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Distributive Properties of Main Overlap and Noise Terms in Autoassociative Memory Networks

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Abstract—*The distributive properties of both main overlap and noise terms of an autoassociative memory network are analyzed with computer simulations. Some interesting statistical properties of the main overlap and noises have been exposed. The results provide an important basis for the establishment of a mathematical model that is capable of describing the dynamics of the neural network.*

Keywords—Autoassociative memory network, Main overlap, Noise terms, Distributive properties.

1. INTRODUCTION

Artificial neural networks are composed of an enormous number of mutually connected elements whose nature remains to be defined. This lack of definition is the main factor that prevents the establishment of a mathematical model for such a system using nonstatistical methods. Since the 1970s, various methods based on statistical theories have been developed and used to analyze the behaviour of such systems (see, e.g., Amari & Maginu, 1988; Zagrebnov & Chvyrov, 1989; Patrick & Valentin Zagrebnov, 1991a,b; Nishimori & Ozaki, 1993). In all of these studies the main overlap [see eqn (4)], which is a quantity for measuring the closeness between the target memory stored in the neural network and the current states of the network, is considered to be a deterministic function. However, this is only true for a system with a small pattern ratio ($r < 0.16$), when the expectation of the initial main overlap is larger than a certain critical value. The pattern ratio is defined as m/n, where m is the number of the memory pattern stored in the neural network and n is the number of the neurons to form the network. The computer simulation confirms that the main overlap is a random process rather than a deterministic function (see Fu, 1993a, 1994). Some statistical properties of a main overlap have been considered by Amit (1989). Based on the fact that the main overlap and noise terms are random variables, Coolen and Sherrington (1993) propose a dynamical theory for describing the dynamics of the system with a small pattern ratio (<0.16).

Obviously, to establish a suitable mathematical model for a randomly generated neural network, it is necessary to consider both the main overlap and noise as random processes. The analysis of the dynamics of a randomly generated neural network is actually reduced to the studies of the statistical properties of the main overlap and noise and their correlation. These problems are still not fully solved.

This paper will introduce some distributive properties of the main overlap and noise terms in a Hopfield network with a Hebb rule by means of computer simulation. The results provide valuable evidences for forming the mathematical model of an autoassociative memory network.

2. AUTOASSOCIATIVE MEMORY NETWORK

We will consider a neural network containing n neurons that satisfies the following assumptions:

1. Neurons in the network change their states synchronously.
2. Each neuron has only two states at any time t: the excited state [denoted by $X_i(t) = 1$] and the quiescent state [denoted by $X_i(t) = -1$].
3. There are m memory patterns $\mathbf{S}^i$, $i = 1, \ldots, m$ stored in the neural network and only one of them is a target memory that will be recalled. Without loss of the generality, $\mathbf{S}^1$ is assumed to be the target memory, the other memory patterns are generated randomly.

A typical model of an autoassociative memory network is illustrated in Figure 1. The current state of the

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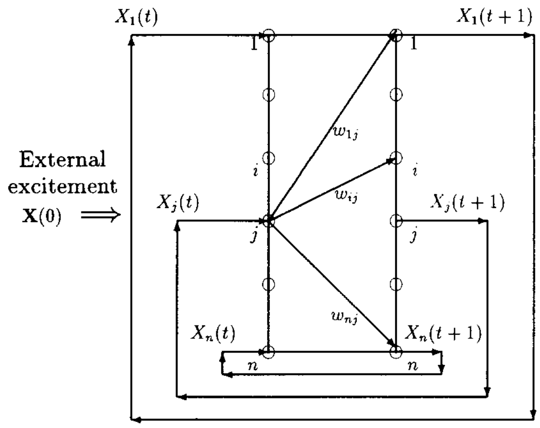

FIGURE 1. Autoassociative memory network.

system is represented by $\mathbf{X}(t) = (X_1(t), \ldots, X_n(t))^T$, where T denotes the transpose and $X_i(t)$ is the state of the ith neuron. We consider $\mathbf{X}(t)$ being the input and $\mathbf{X}(t+1)$ being the output of the system at time t. The ith component of $\mathbf{X}(t+1)$ depends on w_{ij}, the strength of the connection between the ith and jth neuron, and is given by

$$X_i(t+1) = Sgn\left(\sum_{j=1}^{n} w_{ij}X_j(t)\right) = Sgn(s_i^1 A(t) + N_i(t)), \quad i = 1, \ldots, n, \tag{1}$$

