5. Fundamentals of Dynamic Circuits

Ivan Maio and Igor Stievano

maio@polito.it

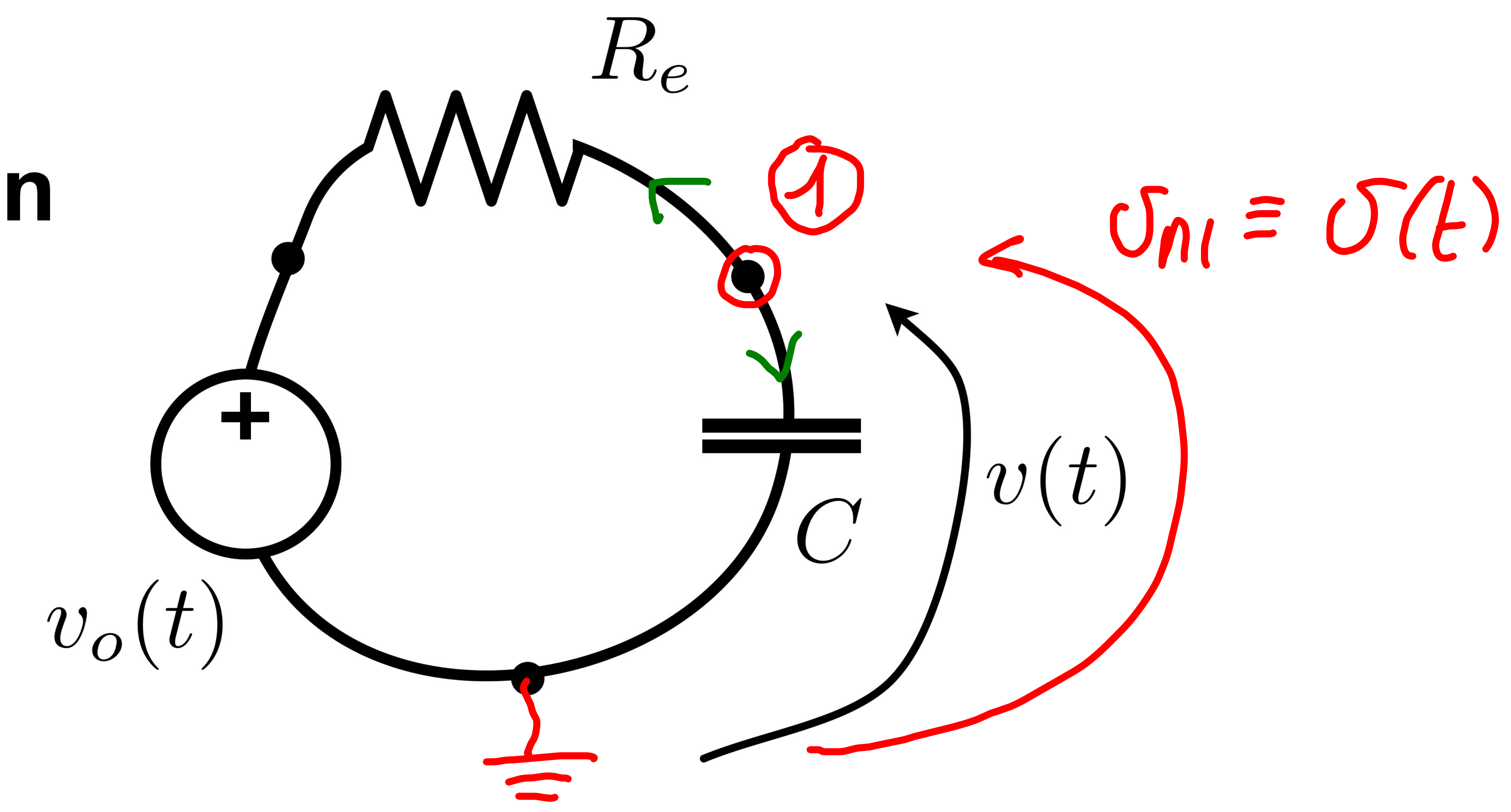
www.emc.polito.it

Part 5: Fundamentals of Dynamic Circuits

1. **Dynamic elements**
2. **Analysis of Dynamic Circuits**
3. **First-Order Dynamic Circuits**
   - RC step response
   - **Circuits with either one capacitor or one inductor, piecewise constant sources and switches**

3. First-order Dynamic Circuits

RC equation



$$G_e\left(v(t) - v_o(t)\right) + C\frac{dv}{dt} = 0$$

$$\frac{dv}{dt} + \frac{G_e}{C} v = \frac{G_e}{C} v_o(t)$$

$$\tau \triangleq \frac{C}{G_e} = R_e C$$

time constant

$$\lambda \triangleq -\frac{1}{\tau} = -\frac{G_e}{C}$$

natural frequency

$$\frac{dv}{dt} + \frac{1}{\tau} v = \frac{1}{\tau} v_o(t)$$

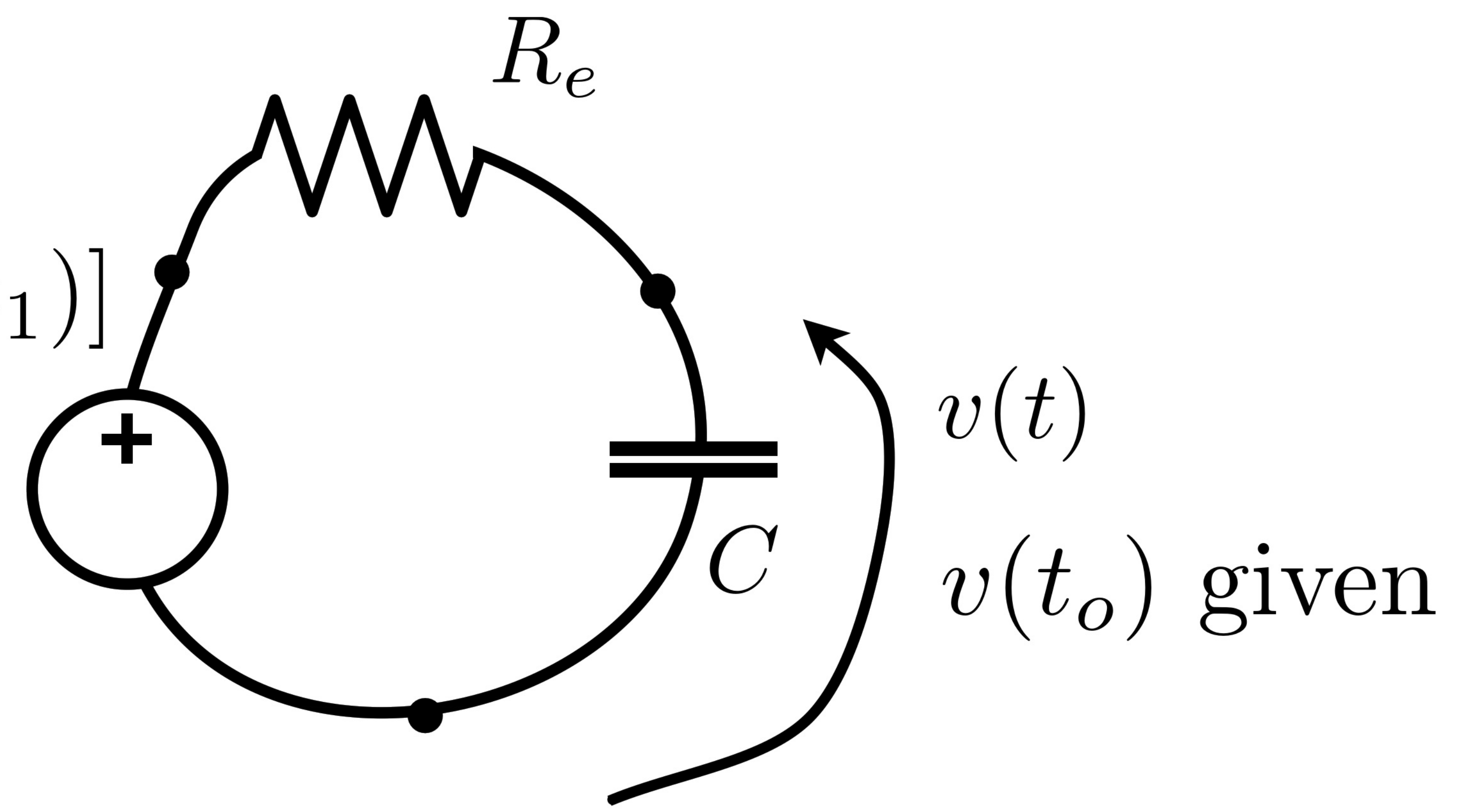
3. First-order Dynamic Circuits, cont'd

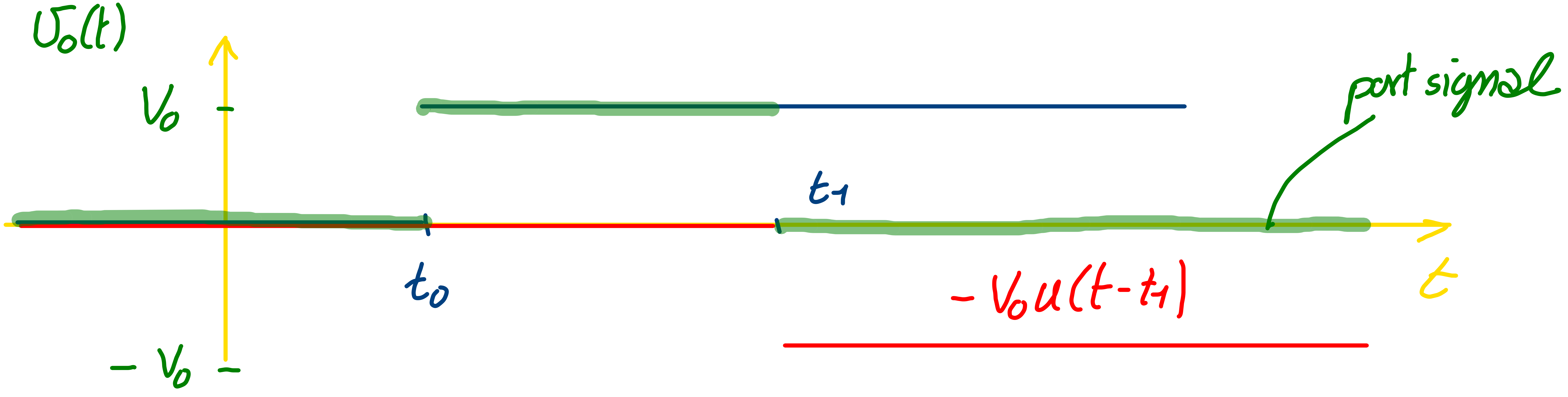
RC response to a port signal for non zero I.C.

$0 \le t_o \le t_1$

$$v_o(t) = V_o[u(t - t_o) - u(t - t_1)]$$

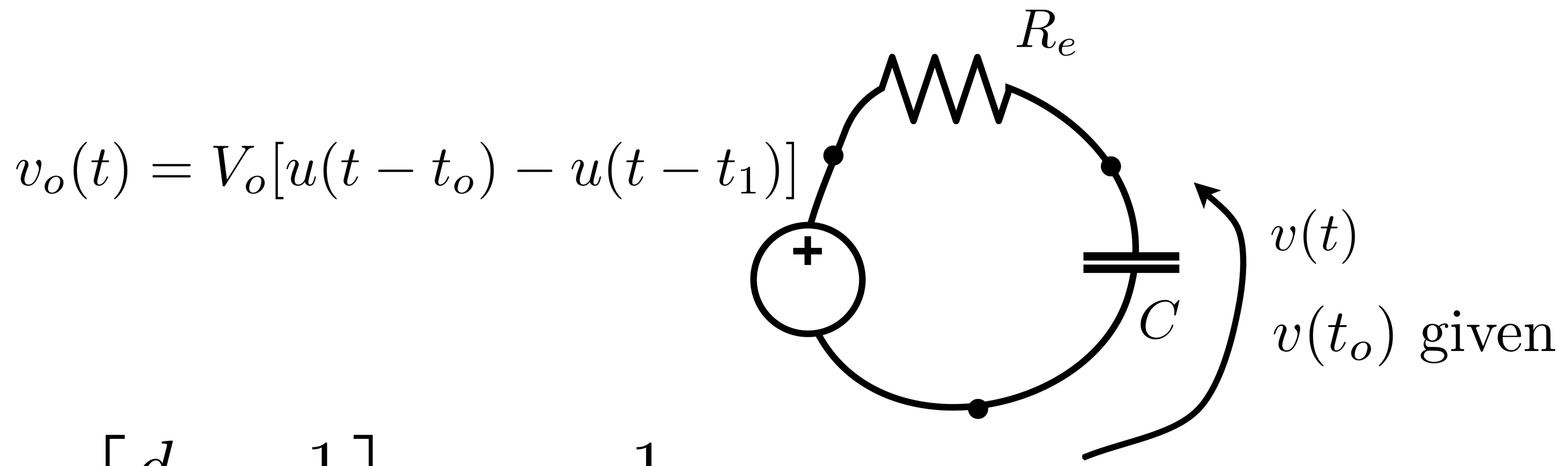
$$u(t-t_o) = \begin{cases} 0 & t-t_o<0 \\ 1 & t-t_o>0 \end{cases}$$





3. First-order Dynamic Circuits, cont'd

RC response to a port signal for non zero I.C.



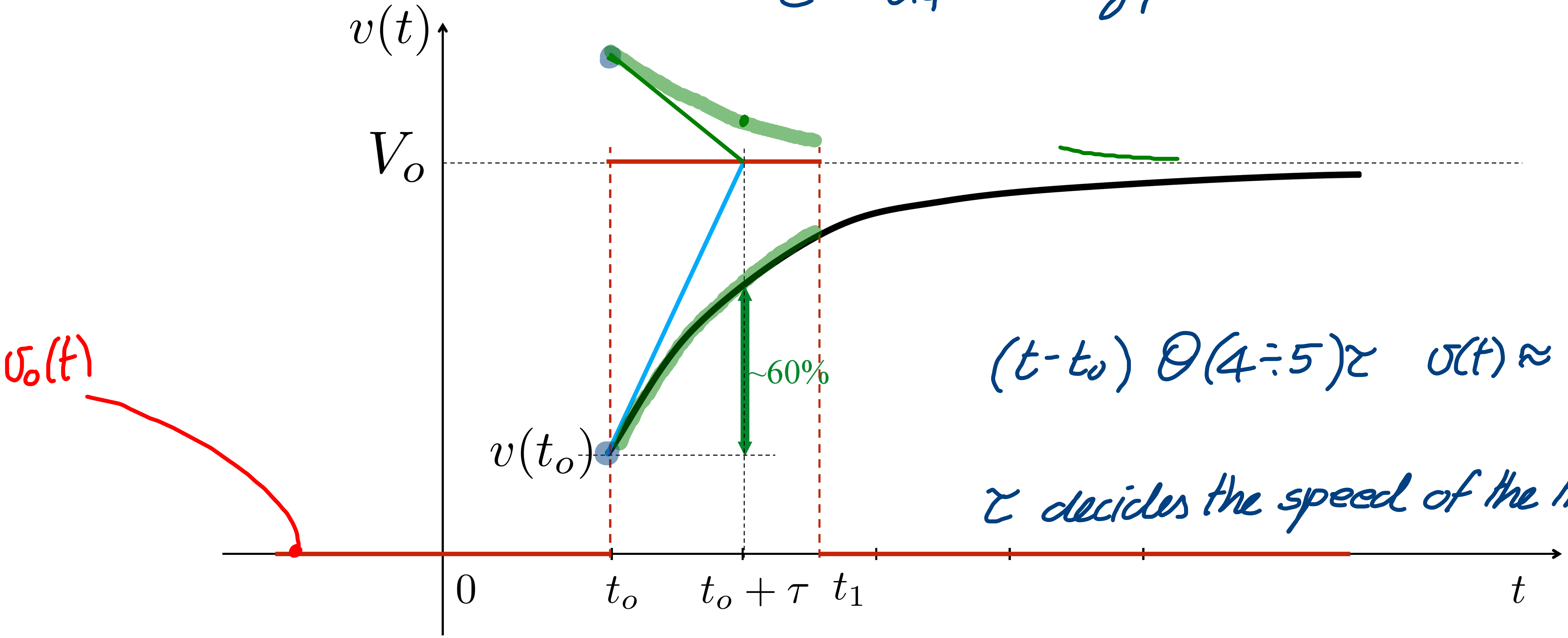
$$\left[\frac{d}{dt} + \frac{1}{\tau}\right] v(t) = \frac{1}{\tau} V_o \quad t \geq t_o, \ v(t_o) \text{ given}$$

$$v(t) = [v(t_o) - V_o] e^{-(t - t_o)/\tau} + V_o, \quad t_o \leq t \leq t_1$$

3. First-order Dynamic Circuits, cont'd

$$v(t) = \overbrace{[v(t_o) - V_o]e^{-(t-t_o)/\tau}}^{\text{transient part}} + \overbrace{V_o}^{\text{final value}}, \qquad t_o \le t \le t_1$$

