Linear Algebra

CS425: Computer Graphics I

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UIC COMPUTER SCIENCE

Overview

- Basic linear algebra concepts
- Coordinate systems
- Coordinate frame

Euclidian space

- A n-dimensional real Euclidian space is denoted $\mathbb{R}^n$.
- A vector $\mathbf{v}$ in this space is an n-tuple (an ordered list of real numbers).

$$\mathbf{v} \in \mathbb{R}^n \Leftrightarrow \mathbf{v} = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}, \text{with } v_i \in \mathbb{R}, i = 0, \dots, n-1$$

- $v_0, \dots, v_{n-1}$: elements, coefficients, or components of vector **v**.

Operations

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{pmatrix} + \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} u_0 + v_0 \\ u_1 + v_1 \\ \vdots \\ u_{n-1} + v_{n-1} \end{pmatrix} \in \mathbb{R}$$

Addition

$$\alpha \mathbf{u} = \begin{pmatrix} \alpha u_0 \\ \alpha u_1 \\ \vdots \\ \alpha u_{n-1} \end{pmatrix} \in \mathbb{R}$$

Multiplication by scalar

Properties

i. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ (associativity)
ii. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ (commutativity)
iii. $\mathbf{0}+\mathbf{v}=\mathbf{v}$ (zero identity)
iv. $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$ (additive inverse)
v. $(ab)\mathbf{u}=a(b\mathbf{u})$
vi. $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$ (distributive law)
vii. $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$ (distributive law)
viii. $1\mathbf{u}=\mathbf{u}$

Dot product

$\mathbf{u} \cdot \mathbf{v} = \sum_{i=0}^{n-1} u_i \, v_i$ (dot product)

Properties:

i. $\mathbf{u} \cdot \mathbf{u} \ \geq 0$, with $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = (0,0, \dots, 0) = \mathbf{0}$

ii. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

iii. $(\alpha \mathbf{u}) \cdot \mathbf{v} = \alpha(\mathbf{u} \cdot \mathbf{v})$

iv. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (commutative)

v. $\mathbf{u} \cdot \mathbf{v} = 0 \Leftrightarrow \mathbf{u} \perp \mathbf{v}$

Norm of a vector

$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{\sum_{i=0}^{n-1} u_i^2}$ (norm)

Properties:

i. $\|\mathbf{u}\| = 0 \Leftrightarrow \mathbf{u} = (0,0,\dots,0) = \mathbf{0}$

ii. $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$

iii. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (triangle inequality)

iv. $\|\mathbf{u} \cdot \mathbf{v}\| \leq \|\mathbf{u}\|\|\mathbf{v}\|$ (Cauchy-Schawrtz inequality)

Geometrical interpretation

- A set of vectors $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_{n-1}}$ is *linearly independent* if:

$\alpha_0 \mathbf{v}_0 + \alpha_1 \mathbf{v}_1 + \cdots + \alpha_{n-1} \mathbf{v}_{n-1} = 0 \text{ if and only if } \alpha_0 = \alpha_1 = \cdots = \alpha_{n-1} = 0$

- If a set of vectors is linearly independent, we cannot represent one in terms of the other.
- If a set of vectors is linearly dependent, at least one can be written in terms of the other.

Geometrical interpretation

- Example: $\mathbf{v_0} = (4,3)$ and $\mathbf{v_1} = (8,6)$ are not linearly independent, since $\alpha_0 = 2$ and $\alpha_1 = -1$ satisfy.
- Example: $\mathbf{v_0} = (4,3)$ and $\mathbf{v_1} = (2,6)$ are linearly independent, since the only scalars to satisfy previous equation are $\alpha_0 = 0$ and $\alpha_1 = 0$.

Geometrical interpretation

If a set of vectors $\mathbf{v_0}, \mathbf{v_1}, \dots, \mathbf{v_{n-1}} \in \mathbb{R}^n$ is linearly independent and any vector $\mathbf{u} \in \mathbb{R}^n$ can be written as a combination of:

$$\mathbf{u} = \sum_{i=0}^{n-1} u_i \mathbf{v}_i$$

Then we can say that the vectors $\mathbf{v}_0, \dots, \mathbf{v}_{n-1}$ span the Euclidian space $\mathbb{R}^n$.