where $Sgn(z) = 1$ if $z > 0$, and $Sgn(z) = -1$ if $z < 0$. s_i^1 is ith component of the target memory pattern stored in the neural network,

$$w_{ij} = \frac{1}{n}\sum_{\alpha=1}^{m} s_i^\alpha s_j^\alpha, \quad \text{for } j \neq i \tag{2}$$

and

$$w_{ii} = 0, \quad i = 1, 2, \ldots, n. \tag{3}$$

$$A(t) = \frac{1}{n}\sum_{j=1}^{n} s_j^1 X_j(t), \tag{4}$$

is the main overlap between the current state $\mathbf{X}(t)$ and the memorized pattern $\mathbf{S}^1$.

If $\mathbf{X}(t)$ and $\mathbf{S}^1$ are considered to be two vectors in an n-dimensional space, then the main overlap in eqn (4) represents the direct-cosine between them.

Amari and Maginu (1988) have pointed out that the main overlap $A(t)$ is related to the normalized Hamming distance $D(X)$ between $\mathbf{X}(t)$ and $\mathbf{S}^1$ by

$$A(t) = 1 - 2D(X),$$
$$D(X) = \frac{1}{2n}\sum_{j=1}^{n} |X_i(t) - S_j^1|;$$

therefore, the main overlap is a quantity measuring the closeness between $\mathbf{S}^1$ and $\mathbf{X}(t)$.

The n governing eqns (1) of the system can be expressed by an n-dimensional vector equation

$$\mathbf{X}(t+1) = Sgn(A(t)\mathbf{S}^1(t) + \mathbf{N}(t)), \tag{5}$$

where $\mathbf{N}(t) = (N_1(t), \ldots, N_n(t))^T$ is a vector, corresponding to noise terms.

This vector equation establishes the relationship between the future state vector $\mathbf{X}(t+1)$, the target memory, the current main overlap $A(t)$, and the noise $\mathbf{N}(t)$. Notice that the target memory $\mathbf{S}^1$ is fixed in a recalling process; therefore, the analysis of the dynamics of the system is completely reduced to the studies of $A(t)$, $N_i(t)$, and their correlation.

$$N_i(t) = \frac{1}{n}\sum_{\alpha\neq 1}^{m}\sum_{j\neq i}^{n} s_i^\alpha s_j^\alpha X_j(t) \tag{6}$$

denotes the noise term. Here s_i^α, $i = 1, 2, \ldots, n$, $\alpha = 1, \ldots, m$ are assumed to be generated randomly with probability distribution

$$P(s_i^\alpha = 1) = \beta, \tag{7}$$

$$P(s_i^\alpha = -1) = 1 - \beta, \tag{8}$$

where $0 < \beta < 1$ is a constant.

Based on the previous assumptions, the main overlap $A(t)$ and noise terms $N_i(t)$, $i = 1, \ldots, n$ are the random processes defined in the probability space (Ω, $\Im$, P), where $\Omega = \{-1, 1\}^n$ is an n-dimensional Bernoulli space, $\Im$ is the minimal σ-algebra generated by the basis of Ω, that is, the set of subset $\{\Lambda\}$ for which element Λ is defined as

$$\Lambda = \{\omega : \omega = (\omega_1, \omega_2, \ldots, \omega_n)\}, \tag{9}$$

ω_i taking $+1$ or -1, $i = 1, \ldots, n$, and the probability of Λ is given by

$$P(\Lambda) = \beta^{\sum_{i=1}^{n}(1+\omega_i)/2}(1-\beta)^{\sum_{i=1}^{n}(1-\omega_i)/2}. \tag{10}$$

Notice that the memory patterns $\mathbf{S}^k \epsilon \{\Lambda\}$, $k = 1, \ldots, m$ and when they are fixed in a specified simulation, one obtains the sample functions, that is, the realizations of $A(t)$ and $N_i(t)$ given by eqns (4) and (6), respectively. On other hand, $\mathbf{S}^k$, $k = 1, \ldots, m$ are random variables and so are $X_i(t)$, $i = 1, \ldots, n$. Therefore, $A(t)$ and $N_i(t)$ are random variables for fixed t. In view of the definition of a random process (see Doob, 1953), main overlap $A(t)$ and noise terms $N_i(t)$ are random processes defined in the probability space (Ω, $\Im$, P). A more general probability space for describing such neural network has been discussed by Patrick and Zagrebnov (1991a).