$e^{-1} \sim 0.4$ crossing point

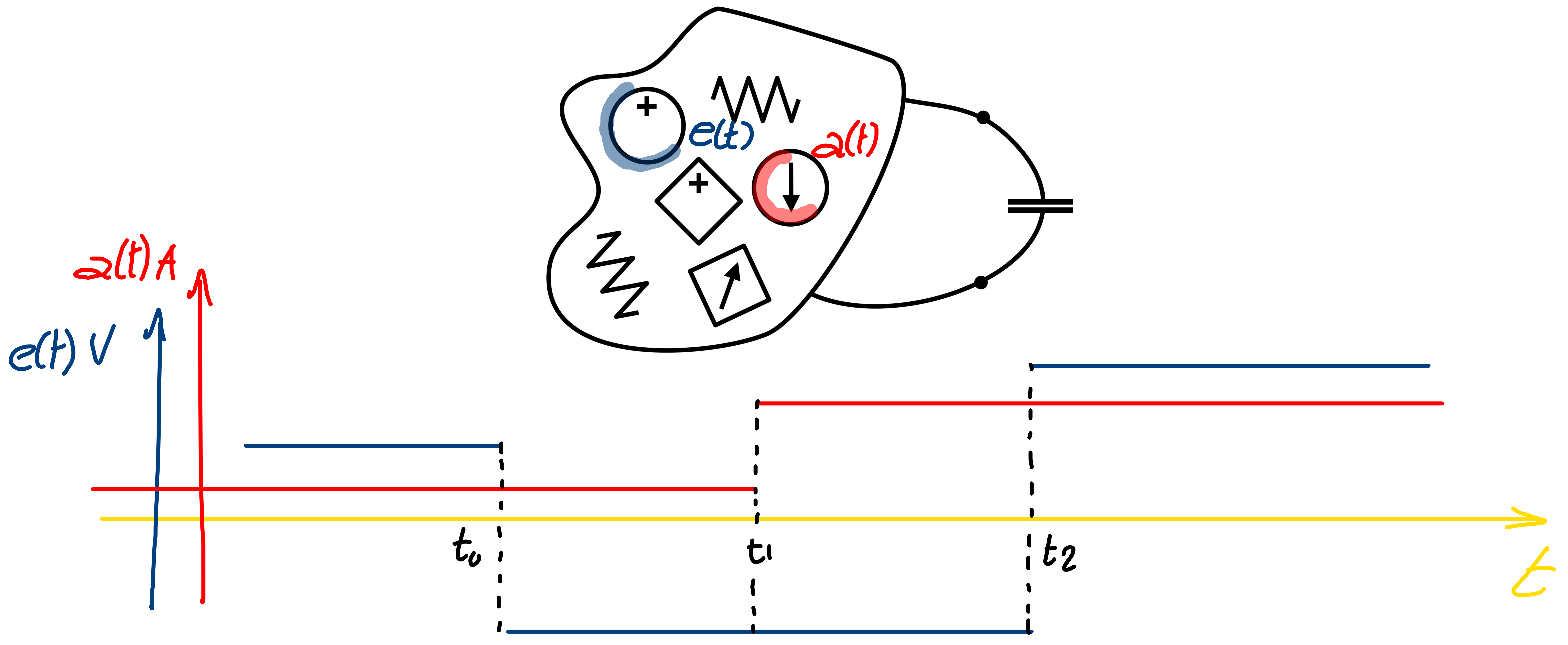


$(t - t_o)\ \Theta(4 \div 5)\tau \quad v(t) \approx V_o$

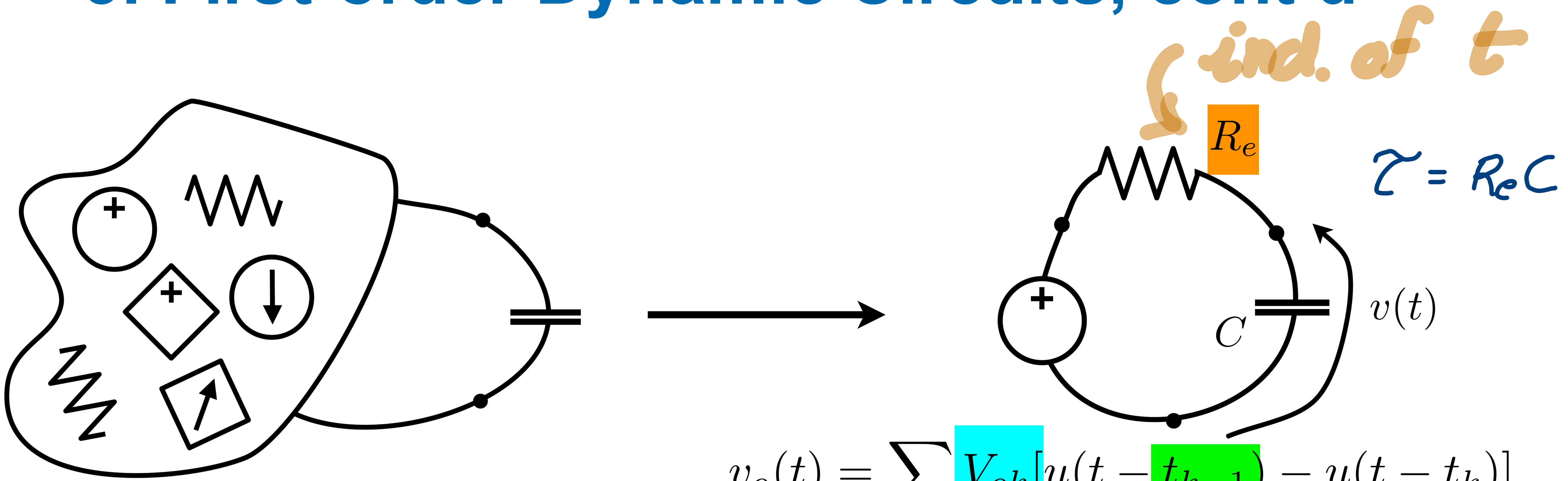
τ decides the speed of the transition

3. First-order Dynamic Circuits, cont'd

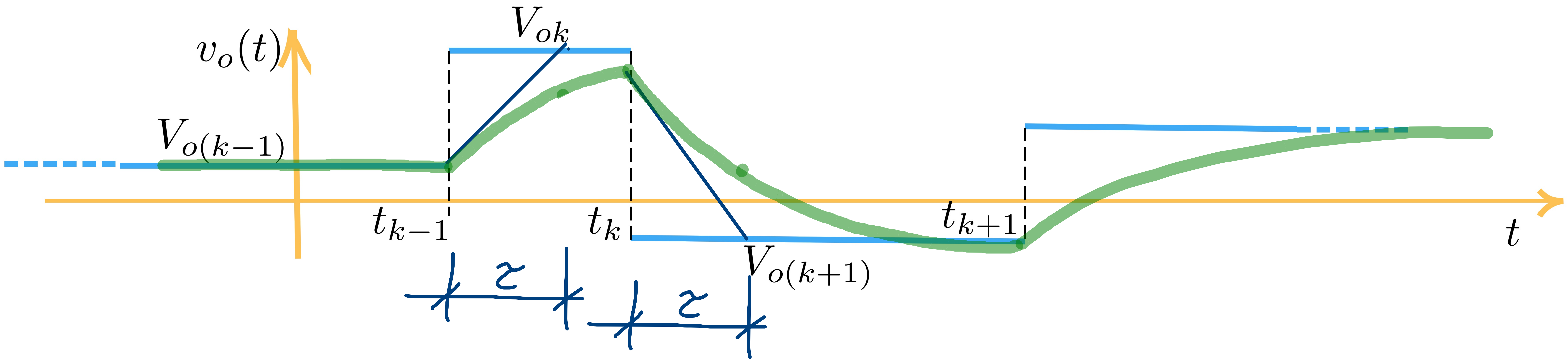
Response of a LTI circuit with one capacitor and piecewise constant sources (discontinuities at t_k, k=0,1,…)



3. First-order Dynamic Circuits, cont'd



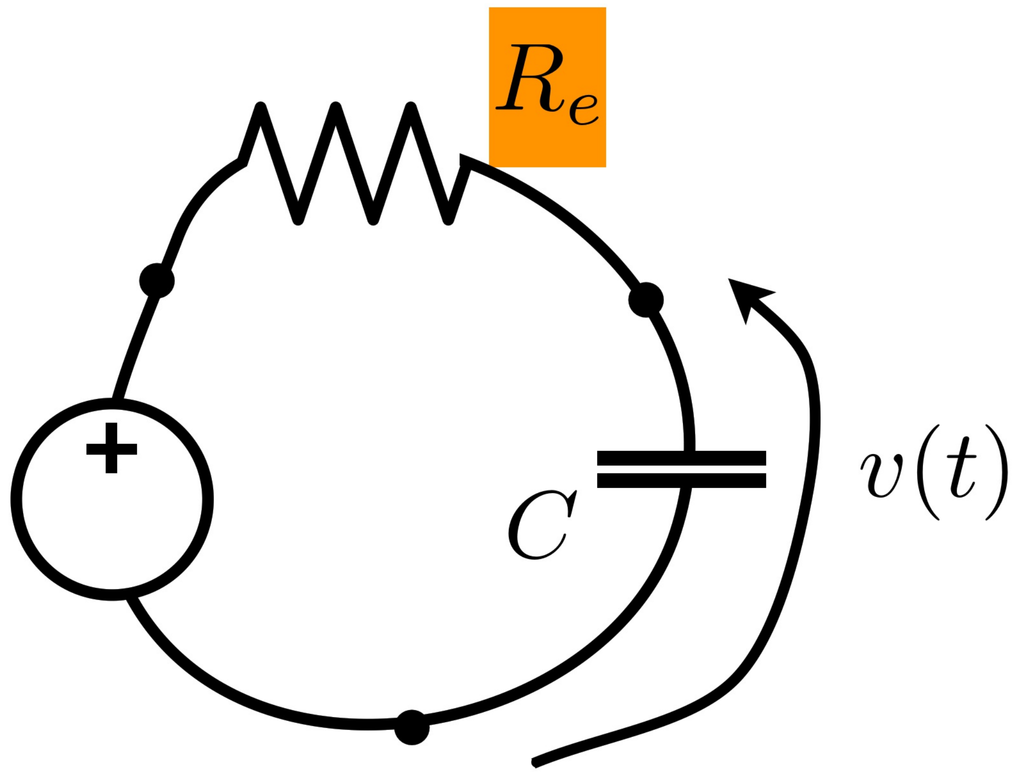
$$v_o(t) = \sum_k V_{ok}[u(t - t_{k-1}) - u(t - t_k)]$$

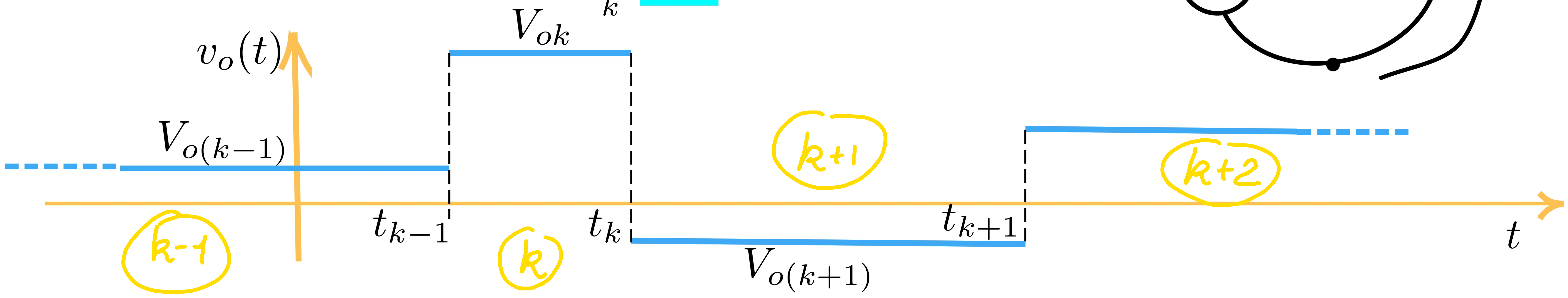


3. First-order Dynamic Circuits, cont’d

Response of a LTI circuit with one capacitor and piecewise constant sources (discontinuities at t_k, k=0,1,…)

$$v_o(t) = \sum_k V_{ok}[u(t - t_{k-1}) - u(t - t_k)]$$





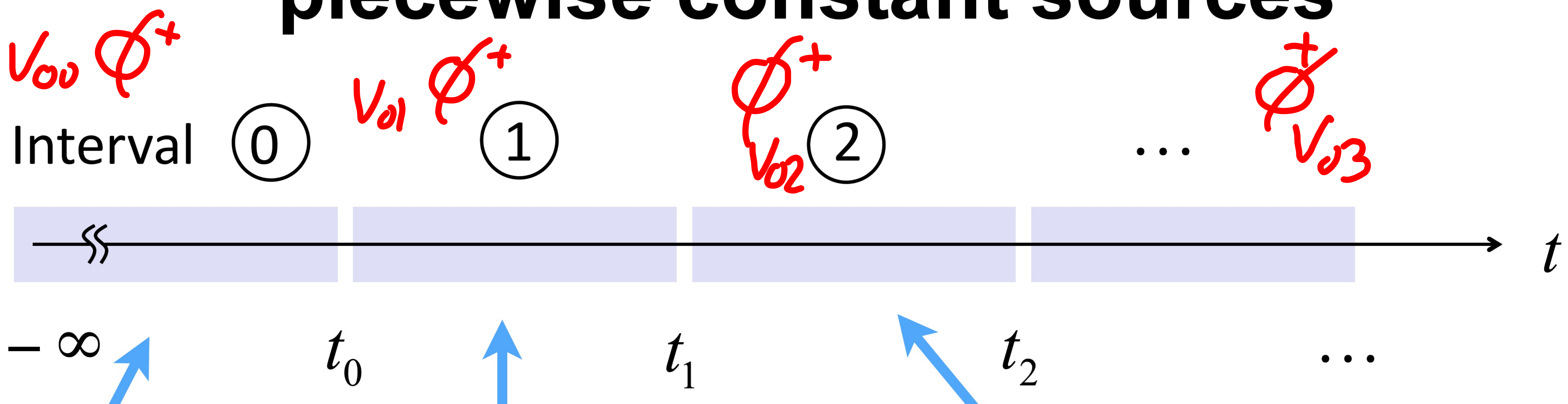
$$v_{(k)} = [v_{(k-1)}(t_{k-1}) - V_{ok}]e^{-(t-t_{k-1})/\tau} + V_{ok}, \quad t_{k-1} \le t \le t_k$$

k-th interval

$$\tau = R_e C$$

3. First-order Dynamic Circuits, cont'd

Response of a LTI circuit with one capacitor and piecewise constant sources



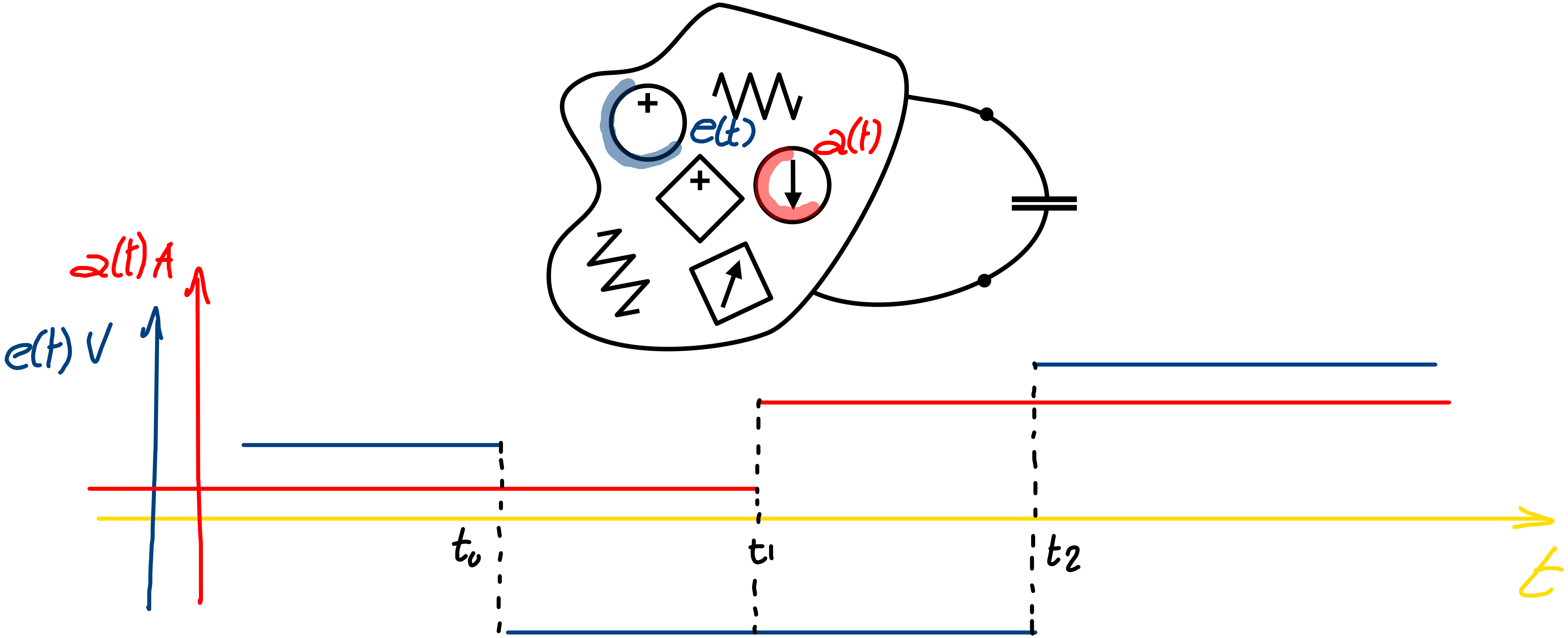
$$v_{(o)}(t) = V_{oo}$$

$$v_{(1)}(t) = [V_{oo} - V_{o1}]e^{-(t-t_o)/\tau} + V_{o1}$$

$$v_{(2)}(t) = [v_{(1)}(t_1) - V_{o2}]e^{-(t-t_1)/\tau} + V_{o2}$$

3. First-order Dynamic Circuits, cont’d

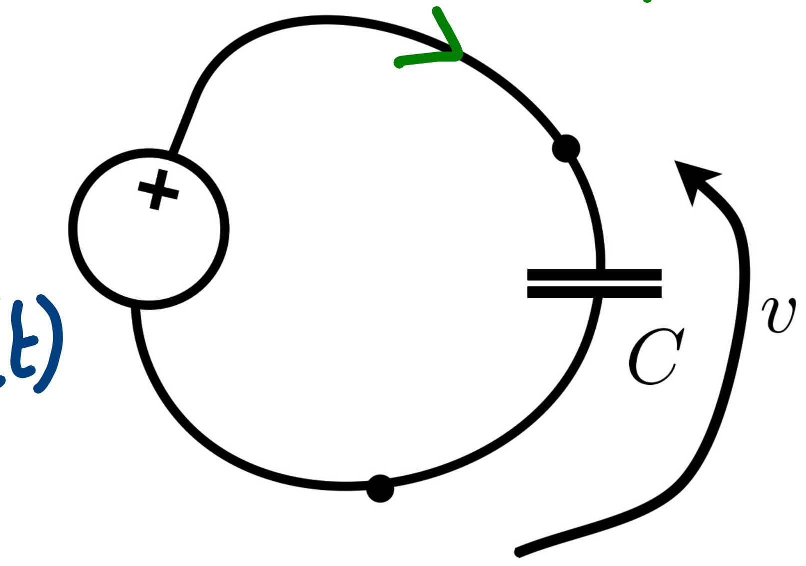
Response of a LTI circuit with one capacitor and piecewise constant sources (discontinuities at t_k, k=0,1,…)



