

A kinetic theory for quantum information transport

Fabio Anza*

*Complexity Sciences Center and Physics Department,
University of California at Davis, One Shields Avenue, Davis, CA 95616*

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In this work we develop a theoretical framework for the transport of information in quantum systems. This is a framework aimed at describing how out of equilibrium open quantum systems move information around their state space, using an approach inspired by transport theories.

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INTRODUCTION.

The study of transport phenomena is crucial to harness the dynamical properties of natural systems. In this sense, transport theories are simplified phenomenological descriptions of the nonequilibrium behavior of a system. Well-known examples are the theories of transport of charge, mass and heat. Most of them were originally formulated to understand macroscopic phenomena like, in the case of mass transport, the tendency of a system to transfer mass to minimize the concentration difference. Over time, they were put on more rigorous ground using statistical mechanics and kinetic theory. Most notably, Boltzmann used this approach to describe the behavior of gases, and explain the rise of macroscopic irreversibility from the underlying time-symmetric classical mechanics. After Boltzmann's work, the core idea of treating motion and interaction in a statistical way has led to innumerable advances, both of fundamental and applied nature. With the rise of quantum theory, these techniques have also been adapted to include quantum fluctuations, leading to improved descriptions of transport phenomena at the nanoscale. While nowadays the use of kinetic and transport theories is ubiquitous, these techniques have not been leveraged to study quantum information in closed and open quantum systems. Thus, in this work, we attempt to lay a bridge between the field of quantum information and that of transport theories. More accurately, here we provide a self-consistent theory to track the dynamical evolution of a quantum state (and its probability distribution) in nonequilibrium quantum systems. We do so by building a kinetic theory of quantum state evolution, on the quantum state space, both in closed and open configurations.

At the technical level, this is done by leveraging recent work [1, 2] on Geometric Quantum Mechanics: a differential-geometric approach to quantum mechanics that gets rid of the phase redundancy of the Hilbert space. The geometric formalism maps the Hilbert space onto a space of pure states that has the structure of a

classical phase-space, with canonically conjugated coordinates.

The paper is organized as follows. In Section I we give a brief summary of GQM, and of the tools introduced in [1, 2], both of which are needed. In Section II we give the first result: a microscopic derivation of the continuity equation for the information transport in quantum systems, with the appropriate microscopic definitions for phenomenological quantities such as information flux and sink/sources. Section III contains our second result: a general equation of motion for the probability distribution to find an open quantum system in one of its pure states states. In Sections IV and V we use the theoretical framework developed to analyze a few concrete examples, supported by numerical analysis. Eventually, in Section VI we provide a quick summary and draw some conclusions.

I. GEOMETRIC QUANTUM MECHANICS

References [3–21] give a comprehensive introduction to GQM. Here, we briefly summarize only the elements we need, working with Hilbert spaces \mathcal{H} of finite dimension D . For the details of the derivations, we send the reader to the literature cited above.

Given an arbitrary basis $\{|e_n\rangle\}_{n=0}^{D-1}$, a pure state is parametrized by D complex homogeneous coordinates $Z = \{Z^n\}$, up to normalization and an overall phase:

$$|\psi\rangle = \sum_{n=0}^{D-1} Z^n |e_n\rangle .$$

Here, and throughout the paper, we will always use upper indices to identify different coordinates of the same point and lower indices to identify different point. This description is redundant since we can renormalize Z by multiplying it for a real number and a phase (hence a complex number) and we would get the same physical state. Therefore, $Z \in \mathbb{C}^D$, $Z \sim \lambda Z$, with $\lambda \in \mathbb{C}/\{0\}$. This equivalence relation means pure states of a quantum system are points in the complex projective space $\mathcal{P}(\mathcal{H}) = \mathbb{C}\mathbb{P}^{D-1}$. We will often refer to $\mathcal{P}(\mathcal{H})$ as the

* fanza@ucdavis.edu

quantum state space. One can always use probability-phase coordinates which, in the case of a single qubit are $Z = (\sqrt{1-p}, \sqrt{p}e^{i\phi})$. We will often refer to this set of coordinates as they play a particular useful role.

Observables and POVMs. In GQM, an *observable* is a function $\mathcal{O}(Z) \in \mathbb{R}$ that associates to each point $Z \in \mathcal{P}(\mathcal{H})$ the expectation value $\langle \psi | \mathcal{O} | \psi \rangle / \langle \psi | \psi \rangle$ of the corresponding operator \mathcal{O} on state $|\psi\rangle$ with coordinates Z :

$$\mathcal{O}(Z) = \frac{\sum_{\alpha,\beta} \mathcal{O}_{\alpha,\beta} Z^\alpha \bar{Z}^\beta}{\sum_\gamma |Z^\gamma|^2}, \quad (1)$$

where $\mathcal{O}_{\alpha\beta}$ is Hermitian $\mathcal{O}_{\beta,\alpha} = \bar{\mathcal{O}}_{\alpha,\beta}$.

Measurement outcome probabilities are determined by *positive operator-valued measurements* (POVMs) $\{E_j\}_{j=1}^D$ applied to a state [22, 23]. They are nonnegative operators $E_j \geq 0$, called *effects*, that sum up to the identity: $\sum_{j=1}^D E_j = \mathbb{I}$. In GQM they consist of nonnegative real functions $E_j(Z) \geq 0$ on $\mathcal{P}(\mathcal{H})$ whose sum is always unity:

$$E_j(Z) = \frac{\sum_{m,n} (E_j)_{m,n} Z^m \bar{Z}^n}{\sum_k |Z^k|^2}, \quad (2)$$

where $\sum_{j=1}^D E_j(Z) = 1$.

The quantum state space $\mathcal{P}(\mathcal{H})$ has a preferred metric g_{FS} —the *Fubini-Study metric* [13]—and an associated volume element dV_{FS} that is coordinate-independent and invariant under unitary transformations. The geometric derivation of dV_{FS} is beyond our immediate goals here. That said, it is sufficient to give its explicit form in the probability-phase coordinate system $Z^n = \sqrt{p_n}e^{i\phi_n}$ that we are going to use for explicit calculations:

$$\begin{aligned} dV_{FS} &= \sqrt{\det g_{FS}} \prod_{n=0}^{D-1} dZ^n d\bar{Z}^n \\ &= \prod_{n=1}^{D-1} \frac{dp_n d\phi_n}{2}. \end{aligned}$$

Notice how p_0 and ϕ_0 are not involved. This is due to $\mathcal{P}(\mathcal{H})$'s projective nature which guarantees that we can choose a coordinate patch in which $p_0 = 1 - \sum_{n=1}^{D-1} p_n$ and $\phi_0 = 0$. As we see now, the probability-phase coordinates play a particular role: they are canonically conjugated.

State-space structure. Beyond the Riemannian structures, it can be shown that $\mathcal{P}(\mathcal{H})$ also has another interesting geometric feature: a symplectic structure. This is the hallmark of classical state spaces and justifies the use of the term *quantum state space* for $\mathcal{P}(\mathcal{H})$. With a more common jargon, a symplectic structure is the geometric entity allowing us to define “Poisson Brackets” and the existence of canonically conjugated coordinates. In particular, using probabilities and phases coordinates

$\{(p_n, \phi_n)\}$ one has that $\{p_n, \phi_n\} = \frac{1}{\hbar} \delta_{nm}$. Thus, for arbitrary functions A and B on $\mathcal{P}(\mathcal{H})$ one has

$$\{A, B\} := \frac{1}{\hbar} \sum_{n=1}^{D-1} \frac{\partial A}{\partial p_n} \frac{\partial B}{\partial \phi_n} - \frac{\partial B}{\partial p_n} \frac{\partial A}{\partial \phi_n} \quad (3)$$

Unitary evolution. In QM, an isolated quantum system evolves with a unitary propagator $U(t, t_0) = e^{-\frac{i}{\hbar} H(t-t_0)}$, where the generator H is the (time-independent) Hamiltonian of the system. Surprisingly, it can be shown [13] that this evolution is equivalent to a classical Hamiltonian dynamics, with geometric coordinates. In particular, calling $E(p_n, \phi_n) = \langle \psi(p_n, \phi_n) | H | \psi(p_n, \phi_n) \rangle$ the expectation value of H on a generic state $|\psi(p_n, \phi_n)\rangle = \sum_{n=0}^{D-1} \sqrt{p_n} e^{i\phi_n} |e_n\rangle$ parametrized by $(p_n, \phi_n)_n$, the unitary evolution of a generic function A on $\mathcal{P}(\mathcal{H})$ is given by

$$\frac{\partial A}{\partial t} = \{A, E\} \quad (4)$$

Or, equivalently, a generic state $(p_n, \phi_n)_n$ evolves according to Hamilton's equations of motion:

$$\frac{dp_n}{dt} = \frac{1}{\hbar} \frac{\partial E}{\partial \phi_n} \quad (5a)$$

$$\frac{d\phi_n}{dt} = -\frac{1}{\hbar} \frac{\partial E}{\partial p_n} \quad (5b)$$

Here the analogies with classical Hamiltonian mechanics are particularly evident. While quantum mechanics is clearly very different from its classical counterpart, at a certain descriptive level we can still use the intuition of classical mechanics to understand the dynamical evolution of quantum systems. We now proceed in our summary by looking at situations in which the state of the system can not be characterized by a single point, thus needing a probability distribution over the whole state space.

Geometric quantum states. The geometric framework makes it very natural to view a quantum state as a functional encoding that associates expectation values to observables, paralleling the C^* -algebra formulation of quantum mechanics [24]. The idea is that one considers probability density functions on $\mathcal{P}(\mathcal{H})$, together with observable functions.

