Efficient convolution using the DFT

Digital Signal Processing with a focus on audio signals

Fabio Antonacci

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Background: convolution theorem

- The convolution operation is at the basis of many signal processing tasks
- For instance, it enables the computation of the output of digital systems, provided the input signal and the impulse response:

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

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$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

- An important theorem states that convolution is equivalent to a product in the frequency domain:

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$

Circular convolution

- A similar theorem exists also for the DFT (sampled version of the DTFT)

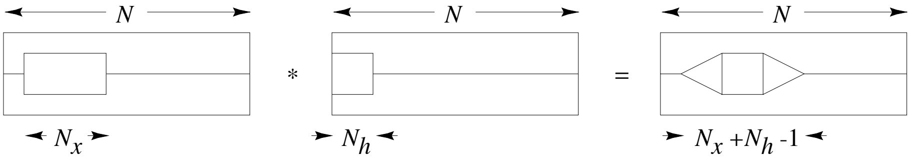
- As the DFT is implicitly periodic in both time and frequency domains, it turns that the convolution theorem involves the circular (cyclic) convolution. For N-points DFT we have, indeed:

$$Y(k) = X(k) \cdot H(k) \qquad \longleftrightarrow \qquad y(n) = \sum_{k=0}^{N-1} x(k) h(n-k)_N$$

Convolution of finite length signals

- In practical cases, performing the convolution in the frequency domain may be very efficient
- Let us consider the simplest case, i.e. that of convolving two finite-length signals:
  - $x(n)$, input signal with length N_x
  - $h(n)$, impulse response with length N_h

- The output signal $y(n) = x(n) * h(n)$ will be time-limited, too, with total length $N = N_x + N_h - 1$ samples, i.e.:



Convolution of finite length signals: computational complexity

- Computing the output signal means calculating the convolution sum:

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- The complexity order of time-domain convolution can therefore be approximated as $O(N^2)$
- **Idea**: what about computing the convolution in the frequency domain (simple product) instead of working in the time domain?

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- We can therefore exploit the (circular) convolution theorem to speed-up the computation of convolution:
  - Compute FFT of $x(n)$ and $h(n)$: $O(N \log N)$
  - Compute the products $Y(k) = X(k)H(k)$: $O(N)$
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Result: overall complexity of FFT-based convolution is $O(N \log N)$ (lower than in the time domain!)

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&= \sum_{k=0}^{n} x(k)h(n-k) + \sum_{k=n+1}^{N-1} x(k)h(n-k+N) \ , \quad \text{for } 0 \leq n \leq N-1
\end{aligned}
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- If we want the circular convolution to be equal to the linear a-cyclic convolution between $x(n)$ and $h(n)$, the second summation must be null, i.e.

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c(n) = \sum_{k=n+1}^{N-1} x(k)h(n-k+N) = 0 \ , \quad \text{for } 0 \leq n \leq N-1
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Dealing with circular convolution (cont')

- We know that:
  - $x(n)$ has duration N_x, so we set $x(n) = 0$ for $n \geq N_x$ and $n < 0$
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- Therefore, we can force the summation $c(n)$ to be identically zero if we select

$$N \geq N_x + N_h - 1$$

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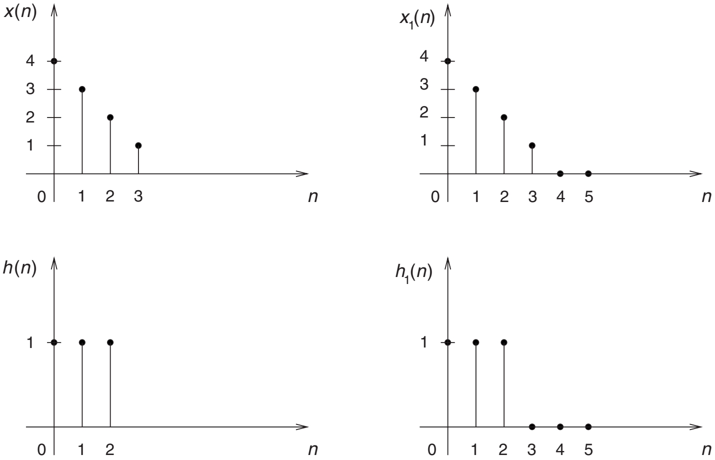
- Considering N-length signals in place of their original lengths correspond to **zero-padding**, i.e. appending trailing zeros to reach the length N
- Remark: to fully exploit the efficiency of the FFT algorithm, N should be selected as a power of 2

Zero-padding

Example of zero-padding of the two sequences to be convolved:



Zero-padding and time-domain aliasing

- Note that the condition $N \geq N_x + N_h - 1$ can be interpreted as an anti-aliasing one
- Indeed, if we add an insufficient number of zeros (i.e., the final signals length is $N < N_x + N_h - 1$), we have $c(n) \neq 0$ in general
- Recall that the DFT corresponds to sampling the DTFT: we must sample it at least at $N_x + N_h - 1$ points to avoid overlapping replicas in the time domain!
- If we don't add enough zeros, some of our convolution terms "wrap around" and add back upon others (due to modulo indexing): this generates **time domain aliasing**

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3. Obtain the output signal via N-points IDFT:

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{j\frac{2\pi kn}{N}} \ , \quad n = 0, 1, \dots N-1$$

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- There are some situations where it will not be practical to perform the convolution of two signals using a single FFT:
  - N_x is extremely large
  - Real time operation (we can't wait until the signal ends)
- Theoretically, there is no problem doing this with direct convolution: since $h(n)$ is finite-length, we only need to store the past N_h samples of the input signal
- However, this would require N_h products for each output sample

