Discrete-time systems
Frequency response
Discrete-Time Fourier transform

Digital Signal Processing with a focus on audio signals

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Discrete-time systems

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$$y(n) = \mathcal{H}\left\{x(n)\right\}$$

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- Discrete-time systems are typically depicted as follows:

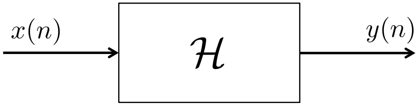

Figure: Sampling a continuous-time signal

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$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n)$$

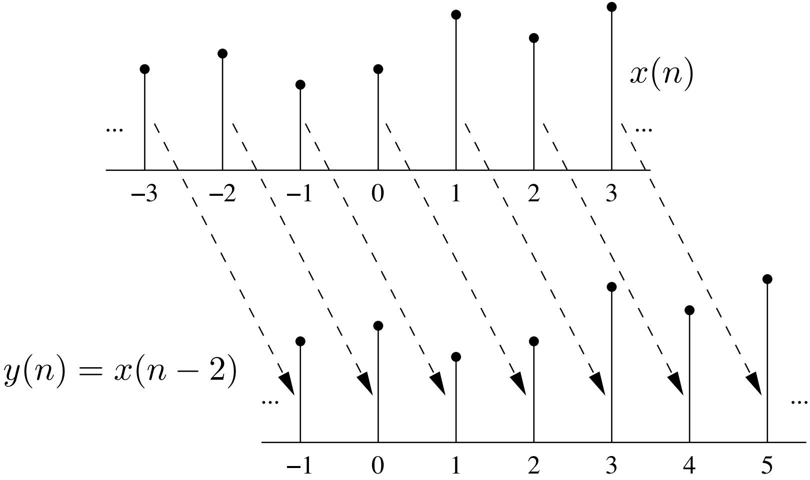
Figure: Sampling a continuous-time signal

- We will see that, depending on the properties of $\mathcal{H}\left\{\cdot\right\}$, the system can be classified in several ways
  - memoryless / with memory
  - linear / non-linear
  - time invariant / time variant
  - causal / non-causal

Discrete-time systems

Example: ideal delay. Shift the input sequence later by n_d samples.

$$y(n) = x(n - n_d)$$



Discrete-time systems

Example: moving average.

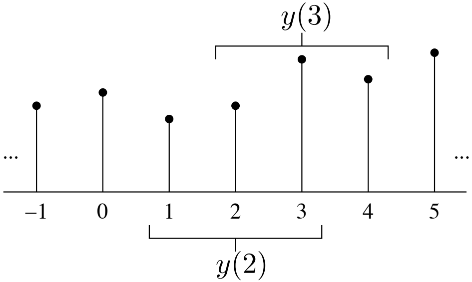
$$y(n) = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x(n-k)$$

Discrete-time systems

Example: moving average.

$$y(n) = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x(n-k)$$

Considering $M_1 = M_2 = 1$, we get



$$\vdots$$
$$y(2) = \frac{1}{3}\left[x(1) + x(2) + x(3)\right]$$
$$y(3) = \frac{1}{3}\left[x(2) + x(3) + x(4)\right]$$
$$\vdots$$

Figure: Sampling a continuous-time signal

Memoryless systems

- A system is **memoryless** if the output $y(n)$ depends only on $x(n)$ at the same n.
- Example:

$$y(n) = [x(n)]^2$$

- The ideal delay $y(n) = x(n - n_d)$ is not memoryless, unless $n_d = 0$

Linearity

- A system is **linear** if the superposition principle applies.
- Provided that

$$y_1(n) = \mathcal{H}\left\{x_1(n)\right\} \quad \text{and} \quad y_2(n) = \mathcal{H}\left\{x_2(n)\right\}$$

  linearity implies:

  - additivity: $\mathcal{H}\left\{x_1(n) + x_2(n)\right\} = \mathcal{H}\left\{x_1(n)\right\} + \mathcal{H}\left\{x_2(n)\right\}$
  - scaling: $\mathcal{H}\left\{a x_1(n)\right\} = a\mathcal{H}\left\{x_1(n)\right\}$

Linearity: principle of superposition

- The two properties combine to form the **principle of superposition**:

$$\begin{aligned}\mathcal{H}\{ax_1(n)+bx_2(n)\} &= a\mathcal{H}\{x_1(n)\}+b\mathcal{H}\{x_2(n)\}\\ &= ay_1(n)+by_2(n)\end{aligned}$$