Geometric Quantum States are functionals $P_q[\mathcal{O}]$ from the algebra of observables \mathcal{A} to the real line:

$$P_q[\mathcal{O}] = \int_{\mathcal{P}(\mathcal{H})} q(Z) \mathcal{O}(Z) dV_{FS}, \quad (6)$$

where $\mathcal{O} \in \mathcal{A}$, $q(Z) \geq 0$ is the normalized distribution associated with the functional P_q :

$$P_q[1] = \int_{\mathcal{P}(\mathcal{H})} q(Z) dV_{FS} = 1,$$

and $P_q[\mathcal{O}] \in \mathbb{R}$. Thus, kets $|\psi_0\rangle$ are described by functionals with a Dirac-delta distribution $q_0(Z) = \tilde{\delta}[Z - Z_0]$:

$$\begin{aligned} P_0[\mathcal{O}] &= \int_{\mathcal{P}(\mathcal{H})} \tilde{\delta}(Z - Z_0) \mathcal{O}(Z) dV_{FS} \\ &= \mathcal{O}(Z_0) = \langle \psi_0 | \mathcal{O} | \psi_0 \rangle . \end{aligned}$$

Here, $\tilde{\delta}(Z - Z_0)$ is shorthand for a coordinate-covariant Dirac-delta in arbitrary coordinates. In homogeneous coordinates this reads:

$$\tilde{\delta}(Z - Z_0) := \frac{1}{\sqrt{\det g_{FS}}} \prod_{n=0}^{D-1} \delta(X^n - X_0^n) \delta(Y^n - Y_0^n) ,$$

where $Z^n = X^n + iY^n$. In (p_n, ϕ_n) coordinates this becomes simply:

$$\tilde{\delta}(Z - Z_0) = \prod_{n=1}^{D-1} 2\delta(p_n - p_n^0)\delta(\phi_n - \phi_n^0) ,$$

where the coordinate-invariant nature of the functionals $P_q[\mathcal{O}]$ is now apparent. Extending by linearity, a specific decomposition $\{\lambda_j, |\lambda_j\rangle\}$ of a density matrix

$$\rho = \sum_j \lambda_j |\lambda_j\rangle \langle \lambda_j|$$

corresponds to a convex combinations of these Dirac-delta functionals:

$$q_{\text{mix}}(Z) = \sum_j \lambda_j \tilde{\delta}(Z - Z_j) .$$

Thus, expressed as functionals from observables to the real line, mixed states are:

$$P_{\text{mix}}[\mathcal{O}] = \sum_j \lambda_j \langle \lambda_j | \mathcal{O} | \lambda_j \rangle . \quad (7)$$

Equipped with this tools, one identifies the distribution $q(Z)$ of Eq. (6) as a system's *geometric quantum state*. This is a generalized notion of quantum state.

A simple example of a geometric quantum state is the *geometric canonical ensemble*:

$$q(Z) = \frac{1}{Q_\beta} e^{-\beta h(Z)} ,$$

where:

$$\begin{aligned} Q_\beta &= \int dV_{FS} e^{-\beta h(Z)} , \\ h(Z) &= \langle \psi(Z) | H | \psi(Z) \rangle , \end{aligned}$$

and H is the system's Hamiltonian operator. This was introduced in Refs. [25]. References [26] and [2] investigated its potential role in establishing a quantum foun-

dation of thermodynamics that is an alternative to that based on Gibbs ensembles and von Neumann entropy.

Density matrix. The connection between geometric quantum states and density matrices is fairly straightforward. Since density matrices are a synthetic way of collecting probability outcomes about POVMs, which are functions of the form $A(Z) \propto \sum_{n,m} A_{nm} Z^n \bar{Z}^m$, given a generic geometric quantum state $q(Z)$ the associated density matrix ρ^q can be simply computed as follows:

$$\begin{aligned} \rho_{mn}^q &= P_q[Z^m \bar{Z}^n] \\ &= \int_{\mathcal{P}(\mathcal{H})} dV_{FS} q(Z) Z^m \bar{Z}^n . \end{aligned} \quad (8)$$

Owing to the specific form of POVMs on $\mathcal{P}(\mathcal{H})$, recall Eq. (2), they are sensitive to $q(Z)$ only via ρ^q . Therefore, if two geometric quantum states q_1 and q_2 induce the same density matrix $\rho^{q_1} = \rho^{q_2}$, then all POVMs produce the same outcomes.

The GQS of an Open Quantum System. As shown above, if a system is in a pure state its GQS is simply a Dirac delta concentrated on a single point. However, if the system is in contact with an environment, this is not true anymore, unless they are exactly in a product state. In general, contact with the environment causes the loss of information about which region of the state space the system inhabits. Hence, we need a probability distribution for the state of our system: a geometric quantum state. The explicit derivation of the generic geometric quantum state for an open quantum system can be found in Ref.[1]. Here we simply give the result, without discussing the derivation. Given a system and its environment, we call $\{|a_j\rangle\}_{j=0}^{d_S-1}$ a basis for the Hilbert space \mathcal{H}_S of the system (dimension d_S) and $\{|b_n\rangle\}_{n=0}^{d_E}$ a basis for the Hilbert space \mathcal{H}_E of the environment (dimension d_E). Assuming system and environment are in a pure state $|\psi_{SE}\rangle = \sum_{j,\alpha} \psi_{j\alpha} |a_j\rangle |b_\alpha\rangle$, the geometric quantum state of the system is

$$q(Z) = \sum_{\alpha=0}^{d_E-1} x_\alpha \tilde{\delta}(Z - Z(\chi_\alpha)) , \quad (9)$$

where

$$x_\alpha := \sum_j |\psi_{j\alpha}|^2 , \quad (10a)$$

$$Z^j(\chi_\alpha) = \frac{\psi_{j\alpha}}{\sqrt{x_\alpha}} \rightarrow |\chi_\alpha\rangle = \sum_j \frac{\psi_{j\alpha}}{\sqrt{x_\alpha}} |a_j\rangle \quad (10b)$$

Here x_α is the probability that the environment is in state $|b_\alpha\rangle$ and $\{|\chi_\alpha\rangle\}$ is a set of d_E states of the system which constitutes the discrete support of the geometric quantum state. The geometric quantum state in Eq.(9) provides the correct reduced density matrix of the system $\rho^S(\psi_{SE}) = \sum_\alpha x_\alpha |\chi_\alpha\rangle \langle \chi_\alpha|$.

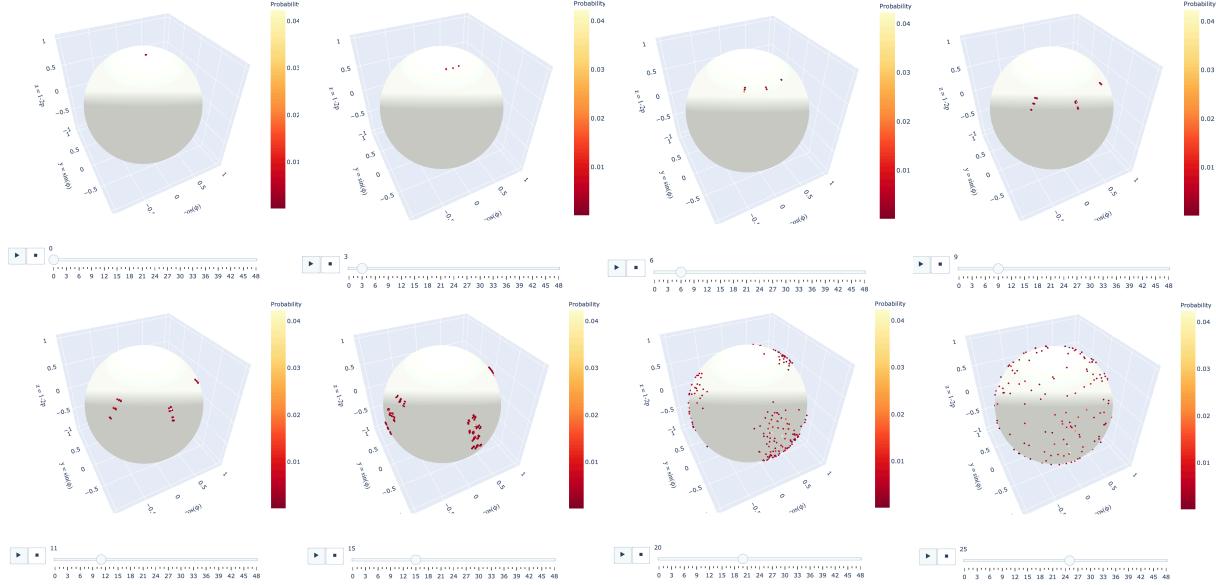


Figure 1. Evolution of the geometric quantum state of one qubit interacting with nine others via an Ising model with transverse field, visualized on the surface of the Bloch sphere. Each particle is $\Gamma_\alpha(t)$, represented in spherical coordinates $\Gamma_\alpha(t) = (\theta_\alpha(t), \phi_\alpha(t))$, with $|\psi(\theta, \phi)\rangle = \cos \theta / 2 |0\rangle + \sin \theta / 2 e^{i\phi} |1\rangle$. The color of the point encodes the probability $x_\alpha(t)$. Time increases left to right and top to bottom. Each particle carries a probability mass x_α , which is the probability to find the system in a state Γ_α . Thus, we think of it as an information carrier, moving the information about the state of a quantum system around the quantum state space. The position of each particle is determined by the state of the environment and, indeed, there are 2^9 particles. The interactive html file from which these snapshots were taken can be found at <http://csc.ucdavis.edu/~cmg/papers/GeoStateEvolution.html>

Probability mass in a region of the state space. Before we proceed with the main result, we introduce here a notation that will be useful later. This is imported from measure theory. Given a region $A \subseteq \mathcal{P}(\mathcal{H})$, we have

$$\mu_t(A) = \int q_t(Z) dV_{FS} \quad (11)$$

$\mu_t(A)$ quantifies the probability that, at time t , our quantum system is in a state Z that belongs to the region A of the quantum state space $\mathcal{P}(\mathcal{H})$. In this sense, this is the core quantity that conveys how much information about the system is contained in a certain region A of the state space.

