Viewing Transformations

CS425: Computer Graphics I

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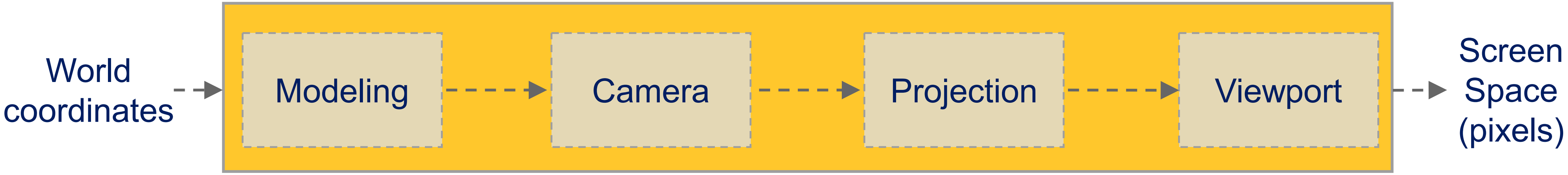
Overview

- Coordinate systems
- Camera
- Viewing transformations
- Orthographic and perspective projections
- Hidden surface removal

Viewing transformations

- Viewing transformation is the mapping of coordinates of points and lines from world coordinates into screen space pixels.



Viewing transformations

- Previously, we saw how transformations can manipulate primitives (points, vectors) in space.
- Today, we will see how transformations can manipulate primitives to 2D screen coordinates (that will be later rasterized).

Perspective projection

Distant objects appear smaller

Parallel lines converge at the horizon

Early paintings

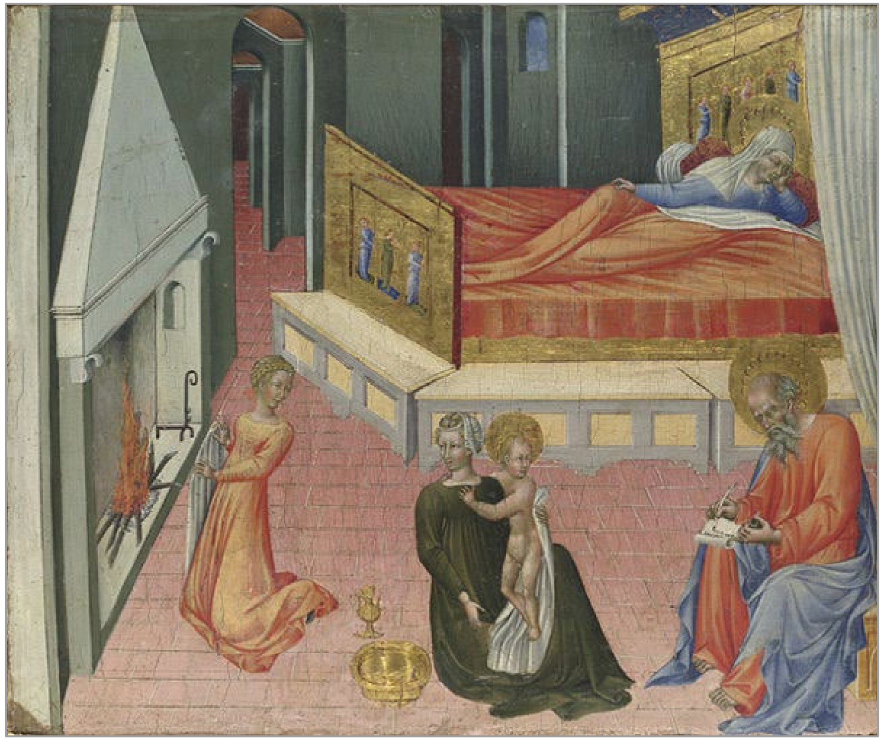
Lorsch Gospels (8th century)

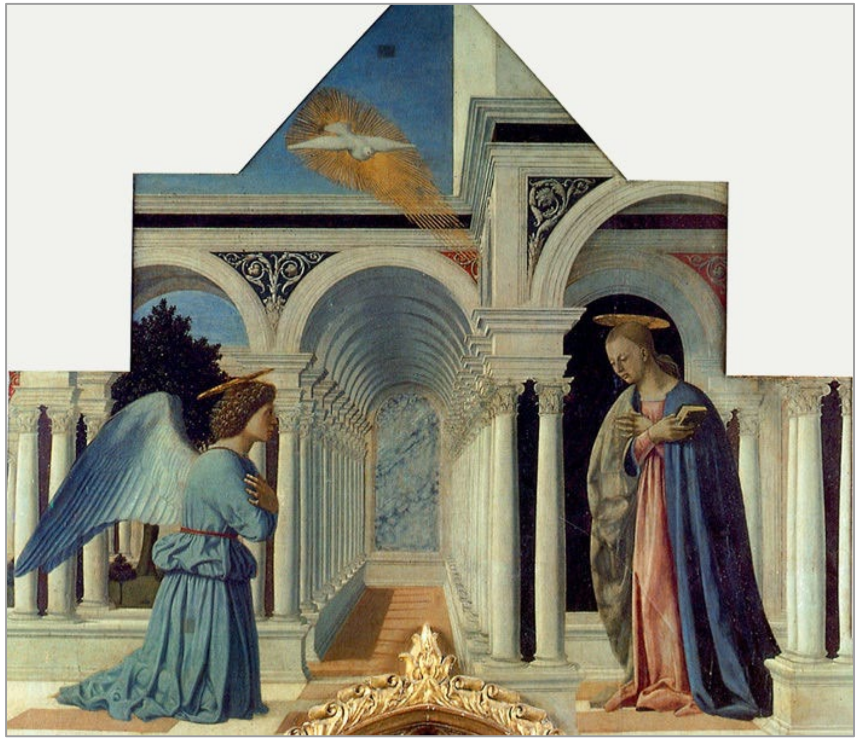
Perspective in art

Giotto di Bondone (14th century)

Giotto di Bondone (14th century)

Birth of perspective in art: One-point perspective


Giovanni di Paolo (15th century)


Piero della Francesca (15th century)

Birth of perspective in art: One-point perspective

Perugino (15th century)

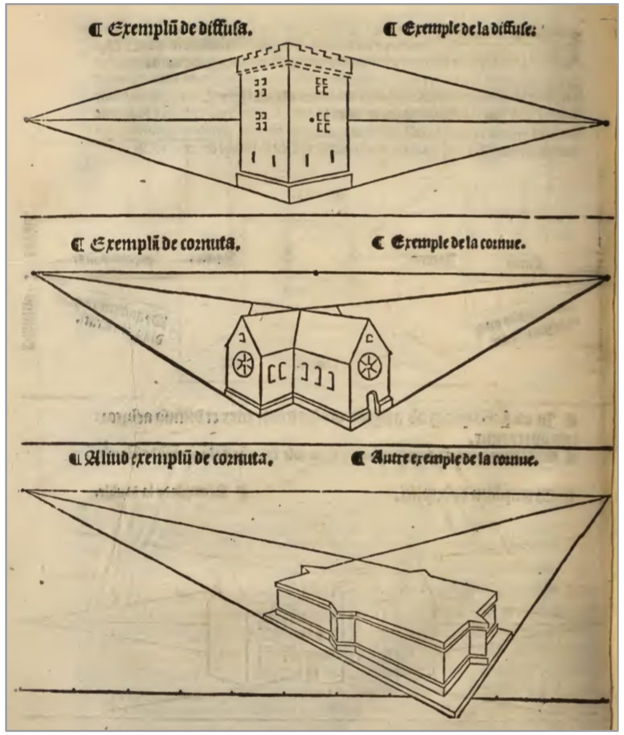
Birth of perspective in art: One-point perspective

The Ideal City (15th century)

Birth of perspective in art: One-point perspective

Rafael (16th century)

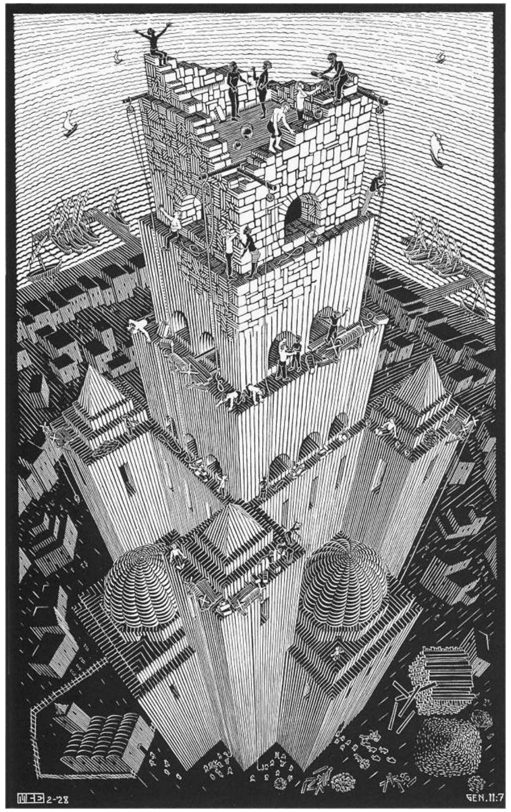
Birth of perspective in art: Two-point perspective



Jean Pélérin (16th century)

Gustave Caillebotte (19th century)