3. First-order Dynamic Circuits, cont'd

Step response of a LTI circuit with one capacitor and piecewise constant sources: special cases

$$R_e = 0$$

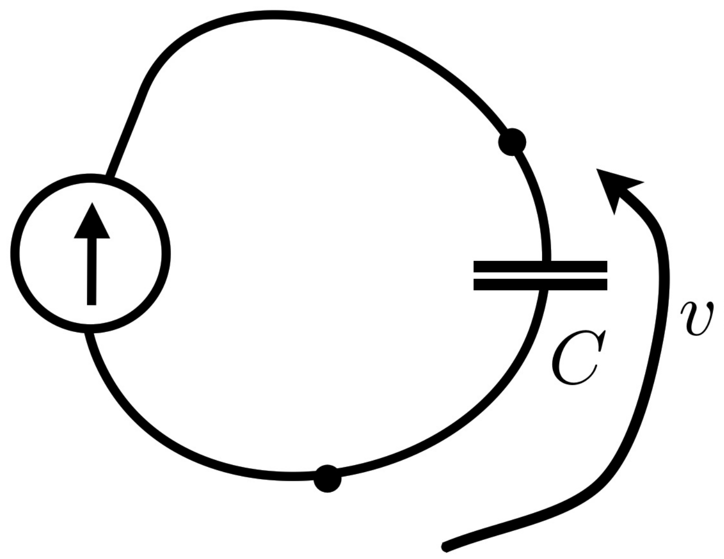


$v_0(t)$

$i(t) = C\,\frac{dv_0}{dt}$

$v(t) = v_0(t)$

$$G_e = 0$$

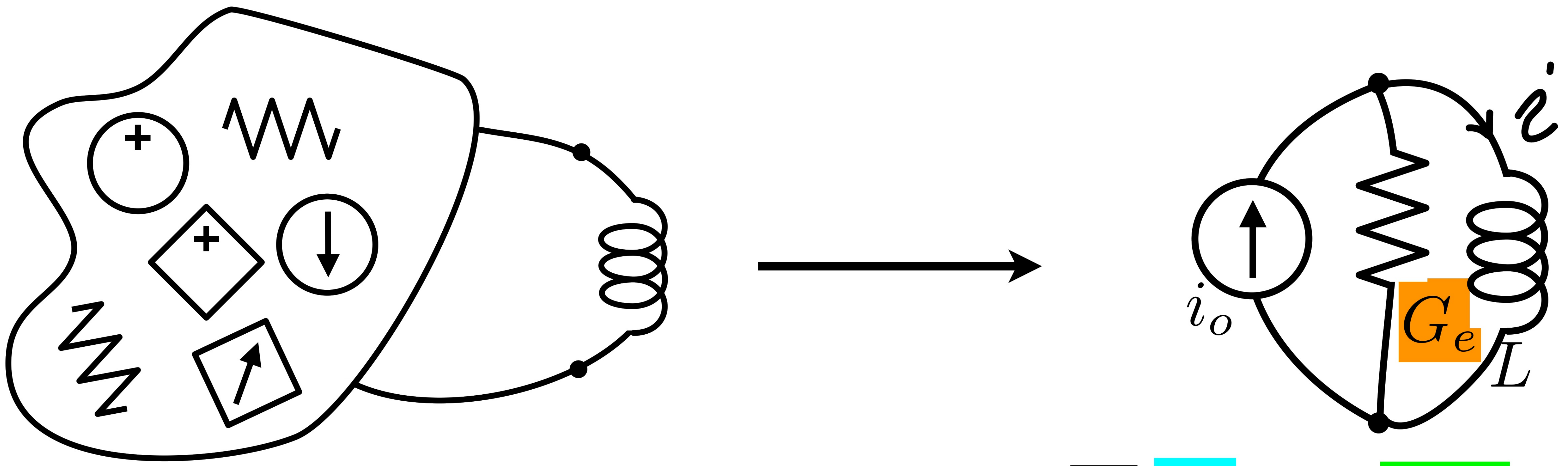


$i_0(t)$

$v(t) = \frac{1}{C}\int_{-\infty}^{t} i_0(t')\,dt'$

3. First-order Dynamic Circuits, cont'd

Response of a LTI circuit with one inductor and piecewise constant sources (discontinuities at t_k, k=0,1,…)



$$i_o(t) = \sum_k I_{ok}[u(t - t_{k-1}) - u(t - t_k)]$$

$$i_{(k)} = [i_{(k-1)}(t_{k-1}) - I_{ok}]e^{-(t-t_{k-1})/\tau} + I_{ok}, \quad t_{k-1} \le t \le t_k$$

$$\tau = G_e L$$

3. First-order Dynamic Circuits, cont’d

RECIPE

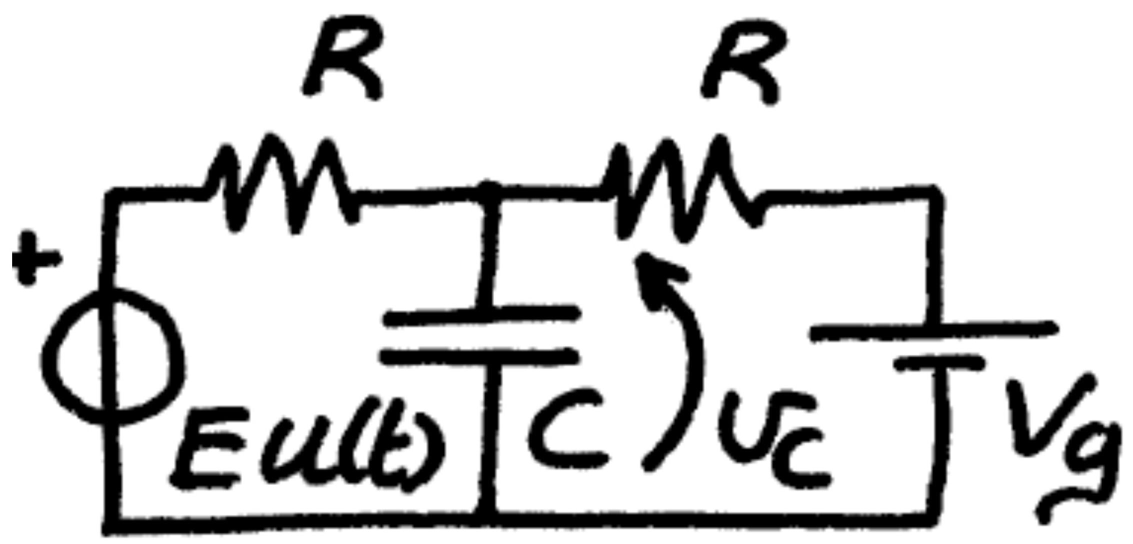
1. **Build the series (parallel) equivalent with piecewise constant voltage (current) source of the resistive element driving the capacitor (inductor)**

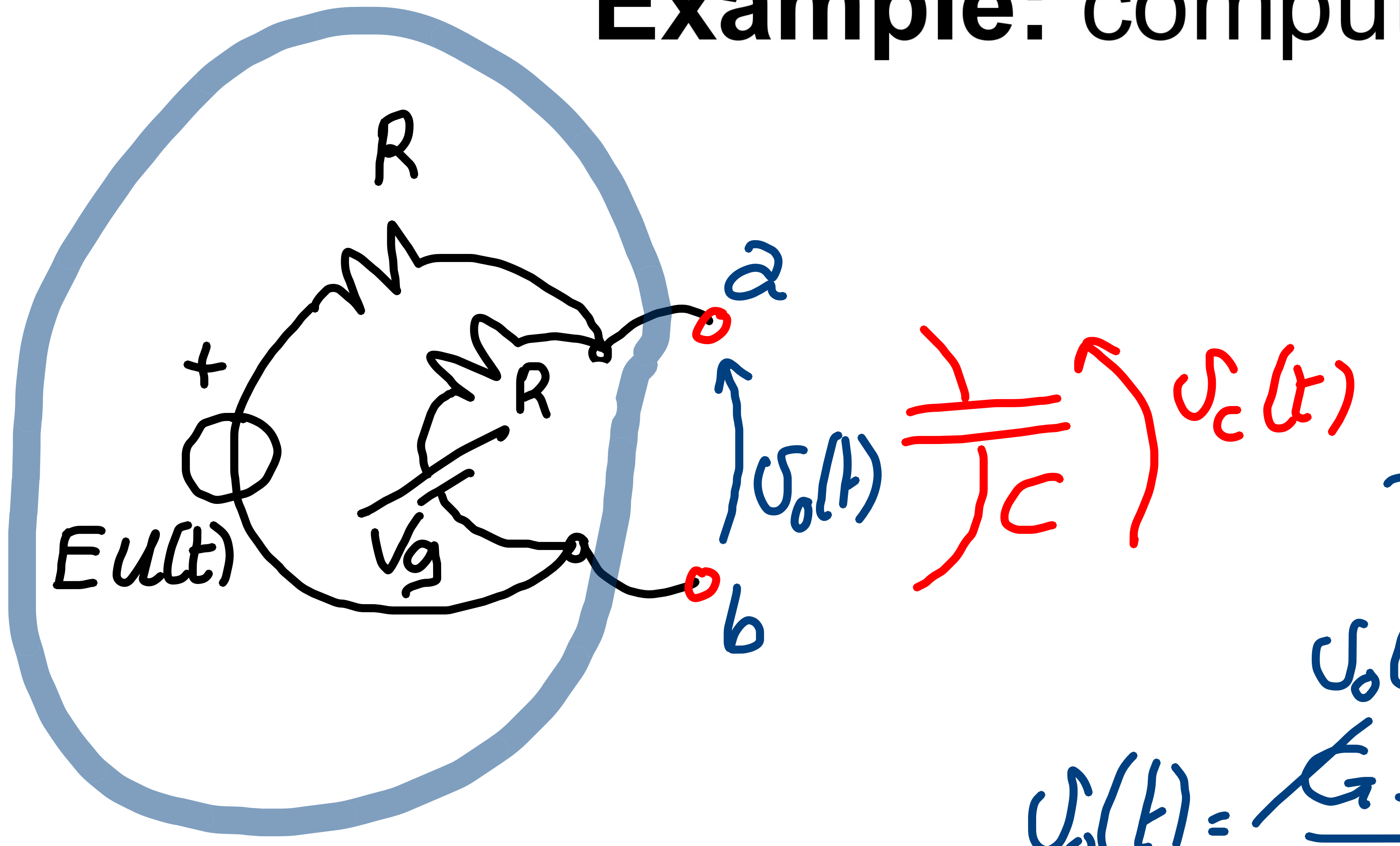
2. **Plot the piecewise constant source, dividing the time axis in intervals with constant source and exponential solution**

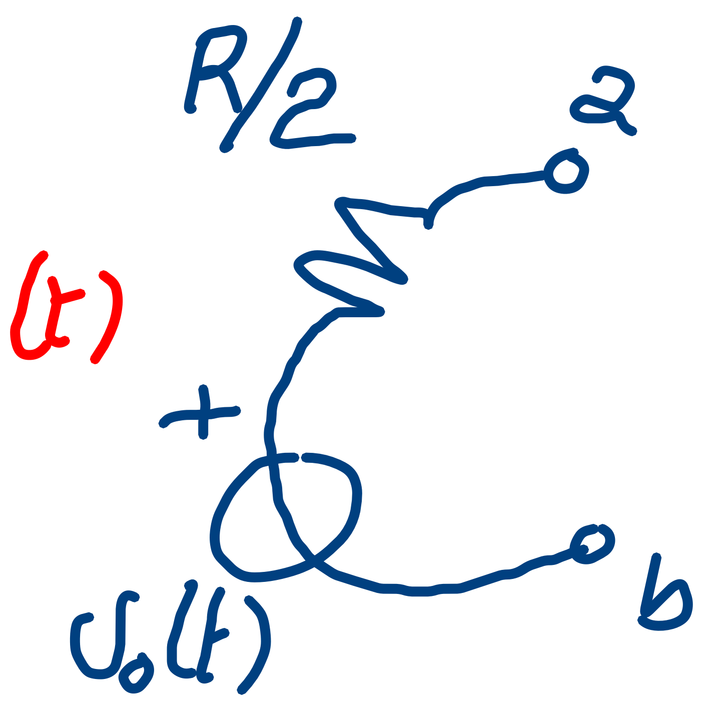
3. **For the** ***k*****-th time interval, write the solution by using $\tau, V_{ok}(I_{ok})$ and the proper initial time and initial value**

3. First-order Dynamic Circuits, cont’d

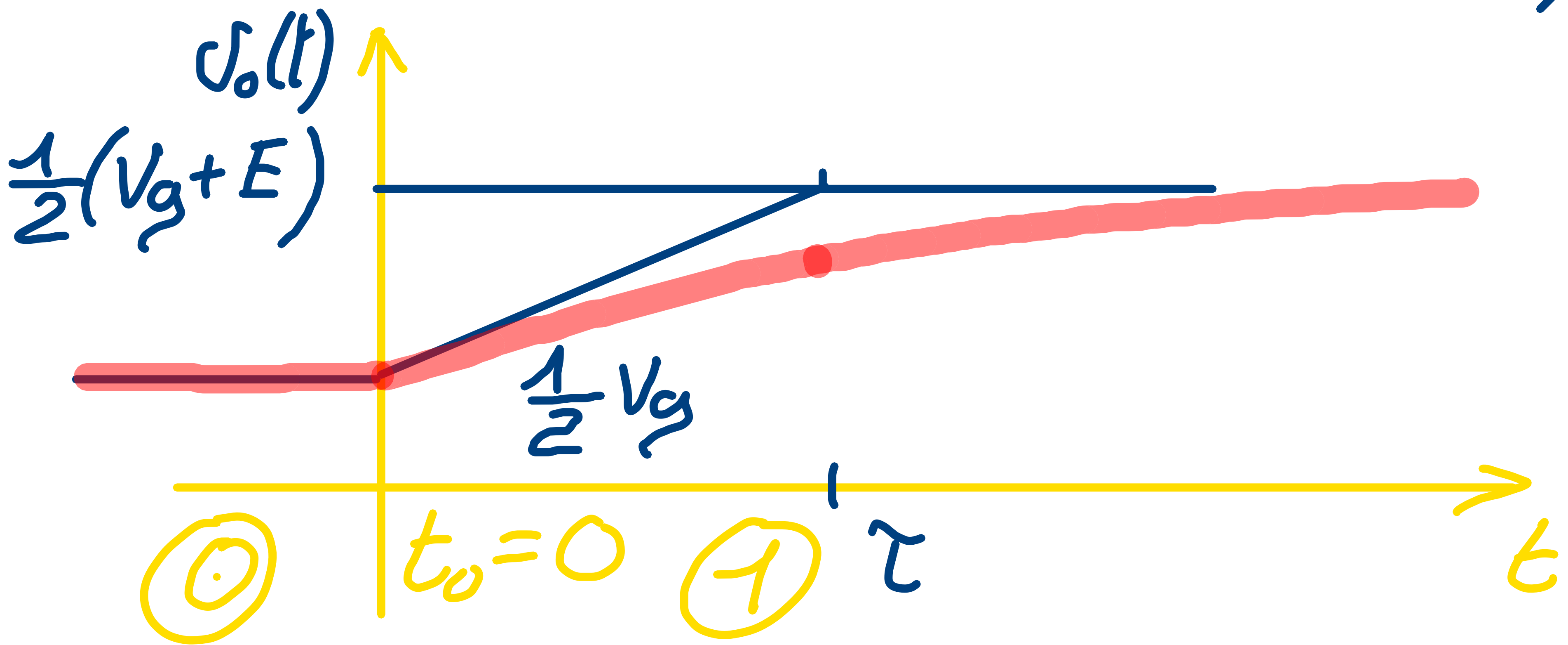
Example: compute and plot $v_c(t)$







$$v_0(t) = \frac{\cancel{G} E u(t) + \cancel{G} V_g}{2\cancel{G}} = \frac{1}{2} E u(t) + \frac{1}{2} V_g$$

