y(x) as follows: $y(x) = e^{\beta x} \Rightarrow \frac{dy}{dx} = \beta e^{\beta x}, \; \frac{d^2y}{dx^2} = \beta^2 e^{\beta x}$
- Substituting this into the ODE:

$$(\beta^2 + k^2)e^{\beta x} = 0$$
$$\Rightarrow \beta^2 + k^2 = 0$$
$$\Rightarrow \beta = \pm jk \quad \text{where } j = \sqrt{-1}$$

That the trial-and-error method works is attributed to the Uniqueness Theorem for linear ODE.

- Since this is a 2[nd] order ODE, we need to introduce 2 unknown constants, A and B, and a general solution is:

$$y(x) = Ae^{jkx} + Be^{-jkx} \qquad (1)$$

Obtaining the Expressions for E and H (2)

Extra

- To find A and B, we need to use the boundary conditions.

$$y(0) = C_1 \Rightarrow A + B = C_1 \quad \text{(2a)}$$

$$y(b) = C_2 \Rightarrow Ae^{jkb} + Be^{-jkb} = C_2 \quad \text{(2b)}$$

- Solving (2a) and (2b) for A and B:

$$(C_1 - B)e^{jkb} + Be^{-jkb} = C_2$$

$$\Rightarrow B = \frac{C_1 e^{jkb} - C_2}{2j\sin(kb)} \quad \text{(3a)}$$

$$A = C_1 - B = \frac{C_1 e^{-jkb} - C_2}{2j\sin(kb)} \quad \text{(3b)}$$

- So the unique solution is:

$$\boxed{y(x) = \left(\frac{C_1 e^{-jkb} - C_2}{2j\sin(kb)}\right)e^{jkx} + \left(\frac{C_1 e^{jkb} - C_2}{2j\sin(kb)}\right)e^{-jkx}} \quad \text{(q.e.d.)}$$

Obtaining the Expressions for E and H (3)

Extra

- The same approach can be applied to Wave Equations or Maxwell Equations for Tline. Consider the Wave Equations (A.1) in time-harmonic form.
- The unknown functions are vector phasors **E**(x,y,z) and **H**(x,y,z). The differential equation for **E** in Cartesian coordinate is:

$$\left(\nabla^2 + k_o^2\right)\vec{E} = 0$$

$$\Rightarrow \boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_o^2\right)E_x\hat{x}} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_o^2\right)E_y\hat{y} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_o^2\right)E_z\hat{z} = 0$$

- This is called a Partial Differential Equation (PDE) as each E_x, E_y and E_z depends on 3 variables, with the differentiation substituted by partial differential. There are 3 PDEs if you observed carefully. For x-component this is:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_o^2\right)E_x = 0$$

- Based on the previous ODE example, and also the fact that we expect the E field to travel along the z-axis, the following form is suggested:

$$\boxed{E_x(x,y,z) = e_x(x,y)e^{-j\beta z} \quad \text{or} \quad e_x(x,y)e^{j\beta z}}$$

A function of x and y

The exponent e !!!

Obtaining the Expressions for E and H (4)

Extra

- Carrying on in this manner for y and z-components, we arrived at the following form for E field.

$$\vec{E}^{+} = e_x(x,y)e^{-j\beta z}\hat{x} + e_y(x,y)e^{-j\beta z}\hat{y} + e_z(x,y)e^{-j\beta z}\hat{z}$$
$$= \left(\vec{e}_t(x,y) + e_z(x,y)\hat{z}\right)e^{-j\beta z} \qquad \textit{(A.1a)}$$

- Notice that up to now we have not solve the Wave Equations, but merely determine the functional form of its solution.
- We still need to find out what is $e_x(x,y)$, $e_y(x,y)$, $e_z(x,y)$ and β.
- Using similar approach on $\left(\nabla^2 + k_o^2\right)\vec{H} = 0$ will yield similar expression for H field.

$$\vec{H}^{+} = h_x(x,y)e^{-j\beta z}\hat{x} + h_y(x,y)e^{-j\beta z}\hat{y} + h_z(x,y)e^{-j\beta z}\hat{z}$$
$$= \left(\vec{h}_t(x,y) + h_z(x,y)\hat{z}\right)e^{-j\beta z} \qquad \textit{(A.1b)}$$

E and H fields Expressions (1)

- Thus the propagating EM fields guided by Tline can be written as:

Transverse component

$$\vec{E}^+ = \underbrace{e_x(x,y)e^{-j\beta z}\hat{x} + e_y(x,y)e^{-j\beta z}\hat{y}} + e_z(x,y)e^{-j\beta z}\hat{z}$$
$$= \left(\boxed{\vec{e}_t(x,y)} + \boxed{e_z(x,y)}\hat{z}\right)e^{-j\beta z}$$

Axial component

$$\vec{H}^+ = h_x(x,y)e^{-j\beta z}\hat{x} + h_y(x,y)e^{-j\beta z}\hat{y} + h_z(x,y)e^{-j\beta z}\hat{z}$$
$$= \left(\vec{h}_t(x,y) + h_z(x,y)\hat{z}\right)e^{-j\beta z}$$

superscript indicates propagation direction

(A.2a)

EM fields Propagating In +z direction

(A.2b)

$$\vec{E}^- = e_x(x,y)e^{+j\beta z}\hat{x} + e_y(x,y)e^{+j\beta z}\hat{y} - e_z(x,y)e^{+j\beta z}\hat{z}$$
$$= \left(\vec{e}_t(x,y) - e_z(x,y)\hat{z}\right)e^{+j\beta z}$$

$$\vec{H}^- = -h_x(x,y)e^{+j\beta z}\hat{x} - h_y(x,y)e^{+j\beta z}\hat{y} + h_z(x,y)e^{+j\beta z}\hat{z}$$
$$= \left(-\vec{h}_t(x,y) + h_z(x,y)\hat{z}\right)e^{+j\beta z}$$

(A.3a)

EM fields Propagating In -z direction

(A.3b)

E and H fields Expressions (2)

- We can convert the phasor form into time-domain form, for instance for E field propagating in +z direction:

$$\vec{E}^{+}(x,y,z,t) = \mathrm{Re}\left\{\overrightarrow{E}^{+}(x,y,z)e^{j\omega t}\right\}$$
$$= e_x(x,y)\cos(\omega t - \beta z)\hat{x} + e_y(x,y)\cos(\omega t - \beta z)y + e_z(x,y)\cos(\omega t - \beta z)\hat{z} \qquad \textbf{(A.4)}$$

- Where

$$\vec{E}^{+}(x,y,z,t) = E_x{}^{+}(x,y,z,t)\hat{x} + E_y{}^{+}(x,y,z,t)\hat{y} + E_z{}^{+}(x,y,z,t)\hat{z} \qquad \textbf{(A.5a)}$$

Unit vector

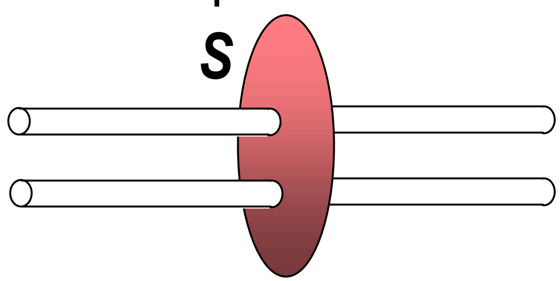
$$E_x{}^{+}(x,y,z,t) = e_x(x,y)\cos(\omega t - \beta z)$$
$$E_y{}^{+}(x,y,z,t) = e_y(x,y)\cos(\omega t - \beta z) \qquad \textbf{(A.5b)}$$
$$E_z{}^{+}(x,y,z,t) = e_z(x,y)\cos(\omega t - \beta z)$$

E and H fields Expressions (3)

- Usually one only solves for E field, the corresponding H field phasor can be obtained from:

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$
$$\Rightarrow \vec{H} = \frac{1}{j\omega\mu}\nabla \times \vec{E} \qquad \textbf{\textit{(A.6)}}$$

- The power carried by the EM fields is given by Poynting Theorem:



Positive Z direction…

$$P = \frac{1}{2}\mathrm{Re}\iint_S \vec{E}^+ \times \left(\vec{H}^+\right)^* \cdot d\vec{s} = \frac{1}{2}\mathrm{Re}\iint_{S_z} \vec{e}_t \times \vec{h}_t^{\,*} \cdot \overrightarrow{ds}$$

Negative Z direction…

$$P = \frac{1}{2}\mathrm{Re}\iint_S \vec{E}^- \times \left(\vec{H}^-\right)^* \cdot d\vec{s} = \frac{1}{2}\mathrm{Re}\iint_{S_z} -\vec{e}_t \times \vec{h}_t^{\,*} \cdot \left(-\overrightarrow{ds}\right)$$

$$= \frac{1}{2}\mathrm{Re}\iint_{S_z} \vec{e}_t \times \vec{h}_t^{\,*} \cdot \overrightarrow{ds}$$

Positive value means that power is carried along the propagation direction.

E and H fields Expressions (4)

- There are 2 reasons for choosing the sign conventions for +ve and -ve propagating waves as in (A.2) and (A.3).
  - So that $\nabla \cdot \vec{E} = 0$ for both +ve and -ve propagating E field (consistency with Maxwell's Equations).
  - The transverse magnetic field must change sign upon reversal of the direction of propagation to obtain a change in the direction of energy flow.

$$\nabla \cdot \vec{E}^{+} = 0$$
$$\Rightarrow \nabla_t \cdot \vec{e}_t - j\beta e_z = 0$$

For +ve direction

$$\nabla \cdot \vec{E}^{-} = 0$$
$$\Rightarrow \nabla_t \cdot \vec{e}_t + j\beta(-e_z) = 0$$

For -ve direction

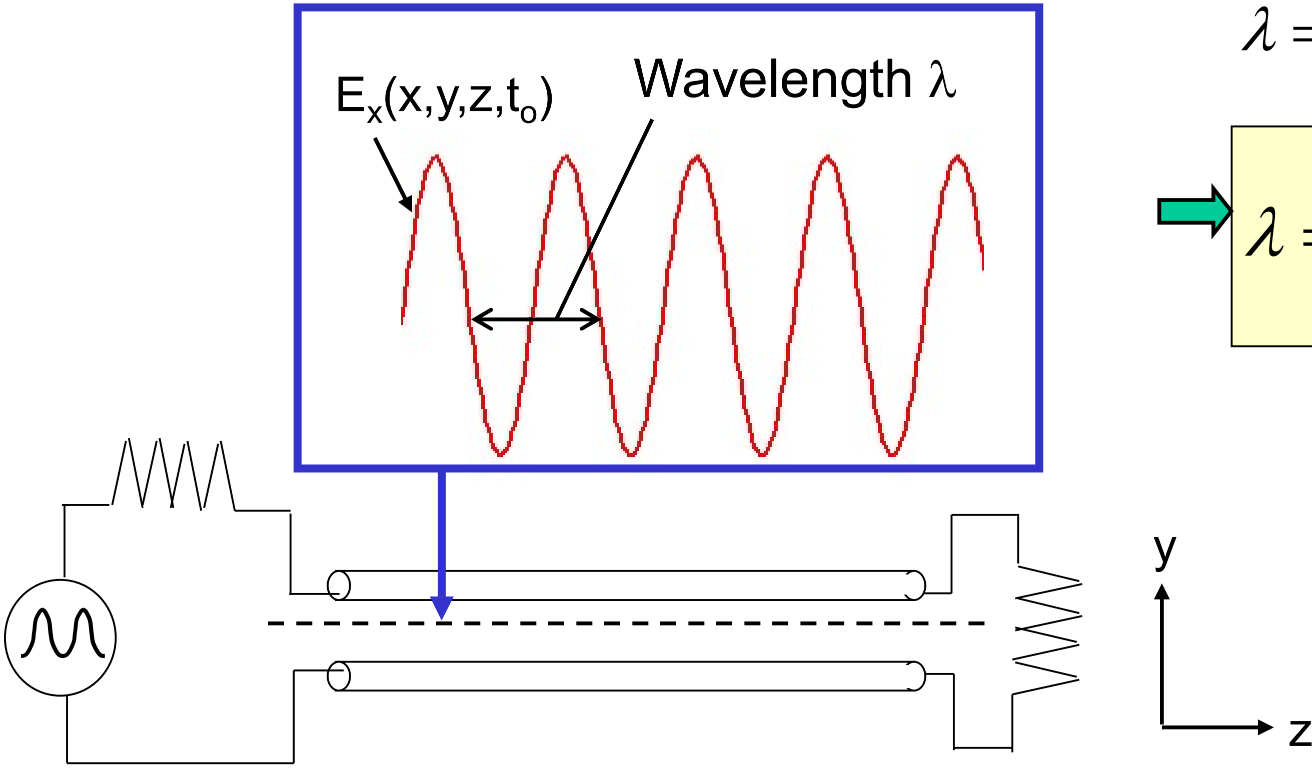
Phase Velocity

- It is easy to show that equation (A.2a) and (A.2b) describes traveling E field waves (also for H).
- The speed where the E and H fields travel is called the Phase Velocity, v_p.
- Phase Velocity depends on the propagation mode (to be discussed later), the frequency and the physical properties of the interconnect.

$$v_p = \frac{\omega}{\beta} \qquad \textbf{(A.7)}$$

Wavelength

- For interconnect excited by sinusoidal source, if we freeze the time at a certain instant, say $t = t_o$, the **E** and **H** fields profile will vary in a sinusoidal manner along z-axis.