Birth of perspective in art: Three-point perspective

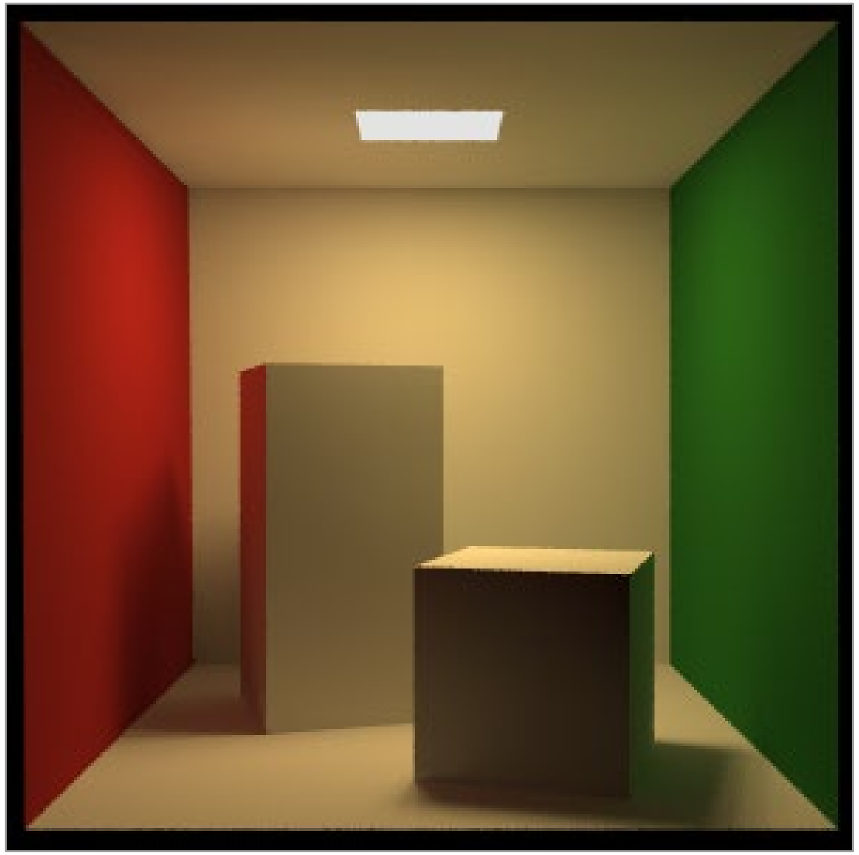


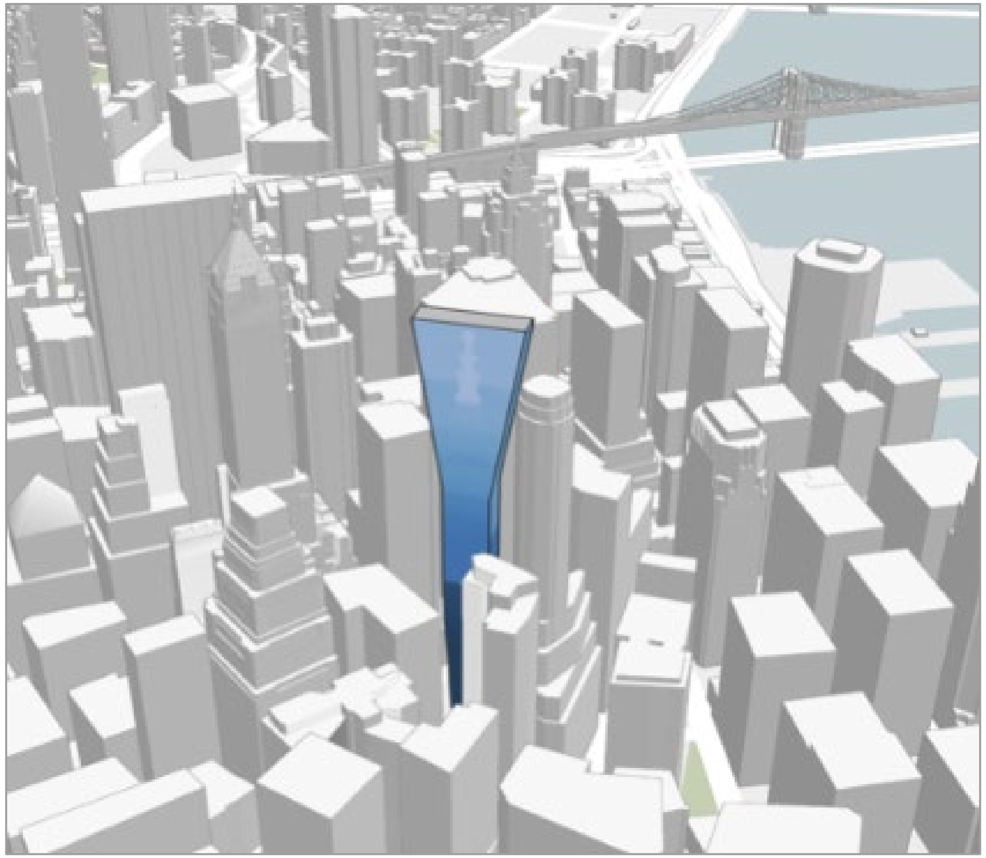
M.C. Escher (1928)

Multi perspective

The Frozen City by Matthias A. K. Zimmermann (2006)

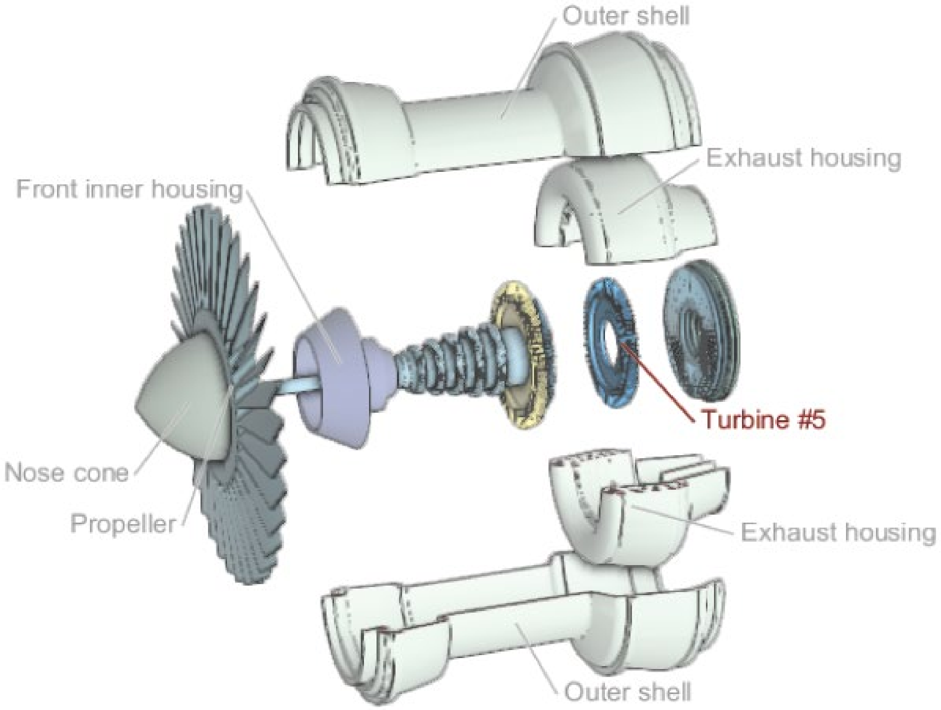
Perspective in computer graphics





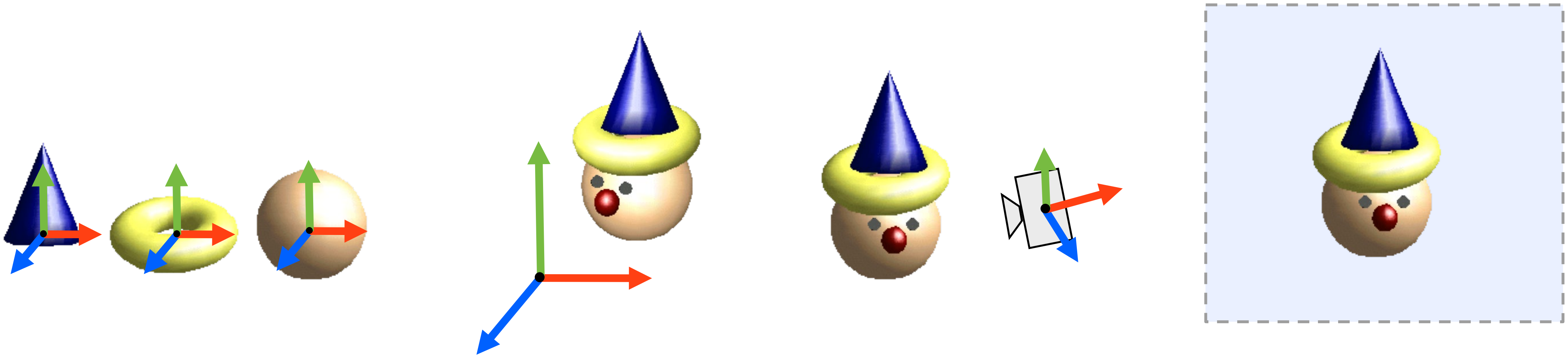
Rejection of perspective in CG

SimCity 2000



“Automated Generation of Interactive 3D Exploded Views”

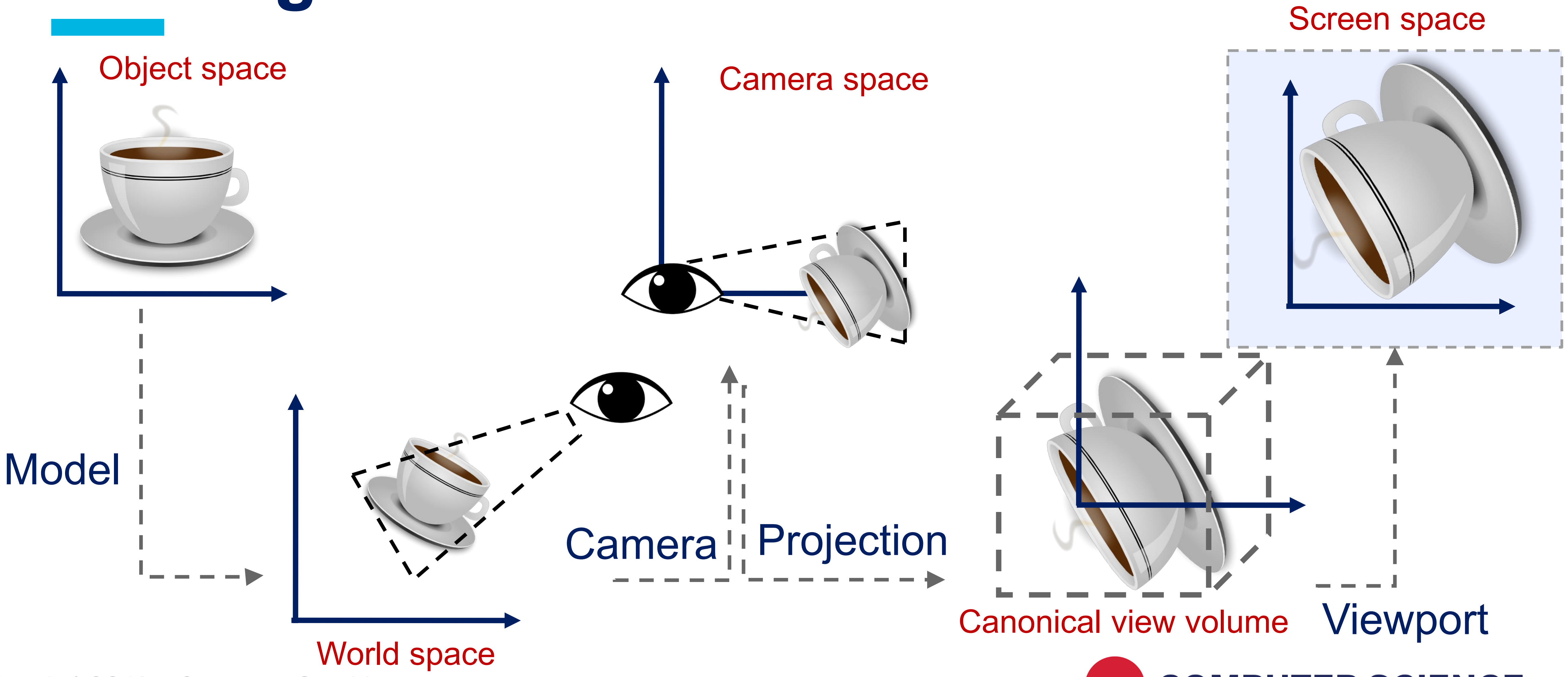
Coordinate spaces



Object coordinates World coordinates Camera coordinates Screen coordinates

From: Mark Pauly

Viewing transformation



Camera transformation

- Construct the camera reference system:
  - The eye position $\mathbf{e}$.
  - The forward direction $\mathbf{d}$.
  - The view-up vector $\mathbf{u}$.
- A view matrix transform all coordinates into view coordinates.