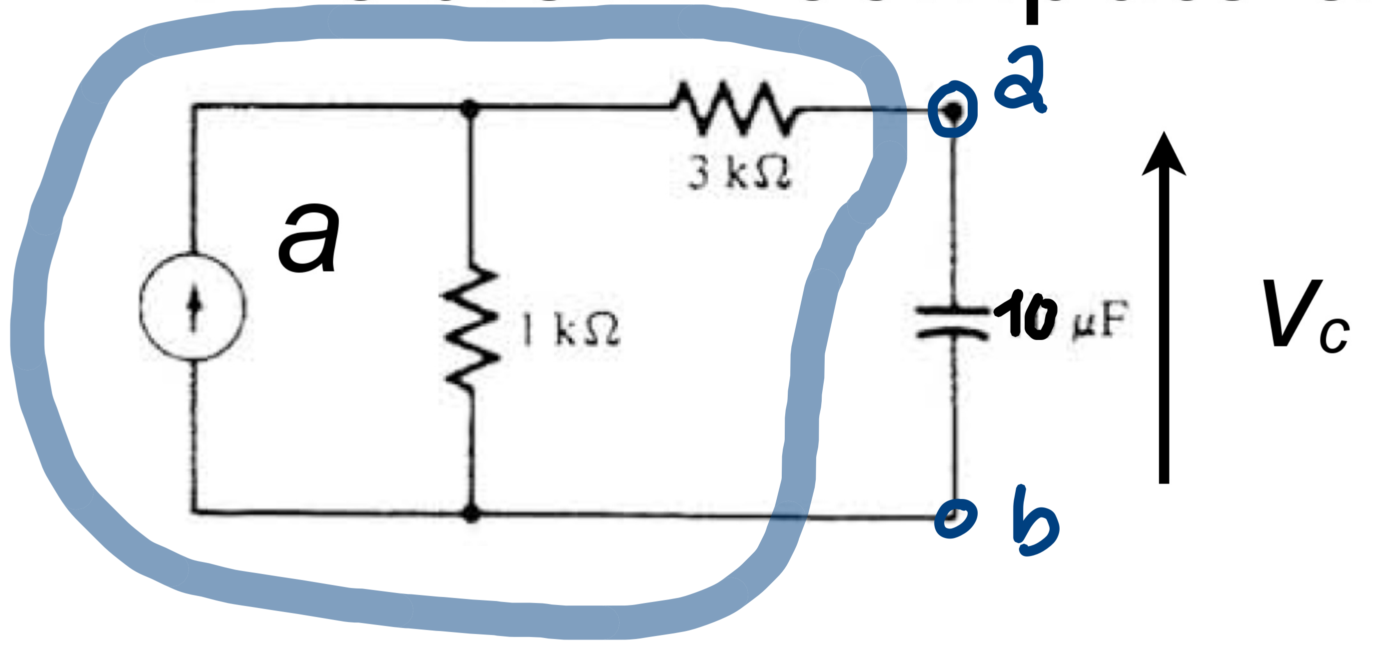


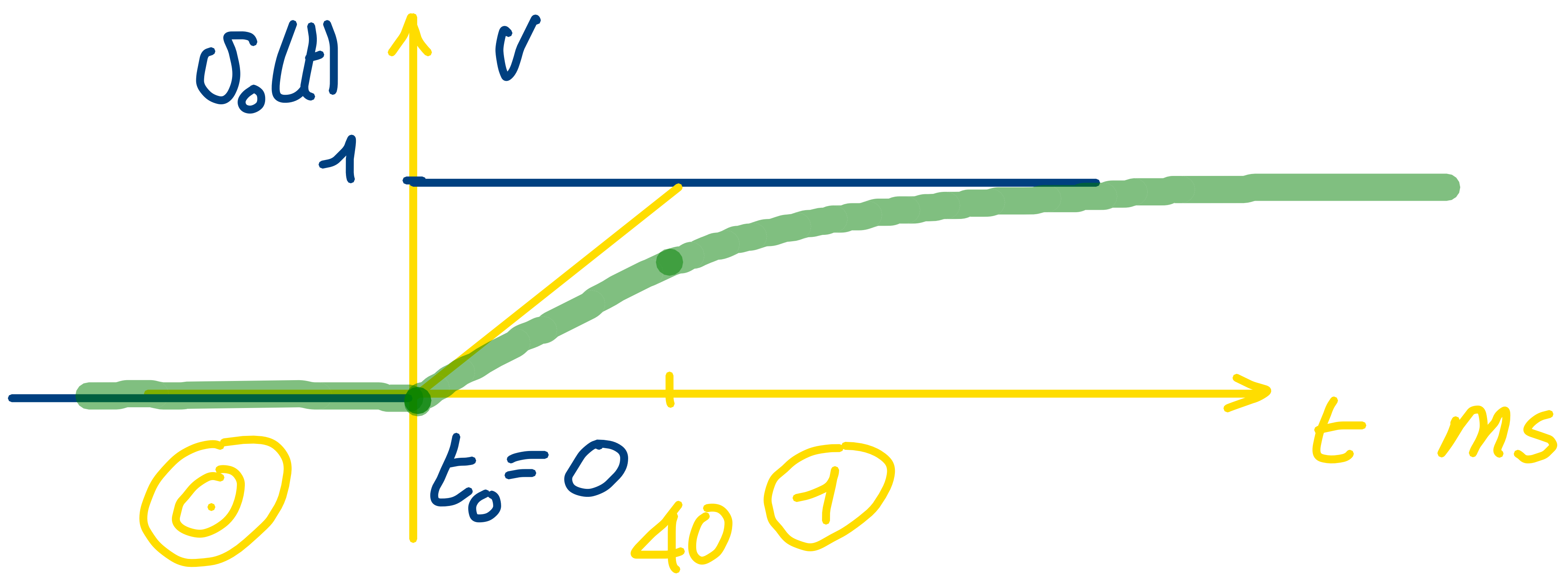
$$\tau = \frac{R}{2} C$$

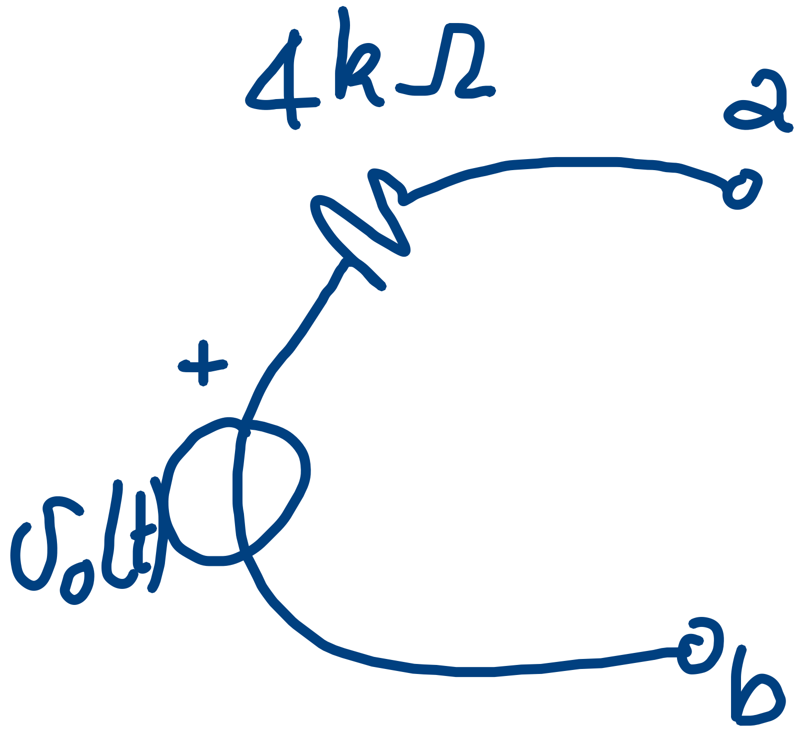
$$v_c(t) = \begin{cases} \frac{1}{2} V_g & t \in 0 \\ \left(\cancel{\tfrac{1}{2} V_g} - \tfrac{1}{2}(\cancel{V_g} + E)\right) e^{-t/\tau} + \tfrac{1}{2}(V_g + E) & t \in 1 \end{cases}$$

3. First-order Dynamic Circuits, cont’d

Problem: compute and plot $v_c(t)$ for $a = u(t)$ mA







$$\tau = 4\,10^{3} \cdot 10 \cdot 10^{-6} = 4 \cdot 10^{-2}\,s = 40\,ms$$

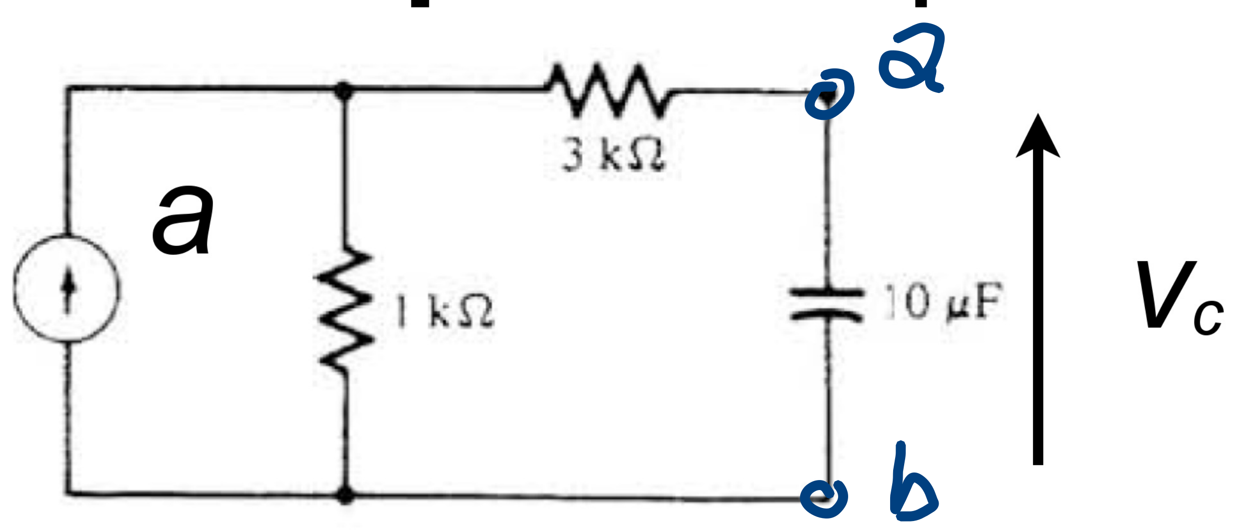
$$v_c(t) = \begin{cases} 0 & t<0 \quad (\text{interval } 0) \\ (0-1)e^{-t/40}+1 & t>0 \quad (\text{interval } 1) \end{cases}$$

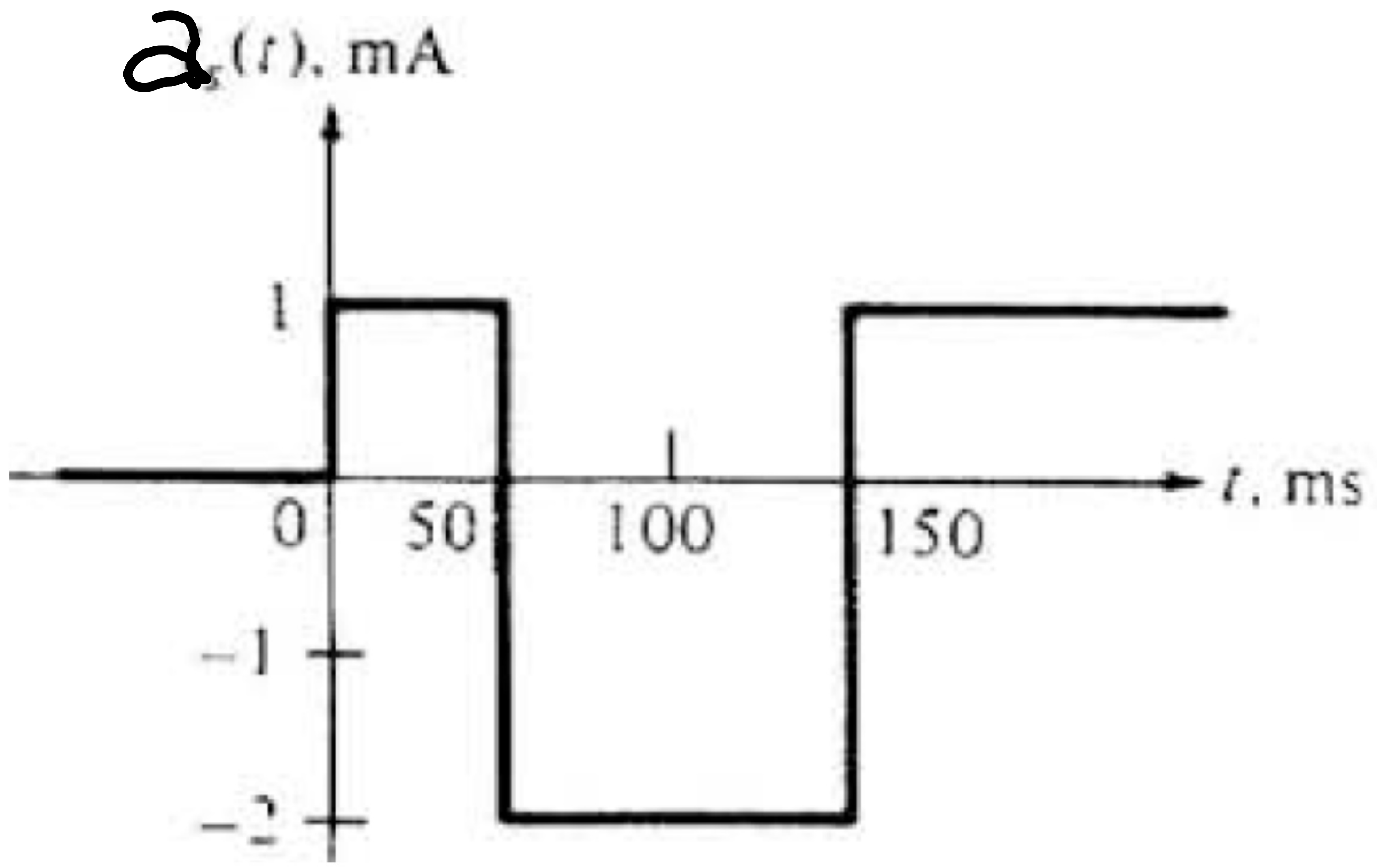
V, ms

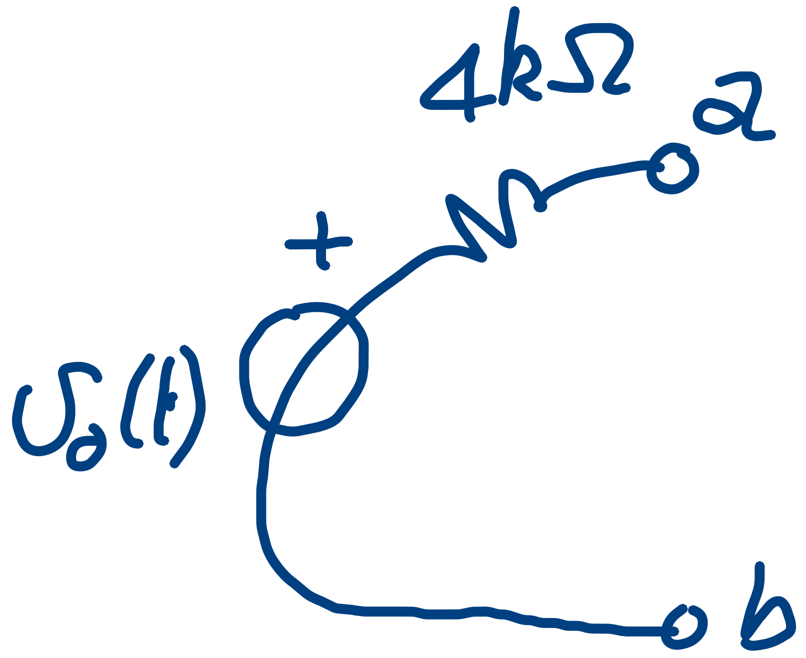
$$v_0(t) = 10^{3} \cdot 10^{-3} u(t) = u(t) \quad V, s$$