II. INFORMATION TRANSPORT: CONTINUITY EQUATION

In the past section we have summarized previous results about GQM. We now build on them and, by bringing in the dynamics of the system, we derive a continuity equation which dictates how the geometric quantum state of a non-equilibrium open quantum system evolves, under very general assumptions. This is the fundamental kinetic equation governing how the information about the state of a quantum system changes as a result of its interaction with an environment. Throughout this section

we will try to maintain a fairly general language, to emphasize how the treatment applies to quantum systems under very general assumptions. However, for concrete examples we will always refer to the simple case of a qubit.

General treatment

The following treatment, and its results, pertains quantum systems which are finite-dimensional, and interact with finite-dimensional environments but are, otherwise, arbitrary. A geometric quantum state is specified by two sets of quantities: $\{x_\alpha\}_{\alpha=0}^{d_E-1}$ and $\{\Gamma_\alpha\}_{\alpha=0}^{d_E-1}$ (see Eq. (10)) which is a short-hand notation for $\Gamma_\alpha = Z(|\chi_\alpha\rangle)$. The first one is a classical probability distribution resulting from measuring the environment on a generic eigenbasis $\{|b_\alpha\rangle\}$. For each of the x_α there is a corresponding pure state that the system inhabits: $\Gamma_\alpha \in \mathcal{P}(\mathcal{H}_S)$, which corresponds to the ket $|\chi_\alpha\rangle \in \mathcal{H}_S$. Since these $\Gamma_\alpha(t)$ are points moving in a state space, we can think of them as particles on a classical phases space, which interact in a non-trivial way. These are the “carriers of information”, in the sense that each of these particles carries a probability mass $x_\alpha(t)$ that the system will be found in $\Gamma_\alpha(t)$. A clarifying example of how the geometric quantum state of a qubit evolves when the system interacts with an environment is given in Figure 1. Since

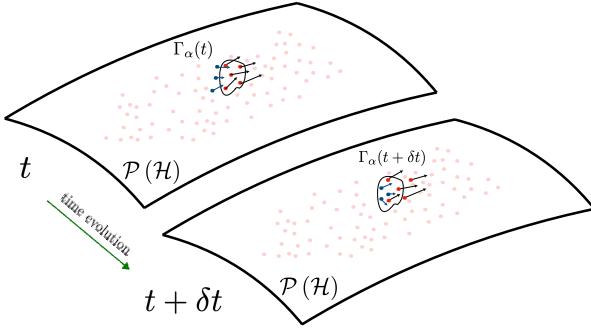


Figure 2. Kinetic interpretation of the Information flux $J_t(Z)$. If we look at a small region of the quantum state space, due to the underlying dynamics we see that information is locally conserved in the sense that there is a certain number of points which enters and leaves this region. As a result of this local process, probability is moved around and can concentrate in a certain region or get scrambled across the quantum state space.

the total amount of information has to be preserved, $\mu_t(\mathcal{P}(\mathcal{H})) = \int_{\mathcal{P}(\mathcal{H})} q_t(Z) dV_{FS} = \sum_\alpha x_\alpha(t) = 1$, the geometric quantum state $q_t(Z)$ must satisfy a continuity equation. Conceptually, this is the starting point of virtually all transport theories, which deal with quantities that are globally conserved, but that are moved around as a result of the underlying dynamics.

Thus, in the dynamics of a geometric quantum state we identify two different terms, arising from the time-evolution of these two sets of quantities. Loosely speaking, by looking at Figure 1 we can see that there are two different ways in which the geometric quantum state changes in time. First, there is the color of each particle, which changes over time: $\dot{x}_\alpha \neq 0$. Second, there is the movements of each point: $\Gamma_\alpha \neq 0$. By summing each term over all particles, and then summing the two terms, we get the following evolution equation for the geometric quantum state:

$$\frac{\partial q_t(Z)}{\partial t} = \sum_\alpha \dot{x}_\alpha \tilde{\delta}(Z - \Gamma_\alpha) - x_\alpha \tilde{\delta}'(Z - \Gamma_\alpha) \dot{\Gamma}_\alpha . \quad (12)$$

Let's start by analyzing the second term. There, $\tilde{\delta}'$ is the first distributional derivative of $\tilde{\delta}(Z - \Gamma_\alpha)$ and $\dot{\Gamma}_\alpha(t)$ is the velocity of each particle $\Gamma_\alpha(t)$. Thus, the second term in the right-hand side looks like the divergence of a velocity field or, in other words, as a flux term in a continuity equation. Indeed, since all the dependence on the coordinate Z lies inside the $\tilde{\delta}'$, the whole term can

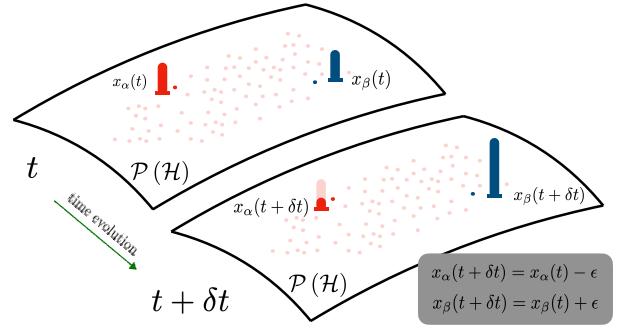


Figure 3. Kinetic interpretation of the sink/source term σ_t . Even if the points do not move around the quantum state space $\dot{\Gamma}_\alpha = 0$, the term σ_t allows the exchange of information between different regions. Note that this can be a non-local effect in state space, as depicted above. In the example above we have two states Γ_α and Γ_β fixed in time, but exchanging a certain amount ϵ of probability, thus moving information from one region of the state space to another one: $x_\beta(t + \delta t) = x_\beta(t) + \epsilon$, $x_\alpha(t + \delta t) = x_\alpha(t) - \epsilon$.

be written as a covariant divergence¹

$$\sum_\alpha x_\alpha \tilde{\delta}'(Z - \Gamma_\alpha) \dot{\Gamma}_\alpha = (\nabla \cdot J_t)(Z) , \quad (13)$$

where we have identified the *Information Flux* $J_t(Z)$ as the “single-particle” flux $J_\alpha(Z, t)$, averaged with the probability mass it is carrying $x_\alpha(t)$:

$$J_\alpha(Z, t) = \tilde{\delta}(Z - \Gamma_\alpha(t)) \dot{\Gamma}_\alpha(t) , \quad (14a)$$

$$J_t(Z) = \sum_\alpha x_\alpha(t) J_\alpha(Z, t) . \quad (14b)$$

We now look at the first term in the right-hand side of Eq.(12): $\sum_\alpha \dot{x}_\alpha \tilde{\delta}(Z - \Gamma_\alpha(t))$. We recognize a source/sink term, which is independent on the underlying movement of the points in the quantum state space. This identifies the other functional term in our continuity equation: *sinks and sources of information* $\sigma_t(Z)$:

$$\sigma_t(Z) = \sum_{\alpha=0}^{d_E} \dot{x}_\alpha(t) \tilde{\delta}(Z - \Gamma_\alpha(t)) . \quad (15)$$

Eventually, we obtain the following continuity equation for the geometric quantum state of a finite-dimensional quantum system interacting with a finite-dimensional en-

¹ With an arbitrary coordinate system we have $\text{div} \vec{f} = \frac{1}{\sqrt{g}} \partial_\alpha (\sqrt{g} f^\alpha)$, where $g = |\det g_{FS}|$ is the absolute value of the determinant of the metric which, in our case, is the Fubini-Study metric.

vironment:

$$\frac{\partial q_t}{\partial t} + \nabla \cdot J_t = \sigma_t \quad (16)$$

This is our first result. A crucial aspect here is that this theoretical framework allows us to talk about information as a physically localized (and overall conserved) quantity, which is carried around by points in quantum state space. This is in analogy with most classical theories of transport which deal with properties of particles, such as mass or charge, which are carried around by particles, represented as points wondering in a classical phase-space.

To support this theoretical framework, the physical interpretation of the flux J_t and source/sink σ_t term is essentially the same as in other transport theories. We now describe them explicitly, using their kinetic interpretation, drawn in Figures 2 and 3. First, J_t is an information flux, in the proper sense of flux. This flux is associated to an overall conserved quantity, probability mass, localized in its carriers Γ_α , which is moved around the quantum state space. To emphasize this point, we look at the simpler situation in which $\dot{x}_\alpha = 0$. In this case, $\sigma_t = 0$ and the information carried around by the points in state space is a constant of motion ($x_\alpha(t) = x_\alpha(t_0)$). Thus, the analogy with a model of point particles carrying around a physical property, like charge or mass, becomes exact. As the carriers move around, the information is dispersed across the quantum state space with a mechanisms that ensures it is locally preserved, and therefore well described with a continuity equation that equates the local time derivative of the probability distribution with the divergence of the local flux of particles.

Second, the term σ_t is a proper source/sink term, with the usual interpretation. Indeed, first, we note that it can not be written as a divergence term. Second, we look at the simpler dynamics in which the position of the points in $\mathcal{P}(\mathcal{H})$ does not change: $\dot{\Gamma}_\alpha = 0$. In this case the support of the distribution is fixed and the particles don't move around $\Gamma_\alpha(t) = \Gamma_\alpha(t_0)$. However, there can still be a non-trivial dynamics, due to $\dot{x}_\alpha(t) \neq 0$. Depending on the actual support of the distribution, and how spread it is, as depicted in Figure 3, the time-evolution generated by this term is generally non-local in the quantum state space. Thus, σ_t possesses all the hallmarks of the standard sinks and sources terms in transport theories: it is not associated with particles moving around the state space and it can not be rewritten as a divergence term.