Change of frame

- Given a vector $\mathbf{v} = \mathrm{x}\mathbf{i} + \mathrm{y}\mathbf{j}$ in the standard frame, to take a vector $\mathbf{v}' = \mathrm{x}'\mathbf{b_1} + \mathrm{y}'\mathbf{b_2}$ to the standard, we solve the linear system:
$$\mathrm{x}\mathbf{i} + \mathrm{y}\mathbf{j} = \mathrm{x}'\mathbf{b_1} + \mathrm{y}'\mathbf{b_2}$$

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$$\mathrm{x}\mathbf{i} + \mathrm{y}\mathbf{j} = \mathrm{x}'\mathbf{b_1} + \mathrm{y}'\mathbf{b_2}$$

- In matrix form:

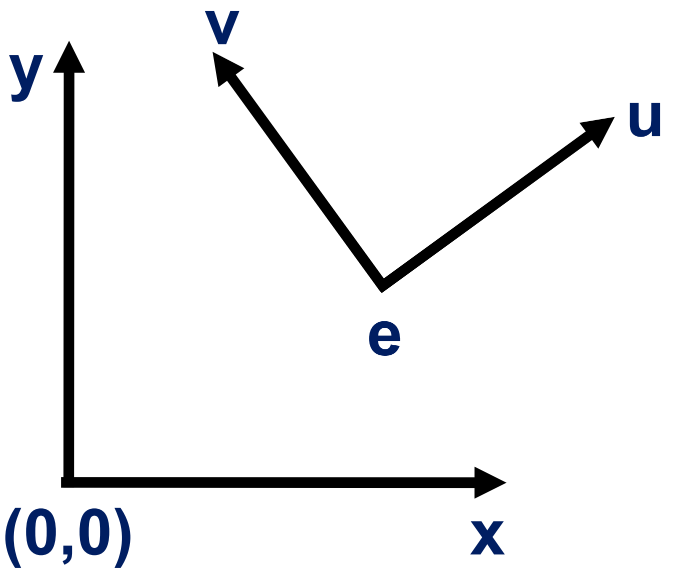
$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mathbf{b_1} & \mathbf{b_2} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

- Other way around, find inverse of the basis matrix

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$$

Change of frame



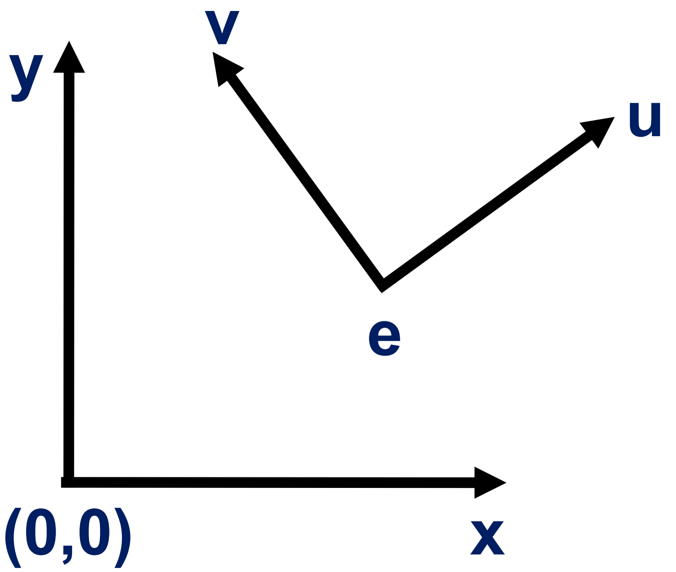
Rotation: $\mathbf{R} = \begin{pmatrix} \mathbf{u} & \mathbf{v} & \boldsymbol{w} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Translate: $\mathbf{T} = \begin{pmatrix} 1 & 0 & e_x \\ 0 & 1 & e_y \\ 0 & 0 & 1 \end{pmatrix}$

$\mathbf{M}_{e \to o} = \mathbf{TR}$

$\mathbf{M}_{o \to e} = (\mathbf{TR})^{-1} = \mathbf{R}^{-1}\mathbf{T}^{-1}$

Change of frame



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Translate: $\mathbf{T} = \begin{pmatrix} 1 & 0 & \mathrm{e_x} \\ 0 & 1 & \mathrm{e_y} \\ 0 & 0 & 1 \end{pmatrix}$

$\mathbf{M}_{e\to o} = \mathbf{TR}$

$\mathbf{M}_{o\to e} = (\mathbf{TR})^{-1} = \mathbf{R}^{-1}\mathbf{T}^{-1}$

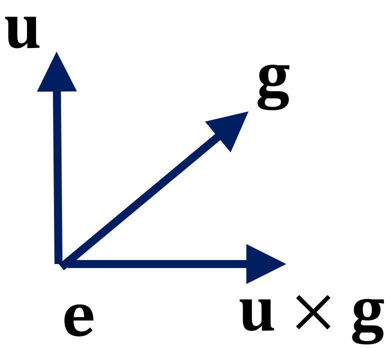
We know:

$\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$ (Pure rotation is orthogonal)

$\mathbf{T}^{-1} = \begin{pmatrix} 1 & 0 & -\mathrm{e_x} \\ 0 & 1 & -\mathrm{e_y} \\ 0 & 0 & 1 \end{pmatrix}$

Camera transformation

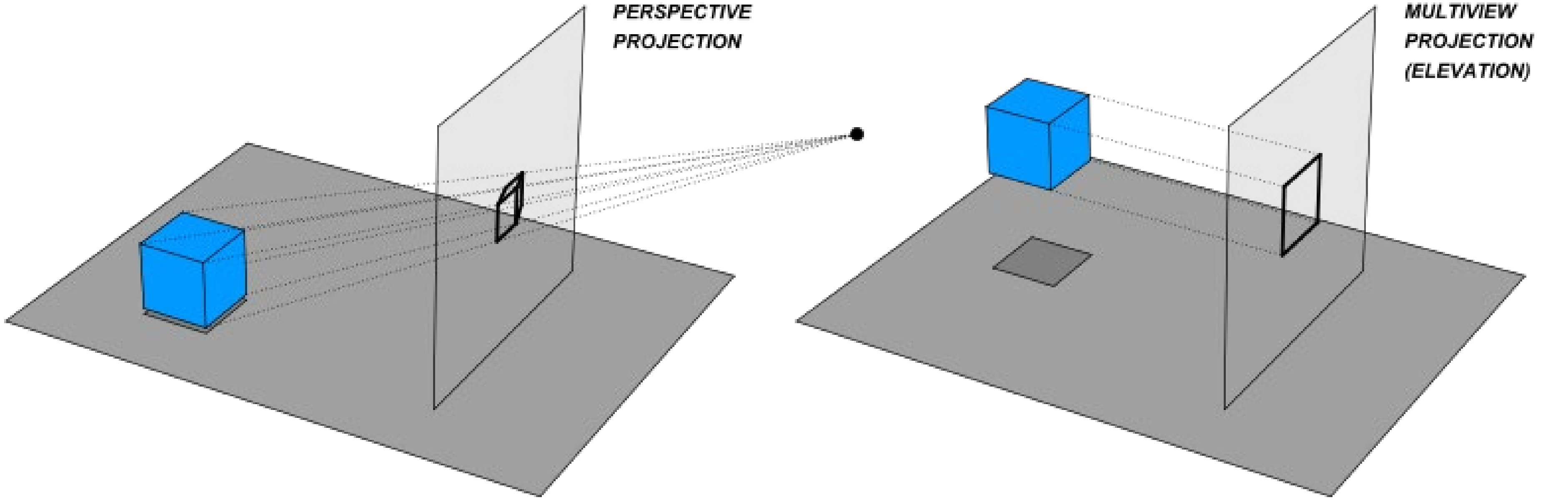
- Construct the camera reference system:
  - The eye position $\mathbf{e}$.
  - The forward direction $\mathbf{d}$.
  - The view-up vector $\mathbf{u}$.
- A view matrix transform all coordinates into view coordinates.



$$\mathbf{M}_{camera2world} = \begin{pmatrix} \mathbf{u}\times\mathbf{g} & \mathbf{u} & \mathbf{d} & \mathbf{e} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

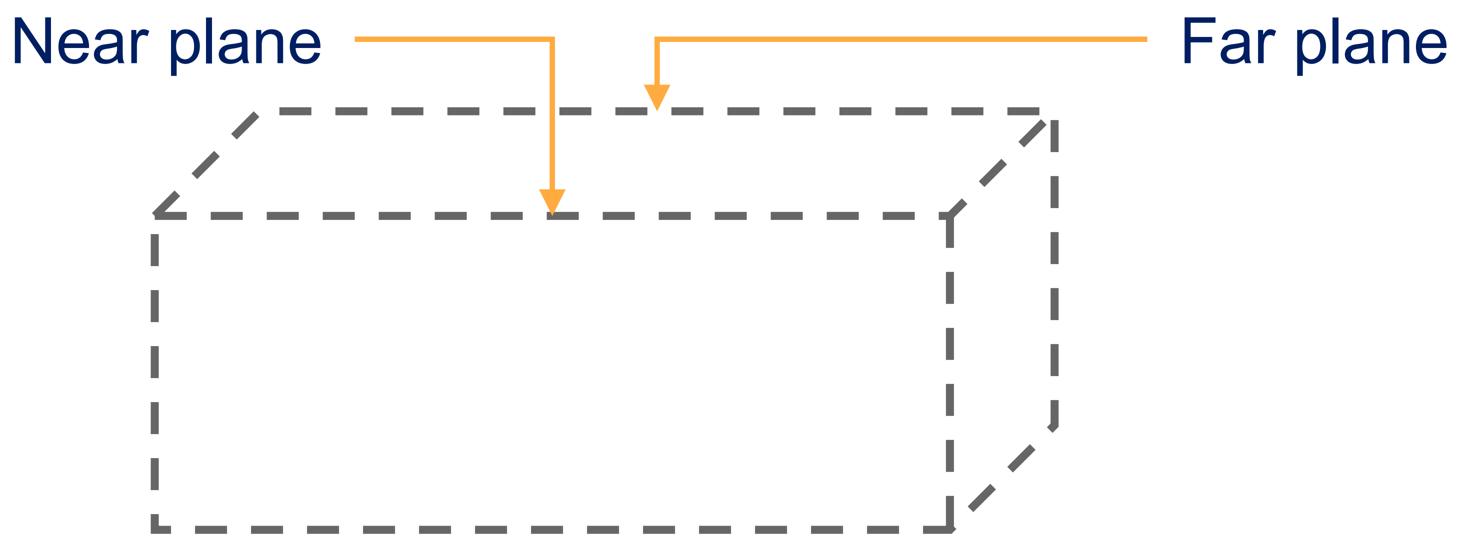
$$\mathbf{M}_{world2camera} = \begin{pmatrix} \mathbf{u}\times\mathbf{g} & \mathbf{u} & \mathbf{d} & \mathbf{e} \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$