- the terms a and b are arbitrary constants

- the superposition principle can be generalized to many inputs:

$$x(n)=\sum_k a_k x_k(n) \quad \rightarrow \quad y(n)=\mathcal{H}\{x(n)\}=\sum_k a_k y_k(n)$$

Time invariance

- A system is **time invariant** if a time shift, or delay, of the input sequence causes a corresponding shift in the output sequence.
- In formulas, a system $\mathcal{H}$ is time invariant when

$$y(n-n_0) = \mathcal{H}\left\{x(n-n_0)\right\}$$

- the ideal delay $y(n) = x(n-n_d)$ is a time invariant system
- the system $y(n) = x(Mn)$ is not time invariant (with M integer, it extracts a sample from the input sequence every M samples)

Causality

- A system is **causal** if the output at n depends only on the input at n and earlier inputs.
- Formal definition: a system is causal if and only if, for each pair of sequences $x_1(n) = x_2(n) \;\; \forall n < n_0$, then:

$$\mathcal{H}\{x_1(n)\} = \mathcal{H}\{x_2(n)\} \;\;\; \forall n < n_0$$

- Causal system example: $y(n) = x(n) - x(n-1)$
- Non-causal system example: $y(n) = x(n+1) - x(n-1)$

- Remark: non-causal system can't be implemented in real time. Anyway, off-line processing is always possible, i.e. non-causal systems are realizable.

Stability

- A system is **stable** if every bounded input sequence produces a bounded output sequence
  - bounded input: $|x(n)| \leq B_x < \infty \ , \ \forall n$
  - bounded output: $|y(n)| \leq B_y < \infty \ , \ \forall n$

- For instance, the accumulator

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

is an example of *unbounded* system, since its response to the unit step $u(n)$ is

$$y(n) = \sum_{k=-\infty}^{n} x(k) = \begin{cases} 0 & n < 0 \\ n+1 & n \geq 0 \end{cases} ,$$

which has no finite upper bound.

Linear time-invariant systems

- If the linearity property is combined with the representation of a general sequence as a linear combination of delayed impulses, then it follows that a linear time-invariant (LTI) system can be completely characterized by its impulse response.
- Consider an LTI system $\mathcal{H}$. We know that $x(n)$ can be expressed as

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

- Thus, we can compute its output as

$$y(n) = \mathcal{H}\left\{\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right\} = \sum_{k=-\infty}^{\infty} \mathcal{H}\left\{x(k)\delta(n-k)\right\}$$
$$= \sum_{k=-\infty}^{\infty} x(k)\mathcal{H}\left\{\delta(n-k)\right\} = \sum_{k=-\infty}^{\infty} x(k)h_k(n)$$

Linear time-invariant systems

- We obtained

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h_k(n) \ ,$$

where the term $h_k(n)$ is the response of the system to an impulse at $n = k$

- Let us define $h(n) \triangleq \mathcal{H}\left\{\delta(n)\right\}$
- For the time invariance, we must have $\mathcal{H}\left\{\delta(n-k)\right\} = h(n-k)$, thus

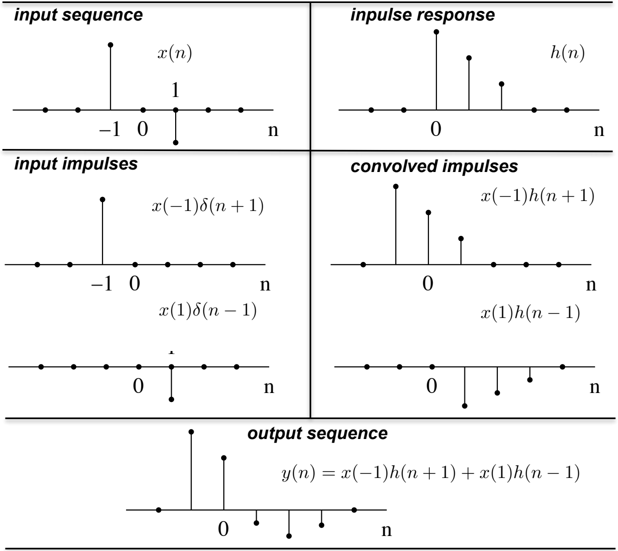
$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) = x(n) * h(n)}$$

- The last equation is known as **discrete-time convolution**, and $h(n)$ is the impulse response of the LTI system

Discrete-time convolution

- The above equation tells that any LTI system is fully characterized by its unit impulse response $h(n)$
- Interpretation 1: the input sample at $n = k$, represented by $x(k)\delta(n-k)$, is transformed by the system into an output sequence $x(k)h(n-k)$:
  - **Input** sequence:
$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$
  - **Output** sequence:
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$
- For each k, these sequences are superimposed to yield the overall output sequence

