EE613
Machine Learning for Engineers

NONLINEAR REGRESSION I

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Outline

- Properties of multivariate Gaussian distributions
  Matlab codes: *demo_Gaussian_product01.m, demo_Gaussian_conditioning01.m, demo_Gaussian_conditioning_noisyInput01.m, demo_Gaussian_lawTotalCov01.m*

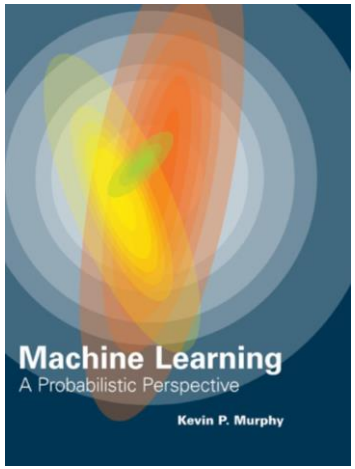
- Locally weighted regression (LWR)
  Matlab code: *demo_LWR01.m,*

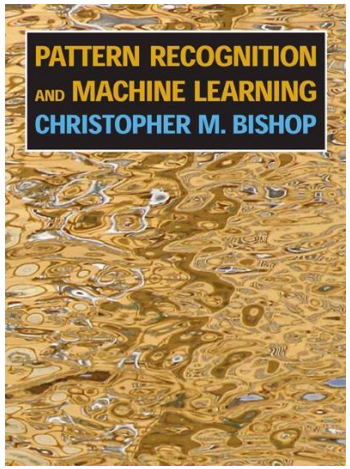
- Dynamical movement primitives (DMP)
  Matlab code: *demo_DMP01.m*

- Gaussian mixture regression (GMR)
  Matlab codes: *demo_GMR01.m, demo_GMR_polyFit01.m, demo_DMP_GMR01.m*

Properties of multivariate Gaussian distributions

Matlab codes: demo_Gaussian_product01.m, demo_Gaussian_conditioning01.m, demo_Gaussian_conditioning_noisyInput01.m, demo_Gaussian_lawTotalCov01.m

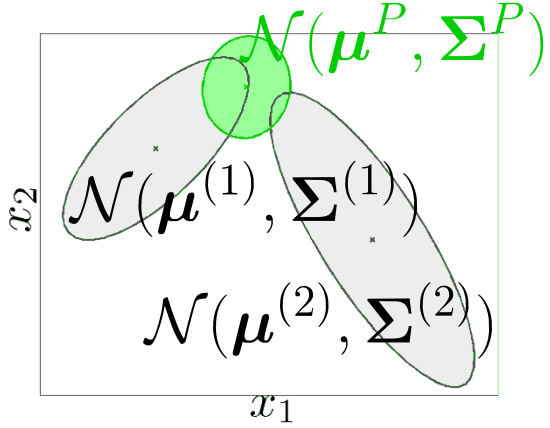


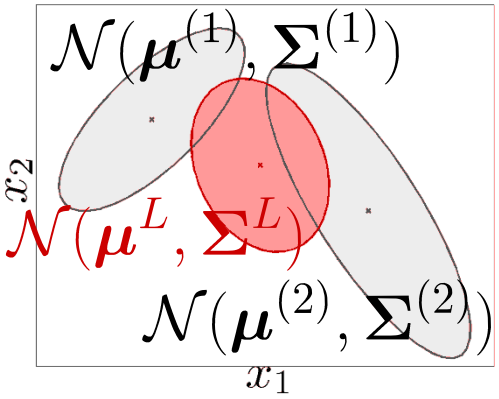


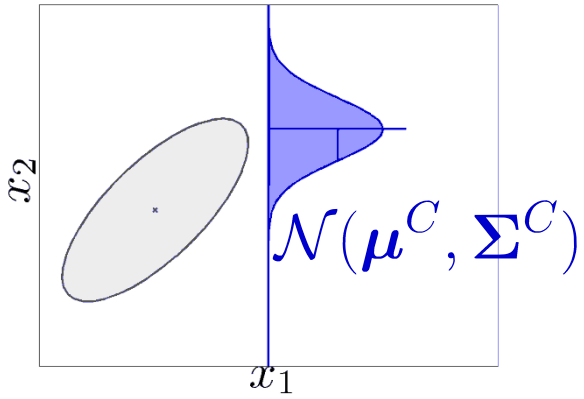
The Matrix Cookbook

Kaare Brandt Petersen
Michael Syskind Pedersen

Some very useful properties







Product of Gaussians:

$$\mathcal{N}(\boldsymbol{\mu}^P, \boldsymbol{\Sigma}^P) \sim \mathcal{N}(\boldsymbol{\mu}^{(1)}, \boldsymbol{\Sigma}^{(1)}) \cdot \mathcal{N}(\boldsymbol{\mu}^{(2)}, \boldsymbol{\Sigma}^{(2)})$$

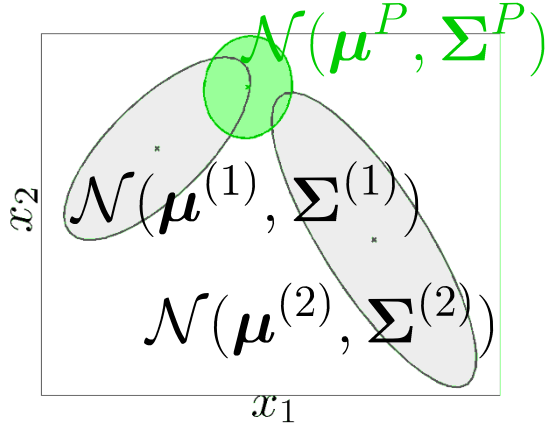
Linear combination:

$$\mathcal{N}(\boldsymbol{\mu}^L, \boldsymbol{\Sigma}^L) \sim \mathcal{N}(\boldsymbol{\mu}^{(1)}, \boldsymbol{\Sigma}^{(1)}) + \mathcal{N}(\boldsymbol{\mu}^{(2)}, \boldsymbol{\Sigma}^{(2)})$$

Conditional probability:

$$\mathcal{N}(\boldsymbol{\mu}^C, \boldsymbol{\Sigma}^C) \sim \mathcal{P}(\boldsymbol{x}_2 | \boldsymbol{x}_1)$$

Product of Gaussians

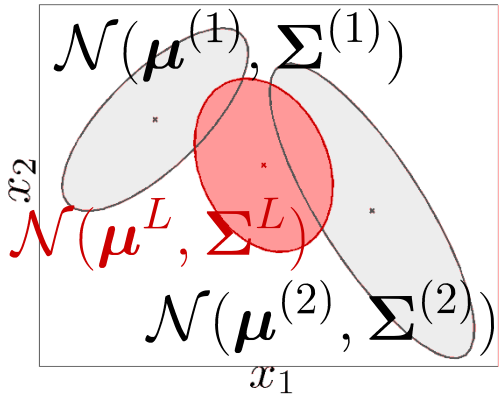


The product of two Gaussian distributions $\mathcal{N}(\boldsymbol{\mu}^{(1)}, \boldsymbol{\Sigma}^{(1)})$ and $\mathcal{N}(\boldsymbol{\mu}^{(2)}, \boldsymbol{\Sigma}^{(2)})$ is defined by

$$c\,\mathcal{N}(\boldsymbol{\mu}^P, \boldsymbol{\Sigma}^P) = \mathcal{N}(\boldsymbol{\mu}^{(1)}, \boldsymbol{\Sigma}^{(1)}) \cdot \mathcal{N}(\boldsymbol{\mu}^{(2)}, \boldsymbol{\Sigma}^{(2)}),$$

$$\begin{aligned} \text{with} \qquad c &= \mathcal{N}(\boldsymbol{\mu}^{(1)} | \boldsymbol{\mu}^{(2)}, \boldsymbol{\Sigma}^{(1)} + \boldsymbol{\Sigma}^{(2)}), \\ \boldsymbol{\Sigma}^P &= \left({\boldsymbol{\Sigma}^{(1)}}^{-1} + {\boldsymbol{\Sigma}^{(2)}}^{-1} \right)^{-1}, \\ \boldsymbol{\mu}^P &= \boldsymbol{\Sigma}^P \left({\boldsymbol{\Sigma}^{(1)}}^{-1} \boldsymbol{\mu}^{(1)} + {\boldsymbol{\Sigma}^{(2)}}^{-1} \boldsymbol{\mu}^{(2)} \right). \end{aligned}$$

Linear combination



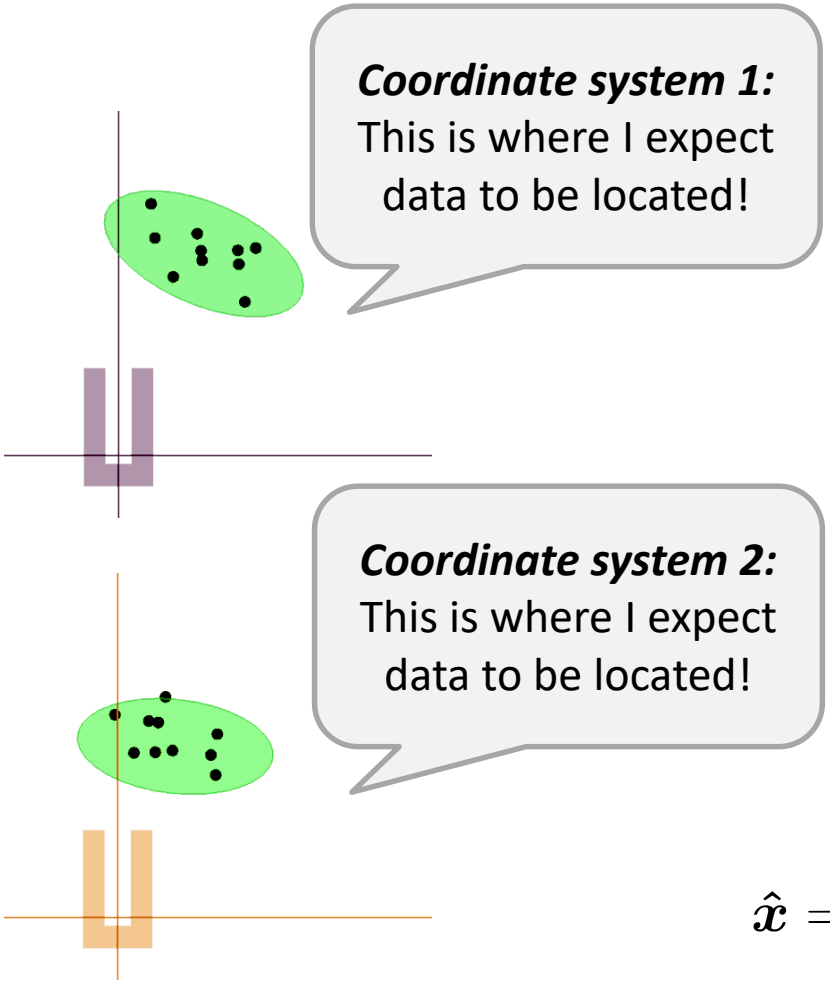
If $\boldsymbol{x}^{(1)} \sim \mathcal{N}(\boldsymbol{\mu}^{(1)}, \boldsymbol{\Sigma}^{(1)})$ and $\boldsymbol{x}^{(2)} \sim \mathcal{N}(\boldsymbol{\mu}^{(2)}, \boldsymbol{\Sigma}^{(2)})$, the linear transformation $\boldsymbol{A}^{(1)}\boldsymbol{x}^{(1)} + \boldsymbol{A}^{(2)}\boldsymbol{x}^{(2)} + \boldsymbol{c}$ follows the distribution

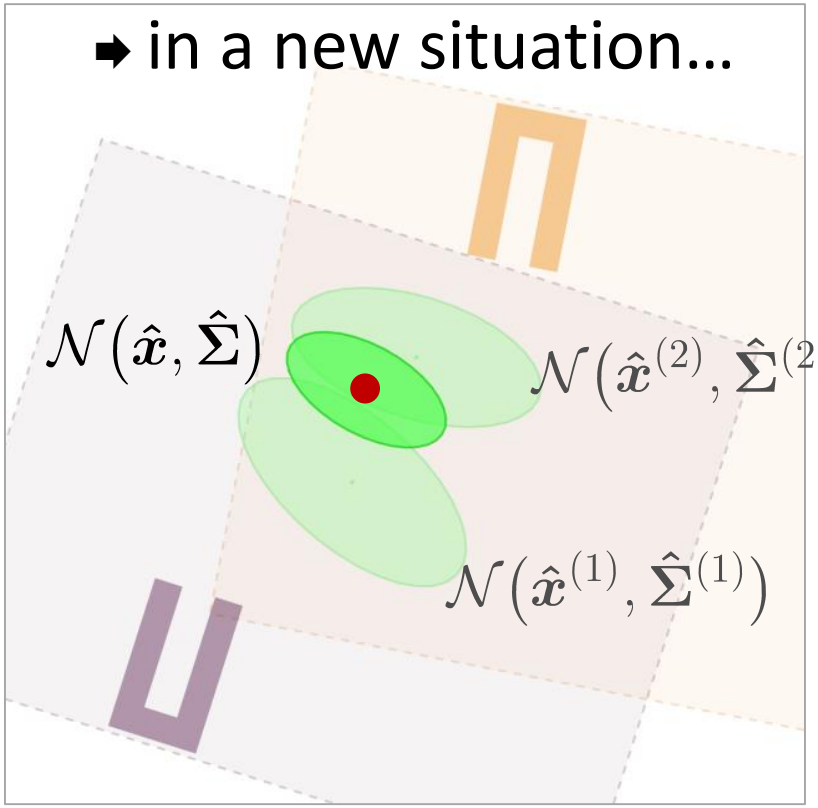
$$\boldsymbol{A}^{(1)}\boldsymbol{x}^{(1)} + \boldsymbol{A}^{(2)}\boldsymbol{x}^{(2)} + \boldsymbol{c} \sim \mathcal{N}(\boldsymbol{\mu}^{L}, \boldsymbol{\Sigma}^{L}),$$

with

$$\begin{aligned} \boldsymbol{\mu}^{L} &= \boldsymbol{A}^{(1)}\boldsymbol{\mu}^{(1)} + \boldsymbol{A}^{(2)}\boldsymbol{\mu}^{(2)} + \boldsymbol{c}, \\ \boldsymbol{\Sigma}^{L} &= \boldsymbol{A}^{(1)}\boldsymbol{\Sigma}^{(1)}\boldsymbol{A}^{(1)\top} + \boldsymbol{A}^{(2)}\boldsymbol{\Sigma}^{(2)}\boldsymbol{A}^{(2)\top}. \end{aligned}$$