A. Isolated quantum systems

What happens when the system is closed? Is there a specific form for the information flux J_t and source σ_t ? Here we answer both these questions in a detailed manner. Since the evolution is Hamiltonian, we can use Hamilton's equations of motion to derive the evolution equation of μ_t or, equivalently, of $q_t(Z)$. First, the sink/sources

terms vanishes, since the evolution is Hamiltonian, thus preserving the local probabilities. Thus proves that a $\sigma_t \neq 0$ is related to the existence of a dissipative dynamics. Second, starting from the definition of the flux in Eq.(14), we can explicitly write the form of the flux J_t , using Hamilton's equations of motion. Writing Γ_α in canonical coordinates $(p_n(\Gamma_\alpha), \phi_n(\Gamma_\alpha))$, we have

$$\begin{aligned} \dot{\Gamma}_\alpha &= (\dot{p}_n(\Gamma_\alpha), \dot{\phi}_n(\Gamma_\alpha))_n \\ &= \left(\frac{1}{\hbar} \frac{\partial E}{\partial \phi_n}, - \frac{1}{\hbar} \frac{\partial E}{\partial p_n} \right)_n = v_H(p_n, \phi_n) \end{aligned} \quad (17)$$

Note how the right-hand side, which we called v_H does not depend on the index α anymore: all the points evolve in the same way, following the same Hamiltonian flow. Inserting this into the definition of the flux we obtain that the flux is equal to the product between the distribution and the Hamiltonian velocity field v_H :

$$J_t(Z) = q_t(Z)v_H(Z), \quad (18)$$

thus providing the following continuity equation:

$$\frac{\partial q_t}{\partial t} = -v_H \cdot \nabla q_t - q_t \nabla \cdot v_H \quad (19)$$

This can be further simplified by noting that the Hamiltonian vector field is divergence free, $\nabla \cdot v_H = 0$. This is due to the smoothness of $E(p_n, \phi_n)$ which, via Schwarz's theorem, implies that the Hessian of $E(p_n, \phi_n)$ is symmetric.

$$\begin{aligned} \nabla \cdot v_H &= \sum_n \left(\frac{\partial}{\partial p_n}, \frac{\partial}{\partial \phi_n} \right) \cdot \frac{1}{\hbar} \left(\frac{\partial E}{\partial \phi_n}, - \frac{\partial E}{\partial p_n} \right) \\ &= \frac{1}{\hbar} \sum_n \left(\frac{\partial^2 E}{\partial p_n \partial \phi_n} - \frac{\partial^2 E}{\partial \phi_n \partial p_n} \right) = 0 \end{aligned}$$

This leads us to the final form of the continuity equation of an isolated quantum system:

$$\frac{\partial q_t}{\partial t} = -v_H \cdot \nabla q_t = -\{q_t, E\} \quad (20)$$

Liouville's theorem for GQM. As a further point of contact with the techniques of classical statistical mechanics and kinetic theory, we now show that a generic Hamiltonian dynamics for the geometric quantum state satisfies Liouville's theorem [27]. Indeed, by writing explicitly the total derivative of $q_t(Z)$ with respect to time we get

$$\frac{dq}{dt} = \sum_\alpha \frac{\partial q_t}{\partial p_\alpha} \frac{dp_\alpha}{dt} + \frac{\partial q_t}{\partial \phi_\alpha} \frac{d\phi_\alpha}{dt} + \frac{\partial q_t}{\partial t} \quad (21)$$

Inserting Hamilton's equations of motion (Eq. (5)), and then using the continuity equation

$$\begin{aligned} \frac{dq}{dt} &= \sum_{\alpha} \frac{\partial q_t}{\partial p_{\alpha}} \frac{dp_{\alpha}}{dt} + \frac{\partial q_t}{\partial \phi_{\alpha}} \frac{d\phi_{\alpha}}{dt} + \frac{\partial q_t}{\partial t} \\ &= \{q_t, E\} - \nabla \cdot q_t v_H = 0 \end{aligned} \quad (22)$$

This concludes the section about the general approach, and the treatment of isolated quantum systems. We now turn to the more convoluted case of an open quantum system.

III. KINETIC EQUATIONS OF INFORMATION TRANSPORT IN OPEN QUANTUM SYSTEMS

As we move to analyze how open quantum systems scramble information around the quantum state space, the goal of this section is to provide a microscopic approach to the transport of information: a kinetic theory of how the information about the state of a quantum system changes as a result of its interaction with a structured, non-thermal, environment. Thus, the main outcome of this section is a concrete set of microscopic equations that determines the evolution of the geometric quantum state in a non-Hamiltonian setting.

Calling H_S and H_E the Hamiltonian operators of the system and environment, respectively, the total Hamiltonian of the joint system is $H = H_S + H_E + H_{\text{int}}$. Since H_{int} is the interaction term between system and environment, we can always put it in the form

$$H_{\text{int}} = \sum_{k=1}^M A^{(k)} \otimes B^{(k)}, \quad (23)$$

where $A^{(k)}$ and $B^{(k)}$ are operators with support on \mathcal{H}_S and \mathcal{H}_E , respectively. With a slight abuse of notation we will often conflate $A^{(k)}$ with $A^{(k)} \otimes \mathbb{I}_E$ and $B^{(k)}$ with $\mathbb{I}_S \otimes B^{(k)}$, where \mathbb{I}_S and \mathbb{I}_E are, respectively, the identity operator on \mathcal{H}_S and \mathcal{H}_E . Here we do not impose a specific choice for the basis of the system and environment. While a natural one is to choose the bases that diagonalize the non-interacting Hamiltonians H_S and H_E , there are cases where a different choice is more appropriate. For example, in the case of a spin-1/2 chain one might be interested in using the computational basis. Thus, we keep things general and use a generic basis, with no particular properties with respect to the algebra of observables: $\{|a_j\rangle\}_{j=0}^{d_S}$ and $\{|b_{\alpha}\rangle\}_{\alpha=0}^{d_E-1}$.

Since the goal is to derive a dynamic equation for $\{x_{\alpha}(t)\}$ and $\{\Gamma_{\alpha}(t)\}$, we begin with the overall Schroedinger equation for the pure state $|\psi(t)\rangle$ of the joint system+environment, written in the generic tensor product

basis defined above, with $\psi_{j\alpha}(t) = \langle a_j, e_{\alpha} | \psi(t) \rangle$:

$$i\hbar \frac{d\psi_{j\alpha}(t)}{dt} = \sum_{k,\beta} H_{j\alpha;k\beta} \psi_{k\beta}(t), \quad (24)$$

where

$$\begin{aligned} H_{j\alpha;k\beta} &:= \langle a_j, b_{\alpha} | H | a_k, b_{\beta} \rangle \\ &= (H_S)_{jk} \delta_{\alpha\beta} + \delta_{jk} (H_E)_{\alpha\beta} + \sum_{n=1}^M A_{jk}^{(n)} B_{\alpha\beta}^{(n)}. \end{aligned} \quad (25)$$

We now collect the probability $x_{\alpha}(t)$ and the states $|\chi_{\alpha}(t)\rangle$ into a single quantity: a non-normalized ket $|\Phi_{\alpha}(t)\rangle := \sqrt{x_{\alpha}} |\chi_{\alpha}\rangle \in \mathcal{H}_S$. By plugging Schroedinger's equation into the time-derivative of $|\Phi_{\alpha}(t)\rangle$ we get the following set of d_E coupled linear equations:

$$i\hbar \frac{d|\Phi_{\alpha}\rangle}{dt} = H_S |\Phi_{\alpha}\rangle + \sum_{\beta} (H_E)_{\alpha\beta} |\Phi_{\beta}\rangle + \hat{M}_{\alpha\beta} |\Phi_{\beta}\rangle, \quad (26)$$

where $\hat{M}_{\alpha\beta}$ is a set of operators acting on the system, defined as

$$\hat{M}_{\alpha\beta} := \sum_{k=1}^M B_{\alpha\beta}^{(k)} A^{(k)} \quad (27)$$

This can be further manipulated to separate the non-interacting part, acting on each vector $|\Phi_{\alpha}\rangle$, from the interacting part, acting on $|\Phi_{\beta}\rangle \neq |\Phi_{\alpha}\rangle$. The final form of our kinetic equation is

$$i\hbar \frac{d|\Phi_{\alpha}\rangle}{dt} = \hat{H}_{\alpha} |\Phi_{\alpha}\rangle + \sum_{\beta \neq \alpha} \hat{V}_{\alpha\beta} |\Phi_{\beta}\rangle, \quad (28)$$

where the single-particle Hamiltonian is \hat{H}_{α} is

$$\hat{H}_{\alpha} := H_S + (H_E)_{\alpha\alpha} \mathbb{I}_S + \sum_{k=1}^M B_{\alpha\alpha}^{(k)} A^{(k)}, \quad (29)$$

and the interaction between particles is vehiculated by the set of operators $\hat{V}_{\alpha\beta}$

$$\hat{V}_{\alpha\beta} := (H_E)_{\alpha\beta} \mathbb{I}_S + \sum_{k=1}^M B_{\alpha\beta}^{(k)} A^{(k)}. \quad (30)$$

The remaining step is to connect this with the quantities that determine the geometric quantum state of the system: x_{α} and Γ_{α} . This is easily done by using the original definition $|\Phi_{\alpha}\rangle = \sqrt{x_{\alpha}} |\chi_{\alpha}\rangle$. Since $\Gamma_{\alpha}(t) \leftrightarrow |\chi_{\alpha}\rangle$, we have $\Gamma_{\alpha}^j(t) = \frac{\langle a_j | \Phi_{\alpha}(t) \rangle}{\sqrt{x_{\alpha}(t)}}$ and $x_{\alpha}(t) = \langle \Phi_{\alpha}(t) | \Phi_{\alpha}(t) \rangle$.

Equation (28) provides a way to describe the dynamics of an open quantum system as a set of d_E linear, coupled, differential equations, for the non-normalized states $|\Phi_{\alpha}\rangle$. The emergence of this set of coupled equation

also reinforces the interpretation of $\Gamma_\alpha(t)$ as behaving as “particle-like”, with their own local notion of energy, represented by H_α and their pair-wise interaction $\hat{V}_{\alpha\beta}$. While not treated here, by using the approach developed in Ref.[1] these results can be extended to include finite-systems interacting with infinite-dimensional environments.