$$\lambda = \frac{v_p}{f}$$

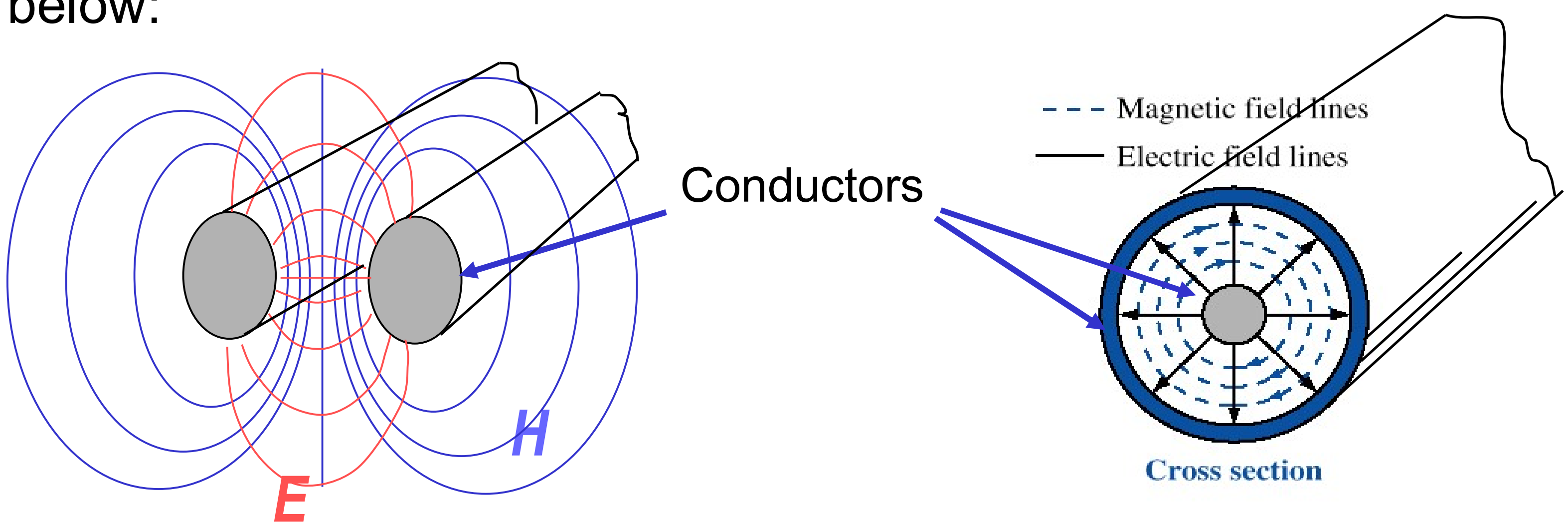
$$\boxed{\lambda = \frac{\frac{\omega}{\beta}}{\frac{\omega}{2\pi}} = \frac{2\pi}{\beta}} \quad \textbf{(A.8)}$$

Superposition Theorem

- At any instant of time, there are E and H fields propagating in the positive and negative direction along the transmission line. The total fields are a superposition of positive and negative directed fields:

$$\vec{E} = \vec{E}^{+} + \vec{E}^{-} \qquad \vec{H} = \vec{H}^{+} + \vec{H}^{-}$$

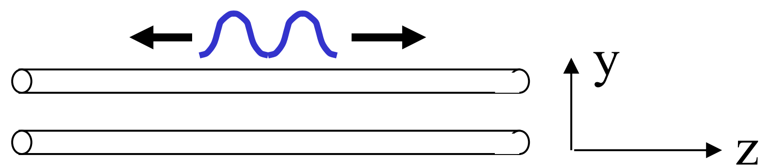
- A typical field distribution at a certain instant of time for the cross section of two interconnects (two-wire and co-axial cable) is shown below:



Field Solution (1)

- To find the value of β and the functions e_x, e_y, e_z, h_x, h_y, h_z, we substitute equations (A.2a) and (A.2b) into Maxwell or Wave equations.

$$\vec{E}^+ = e_x(x,y)e^{-j\beta z}\hat{x} + e_y(x,y)e^{-j\beta z}\hat{y} + e_z(x,y)e^{-j\beta z}\hat{z}$$
$$= \left(\vec{e}_t(x,y) + e_z(x,y)\hat{z}\right)e^{-j\beta z}$$
$$\vec{H}^+ = h_x(x,y)e^{-j\beta z}\hat{x} + h_y(x,y)e^{-j\beta z}\hat{y} + h_z(x,y)e^{-j\beta z}\hat{z}$$
$$= \left(\vec{h}_t(x,y) + h_z(x,y)\hat{z}\right)e^{-j\beta z}$$



Maxwell Equations

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$
$$\nabla \times \vec{H} = \vec{J} + j\omega\varepsilon\vec{E}$$
$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$$
$$\nabla \cdot \vec{H} = 0$$

+

Boundary conditions

+

Wave Equations

$$\nabla^2\vec{E} + k_o^2\vec{E} = 0$$
$$\nabla^2\vec{H} + k_o^2\vec{H} = 0$$
$$k_o = \omega\sqrt{\varepsilon\mu}$$

$e_x(x,y)$
$e_y(x,y)$
$e_z(x,y)$
$h_x(x,y)$
$h_y(x,y)$
$h_z(x,y)$
β

Field Solution (2)

- The procedure outlined here follows those from Pozar [2]. Assume the Tline or waveguide dielectric region is source free. From Maxwell's Equations:

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad \nabla \times \vec{H} = j\omega\varepsilon\vec{E} \qquad \textbf{(A.9)}$$

$$\vec{J} = 0$$

$$\nabla \times \widetilde{B} = \mu\widetilde{J} + j\omega\mu\varepsilon\widetilde{E}$$

- Substituting the suggested solution for **E**$^+(x,y,z,\beta)$ of (A.2a) into (A.9), and expanding the differential equations into *x*, *y* and *z* components:

$$\frac{\partial e_z}{\partial y} + j\beta e_y = -j\omega\mu h_x \qquad \textbf{(A.10a)}$$

$$\frac{\partial h_z}{\partial y} + j\beta h_y = j\omega\varepsilon e_x \qquad \textbf{(A.10d)}$$

$$-j\beta e_x - \frac{\partial e_z}{\partial x} = -j\omega\mu h_y \qquad \textbf{(A.10b)}$$

$$-j\beta h_x - \frac{\partial h_z}{\partial x} = j\omega\varepsilon e_y \qquad \textbf{(A.10e)}$$

$$\frac{\partial e_y}{\partial x} - \frac{\partial e_x}{\partial y} = -j\omega\mu h_z \qquad \textbf{(A.10c)}$$

$$\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y} = j\omega\varepsilon e_z \qquad \textbf{(A.10f)}$$

Field Solution (3)

- From (A.10a)-(A.10f), we can express e_x, e_y, h_x, h_y in terms of e_z and h_z:

These equations describe the *x*,*y* components of general EM wave propagation in a waveguiding system. The unknowns are $e_z(x,y)$ and $h_z(x,y)$, called the **Potential** in the literature.

$$h_x = \frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial e_z}{\partial y} - \beta\frac{\partial h_z}{\partial x}\right) \quad \textbf{(A.11a)}$$

$$h_y = \frac{-j}{k_c^2}\left(\omega\varepsilon\frac{\partial e_z}{\partial x} + \beta\frac{\partial h_z}{\partial y}\right) \quad \textbf{(A.11b)}$$

$$e_x = \frac{-j}{k_c^2}\left(\beta\frac{\partial e_z}{\partial x} + \omega\mu\frac{\partial h_z}{\partial y}\right) \quad \textbf{(A.11c)}$$

$$e_y = \frac{j}{k_c^2}\left(-\beta\frac{\partial e_z}{\partial y} + \omega\mu\frac{\partial h_z}{\partial x}\right) \quad \textbf{(A.11d)}$$

$$k_c^2 = k_o{}^2 - \beta^2 \ , \ k_o = \omega\sqrt{\mu\varepsilon} = \frac{2\pi}{\lambda} \quad \textbf{(A.11e)}$$

See the book by Collin [1], Chapter 3 for alternative derivation

TE Mode Summary (1)

- For TE mode, $e_z = 0$ (Sometimes this is called the H mode).
- We could characterize the Tline in TE mode, by EM fields:

$$\vec{H}^{\pm} = \left(\pm \vec{h}_t + h_z \hat{z}\right) e^{\mp j\beta z} \qquad \vec{E}^{\pm} = \vec{e}_t e^{\mp j\beta z}$$

- From wave equation for H field:

$$\nabla^2 \vec{H} + k_o^2 \vec{H} = 0$$

$$k_o = \omega\sqrt{\varepsilon\mu}$$

Note

$$\nabla^2 = \nabla_t^2 - \beta^2$$

$$\left(\nabla_t^2 + \frac{\partial^2}{\partial z^2} + k_o^2\right)\left(\left(\vec{h}_t + h_z \hat{z}\right) e^{-j\beta z}\right) = 0$$

Using the fact that $\frac{\partial^2}{\partial z^2}\left(e^{\mp j\beta z}\right) = -\beta^2 e^{\mp j\beta z}$

From (A.11e)

$$\boxed{\nabla_t^2 h_z + k_c^2 h_z = 0}$$

$$\nabla_t^2 \vec{h}_t + k_c^2 \vec{h}_t = 0$$

$$\boxed{k_c^2 = k_o^2 - \beta^2}$$

Only these are needed. The other transverse field components can be derived from h_z

TE Mode Summary (2)

- Setting $e_z = 0$ in (A.11a)-(A.11d):

$$h_x = \frac{-j\beta}{k_c^2}\frac{\partial h_z}{\partial x} \qquad h_y = \frac{-j\beta}{k_c^2}\frac{\partial h_z}{\partial y}$$

$$e_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial y} \qquad e_y = \frac{j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial x}$$

(A.12a)

- These equations plus the previous wave equation for h_z enable us to find the complete field pattern for TE mode.

$$\boxed{\begin{aligned} &\nabla_t{}^2 h_z + k_c^2 h_z = 0 \\ &k_c{}^2 = k_o{}^2 - \beta^2 \end{aligned}}$$

From previous slide

+ boundary conditions for E and H fields

(A.12b)

TE Mode Summary (3)

- From (A.12a) and (A.12b), we can show that:

$$\nabla_t \times \vec{e}_t = \left[\frac{\partial e_y}{\partial x} - \frac{\partial e_x}{\partial y}\right]\hat{z} = \frac{j\omega\mu}{k_c^2}\left[\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right]\hat{z}$$

$$= \frac{j\omega\mu}{k_c^2}\left(-k_c^2 h_z\right)\hat{z} = -j\omega\mu h_z \hat{z}$$

$$\nabla_t \times \vec{e}_t = -j\omega\mu h_z \hat{z} \neq 0$$

$$\nabla_t \times \vec{h}_t = 0$$

- Therefore we cannot define a unique voltage by (but we can define a unique current):

$$V_t = -\int_{C_1}^{C_2} \vec{e}_t \cdot d\vec{l}$$

- Also from (A.12a) we can define a wave impedance for the TE mode.

$$\boxed{Z_{TE} = \frac{e_x}{h_y} = \frac{k_o Z_o}{\beta} = \frac{-e_y}{h_x}} \qquad \textbf{\textit{(A.13)}}$$

TM Mode Summary (1)

- For TM mode, $h_z = 0$ (Sometimes this is called the E mode).
- We could characterize the Tline in TM mode, by EM fields:

$$\vec{H}^{\pm} = \pm\vec{h}_t e^{\mp j\beta z} \qquad \vec{E}^{\pm} = \left(\vec{e}_t \pm e_z\hat{z}\right)e^{\mp j\beta z}$$

- From wave equation for E field:

$$\nabla^2\vec{E} + k_o{}^2\vec{E} = 0$$

$$k_o = \omega\sqrt{\varepsilon\mu}$$

$$\left(\nabla_t{}^2 + \frac{\partial^2}{\partial Z^2} + k_o^2\right)\left(\left(\vec{e}_t + e_z\hat{z}\right)e^{-j\beta z}\right) = 0$$

$$\boxed{\nabla_t{}^2 e_z + k_c{}^2 e_z = 0}$$

$$\nabla_t{}^2\vec{e}_t + k_c{}^2\vec{e}_t = 0$$

$$\boxed{k_c{}^2 = k_o^2 - \beta^2}$$

Only these are needed. The other transverse field components can be derived from e_z

TM Mode Summary (2)

- Setting $h_z = 0$ in (A.11a)-(A.11d):

$$h_x = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial e_z}{\partial y} \qquad h_y = \frac{-j\omega\varepsilon}{k_c^2}\frac{\partial e_z}{\partial x}$$

$$e_x = \frac{-j\beta}{k_c^2}\frac{\partial e_z}{\partial x} \qquad e_y = \frac{-j\beta}{k_c^2}\frac{\partial e_z}{\partial y}$$

(A.14a)

- These equations plus the previous wave equation for e_z enable us to find the complete field pattern for TM mode.

$$\boxed{\begin{aligned} &\nabla_t{}^2 e_z + k_c^2 e_z = 0 \\ &k_c{}^2 = k_o{}^2 - \beta^2 \end{aligned}}$$

+ boundary conditions for E and H fields

(A.14b)

From previous slide

TM Mode Summary (3)

- Similarly from (A.14a) and (A.14b), we can show that

$$\nabla_t \times \vec{h}_t = j\omega\varepsilon e_z \hat{z} \neq 0$$

$$\nabla_t \times \vec{e}_t = 0$$

- We cannot define a unique current by (but we can define a unique voltage):

$$I_t = \oint_C \vec{h}_t \cdot d\vec{l}$$

- Also from (A.14a) we can define a wave impedance for the TM mode.

$$Z_{TM} = \frac{e_x}{h_y} = \frac{\beta}{\omega\varepsilon} = \frac{-e_y}{h_x} \qquad \textbf{\textit{(A.15)}}$$

TEM Mode (1)

- TEM mode is particularly important, characterized by $e_z = h_z = 0$. For +ve propagating waves:

$$\vec{E}^{\pm} = \vec{e}_t e^{\mp j\beta z} \qquad \vec{H}^{\pm} = \pm \vec{h}_t e^{\mp j\beta z} \qquad \textbf{(A.16a)}$$

- Setting $e_z = h_z = 0$ in (A.11a)-(A.11d), we observe that $k_c = 0$ in order for a non-zero solution to exist. This implies:

$$\beta = k_o = \omega\sqrt{\mu\varepsilon} \qquad \textbf{(A.16b)}$$

- Applying Hemholtz Wave Equation to E field:

$$\left(\nabla^2 + k_o^2\right)\vec{e}_t e^{-j\beta z} = \left(\nabla_t^2 + \frac{\partial^2}{\partial z^2}\hat{z} + k_o^2\right)\vec{e}_t e^{-j\beta z} = 0$$

$$\boxed{\nabla_t = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}}$$

$$\Rightarrow \boxed{\nabla_t^2\left(\vec{e}_t e^{-j\beta z}\right)} + \underbrace{\left(\frac{\partial^2}{\partial z^2} e^{-j\beta z} + k_o^2 e^{-j\beta z}\right)}_{0}\vec{e}_t = 0$$

$$\Rightarrow \nabla_t^2 \vec{e}_t = 0 \qquad \textbf{(A.17a)}$$

TEM Mode (2)

- The same can be shown for h_t:

$$\nabla_t^2 \vec{h}_t = 0 \qquad \textbf{(A.17b)}$$

- The transverse fields e_t and h_t are similar to the static fields that can exist between conductors, so we could define a transverse scalar potential Φ:

See [1], Chapter 3 for alternative derivation

$$\vec{e}_t = -\nabla_t \boxed{\Phi(x,y)}$$ Transverse potential

$$\Rightarrow \nabla_t{}^2 \Phi(x,y) = 0 \qquad \textbf{(A.18)}$$

- Also note that:

$$\nabla_t \times \vec{e}_t = \nabla_t \times \left(-\nabla_t \Phi\right) = 0 \qquad \textbf{(A.19)}$$

Using an important identity in vector calculus

$\nabla \times \left(\nabla F\right) = 0$ Where F is arbitrary function of position, i.e. F = F(x,y,z).