Example: Fusion of sensor/control information





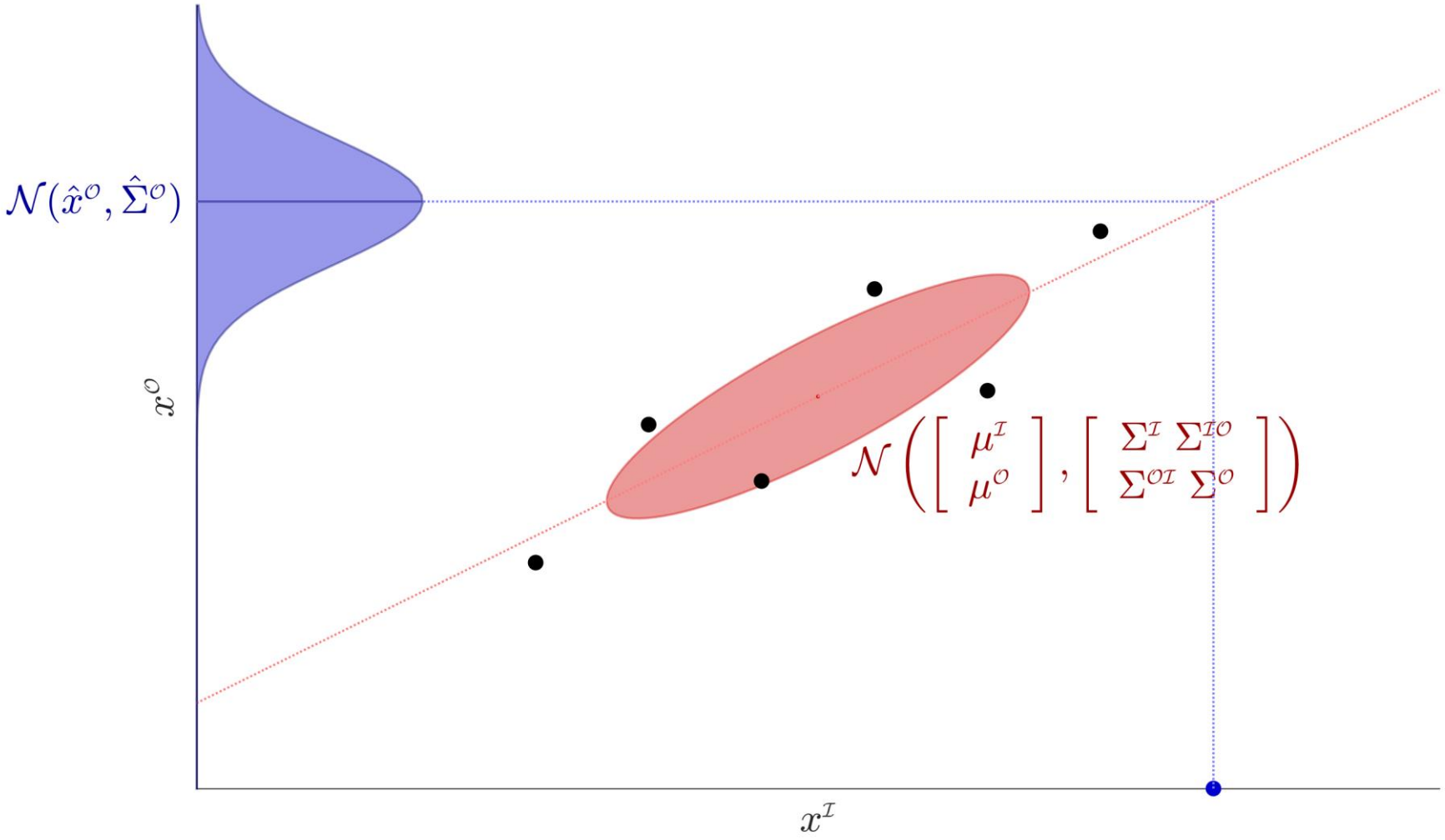
$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \sum_{j=1}^{2} \left(\boldsymbol{x} - \hat{\boldsymbol{x}}^{(j)}\right)^{\top} \hat{\boldsymbol{\Sigma}}^{(j)-1} \left(\boldsymbol{x} - \hat{\boldsymbol{x}}^{(j)}\right)$$

→ Product of linearly transformed Gaussians

[Calinon, S. (2016). A Tutorial on Task-Parameterized Movement Learning and Retrieval. Intelligent Service Robotics 9(1):1–29]

Conditional probability

$$\hat{\boldsymbol{A}} = \arg\min_{\boldsymbol{A}} (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{A})^\top (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{A})$$
$$= (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{Y} = \boldsymbol{X}^\dagger \boldsymbol{Y}$$



→ Linear regression from joint distribution

Conditional probability

We consider multivariate datapoints $\boldsymbol{x}$ and multivariate Gaussian distributions characterized by centers $\boldsymbol{\mu}$ and covariances $\boldsymbol{\Sigma}$, that can be partitioned as

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^{\mathcal{I}} \\ \boldsymbol{x}^{\mathcal{O}} \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}^{\mathcal{I}} \\ \boldsymbol{\mu}^{\mathcal{O}} \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}^{\mathcal{I}} & \boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}} \\ \boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}} & \boldsymbol{\Sigma}^{\mathcal{O}} \end{bmatrix}.$$

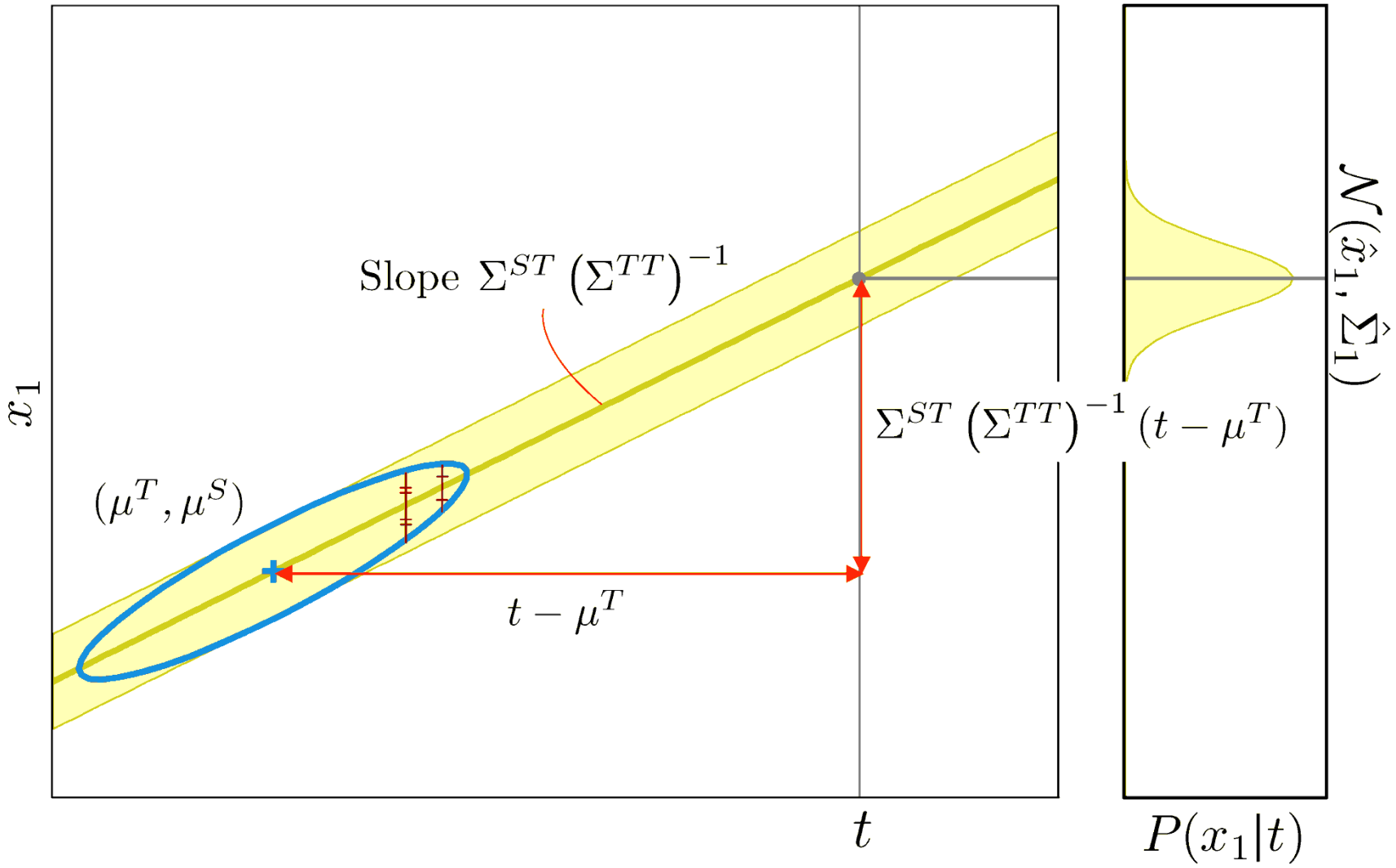
If $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we have that $\boldsymbol{x}^{\mathcal{O}} | \boldsymbol{x}^{\mathcal{I}} \sim \mathcal{N}\left(\hat{\boldsymbol{x}}^{\mathcal{O}}, \hat{\boldsymbol{\Sigma}}^{\mathcal{O}}\right)$,
with parameters

$$\begin{aligned} \hat{\boldsymbol{x}}^{\mathcal{O}} &= \boldsymbol{\mu}^{\mathcal{O}} + \boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}} \boldsymbol{\Sigma}^{\mathcal{I}-1} (\boldsymbol{x}^{\mathcal{I}} - \boldsymbol{\mu}^{\mathcal{I}}), \\ \hat{\boldsymbol{\Sigma}}^{\mathcal{O}} &= \boldsymbol{\Sigma}^{\mathcal{O}} - \boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}} \boldsymbol{\Sigma}^{\mathcal{I}-1} \boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}}. \end{aligned}$$

We can see that $\hat{\boldsymbol{x}}^{\mathcal{O}}$ is linearly dependent on $\boldsymbol{x}^{\mathcal{I}}$, and that $\hat{\boldsymbol{\Sigma}}^{\mathcal{O}}$ is independent of $\boldsymbol{x}^{\mathcal{I}}$.

We can also notice that for full joint covariance, the conditional covariance $\hat{\boldsymbol{\Sigma}}^{\mathcal{O}}$ will typically be smaller than the marginal $\boldsymbol{\Sigma}^{\mathcal{O}}$.

Conditional probability - Geometric interpretation



Conditional probability - Proof

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^{\mathcal{I}} \\ \boldsymbol{x}^{\mathcal{O}} \end{bmatrix} \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}^{\mathcal{I}} \\ \boldsymbol{\mu}^{\mathcal{O}} \end{bmatrix} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}^{\mathcal{I}} & \boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}} \\ \boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}} & \boldsymbol{\Sigma}^{\mathcal{O}} \end{bmatrix}$$

We want to find the distribution of $\boldsymbol{x}^{\mathcal{O}}$ that maximizes the log-likelihood

$$\begin{aligned} f(\boldsymbol{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) &= \log\left(\mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})\right) \\ &= -\frac{1}{2}\log|\boldsymbol{\Sigma}| - \frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{\top}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}) - \frac{D}{2}\log(2\pi), \end{aligned}$$

when $\boldsymbol{x}^{\mathcal{I}}$ is known and acts as a constant.

This can be computed by deriving the above equation and equating to zero, namely

$$\frac{\partial f}{\partial \boldsymbol{x}^{\mathcal{O}}} = 0.$$

Conditional probability - Proof

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}^{\mathcal{I}} & \boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}} \\ \boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}} & \boldsymbol{\Sigma}^{\mathcal{O}} \end{bmatrix}$$

To do this, we first note that $\boldsymbol{\Sigma}^{-1}$ can be partitioned as

$$\begin{aligned} \boldsymbol{\Sigma}^{-1} = \boldsymbol{\Gamma} &= \begin{bmatrix} \boldsymbol{\Gamma}^{\mathcal{I}} & \boldsymbol{\Gamma}^{\mathcal{I}\mathcal{O}} \\ \boldsymbol{\Gamma}^{\mathcal{O}\mathcal{I}} & \boldsymbol{\Gamma}^{\mathcal{O}} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{\Sigma}^{\mathcal{I}-1}\boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma}^{\mathcal{I}-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{S}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}\boldsymbol{\Sigma}^{\mathcal{I}-1} & \boldsymbol{I} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{\Sigma}^{\mathcal{I}-1} + \boldsymbol{\Sigma}^{\mathcal{I}-1}\boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}}\boldsymbol{S}^{-1}\boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}\boldsymbol{\Sigma}^{\mathcal{I}-1} & -\boldsymbol{\Sigma}^{\mathcal{I}-1}\boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}}\boldsymbol{S}^{-1} \\ -\boldsymbol{S}^{-1}\boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}\boldsymbol{\Sigma}^{\mathcal{I}-1} & \boldsymbol{S}^{-1} \end{bmatrix}, \end{aligned}$$

where $\boldsymbol{S} = \boldsymbol{\Sigma}^{\mathcal{O}} - \boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}\boldsymbol{\Sigma}^{\mathcal{I}-1}\boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}}$ is the **Schur complement** of $\boldsymbol{\Sigma}$.

The above result can be shown by using a LDU decomposition of $\boldsymbol{\Sigma}$, where D is a diagonal matrix and L and U are atomic triangular matrices (lower and upper, respectively), and then computing its inverse by exploiting the inversion properties of diagonal and atomic triangular matrices.