Discrete-time convolution: example 1



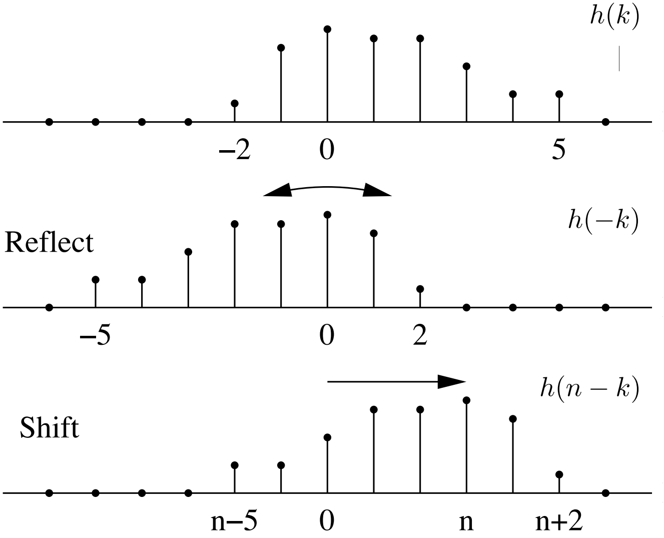
Discrete-time convolution

- Interpretation 2: the sample $y(n)$ is obtained by multiplying the input sequence (expressed as a function of k) by the sequence with values $h(n-k)$, and then summing the results of all the products

- The key point is how to form the sequence $h(n-k)$ for all values n of interest:
  - Note that $h(n-k) = h(-(k-n))$. Thus, the sequence $h(-k)$ is seen to be equivalent to $h(k)$ reflected around the origin
  - Then, $h(n-k)$ is obtained by shifting the origin of the sequence to $k = n$

- To summarize, the sequences $x(k)$ and $h(n-k)$ are multiplied together for $-\infty < k < \infty$, and the products summed to obtain the value of output sample $y(n)$. To obtain another output sample, the procedure is repeated with the origin shifted to the new sample position, and so on.

Discrete-time convolution: example 2



Example: analytical evaluation of the convolution sum (1)

- System impulse response:

$$h(n) = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

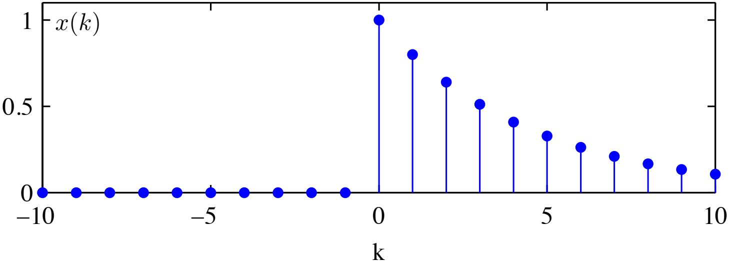
- Input sequence:

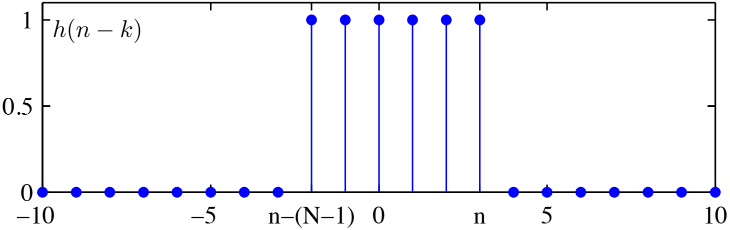
$$x(n) = a^n u(n) \ ,$$

$u(n)$ being the unit step sequence.

Example: analytical evaluation of the convolution sum (2)

- To find the output at n, we must form the sum over all k of the product $x(k)h(n-k)$





Example: analytical evaluation of the convolution sum (3)

- Since the sequences are non-overlapping for all negative n, the output must be zero:

$$y(n) = 0\ , \quad n < 0$$

- For $0 \leq n \leq N-1$, the product terms in the sum are $x(k)h(n-k) = a^k$, so it follows that

$$y(n) = \sum_{k=0}^{n} a^k\ , \quad 0 \leq n \leq N-1$$

- Finally, for $n > N-1$, the product terms are $x(k)h(n-k) = a^k$ as before, but the lower limit of the summation is now $n - N + 1$. Therefore

$$y(n) = \sum_{k=n-N+1}^{n} a^k\ , \quad n > N-1$$

Properties of LTI systems

Some properties of LTI systems can be found considering the properties of the discrete convolution:

- **Commutative** (easy to prove replacing the summation index as $l = n - k$):

$$x(n) * h(n) = h(n) * x(n)$$

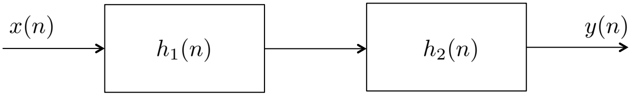
- **Distributive** over addition:

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

Combination of LTI systems

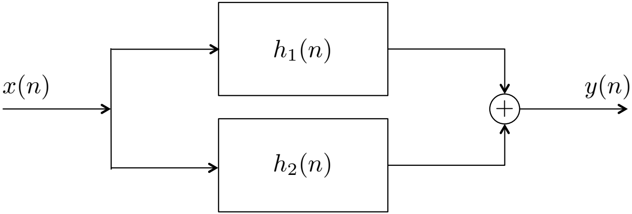
It directly follows that LTI systems can be easily combined in two ways:

- **Cascade connection**:



$$y(n) = x(n) * h_1(n) * h_2(n) = x(n) * h_C(n)$$

- **Parallel connection**:



$$y(n) = [h_1(n) + h_2(n)] * x(n) = x(n) * h_P(n)$$

Stability of LTI systems

- A sufficient condition for a LTI system to be bounded-input bounded-output (BIBO) stable is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

- Examples of stable systems:
  - Ideal delay $h(n) = \delta(n - n_d)$
  - Backward difference $h(n) = \delta(n) - \delta(n-1)$
  - Forward difference $h(n) = \delta(n+1) - \delta(n)$
  - Moving average

Causality of LTI systems

- A LTI system with impulse response $h(n)$ is causal if and only if

$$h(n) = 0 \quad \forall n < 0$$

- Examples:
  - The ideal delay $h(n) = \delta(n - n_d)$ is causal if $n_d \geq 0$
  - Backward difference $h(n) = \delta(n) - \delta(n-1)$ is causal
  - Forward difference $h(n) = \delta(n+1) - \delta(n)$ is non-causal

Linear constant coefficient difference equations

- Discrete time-invariant systems are often represented in terms of discrete difference equations
- A convenient representation is the **advance operator form**:

$$a_N y(n+N) + a_{N-1} y(n+N-1) + \cdots + a_1 y(n+1) + a_0 y(n) = \\ b_M x(n+M) + b_{M-1} x(n+M-1) + \cdots + b_1 x(n+1) + b_0 x(n)$$

  - it is evident how the **causality condition** is $M \leq N$
- For a general causal system with $M = N$ we have

$$a_N y(n+N) + a_{N-1} y(n+N-1) + \cdots + a_1 y(n+1) + a_0 y(n) = \\ b_N x(n+N) + b_{N-1} x(n+N-1) + \cdots + b_1 x(n+1) + b_0 x(n)$$

- The difference equation has an infinite number of solutions $y(n)$ (like the solutions of continuous differential equations).

Linear constant coefficient difference equations (cont')

- In case $M = N$, we can replace n by $n - N$ (safe operation, as the system is time-invariant), thus obtaining the **delay operator form**:

$$a_N y(n) + a_{N-1} y(n-1) + \cdots + a_1 y(n-N+1) + a_0 y(n-N) =$$
$$b_N x(n) + b_{N-1} x(n-1) + \cdots + b_1 x(n-N+1) + b_0 x(n-N)$$

- Example: the accumulator system

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

can be rewritten as

$$y(n) = x(n) + y(n-1) \quad \rightarrow \quad y(n) - y(n-1) - x(n) = 0\ .$$

Iterative solution of difference equations

- From the delay operator form, assuming $a_N = 1$, we readily obtain

$$
\begin{aligned}
y(n) = -a_{N-1}y(n-1) - a_{N-2}y(n-2) - \cdots - a_0 y(n-N) \\
+ b_N x(n) + b_{N-1}x(n-1) + \cdots + b_0 x(n-N)
\end{aligned}
$$

- To solve it, one needs to know:
  - the past N values of the output $y(n-k)$, $k = 1, 2, \ldots, N$
  - the past N values of the input $x(n-k)$, $k = 1, 2, \ldots, N$
  - the current value of the input $x(n)$
- In other words, beyond the current input $x(n)$, we need N initial conditions on the input and N on the output

Iterative solution of difference equations (cont)