TEM Mode (3)

- Normally we would find e_t from (A.17a), then we derive h_t from e_t :

In free space $\Longrightarrow \nabla^2 \vec{h}_t = 0 \;,\; \nabla \times \vec{h}_t = 0$

$$\vec{H} = \frac{-1}{j\omega\mu}\nabla \times \vec{E}$$

$$\vec{H}_t = \frac{-1}{j\omega\mu}\left[-\frac{\partial E_y}{\partial z}\hat{x} + \frac{\partial E_x}{\partial z}\hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{z}\right]$$

$$= \frac{-1}{j\omega\mu}\left[j\beta E_y\hat{x} - j\beta E_x\hat{y} + \left(\frac{\partial e_y}{\partial x} - \frac{\partial e_x}{\partial y}\right)e^{-j\beta z}\hat{z}\right]$$

$\nabla_t \times \vec{e}_t = 0$

Exercise: see if you can derive this equation

$$\Rightarrow \vec{h}_t = \sqrt{\frac{\mu}{\varepsilon}}^{\,-1}\left(e_y\hat{x} - e_x\hat{y}\right)$$

Z_o = Intrinsic impedance of free space

Z_{TEM} = Wave impedance of TEM mode

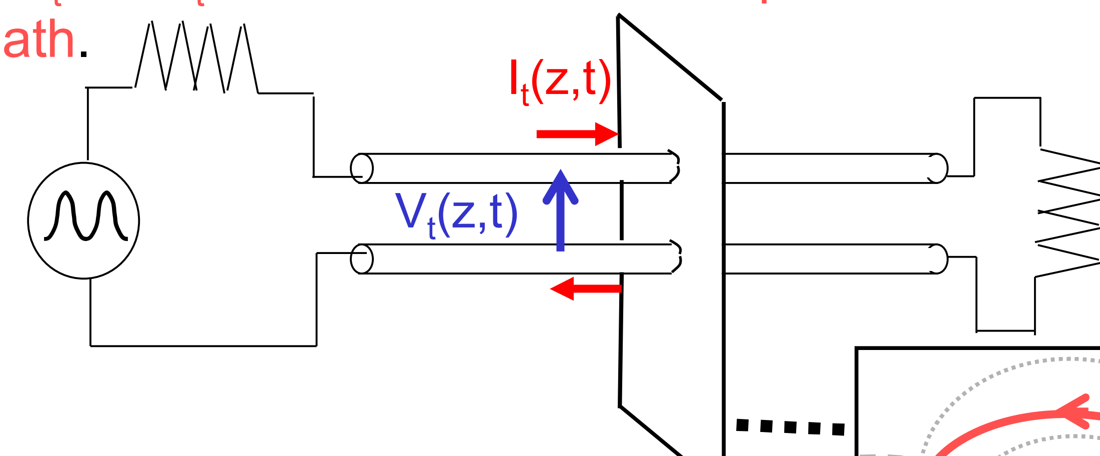
$$\Rightarrow \vec{h}_t = \frac{1}{Z_o}\left(\hat{z} \times \vec{e}_t\right) \quad \textbf{(A.20a)}$$

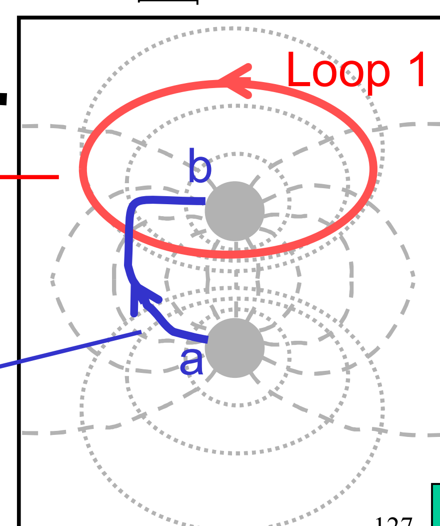
$$Z_o = \sqrt{\frac{\mu}{\varepsilon}} = \frac{e_x}{h_y} = \frac{-e_y}{h_x} = Z_{TEM} \quad \textbf{(A.20b)}$$

- An important observation is that under TEM mode the transverse field components e_t and h_t fulfill similar equations as in electrostatic.

Voltage and Current under TEM Mode

- Due to $\nabla_t \times \vec{e}_t = 0$ and $\nabla_t \times \vec{h}_t = 0$ in the space surrounding the conductors, we could define unique transverse voltage (V_t) and transverse current (I_t) for the system following the standard definitions for V and I. The V_t and I_t so defined does not depends on the shape of the integration path.



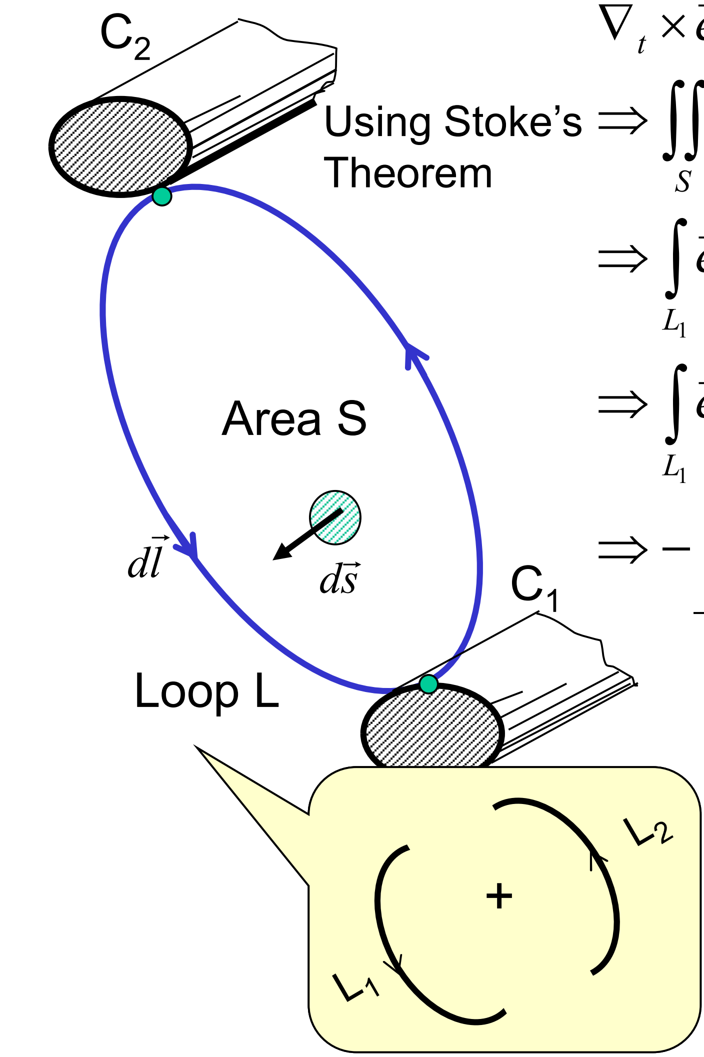


$$I_t(z,t) = \oint_{\text{Loop 1}} \vec{H} \cdot d\vec{l}$$

$$V_t(z,t) = -\int_a^b \vec{E} \cdot d\vec{l}$$

See the more detailed version of this note or see references [1] & [2].

Extra: Independence of V_t and I_t from Integration Path under TEM Mode



$$\nabla_t \times \vec{e}_t = 0$$

Using Stoke's Theorem $\Rightarrow \iint_S \nabla_t \times \vec{e}_t \cdot d\vec{s} = \int_L \vec{e}_t \cdot d\vec{l} = 0$

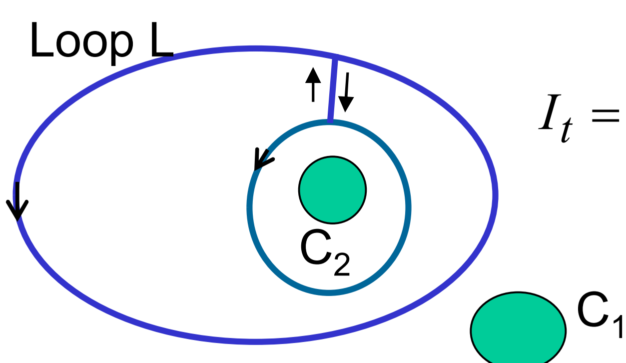
$$\Rightarrow \int_{L_1} \vec{e}_t \cdot d\vec{l} + \int_{L_2} \vec{e}_t \cdot d\vec{l} = 0$$

$$\Rightarrow \int_{L_1} \vec{e}_t \cdot d\vec{l} = -\int_{L_2} \vec{e}_t \cdot d\vec{l}$$

$$\Rightarrow -\int_{-L_1} \vec{e}_t \cdot d\vec{l} = -\int_{L_2} \vec{e}_t \cdot d\vec{l} = V_t$$

Since the shape of loop L is arbitrary, as long as it stays in the transverse plane, paths L_1, L_2 and hence the integration path for V_t is arbitrary.

Similar proof can be carried out for I_t, using the loop as shown and $\nabla_t \times \vec{h}_t = 0$



$$I_t = \oint_{C_1 \text{ or } C_2} \vec{h}_t \cdot d\vec{l}$$

TEM Mode Summary (1)

- For TEM mode, $h_z = e_z = 0$.
- To find EM fields for TEM mode:
  - Solve $\nabla_t^2 \Phi(x,y) = 0$ with boundary conditions.
  - Find E from $\vec{E}_t^{\pm} = [-\nabla_t \Phi(x,y)] e^{\mp j\beta z} = \vec{e}_t e^{\mp j\beta z}$
  - Find H from $\vec{H}_t^{\pm} = \dfrac{1}{Z_o} (\hat{z} \times \vec{e}_t) e^{\pm j\beta z}$
- We could characterize the Tline in TEM mode, by EM fields:

$$\vec{E}^{\pm} = \vec{e}_t e^{\mp j\beta z} \qquad \vec{H}^{\pm} = \pm \vec{h}_t e^{\mp j\beta z}$$

$$\beta = k_o = \omega\sqrt{\mu\varepsilon}$$

TEM Mode Summary (2)

- Or through auxiliary circuit theory quantities:

$$V^{\pm} = V_t e^{\mp j\beta z} \qquad I^{\pm} = \pm I_t e^{\mp j\beta z}$$

- The power carried by the EM wave along the Tline is given by Poynting Theorem:

$$P = \tfrac{1}{2}\mathrm{Re}\iint_S \left(\vec{E} \times \vec{H}^*\right)\cdot \overrightarrow{ds} = \tfrac{1}{2}\mathrm{Re}\, VI^*$$
$$\Rightarrow P = \tfrac{1}{2}\mathrm{Re}\left(Z_c II^*\right) = \tfrac{1}{2}\mathrm{Re}\left(Z_c^{-1} VV^*\right)$$

(A.21)

See extra note by F.Kung for the proof

- Because β is always real or complex (when dielectric is lossy) for all frequencies, the TEM mode always exist from near d.c. to extremely high frequencies.

$$\beta = k_o = \omega\sqrt{\mu\varepsilon}$$

Non-TEM Modes and V_t, I_t (1)

- For non-TEM modes, we cannot define both the auxiliary quantities V_t and I_t uniquely using the standard definition for voltage and current (because $\nabla_t \times \vec{e}_t \neq 0$ or $\nabla_t \times \vec{h}_t \neq 0$).
- For instance in TE mode: $\nabla_t \times \vec{e}_t = -j\omega\mu h_z$
- Thus $V_t = -\int_{C_1}^{C_2} \vec{e}_t \cdot \overrightarrow{dl}$ will not be unique and will depends on the line integration path. This means if we attempt to measure the "voltage" across the Tline using an instrument, the reading will depend on the wires and connection of the probe !
- Furthermore for non-TEM modes: $P = \frac{1}{2}\mathrm{Re}\iint_S \left(\vec{E} \times \vec{H}^*\right) \cdot \overrightarrow{ds} \neq \frac{1}{2}\mathrm{Re}\, V_t I_t^*$
- Thus we cannot characterize a Tline supporting non-TEM modes using auxiliary quantities such as V_t and I_t.

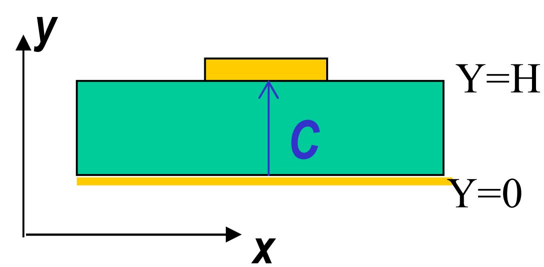
Extra

Non-TEM Modes and V_t, I_t (2)

- As another example consider the TM mode in microstrip line:

$$\vec{e}_t = -\frac{j\beta}{k_c^2}\nabla_t e_z = -\frac{j\beta}{k_c^2}\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}\right)e_z(x,y)$$

$$V_t = -\int_C \vec{e}_t\cdot\vec{dl} = \frac{j\beta}{k_c^2}\left(\int_C \frac{\partial e_z}{\partial x}dx + \int_C \frac{\partial e_z}{\partial y}dy\right)$$



- Using path C as shown in figure:

Under quasi-TEM condition, when $e_z \rightarrow 0$, k_c also $\rightarrow 0$, then V_t will be a non-zero value.

$$V_t = \frac{j\beta}{k_c^2}\left(\int_0^H \frac{\partial e_z}{\partial y}dy\right) = \frac{j\beta}{k_c^2}\left[e_z(x,H) - e_z(x,0)\right] = 0$$

0 because of boundary condition

- In general this is true for arbitrary Tline and waveguide cross section. If we choose integration path other than C, we still obtain V_t= 0 due to $\nabla_t \times \vec{e}_t = 0$ in TM mode.

Cut-off Frequency for TE/TM Mode

- Because $\beta = \sqrt{k_o^2 - k_c^2}$ for TE and TM modes:
- There is a possibility that β becomes imaginary when $k_c > k_o$. When this occur the TE or TM mode EM fields will decay exponentially from the source. These modes are known as Evanescent and are non-propagating.
- Thus for TE or TM mode, there is a possibility of a cut-off frequency f_c, where for signal frequency $f < f_c$, no propagating EM field will exist.