Conditional probability - Proof

$$\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{\Gamma}^{\mathcal{I}} & \boldsymbol{\Gamma}^{\mathcal{I}\mathcal{O}} \\ \boldsymbol{\Gamma}^{\mathcal{O}\mathcal{I}} & \boldsymbol{\Gamma}^{\mathcal{O}} \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^{\mathcal{I}} \\ \boldsymbol{x}^{\mathcal{O}} \end{bmatrix} \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}^{\mathcal{I}} \\ \boldsymbol{\mu}^{\mathcal{O}} \end{bmatrix} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}^{\mathcal{I}} & \boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}} \\ \boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}} & \boldsymbol{\Sigma}^{\mathcal{O}} \end{bmatrix}$$

With this partitioning, we can see that

$$\begin{aligned} -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top \boldsymbol{\Gamma}(\boldsymbol{x}-\boldsymbol{\mu}) = &-\frac{1}{2}(\boldsymbol{x}^{\mathcal{I}}-\boldsymbol{\mu}^{\mathcal{I}})^\top \boldsymbol{\Gamma}^{\mathcal{I}}(\boldsymbol{x}^{\mathcal{I}}-\boldsymbol{\mu}^{\mathcal{I}}) \\ &-\frac{1}{2}(\boldsymbol{x}^{\mathcal{I}}-\boldsymbol{\mu}^{\mathcal{I}})^\top \boldsymbol{\Gamma}^{\mathcal{I}\mathcal{O}}(\boldsymbol{x}^{\mathcal{O}}-\boldsymbol{\mu}^{\mathcal{O}}) \\ &-\frac{1}{2}(\boldsymbol{x}^{\mathcal{O}}-\boldsymbol{\mu}^{\mathcal{O}})^\top \boldsymbol{\Gamma}^{\mathcal{O}\mathcal{I}}(\boldsymbol{x}^{\mathcal{I}}-\boldsymbol{\mu}^{\mathcal{I}}) \\ &-\frac{1}{2}(\boldsymbol{x}^{\mathcal{O}}-\boldsymbol{\mu}^{\mathcal{O}})^\top \boldsymbol{\Gamma}^{\mathcal{O}}(\boldsymbol{x}^{\mathcal{O}}-\boldsymbol{\mu}^{\mathcal{O}}). \end{aligned}$$

With the symmetry of precision matrices ($\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^\top$), we have

$$\begin{aligned} -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top \boldsymbol{\Gamma}(\boldsymbol{x}-\boldsymbol{\mu}) &= -\frac{1}{2}\boldsymbol{x}^\top \boldsymbol{\Gamma}(\boldsymbol{x}-\boldsymbol{\mu}) + \frac{1}{2}\boldsymbol{\mu}^\top \boldsymbol{\Gamma}(\boldsymbol{x}-\boldsymbol{\mu}) \\ &= -\frac{1}{2}\boldsymbol{x}^\top \boldsymbol{\Gamma}\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{\Gamma}\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{\mu}^\top \boldsymbol{\Gamma}\boldsymbol{x} - \frac{1}{2}\boldsymbol{\mu}^\top \boldsymbol{\Gamma}\boldsymbol{\mu} \\ &= -\frac{1}{2}\boldsymbol{x}^\top \boldsymbol{\Gamma}\boldsymbol{x} + \boldsymbol{x}^\top \boldsymbol{\Gamma}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{\mu}^\top \boldsymbol{\Gamma}\boldsymbol{\mu}. \end{aligned}$$

Conditional probability - Proof

$$f(\boldsymbol{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = -\frac{1}{2}\log|\boldsymbol{\Sigma}| - \frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}) - \frac{D}{2}\log(2\pi)$$

By using the linear algebra relations

$$-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top \boldsymbol{\Gamma}(\boldsymbol{x}-\boldsymbol{\mu}) = -\frac{1}{2}\boldsymbol{x}^\top\boldsymbol{\Gamma}\boldsymbol{x} + \boldsymbol{x}^\top\boldsymbol{\Gamma}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{\mu}^\top\boldsymbol{\Gamma}\boldsymbol{\mu}$$

$$\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{x}^\top\boldsymbol{A} = \frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{A}^\top\boldsymbol{x} = \boldsymbol{A}, \qquad \frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x} = (\boldsymbol{A}+\boldsymbol{A}^\top)\boldsymbol{x},$$

and by exploiting the derivation chain rule and the symmetry of covariances, we obtain

$$-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top \boldsymbol{\Gamma}(\boldsymbol{x}-\boldsymbol{\mu}) = -\frac{1}{2}(\boldsymbol{x}^{\mathcal{I}}-\boldsymbol{\mu}^{\mathcal{I}})^\top \boldsymbol{\Gamma}^{\mathcal{I}}(\boldsymbol{x}^{\mathcal{I}}-\boldsymbol{\mu}^{\mathcal{I}}) - \frac{1}{2}(\boldsymbol{x}^{\mathcal{I}}-\boldsymbol{\mu}^{\mathcal{I}})^\top \boldsymbol{\Gamma}^{\mathcal{I}\mathcal{O}}(\boldsymbol{x}^{\mathcal{O}}-\boldsymbol{\mu}^{\mathcal{O}})$$
$$-\frac{1}{2}(\boldsymbol{x}^{\mathcal{O}}-\boldsymbol{\mu}^{\mathcal{O}})^\top \boldsymbol{\Gamma}^{\mathcal{O}\mathcal{I}}(\boldsymbol{x}^{\mathcal{I}}-\boldsymbol{\mu}^{\mathcal{I}}) - \frac{1}{2}(\boldsymbol{x}^{\mathcal{O}}-\boldsymbol{\mu}^{\mathcal{O}})^\top \boldsymbol{\Gamma}^{\mathcal{O}}(\boldsymbol{x}^{\mathcal{O}}-\boldsymbol{\mu}^{\mathcal{O}})$$

$$\begin{aligned}
&\frac{\partial f}{\partial \boldsymbol{x}^{\mathcal{O}}} = -\boldsymbol{\Gamma}^{\mathcal{O}}\boldsymbol{\mu}^{\mathcal{O}} + \boldsymbol{\Gamma}^{\mathcal{O}\mathcal{I}}(\boldsymbol{x}^{\mathcal{I}}-\boldsymbol{\mu}^{\mathcal{I}}) + \boldsymbol{\Gamma}^{\mathcal{O}}\boldsymbol{x}^{\mathcal{O}} = 0 \\
\iff \quad &\hat{\boldsymbol{x}}^{\mathcal{O}} = \boldsymbol{\mu}^{\mathcal{O}} - \boldsymbol{\Gamma}^{\mathcal{O}-1}\boldsymbol{\Gamma}^{\mathcal{O}\mathcal{I}}(\boldsymbol{x}^{\mathcal{I}}-\boldsymbol{\mu}^{\mathcal{I}}).
\end{aligned}$$

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \boldsymbol{\Sigma}^{\mathcal{I}-1} + \boldsymbol{\Sigma}^{\mathcal{I}-1}\boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}}\boldsymbol{S}^{-1}\boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}\boldsymbol{\Sigma}^{\mathcal{I}-1} & -\boldsymbol{\Sigma}^{\mathcal{I}-1}\boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}}\boldsymbol{S}^{-1} \\ -\boldsymbol{S}^{-1}\boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}\boldsymbol{\Sigma}^{\mathcal{I}-1} & \boldsymbol{S}^{-1} \end{bmatrix}$$
$$\boldsymbol{S} = \boldsymbol{\Sigma}^{\mathcal{O}} - \boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}\boldsymbol{\Sigma}^{\mathcal{I}-1}\boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}}$$

By using the Schur decomposition, we can see that

$$\begin{aligned}
\hat{\boldsymbol{x}}^{\mathcal{O}} &= \boldsymbol{\mu}^{\mathcal{O}} - \boldsymbol{S}(-\boldsymbol{S}^{-1}\boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}\boldsymbol{\Sigma}^{\mathcal{I}-1})(\boldsymbol{x}^{\mathcal{I}}-\boldsymbol{\mu}^{\mathcal{I}}) \\
&= \boldsymbol{\mu}^{\mathcal{O}} + \boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}\boldsymbol{\Sigma}^{\mathcal{I}-1}(\boldsymbol{x}^{\mathcal{I}}-\boldsymbol{\mu}^{\mathcal{I}}).
\end{aligned}$$

Conditional probability - Proof

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \boldsymbol{\Sigma}^{\mathcal{I}-1} + \boldsymbol{\Sigma}^{\mathcal{I}-1}\boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}}\boldsymbol{S}^{-1}\boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}\boldsymbol{\Sigma}^{\mathcal{I}-1} & -\boldsymbol{\Sigma}^{\mathcal{I}-1}\boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}}\boldsymbol{S}^{-1} \\ -\boldsymbol{S}^{-1}\boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}\boldsymbol{\Sigma}^{\mathcal{I}-1} & \boldsymbol{S}^{-1} \end{bmatrix}$$

$$\boldsymbol{S} = \boldsymbol{\Sigma}^{\mathcal{O}} - \boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}\boldsymbol{\Sigma}^{\mathcal{I}-1}\boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}}$$

The associated covariance matrix $\hat{\boldsymbol{\Sigma}}^{\mathcal{O}}$ measuring the error of this estimate is given by the inverse of the Hessian matrix $\boldsymbol{H}$. We have

$$\boldsymbol{H} = \frac{\partial^2 f}{\partial \boldsymbol{x}^{\mathcal{O}} \boldsymbol{x}^{\mathcal{O}\top}} = \boldsymbol{\Gamma}^{\mathcal{O}} \quad \Rightarrow \quad \hat{\boldsymbol{\Sigma}}^{\mathcal{O}} = \boldsymbol{\Gamma}^{\mathcal{O}-1}.$$

We can then see that

$$\hat{\boldsymbol{\Sigma}}^{\mathcal{O}} = \boldsymbol{S} = \boldsymbol{\Sigma}^{\mathcal{O}} - \boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}\boldsymbol{\Sigma}^{\mathcal{I}-1}\boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}}.$$

Note that in many cases, evaluating the conditional distribution with precision matrices is computationally more efficient than with covariance matrices.

Conditional probability - Summary

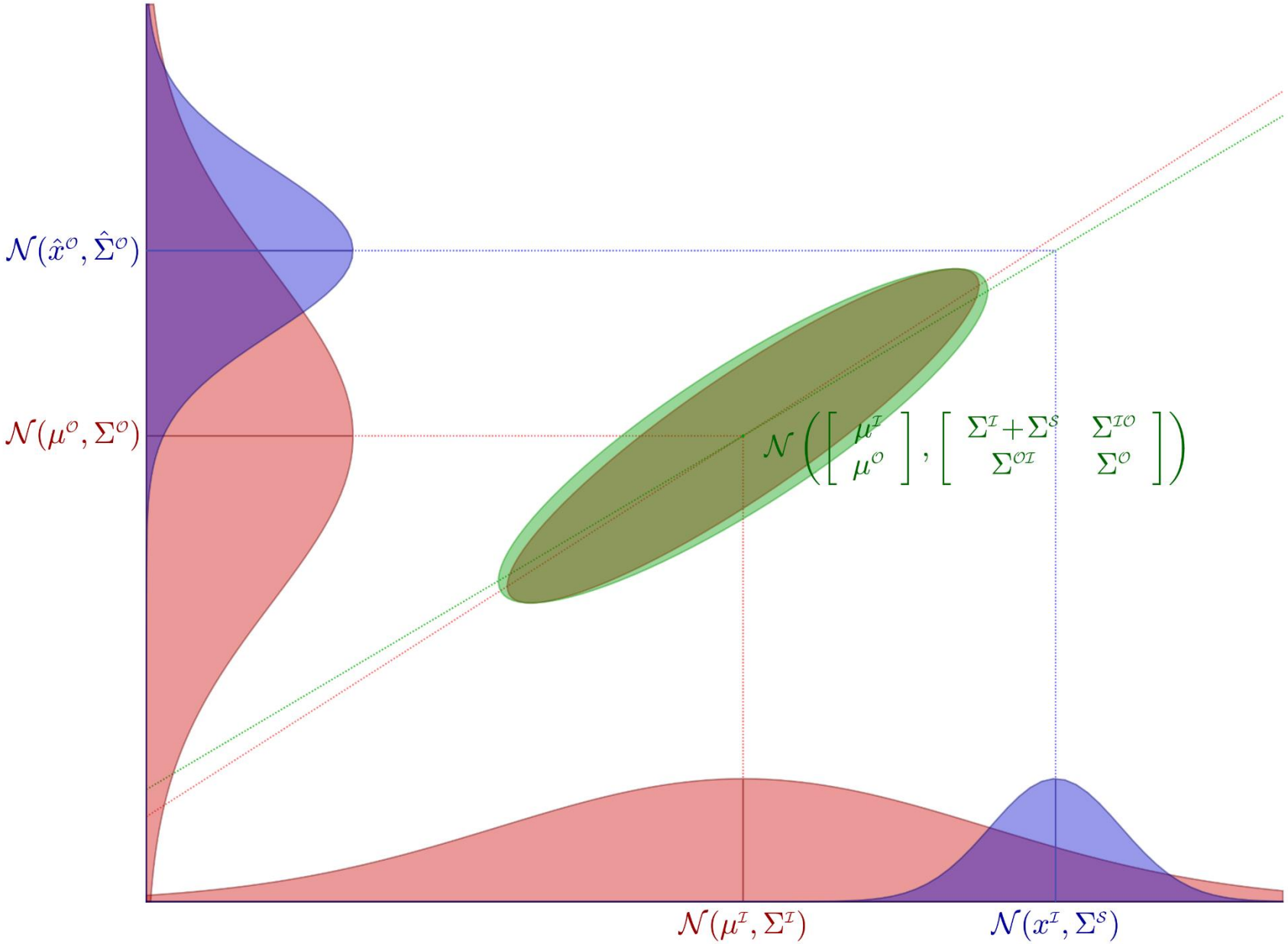
If $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we have that $\boldsymbol{x}^{\mathcal{O}}|\boldsymbol{x}^{\mathcal{I}} \sim \mathcal{N}\left(\hat{\boldsymbol{x}}^{\mathcal{O}}, \hat{\boldsymbol{\Sigma}}^{\mathcal{O}}\right)$, with parameters

$$\begin{aligned}\hat{\boldsymbol{x}}^{\mathcal{O}} &= \boldsymbol{\mu}^{\mathcal{O}} + \boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}\boldsymbol{\Sigma}^{\mathcal{I}-1}(\boldsymbol{x}^{\mathcal{I}} - \boldsymbol{\mu}^{\mathcal{I}}),\\ \hat{\boldsymbol{\Sigma}}^{\mathcal{O}} &= \boldsymbol{\Sigma}^{\mathcal{O}} - \boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}\boldsymbol{\Sigma}^{\mathcal{I}-1}\boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}}.\end{aligned}$$