3. COMPUTER SIMULATION

An autoassociative memory network is determined by three parameters n, r, and β. Without loss of the generality, we assume that the components of the target memory pattern $\mathbf{S}^1$ are all $+1$. The other $m - 1$ memory patterns are generated randomly based on the probability distribution in eqns (7) and (8). The connectivity matrix $\mathbf{W}$ is given by eqns (2) and (3).

To simulate the dynamics of the neural network, the initial external excitement, the initial state, $\mathbf{X}(0)$, of the neural network must be given first. To simplify the analysis of the dynamics, $\mathbf{X}(0)$ can be constructed deterministically. Without loss of generality, $\mathbf{X}(0)$ may be expressed in the following form:

$$\mathbf{X}(0) = (\underbrace{1, 1, \ldots, 1}_{n'}, \underbrace{-1, -1, \ldots, -1}_{n-n'}), \quad (11)$$

where $0 \le n' \le n$ is a positive integer, which is a distortion of the target memory

$$\mathbf{S}^1 = (1, 1, \ldots, 1). \quad (12)$$

The updated states $X_i(1)$, $i = 1, 2, \ldots, n$ are obtained from eqn (1). This process is repeated with $\mathbf{X}(1)$ as the new initial state, and thus the progressive recall sequence $\mathbf{X}(0), \mathbf{X}(1), \mathbf{X}(2), \ldots$ is generated.

To analyze the dynamics of a neural network, a set of statistical quantities concerning the main overlap and noise terms is required. To obtain these statistical quantities, it is necessary to collect a set of samples of the data for the main overlap and noise terms by computer simulation. The computer simulation has to be performed repeatedly. In our case, the samples were obtained by 500 computer simulations of the neural network with $n = 100$, $\beta = 0.5$, initial expectation of main overlap $a(0) = 0.7$, and various values of memory pattern ratio r in evaluating the statistical quantities of the main overlap and noise terms (Figures 2–5). The realizations of main overlap (Figures 6 and 7) were resulted from the same system except the number of neurons in the system is changed from $n = 100$ to $n = 400$. Each simulation starts with the same initial expectation of main overlap and same basic parameters but different sequences of the randomly generated numbers in constructing the memorized patterns $\mathbf{S}^2, \ldots, \mathbf{S}^m$ and the connectivity matrix $\mathbf{W}$. It is clear that the realizations of $A(t)$ and $N_i(t)$ are obtained in a simulation. The different realizations are corresponding to different simulations.

Figure 2a–c shows the density functions of the main overlap of the neural network with pattern ratios 0.08, 0.16, and 0.3, respectively, at the second step in a recalling process. The graphs show that the density function of the main overlap is a δ-function in the system with $r = 0.08$ and is approximately normal in the system with $r = 0.3$, but it is not close to normal in the case of $r = 0.16$.

Figure 3a–c shows the density functions of the noise of the system with the same parameters as in Figure 2a–c, respectively, at the second step. Figure 3d–f shows the density functions of the noise of the system with the same r as in Figure 2a–c at the sixth step. The graphs show that the density functions of the noise are approximately normally distributed up to the sixth step for the system with $r = 0.08$ and $r = 0.16$, but not with