Phase Velocity for TEM, TE and TM Modes

Extra

- Phase velocity is the propagation velocity of the EM field supported by the tline. It is given by: $V_p = \frac{\omega}{\beta}$

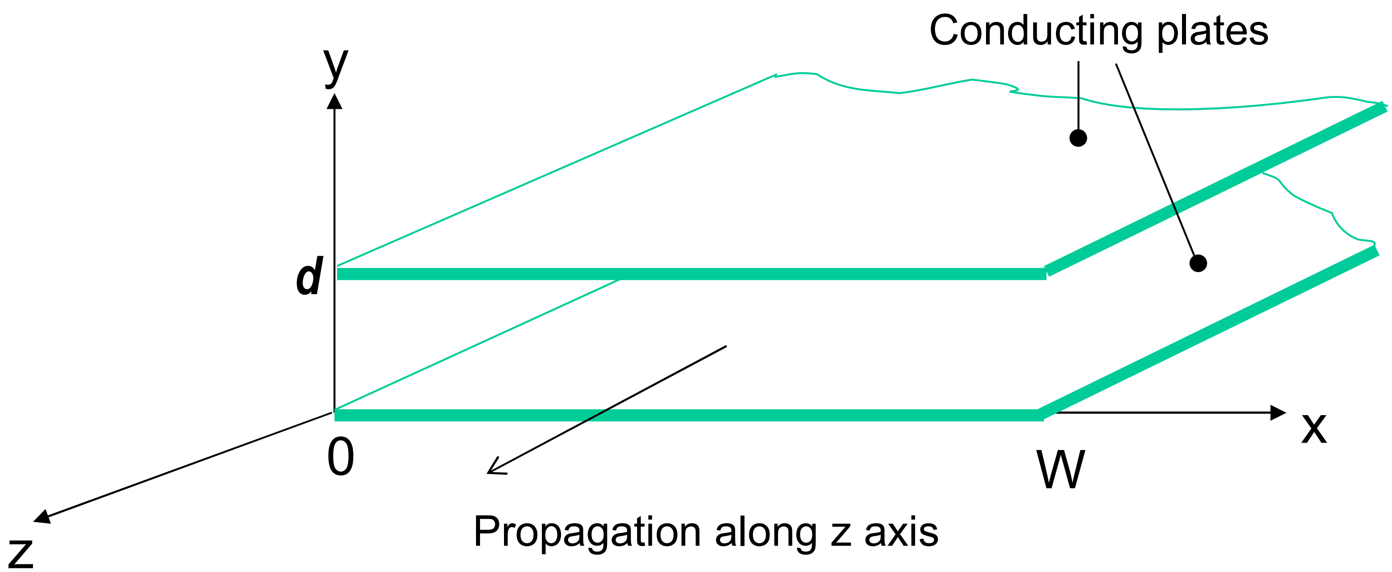
- For TEM mode: $V_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}$ ***(A.22)***

- For TE & TM mode: $V_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{k_o{}^2 - k_c{}^2}} = \frac{1}{\sqrt{\omega^2\mu\varepsilon - k_c^2}}$ ***(A.23)***

- Thus we observe that TEM mode is intrinsically non-dispersive, while TE and TM mode are dispersive.

Example A1.1 - Parallel Plate Waveguide/Tline

- The parallel plate waveguide is the simplest type of waveguide that can support TEM, TE and TM modes. Here we assume that W >> d so that fringing field and variation along *x* can be ignored. $\Rightarrow$ $\dfrac{\partial}{\partial x} = 0$



Example A1.1 Cont...

- Derive the EM fields for TEM, TE and TM modes for parallel plate waveguide.
- Show that TEM mode can exist for all frequencies.
- Show that TE and TM modes possess cut-off frequency f_c , where for operating frequency f less than f_c , the resulting EM field cannot propagate.

Example A1.1 – Solution for TEM Mode (1)

TEM mode Solution:

$$\nabla_t^{\,2}\Phi(x,y)=0 \ \text{ for } \ 0\le x\le W \ , \ 0\le y\le d$$

$$\Phi(x,0)=0$$

$$\Phi(x,d)=V_o$$

Boundary conditions

Solution for the transverse Laplace PDE:

$$\nabla_t^2\Phi=\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0 \quad \Rightarrow \frac{\partial^2\Phi}{\partial y^2}\cong 0 \qquad \text{since} \quad \Phi(x,y)=\Phi(y)$$

$$\Rightarrow \Phi(x,y)=A+By$$

General Solution

$$\Phi(x,0)=A=0 \quad \Rightarrow \quad A=0$$

$$\Phi(x,d)=Bd=V_o \quad \Rightarrow \quad B=\frac{V_o}{d}$$

Thus $\boxed{\Phi(x,y)=\frac{V_o}{d}y}$

Unique solution

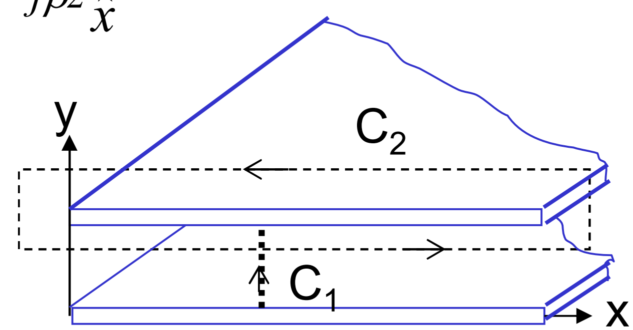
Example A1.1 – Solution for TEM Mode (2)

Computing the E and H fields:

$$\vec{e}_t = -\nabla_t \Phi = -\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}\right)\left(\frac{V_o}{d}y\right) = -\frac{V_o}{d}\hat{y}$$

$$\vec{E}_t = -\frac{V_o}{d}e^{-j\beta z}\hat{y}$$

$$\vec{H}_t = \frac{-1}{j\omega\mu}\nabla\times\vec{E}_t = \frac{1}{\sqrt{\frac{\mu}{\varepsilon}}}\left(\hat{z}\times\left(-\frac{V_o}{d}e^{-j\beta z}\hat{y}\right)\right) = \sqrt{\frac{\varepsilon}{\mu}}\frac{V_o}{d}e^{-j\beta z}\hat{x}$$



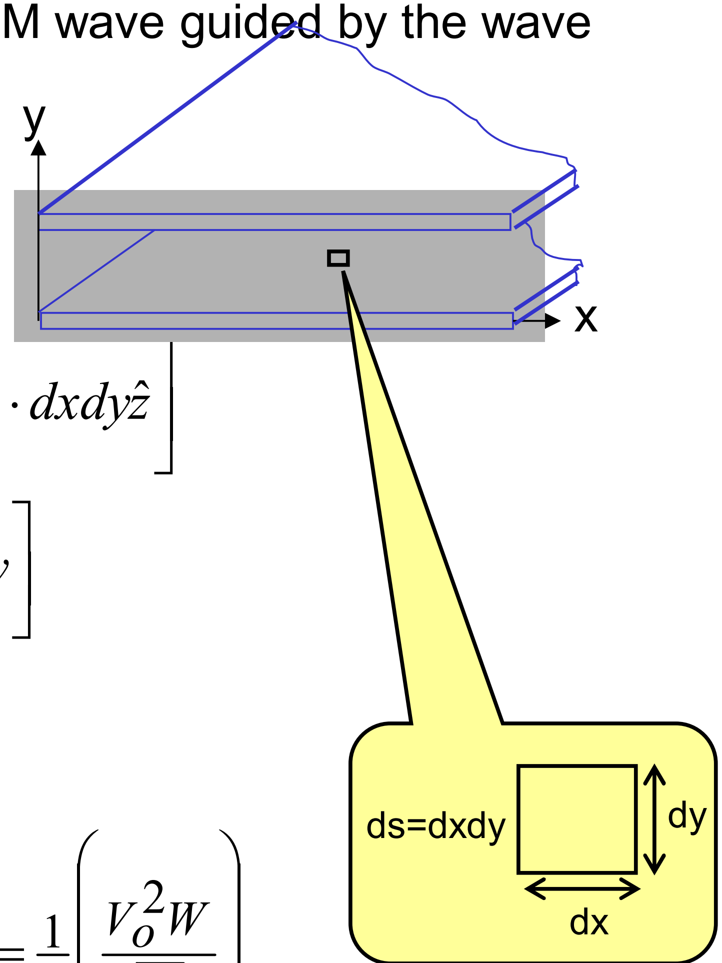
Computing the transverse voltage and current:

$$V_t = -\int_{C_1}\vec{E}_t\cdot d\vec{l} = -\int_0^d\left(-\frac{V_o}{d}e^{-j\beta z}\hat{y}\right)\cdot dy\hat{y} = V_o e^{-j\beta z}$$

$$I_t = \oint_{C_2}\vec{H}_t\cdot d\vec{l} = \int_0^W\left(\sqrt{\frac{\varepsilon}{\mu}}\frac{V_o}{d}e^{-j\beta z}\hat{x}\right)\cdot dx\hat{x} = \sqrt{\frac{\varepsilon}{\mu}}\frac{V_o}{d}We^{-j\beta z}$$

Example A1.1 – Solution for TEM Mode (3)

Computing the power flow (power carried by the EM wave guided by the wave -guide):



$$P = \frac{1}{2}\mathrm{Re}\left[\iint_S \vec{E}_t \times \vec{H}_t^{\,*} \cdot d\vec{s}\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\int_0^W \int_0^d \left(-\frac{V_o}{d}e^{-j\beta z}\hat{y}\right) \times \left(\sqrt{\frac{\varepsilon}{\mu}}\frac{V_o}{d}e^{+j\beta z}\hat{x}\right) \cdot dxdy\hat{z}\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\int_0^W \int_0^d \left(-\frac{V_o}{d}e^{-j\beta z}\right)\left(\sqrt{\frac{\varepsilon}{\mu}}\frac{V_o}{d}e^{+j\beta z}\right) dxdy\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\left(\int_0^d \frac{V_o}{d}dy\right)e^{-j\beta z}\left(\int_0^W \sqrt{\frac{\varepsilon}{\mu}}\frac{V_o}{d}dx\right)e^{+j\beta z}\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[V_o e^{-j\beta z} \cdot \sqrt{\frac{\varepsilon}{\mu}}\frac{V_o W}{d}e^{j\beta z}\right] = \frac{1}{2}\mathrm{Re}\left[V_t I_t^*\right] P = \frac{1}{2}\left(\frac{V_o^2 W}{\sqrt{\frac{\mu}{\varepsilon}}d}\right)$$

Example A1.1 – Solution for TEM Mode (4)

Phase velocity v_p for TEM mode:

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

The phase velocity is equal to speed-of-light in the dielectric.

Example A1.1 – Solution for TM Mode (1)

$$\left(\nabla_t^{\,2} + k_c^2\right)e_z(x,y) = 0 \text{ for } 0 \le x \le W \;,\; 0 \le y \le d$$

$$k_c^2 = k_o^2 - \beta^2$$

$$e_z(x,0) = e_z(x,d) = 0$$ ← Boundary conditions

Solution for e_z:

$$\left(\nabla_t^{\,2} + k_c^2\right)e_z \cong \frac{\partial^2 e_z}{\partial y^2} + k_c^2 e_z = 0$$ since $e_z(x,y) = e_z(y)$

$$\Rightarrow \mathrm{e}_z(x,y) = A\sin(k_c y) + B\cos(k_c y)$$ ← General solution

Applying boundary conditions:

$$\mathrm{e}_z(x,0) = A \cdot 0 + B = 0 \;\Rightarrow\; B = 0$$

$$\mathrm{e}_z(x,d) = A\sin(k_c d) = 0$$

$$\Rightarrow A \ne 0 \text{ and } k_c d = n\pi \;,\; n = 1,2,3\cdots$$

$$\Rightarrow k_c = \frac{n\pi}{d}$$

Thus:

$$\boxed{\mathrm{e}_z(x,y) = A_n \sin\left(\frac{n\pi}{d}y\right)}$$

or

$$\boxed{E_z(x,y) = A_n \sin\left(\frac{n\pi}{d}y\right)e^{-j\beta z}}$$

Example A1.1 – Solution for TM Mode (2)

Computing the transverse EM fields using (A.20a):

$$\vec{e}_t = \frac{-j\beta}{k_c^2}\frac{\partial e_z}{\partial x}\hat{x} + \frac{-j\beta}{k_c^2}\frac{\partial e_z}{\partial y}\hat{y} = -\frac{j\beta}{k_c^2}\frac{\partial}{\partial y}\left(A_n \sin\left(\frac{n\pi}{d}y\right)\right)\hat{y}$$

$$\Rightarrow \vec{e}_t = -\frac{j\beta A_n}{\left(\frac{n\pi}{d}\right)}\cos\left(\frac{n\pi}{d}y\right)\hat{y}$$

or $$\vec{E}_t = -\frac{j\beta A_n}{\left(\frac{n\pi}{d}\right)}\cos\left(\frac{n\pi}{d}y\right)e^{-j\beta z}\hat{y}$$

$$\vec{H}_t = \left(\frac{j\omega\varepsilon}{k_c^2}\frac{\partial e_z}{\partial y}\hat{x} - \frac{j\omega\varepsilon}{k_c^2}\frac{\partial e_z}{\partial x}\hat{y}\right)e^{-j\beta z}$$

$$= \frac{jk_o A_n}{\sqrt{\frac{\mu}{\varepsilon}}\left(\frac{n\pi}{d}\right)}\cos\left(\frac{n\pi}{d}y\right)e^{-j\beta z}\hat{x}$$

Since n is an arbitrary integer, the TM mode is usually called the TM_n mode.

Example A1.1 – Solution for TM Mode (3)

We can now determine β knowing k_c:

$$\beta = \sqrt{k_o^2 - k_c^2} = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{n\pi}{d}\right)^2}$$

Since the TM mode can only propagate if β is real, and the smallest value for β Is 0, then when β=0:

$$\sqrt{\omega^2 \mu\varepsilon - \left(\frac{n\pi}{d}\right)^2} = 0$$

$$\Rightarrow \omega = \frac{n\pi}{d\sqrt{\mu\varepsilon}} = 2\pi f$$

$$\Rightarrow f = \frac{n}{2d\sqrt{\mu\varepsilon}}$$

When n = 1, this represent the cut-off frequency for TM mode.

$$\boxed{f_{cutoff\_TM} = \frac{1}{2d\sqrt{\mu\varepsilon}}}$$

Example A1.1 – Solution for TM Mode (4)

For arbitrary n, phase velocity v_p for TM_n mode:

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega^2 \mu\varepsilon - \left(\frac{n\pi}{d}\right)^2}}$$

For f > f_{cutoff} , we observe that phase velocity v_p is actually greater than the Speed of light!!!