As a consistency check, we note that the preservation of information within the quantum state space of the system is inherited by the unitarity of the global evolution. This is manifested in the relations $\hat{H}_\alpha^\dagger = \hat{H}_\alpha$ and $\hat{V}_{\alpha\beta}^\dagger = \hat{V}_{\beta\alpha}$, both of which can be easily shown to be true from the definitions above.

So far, we have derived the microscopic, kinetic, equations regulating the information transport across the quantum state space. In the next few sections, using both analytical and numerical approaches, we explore the theoretical framework developed so far to look at the phenomenology of information transport in concrete physical systems. In particular, in our examples we look at the information transport in the state space of a qubit, both in isolated and open configurations. This choice is made to support the theoretical calculations with visual representations, both of which are simpler in the case of a qubit.

IV. SYSTEM 1: ISOLATED QUBIT

Before we begin our case studies, here we collect useful technical details about the geometric quantum mechanics of a qubit. Then, using as a reference basis an arbitrary basis $\{|0\rangle, |1\rangle\}$ the (probability,phase) coordinates are identified via the scalar product operation on the Hilbert space $\langle s|\psi\rangle = \sqrt{p_s}e^{i\phi_s}$, with $s = 0, 1$, giving $(p_0, \phi_0, p_1, \phi_1)$. However, the quantum state space gets rid of two fundamental redundancies in this description. First, we can always assume that $\phi_0 = 0$. Second, due to normalization we have $p_0 = 1 - p_1$. Thus, we only have a pair of independent coordinates: (p_1, ϕ_1) . We can therefore drop the index and say that a general point on the state space of a qubit is uniquely identified by $(p, \phi) \in [0, 1] \times [0, 2\pi[$, with an embedding on the Hilbert space defined by $|\psi(p, \phi)\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}e^{i\phi}|1\rangle$. In analogy with the Bloch sphere, we call the square representation of $\mathbb{C}P^1$ the “Bloch square”. Note that the embedding is smooth everywhere, aside for two isolated points $|0\rangle$ and $|1\rangle$, where ϕ is not defined and which are represented only by $p = 0$ and $p = 1$, respectively. These coordinates (p, ϕ) are canonically conjugated: $\{p, \phi\}$ and the determinant of the Fubini-Study metric is simply $g = \frac{1}{2}$. Moreover, since the total volume of the state space is π [13], we renormalize the Fubini-Study volume element with the total volume, so that $dV = \frac{dp d\phi}{2\pi} = \frac{dV_{FS}}{\text{vol}(\mathbb{C}P^1)}$.

Analytical solution

Our first case study is an isolated qubit, with an arbitrary initial geometric quantum state $f_0(p, \phi)$ and Hamiltonian operator $H = \vec{B} \cdot \vec{m} = E_0|E_0\rangle\langle E_0| + E_1|E_1\rangle\langle E_1|$, with \vec{B} an arbitrary 3D magnetic vector field, $\vec{m} = \frac{\hbar}{2}\vec{\sigma}$ and $\vec{\sigma}$ are the Pauli matrices. Thus, the generator of the geometric evolution is:

$$\begin{aligned} \mathcal{E}(p, \phi) &= \langle \psi(p, \phi) | H | \psi(p, \phi) \rangle = (1-p)H_{00} + pH_{11} \\ &\quad + \sqrt{p(1-p)} (e^{i\phi}H_{01} + e^{-i\phi}H_{10}) \\ &= \vec{B} \cdot \vec{S}, \end{aligned} \quad (31)$$

where

$$\vec{S} = \langle \vec{m} \rangle = \frac{\hbar}{2} (2\sqrt{p(1-p)} \cos \phi, 2\sqrt{p(1-p)} \sin \phi, p)$$

Since the evolution is Hamiltonian, there is no source or sink term, so the continuity equation is

$$\frac{\partial f_t(p, \phi)}{\partial t} + \nabla \cdot J_t = 0. \quad (32)$$

In canonical coordinates, the ∇ operator is simply $\nabla = (\frac{\partial}{\partial p}, \frac{\partial}{\partial \phi})$. Writing explicitly, we get

$$\frac{\partial f_t(p, \phi)}{\partial t} + \frac{\partial J_t^p}{\partial p} + \frac{\partial J_t^\phi}{\partial \phi} = 0. \quad (33)$$

We now come to writing explicitly the flux $J_t = (J_t^p, J_t^\phi)$. As argued above, with Hamiltonian evolution the flux is the product between the Hamiltonian vector field v_H and the geometric quantum state $f_t(p, \phi)$. Explicitly:

$$J_t = \frac{f_t(p, \phi)}{\hbar} \left(\frac{\partial \mathcal{E}(p, \phi)}{\partial \phi}, -\frac{\partial \mathcal{E}(p, \phi)}{\partial p} \right) \quad (34)$$

Plugging this into the continuity equation and remembering that a Hamiltonian vector field is divergence-free, we the following equation for $f_t(p, \phi)$

$$\frac{\partial f_t}{\partial t} + A(p, \phi) \frac{\partial f_t}{\partial p} + B(p, \phi) \frac{\partial f_t}{\partial \phi} = 0, \quad (35)$$

with $A(p, \phi) = \frac{1}{\hbar} \frac{\partial \mathcal{E}}{\partial \phi}$ and $B(p, \phi) = -\frac{1}{\hbar} \frac{\partial \mathcal{E}}{\partial p}$. Alternatively, using the poisson brackets, we get

$$\frac{\partial f_t}{\partial t} + \{f_t, \mathcal{E}\} = 0 \quad (36)$$

Its solution is given by the propagator $P(t, t_0) = \text{Exp}[-(t-t_0)\{\mathcal{E}, \cdot\}]$. Since $\mathcal{E} = \vec{h} \cdot \vec{m}$ and, by using the algebra of poisson brackets, one can prove that $\vec{m} = \frac{\hbar}{2}(S_x(p, \phi), S_y(p, \phi), S_z(p, \phi))$ are the generators of $SO(3)$:

$$\{m_a, m_b\} = \epsilon_{abc} m_c, \quad (37)$$

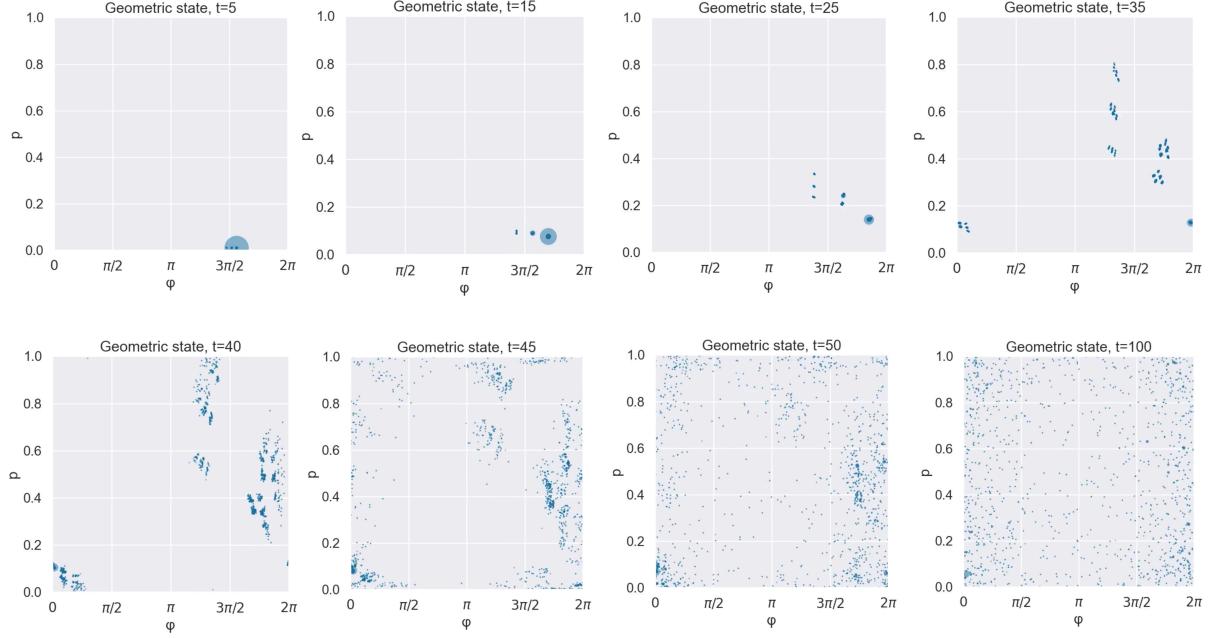


Figure 4. Evolution of the geometric quantum state of one qubit interacting with eleven others via an antiferromagnetic Ising model with both transverse and longitudinal field $\vec{B} = (1, 0, 0.5)$, visualized on the Bloch square. Each particle is $\Gamma_\alpha(t)$, represented in canonically conjugated coordinates $\Gamma_\alpha(t) = (p = p_\alpha(t), \phi = \phi_\alpha(t))$, with $|\psi(p, \phi)\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}e^{i\phi}|1\rangle$. The radius of the particle encodes its probability mass $x_\alpha(t)$. Time increases left to right and top to bottom. Each particle carries a probability mass x_α , which is the probability to find the system in a state Γ_α . Thus, we think of it as an information carrier, moving the information about the state of a quantum system around the quantum state space. After 60/70 time steps the distribution settles on an equilibrium distribution. A full video of the dynamical evolution is available at the following link: https://github.com/fabioanza/GeomQuantMech/blob/main/video_g1_h05_L12.mp4

here we recognize that the propagator $P(t, t_0)$ implements a 3D rotation of the generic expectation value $\vec{m}(t)$ around the axis \vec{B} , with angle $||\vec{B}||(t - t_0)$. In more physics-agnostic terms, this is a linear, first-order, partial differential equation, which can be solved in full generality by using the method of the characteristics. Here we simplify the treatment by switching to coordinates aligned to the convenient basis in which the Hamiltonian is diagonal. This leads to

$$\frac{\partial f_t}{\partial t} - \omega \frac{\partial f_t}{\partial \phi} = 0, \quad (38)$$

where $\omega = \frac{E_1 - E_0}{\hbar}$. This is equivalent to the transport equation for advection, or heat equation, which leads to the analytical solution for the time-dependent geometric quantum state: $f_t(p, \phi) = f_0(p, \phi + \omega t)$, for any valid initial distribution $f_0(p, \phi)$.