Geometrical interpretation

- In addition, if $u_0, \ldots, u_{n-1}$ are uniquely determined by $\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^n$, then $\mathbf{v}_0, \ldots, \mathbf{v}_{n-1}$ is called a basis of $\mathbb{R}^n$.
- What this means: every vector can be described **uniquely** by *n* scalars $(u_0, \ldots, u_{n-1})$, and the basis vectors $\mathbf{v}_0, \ldots, \mathbf{v}_{n-1}$.

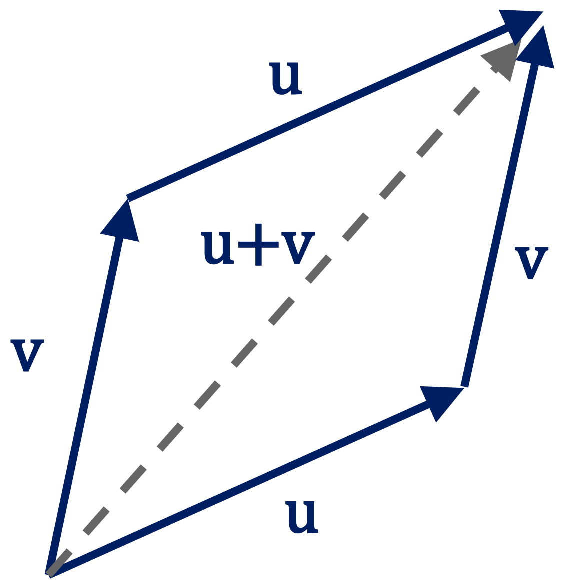
- $(1,0), (0,1)$
  - Linearly independent? Yes.
  - Spans $\mathbb{R}^2$? Yes.
  - Basis for $\mathbb{R}^2$? Yes.
- $(1,0), (1,1), (0,2)$
  - Linearly independent? No.
  - Spans $\mathbb{R}^2$? Yes.
  - Basis for $\mathbb{R}^2$? No.
- $(1,1,0), (0,1,1)$
  - Linearly independent? Yes.
  - Spans $\mathbb{R}^3$? No.
  - Basis for $\mathbb{R}^3$? No.
- $(1,1), (1,-1)$
  - Linearly independent? Yes.
  - Spans $\mathbb{R}^2$? Yes.
  - Basis for $\mathbb{R}^2$? Yes.

Coordinate systems

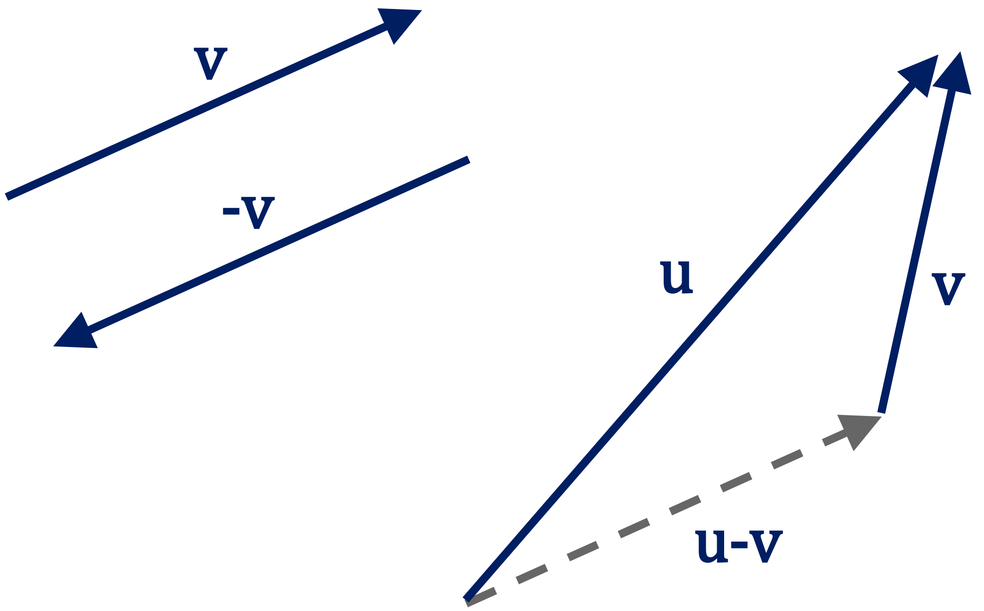
- Consider a basis $\mathbf{v}_0, \ldots, \mathbf{v}_{n-1}$
- A vector can be written as $\mathbf{v} = \alpha_0\mathbf{v}_0 + \alpha_1\mathbf{v}_1 + \cdots + \alpha_{n-1}\mathbf{v}_{n-1}$
- List of scalars is the representation of $\mathbf{v}$ with respect to given basis.
- A vector describes a direction and a length (norm).
- We can write this as:

$$\mathbf{v} = (\alpha_0 \quad \alpha_1 \quad \ldots \quad \alpha_{n-1})^T = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \ldots \\ \alpha_{n-1} \end{pmatrix}$$

Operations

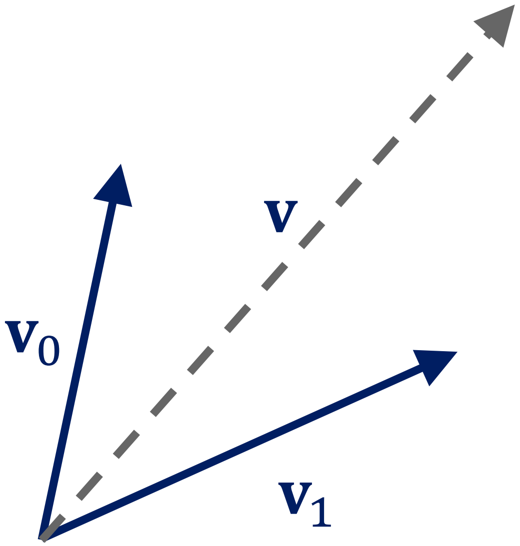


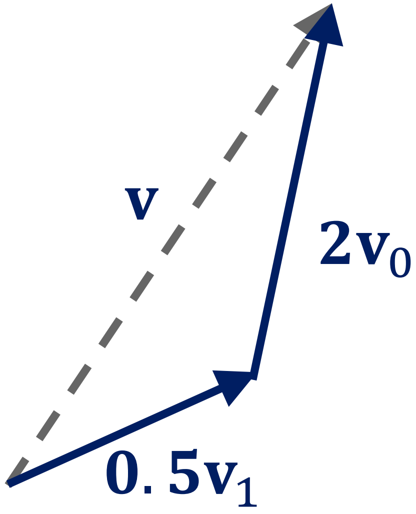
$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity)

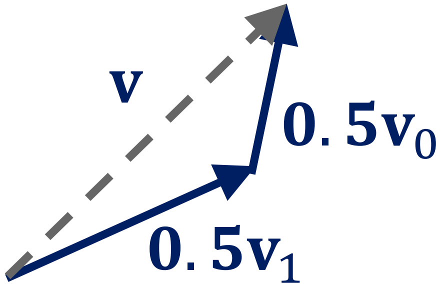


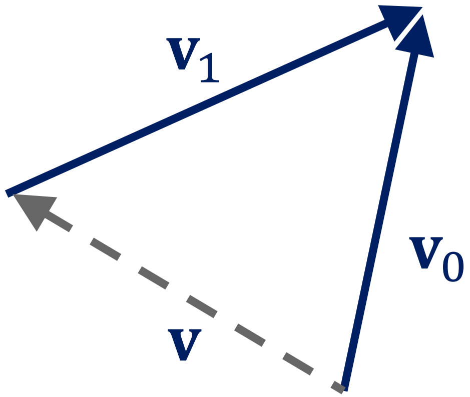
$\mathbf{u} - \mathbf{v} = -\mathbf{v} + \mathbf{u}$