NOTE:
The EM fields can travel at speed greater than light, however we can show that the rate of energy flow is less than the speed-of-light. This rate of energy flow corresponds to the speed of the photons if the propagating EM wave is treated as a cluster of photons. See the extra notes for the proof.

Example A1 – Solution for TE Mode

The EM fields for TE mode are shown below:

$$\vec{E}_t(x,y) = \frac{jk_oZ_oB_n}{\left(\frac{n\pi}{d}\right)} \sin\left(\frac{n\pi}{d} y\right) e^{-j\beta z}\hat{x}$$

$$\vec{H}_t(x,y) = \frac{j\beta B_n}{\left(\frac{n\pi}{d}\right)} \sin\left(\frac{n\pi}{d} y\right) e^{-j\beta z}\hat{y}$$

$$H_z(x,y) = B_n \cos\left(\frac{n\pi}{d} y\right) e^{-j\beta z}$$

Since *n* is an arbitrary integer, the TE mode is usually called the TE_n mode.

$$k_o = \omega\sqrt{\mu\varepsilon}$$

$$Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}}$$

Dominant Propagation Mode

- For the various transmission line topology, there is a dominant mode. This dominant mode of propagation is the first mode to exist at the lowest operating frequency. The secondary modes will come into existent at higher frequencies.
- The propagation modes on Tline line depends on the dielectric and the cross section of the transmission line.
- For Tline that can support TEM mode, the TEM mode will be the dominant mode as it can exist at all frequencies (there is no cut-off frequency).

Transmission Lines Dominant Propagation Mode

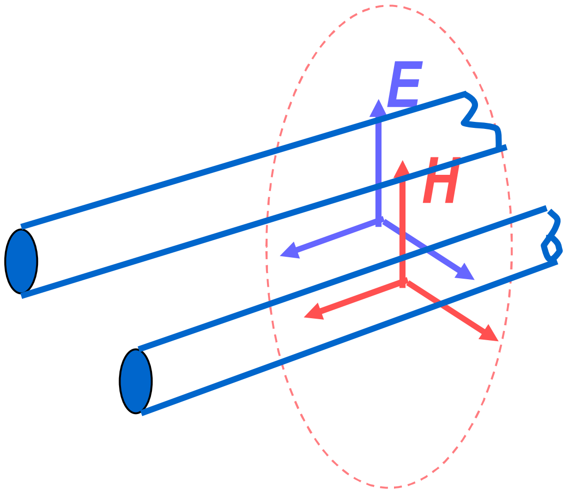
- Coaxial line - TEM.
- Microstrip line - quasi-TEM.
- Stripline - TEM.
- Parallel plate line - TEM or TM (depends on homogenuity of the dielectric).
- Co-planar line - quasi-TEM.
- Note: Generally for Tline with non-homogeneous dielectric, the Tline cannot support TEM propagation mode.

Quasi TEM Mode (1)

- Luckily for planar Tline configuration whose dominant mode is not TEM, the TM or TE dominant modes can be approximated by TEM mode at ‘low frequency’.
- For instance microstrip line does not support TEM mode. The actual mode is TM. However at a few GHz, e_z is much smaller than e_t and h_t that it can be ignored. We can assume the mode to be TEM without incurring much error. Thus it is called quasi-TEM mode.
- Low frequency approximation is usually valid when wavelength >> distance between two conductors. For microstrip/stripline, this can means frequency below 20 GHz or higher.
- The E_z and H_z components approach zero at ‘low frequency’, and the propagation mode approaches TEM, hence known as quasi-TEM. When this happens we can again define unique voltage and current for the system.

See Collin [1], Chapter 3 for more mathematical illustration on this.

Quasi TEM Mode (2)



$\nabla_t \times \vec{e}_t \neq 0$

or $\nabla_t \times \vec{h}_t \neq 0$

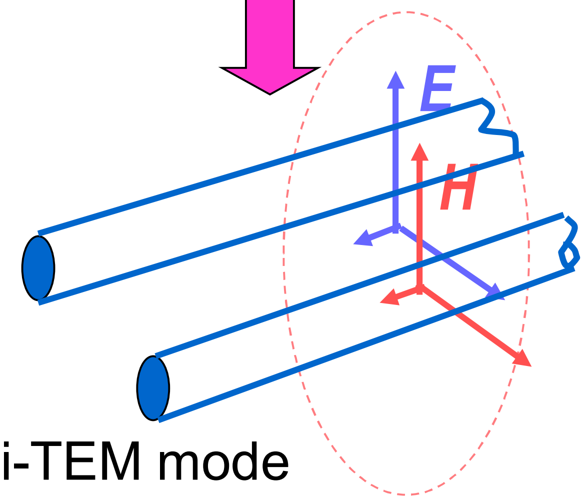
V_t and I_t
No

Non-TEM mode

$\nabla_t \times \vec{e}_t \cong 0$

or $\nabla_t \times \vec{h}_t \cong 0$

V_t and I_t
Yes

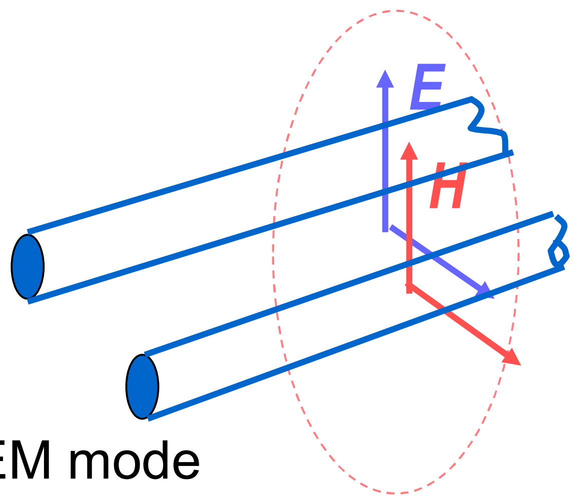


Quasi-TEM mode

$\nabla_t \times \vec{e}_t = 0$

or $\nabla_t \times \vec{h}_t = 0$

V_t and I_t
Yes



TEM mode

Extra: Why Inhomogeneous Structures Does Not Support Pure TEM Mode (1)

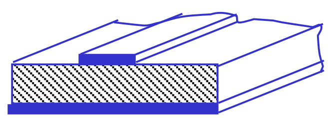
Extra

- We will use Proof by Contradiction. Suppose TEM mode is supported. The propagation factor in air and dielectric would be:

$$\beta_{air} = \omega\sqrt{\mu\varepsilon_o} \qquad \beta_{die} = \omega\sqrt{\mu\varepsilon_o\varepsilon_r}$$

- EM fields in air will travel faster than in the dielectric.

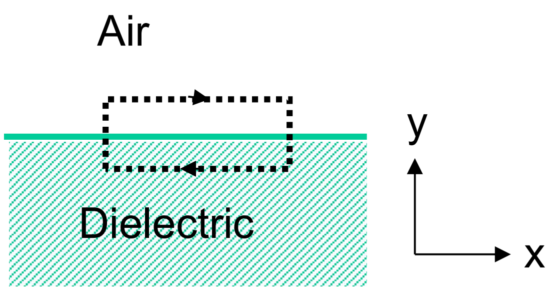
For TEM mode



$$\beta_{air} < \beta_{die}$$

$$\Rightarrow v_{p(air)} = \frac{\omega}{\beta_{air}} > v_{p(die)} = \frac{\omega}{\beta_{die}}$$

- No consider the boundary condition at the air/dielectric interface. The E field must be continuous across the boundary from Maxwell's equation. Examining the x component of E field:



$$E_{x(air)} e^{-j\beta_{air} z} = E_{x(die)} e^{-j\beta_{die} z}$$

$$\Rightarrow \frac{E_{x(air)}}{E_{x(die)}} = \frac{e^{-j\beta_{die} z}}{e^{-j\beta_{air} z}} = e^{-j(\beta_{die}-\beta_{air})z}$$

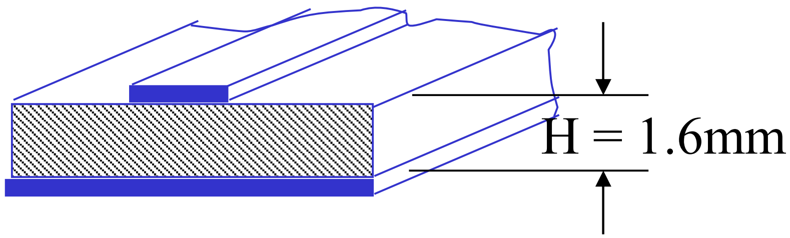
Extra: Why Inhomogeneous Structures Does Not Support Pure TEM Mode (2)

Extra

- Since the left hand side is a constant while the right hand side depends on distance z, the previous equation cannot be fulfilled.
- What this conclude is that our initial assumption of TEM propagation mode in inhomogeneous structure is wrong. So pure TEM mode cannot be supported in inhomogeneous dielectric Tline.

Example A2 – Minimum Frequency for Quasi-TEM Mode in Microstrip Line

- Estimate the low frequency limit for microstrip line.



$C \approx 3.0\times10^8$

$\lambda = C/f > 20H = 32.0\text{mm}$

$f < C/0.032 = 9.375\text{GHz}$

$f_{critical} = 9.375\text{GHz}$

Here we replace >> sign with the requirement that wavelength > 20H. You can use larger limit, as this is basically a rule of thumb.

Thus beyond $f_{critical}$ quasi-TEM approximation cannot be applied. The propagation mode beyond $f_{critical}$ will be TM. A more conser-vative limit would be to use 30H or 40H.

Summary for TEM, Quasi-TEM, TE and TM Modes

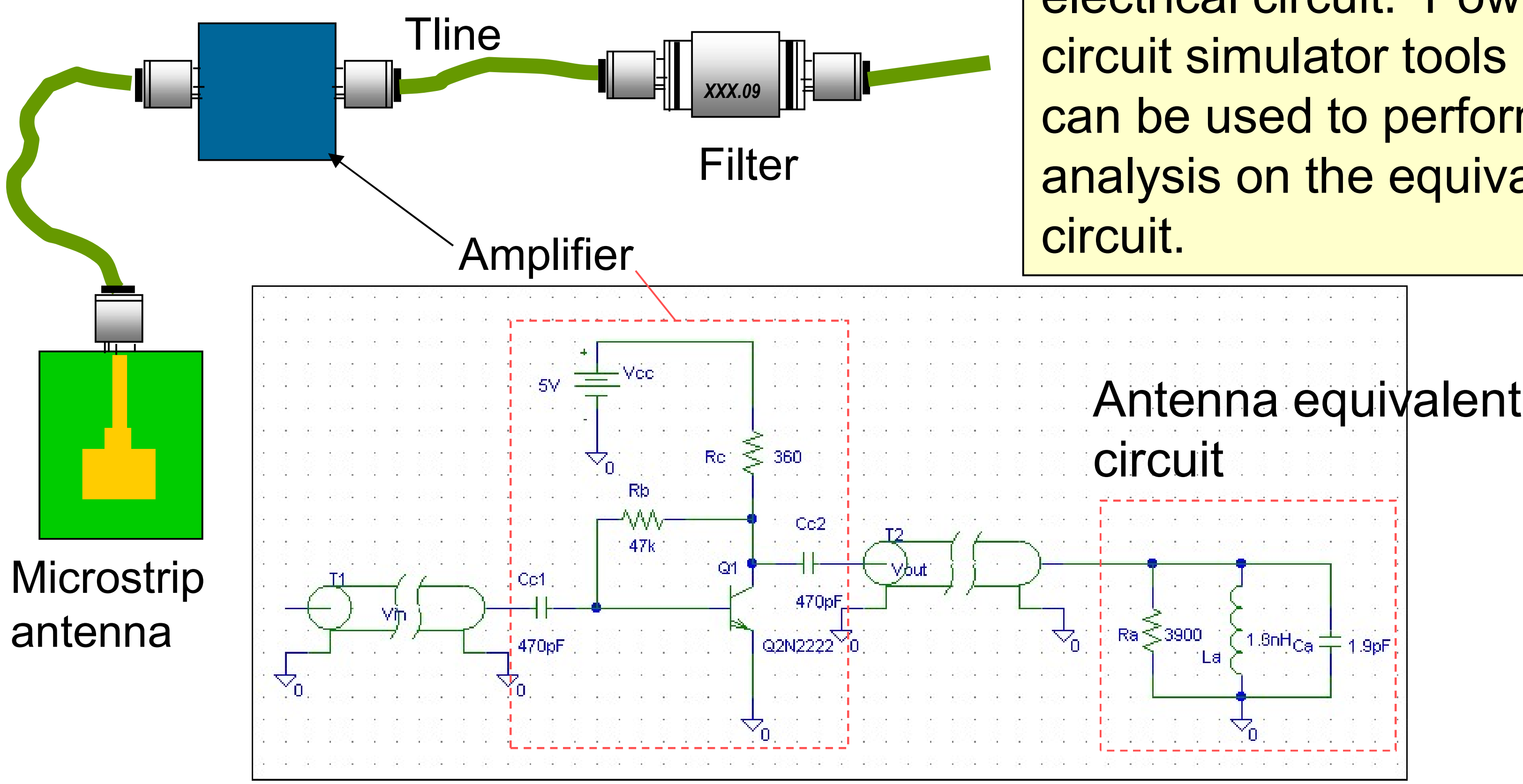
TEM:	Quasi-TEM:	TE:	TM:
$E_z = H_z = 0$	$E_z \approx 0$, $H_z \approx 0$	$E_z = 0$, $H_z \neq 0$	$E_z \neq 0$, $H_z = 0$
$\vec{E}^{\pm} = \vec{e}_t e^{\mp j\beta z}$	$\vec{E}^{\pm} \cong \vec{e}_t e^{\mp j\beta z}$	$\vec{H}^{\pm} = \left(\pm \vec{h}_t + h_z \hat{z}\right) e^{\mp j\beta z}$	$\vec{H}^{\pm} = \pm \vec{h}_t e^{\mp j\beta z}$
$\vec{H}^{\pm} = \pm \vec{h}_t e^{\mp j\beta z}$	$\vec{H}^{\pm} \cong \pm \vec{h}_t e^{\mp j\beta z}$	$\vec{E}^{\pm} = \vec{e}_t e^{\mp j\beta z}$	$\vec{E}^{\pm} = \left(\vec{e}_t \pm e_z \hat{z}\right) e^{\mp j\beta z}$
Can defined unique V_t and I_t.	Can defined unique V_t and I_t.	Cannot defined unique I_t.	Cannot defined unique V_t.
Physical Tline can be Modeled by equivalent Electrical circuit.	Physical Tline can be Modeled by equivalent Electrical circuit.	Physical Tline cannot be modeled by equivalent electrical circuit.	Physical Tline cannot be modeled by equivalent electrical circuit.
Phase velocity. $V_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}$	Phase velocity. $V_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{\mu\varepsilon_{eff}}}$	Phase velocity. $v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\omega^2\mu\varepsilon - k_c^2}}$	Phase velocity. $v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\omega^2\mu\varepsilon - k_c^2}}$
No cut-off frequency.	No cut-off frequency.	Cut-off frequency. $f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}}$	Cut-off frequency. $f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}}$
Non-dispersive	Non-dispersive	Dispersive	Dispersive

Why V_t and I_t is so Important ? (1)

- When we can define voltage and current along Tline or high-frequency circuit for that matter, then we can analyze the system using circuit theory instead of field theory.
- A system of equations using circuit theories such as KVL, KCL, 2-port network theory are much easier to solve than Maxwell equations or wave equations.
- High-frequency circuits usually consist of components which are connected by Tlines. Thus the microwave system can be modeled by an equivalent electrical circuit when dominant mode in the system is TEM or quasi-TEM. For this reason Tline which can support TEM or quasi-TEM is very important.