If $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Gamma}^{-1})$, we have that $\boldsymbol{x}^{\mathcal{O}}|\boldsymbol{x}^{\mathcal{I}} \sim \mathcal{N}\left(\hat{\boldsymbol{x}}^{\mathcal{O}}, \hat{\boldsymbol{\Gamma}}^{\mathcal{O}-1}\right)$, with parameters

$$\begin{aligned}\hat{\boldsymbol{x}}^{\mathcal{O}} &= \boldsymbol{\mu}^{\mathcal{O}} - \boldsymbol{\Gamma}^{\mathcal{O}-1}\boldsymbol{\Gamma}^{\mathcal{O}\mathcal{I}}(\boldsymbol{x}^{\mathcal{I}} - \boldsymbol{\mu}^{\mathcal{I}}),\\ \hat{\boldsymbol{\Sigma}}^{\mathcal{O}} &= \boldsymbol{\Gamma}^{\mathcal{O}-1}.\end{aligned}$$

Gaussian conditioning with uncertain inputs



Gaussian conditioning with uncertain inputs

For inputs corrupted by noise $\boldsymbol{\epsilon}^{\mathcal{I}} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}^{\mathcal{S}})$, we have that $\boldsymbol{x}^{\mathcal{O}} | \boldsymbol{x}^{\mathcal{I}} + \boldsymbol{\epsilon}^{\mathcal{I}} \sim \mathcal{N}\left(\hat{\boldsymbol{x}}^{\mathcal{O}}, \hat{\boldsymbol{\Sigma}}^{\mathcal{O}}\right)$ with parameters

$$\begin{aligned} \hat{\boldsymbol{x}}^{\mathcal{O}} &= \boldsymbol{\mu}^{\mathcal{O}} + \boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}(\boldsymbol{\Sigma}^{\mathcal{I}} + \boldsymbol{\Sigma}^{\mathcal{S}})^{-1}(\boldsymbol{x}^{\mathcal{I}} - \boldsymbol{\mu}^{\mathcal{I}}), \\ \hat{\boldsymbol{\Sigma}}^{\mathcal{O}} &= \boldsymbol{\Sigma}^{\mathcal{O}} - \boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}}(\boldsymbol{\Sigma}^{\mathcal{I}} + \boldsymbol{\Sigma}^{\mathcal{S}})^{-1}\boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}}. \end{aligned}$$

This can for example be shown by redefining the joint distribution $\boldsymbol{\Sigma}$ with

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}^{\mathcal{I}} + \boldsymbol{\Sigma}^{\mathcal{S}} & \boldsymbol{\Sigma}^{\mathcal{I}\mathcal{O}} \\ \boldsymbol{\Sigma}^{\mathcal{O}\mathcal{I}} & \boldsymbol{\Sigma}^{\mathcal{O}} \end{bmatrix},$$

and applying the same formulas as for standard Gaussian conditioning.

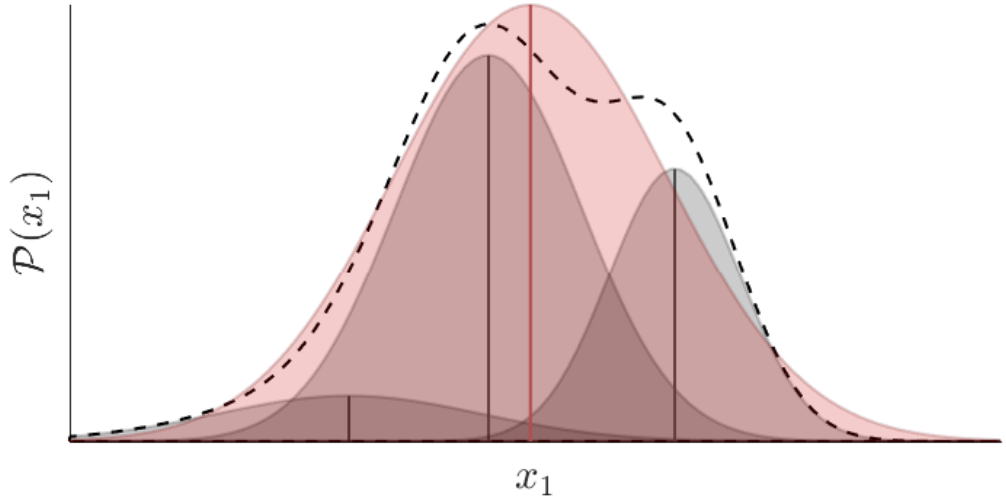
Gaussian estimate of a mixture of Gaussians

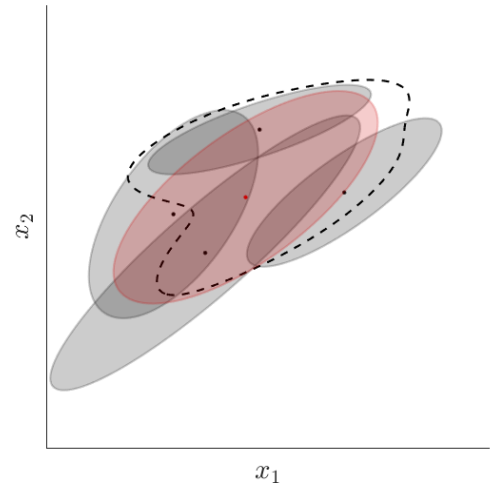
We can approximate a mixture of Gaussians $\sum_{i=1}^{K} h_i \mathcal{N}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$ with a single Gaussian $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, by **moment matching of the means (first moments) and covariances (second moments)** with

$$\boldsymbol{\mu} = \sum_{i=1}^{K} h_i \, \boldsymbol{\mu}_i,$$

$$\boldsymbol{\Sigma} = \sum_{i=1}^{K} h_i \left(\boldsymbol{\Sigma}_i + \boldsymbol{\mu}_i \boldsymbol{\mu}_i^\top \right) - \boldsymbol{\mu} \boldsymbol{\mu}^\top,$$

also referred to as the **law of total mean and (co)variance**.





Gaussian estimate of a mixture - Proof

The result can be demonstrated by developing the expressions

$$\mathbb{E}(\boldsymbol{x}) = \boldsymbol{\mu}, \quad \boldsymbol{\Sigma} = \mathrm{cov}(\boldsymbol{x}) = \mathbb{E}(\boldsymbol{x}\boldsymbol{x}^\top) - \mathbb{E}(\boldsymbol{x})\mathbb{E}(\boldsymbol{x}^\top) = \mathbb{E}(\boldsymbol{x}\boldsymbol{x}^\top) - \boldsymbol{\mu}\boldsymbol{\mu}^\top$$
$$\iff \mathbb{E}(\boldsymbol{x}\boldsymbol{x}^\top) = \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^\top.$$

By considering datapoints $\boldsymbol{x}$ distributed with a mixture of Gaussians

$$\mathcal{P}(\boldsymbol{x}) = \sum_{i=1}^{K} \mathcal{P}(z_i)\mathcal{P}(\boldsymbol{x}|z_i) = \sum_{i=1}^{K} h_i \, \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i) \,,$$

the mean is computed as

$$\boldsymbol{\mu} = \mathbb{E}(\boldsymbol{x}) = \int \boldsymbol{x} \, \mathcal{P}(\boldsymbol{x}) \, d\boldsymbol{x} = \int \boldsymbol{x} \sum_{i=1}^{K} h_i \, \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i) \, d\boldsymbol{x}$$
$$= \sum_{i=1}^{K} h_i \int \boldsymbol{x} \, \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i) \, d\boldsymbol{x}$$

$\mathbb{E}(\boldsymbol{x}) = \int \boldsymbol{x} \, \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i) \, d\boldsymbol{x}$

$$= \sum_{i=1}^{K} h_i \, \boldsymbol{\mu}_i.$$

Gaussian estimate of a mixture of Gaussians

By noting that

$$\mathbb{E}(\boldsymbol{x}\boldsymbol{x}^\top) = \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^\top$$

$$\begin{aligned}
\mathbb{E}(\boldsymbol{x}\boldsymbol{x}^\top) &= \int \boldsymbol{x}\boldsymbol{x}^\top \mathcal{P}(\boldsymbol{x})\, d\boldsymbol{x} \\
&= \int \sum_{i=1}^{K} h_i\, \boldsymbol{x}\boldsymbol{x}^\top \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)\, d\boldsymbol{x} \\
&= \sum_{i=1}^{K} h_i \int \boldsymbol{x}\boldsymbol{x}^\top \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)\, d\boldsymbol{x} \\
&= \sum_{i=1}^{K} h_i \left(\boldsymbol{\Sigma}_i + \boldsymbol{\mu}_i\boldsymbol{\mu}_i^\top\right),
\end{aligned}$$

$$\mathbb{E}(\boldsymbol{x}\boldsymbol{x}^\top) = \int \boldsymbol{x}\boldsymbol{x}^\top \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)\, d\boldsymbol{x}$$

the covariance is then computed as

$$\boldsymbol{\Sigma} = \sum_{i=1}^{K} h_i \left(\boldsymbol{\Sigma}_i + \boldsymbol{\mu}_i\boldsymbol{\mu}_i^\top\right) - \boldsymbol{\mu}\boldsymbol{\mu}^\top.$$

Locally weighted regression (LWR)

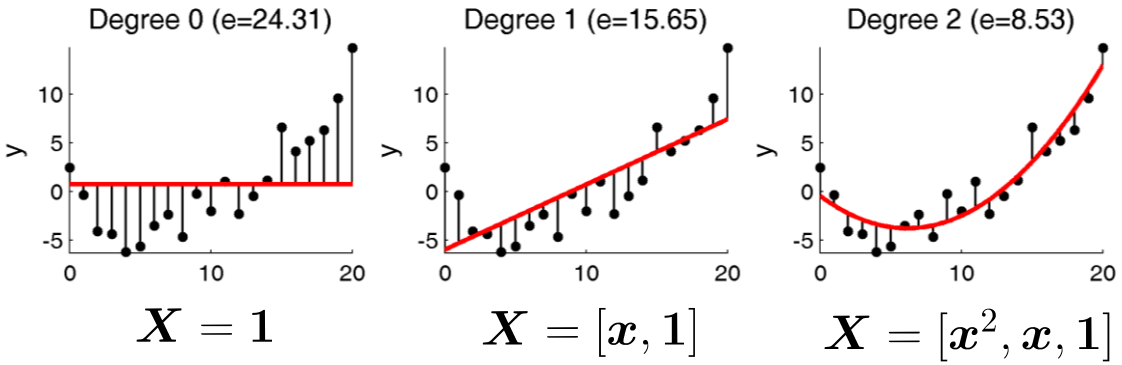
Matlab codes: demo_LWR01.m, demo_DMP01.m

[C.G. Atkeson, A.W. Moore, and S. Schaal. Locally weighted learning for control. Artificial Intelligence Review, 11(1-5):75–113, 1997]

[W.S. Cleveland. Robust locally weighted regression and smoothing scatterplots. American Statistical Association 74(368):829–836, 1979]

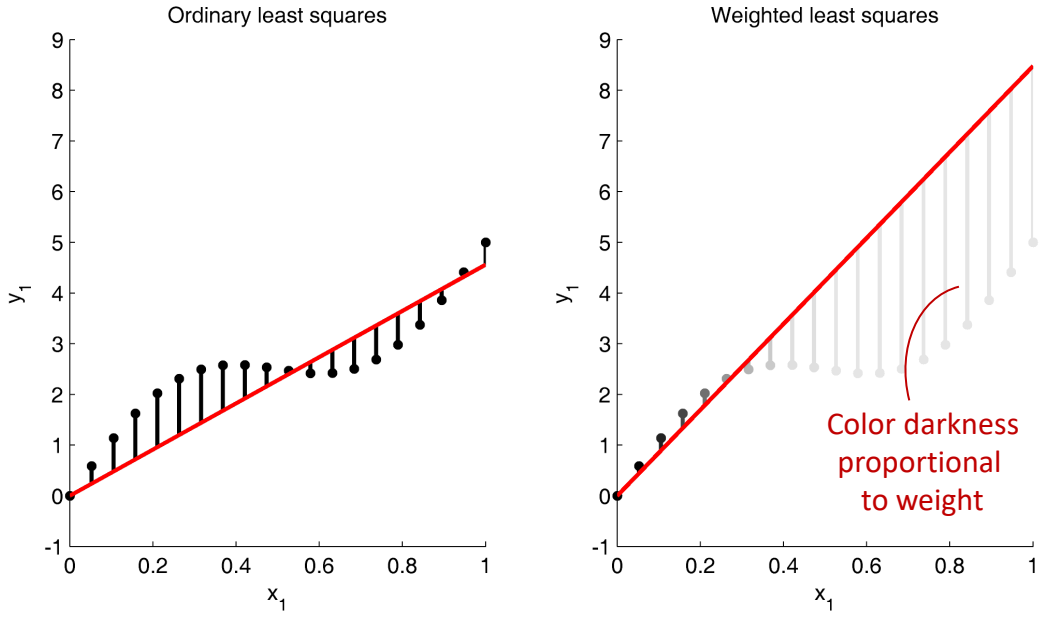
Previous lecture on linear regression

$$\hat{\boldsymbol{A}} = \arg\min_{\boldsymbol{A}} (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{A})^\top (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{A})$$
$$= (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{Y} = \boldsymbol{X}^\dagger \boldsymbol{Y}$$



$\boldsymbol{X} = \mathbf{1}$ $\quad \boldsymbol{X} = [\boldsymbol{x}, \mathbf{1}]$ $\quad \boldsymbol{X} = [\boldsymbol{x}^2, \boldsymbol{x}, \mathbf{1}]$

$$\hat{\boldsymbol{A}} = \arg\min_{\boldsymbol{A}} (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{A})^\top \boldsymbol{W} (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{A})$$
$$= (\boldsymbol{X}^\top \boldsymbol{W} \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{W} \boldsymbol{Y}$$



Locally weighted regression (LWR)

Locally weighted regression (LWR) is a direct extension of the weighted least squares formulation in which K weighted regressions are performed on the same dataset $\{\boldsymbol{X}^{\mathcal{I}}, \boldsymbol{X}^{\mathcal{O}}\}$.