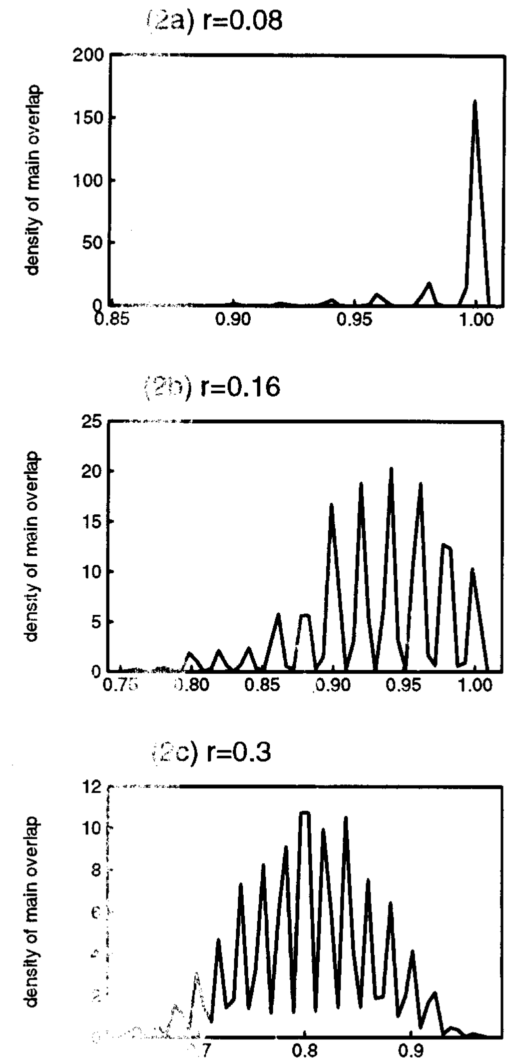

FIGURE 2. Density functions of main overlap at second step with $a(0) = 0.7$

$r = 0.3$. It has been found that the density of the noise will deviate from normal as the number of recalling steps increases for the system with an intermediate and large pattern ratio.

Figure 4 shows the standard deviation of the main overlap as a function of time with various values of memory pattern ratio. The graph shows that the standard deviation of the main overlap has the following main features:

i) For the system with a small memory pattern ratio ($r < 0.16$), the standard deviation of the main overlap is almost zero if the initial expectation of the main overlap is larger than a certain critical value.
ii) For the system with a large memory pattern ratio ($r > 0.2$), the standard deviation of the main overlap is small when the initial expectation of main overlap is larger than a certain critical value.
iii) For the system with an intermediate memory pattern ratio ($0.16 \le r \le 0.2$), the standard deviation of the main overlap is large.

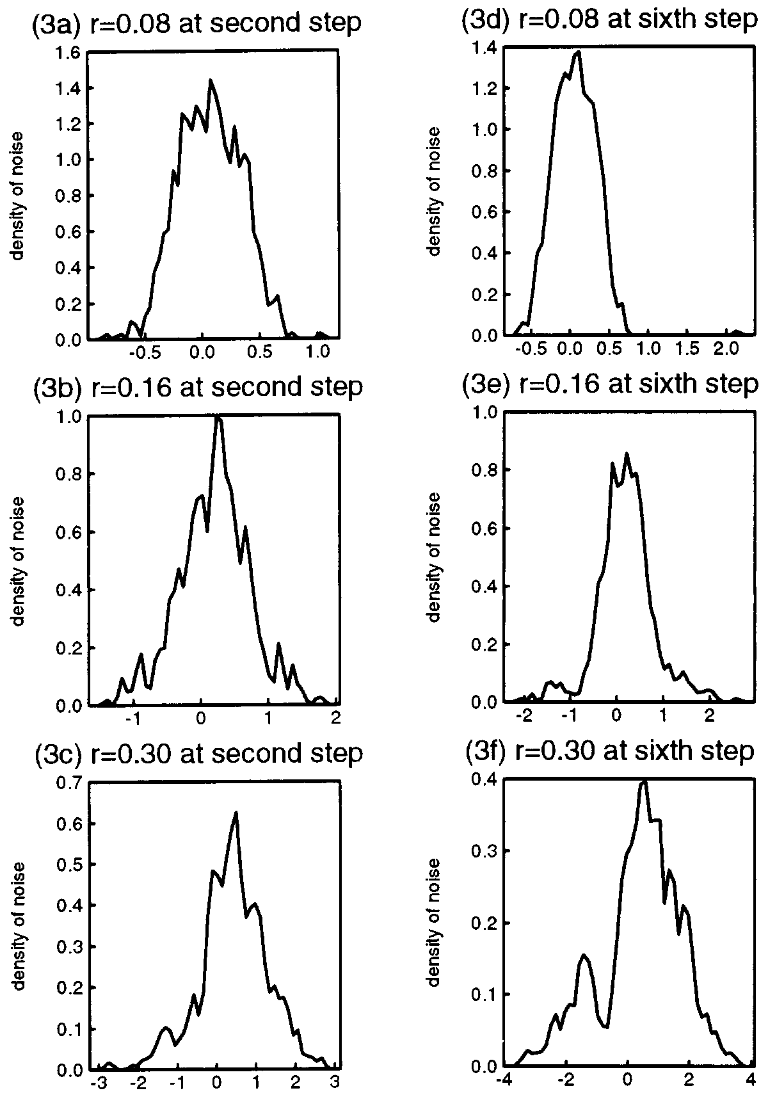

FIGURE 3. Density functions of noise terms at the second and sixth steps with $a(0) = 0.7$.

Figure 5 shows the correlation coefficients between the main overlap and the noise terms as functions of time with different memory pattern ratios. The graph indicates that the relatively strong dependence between the main overlap and noise terms arises in the case of the system with an intermediate or small pattern ratio, but the weak dependence in the case of the system with a large pattern ratio.