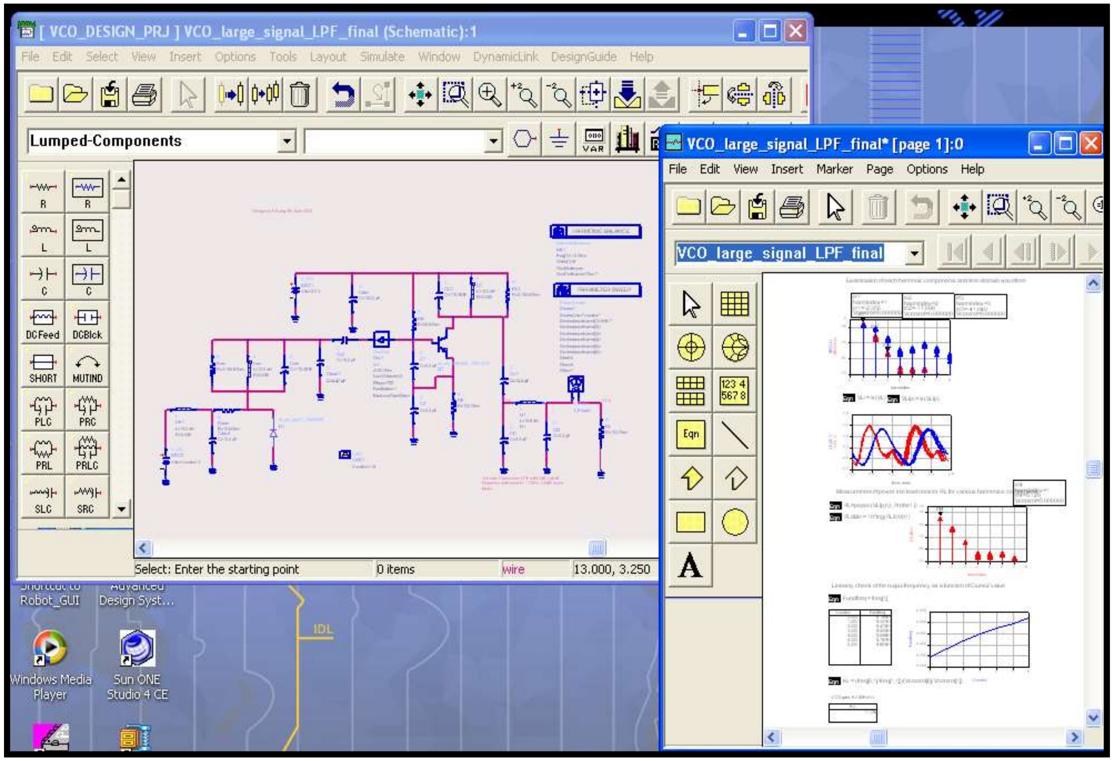
Why V and I is so Important ? (2)

A complex physical system can be cast into equivalent electrical circuit. Powerful circuit simulator tools can be used to perform analysis on the equivalent circuit.

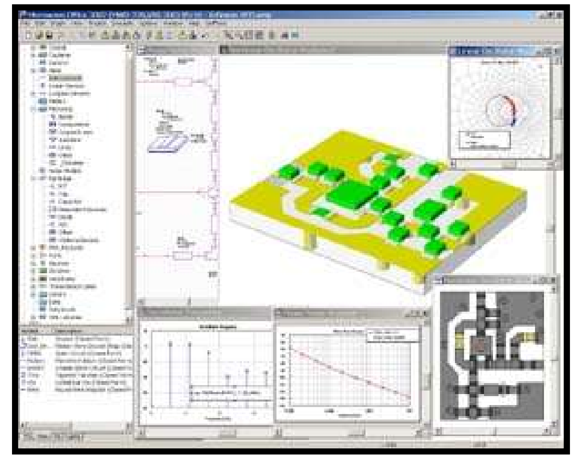


Examples of Circuit Analysis* Based Microwave/RF CAD Software

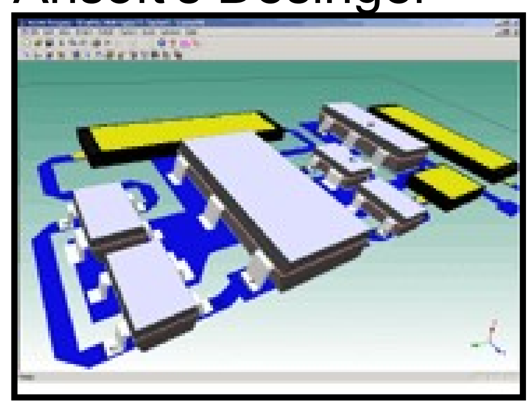
Agilent's Advance Design System™



Applied Wave Research's Microwave Office™



Ansoft's Desinger™

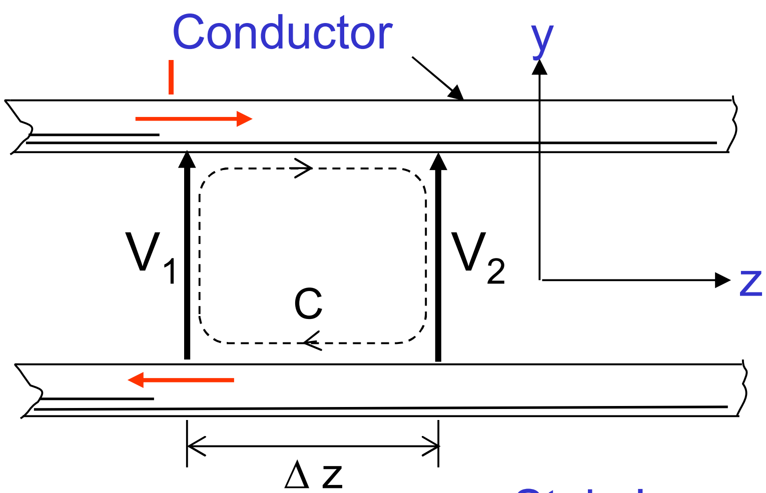


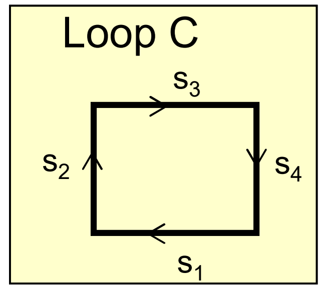
*The software shown here also have
optional numerical EM field solver capability, from 2D, 2.5D to full 3D, that can be integrated with the circuit simulator.

Appendix 2
Advanced Concepts – Distributed RLCG Model for Transmission Line and Telegraphic Equations

Distributed Parameters Model (1)

- For TEM or quasi-TEM mode propagation along z direction:





$$-\oint_C \vec{E}\cdot\vec{dl} = -\iint_S \nabla\times\vec{E}\cdot\vec{ds}$$

Stoke's Theorem

$$= -\int_{s1}\vec{E}\cdot\vec{dl} - \int_{s2}\vec{E}\cdot\vec{dl} - \int_{s3}\vec{E}\cdot\vec{dl} - \int_{s4}\vec{E}\cdot\vec{dl}$$

$$\Rightarrow -\iint_S \nabla\times\vec{E}\cdot\vec{ds} = -\int_{s2}\vec{E}\cdot\vec{dl} - \left(-\int_{-s4}\vec{E}\cdot\vec{dl} \right)$$

$$\Rightarrow -\iint_S \left(-j\mu\omega\vec{H}\right)\cdot\vec{ds} = V_1 - V_2$$

Flux linkage (Definition for Inductance)

$$-V_1 + V_2 = -j\omega\left(\boxed{\mu\iint_S \vec{H}\cdot\vec{ds}} \right)$$

$$\Rightarrow V_2 - V_1 = -j\omega(L\Delta z I)$$

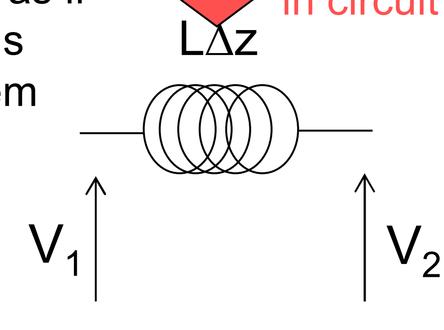
When Δz is small as compared to wavelength

$$\vec{H}^+ \cong \vec{h}_t(x,y) + h_z(x,y)\hat{z}$$

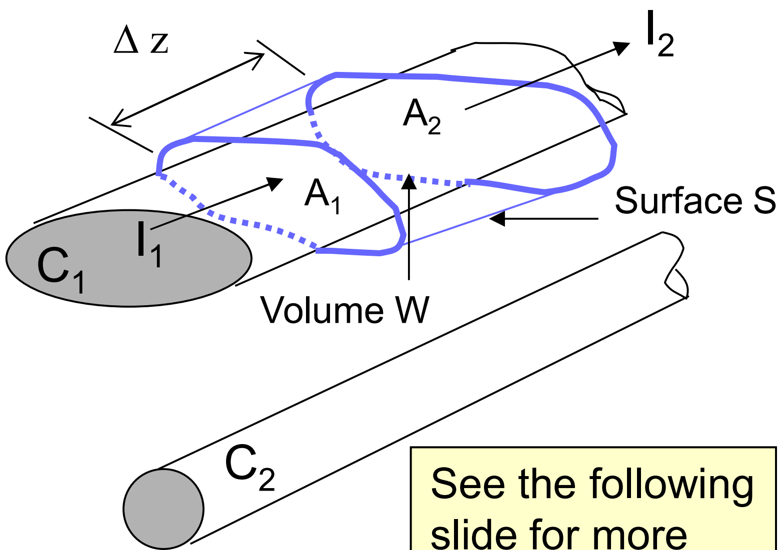
This means the relation between V_1 and V_2 is as if an inductor is between them

L is the inductance per meter

Can be represented in circuit as:



Distributed Parameters Model (2)



$$\iiint_W \nabla\cdot\vec{E}\,dW = \iiint_W \frac{\rho}{\varepsilon}dW$$

Using Divergence Theorem

$$\Rightarrow \iint_S \vec{E}\cdot d\vec{S} = \iiint_W \frac{\rho}{\varepsilon}dW$$

$$\Rightarrow \frac{\partial}{\partial t}\iint_S \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon}\iiint_W \frac{\partial\rho}{\partial t}dW$$

$$\Rightarrow \frac{\partial}{\partial t}\iint_S \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon}\iiint_W -\nabla\cdot\vec{J}dW$$

$$\Rightarrow \varepsilon\frac{\partial}{\partial t}\iint_S \vec{E}\cdot d\vec{S} = -\iint_S \vec{J}\cdot d\vec{S} = -\iint_{A_1} \vec{J}\cdot d\vec{S} - \iint_{A_2} \vec{J}\cdot d\vec{S}$$

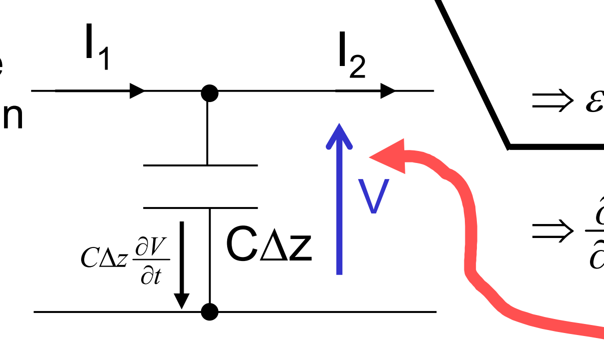
See the following slide for more proof.

$$\Rightarrow \varepsilon\frac{\partial}{\partial t}\iint_S \vec{E}\cdot d\vec{S} = -I_1 + I_2$$

$$\Rightarrow \frac{\partial}{\partial t}\left((C\Delta z)V\right) = \boxed{(C\Delta z)\frac{\partial V}{\partial t} = I_2 - I_1}$$

When Δz is small as compred to wavelength

This means the relation between I_1 and I_2 is as if a capacitor is between them

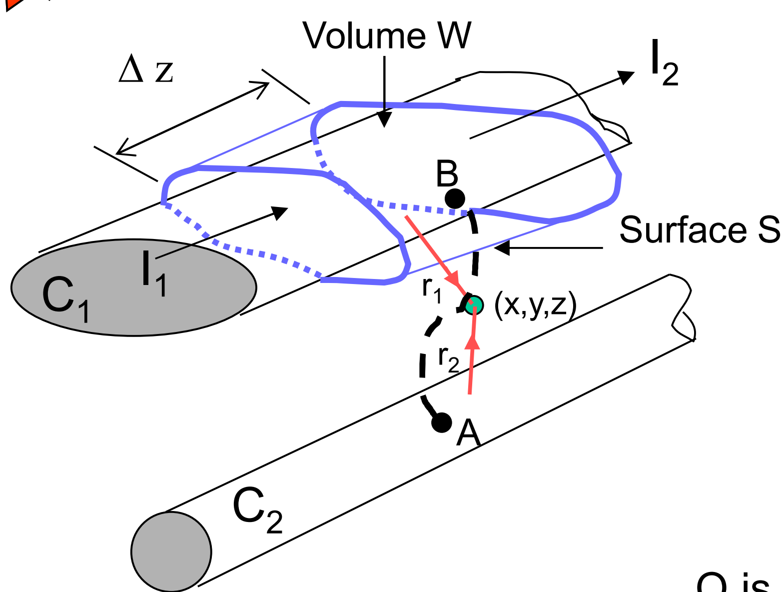


Can be represented in circuit as

- C is the per unit length capacitance between the 2 conductors of the Tline.