Orthographic and perspective projections



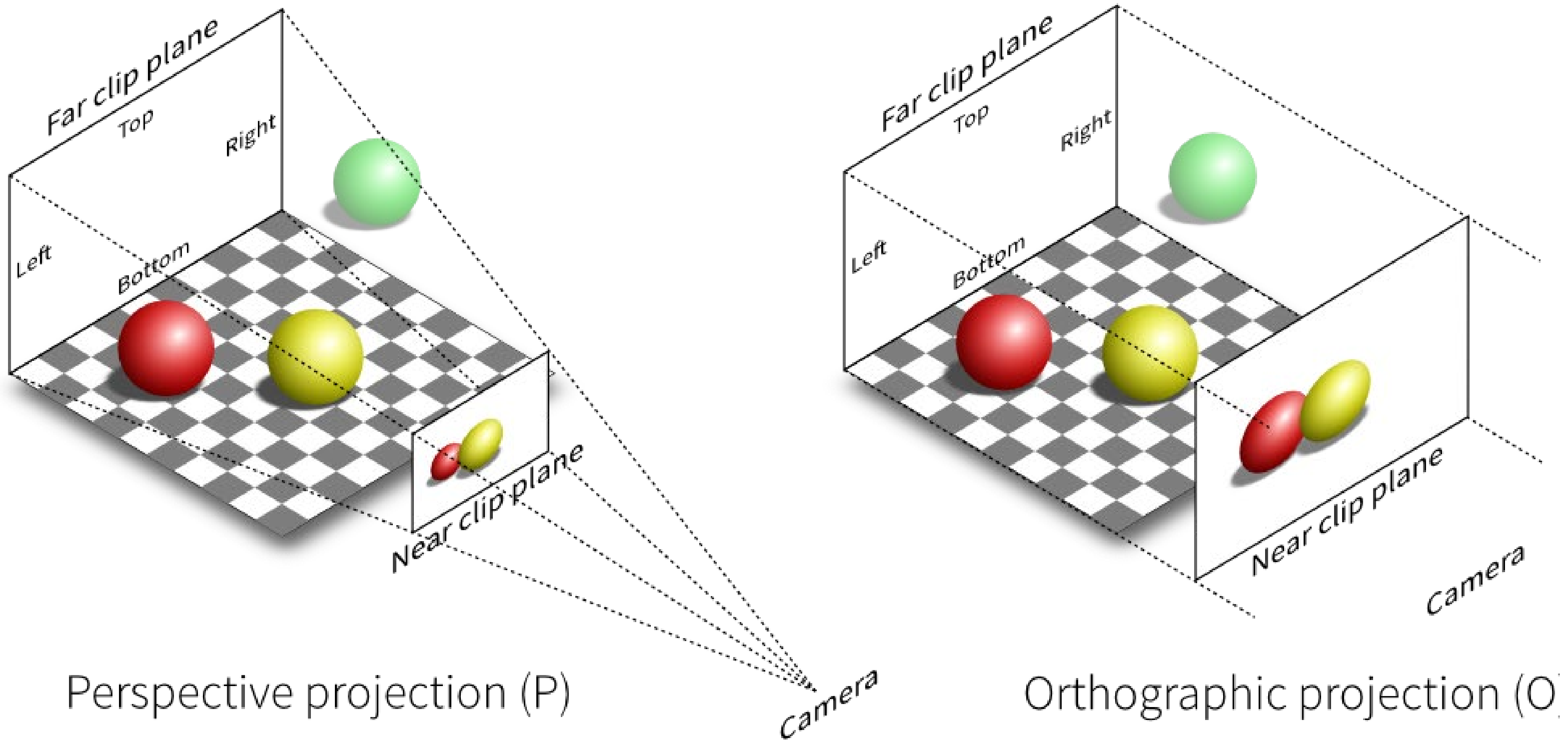
https://en.wikipedia.org/wiki/File:Various_projections_of_cube_above_plane.svg

View frustum

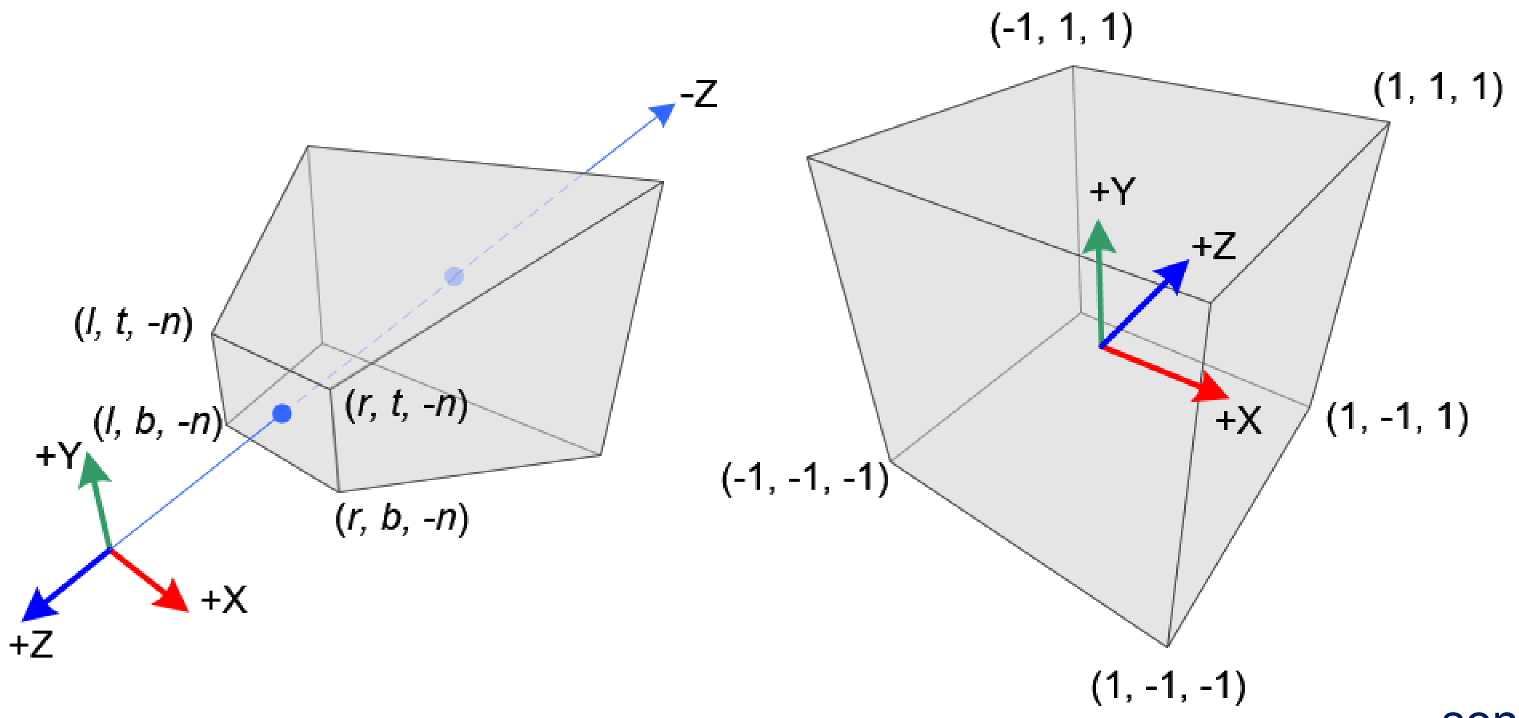


Camera space
$(x_{left}, y_{botton}, z_{near}) \times (x_{right}, y_{top}, z_{far})$