3. First-order Dynamic Circuits, cont'd

Example: compute and plot $v_c(t)$

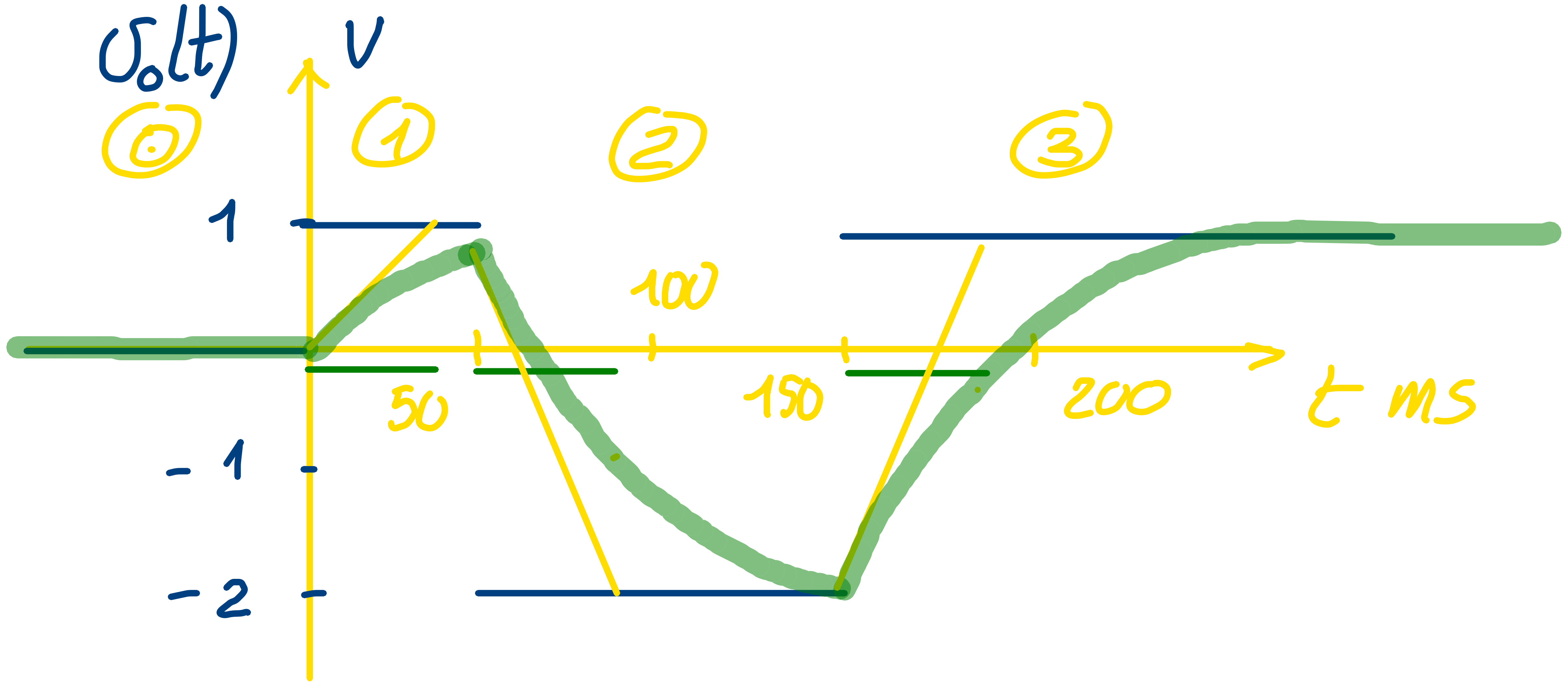


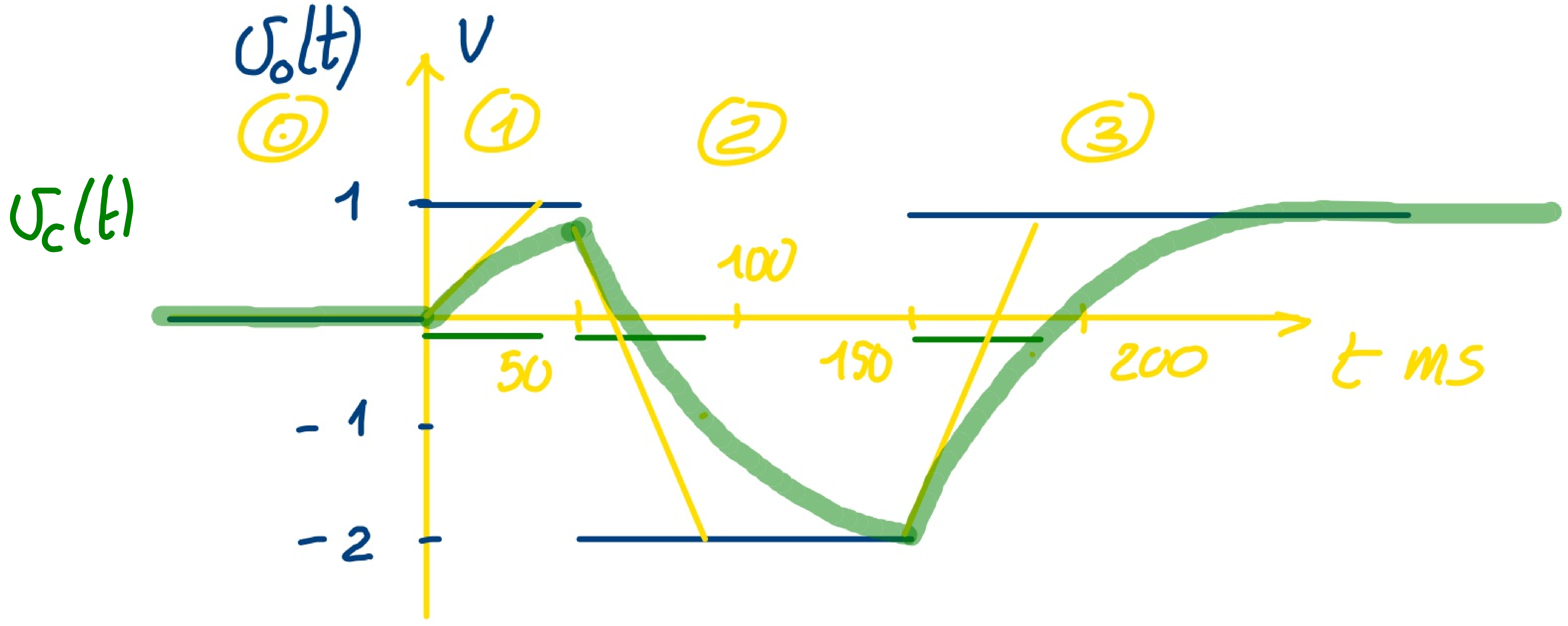




$\tau = 40\text{ms}$

$$v_o(t) = \begin{cases} 0 & t < 0 \\ 1V & 0 < t < 50\text{ms} \\ -2V & 50 < t < 150\text{ms} \\ 1V & t > 150\text{ms} \end{cases}$$





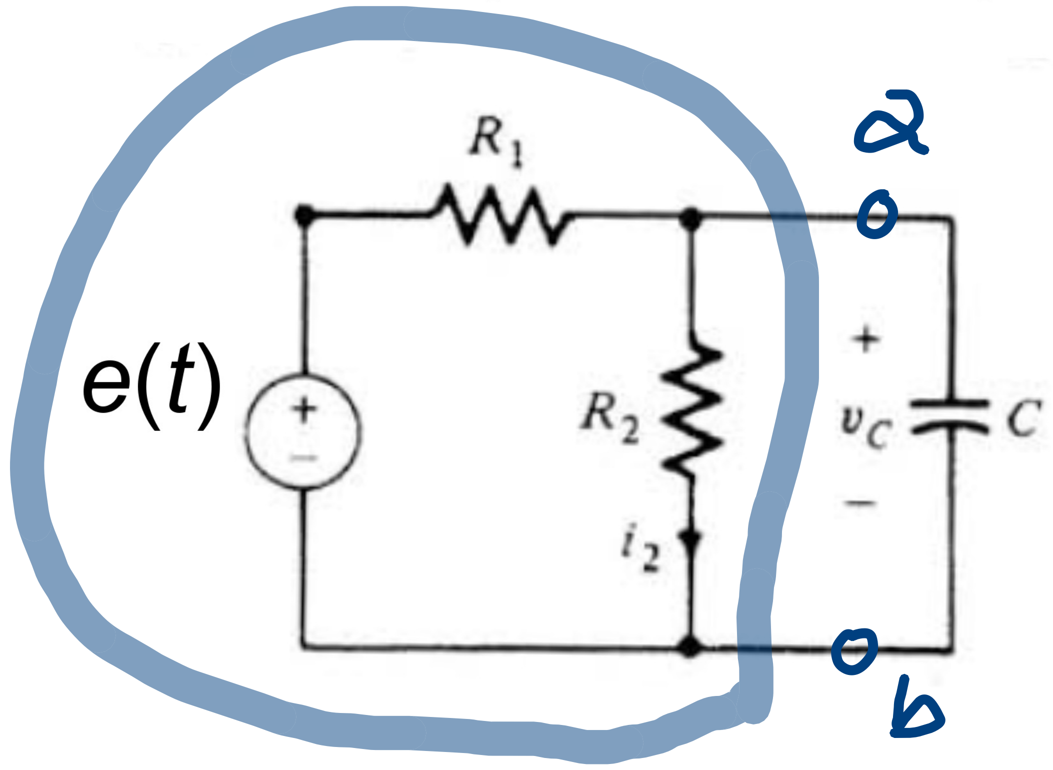
$$v_c(t) = \begin{cases} 0 & t \le 0 \\ (0-1)e^{-t/40}+1 & 0 \le t \le 50\,ms \\ \left(\left(-e^{-5/4}+1\right)-(-2)\right)e^{-\frac{t-50}{40}}-2 & 50\,ms \le t \le 150\,ms \\ \left(v_{(2)}(150)-1\right)e^{-\frac{t-150}{40}}+1 & t \ge 150\,ms \end{cases}$$

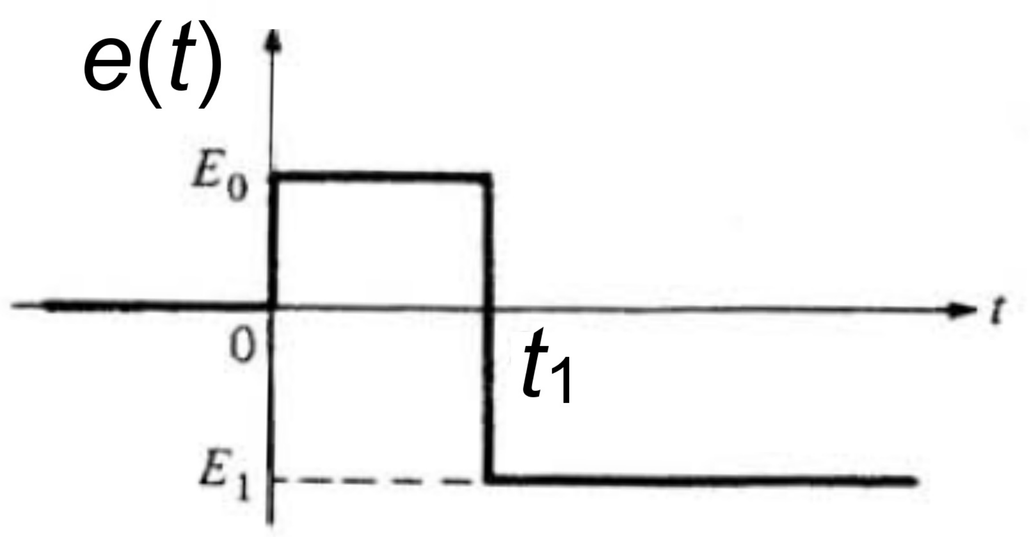
$v_{(2)}$

V, ms

3. First-order Dynamic Circuits, cont'd

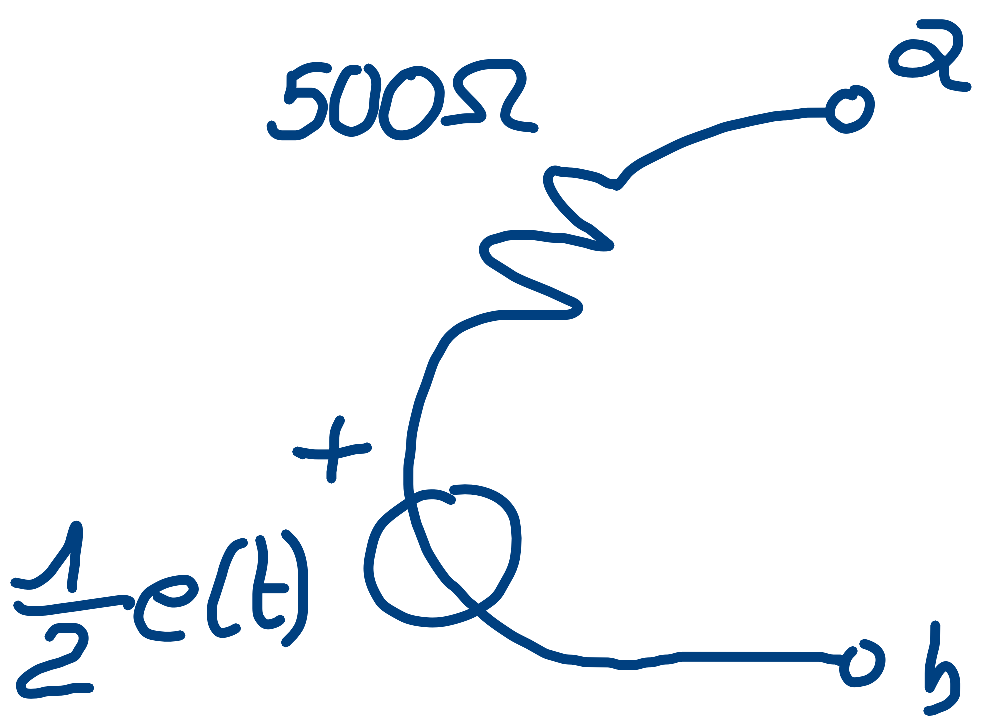
Problem: R_1=R_2= 1kΩ, C=4μF, E_o=2V, E_1=-3V and t_1=3ms, compute and plot $v_c(t)$



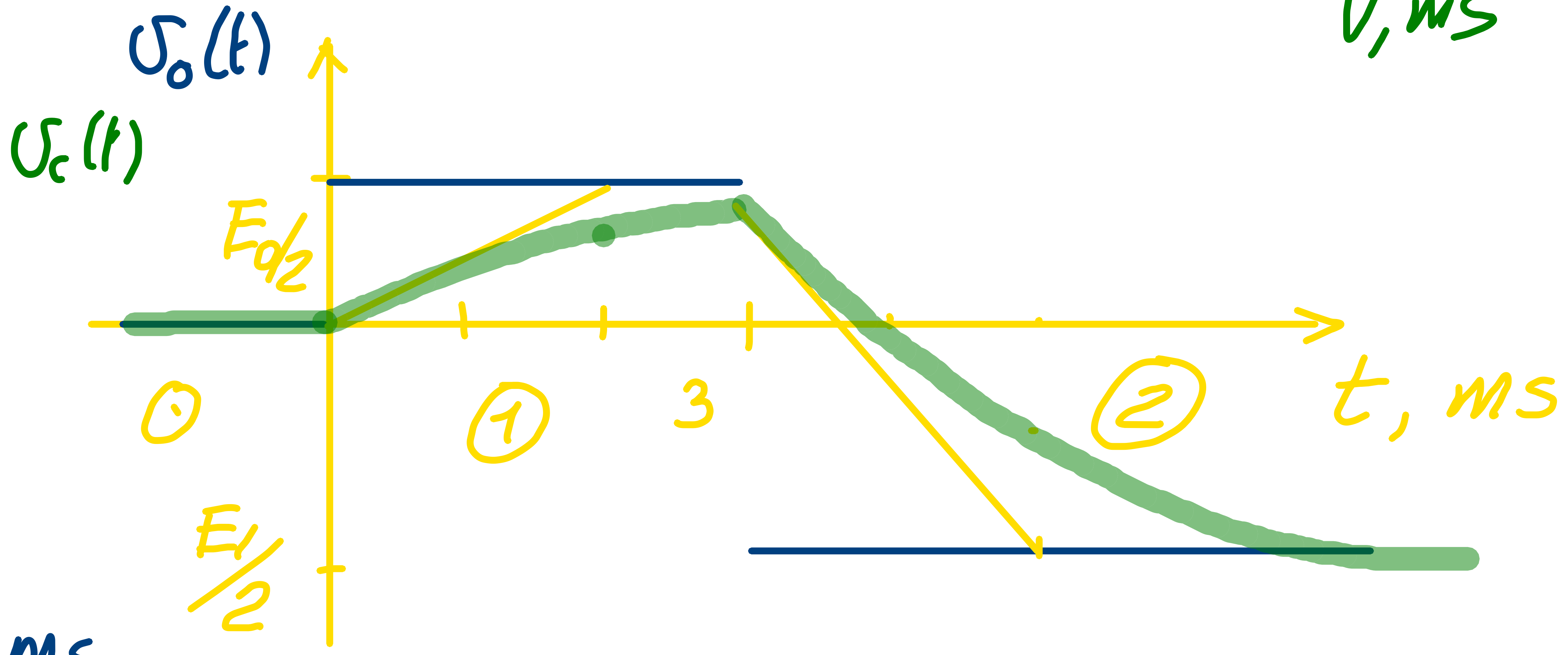


$$v_c(t) = \begin{cases} 0 & t \in (0) \\ \left(0 - \frac{E_0}{2}\right) e^{-t/2} + \frac{E_0}{2} & t \in (1) \\ \left(\frac{E_0}{2}\left(1 - e^{-3/2}\right) - \frac{E_1}{2}\right) e^{-\frac{t-3}{2}} + \frac{E_1}{2} & t \in (2) \end{cases}$$