Coordinates









$$\mathbf{v} = \alpha_0\mathbf{v}_0 + \alpha_1\mathbf{v}_1$$

$\mathbf{v}_0$ and $\mathbf{v}_1$ form a 2D basis

$$\mathbf{v} = 2\mathbf{v}_0 + \mathbf{0.5v}_1$$

$$\mathbf{v} = 0.5\mathbf{v}_0 + \mathbf{0.5v}_1$$

$$\mathbf{v} = \mathbf{v}_0 - \mathbf{v}_1$$

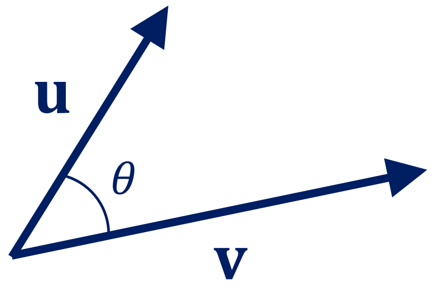
Length

- Length of a vector is denoted as $||\mathbf{v}||$
- A vector can be normalized, to change its length to 1, without affecting its direction: $\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}$

Dot product

- The dot product is related to the length of two vectors and the angle between them.

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| cos\theta$$

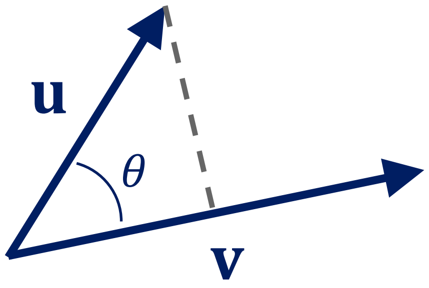


- If both are normalized, it is directly the cosine of the angle between them.

Projection

- The length of the projection of $\mathbf{u}$ onto $\mathbf{v}$ can be computed using the dot product

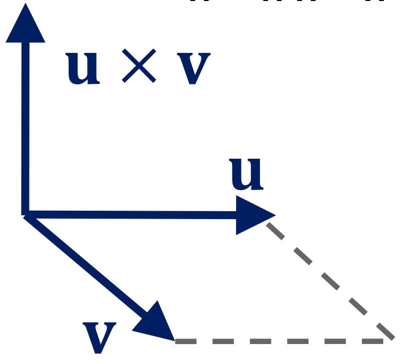
$$\mathbf{u} \rightarrow \mathbf{v} = \|\mathbf{u}\| cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|}$$



Cross product

- Defined only for 3D vectors.
- The resulting vector is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, the direction depends on the right hand rule.
- The magnitude is equal to the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = ||\mathbf{u}||||\mathbf{v}||sin\theta$$



Properties

i. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ (anti-commutativity)

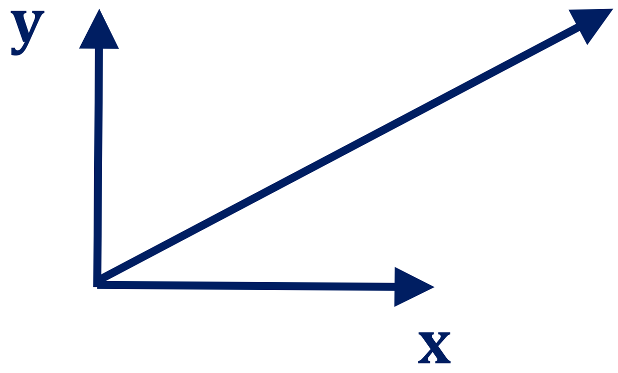
ii. $(a\mathbf{u} + b\mathbf{v}) \times \mathbf{w} = a(\mathbf{u} \times \mathbf{w}) + b(\mathbf{v} \times \mathbf{w})$ (linearity)

iii. $\mathbf{0} + \mathbf{v} = \mathbf{v}$ (zero identity)