View frustum

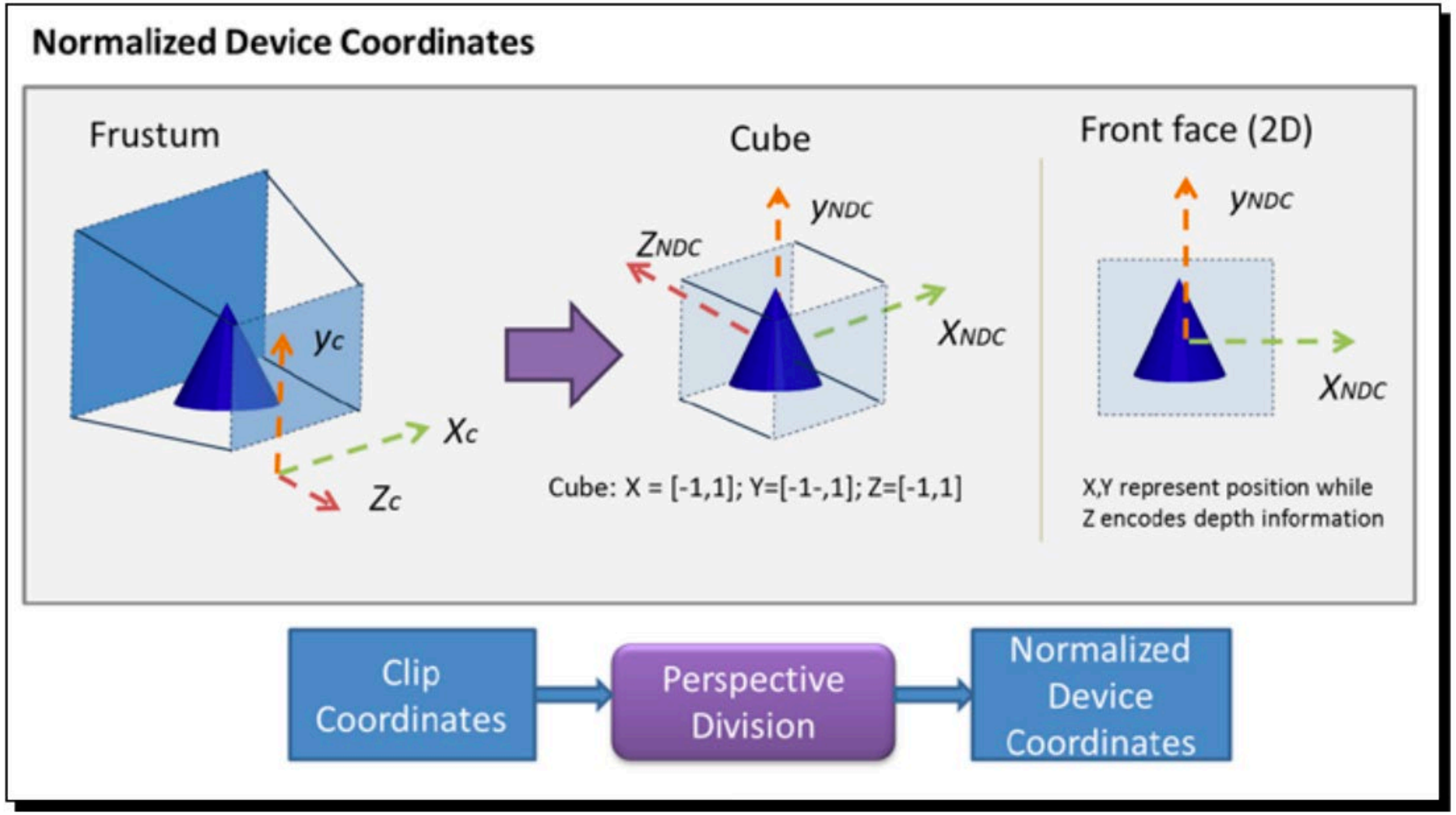


View frustum

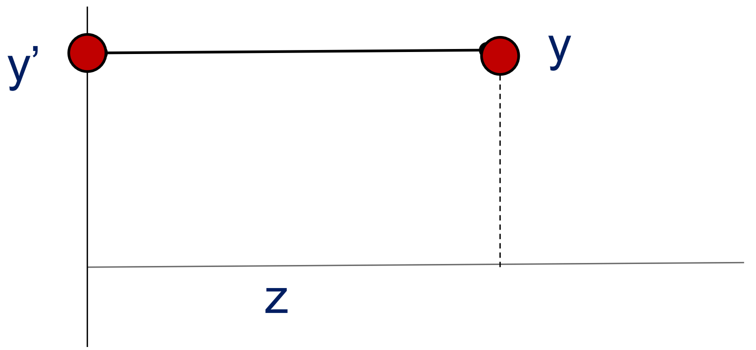


songho.ca

View frustum



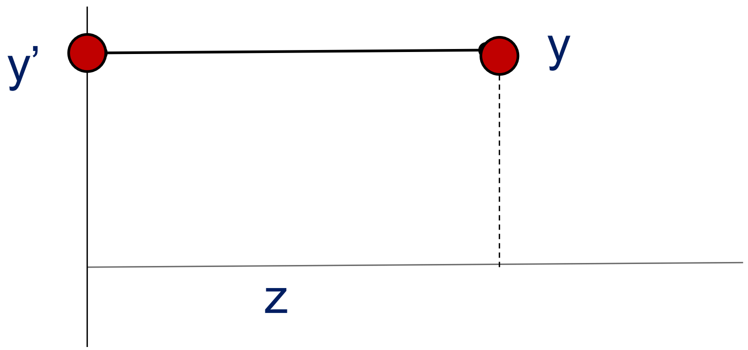
Basic orthographic projection



Projection plane
(z = 0)

$$\mathbf{x}' = \mathbf{x}, \mathbf{y}' = \mathbf{y}$$

Basic orthographic projection

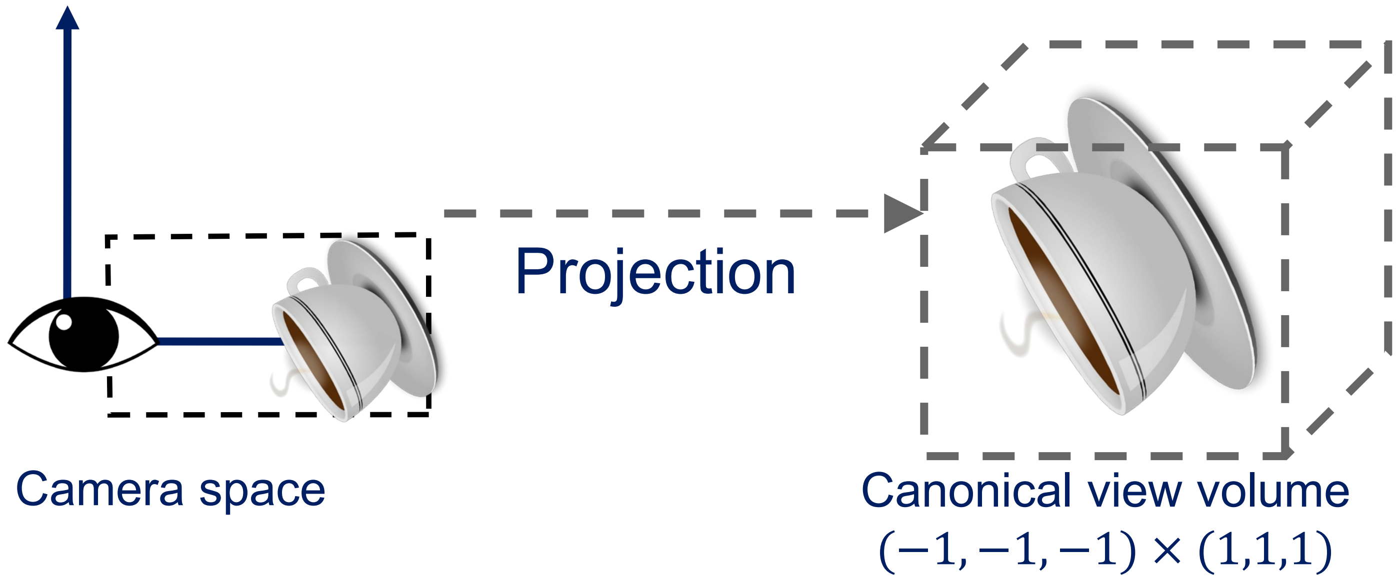


Projection plane
(z = 0)

$$\mathbf{x}' = \mathbf{x}, \mathbf{y}' = \mathbf{y}$$

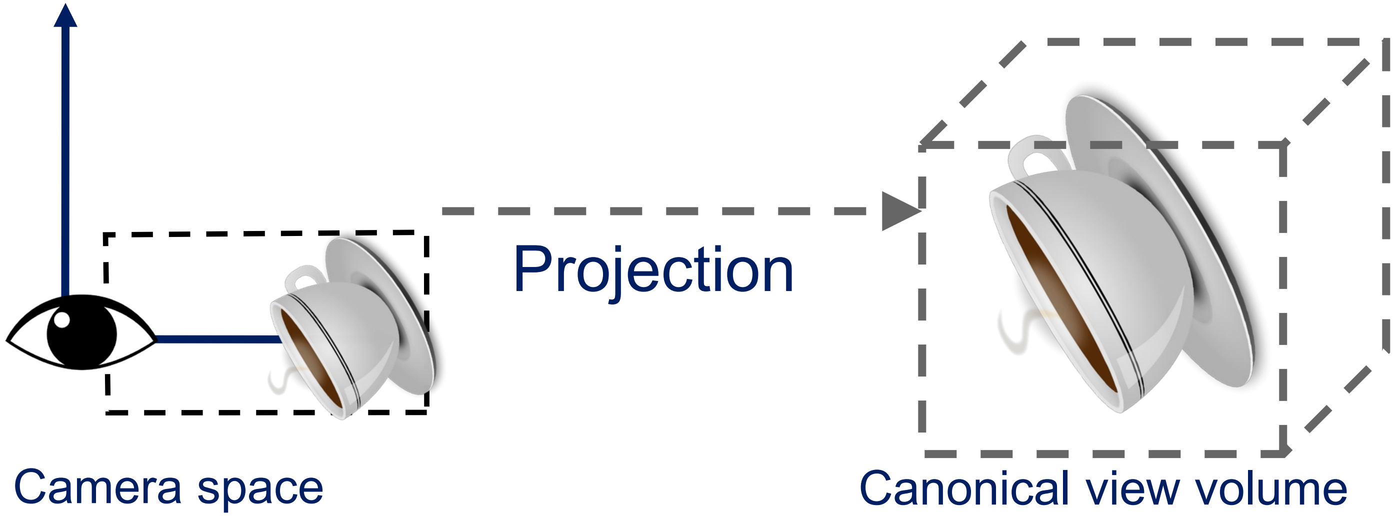
$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Orthographic transformation



1. Translate to the origin.
2. Scale the volume to a 2-by-2 unit square: (-1,-1) to (+1, +1).
3. Switch coordinate system.