V, ms

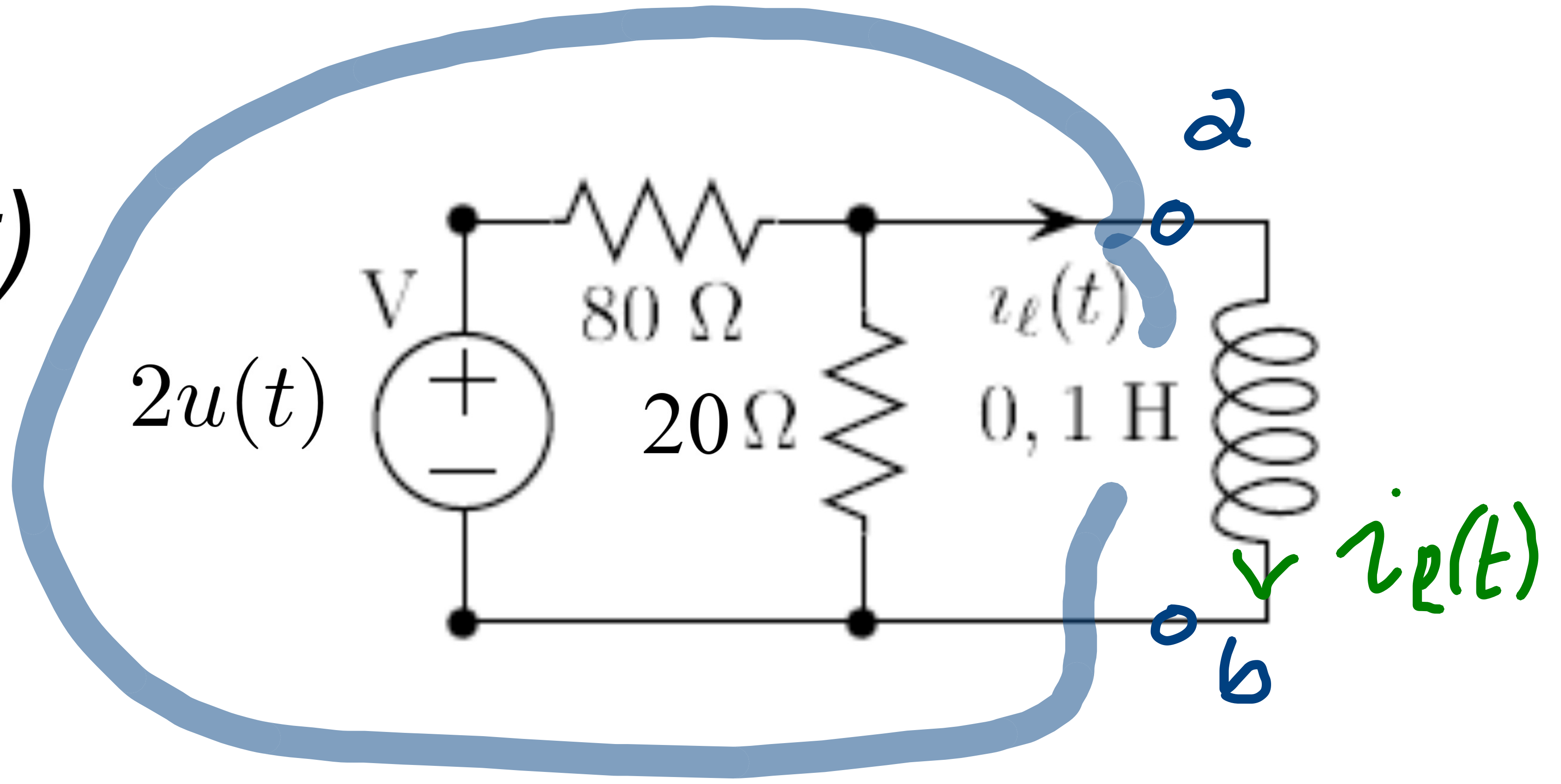


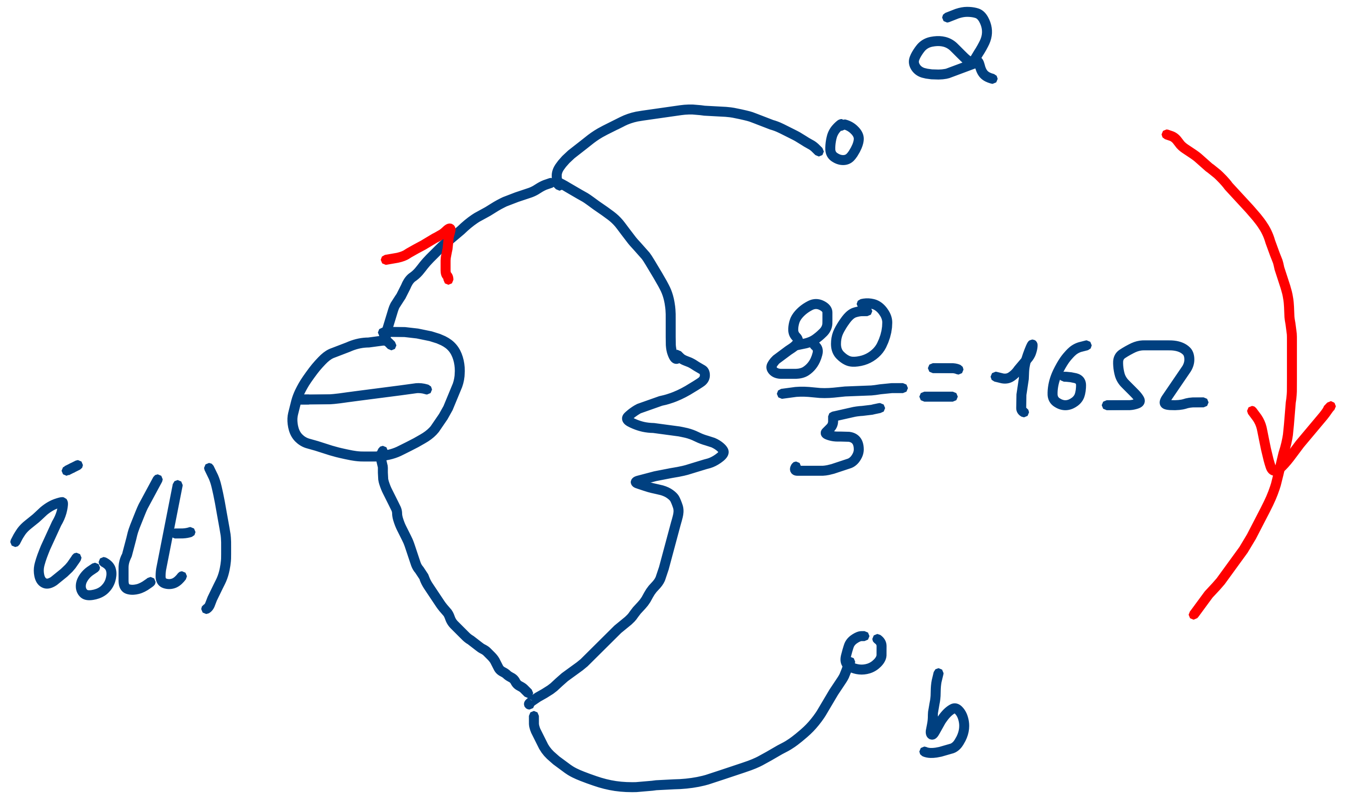
$$\tau = 500 \cdot 4\,10^{-6} = 2\,ms$$



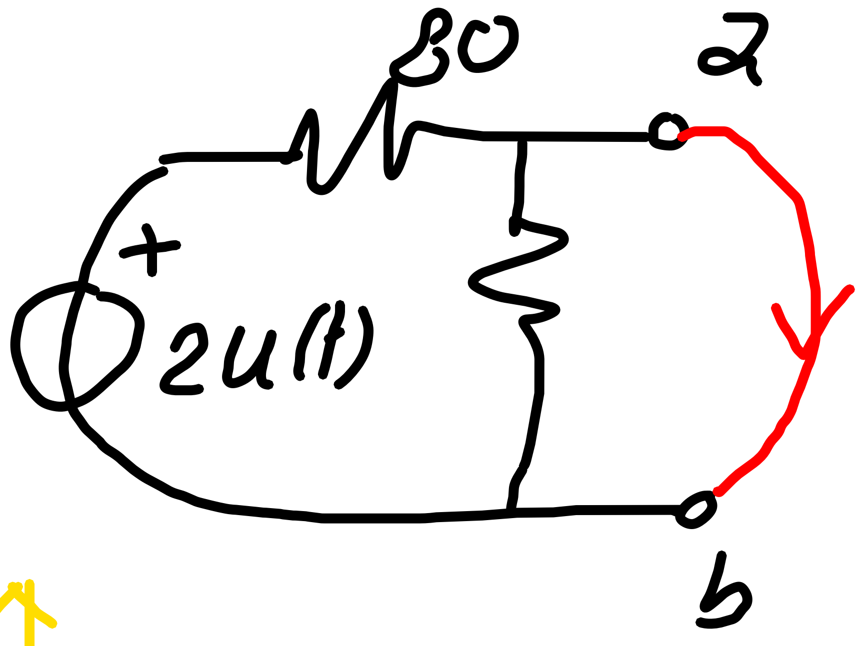
3. First-order Dynamic Circuits, cont'd

Example: compute and plot $i_L(t)$

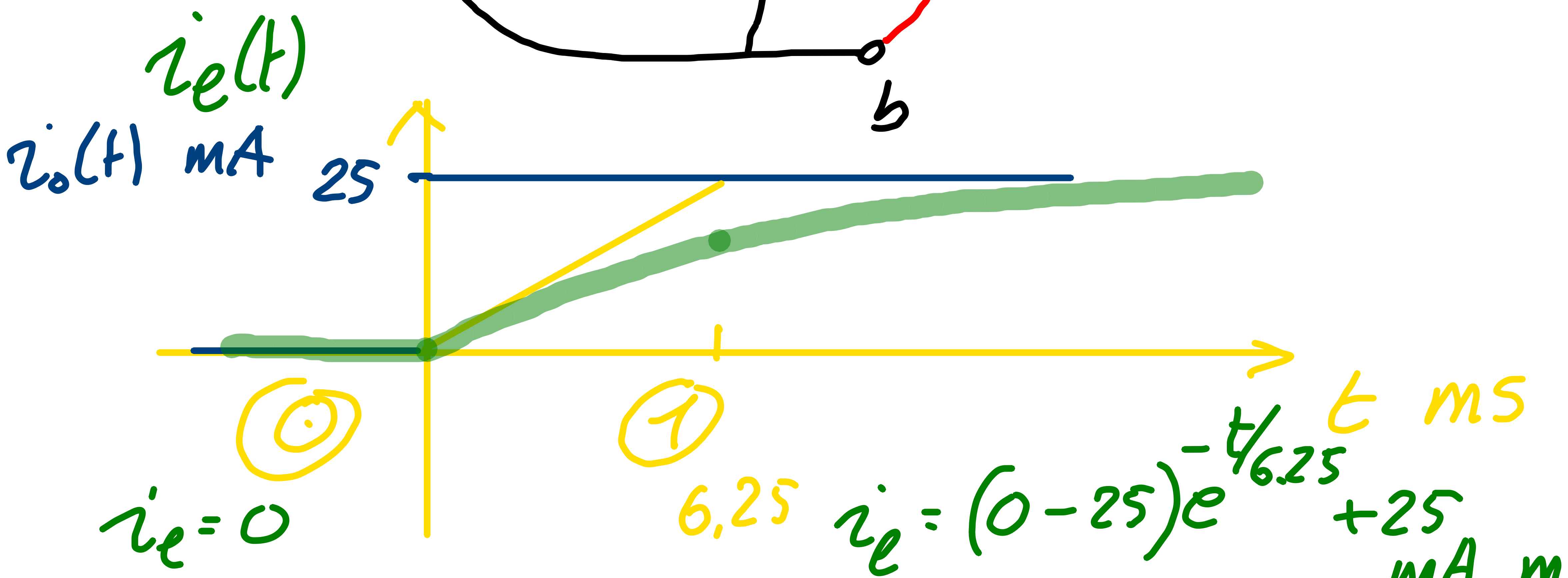




$$\tau = \frac{0.1}{16} = 6.25 \text{ ms}$$



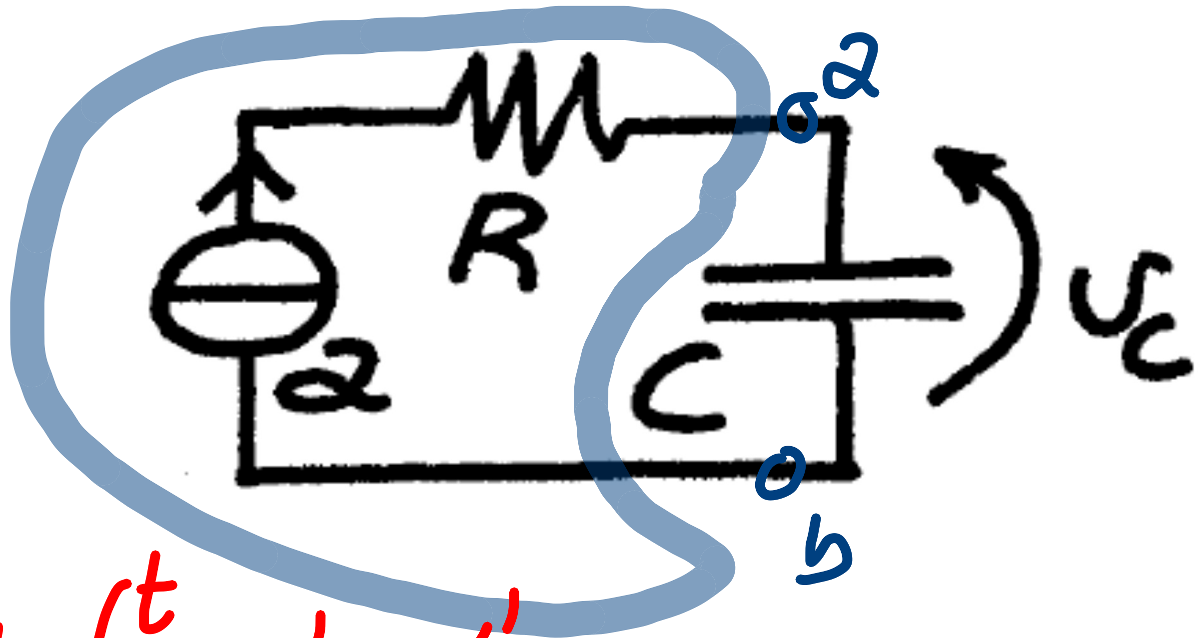
$$i_0(t) = \frac{2}{80}u(t)$$
$$= \frac{1}{40}u(t) = \frac{100}{40}10^{-2}u(t)$$
$$= 2.5\,10^{-2}u(t)$$
$$= 25\,u(t) \text{ mA, s}$$

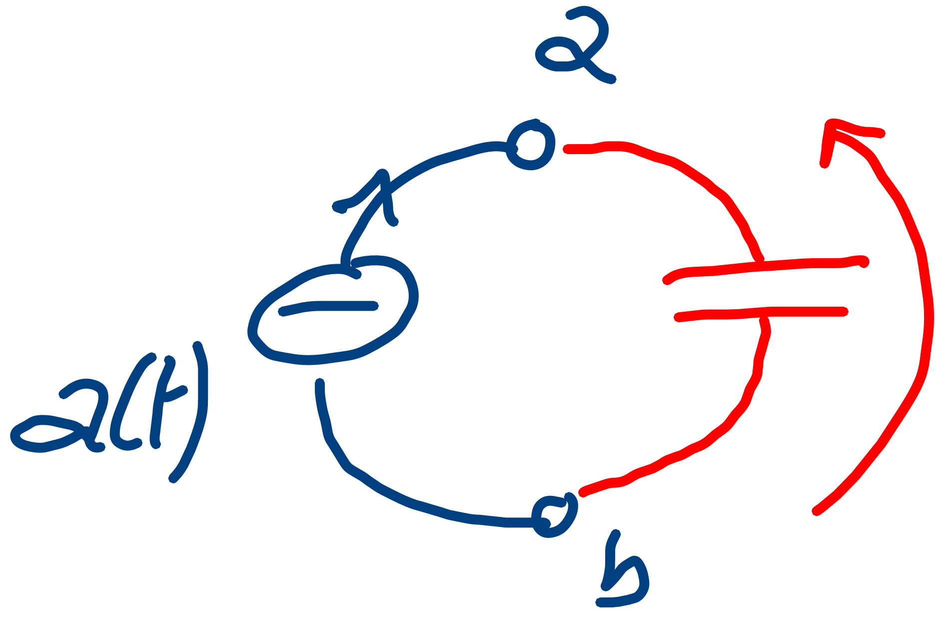


$$i_\ell = (0-25)e^{-t/6.25} + 25 \quad \text{mA, ms}$$

3. First-order Dynamic Circuits, cont'd

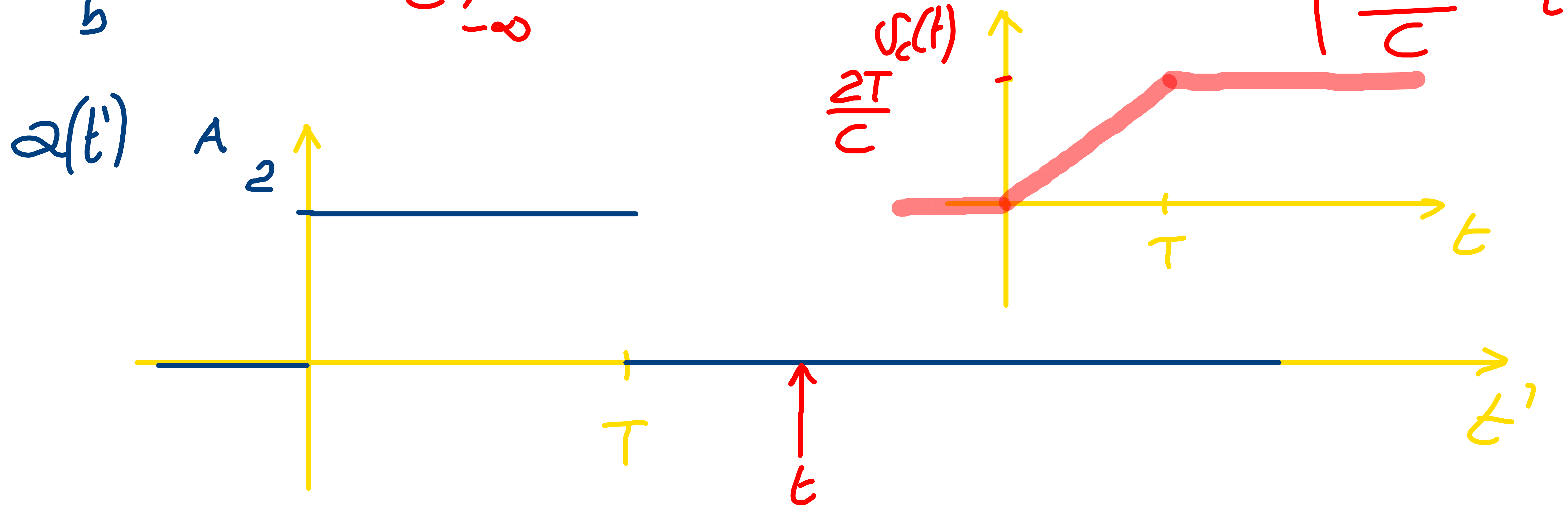
Problem: for $a=2(u(t)-u(t-T))$ A, $T>0$, compute and plot $v_c(t)$