Distributed Parameters Model (3)

Extra



Consider the ratio:
$$\frac{\varepsilon \iint_S \vec{E}\cdot\vec{ds}}{-\int_A^B \vec{E}\cdot\vec{dl}} = \frac{Q}{-\int_A^B \vec{E}\cdot\vec{dl}}$$

For static or quasi-static condition, the E field is given by:

$$\vec{E} = \iint_{\text{surface on } C_1} \frac{\sigma_{s1}(x,y,z)}{4\pi\varepsilon}\cdot\frac{\hat{r}_1}{r_1}ds + \iint_{\text{surface on } C_2} \frac{\sigma_{s2}(x,y,z)}{4\pi\varepsilon}\cdot\frac{\hat{r}_2}{r_2}ds =$$
$$= Q\left[\iint_{\text{surface on } C_1} \frac{f_{s1}(x,y,z)}{4\pi\varepsilon}\cdot\frac{\hat{r}_1}{r_1}ds + \iint_{\text{surface on } C_2} \frac{f_{s2}(x,y,z)}{4\pi\varepsilon}\cdot\frac{\hat{r}_2}{r_2}ds\right]$$

Q is the total charge on conductors C_1 or C_2, σ_s is the surface charge density, while f_s is the normalized surface charge density with respect to Q.

Hence the ratio becomes:

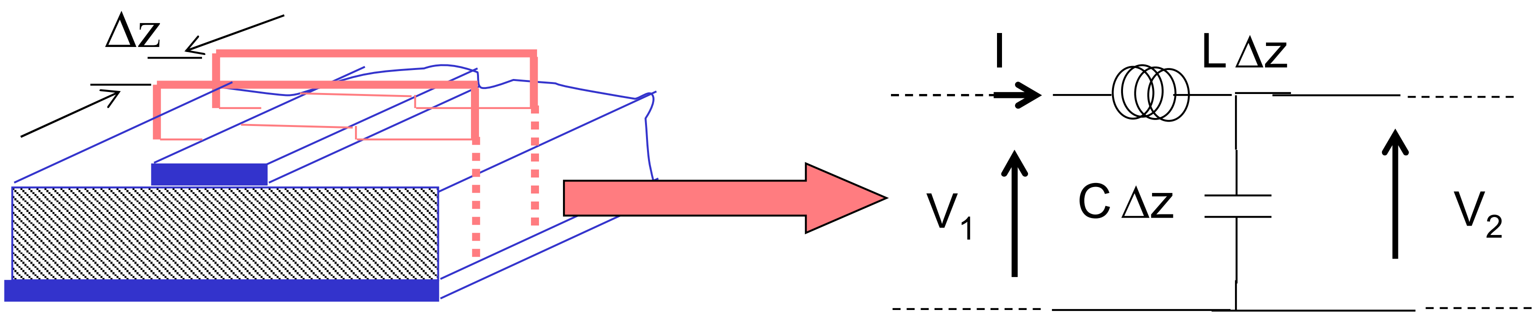
$$\frac{\varepsilon \iint_S \vec{E}\cdot\vec{ds}}{-\int_A^B \vec{E}\cdot\vec{dl}} = \frac{Q}{-Q\int_A^B\left[\iint_{\text{surface on } C_1} \frac{f_{s1}(x,y,z)}{4\pi\varepsilon}\cdot\frac{\hat{r}_1}{r_1}ds + \iint_{\text{surface on } C_2} \frac{f_{s2}(x,y,z)}{4\pi\varepsilon}\cdot\frac{\hat{r}_2}{r_2}ds\right]\cdot\vec{dl}}$$
$$= \frac{1}{-\int_A^B\left[\iint_{\text{surface on } C_1} \frac{f_{s1}(x,y,z)}{4\pi\varepsilon}\cdot\frac{\hat{r}_1}{r_1}ds + \iint_{\text{surface on } C_2} \frac{f_{s2}(x,y,z)}{4\pi\varepsilon}\cdot\frac{\hat{r}_2}{r_2}ds\right]\cdot\vec{dl}} = C$$

Potential difference between A and B

C does not depend on the total charge on the conductors, we called this constant the 'Capacitance'. $\varepsilon \iint_S \vec{E}\cdot\vec{ds} = CV$

Distributed Parameters (4)

- Combining the relationship between the L, C and transverse voltages and currents, the equivalent circuit for a short section of transmission line supporting TEM or quasi-TEM propagating EM field can be represented by the equivalent circuit:

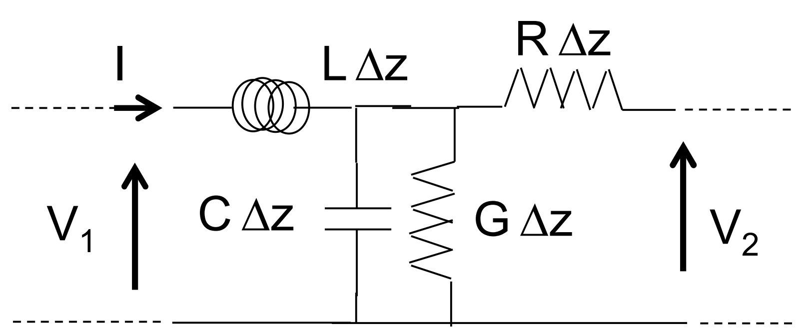


- Thus a long Tline can be considered as a cascade of many of these equivalent circuit sections. Working with circuit theory and circuit elements are much easier than working with E and H fields using Maxwell equations.

Distributed Parameters Model (5)

- When the conductor has small conductive loss a series resistance $R\Delta z$ can be added to the inductance.
- When the dielectric has finite conductivity, a shunt conductance $G\Delta z$ can be added in parallel to the capacitance.
- The inclusion of constant R and G in the Tline distributed model is only accurate for very small losses*. This is true most of the time as Tline is usually made of very good conductive material.
- The equations for finding L, C, R, G under low loss condition are given in the following slide.

*Under lossy condition, R, L and G are usually function of frequency, hence the Tline is dispersive.



Extra: Lossy Dielectric

Extra

- Assuming the dielectric is non-magnetic, then the dielectric loss is due to leakage (non-zero conductivity) and polarization loss*.
- Polarization loss is due to the vibration of the polarized molecules in the dielectric when an a.c. electric field is imposed.
- Both mechanisms can be modeled by considering an effective conductivity σ_d for the dielectric at the operating frequency. This is usually valid for small electric field.

Loss current density

$$\nabla \times \vec{H} = \vec{J} + j\omega\varepsilon_r\varepsilon_o\vec{E} \Rightarrow \nabla \times \vec{H} = \sigma_d\vec{E} + j\omega\varepsilon_r\varepsilon_o\vec{E}$$

*We should also include hysterisis loss in ferromagnetic material.

$$\Rightarrow \nabla \times \vec{H} = j\omega\varepsilon_r\varepsilon_o\left(1 + \frac{\sigma_d}{j\omega\varepsilon_r\varepsilon_o}\right)\vec{E}$$

$$\Rightarrow \nabla \times \vec{H} = j\omega\varepsilon_r\varepsilon_o(1 - j\tan\delta)\vec{E}$$

$$\nabla \times \vec{H} = j\omega\left(\varepsilon' - j\varepsilon''\right)\vec{E}$$

$$\varepsilon' = \varepsilon_r\varepsilon_o \qquad \varepsilon'' = \varepsilon_r\varepsilon_o\tan\delta$$

$$\tan\delta = \frac{\sigma_d}{\omega\varepsilon_r\varepsilon_o}$$

This is called Loss Tangent

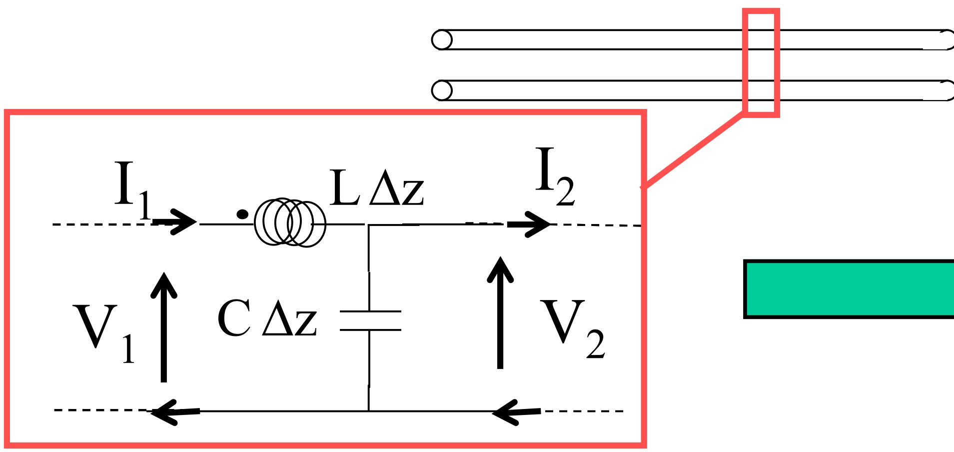
Note that σ_d is a function of frequency

Finding Distributed Parameters for Low Loss Practical Transmission Line

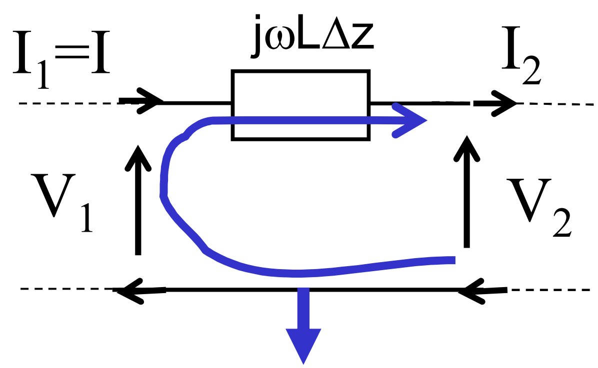
- When loss is present, the propagation mode will not be TEM anymore (Can you explain why this is so?).
- However if the loss is very small, we can assume the propagating EM field to be similar to the EM fields under lossless condition. From the **E** and **H** fields, we could derived the RLCG parameters from equations (3.1a) and (3.1b). Although the RLCG parameters under this condition is only an approximation, the error is usually small.
- This approach is known as perturbation method.

Derivation of Telegraphic Equations (1)

- For Tline supporting TEM and quasi-TEM modes, the V and I on the line is the solution of a hyperbolic partial differential equation (PDE) known as telegraphic equations. Consider first lossless line:



Use Kirchoff's Voltage Law (KVL):



$$V_2 = V_1 - j\omega L \Delta z I$$

$$\Rightarrow \frac{V_2 - V_1}{\Delta z} = -j\omega L I$$

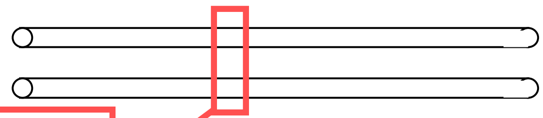
Observing that:

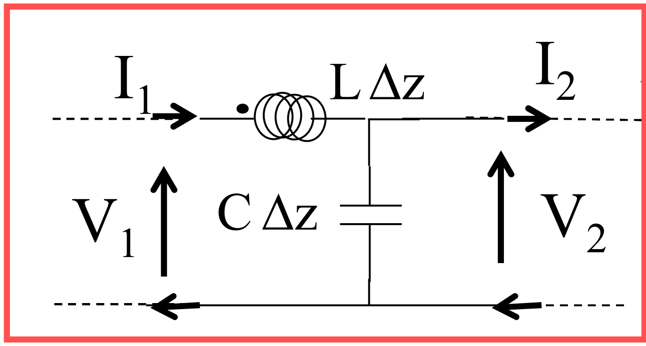
$$\lim_{\Delta z \to 0}\left(\frac{V_2 - V_1}{\Delta z}\right) = \frac{\partial V}{\partial z}$$

$$\Rightarrow \frac{\partial V}{\partial z} = -j\omega L I$$

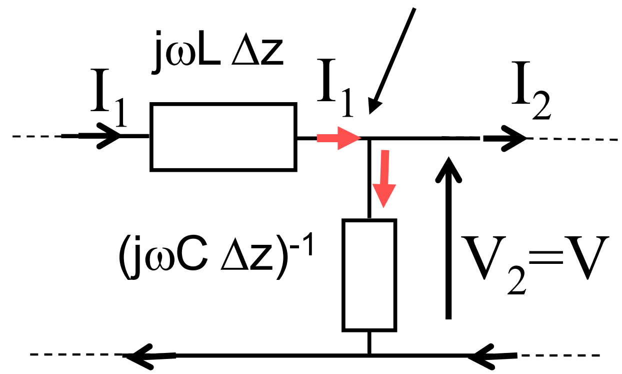
Derivation of Telegraphic Equations (2)

- Now considering the current on both ends, and using Kirchoff's Current Law (KCL):





Using KCL here:



$$I_1 - (j\omega C \Delta z)V = I_2$$

$$\Rightarrow \frac{I_2 - I_1}{\Delta z} = -j\omega C V$$

Again observing that:

$$\frac{I_2 - I_1}{\Delta z} = -C\frac{\partial V_2}{\partial t}$$

$$\boxed{\frac{\partial I}{\partial z} = -j\omega C V}$$

Lossless Telegraphic Equations

Solution of Telegraphic Equations

- Consider the set of lossless telegraphic equations:

$$\frac{\partial I}{\partial z} = -j\omega CV \qquad\qquad \frac{\partial V}{\partial z} = -j\omega LI$$

- We can decouple them by increasing the order of differentiation. For instance we do this for the differential equation involving ***I***:

$$\frac{\partial^2 I}{\partial z^2} = -j\omega C\left(\frac{\partial V}{\partial z}\right) = -j\omega C\left(-j\omega LI\right) = -\omega^2 LCI$$