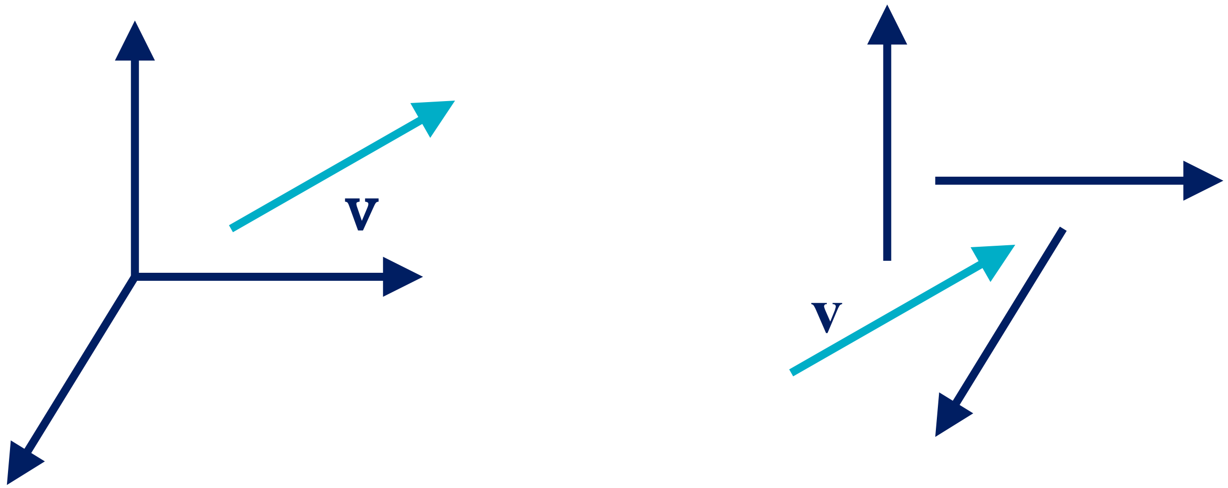
iv. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ (vector triple product)

Cartesian coordinates

- $\mathbf{x} = (1,0)$ and $\mathbf{y} = (0,1)$ form a canonical, Cartesian basis.



Coordinate systems



Which one is correct? Both, vectors don’t have fixed location.

Coordinate systems

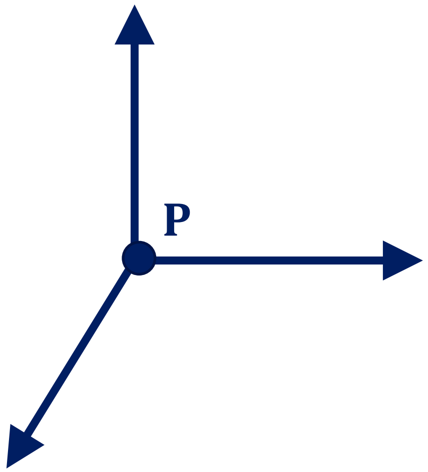
- We will always use orthonormal bases, which are formed by pairwise orthogonal unit vectors:

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases}$$

- This means that every basis vector must have a length of one, and also that each pair of basis vectors must be orthogonal.

Coordinate frames

- Note that a coordinate system is insufficient to represent points.
- We can add an origin to the basis vectors to form a frame.



Coordinate frames

- Frame determined by $(\mathbf{P}_0, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$.
- Within this frame, every vector can be written as:
$$\mathbf{v} = \alpha_0 \mathbf{v}_0 + \alpha_1 \mathbf{v}_1 + \cdots + \alpha_{n-1} \mathbf{v}_{n-1}$$
- Within this frame, every point can be written as:
$$\mathbf{P} = \mathbf{P}_0 + \beta_0 \mathbf{v}_0 + \beta_1 \mathbf{v}_1 + \cdots + \beta_{n-1} \mathbf{v}_{n-1}$$

Points and vectors

- Consider the point and the vector:

$$\mathbf{v} = \alpha_0 \mathbf{v}_0 + \alpha_1 \mathbf{v}_1 + \cdots + \alpha_{n-1} \mathbf{v}_{n-1}$$
$$\mathbf{P} = \mathbf{P}_0 + \beta_0 \mathbf{v}_0 + \beta_1 \mathbf{v}_1 + \cdots + \beta_{n-1} \mathbf{v}_{n-1}$$

- Similar representations:

$$\mathbf{v} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{pmatrix}$$

- But: a vector has no position.

References

- Real-time Rendering, 3rd Ed. by Tomas Akenine-Möller, Eric Haines, and Naty Hoffman (Appendix A)
- Interactive Computer Graphics 7th Ed. by Ed Angel and Dave Shreiner (Chapter 3)