The standard deviation of the main overlap and the correlation coefficients between the main overlap and the noise terms were calculated in terms of the software of NAG (Fox & Wilkinson, 1987).

Figures 6 and 7 show the 50 realizations of the main overlap resulting from 50 independent computer simulations of the network with $n = 400$, initial main overlap $a(0) = 0.7$, and memory pattern ratio $r = 0.3$ and $r = 0.5$, respectively. In each simulation, 50 steps are carried out. The crosses represent 50 computer simulations at each step and the solid line represents the mean of the main overlap.

4. CONCLUSIONS

Based on the results of the computer simulation, the autoassociative memory networks can be classified into three cases:

(i) small pattern ratio: $r < r_c$
(ii) intermediate pattern ratio: $r_c \leq r \leq r_{cc}$
(iii) large pattern ratio: $r > r_{cc}$

where r_c and r_{cc} are about 0.16 and 0.2, respectively (see Fu, 1993a, 1994; Fu & Gibson, 1993). We can summarize the main points as follows.

For a small pattern ratio system, the density functions of both the main overlap and noise terms are normally distributed when the initial expectation $a(0)$ of the main overlap is larger than a certain critical value a_c, which depends on r (for example, $r = 0.08$, and a_c is about 0.4). We have seen (Figure 2a) that the density function of the main overlap is approximately a δ-function [i.e., $A(t) = a(t)$ with a probability one], and therefore we can say that the main overlap is a deter-

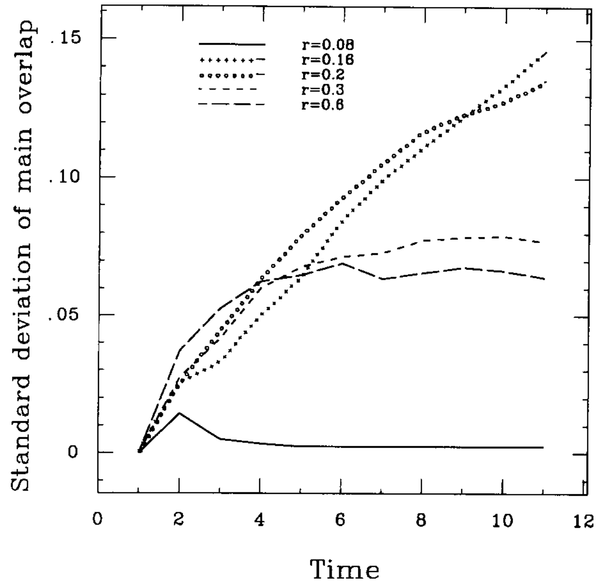

FIGURE 4. Standard deviation of main overlap with $a(0) = 0.7$.

ministic function rather than a random process in this case. This situation has been considered by many authors (see, for example, Amari & Maginu, 1988; Patrick & Zagrebnov, 1991a,b; Nishimori & Ozaki, 1993).

For a large pattern ratio system, the density functions of both the main overlap and noise terms are approximately normally distributed up to the third step. Furthermore, there is a small standard deviation of the main overlap (see Figure 4) and the small correlation coefficient between the main overlap and noise terms (see Figure 5) in this case, so the effect of the dependence between the main overlap and noise terms can be neglected. Therefore, it is acceptable to make the

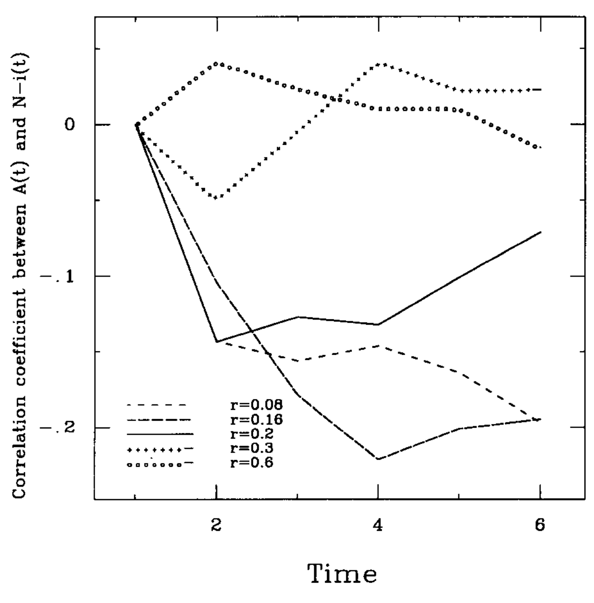

FIGURE 5. Correlation coefficient between main overlap and noise terms with $a(0) = 0.7$.