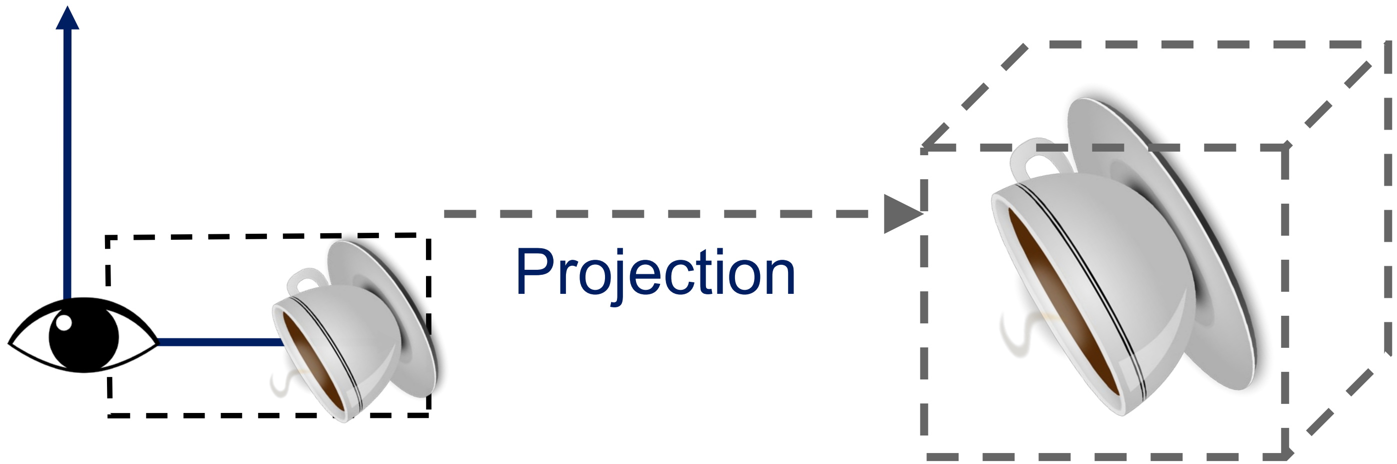
Orthographic transformation



$(-1,-1,-1) \times (1,1,1)$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -x_{mid} \\ 0 & 1 & 0 & -y_{mid} \\ 0 & 0 & 1 & -z_{mid} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Orthographic transformation

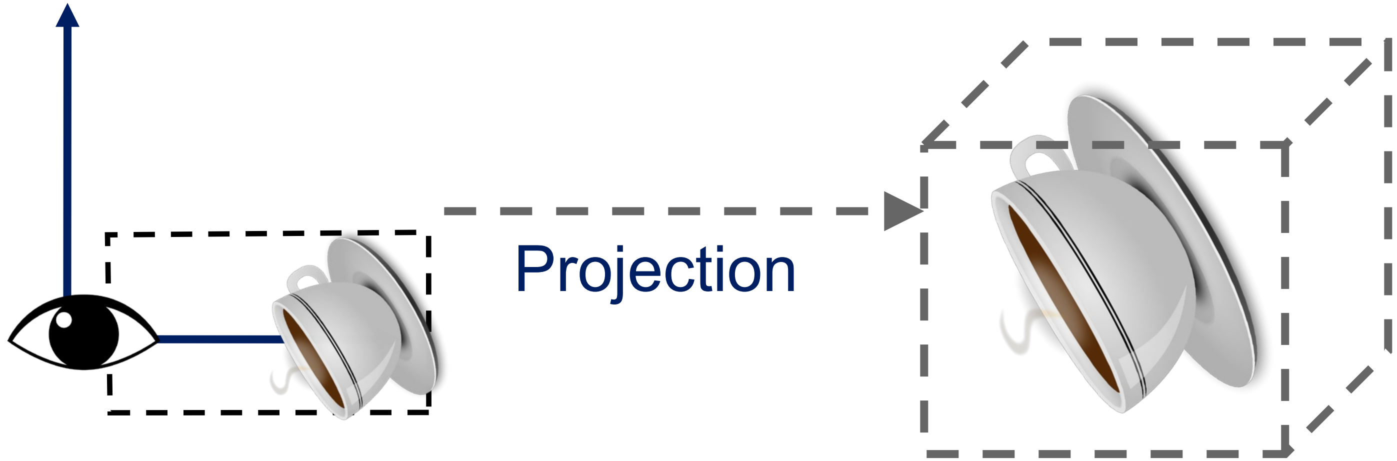


Camera space

Canonical view volume
$(-1,-1,-1) \times (1,1,1)$

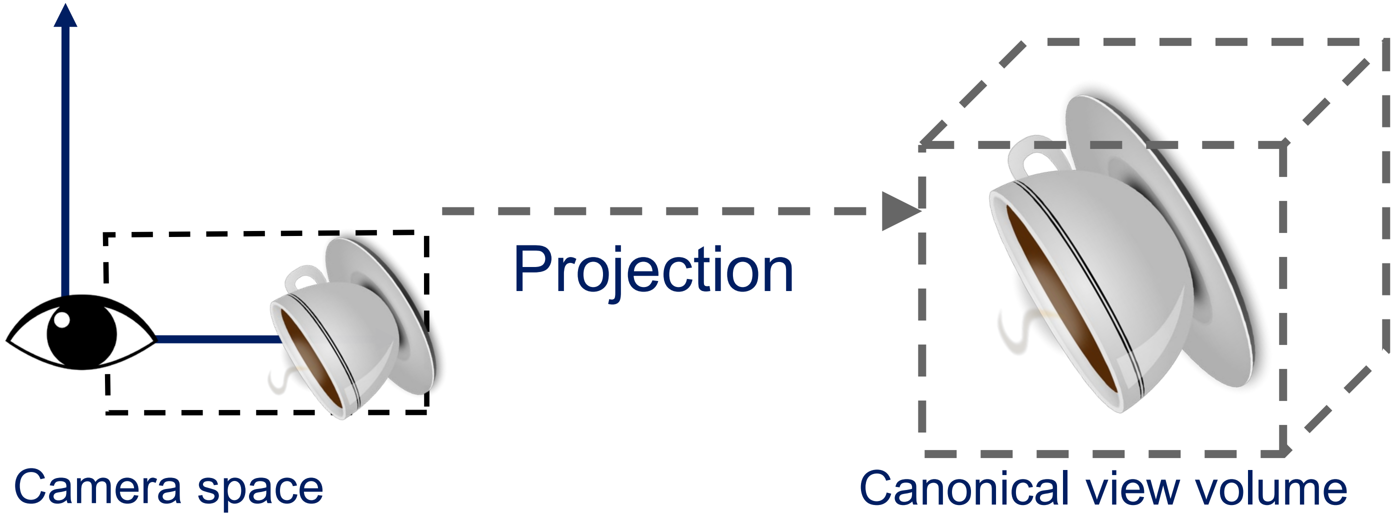
$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{(right-left)} & 0 & 0 & 0 \\ 0 & \frac{2}{(top-bottom)} & 0 & 0 \\ 0 & 0 & \frac{2}{(far-near)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -x_{mid} \\ 0 & 1 & 0 & -y_{mid} \\ 0 & 0 & 1 & -z_{mid} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Orthographic transformation



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{(right-left)} & 0 & 0 & 0 \\ 0 & \frac{2}{(top-bottom)} & 0 & 0 \\ 0 & 0 & \frac{2}{(far-near)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -x_{mid} \\ 0 & 1 & 0 & -y_{mid} \\ 0 & 0 & 1 & -z_{mid} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