Discussion

The evolution of a isolated qubit is periodic in time and this appears in the evolution of its geometric quantum state as $f_0(p, \phi + \omega t)$. Crucially, as the distribution

changes over time with a Hamiltonian evolution, it does so in a rigid fashion, never changing its shape. This can, indeed, be seen explicitly. Say that, at $t = 0$, the geometric quantum state is made by a convex sum of dirac deltas concentrated on a certain number N of points $\Gamma_\alpha = (p_\alpha, \phi_\alpha)$: $q_0(p, \phi) = \sum_{\alpha=1}^N \lambda_\alpha 2\delta[p - p_\alpha] \delta[\phi - \phi_\alpha]$. The evolution of the geometric quantum state is therefore dictated by how the points $\Gamma_\alpha(t)$ change over time. Since, however, the evolution can be written as a linear change in the ϕ coordinate $\phi(t) = \phi(0) + \omega t$, this translates into $\phi_\alpha(t) = \phi_\alpha(0) - \omega t$, with $p_\alpha(t) = p_\alpha(0)$. As a result, the shape of the distribution never changes as the relative distance between the points $\Gamma_\alpha(t)$ does not depend on time. Calling $d_{FS}(\cdot, \cdot)$ the Fubini-Study distance, we have

$$\cos d_{FS}(p_\alpha, \phi_\alpha; p_\beta, \phi_\beta) := \left(1 - (p_\alpha + p_\beta) + p_\alpha p_\beta + \sqrt{p_\alpha(1 - p_\alpha)} \sqrt{p_\beta(1 - p_\beta)} \cos(\phi_\alpha - \phi_\beta) \right)^{1/2}, \quad (39)$$

giving

$$d_{FS}(\Gamma_\alpha(t), \Gamma_\beta(t)) = d_{FS}(\Gamma_\alpha(0), \Gamma_\beta(0)) \quad \forall \alpha, \beta. \quad (40)$$

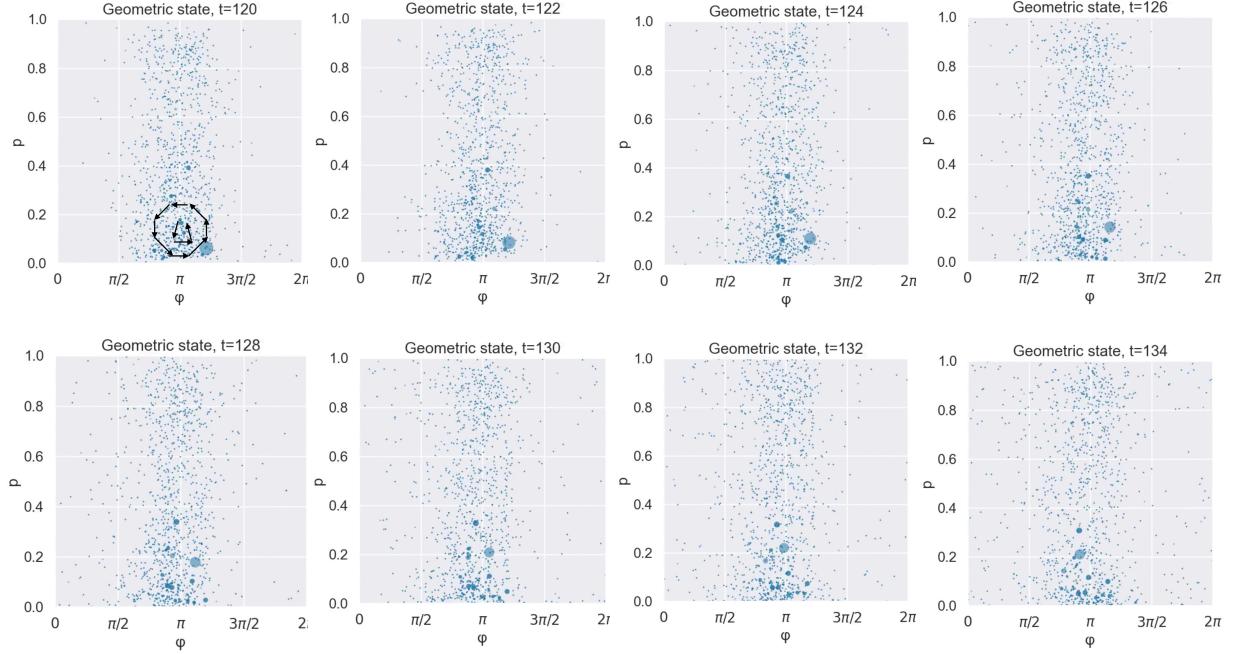


Figure 5. Evolution of the geometric quantum state of one qubit interacting with eleven others via a Ferromagnetic Ising model with longitudinal and transverse field, visualized on the Bloch square. Each particle is $\Gamma_\alpha(t)$, represented via canonically conjugated coordinates $\Gamma_\alpha(t) = (p = p_\alpha(t), \phi = \phi_\alpha(t))$, with $|\psi(p, \phi)\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}e^{i\phi}|1\rangle$. The radius of the particle encodes its probability mass $x_\alpha(t)$. Time increases left to right and top to bottom. Each particle carries a probability mass x_α , which is the probability to find the system in a state Γ_α . Here we show the detail of the evolution of a macroscopic coherent phenomenon that gives rise to sustained fluctuations and maintains the system out of thermal equilibrium. A full video of the dynamical evolution is available https://github.com/fabioanza/GeomQuantMech/blob/main/video_g1_h05_L12_ferro_long.mp4

Since Hamiltonian evolutions can generically support folding and stretching in state space, causing chaotic behavior, we conclude that this is not the most generic Hamiltonian evolution a qubit state space can support. Note that this goes beyond the assumption of a time-independent Hamiltonian. Even with time-dependent Hamiltonian, this aspect of the isolated evolution of a qubit still holds true, thus confirming that a unitary evolution might not be the most general Hamiltonian evolution that can occur in a quantum state space.

V. SYSTEM 2: QUBIT IN ISING ENVIRONMENT

We now look at the case of a single qubit which interacts with an environment made by a $1D$ chain of qubits evolving with an Ising Hamiltonian with periodic boundary conditions $\vec{\sigma}_{k+L} = \vec{\sigma}_k$

$$H_{\text{Ising}} = \sum_{k=0}^L J_z \sigma_k^z \sigma_{k+1}^z + \vec{B} \cdot \vec{\sigma}_k , \quad (41)$$

with \vec{B} a homogeneous magnetic field. Here, since the number of points in quantum state space is $2^{L_{\text{env}}}$, where

L_{env} is the size of the environment, solving the equations of motion analytically becomes unfeasible, and we resort to a numerical approach, in which we can directly evaluate physical quantities of interest, such as the information flux, the sink/source term and the entropies of the various distribution involved. In our numerical analysis, at $t = 0$ the system and the environment begin in a product state, in which the state of the system is $|1\rangle$. Hence, the geometric quantum state is a dirac delta distribution, concentrated on the north pole. However, as time goes by, the overall pure state is not a product anymore and we end up with a non-trivial geometric quantum state, whose evolution is dictated by Eq.(28) or, at the phenomenological level, by the continuity equation. We can map the evolution of the geometric quantum state on the surface of the Bloch sphere, as done in Figure 1 for example, or using (p, ϕ) coordinates, on the Bloch square.

Anti-Ferromagnetic example

In Figure 4 we show the evolution of the geometric quantum state of a qubit with an environment of 12 other qubits, all interacting with an Ising Hamiltonian with $J_z = 1$, $\vec{B} = (1, 0, 0.5)$ and with an initial uniform product state $|\uparrow \dots \uparrow\rangle$. Here we can see that the geometric

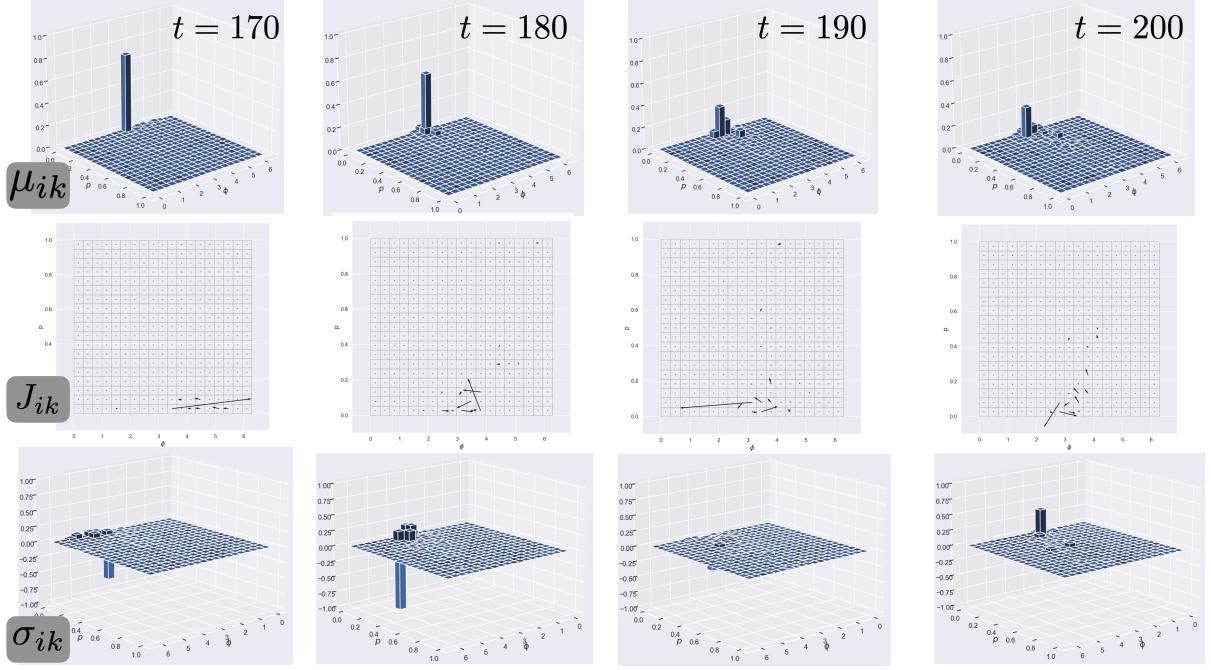


Figure 6. Evolution of the coarse-grained geometric quantum state (Top row), Information Flux (Middle row) and Information sink/source (Bottom row), for one qubit interacting with eleven other ones via a Ferromagnetic Ising mode with longitudinal and transverse field, visualized on the Bloch square. We have sliced the quantum state space in (p, ϕ) coordinates into a grid of 19×19 small uniform cells $\mathcal{C}_{ik} = \left[\frac{i}{20}, \frac{i+1}{20} \right] \times \left[2\pi \frac{k}{20}, 2\pi \frac{k+1}{20} \right]$ of equal volume and integrated the relevant quantities for the study of the continuity equation over \mathcal{C}_{ik} to obtain a time-dependent coarse-grained graphic representation: $\mu_{ik}(t)$, $J_{ik}(t)$ and $\sigma_{ik}(t)$.