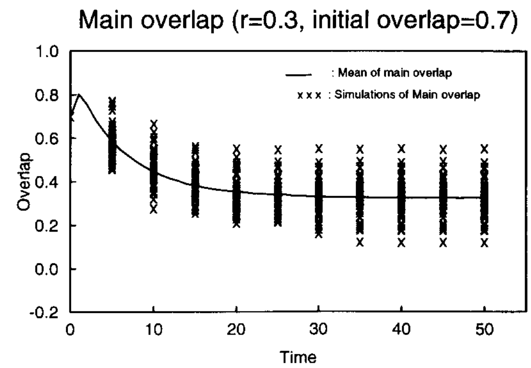

FIGURE 6. The realizations of main overlap of the system with $r = 0.3$ and $a(0) = 0.7$.

assumption that the main overlap and noise terms are independent normal random process in analyzing the dynamics of the system at the several earlier steps in this case. In fact, the theoretical analysis based on this assumption gives good agreement with the computer simulations (see Fu, 1993a, 1994; Fu & Gibson, 1993). However, the density function of the noise tends to diverge from normal as the recalling step increases (see Figure 3f), which is in agreement with the observation made by Zagrebnov and Chvyrov (1989).

For an intermediate pattern ratio system, the noise terms have an approximately normal distribution up to the sixth step (see Figure 3b,e), which agrees with the observation by Zagrebnov and Chvyrov (1989), but the main overlap is not normally distributed even at the first and second steps (see Figure 2b). The large standard deviation of the main overlap and the relatively large correlation coefficient between the main overlap and noise terms exist in this case (see Figures 4 and 5). The assumption that the main overlap and noise terms are independent normal random process is not satisfied and the correlation between the main overlap and noise terms has to be taken into account when analyzing the dynamics of the system at the earlier several steps in this case (see Fu, 1993b, 1994). A statistical analysis based on the assumption that the main overlap

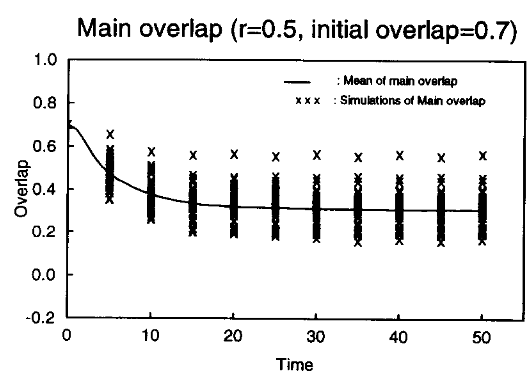

FIGURE 7. The realizations of main overlap of the system with $r = 0.5$ and $a(0) = 0.7$.

and noise terms obey a bivariate normal distribution yields results that are in agreement with the computer simulation at the several earlier steps for any pattern ratio system (see Fu, 1993b).

The main overlap $A(t)$ tends towards a stationary process when time t is increasing (see Figures 6 and 7). $A(t)$ is a stationary process after about 20 steps in a recalling process for the network with $r = 0.3$ and $r = 0.5$. This conclusion is held for the network with other r value when the initial main overlap is larger than certain critical value (see Fu, 1993a).

The results that we propose in this paper are based on a small-size neural network ($n = 100$ and $n = 400$). Of course, the larger the size of the neural network the more accurate the estimation of the properties of $A(t)$ and $N_i(t)$. But the size of the neural network for a large number of simulations is limited by the capacity of the computer available.

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NOMENCLATURE

n	the number of neurons in the neural network
m	the number of memories stored in the neural network
r	the pattern ratio that is defined as $r = m/n$
β	the activity, equal to the probability that a randomly chosen neuron is active
$\mathbf{X}(t)$	the random vector giving the state of the system at discrete time $t = 0, 1, 2, \ldots$. It has dimension n and its ith component is $X_i(t)$
$\mathbf{S}^k$	one of the m memory vectors ($k = 1, 2, \ldots, m$). It has dimension n, and the ith component is s_i^k taking the value -1 or 1
$\mathbf{W}$	the connectivity matrix, for which elements are w_{ij}, $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$
$A(t)$	the main overlap of neural network at time t, which is a random variable
$a(t)$	the expectation of $A(t)$
$N_i(t)$	the noise term related to the ith neuron