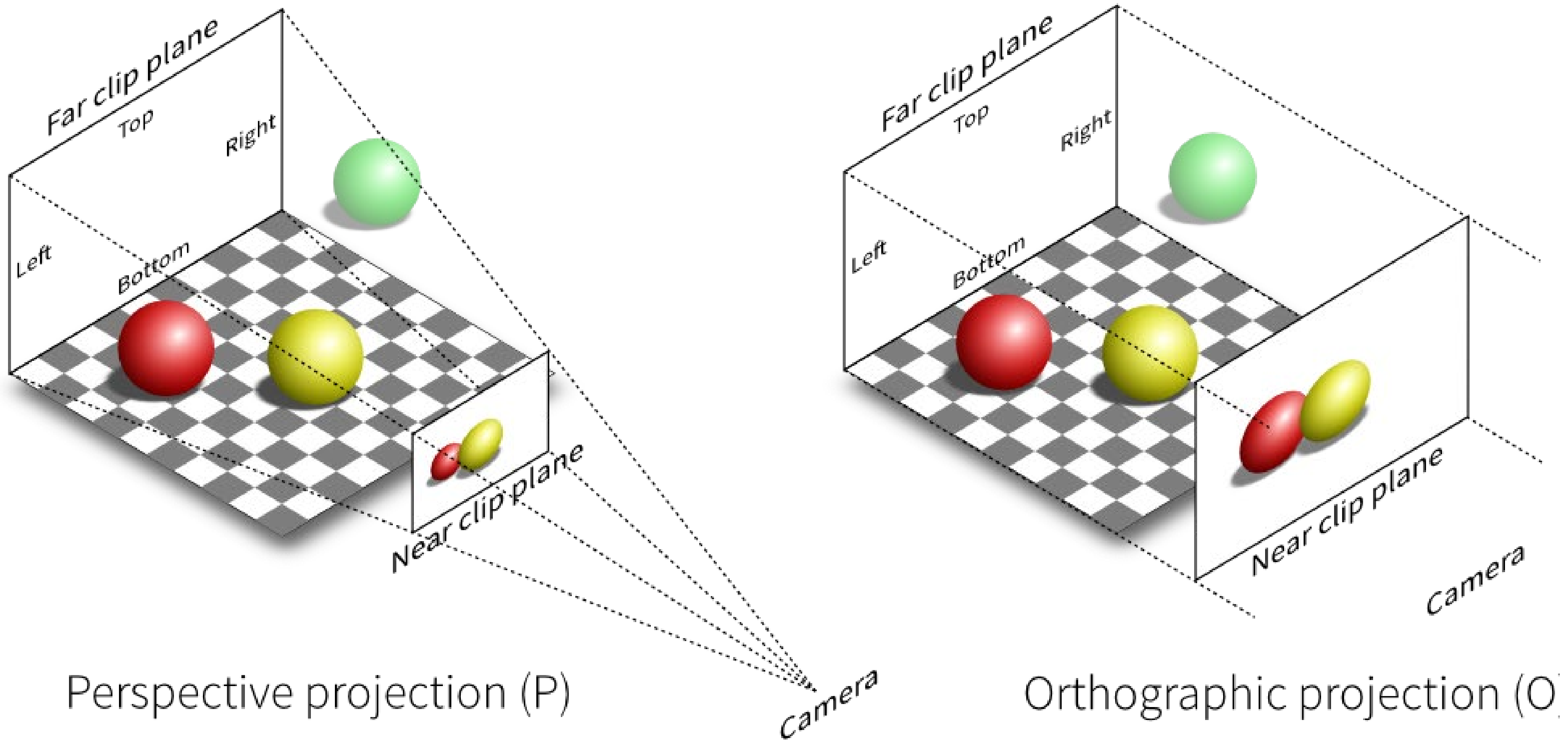
Orthographic transformation



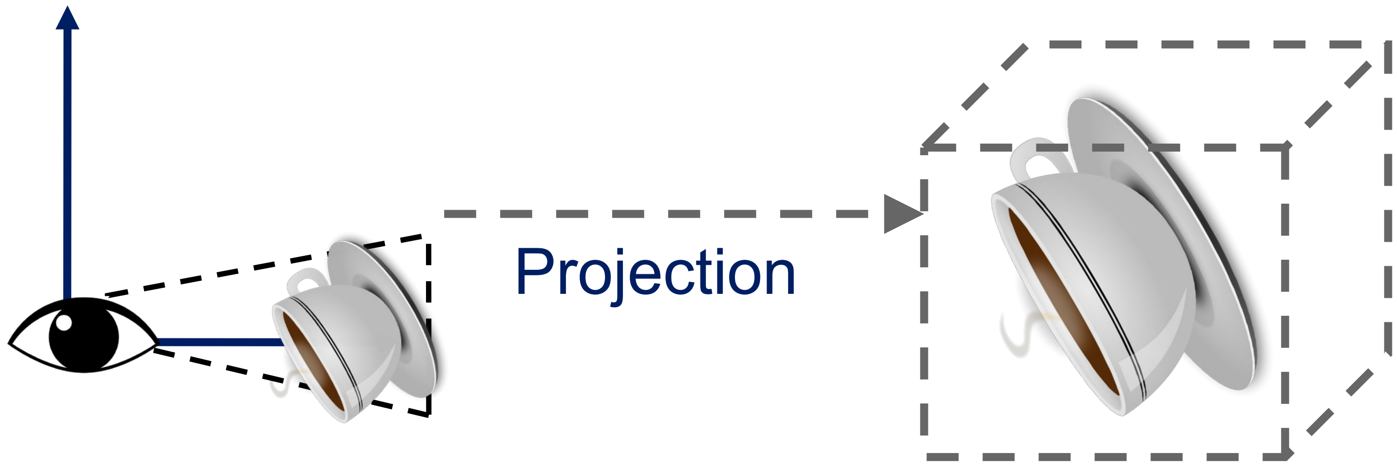
$(-1,-1,-1) \times (1,1,1)$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{(right - left)} & 0 & 0 & \frac{-(right + left)}{(right - left)} \\ 0 & \frac{2}{(top - bottom)} & 0 & \frac{-(top + bottom)}{(top - bottom)} \\ 0 & 0 & \frac{-2}{(far - near)} & \frac{-(far + near)}{(far - near)} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

View frustum



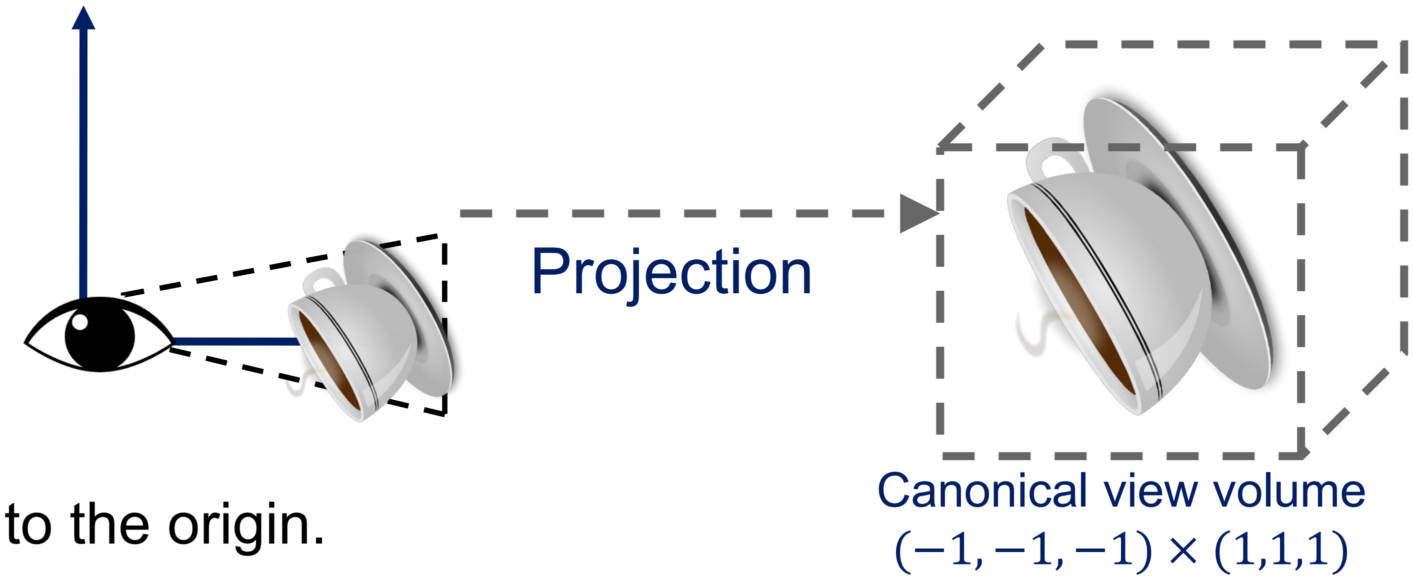
Perspective projection



1. Translate to the origin.

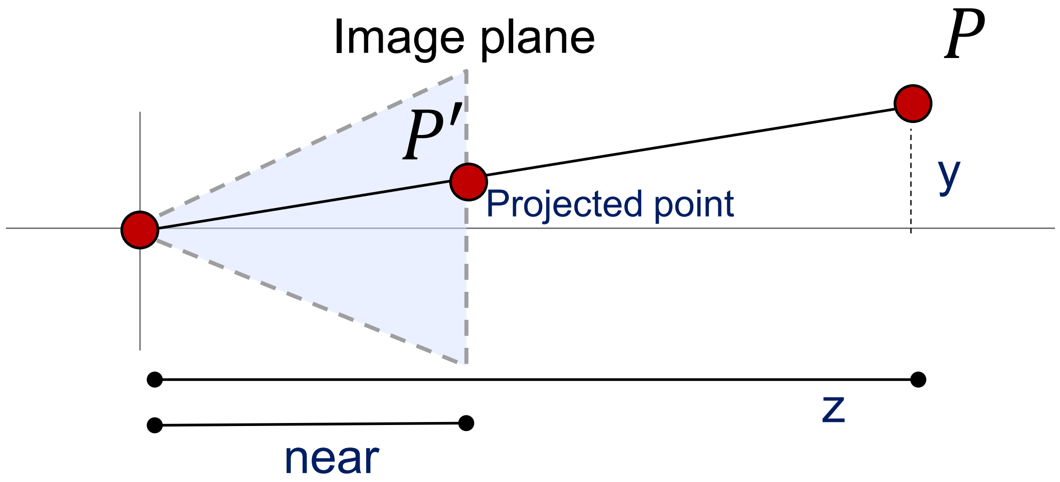
4. Scale the volume to a 2-by-2 unit square: (-1,-1) to (+1, +1).
5. Switch coordinate system.

Perspective projection

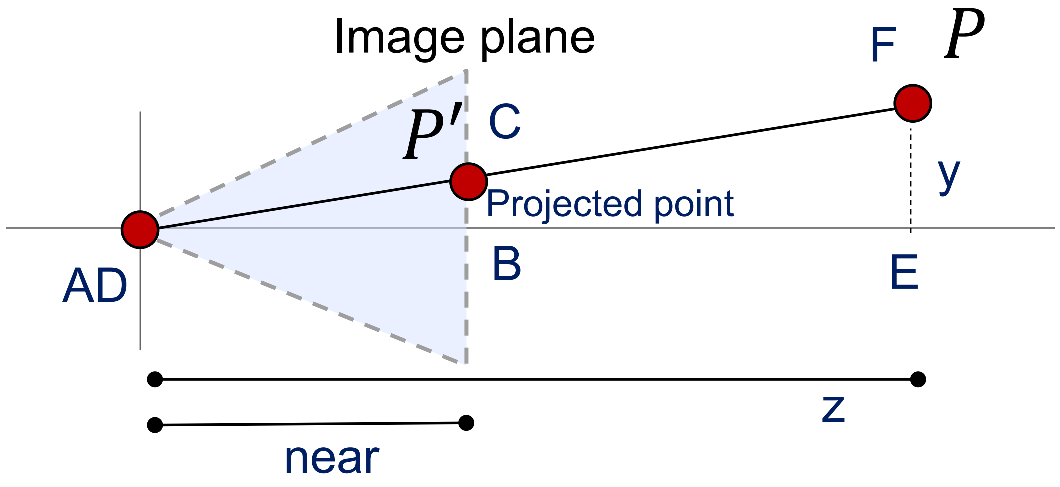


1. Translate to the origin.
2. Scale depth values into normalized range: (-1, +1).
3. Perspective calculation.
4. Scale the volume to a 2-by-2 unit square: (-1,-1) to (+1, +1).
5. Switch coordinate system.

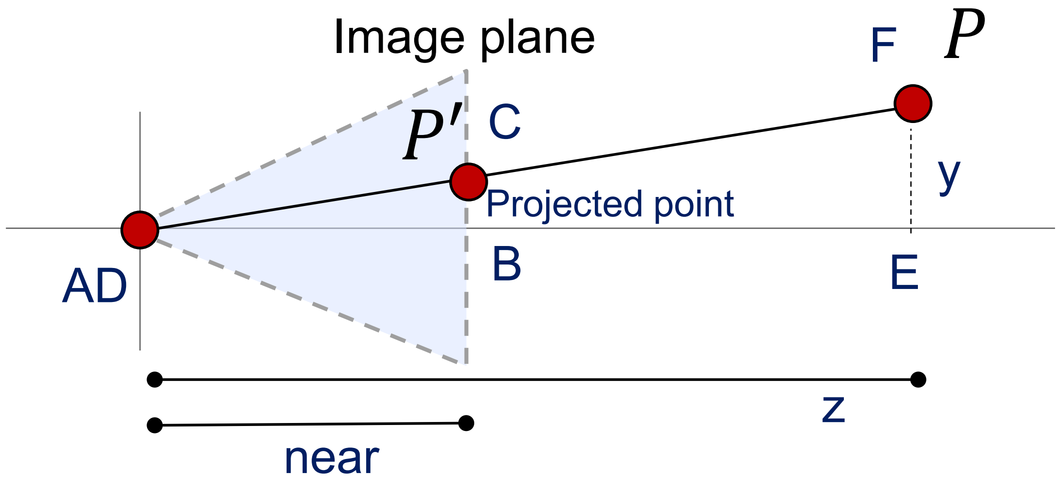
Perspective calculation



Perspective calculation



Perspective calculation



Project point P'?

$$\frac{AB}{DE} = \frac{BC}{EF}$$

$$\frac{near}{-P_z} = \frac{P'_y}{P_y}$$

$$P'_y = \frac{near \ * P_y}{-P_z}$$

$$P'_x = \frac{near \ * P_x}{-P_z}$$

What is the matrix transformation?