- If the input is causal, then

$$x(-1) = x(-2) = \ldots = x(-N) = 0$$

thus we need only N initial conditions on the output, namely the values

$$y(-1)\,,\ y(-2)\,,\ \ldots\,,\ y(-N)$$

- To summarize:
  1. start finding $y(0)$: the right-hand side contains terms $y(-1), \ldots, y(-N)$ and no input terms
  2. using the N initial conditions $y(-1), \ldots, y(-N)$, solve iteratively for $y(0)$,$y(1)$, $y(2)$ and so on

Recursive equations

- Difference equations can be rewritten, without loss of generality, considering that $a_0 = 1$, yielding
$$y(n) = -\sum_{i=1}^{N} a_i y(n-i) + \sum_{l=0}^{M} b_l x(n-l)$$
- The output signal is thus dependent on both the samples of the input $x(n), x(n-1), \ldots, x(n-M)$ and on previous sample of the output $y(n-1), \ldots, y(n-N)$
- In this general case, we say that the system is *recursive*, as we need past samples of the output itself

Finite-duration impulse response

- When $a_1 = a_2 = \ldots = a_N = 0$ the system is called non-recursive:

$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

- It corresponds to a system with impulse response $h(k) = b_k$ for $0 \leq k \leq M$
- This implies that this system has a finite-duration impulse response
- Such systems are often referred to as **finite-duration impulse response** (FIR) filters

Infinite-duration impulse response

- In contrast, when the system is recursive, the impulse response might not be zero when $n \to \infty$
- Therefore, recursive digital systems are often referred to as **infinite-duration impulse response** (IIR) filters

Frequency response

- Discrete LTI systems are often represented by means of their **frequency response**
- The frequency response can be evaluated considering the output of a system $h(n)$ when it is excited by a complex sinusoid:
  - input sequence:
$$x(n) = e^{j\omega n}$$
  - output sequence:
$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\ &= \sum_{k=-\infty}^{\infty} h(k)e^{j\omega(n-k)} \\ &= e^{j\omega n}\left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right] \end{aligned}$$

Frequency response

- The output sequence is thus a complex sinusoid with the same frequency ω, multiplied by the complex factor in the brackets
- We thus define the frequency response of the system $h(n)$ as

$$\boxed{H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}}$$

- Since it is a complex number, it can be expressed in polar form as

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\Theta(\omega)}$$

- It follows that the output sequence has the form

$$y(n) = H(e^{j\omega})e^{j\omega n} = |H(e^{j\omega})|e^{j\omega n + j\Theta(\omega)}$$

Magnitude and phase responses

- The effect of a LTI system characterized by $H(e^{j\omega})$ on a complex sinusoid is to multiply its amplitude by $|H(e^{j\omega})|$ and to add $\Theta(\omega)$ to its phase

- For this reason, the two terms are referred to as:
  - $|H(e^{j\omega})|$: **magnitude response**
  - $\Theta(\omega)$: **phase response**

Example: frequency response of the ideal delay

- Consider the ideal delay system $h(n) = \delta(n - n_d)$
- By definition, the frequency response is given by

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \delta(k - n_d) e^{-j\omega k} = e^{-j\omega n_d}$$

- Note that we can compute it also by directly considering the output of the system, excited with $x(n) = e^{j\omega n}$:

$$y(n) = e^{j\omega(n-n_d)} = e^{-j\omega n_d} e^{j\omega n} = H(e^{j\omega n}) e^{j\omega n}$$

- The magnitude and phase responses are

$$|H(e^{j\omega})| = 1 \quad , \quad \Theta(\omega) = -\omega n_d$$

Periodicity of the frequency response

- The frequency response of a discrete LTI system is *always* periodic in frequency with a period 2π:

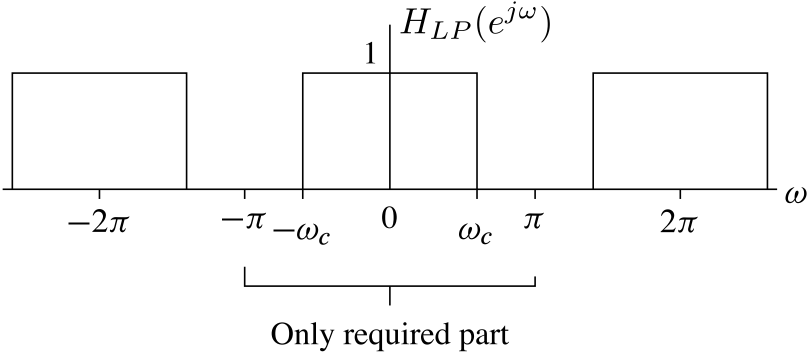
$$
\begin{aligned}
H(e^{j(\omega+2\pi)}) &= \sum_{k=-\infty}^{\infty} h(k)e^{-j(\omega+2\pi)k} \\
&= \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}e^{-j2\pi k} \\
&= \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} = H(e^{j\omega})
\end{aligned}
$$