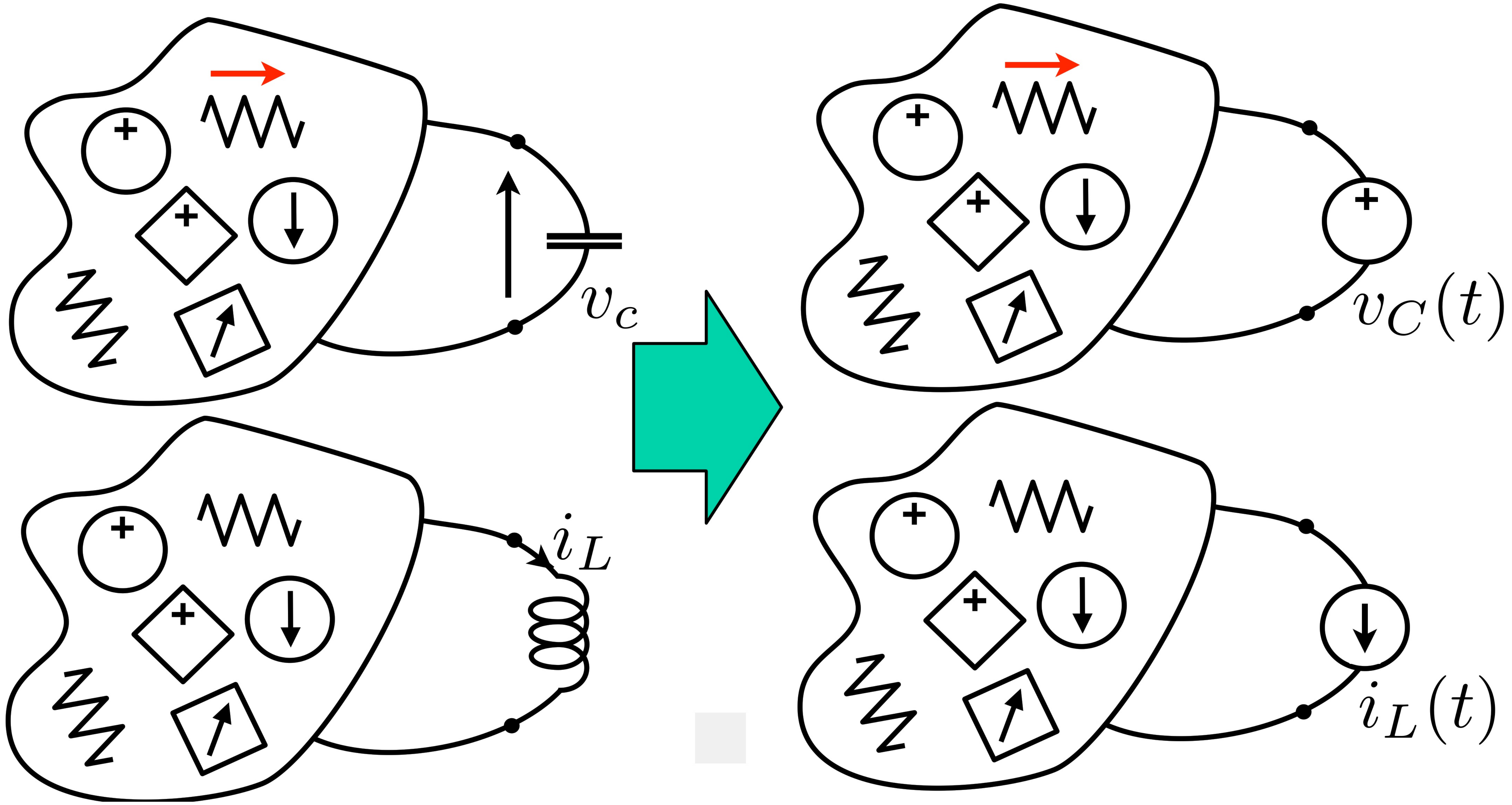
$$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} a(t')\,dt'$$

$$v_c(t) = \begin{cases} 0 & t \leq 0 \\ \dfrac{2t}{C} & 0 \leq t \leq T \\ \dfrac{2T}{C} & t \geq T \end{cases}$$



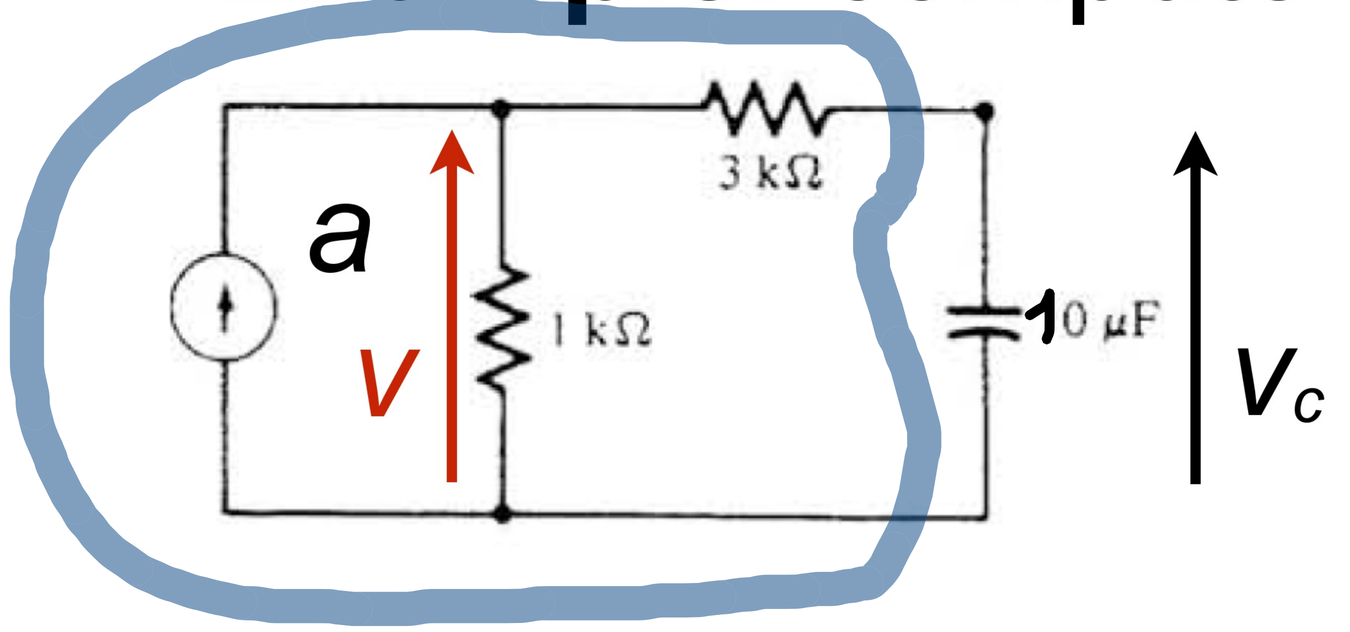
3. First-order Dynamic Circuits, cont'd

Response of a LTI circuit with one capacitor (inductor) and piecewise constant sources:
variables other than $v_C(t)$ ($i_L(t)$)

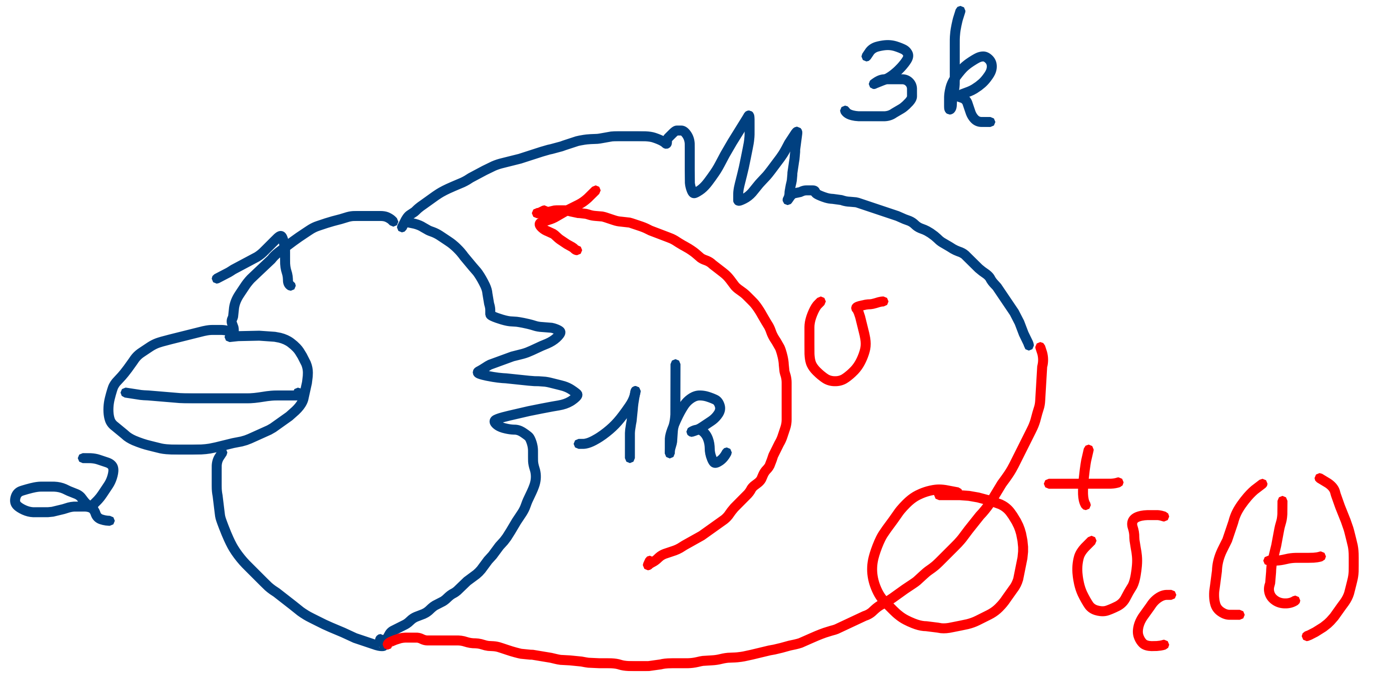


3. First-order Dynamic Circuits, cont'd

Example: compute and plot $v_c(t)$ and $v(t)$ for $a = u(t)$ mA



$$v_c(t) = \begin{cases} 0 & t \leq 0 \\ (0-1)e^{-t/40} + 1 \ \text{V, ms} & t \geq 0 \end{cases}$$



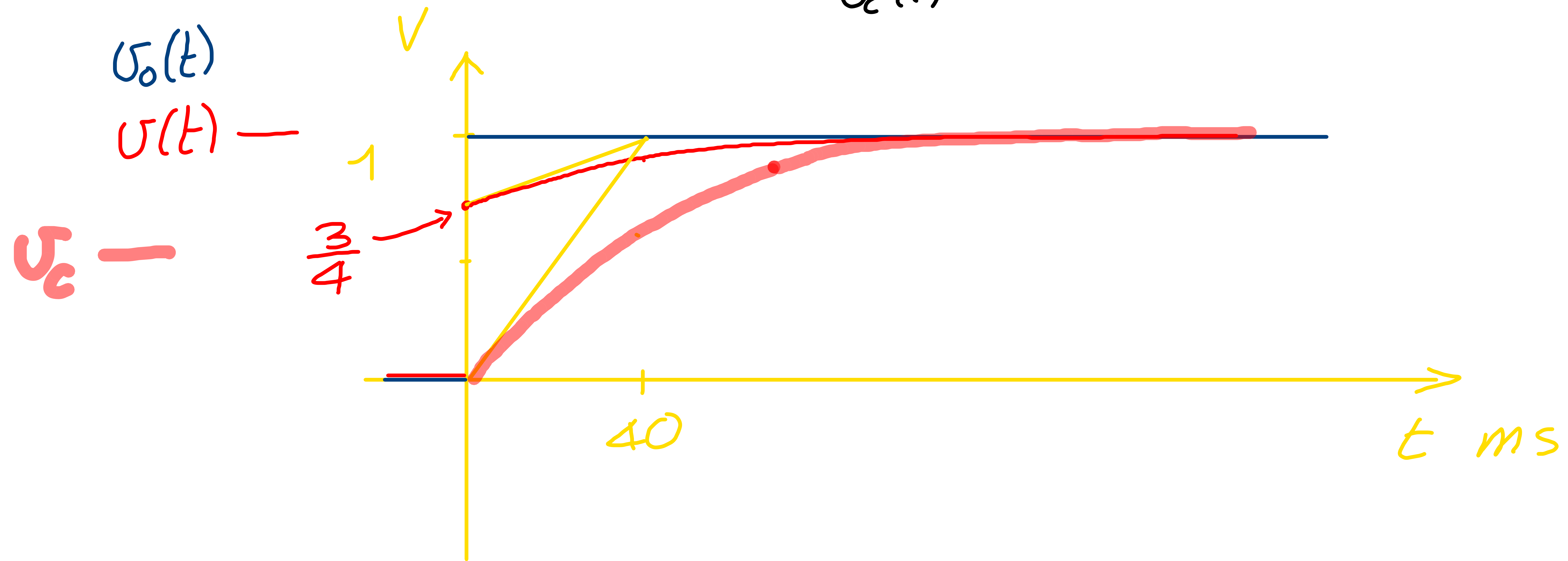
$$v = \frac{\frac{1}{3\,10^3} v_c + a}{\frac{1}{10^3} + \frac{1}{3\,10^3}} \quad \frac{3\,10^3}{3\,10^3}$$

$$= \frac{v_c + 3\,10^3 a}{3+1}$$

$$= \frac{1}{4} v_c(t) + \frac{3}{4} \underbrace{(a\,10^3)}_{u(t)\ \text{V}}$$

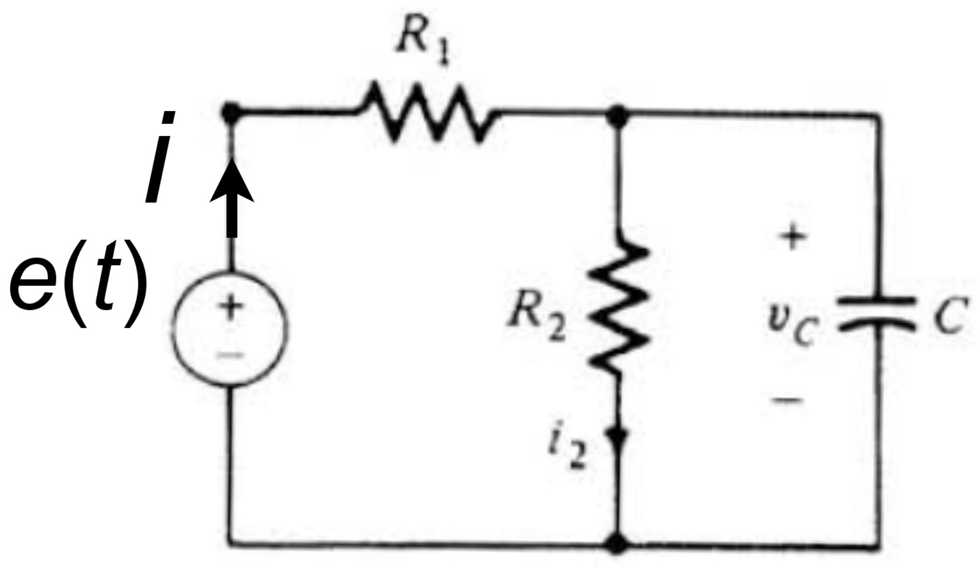
$$v = \begin{cases} 0 & t < 0 \\ \frac{3}{4} + \frac{1}{4}\left(1 - e^{-t/40}\right) \ \text{V, ms} & t > 0 \end{cases}$$

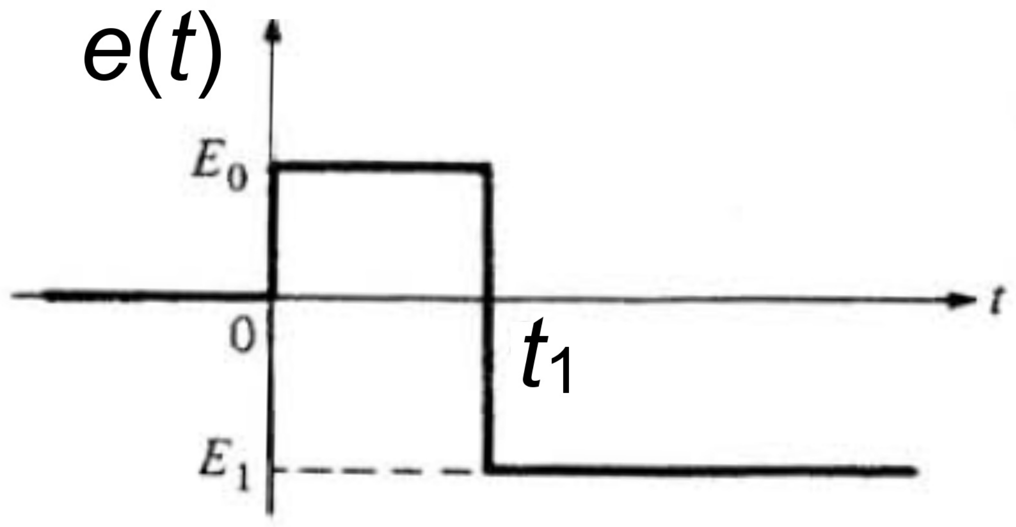
$$v = \begin{cases} 0 & t<0 \\ \frac{3}{4} + \frac{1}{4}\underbrace{\left(1-e^{-t/40}\right)}_{v_c(t) \text{ in } t>0} \; V, ms & t>0 \end{cases}$$



3. First-order Dynamic Circuits, cont'd

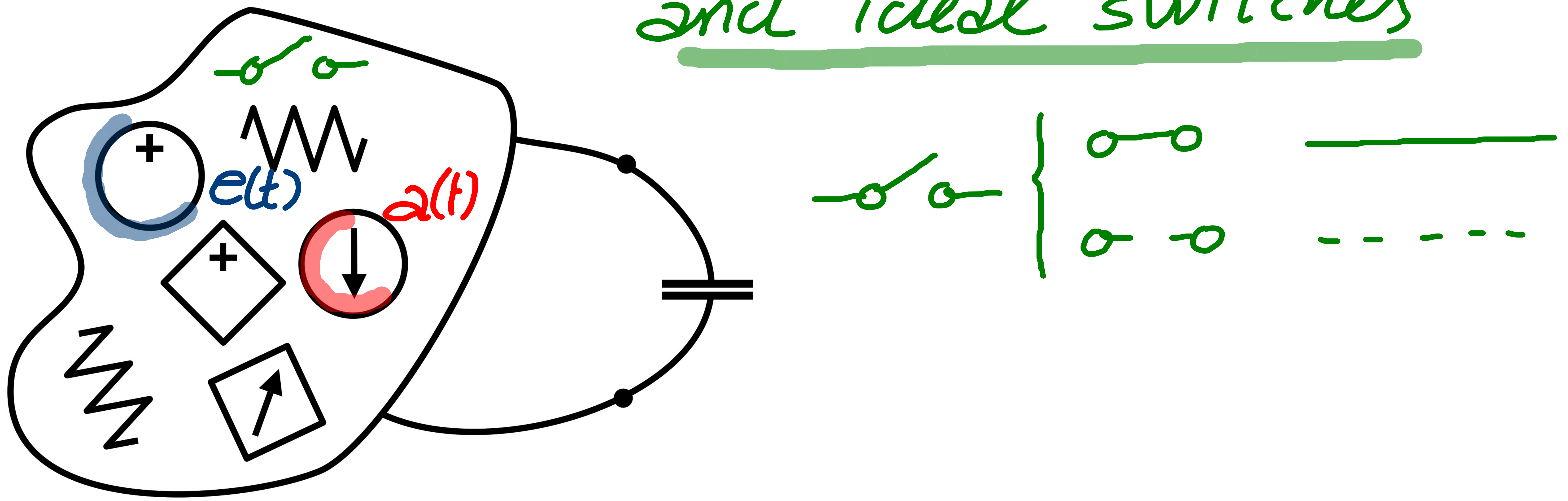
Example: R_1=R_2= 1kΩ, C=4μF, E_o=2V, E_1=-3V and t_1=3ms, compute and plot *i(t)*