quantum state, while initially localized, quickly spreads around to reach an equilibrium distribution. It is, however, interesting to see that it reaches equilibrium with a highly structured fashion. Here we can see that for $t \approx 0$ the distribution of points $\Gamma_\alpha(t)$ is organized in three clusters of states which are quite close to each other. As time goes by, each of these clusters *tri-furcates* into smaller clusters of less points. After this splitting occurs a certain number of time we loose track of the clustering and the distribution reaches an equilibrium configuration. The origin of this behavior has to be tracked to the structure of the operators $\hat{V}_{\alpha\beta}$. In particular, we believe this the tri-furcation to be due to the fact that there are two nearest neighbors directly interacting with our system. Increasing this number would result in a more detailed splitting, increasing the number of sub-clusters. To confirm this, we performed the same exact simulation, with a next-to-nearest neighbor Ising model, observing the formation of 5 sub-clusters at each splitting. A phenomenon we can call “penta-furcation”. This suggests the number of new subclusters n_{sc} to be equal to the number n_{int} of qubits our system interacts directly with: $n_{sc} = n_{int}$. While we do not have a definitive explanation for this phenomenon, it has a distinct, and unexpected, fractal flavor which deserves further investigation.

After a transient of 60/70 time-steps, with $\delta t = 0.005$, we can see that the distribution settles on an equilib-

rium configuration that is almost flat. This can be seen by looking directly at the distribution of information contained in a certain region of the state space. By discretizing the Bloch square, and therefore \mathbb{CP}^1 , via its canonical coordinates, we can integrate the geometric quantum state on the small cells, track their evolution and extract a phenomenological evolution equation. Given a certain number $N_p N_\phi$ of desired cells $\mathcal{C}_{ik} = \left[\frac{i}{N_p}, \frac{i+1}{N_p} \right] \times \left[\frac{2\pi k}{N_\phi}, \frac{2\pi(k+1)}{N_\phi} \right]$, we have $\bigcup_{i=1}^{N_p} \bigcup_{k=1}^{N_\phi} \mathcal{C}_{ik} = \mathcal{P}(\mathcal{H})$, resolution $\delta_p = \frac{1}{N_p}$ and $\delta_\phi = \frac{2\pi}{N_\phi}$. Given a geometric quantum state $q_t(Z)$, the probability mass in each cell is $\mu_{ik}(t) = \int_{\mathcal{C}_{ik}} q_t(Z) dV_{FS}$. By integrating the continuity equation (Eq.(16)) over a cell \mathcal{C}_{ik} we get the following equation:

$$\frac{d\mu_{ik}}{dt} + F_{ik} = \sigma_{ik} , \quad (42)$$

where

$$\sigma_{ik}(t) = \int_{\mathcal{C}_{ik}} dV_{FS} \sigma_t(Z) \quad (43)$$

$$F_{ik} = \int_{\mathcal{C}_{ik}} dV_{FS} \nabla \cdot J_t . \quad (44)$$

Using the divergence theorem, we can turn F_{ik} into an

integral over the surface $\partial\mathcal{C}_{ik}$ of \mathcal{C}_{ik} :

$$F_{ik} = \int_{\partial\mathcal{C}_{ik}} J_t \cdot n dS_{FS} = \Phi_{ik}(J_t) , \quad (45)$$

where $n = (n_p, n_\phi)$ is the vector orthogonal to the boundary of \mathcal{C}_{ik} and dS_{FS} is the surface element. This quantity, $F_{ik} = \Phi_{ik}(J)$ is the flow of probability mass across the boundary of each cell \mathcal{C}_{ik} due to the underlying movement of the points $\Gamma_\alpha(t)$. On the other hand, $\sigma_{ik}(t)$ quantifies the probability mass that is produced or sinked in \mathcal{C}_{ik} . Beyond the initial transient, as the distribution settles on a more regular behavior, we expect a more regular evolution equation to emerge. In particular, we expect two emergent behavior. First, we assume the sink/sources term to be zero after a short transient $\sigma_{ik} \approx 0$. This assumption is motivated by two physical reasons. Since the $x_\alpha(t)$ are the probabilities of finding the environment in one of the elements of its eigenbasis $|b_\alpha\rangle$, for a large environment (the number of points is exponential in the size of the system) it is reasonable to expect that after a short transient these probabilities will settle on “equilibrium” values (as the ones given by max entropy) and, possibly, oscillate around them. Second, we are interested in looking at the system on longer time-scales, where these oscillations can be averaged out. Together, these arguments supports the assumptions for $\sigma_{ik} \approx 0$. This implies that the evolution is guided only by the flux term, leading to a local conservation of probability mass. Thus, we can now build a microscopic, particle-like, model for $\Phi_{ik}(J_t)$. Since probability mass is locally conserved, the only way \mathcal{C}_{ik} can loose or gain probability mass is by exchanging particles with its immediate neighbors: $\mathcal{C}_{i\pm 1,k}$ and $\mathcal{C}_{i,k\pm 1}$. Thus, as first approximation, we assume there is a fixed rate of gain and loss of particles, in each direction (p, ϕ) : γ_p and γ_ϕ . This leads to the following phenomenological model for the evolution of $\mu_{ik}(t)$:

$$\frac{d\mu_{ik}}{dt} = \gamma_p (\mu_{i+1,k} + \mu_{i-1,k}) + \gamma_\phi (\mu_{i,k+1} + \mu_{i,k-1}) - \gamma_{\text{loss}} \mu_{ik} . \quad (46)$$

The first two terms represent the gain due to a certain number of particles entering from each of the four sides of a cell \mathcal{C}_{ik} , either from p or ϕ direction. The last term represents the loss due to a certain number of particles leaving \mathcal{C}_{ik} for one of its four adjacent neighbors. By assuming that there are no special cells, unless otherwise specified, this leads to $\gamma_{\text{loss}} = 2(\gamma_p + \gamma_\phi)$. This model leads to a simple diffusive behavior that, while not entirely accurate, clearly captures the essence of the evolution beyond the transient. While finite-size fluctuations in time are to be expected, as we increase the size of the Ising environment we expect them to be less relevant, and our model more accurate. A full analysis of the phenomenology of this model will be carried elsewhere. Here we are interested in proving that this theoretical framework can lead to tractable phenomenological mod-

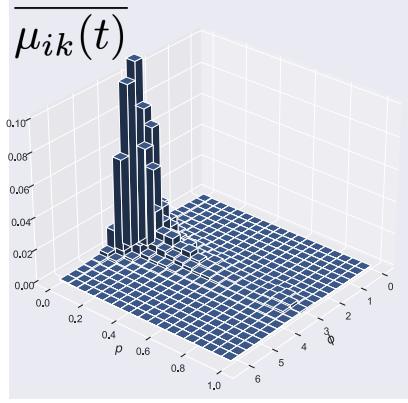


Figure 7. Time-average of the coarse-grained geometric quantum state, $\mu_{ik}(t)$, which conveys the information about how much information about the system is contained in a small cell \mathcal{C}_{ik} of state space. In the case of a ferromagnetic Ising environment the non-equilibrium behavior of the quantum system maintains forces the information to be localized in a certain region of the state space, loosely localized around $Z_{\text{center}} = (p = 0.1, \phi = \pi)$. Here this is shown by plotting the time-average of $\mu_{ik}(t)$, after the first 100 time-steps, to avoid including transient behavior. A localized distribution is evident.

els. Indeed, in the limit of an infinitely fine-grained discretization this model leads to a Fick’s law of information diffusion $\mathcal{P}(\mathcal{H})$, as $J_t^{\text{diff}} = (-\gamma_p \frac{\partial q_t}{\partial p}, -\gamma_\phi \frac{\partial q_t}{\partial \phi}) = -\gamma \cdot \nabla q_t$ and therefore to a diffusion equation in $\mathcal{P}(\mathcal{H})$:

$$\frac{\partial q_t}{\partial t} = \gamma \cdot \nabla^2 q_t , \quad (47)$$

where $\gamma = (\gamma_p, \gamma_\phi)$ are direction-dependent diffusion coefficients.