- The last passage relies on the fact that $e^{\pm j2\pi k} = 1$ for integer k

Periodicity of the frequency response

Example 1: ideal lowpass filter

- The frequency response of an ideal lowpass filter is as follows:



- Due to the periodicity in the response, it is only necessary to consider one frequency cycle, usually chosen to be the range $[-\pi, \pi]$

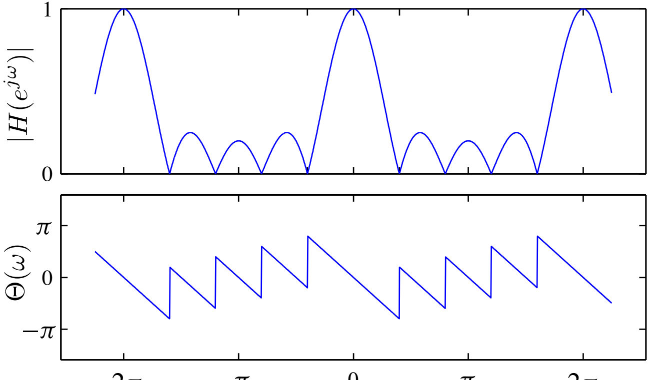
Periodicity of the frequency response

Example 2: moving-average system

- Impulse response:

$$h(n) = \begin{cases} \frac{1}{M_1+M_2+1} & -M_1 \leq n \leq M_2 \\ 0 & \text{otherwise} \end{cases}$$

- Frequency response for $M_1 = 0$ and $M_2 = 4$:



This system attenuates high frequencies (at around $\omega = \pi$), and therefore acts as a lowpass filter

Road map to Fourier Kingdom

	APERIODIC *in time* ***CONTINUOUS*** *in frequency*	***PERIODIC*** *in time* ***DISCRETE*** *in frequency*
CONTINUOUS *in time* ***APERIODIC*** *in frequency*	Fourier Transform $X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$ $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$	Fourier Series $a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j\frac{2\pi kt}{T}}dt$ $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi kt}{T}}$
DISCRETE *in time* ***PERIODIC*** *in frequency*	?	?

The Discrete-Time Fourier Transform and its inverse

- The Fourier transform is a mathematical tool defined for continuous-time signals, suitable for their analysis in the frequency-domain
- To deal with discrete-time signals, we introduce the **Discrete-Time Fourier Transform** (DTFT) of a sequence $x(n)$ as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} ,$$

  which is a **continuous and periodic** function of the frequency ω, with period 2π
- The inverse relationship is the Inverse Discrete-Time Fourier Transform (IDTFT), given by the Fourier integral

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Magnitude, phase, and Fourier spectrum

- The DTFT is generally a complex-valued function of ω:

$$\begin{aligned} X(e^{j\omega}) &= X_{Re}(e^{j\omega}) + jX_{Im}(e^{j\omega}) \\ &= |X(e^{j\omega})| e^{j\angle X(e^{j\omega})} \end{aligned}$$

- The quantities $|X(e^{j\omega})|$ and $\angle X(e^{j\omega})$ are referred to as the **magnitude** and **phase** of the Fourier transform

- The DTFT $X(e^{j\omega})$ is often referred to as the **Fourier spectrum**

DTFT and frequency response

- Since the frequency response of a LTI system is given by

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} \ ,$$

  it is clear that the frequency response coincides with the DTFT of the impulse response $h(n)$
- Moreover, the impulse response can be obtained via IDTFT as

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

Existence of the DTFT

- A sufficient condition for the existence of the Fourier transform of a sequence $x(n)$ is that it be absolutely summable, i.e.

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

- In other words, the DTFT exists if the sum $\sum_{n=-\infty}^{\infty} |x(n)|$ converges
- The DTFT may however exist for sequences where this is not true; a rigorous mathematical treatment can be found in the theory of *generalised functions*, but it is out of the scope of this course

Properties of the DTFT

- **Linearity**

$$k_1 x_1(n) + k_2 x_2(n) \quad \longleftrightarrow \quad k_1 X_1(e^{j\omega}) + k_2 X_2(e^{j\omega})$$

Properties of the DTFT

- **Linearity**

$$k_1 x_1(n) + k_2 x_2(n) \quad \longleftrightarrow \quad k_1 X_1(e^{j\omega}) + k_2 X_2(e^{j\omega})$$