It aims at splitting a nonlinear problem so that it can be solved locally by linear regression.

LWR computes K estimates $\hat{\boldsymbol{A}}_k$, each with a different weighting function $\phi_k(\boldsymbol{x}_n^{\mathcal{I}})$, often defined as the **radial basis functions** (RBF)

$$\tilde{\phi}_k(\boldsymbol{x}_n^{\mathcal{I}}) = \exp\Big(-\frac{1}{2}(\boldsymbol{x}_n^{\mathcal{I}} - \boldsymbol{\mu}_k^{\mathcal{I}})^\top {\boldsymbol{\Sigma}_k^{\mathcal{I}}}^{-1}(\boldsymbol{x}_n^{\mathcal{I}} - \boldsymbol{\mu}_k^{\mathcal{I}})\Big),$$

or in its rescaled form as

$$\phi_k(\boldsymbol{x}_n^{\mathcal{I}}) = \frac{\tilde{\phi}_k(\boldsymbol{x}_n^{\mathcal{I}})}{\sum_{i=1}^{K} \tilde{\phi}_i(\boldsymbol{x}_n^{\mathcal{I}})},$$

where $\boldsymbol{\mu}_k^{\mathcal{I}}$ and $\boldsymbol{\Sigma}_k^{\mathcal{I}}$ are the parameters of the k-th RBF.

Locally weighted regression (LWR)

Often, the centroids $\boldsymbol{\mu}_k^{\mathcal{I}}$ are set to uniformly cover the input space, and $\boldsymbol{\Sigma}_k^{\mathcal{I}} = \boldsymbol{I}\sigma^2$ is used as a common bandwidth shared by all basis functions.

An associated diagonal matrix

$$\boldsymbol{W}_k = \text{diag}\Big(\phi_k(\boldsymbol{x}_1^{\mathcal{I}}), \phi_k(\boldsymbol{x}_2^{\mathcal{I}}), \ldots, \phi_k(\boldsymbol{x}_N^{\mathcal{I}})\Big),$$

can then be used to evaluate $\hat{\boldsymbol{A}}_k$. The result can finally be used to compute

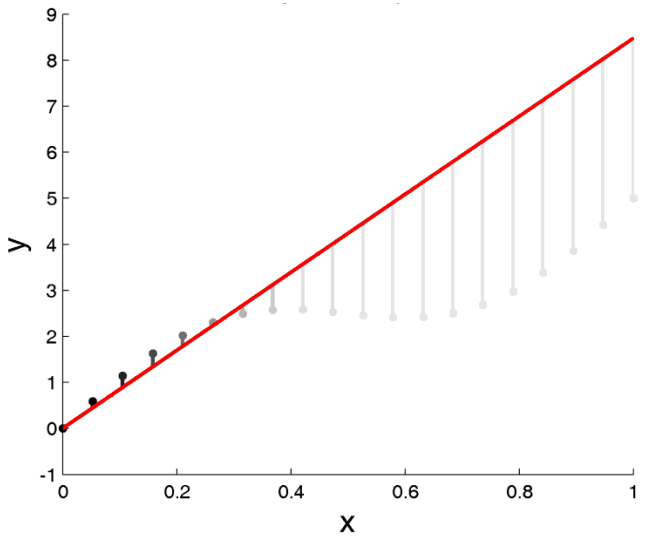
$$\boldsymbol{X}^{\mathcal{O}} = \sum_{k=1}^{K} \boldsymbol{W}_k \boldsymbol{X}^{\mathcal{I}} \hat{\boldsymbol{A}}_k.$$

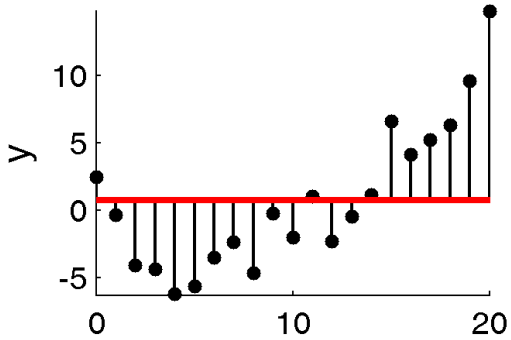
$$\boldsymbol{X}^{\mathcal{I}} = [t_1, t_2, \ldots, t_N]^\top$$
$$\hat{\boldsymbol{A}}_k = (\boldsymbol{X}^{\mathcal{I}\top} \boldsymbol{W}_k \boldsymbol{X}^{\mathcal{I}})^{-1} \boldsymbol{X}^{\mathcal{I}\top} \boldsymbol{W}_k \boldsymbol{X}^{\mathcal{O}}$$

Multiple variants exist, including online estimation within a recursive formulation [Schaal'98], Bayesian treatments of LWR [Ting'08], and also extensions such as **locally weighted projection regression** (LWPR) that exploit partial least squares to handle redundant and/or irrelevant inputs, with an online algorithm to estimate the model parameters [Vijayakumar'05].

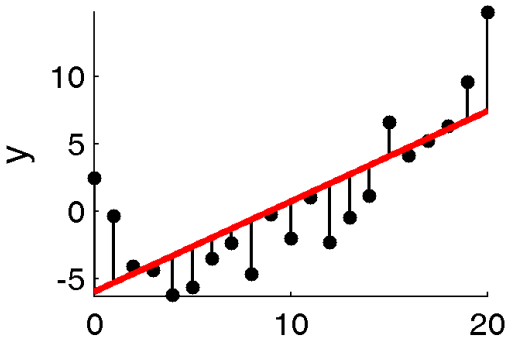
Locally weighted regression (LWR)

$$\hat{A} = (X^\top W X)^{-1} X^\top W Y$$



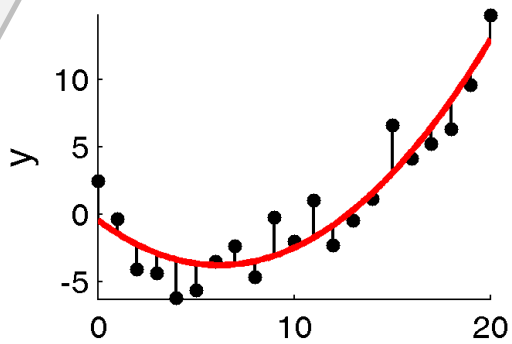


$X = 1$



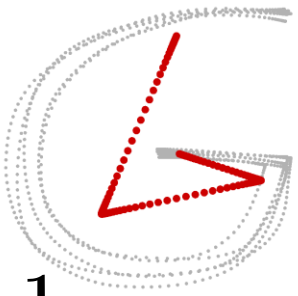
$X = [x, 1]$

LWR can be used for local least squares polynomial fitting by changing the definition of the inputs.

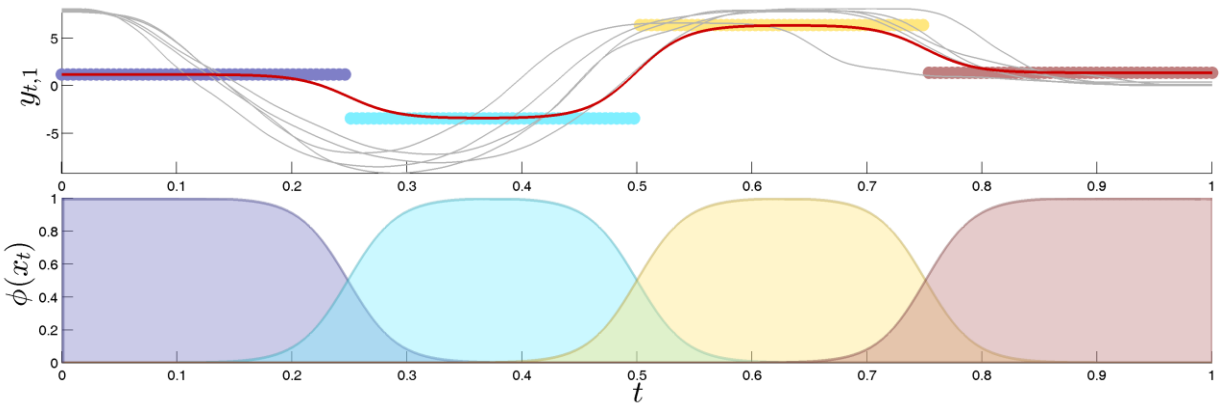


$X = [x^2, x, 1]$

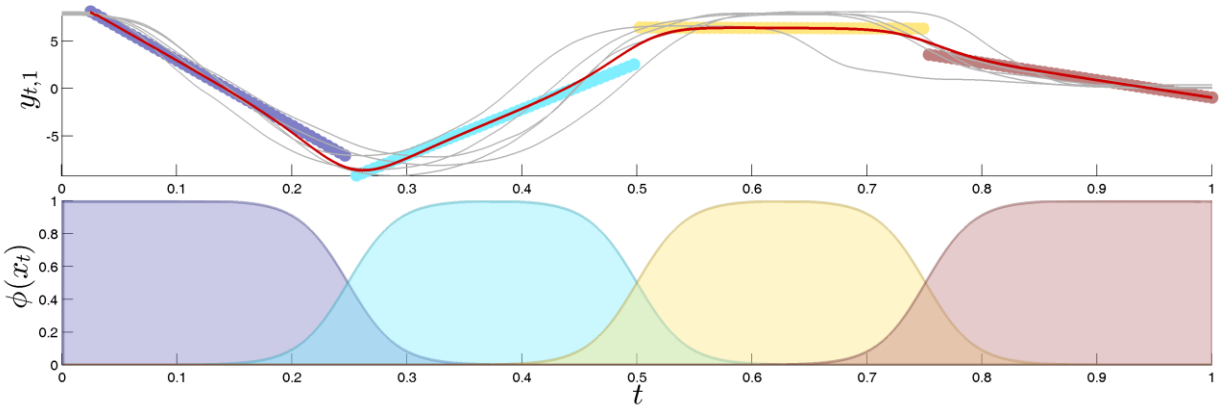
Locally weighted regression (LWR)



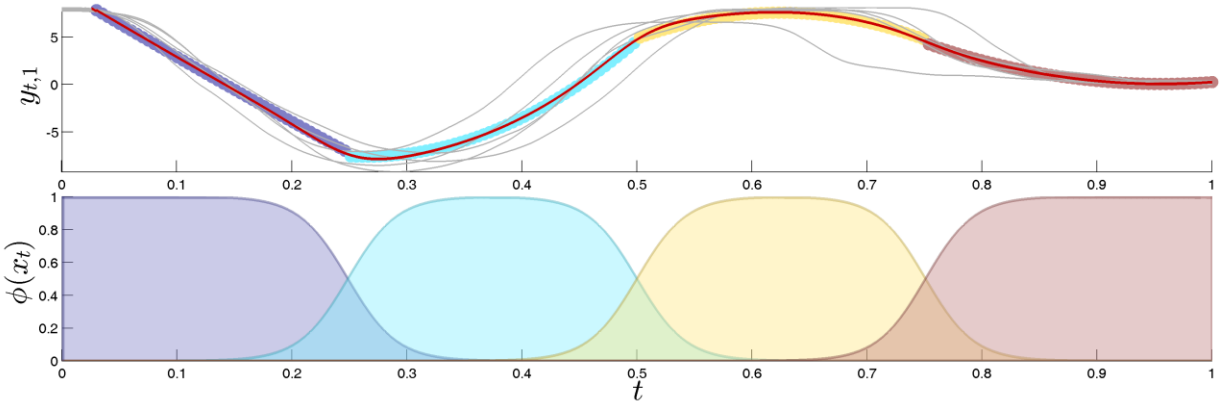
$$X = 1$$



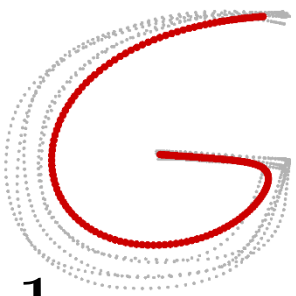
$$X = [x, 1]$$



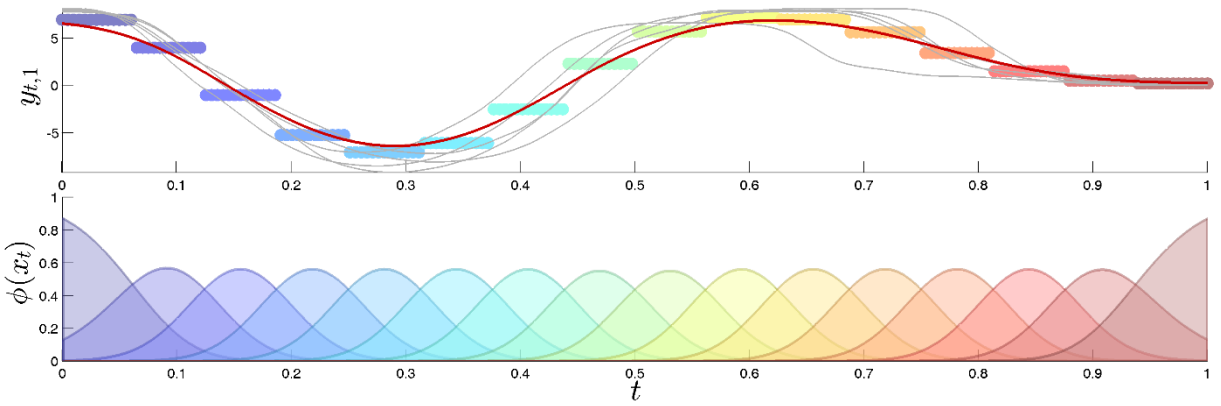
$$X = [x^2, x, 1]$$



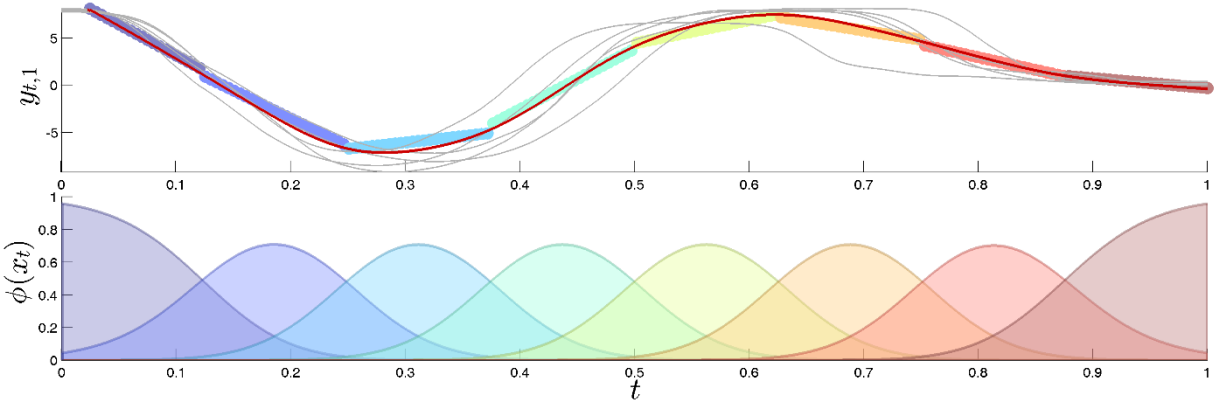
Locally weighted regression (LWR)