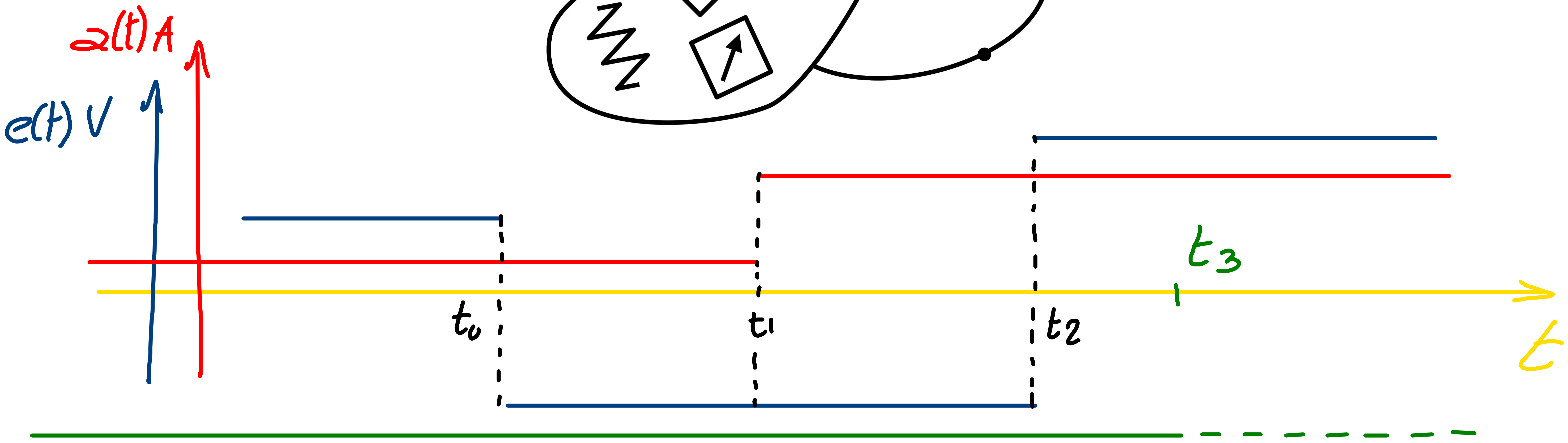




3. First-order Dynamic Circuits, cont'd

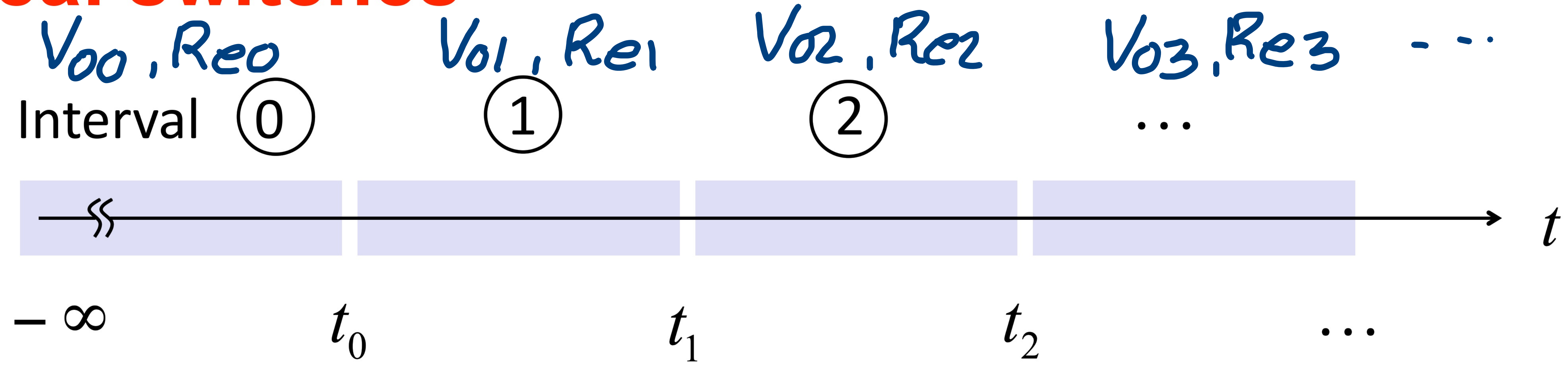
Response of a LTI circuit with one capacitor and piecewise constant sources (discontinuities at t_k, k=0,1,…) and ideal switches





3. First-order Dynamic Circuits, cont'd

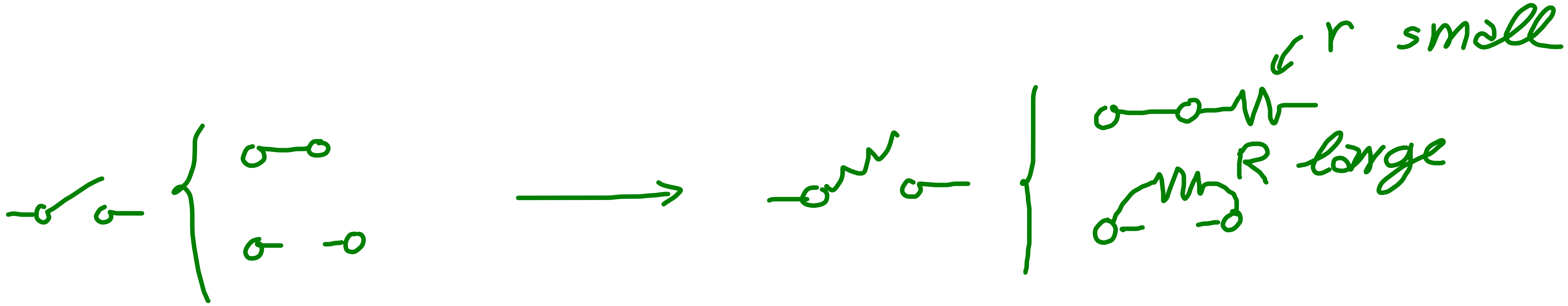
Response of a LTI circuit with one capacitor (inductor) and piecewise constant sources and ideal switches



Divide time axis into intervals with constant source values and switch states

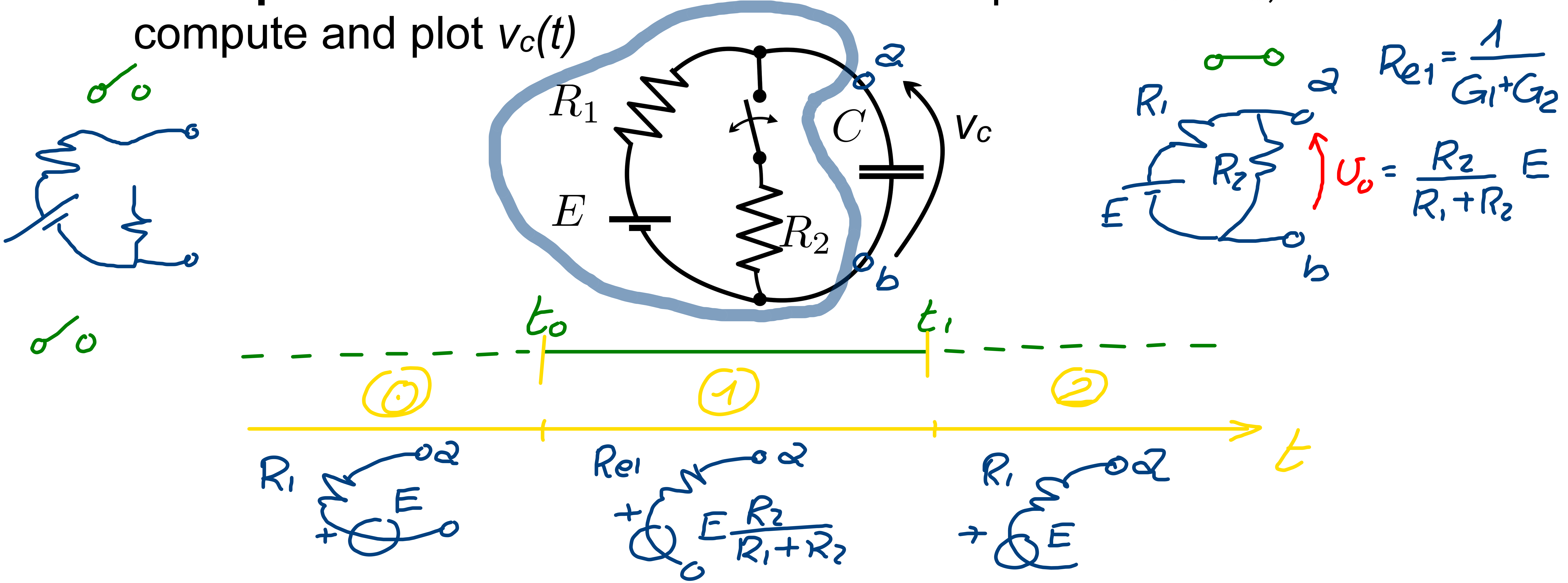
$$v_{(2)}(t) = [v_{(1)}(t_1) - V_{o2}]e^{-(t-t_1)/\tau_2} + V_{o2}$$

$$v_{(1)}(t) = [V_{oo} - V_{o1}]e^{-(t-t_o)/\tau_1} + V_{o1}$$

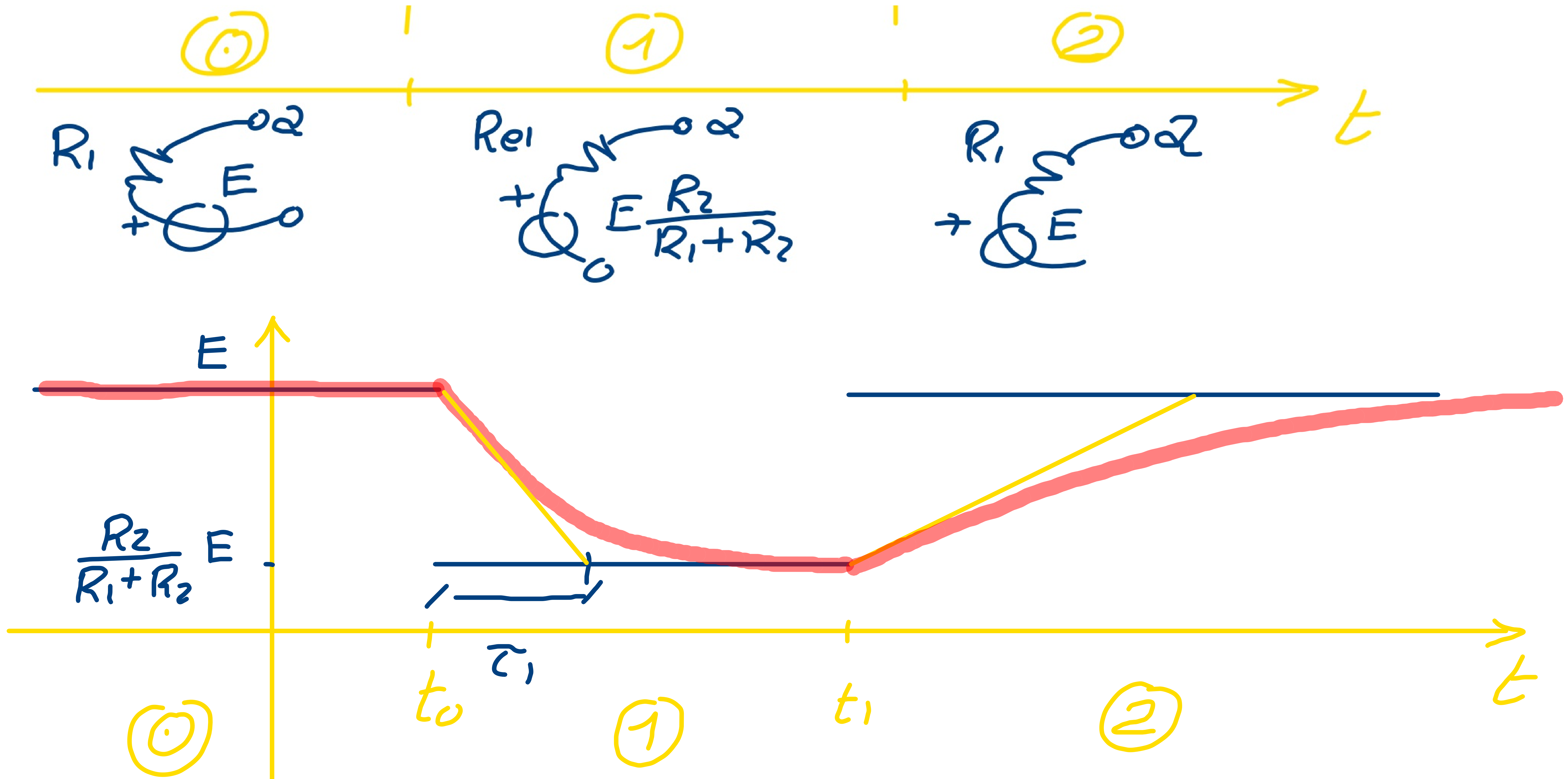

r small
R large

3. First-order Dynamic Circuits, cont'd

Example: the switch closes at $t_o = 0$ and opens at $t_1 > t_o$, compute and plot $v_c(t)$



$v_c(t)$



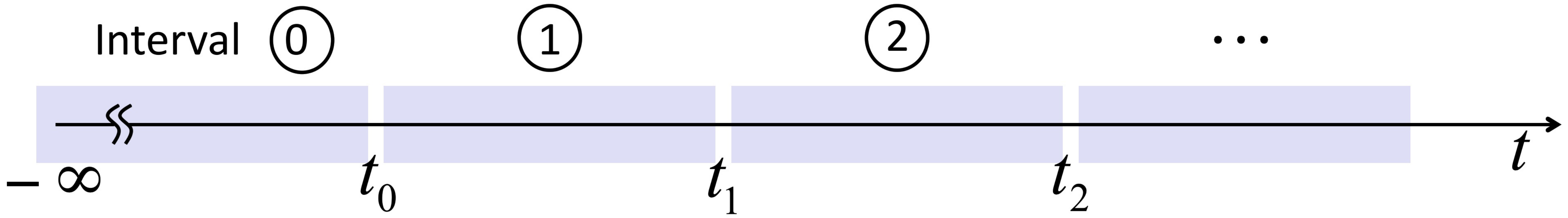
$$\tau_1 = \frac{1}{G_1+G_2}C < \tau_2 = R_1 C$$

$$v_c(t) = \begin{cases} E & t \in 0 \\ \left(E - E\frac{R_2}{R_1+R_2}\right) e^{-\frac{t-t_0}{\tau_1}} + E\frac{R_2}{R_1+R_2} & t \in 1 \\ \left(v_{c1}(t_1) - E\right) e^{-\frac{t-t_1}{\tau_2}} + E & t \in 2 \end{cases}$$

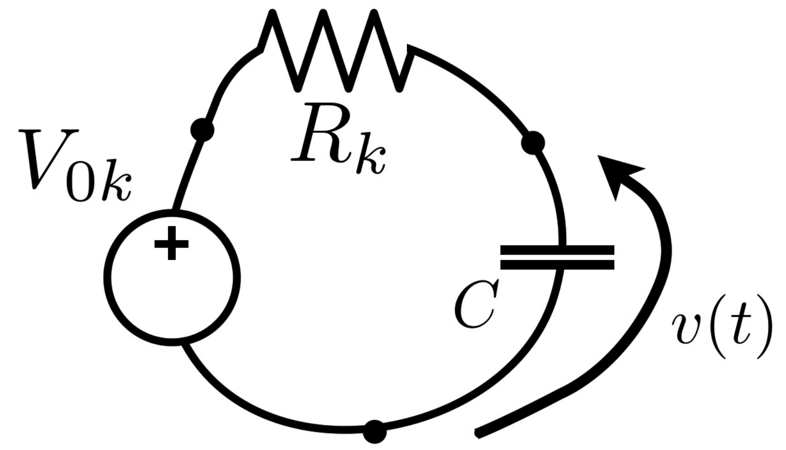
Solution by Inspection of 1st-Order Circuits

- Allowed for linear circuits with one (equivalent) capacitor, piecewise constant sources and ideal switches only
- The single-inductor case is the dual one and must be treated accordingly

(1) Subdivide the time axis in intervals with constant source values and switch states



(2) For each time interval, build the series equivalent of the 2-terminal element driving the capacitor



(3) For the *k*-th interval, if the equivalent doesn't exist or has zero equivalent resistance, the solution is trivial, if it exists with parameters R_k and V_{0k}, then the capacitor voltage is continuous in t_{k-1} and given by:

$$v_k(t) = (v_{k-1}(t_{k-1}) - V_{ok})e^{-(t-t_{k-1})/R_k C} + V_{ok}$$

(4) Replace the capacitor with a voltage sources applying $v(t)$ and solve the resulting resistive circuit for the other variables