- **Time-reversal**

$$x(-n) \quad \longleftrightarrow \quad X(e^{-j\omega})$$

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- **Time-reversal**

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- **Time-shift theorem**

$$x(n+l) \quad \longleftrightarrow \quad e^{j\omega l} X(e^{j\omega})$$

Properties of the DTFT

- **Linearity**

$$k_1 x_1(n) + k_2 x_2(n) \quad \longleftrightarrow \quad k_1 X_1(e^{j\omega}) + k_2 X_2(e^{j\omega})$$

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$$x(-n) \quad \longleftrightarrow \quad X(e^{-j\omega})$$

- **Time-shift theorem**

$$x(n+l) \quad \longleftrightarrow \quad e^{j\omega l} X(e^{j\omega})$$

- **Multiplication by an exponential**

$$e^{j\omega_0 n} x(n) \quad \longleftrightarrow \quad X(e^{j(\omega-\omega_0)})$$

Properties of the DTFT (cont')

- **Complex differentiation**

$$nx(n) \quad \longleftrightarrow \quad -j\frac{dX(e^{j\omega})}{d\omega}$$

Properties of the DTFT (cont')

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$$x_1(n) * x_2(n) \quad \longleftrightarrow \quad X_1(e^{j\omega})X_2(e^{j\omega})$$

Properties of the DTFT (cont')

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- **Convolution theorem**

$$x_1(n) * x_2(n) \quad \longleftrightarrow \quad X_1(e^{j\omega})X_2(e^{j\omega})$$

- **Product of two sequences**

$$x_1(n)x_2(n) \quad \longleftrightarrow \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\Omega})X_2(e^{j(\omega-\Omega)})d\Omega$$

Properties of the DTFT (cont')

- **Parseval's theorem**

$$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\omega}) X_2^*(e^{j\omega}) d\omega$$

Properties of the DTFT (cont')

- **Parseval's theorem**

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\omega})X_2^*(e^{j\omega})d\omega$$

and, for $x_1(n) = x_2(n) = x(n)$

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

Properties of the DTFT (cont')

- **Parseval's theorem**

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\omega})X_2^*(e^{j\omega})d\omega$$

and, for $x_1(n) = x_2(n) = x(n)$

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

- **Real and imaginary sequences**

$$\begin{aligned}
\mathrm{Re}\left\{x(n)\right\} \quad &\longleftrightarrow \quad \frac{1}{2}\left(X(e^{j\omega}) + X^*(e^{-j\omega})\right)\\
\mathrm{Im}\left\{x(n)\right\} \quad &\longleftrightarrow \quad \frac{1}{2j}\left(X(e^{j\omega}) - X^*(e^{-j\omega})\right)
\end{aligned}$$

Symmetry properties

- **For real sequences** (i.e., $x(n)$ such that $\text{Im}\{x(n)\} = 0$), the following properties follow:

$$\begin{array}{rcll} X(e^{j\omega}) & = & X^*(e^{-j\omega}) & \text{conjugate symmetric DTFT} \\ \text{Re}\{X(e^{j\omega})\} & = & \text{Re}\{X(e^{-j\omega})\} & \text{even real part} \\ \text{Im}\{X(e^{j\omega})\} & = & -\text{Im}\{X(e^{-j\omega})\} & \text{odd imaginary part} \\ |X(e^{j\omega})| & = & |X(e^{-j\omega})| & \text{even magnitude} \\ \angle X(e^{j\omega}) & = & -\angle X(e^{-j\omega}) & \text{odd phase} \end{array}$$

- **For imaginary sequences** (i.e., $x(n)$ such that $\text{Re}\{x(n)\} = 0$), similar properties can be deduced

Symmetry properties (cont')

- The following symmetry properties also hold:

$x(n)$		$X(e^{j\omega})$
real and even	$\longleftrightarrow$	real and even
imaginary and even	$\longleftrightarrow$	imaginary and even
real and odd	$\longleftrightarrow$	imaginary and odd
imaginary and odd	$\longleftrightarrow$	real and odd
conjugate symmetric	$\longleftrightarrow$	real

Example 1: DTFT of a discrete rectangle sequence

- Consider the following rectangular function

$$x(n) = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

- The DTFT can be computed as

$$X(e^{j\omega}) = \sum_{n=0}^{M} e^{-j\omega n} = \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} \, ,$$

where we exploited the closed-form expression of converging geometric series:

$$\sum_{n=0}^{M} r^n = \frac{1 - r^{M+1}}{1 - r}$$

Example 1: DTFT of a discrete rectangular sequence (cont')