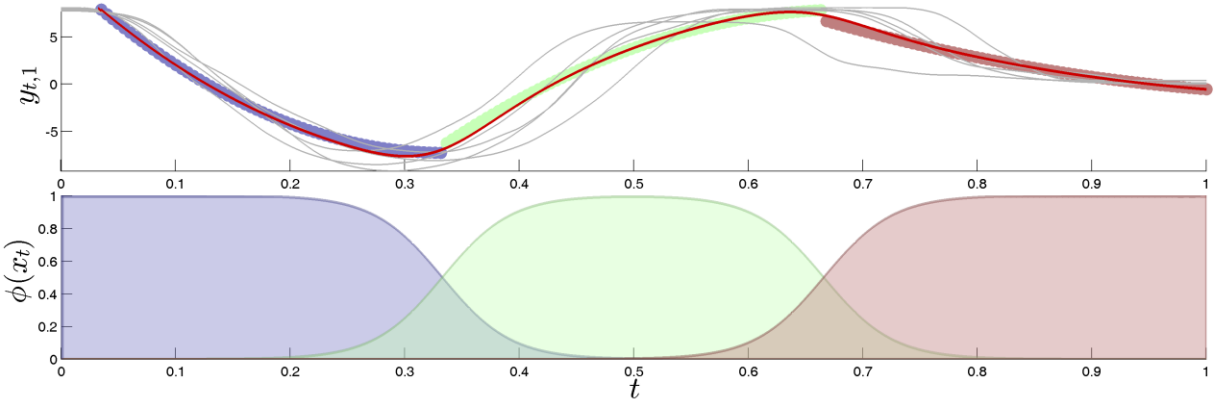
$$X = 1$$



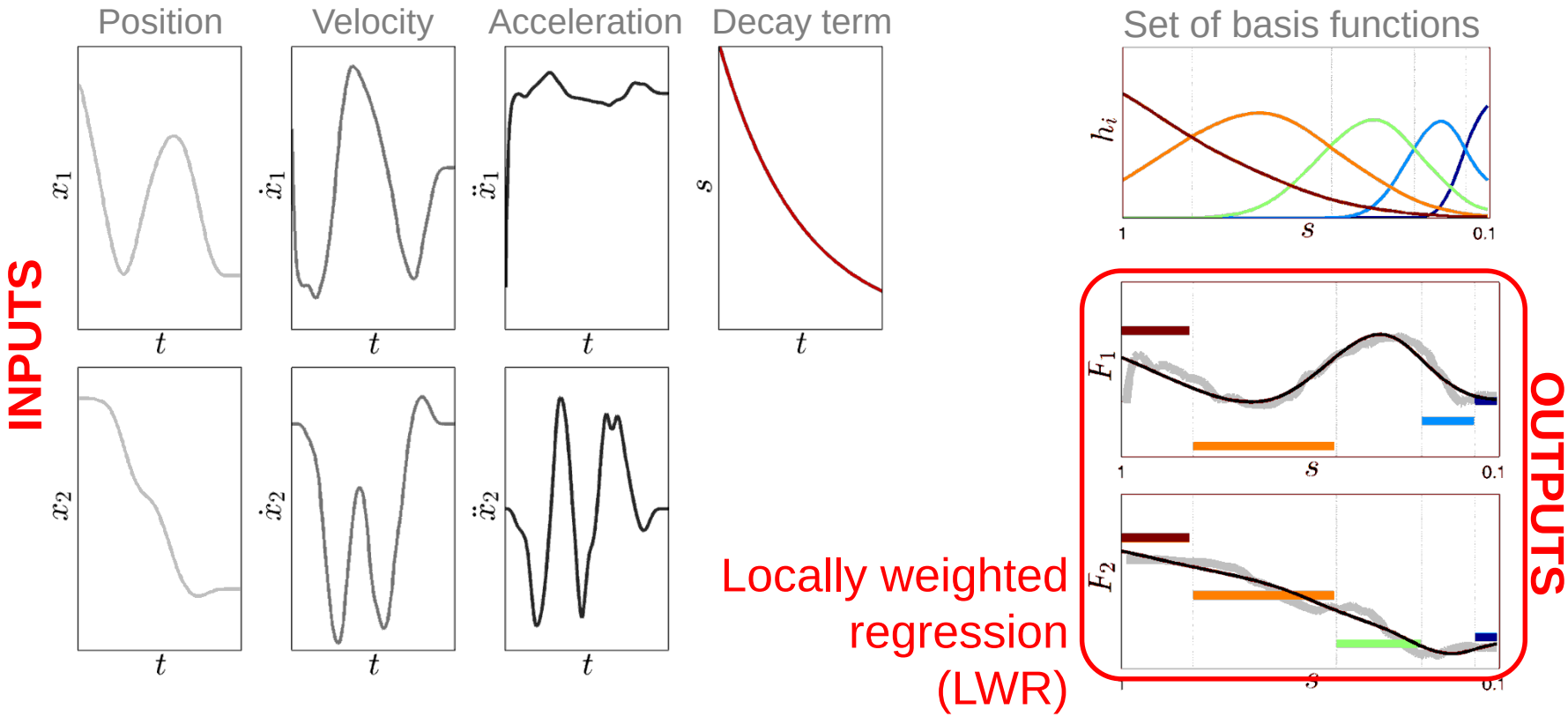
$$X = [x, 1]$$



$$X = [x^2, x, 1]$$



Dynamical movement primitives (DMP)



Locally weighted regression (LWR)

$$\tau\ddot{x} = \kappa^{P}\left[x_T - x\right] - \kappa^{V}\dot{x} + f(s), \quad f(s) = s\sum_{i=1}^{K} h_i(s)\, f_i$$

$$\tau\dot{s} = -\alpha s$$

[Ijspeert, Nakanishi and Schaal, NIPS'2003]

[Ijspeert, Nakanishi, Pastor, Hoffmann and Schaal, Neural Computation 25(2), 2013]

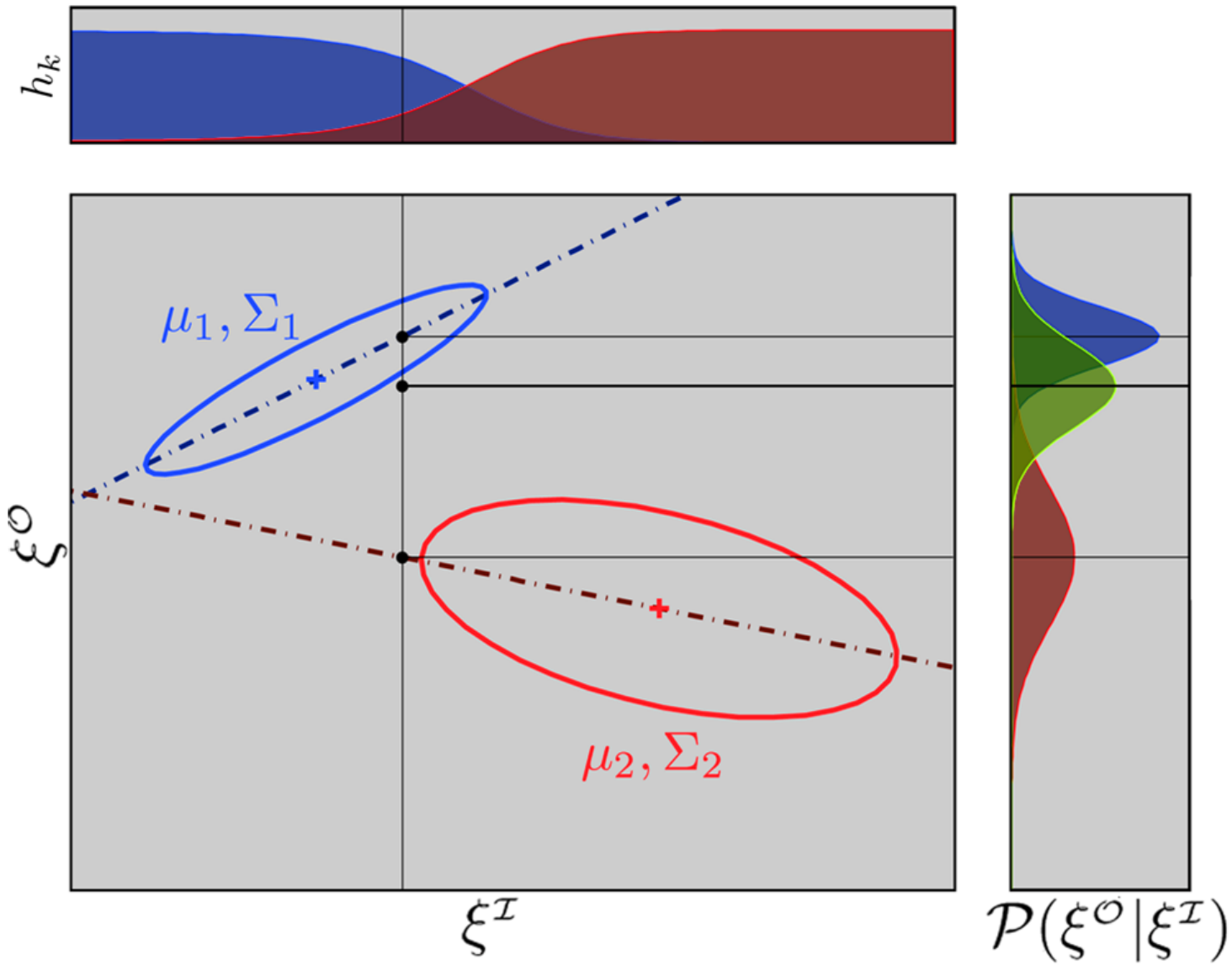
Gaussian mixture regression (GMR)

Matlab codes: demo_GMR01.m demo_GMR_polyFit01.m, demo_DMP_GMR01.m

[Z. Ghahramani and M. I. Jordan. Supervised learning from incomplete data via an EM approach. In Advances in Neural Information Processing Systems (NIPS), volume 6, pages 120–127, 1994]

[S. Calinon S. A tutorial on task-parameterized movement learning and retrieval. Intelligent Service Robotics 9(1):1–29, 2016]

Gaussian mixture regression (GMR)



Gaussian mixture regression (GMR)

- Gaussian mixture regression (GMR) is a nonlinear regression technique that does not model the regression function directly, but instead first models the **joint probability density of input-output data** in the form of a Gaussian mixture model (GMM).
- The computation relies on **linear transformation and conditioning properties** of multivariate normal distributions.
- GMR provides a regression approach in which **multivariate output distributions can be computed in an online manner**, with a computation time **independent of the number of datapoints** used to train the model, by exploiting the learned joint density model.
- In GMR, **both input and output variables can be multivariate**, and after learning, **any subset of input-output dimensions can be selected** for regression. This can for example be exploited to handle different sources of missing data, where expectations on the remaining dimensions can be computed as a multivariate distribution.