Ferromagnetic example

In figure 5 we show the evolution of the geometric quantum state for a qubit in a ferromagnetic Ising environment. We do not show the short-time transient, as it is qualitatively similar to the antiferromagnetic case, still exhibiting the tri-furcation dynamical pattern. Rather, we focus on two aspects of this non-equilibrium evolution, which were absent from the antiferromagnetic example: *sustained fluctuations*. We have highlighted them with the black arrow superimposed to the top left plot, and are evident in the video, which can be found at https://github.com/fabioanza/GeomQuantMech/blob/main/video_g1_h05_L12_ferro_long.mp4. At its core, this is an emergent phenomenon in which the probability mass, instead of diffusing across the whole state space as in the antiferromagnetic example, appears to be rotating around a point loosely localized as $Z_{\text{center}} \approx (p = 0.1, \phi = \pi)$. As a result of this, over time the distribution $\mu_{ik}(t)$ appears to remain localized in a small region around

Z_{center} . This can be seen by plotting the time-averaged information distribution $\mu_{ik}(t) = \frac{1}{400} \sum_{n=101}^{500} \mu_{ik}(t_n)$, in which the time-average has been performed by excluding the first 100 time-steps, to exclude the transient behavior. The result is shown in the two bottom rows of Figure 6. We now use the approach developed in the previous section to understand how this occurs. Indeed, by plotting the coarse-grained flux of information $J_{ik}(t) = \int_{C_{ik}} J_t(Z) dV_{FS}$ and the coarse-grained information production term, in Figure 6 we see that the localization of the information around Z_{center} , and its stability over time, arises from the competition between a sink term, drawing up information and delocalizing it across the state space, and a rotating vector field $J_{ik}(t)$, moving the points away from the sink but maintaining them close to Z_{center} , with an overall rotation effect. Over time, these two effects balance each other out maintaining the information localized around Z_{center} , as it can be seen from the instantaneous distribution $\mu_{ik}(t)$, plotted in the top row of Figure 6.

While a fully analytical description of this phenomenon requires an in-depth investigation of the microscopic details, the analysis given in term of information flux and information sink and source term provides a clear connection between the microscopic, kinetic, evolution and the macroscopic behavior provided by expectation values of observables like $\langle \sigma_z \rangle$.

VI. CONCLUSIONS

We have developed a consistent framework to study how quantum systems move information around their state space. Our main results are the continuity equation (Eq.(16)) and the underlying kinetic equations (Eq.(28)). Together, they regulate how information can be transported around the quantum state space as a result of the interactions between the system and its environment. The kinetic equation represents the microscopic, detailed, model while the goal of the continuity equation formulation is to extract emergent, simplified, phenomenological descriptions. As an example of this logic, in Section IV and V we have looked at the transport of information in the state space of a qubit. In both cases we have seen the emergence of consistent behavior that can be described with sufficiently simple information fluxes. In the case of an isolated qubit, we were able to analytically extract the information flux. The case of an antiferromagnetic Ising environment with both longitudinal and transverse field led us to see how a simple diffusion model can reproduce the macroscopic feature of the kinetic evolution.

While here we focused only on the theoretical framework, we believe we have shown that the study of concrete kinetic models of information transport is phenomenologically quite rich, and deserves to be investigated in depth. The underlying reason behind our efforts in this kind of geometric-oriented research direction is that we believe

these new tools will allow us to (1) gather a more accurate understanding and (2) build improved models for the nonequilibrium behavior of quantum systems. In particular, we believe these tools will reveal to be appropriate to get a more accurate picture of the interplay between the non-equilibrium physical properties of open quantum systems and the information-theoretic features of their dynamics, which emerge from their complex stochastic nature.

Future work in this direction will be focused on extracting more general, approximated, analytical models and developing more powerful numerical techniques, to investigate the behavior of quantum systems with larger environments. An interesting concrete issue is to understand how quantum information moves around the quantum state space when our system is part of a quantum computer performing quantum computation. Indeed, we believe this framework can help better understand how to improve and optimize quantum information processing for future quantum technologies.

We conclude with a general comment, clarifying our main goal behind the paper, and with a series of future directions opened by our work. While our main goal here was to develop the theoretical framework and build the intuition to support it, we believe the general analyses presented in Sections IV and V revealed four features of interest, which deserve attention on their own. First, the dynamical emergence of the fractal pattern in the transient behavior of the antiferromagnetic model was unexpected and one wonders if similar dynamical patterns are typical of quantum models or not. This is clearly related to the Second point: the more general question of describing stationary states, with non-trivial non-equilibrium behavior. A natural route starts with imposing $\frac{\partial g_t}{\partial t} = 0$ in the continuity equation, which in turn leads to a balance equation between information flux and information sinks and sources. While one can expect that $J_t = \sigma_t = 0$ characterizes thermal equilibrium, this will not be true in general, and the continuity equation can be used to characterize stationary states. Third, in the Discussion part of Section IV (see Eq. (40)), we have seen that a unitary dynamics is a very rigid evolution in which there is a unique velocity field (a divergence-free one) and all points in the support of any geometric quantum state have to move in the same way. It is well known from the theory of dynamical systems that this is not a feature of all Hamiltonian evolutions, which are able to support chaotic phenomenology via stretching and folding. Therefore, a quantum state space should be able to support more general Hamiltonian evolutions, exhibiting richer phenomenology as chaos, of quantum nature. Thus, we believe the geometric formalism and the framework developed here to be a concrete route to improve our understanding of quantum chaos. Fourth, in Section II A we have proven a novel form of Liouville's theorem for quantum systems. This poses a conundrum as it is in stark contrast with all other phase space formulation of quantum mechanics, such as Wigner's [28] or varia-

tions of it such as the one based on the Husimi function [29], both of which are known to violate Liouville's theorem [30]. The validity of Liouville's theorem is not just a mathematical fact, but a truly important property of the dynamical evolution, which can help us characterize useful properties of the dynamics, such as the emergence of chaos. Here, again, the geometric formalism and the theoretical framework developed here is bringing to light interesting new insights into the nonequilibrium behavior of quantum systems.

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- [1] F. Anza and J. P. Crutchfield. Beyond density matrices: Geometric quantum states. *arXiv:2008.08682*.
 - [2] F. Anza and J. P. Crutchfield. Geometric quantum thermodynamics. *arXiv:2008.08683*.
 - [3] F. Strocchi. Complex Coordinates and Quantum Mechanics. *Rev. Mod. Phys.*, 38(1):36–40, 1966.
 - [4] B. Mielnik. Geometry of Quantum States. *Comm. Math. Phys.*, 9:55–80, 1968.
 - [5] T. W. B. Kibble. Geometrization of quantum mechanics. *Comm. Math. Physics*, 65(2):189–201, 1979.
 - [6] André Heslot. Quantum mechanics as a classical theory. *Phys. Rev. D*, 31(6):1341–1348, 1985.
 - [7] D. Page. Geometrical description of Berry's phase. *Phys. Rev. A*, 36, 1987.
 - [8] J. Anandan and Y. Ahronov. Geometry of quantum evolution. *Phys. Rev. Lett.*, 65, 1990.
 - [9] G. W. Gibbons. Typical states and density matrices. *J. Geom. Physics*, 8(1-4):147–162, 1992.
 - [10] A. Ashtekar and T. A. Schilling. Geometry of quantum mechanics. In *AIP Conference Proceedings*, volume 342, pages 471–478. AIP, 1995.
 - [11] A. Ashtekar and T. A. Schilling. Geometrical Formulation of Quantum Mechanics. In *On Einstein's Path*, pages 23–65. Springer New York, New York, NY, 1999.
 - [12] D. C. Brody and L. P. Hughston. Geometric quantum mechanics. *J. Geom. Physics*, 38(1):19–53, 2001.
 - [13] I. Bengtsson and K. Zyczkowski. *Geometry of Quantum States*. Cambridge University Press, Cambridge, 2017.
 - [14] J. F. Cariñena, J. Clemente-Gallardo, and G. Marmo. Geometrization of quantum mechanics. *Theo. Math. Physics*, 152(1):894–903, jul 2007.
 - [15] D. Chruściński. Geometric Aspects of Quantum Mechanics and Quantum Entanglement. *J. Physics: Conf. Ser.*, 30:9–16, 2006.
 - [16] G. Marmo and G. F. Volkert. Geometrical description of quantum mechanics—transformations and dynamics. *Physica Scripta*, 82(3):038117, 2010.
 - [17] J. Avron and O. Kenneth. An elementary introduction to the geometry of quantum states with pictures. *Rev. Math. Physics*, 32(02):2030001, 2020.
 - [18] D. Pastorello. A geometric Hamiltonian description of composite quantum systems and quantum entanglement. *Intl. J. Geo. Meth. Mod. Physics*, 12(07):1550069, 2015.
 - [19] D. Pastorello. Geometric Hamiltonian formulation of quantum mechanics in complex projective spaces. *Intl. J. Geom. Meth. Mod. Physics*, 12(08):1560015, 2015.
 - [20] D. Pastorello. Geometric Hamiltonian quantum mechanics and applications. *Intl. J. Geo. Meth. Mod. Physics*, 13(Supp. 1):1630017, 2016.
 - [21] J. Clemente-Gallardo and G. Marmo. The Ehrenfest picture and the geometry of Quantum Mechanics. *Il Nuovo Cimento C*, 3:35–52, 2013.
 - [22] M. A. Nielsen and I. L. Chuang. *Quantum computation and quantum information*. Cambridge University Press, 2010.
 - [23] T. Heinosaari and M. Ziman. *The mathematical language of quantum theory*. Cambridge University Press, 2012.
 - [24] F. Strocchi. *An introduction to the mathematical structure of quantum mechanics: A short course for mathematicians*. World Scientific, 2008.
 - [25] D. C. Brody and L. P. Hughston. The quantum canonical ensemble. *J. Math. Physics*, 39(12):6502–6508, 1998.
 - [26] D. C. Brody and L. P. Hughston. Thermodynamics of quantum heat bath. *J. Physics A: Math. Theo.*, 49(42):425302, 2016.
 - [27] R. Soto. *Kinetic theory and transport phenomena*. Oxford University Press, New York, US, 2016.
 - [28] E. Wigner. On the quantum corrections for thermodynamic equilibrium. *Phys. Rev.*, 40, 1932.
 - [29] K. Husimi. Some formal properties of the density matrix. *Proceedings of the Physico-Mathematical Society of Japan. 3rd Series*, 22(4), 1940.
 - [30] M. Oliva, D. Kakofengitis, and O. Steuernagel. Anharmonic quantum mechanical systems do not feature phase space trajectories. *Physica A*, 502, 2018.