Overlap and add: overview

- For the above reason, filtering directly in the time domain can be extremely time-consuming when N_h is large
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- **Idea**: use overlap-and-add to process the signal on a frame basis
  - processing one frame (segment) at a time
  - compute FFT convolution on each frame separately
  - put it all back together correctly (we'll see how to)

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  - processing one frame (segment) at a time
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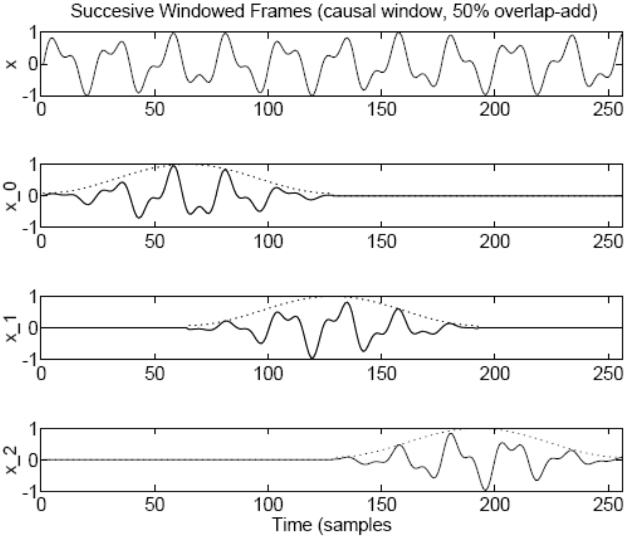
- **Frames** (or **segments**) from the input signal $x(n)$ are extracted as

$$x_m(n) \triangleq x(n)w(n - mR) \, , n \in (-\infty, +\infty) \, ,$$

  where:
  - $w(n)$ is a length M window (rectangular/Hamming/...)
  - m is the frame index
  - R is the hop-size

Overlap-and-add: overview (cont')

Example of frame extraction:



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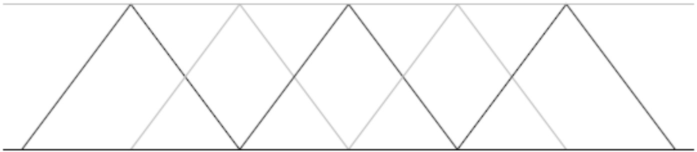
- It follows that

$$x(n) = \sum_{m=-\infty}^{+\infty} x_m(n) \iff \sum_{m=-\infty}^{+\infty} w(n - mR) = 1$$

- The last equality plays a key-role in overlap-and-add convolution, and it is called **Constant Overlap-And-Add** (COLA) **condition** on the analysis window $w(n)$

Constant overlap-and-add condition (cont')

- All windows which obey the COLA will yield perfect reconstruction of the original signal from the data frames by overlap-add
- There is no constraint on window type, only that the window overlap-adds to a constant for the hop size used
- Examples (where M is the generic window length):
  - Rectangular window at 0% overlap ($R = M$)
  - Rectangular window at 50% overlap ($R \approx M/2$, i.e., $R = M/2$ if M is even, $R = (M-1)/2$ if M is odd)
  - Hamming window at 50% overlap ($R \approx M/2$)
  - Hamming window at 75% overlap ($R \approx M/4$)
  - Triangular (Bartlett) window at 50% overlap (see figure below)
  - Any window with $R = 1$



Frequency-domain convolution via OLA

- Once a suitable window has been selected, the convolution process can be expressed as follows:

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{\infty} x_m(n) * h(n)$$

- The last equation tells that the overall convolution operation can be separated into a summation of finite-length convolutions on signal segments
- The segment convolutions can be accomplished in the frequency domain, exploiting the efficiency of the FFT algorithm

Frequency-domain convolution via OLA (cont')

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- To compute the DFT, it is convenient to shift-back the frames to the origin. Let's define:

$$\tilde{x}_m(n) \triangleq \mathrm{SHIFT}_{mR}[x_m(n)] \triangleq x_m(n+mR)$$
$$\tilde{y}_m(n) \triangleq \mathrm{SHIFT}_{mR}[y_m(n)] \triangleq y_m(n+mR) = \tilde{x}_m(n) * h(n)$$

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- This way, the signals are so that:
  - $\tilde{x}_m(n) = 0$ for $n < 0$ and $n \geq M$
  - $\tilde{y}_m(n) = 0$ for $n < 0$ and $n \geq N_h$

Frequency-domain convolution via OLA (cont')

- The mth convolution can be safely performed using N-points DFTs, paying attention to:
  - select $N \geq M + N_h - 1$ for avoiding aliasing problems
  - zero-pad both $\tilde{x}_m(n)$ and $h(n)$ to reach length N

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- The N-points DFTs are thus computed as

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- The overall output signal is finally obtained by superposing all the filtered segments, remembering to shift back each segment to the right position:

$$y(n) = \sum_{m=-\infty}^{\infty} \text{SHIFT}_{-mR}\left[\tilde{y}_m(n)\right] = \sum_{m=-\infty}^{\infty} \tilde{y}_m(n - mR) = \sum_{m=-\infty}^{\infty} y_m(n)$$

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   - Obtain $\tilde{y}_m(n)$ by performing alias-free convolution using the FFT
   - Obtain $y_m(n)$ by shifting $\tilde{y}_m(n)$ back to the original position
   - Overlap and add $y_m(n)$ to the output buffer $y(n)$

Overlap-and-add example