Perspective calculation

- What is the matrix transformation?
- 4-by-4 transformation matrix is a linear combination of terms:
  - We can calculate $a * x + b * y + c * z + d$
  - But not $\frac{a*x}{z} + \cdots$

Perspective calculation

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- 4-by-4 transformation matrix is a linear combination of terms:
  - We can calculate $a * x + b * y + c * z + d$
  - But not $\frac{a*x}{z} + \cdots$
- Solution: homogeneous coordinates.

$$(x, y, z, w) \rightarrow (\frac{x}{w}, \frac{y}{w}, \frac{z}{w}, 1)$$

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$$(x, y, z, w) \rightarrow (\frac{x}{w}, \frac{y}{w}, \frac{z}{w}, 1)$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} near & 0 & 0 & 0 \\ 0 & near & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Perspective calculation

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Perspective calculation

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} near & 0 & 0 & 0 \\ 0 & near & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} x * near \\ y * near \\ z \\ -z \end{pmatrix}$$

Graphics pipeline expects perspective division in the w component

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{x*near}{-z} \\ \frac{y*near}{-z} \\ -\frac{z}{z} \\ \frac{-z}{-z} \end{pmatrix}$$

Exactly what we wanted to get (slide 33):

$$P'_y = \frac{near \ * P_y}{-P_z}$$

$$P'_x = \frac{near \ * P_x}{-P_z}$$

Scaling depth values

- Scaling depth values into a normalized range.
- Objective: mapping between (-near,-far) to (-1,+1).
- Naïve solution:

$$-1 = -near$$
$$+1 = -far$$

- However, float point numbers suffer from round-off errors (difference between 0.12345678 and 0.12345677 can have a visual impact.

Scaling depth values

- Scaling depth values into a normalized range.
- Objective: non-linear mapping between (-near,-far) to (-1,+1).
- More precision for values close to the camera, less precision for values farther from the camera.
- Non-linear mapping with two constants:

$$\frac{c_1}{-z} + c_2$$

Scaling depth values

- Scaling depth values into a normalized range.
- Non-linear mapping with two constants:

$$\frac{c_1}{-z} + c_2$$

- When $z = -near$, mapping should return -1
- When $z = -far$, mapping should return $+1$

$$-1 = \frac{c_1}{-(-near)} + c_2 \text{ and } +1 = \frac{c_1}{-(-far)} + c_2$$

Scaling depth values

- Scaling depth values into a normalized range.
- Non-linear mapping with two constants:

$$\frac{c_1}{-z} + c_2$$

- When $z = -near$, mapping should return -1
- When $z = -far$, mapping should return $+1$

$$-1 = \frac{c_1}{-(-near)} + c_2 \text{ and } +1 = \frac{c_1}{-(-far)} + c_2$$

$$c_1 = 2 * far * \frac{near}{near - far}$$

$$c_2 = \frac{far + near}{far - near}$$

Scaling depth values

- What is the matrix transformation?
- Same problem as before: 4-by-4 transformation matrix is a linear combination of terms:
  - We can calculate $a * x + b * y + c * z + d$
  - But not $\frac{a*x}{z} + \cdots$
- Solution: homogeneous coordinates (again).

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -c_2 & c_1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Perspective projection

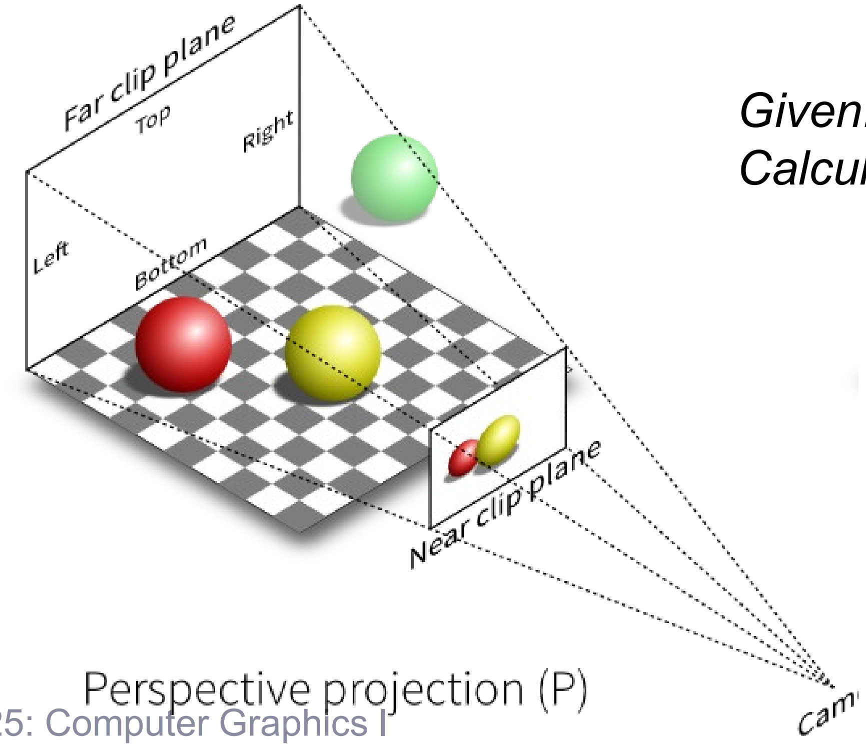
1. Translate to the origin.
2. Scale depth values into normalized range: (-1, +1). (and switch coordinate system)
3. Perspective calculation.
4. Scale the volume to a 2-by-2 unit square: (-1,-1) to (+1, +1).

Note: you only need to perform perspective calculation once.

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{(right - left)} & 0 & 0 & 0 \\ 0 & \frac{2}{(top - bottom)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} near & 0 & 0 & 0 \\ 0 & near & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -c_2 & c_1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -x_{mid} \\ 0 & 1 & 0 & -y_{mid} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Perspective projection

- How to create a perspective transformation matrix given a view of view (FOV) and aspect ratio width:height?



Perspective projection (P)

Given: fov, aspect, near, far
Calculate frustum properties (left, right, bottom, top, near, far):

$$top = near * \tan\left(\frac{fov}{2}\right)$$

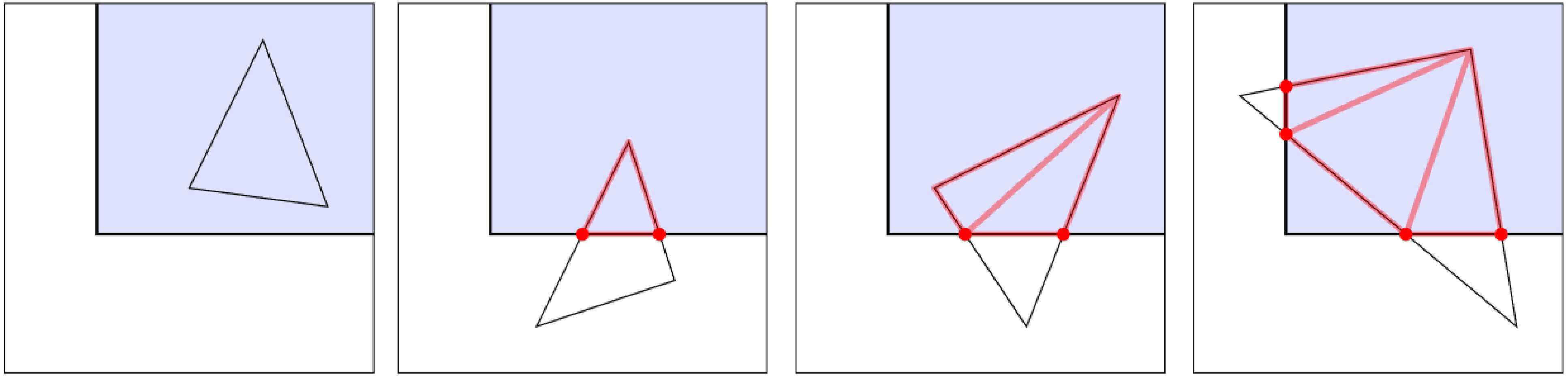
$$bottom = -top$$

$$right = top * aspect$$

$$left = -right$$

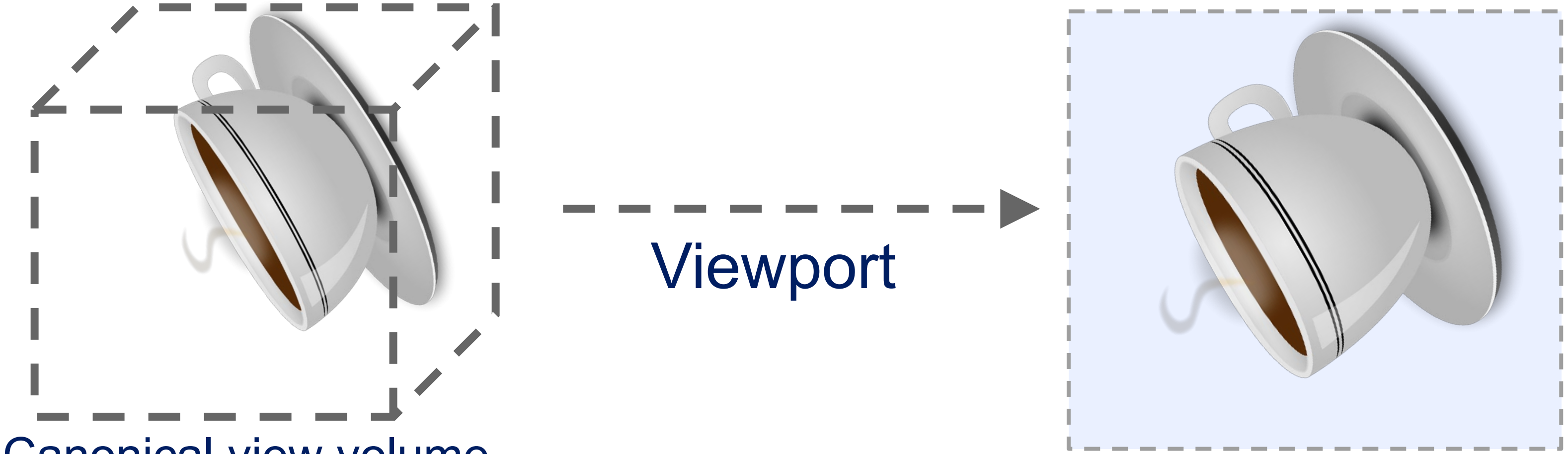
Canonical view volume

- Why?
  - Makes clipping much easier! GL can quickly discard geometry outside -1,1.



From: https://paroj.github.io/gltut/

Viewport transformation



Canonical view volume
$(-1,-1,-1) \times (1,1,1)$

Screen space

$$\begin{pmatrix} x_{screen} \\ y_{screen} \\ z_{depth} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{width}{2} & 0 & 0 & \frac{width}{2} \\ 0 & \frac{height}{2} & 0 & \frac{height}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

```
gl.viewport(0, 0, width, height);
```

Lab

- Draw a cube on screen with size $(-1,-1,-1) \times (1,1,1)$.
  - Apply a model transformation that scales it by 0.5, and tilts it slightly.
  - Apply a projection matrix (orthographic and perspective).