Gaussian mixture regression (GMR)

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^{\mathcal{I}} \\ \boldsymbol{x}^{\mathcal{O}} \end{bmatrix} \quad \boldsymbol{\mu}_i = \begin{bmatrix} \boldsymbol{\mu}_i^{\mathcal{I}} \\ \boldsymbol{\mu}_i^{\mathcal{O}} \end{bmatrix} \quad \boldsymbol{\Sigma}_i = \begin{bmatrix} \boldsymbol{\Sigma}_i^{\mathcal{I}} & \boldsymbol{\Sigma}_i^{\mathcal{I}\mathcal{O}} \\ \boldsymbol{\Sigma}_i^{\mathcal{O}\mathcal{I}} & \boldsymbol{\Sigma}_i^{\mathcal{O}} \end{bmatrix}$$

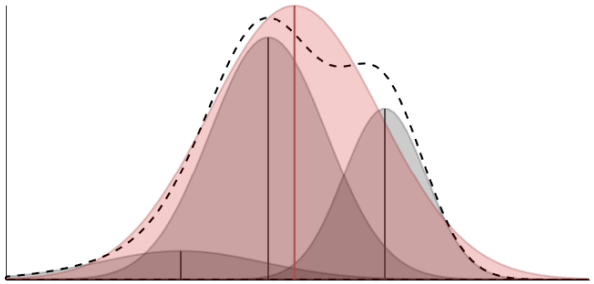
$\mathcal{P}(\boldsymbol{x}^{\mathcal{O}}|\boldsymbol{x}^{\mathcal{I}})$ can be computed as the multimodal conditional distribution

$$\begin{aligned} \mathcal{P}(\boldsymbol{x}^{\mathcal{O}}|\boldsymbol{x}^{\mathcal{I}}) &= \sum_{i=1}^{K} h_i \, \mathcal{N}\left(\boldsymbol{x}^{\mathcal{O}}|\hat{\boldsymbol{\mu}}_i^{\mathcal{O}}, \hat{\boldsymbol{\Sigma}}_i^{\mathcal{O}}\right), \\ \text{with} \quad \hat{\boldsymbol{\mu}}_i^{\mathcal{O}} &= \boldsymbol{\mu}_i^{\mathcal{O}} + \boldsymbol{\Sigma}_i^{\mathcal{O}\mathcal{I}} \boldsymbol{\Sigma}_i^{\mathcal{I}-1} (\boldsymbol{x}^{\mathcal{I}} - \boldsymbol{\mu}_i^{\mathcal{I}}), \\ \hat{\boldsymbol{\Sigma}}_i^{\mathcal{O}} &= \boldsymbol{\Sigma}_i^{\mathcal{O}} - \boldsymbol{\Sigma}_i^{\mathcal{O}\mathcal{I}} \boldsymbol{\Sigma}_i^{\mathcal{I}-1} \boldsymbol{\Sigma}_i^{\mathcal{I}\mathcal{O}} \\ \text{and} \quad h_i &= \frac{\pi_i \, \mathcal{N}(\boldsymbol{x}^{\mathcal{I}}|\, \boldsymbol{\mu}_i^{\mathcal{I}}, \boldsymbol{\Sigma}_i^{\mathcal{I}})}{\sum_k^K \pi_k \, \mathcal{N}(\boldsymbol{x}^{\mathcal{I}}|\, \boldsymbol{\mu}_k^{\mathcal{I}}, \boldsymbol{\Sigma}_k^{\mathcal{I}})}, \end{aligned}$$

computed with the marginal

$$\mathcal{N}(\boldsymbol{x}^{\mathcal{I}}|\, \boldsymbol{\mu}_i^{\mathcal{I}}, \boldsymbol{\Sigma}_i^{\mathcal{I}}) = (2\pi)^{-\frac{D}{2}} |\boldsymbol{\Sigma}_i^{\mathcal{I}}|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}^{\mathcal{I}} - \boldsymbol{\mu}_i^{\mathcal{I}})^{\top} \boldsymbol{\Sigma}_i^{\mathcal{I}-1} (\boldsymbol{x}^{\mathcal{I}} - \boldsymbol{\mu}_i^{\mathcal{I}})\right).$$

Gaussian mixture regression (GMR)



$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^{\mathcal{I}} \\ \boldsymbol{x}^{\mathcal{O}} \end{bmatrix} \quad \boldsymbol{\mu}_i = \begin{bmatrix} \boldsymbol{\mu}_i^{\mathcal{I}} \\ \boldsymbol{\mu}_i^{\mathcal{O}} \end{bmatrix} \quad \boldsymbol{\Sigma}_i = \begin{bmatrix} \boldsymbol{\Sigma}_i^{\mathcal{I}} & \boldsymbol{\Sigma}_i^{\mathcal{I}\mathcal{O}} \\ \boldsymbol{\Sigma}_i^{\mathcal{O}\mathcal{I}} & \boldsymbol{\Sigma}_i^{\mathcal{O}} \end{bmatrix}$$

$$\begin{aligned} \hat{\boldsymbol{\mu}}_i^{\mathcal{O}} &= \boldsymbol{\mu}_i^{\mathcal{O}} + \boldsymbol{\Sigma}_i^{\mathcal{O}\mathcal{I}} \boldsymbol{\Sigma}_i^{\mathcal{I}-1} (\boldsymbol{x}^{\mathcal{I}} - \boldsymbol{\mu}_i^{\mathcal{I}}) \\ \hat{\boldsymbol{\Sigma}}_i^{\mathcal{O}} &= \boldsymbol{\Sigma}_i^{\mathcal{O}} - \boldsymbol{\Sigma}_i^{\mathcal{O}\mathcal{I}} \boldsymbol{\Sigma}_i^{\mathcal{I}-1} \boldsymbol{\Sigma}_i^{\mathcal{I}\mathcal{O}} \end{aligned}$$

An output distribution as a single multivariate Gaussian can be evaluated by moment matching of the means and covariances. The resulting Gaussian distribution $\mathcal{N}(\hat{\boldsymbol{\mu}}^{\mathcal{O}}, \hat{\boldsymbol{\Sigma}}^{\mathcal{O}})$ has parameters

$$\hat{\boldsymbol{\mu}}^{\mathcal{O}} = \sum_{i=1}^{K} h_i \; \hat{\boldsymbol{\mu}}_i^{\mathcal{O}},$$

$$\hat{\boldsymbol{\Sigma}}^{\mathcal{O}} = \sum_{i=1}^{K} h_i \left(\hat{\boldsymbol{\Sigma}}_i^{\mathcal{O}} + \hat{\boldsymbol{\mu}}_i^{\mathcal{O}} \; \hat{\boldsymbol{\mu}}_i^{\mathcal{O}\top} \right) - \hat{\boldsymbol{\mu}}^{\mathcal{O}} \hat{\boldsymbol{\mu}}^{\mathcal{O}\top}.$$

Gaussian mixture regression (GMR) - Proof

The above can be demonstrated by computing

$$\hat{\boldsymbol{\mu}}^{\mathcal{O}} = \mathbb{E}(\boldsymbol{x}^{\mathcal{O}}|\boldsymbol{x}^{\mathcal{I}}),$$
$$\hat{\boldsymbol{\Sigma}}^{\mathcal{O}} = \mathrm{cov}(\boldsymbol{x}^{\mathcal{O}}|\boldsymbol{x}^{\mathcal{I}}) = \mathbb{E}(\boldsymbol{x}^{\mathcal{O}}\boldsymbol{x}^{\mathcal{O}\top}|\boldsymbol{x}^{\mathcal{I}}) - \mathbb{E}(\boldsymbol{x}^{\mathcal{O}}|\boldsymbol{x}^{\mathcal{I}})\mathbb{E}(\boldsymbol{x}^{\mathcal{O}\top}|\boldsymbol{x}^{\mathcal{I}}).$$

The conditional mean can be computed as

$$\hat{\boldsymbol{\mu}}^{\mathcal{O}} = \mathbb{E}(\boldsymbol{x}^{\mathcal{O}}|\boldsymbol{x}^{\mathcal{I}}) = \sum_{i=1}^{K} h_i \, \hat{\boldsymbol{\mu}}_i^{\mathcal{O}}.$$

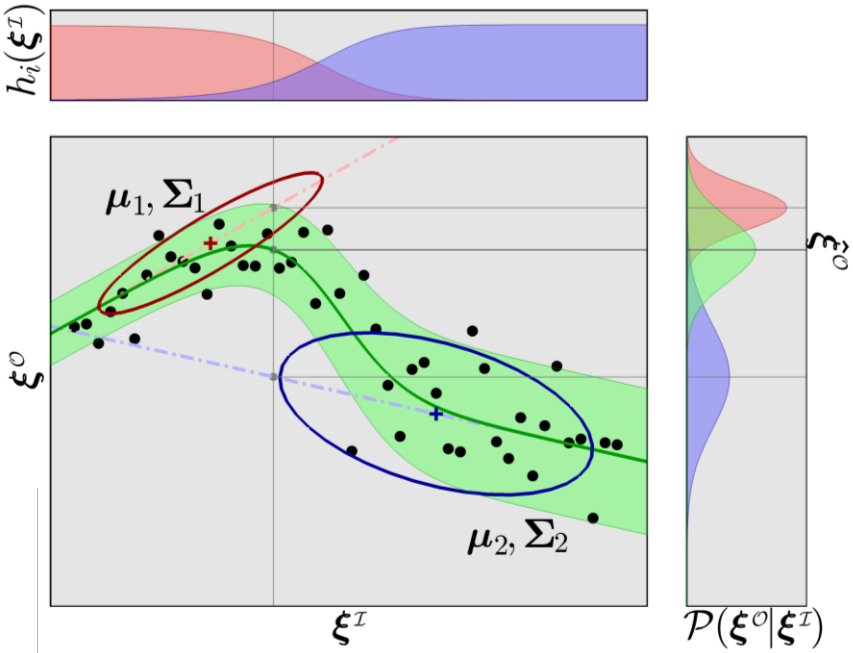
In order to evaluate the covariance, we first note that

$$\mathbb{E}(\boldsymbol{x}^{\mathcal{O}}\boldsymbol{x}^{\mathcal{O}\top}|\boldsymbol{x}^{\mathcal{I}}) = \sum_{i=1}^{K} h_i \hat{\boldsymbol{\Sigma}}_i^{\mathcal{O}} + \sum_{i=1}^{K} h_i \, \hat{\boldsymbol{\mu}}_i^{\mathcal{O}} \, \hat{\boldsymbol{\mu}}_i^{\mathcal{O}\top}.$$

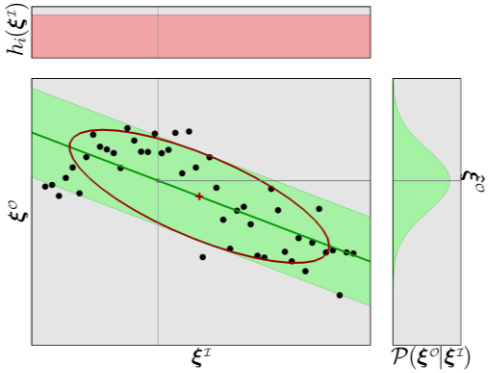
We then have

$$\hat{\boldsymbol{\Sigma}}^{\mathcal{O}} = \sum_{i=1}^{K} h_i \left(\hat{\boldsymbol{\Sigma}}_i^{\mathcal{O}} + \hat{\boldsymbol{\mu}}_i^{\mathcal{O}} \, \hat{\boldsymbol{\mu}}_i^{\mathcal{O}\top} \right) - \hat{\boldsymbol{\mu}}^{\mathcal{O}} \hat{\boldsymbol{\mu}}^{\mathcal{O}\top}.$$

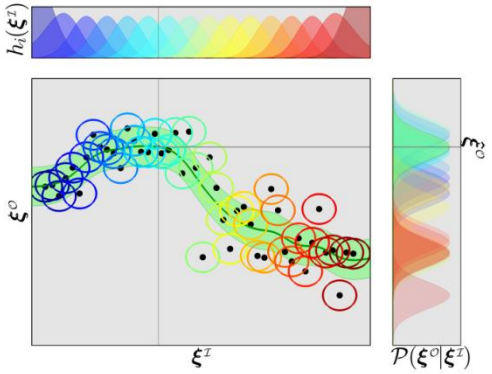
Gaussian mixture regression (GMR)



Least squares linear regression



Nadaraya-Watson kernel regression



GMR can cover a large spectrum of regression mechanisms

Both $\xi^{\mathcal{I}}$ and $\xi^{\mathcal{O}}$ can be multidimensional

$\mathcal{P}(\xi^{\mathcal{I}}, \xi^{\mathcal{O}})$ encoded in **Gaussian mixture model (GMM)**

$\mathcal{P}(\xi^{\mathcal{O}}|\xi^{\mathcal{I}})$ retrieved by **Gaussian mixture regression (GMR)**

GMR with uncertain inputs

For inputs with noise $\boldsymbol{\epsilon}^{\mathcal{I}} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}^{\mathcal{S}})$, we have that $\boldsymbol{x}^{\mathcal{O}}|\boldsymbol{x}^{\mathcal{I}}+\boldsymbol{\epsilon}^{\mathcal{I}} \sim \mathcal{N}\left(\hat{\boldsymbol{\mu}}^{\mathcal{O}}, \hat{\boldsymbol{\Sigma}}^{\mathcal{O}}\right)$, with parameters

$$\hat{\boldsymbol{\mu}}^{\mathcal{O}} = \sum_{i=1}^{K} h_i \; \hat{\boldsymbol{\mu}}_i^{\mathcal{O}},$$

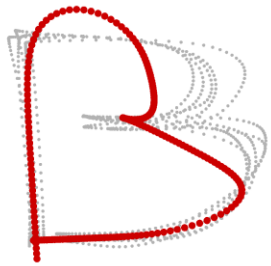
$$\hat{\boldsymbol{\Sigma}}^{\mathcal{O}} = \sum_{i=1}^{K} h_i \left(\hat{\boldsymbol{\Sigma}}_i^{\mathcal{O}} + \hat{\boldsymbol{\mu}}_i^{\mathcal{O}} \; \hat{\boldsymbol{\mu}}_i^{\mathcal{O}\top}\right) - \hat{\boldsymbol{\mu}}^{\mathcal{O}} \hat{\boldsymbol{\mu}}^{\mathcal{O}\top},$$

computed with

$$\begin{aligned} \hat{\boldsymbol{\mu}}_i^{\mathcal{O}} &= \boldsymbol{\mu}_i^{\mathcal{O}} + \boldsymbol{\Sigma}_i^{\mathcal{O}\mathcal{I}}(\boldsymbol{\Sigma}_i^{\mathcal{I}} + \boldsymbol{\Sigma}^{\mathcal{S}})^{-1}(\boldsymbol{x}^{\mathcal{I}} - \boldsymbol{\mu}_i^{\mathcal{I}}) \;, \\ \hat{\boldsymbol{\Sigma}}_i^{\mathcal{O}} &= \boldsymbol{\Sigma}_i^{\mathcal{O}} - \boldsymbol{\Sigma}_i^{\mathcal{O}\mathcal{I}}(\boldsymbol{\Sigma}_i^{\mathcal{I}} + \boldsymbol{\Sigma}^{\mathcal{S}})^{-1}\boldsymbol{\Sigma}_i^{\mathcal{I}\mathcal{O}}, \end{aligned}$$

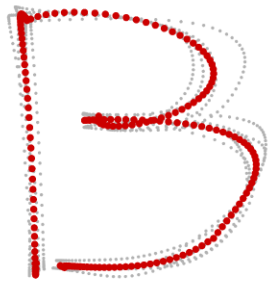
$$\text{and} \quad h_i = \frac{\pi_i \, \mathcal{N}(\boldsymbol{x}^{\mathcal{I}}|\; \boldsymbol{\mu}_i^{\mathcal{I}}, \boldsymbol{\Sigma}_i^{\mathcal{I}} + \boldsymbol{\Sigma}^{\mathcal{S}})}{\sum_k^K \pi_k \, \mathcal{N}(\boldsymbol{x}^{\mathcal{I}}|\; \boldsymbol{\mu}_k^{\mathcal{I}}, \boldsymbol{\Sigma}_k^{\mathcal{I}} + \boldsymbol{\Sigma}^{\mathcal{S}})}.$$