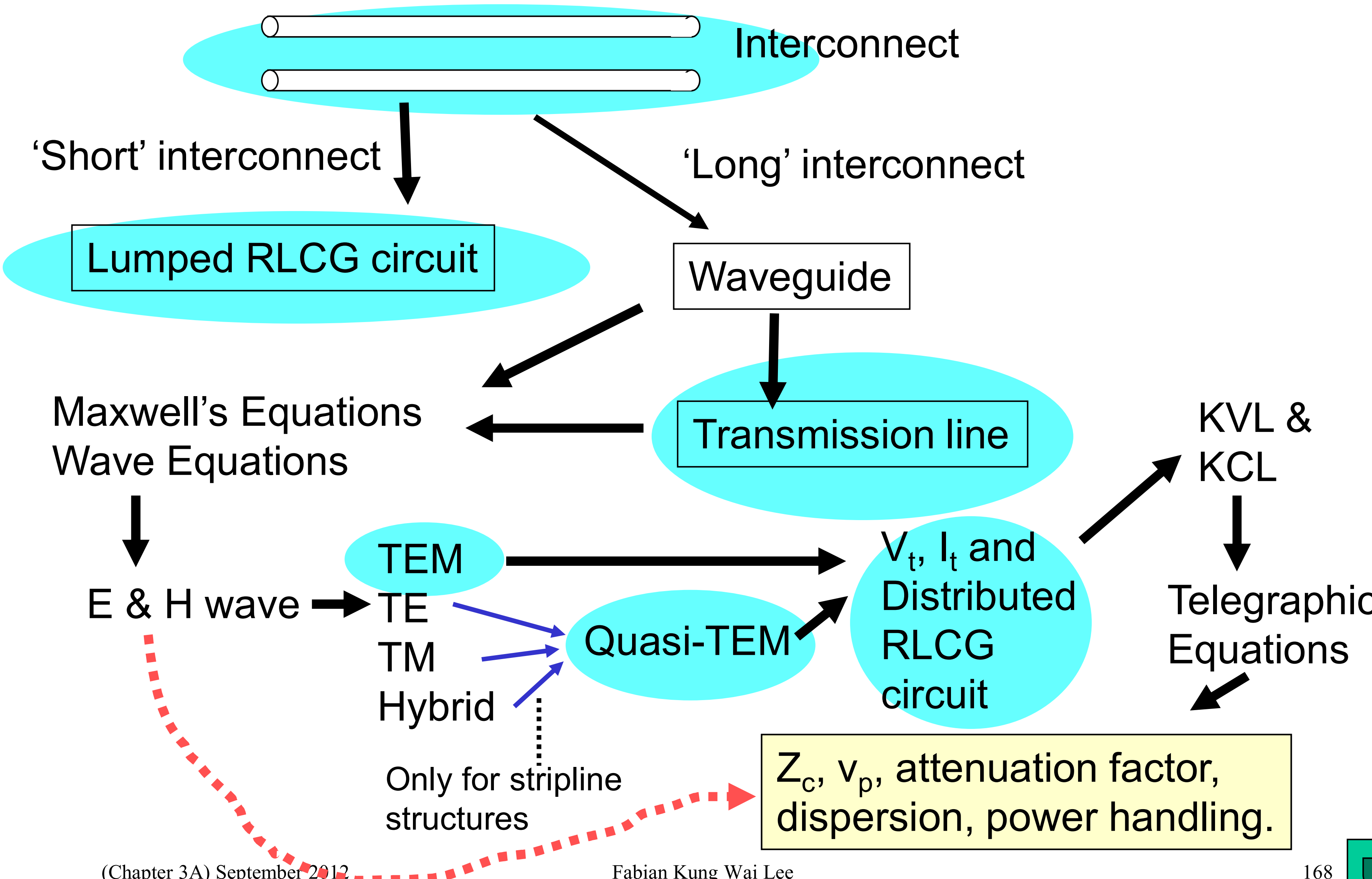
$$\Rightarrow \frac{\partial^2 I}{\partial z^2} + \beta^2 I = 0 \qquad \text{where} \qquad \beta = \omega\sqrt{LC}$$

- We can easily verify (by substitution) that the solution to the 2nd order differential is: $I = I_o^+ e^{-j\beta z} + I_o^- e^{+j\beta z}$

- In a similar manner we can show that $\frac{\partial^2 V}{\partial z^2} + \beta^2 V = 0$

$$V = V_o^+ e^{-j\beta z} + V_o^- e^{+j\beta z}$$

Relationship Between Field Solutions and Telegraphic Equations



Example A2.1

- Find the RLCG parameters of the low loss parallel plate waveguide in Example A1. Assuming the conductivity of the conductor is σ and the dielectric between the plates is complex (this means the dielectric is lossy too):

$$\varepsilon = \varepsilon' - j\varepsilon''$$

- Use the expressions for E_t and H_t as derived in Example A1.

$$L = \frac{\mu d}{W}\ \mathrm{H/m} \qquad C = \frac{\varepsilon' W}{d}\ \mathrm{F/m}$$

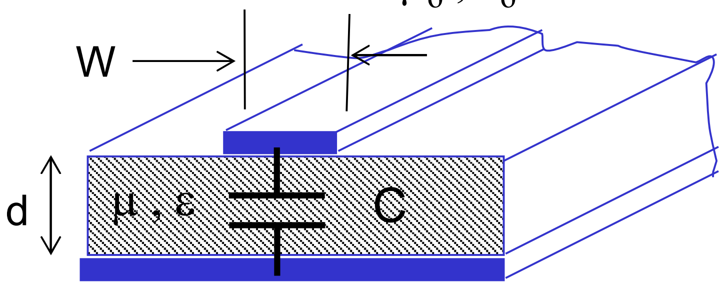
$$R = \frac{2}{\sigma_c \delta_s W}\ \Omega/\mathrm{m} \qquad G = \frac{\omega\varepsilon''}{\varepsilon'}\mathrm{C} = \frac{\sigma_d W}{d}\ \Omega^{-1}/\mathrm{m}$$

Appendix 3

The Origin of Effective Dielectric Constant (ε_{eff})

- The approach can be traced to a paper by Bryant and Weiss[1]. Assuming low loss and quasi-TEM mode:

$$v_p = \frac{1}{\sqrt{LC}} \qquad Z_c = \sqrt{\frac{L}{C}} = \frac{1}{v_p C}$$

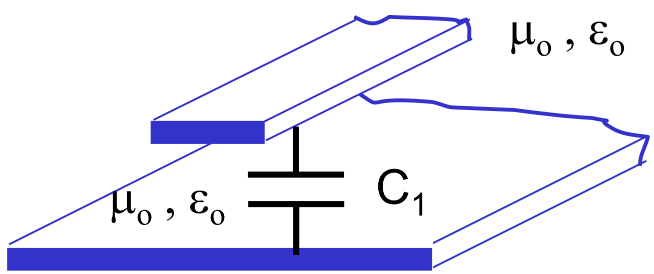


With dielectric, C is the capacitance per meter between the conductors

Define $\varepsilon_{eff} = \frac{C}{C_1}$

Then $v_p = \frac{1}{\sqrt{LC_1}\cdot\sqrt{\varepsilon_{eff}}} = \frac{c}{\sqrt{\varepsilon_{eff}}}$ ← Speed of light in vacuum

$$Z_c = \frac{1}{\frac{c}{\sqrt{\varepsilon_{eff}}}\cdot C} = \frac{1}{\frac{c}{\sqrt{\varepsilon_{eff}}}\cdot C_1 \varepsilon_{eff}}$$



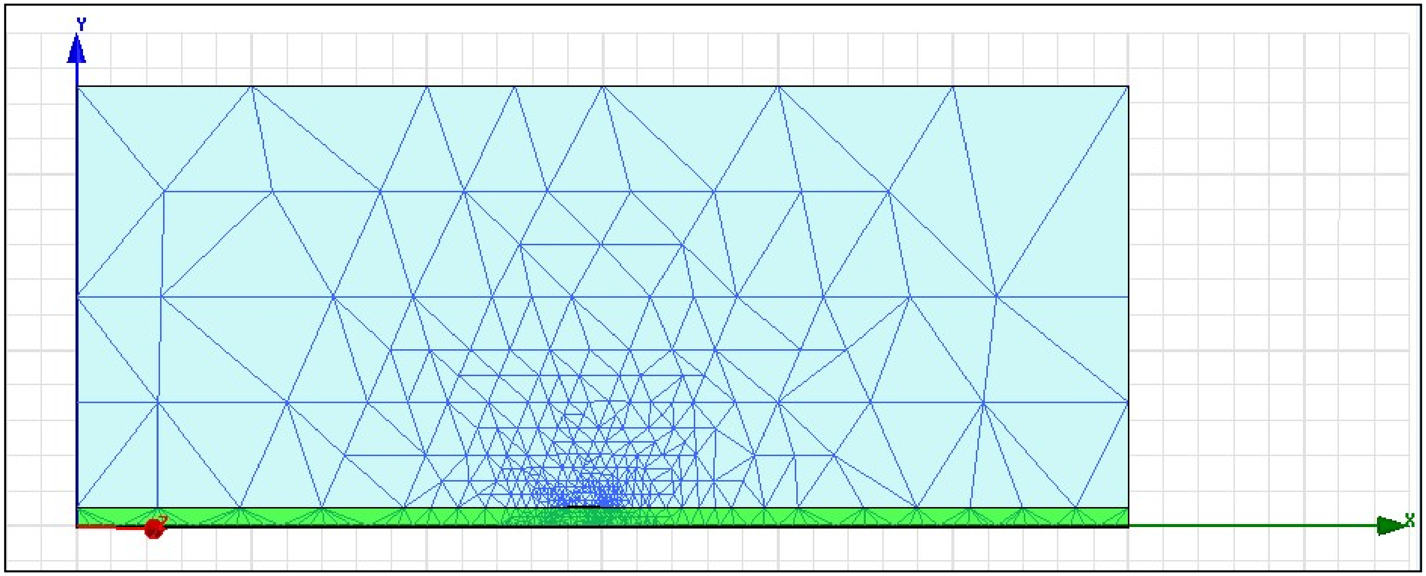
Without dielectric, C_1 is the capacitance per meter between the conductors

$$\Rightarrow Z_c = \frac{1}{c\sqrt{\varepsilon_{eff}}\cdot C_1}$$ ← C_1 is computed via numerical methods for various W and d

Note 1: T. G. Bryant and J. A. Weiss,"Parameters of microstrip transmission lines and coupled pairs of microstrip lines", IEEE Trans. Microwave Theory Tech., vol.MTT-16, pp.1021-1027,1968.

Extra

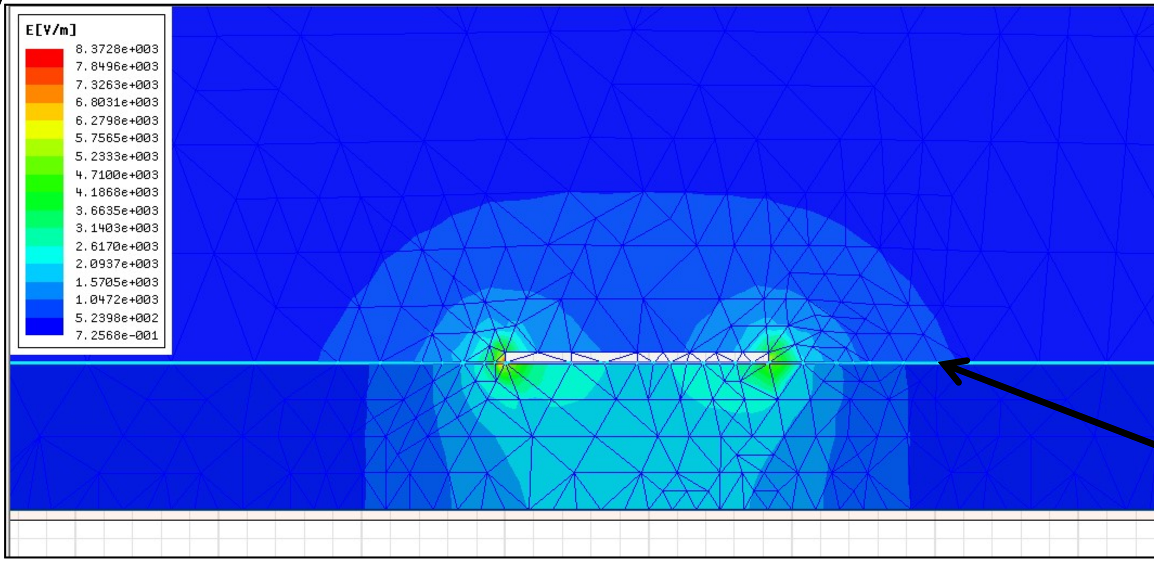
Microstrip Line Modeling Example using Ansoft's Q2D Parametric Extractor



The mesh

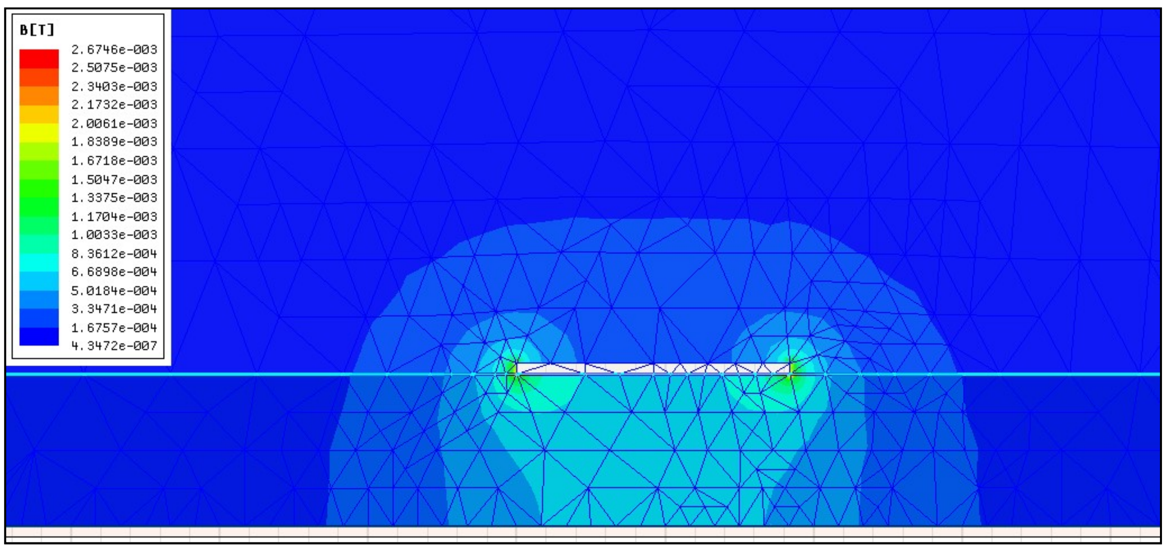
Extra

Microstrip Line Modeling Example using Ansoft's Q2D Parametric Extractor



Distribution of |**E**| over the cross section

Note the discontinuity of electric field across the air/PCB substrate boundary



Distribution of |**B**| over the cross section