- Rearranging the terms, the result can be rewritten in a more interesting form:

$$
\begin{aligned}
X(e^{j\omega}) &= \frac{e^{-j\frac{\omega}{2}}}{e^{-j\frac{\omega}{2}}} \cdot \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} \\
&= \frac{e^{-j\omega\frac{M}{2}} \left[e^{j\omega\left(\frac{M+1}{2}\right)} - e^{-j\omega\left(\frac{M+1}{2}\right)} \right]}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} \\
&= e^{-j\omega\frac{M}{2}} \cdot \frac{\sin\omega\left(\frac{M+1}{2}\right)}{\sin\frac{\omega}{2}} \\
&= e^{-j\omega\frac{M}{2}} \cdot (M+1) \cdot \mathrm{asinc}_{M+1}(\omega)
\end{aligned}
$$

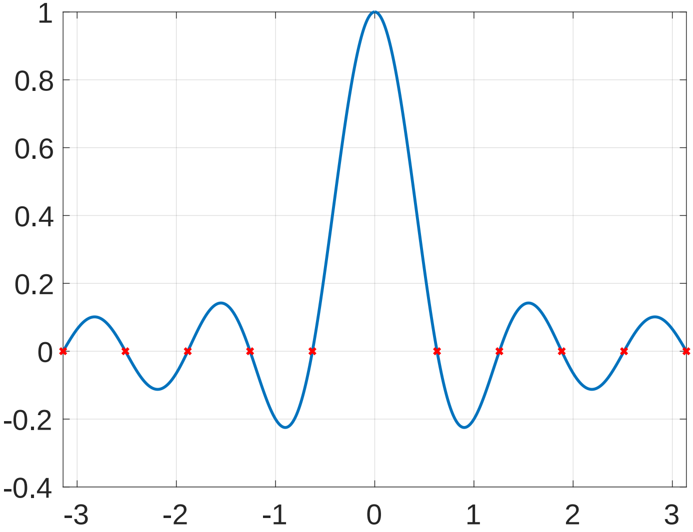
- The transform of a rectangular window is thus proportional to an *aliased sinc function* (asinc), defined as

$$
\mathrm{asinc}_M(\omega) = \frac{\sin M\frac{\omega}{2}}{M \cdot \sin\frac{\omega}{2}}
$$

- Zeros occur at $\omega = 2k\pi/M$, with $k = \pm 1, \pm 2, \ldots, \pm M$

Example 1: DTFT of a discrete rectangular sequence (cont')

- Example with $M = 10$



Example 2: Ideal low-pass filter

- We can specify an ideal low-pass filter in the frequency domain, i.e. defining its frequency response:

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

- The impulse response can be therefore obtained via IDTFT:

$$\begin{aligned} h(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi j n} \left[e^{j\omega_c n} - e^{-j\omega_c n} \right] = \frac{\sin \omega_c n}{\pi n} = \frac{\omega_c}{\pi} \mathrm{sinc}(\omega_c n) \end{aligned}$$

- The impulse response of an ideal low-pass filter is thus proportional to the *sinc function*, defined as:

$$\mathrm{sinc}(n) = \frac{\sin n}{n}$$

Some useful transform pairs

Sequence	DTFT
$\delta(n)$	1
$\delta(n-n_0)$	$e^{-j\omega n_0}$
1	$\displaystyle\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
$u(n)$	$\dfrac{1}{1-e^{-j\omega}} + \displaystyle\sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
$a^n u(n)$, $\lvert a\rvert < 1$	$\dfrac{1}{1-ae^{-j\omega}}$
$(n+1)a^n u(n)$, $\lvert a\rvert < 1$	$\dfrac{1}{(1-ae^{-j\omega})^2}$

Some useful transform pairs

Sequence	DTFT
$x(n) = \dfrac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \leq \pi \end{cases}$
$x(n) = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$	$X(e^{j\omega}) = \dfrac{\sin \frac{\omega(M+1)}{2}}{\sin \frac{\omega}{2}} e^{-j\omega \frac{M}{2}}$
$e^{j\omega_0 n}$	$\displaystyle\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Review: Road map to Fourier Kingdom

	APERIODIC in time *CONTINUOUS in frequency*	*PERIODIC in time* *DISCRETE in frequency*
CONTINUOUS in time *APERIODIC in frequency*	Fourier Transform $X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$ $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$	Fourier Series $a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j\frac{2\pi kt}{T}}dt$ $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi kt}{T}}$
DISCRETE in time *PERIODIC in frequency*	Discrete-Time Fourier Transform (DTFT) $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$ $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$	?