GMR for smooth piecewise polynomial fitting

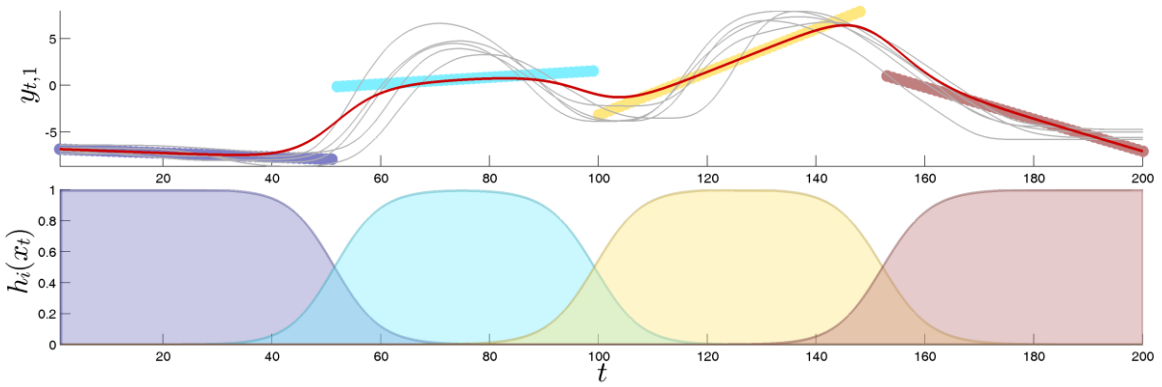


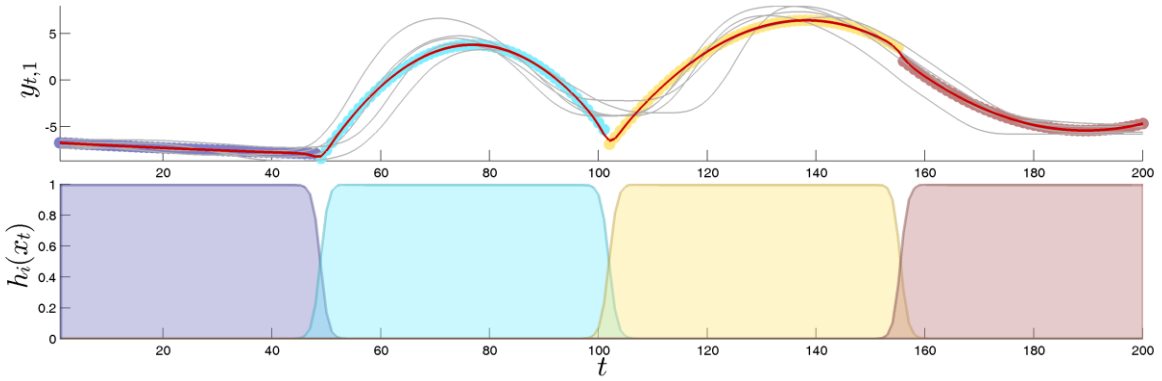
$$\boldsymbol{X} = \boldsymbol{x}$$

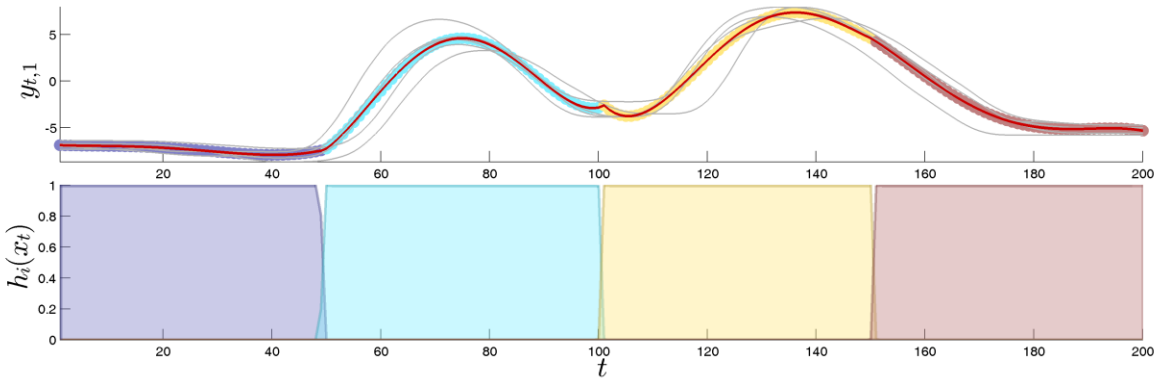
$$\boldsymbol{X} = [\boldsymbol{x}^2, \boldsymbol{x}]$$



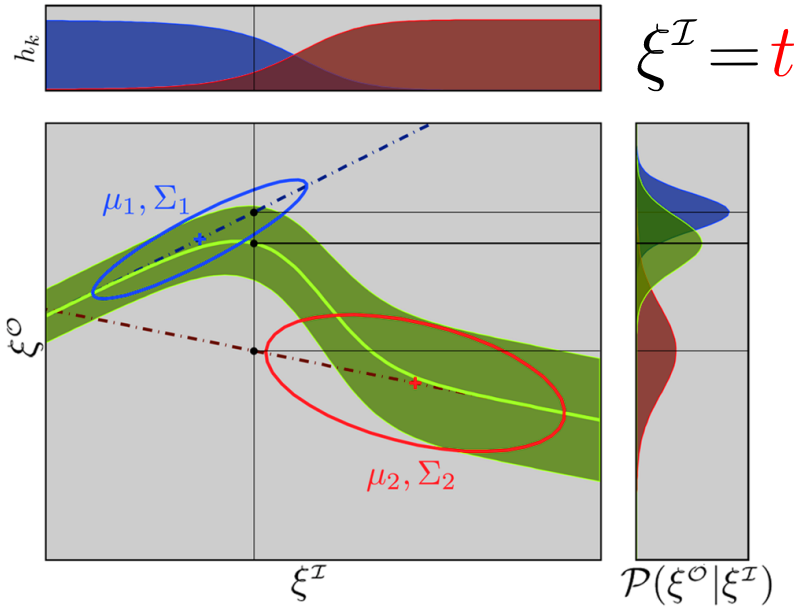
$$\boldsymbol{X} = [\boldsymbol{x}^3, \boldsymbol{x}^2, \boldsymbol{x}]$$





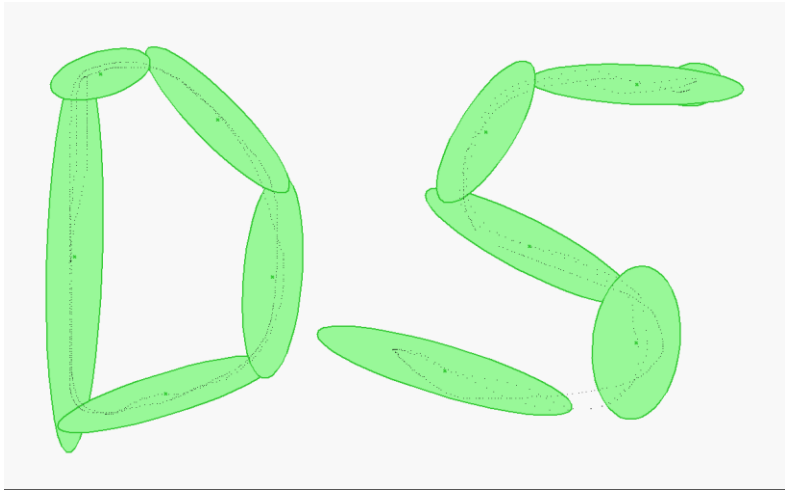


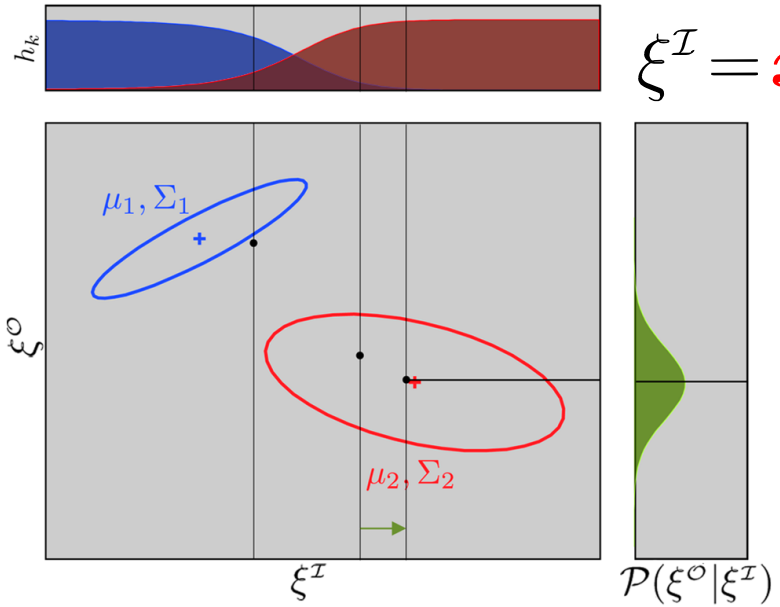
Gaussian mixture regression - Examples



$\xi^{\mathcal{I}} = t,\ \xi^{\mathcal{O}} = \boldsymbol{x}$

[Calinon, Guenter and Billard, IEEE Trans. on SMC-B 37(2), 2007]





$\xi^{\mathcal{I}} = \boldsymbol{x},\ \xi^{\mathcal{O}} = \dot{\boldsymbol{x}}$

With expectation-maximization (EM):
(maximizing log-likelihood)

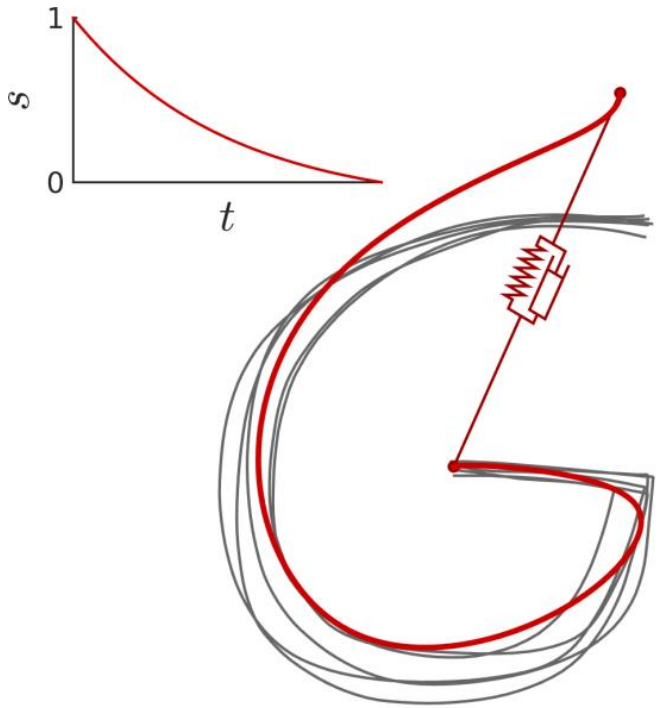
[Hersch, Guenter, Calinon and Billard, IEEE Trans. on Robotics 24(6), 2008]

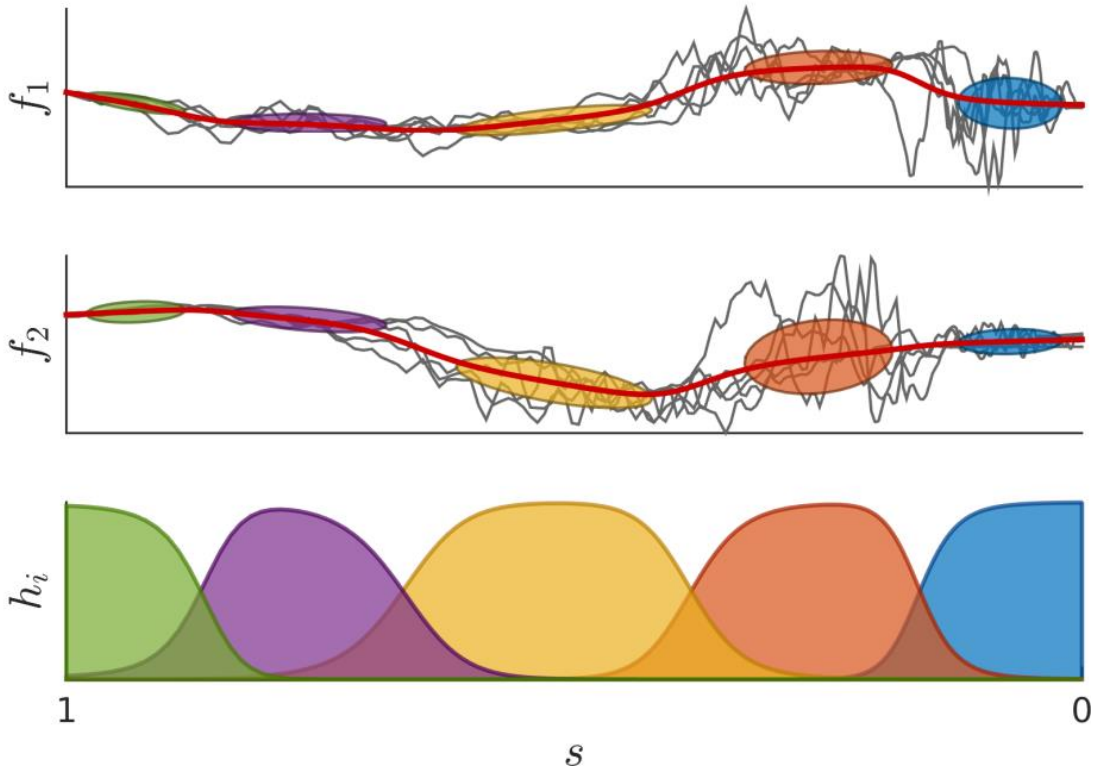
With quadratic programming solver:
(maximizing log-likelihood s.t. stability constraints)

[Khansari-Zadeh and Billard, IEEE Trans. on Robotics 27(5), 2011]

Dynamical movement primitives with GMR

Learning of $\mathcal{P}(s, \boldsymbol{x})$ and retrieval of $\mathcal{P}(\boldsymbol{x}|s)$





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