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operator \mathcal{O} on state $|\psi\rangle$ with coordinates Z :

$$\mathcal{O}(Z) = \frac{\sum_{\alpha,\beta} \mathcal{O}_{\alpha,\beta} Z^\alpha \bar{Z}^\beta}{\sum_\gamma |Z^\gamma|^2}, \quad (1)$$

where $\mathcal{O}_{\alpha\beta}$ is Hermitian $\mathcal{O}_{\beta,\alpha} = \overline{\mathcal{O}_{\alpha,\beta}}$.

Measurement outcome probabilities are determined by *positive operator-valued measurements* (POVMs) $\{E_j\}_{j=1}^n$ applied to a state $|\psi\rangle$. They are nonnegative operators $E_j \geq 0$, called *effects*, that sum up to the identity: $\sum_{j=1}^n E_j = \mathbb{I}$. In GQM they consist of nonnegative real functions $E_j(Z) \geq 0$ on $\mathcal{P}(\mathcal{H})$ whose sum is always unity:

$$E_j(Z) = \frac{\sum_{\alpha,\beta} (E_j)_{\alpha,\beta} Z^\alpha \bar{Z}^\beta}{\sum_\gamma |Z^\gamma|^2}, \quad (2)$$

where $\sum_{j=1}^n E_j(Z) = 1$.

The quantum state space $\mathcal{P}(\mathcal{H})$ has a preferred metric g_{FS} —the *Fubini-Study metric* [?]—and an associated volume element dV_{FS} that is coordinate-independent and invariant under unitary transformations. The geometric derivation of dV_{FS} is beyond our immediate goals here. That said, it is sufficient to give its explicit form in the probability-phase coordinate system $Z^\alpha = \sqrt{p_\alpha} e^{i\phi_\alpha}$ that we use for explicit calculations:

$$\begin{aligned} dV_{FS} &= \sqrt{\det g_{FS}} \prod_{\alpha=0}^{D-1} dZ^\alpha d\bar{Z}^\alpha \\ &= \prod_{\alpha=1}^{D-1} \frac{dp_\alpha d\phi_\alpha}{2}. \end{aligned}$$

Notice how p_0 and ϕ_0 are not involved. This is due to $\mathcal{P}(\mathcal{H})$'s projective nature which guarantees that we can choose a coordinate patch in which $p_0 = 1 - \sum_{\alpha=1}^{D-1} p_\alpha$ and $\phi_0 = 0$. As we will see now, these coordinates play a particular role: they are canonically conjugated.

State-space structure. Beyond the Riemannian structures, it can be shown that $\mathcal{P}(\mathcal{H})$ also has another interesting geometric feature: a symplectic structure. This is the hallmark of classical state spaces and justifies the use of the term *quantum state space* for $\mathcal{P}(\mathcal{H})$. With a more common jargon, a symplectic structure is the geometric entity allowing us to define ‘‘Poisson Brackets’’ and the existence of canonically conjugated coordinates. In particular, using probabilities and phases coordinates $\{(p_\alpha, \phi_\alpha)\}$ one has that $\{p_\alpha, \phi_\beta\} = \frac{1}{\hbar} \delta_{\alpha\beta}$. Thus, for arbitrary functions A and B on $\mathcal{P}(\mathcal{H})$ one has

$$\{A, B\} := \frac{1}{\hbar} \sum_{\alpha=1}^{D-1} \frac{\partial A}{\partial p_\alpha} \frac{\partial B}{\partial \phi_\alpha} - \frac{\partial B}{\partial p_\alpha} \frac{\partial A}{\partial \phi_\alpha} \quad (3)$$

Unitary evolution. In QM, an isolated quantum system evolves with a unitary propagator $U(t, t_0) = e^{-\frac{i}{\hbar} H(t-t_0)}$,

where the generator H is the (time-independent) Hamiltonian of the system. Surprisingly, it can be shown [?] that this evolution is equivalent to a classical Hamiltonian dynamics, with geometric coordinates. In particular, calling $E(p_\alpha, \phi_\alpha) = \langle \psi(p_\alpha, \phi_\alpha) | H | \psi(p_\alpha, \phi_\alpha) \rangle$ the expectation value of H on a generic state $|\psi(p_\alpha, \phi_\alpha)\rangle = \sum_{\alpha=0}^{D-1} \sqrt{p_\alpha} e^{i\phi_\alpha} |e_\alpha\rangle$ parametrized by (p_α, ϕ_α) , the unitary of a generic function A on $\mathcal{P}(\mathcal{H})$ is given by

$$\frac{\partial A}{\partial t} = \{A, E\} \quad (4)$$

Or, equivalently, a generic state $\{(p_\alpha, \phi_\alpha)\}$ evolves according to Hamilton's equations of motion:

$$\frac{dp_\alpha}{dt} = \frac{1}{\hbar} \frac{\partial E}{\partial \phi_\alpha} \quad (5a)$$

$$\frac{d\phi_\alpha}{dt} = -\frac{1}{\hbar} \frac{\partial E}{\partial p_\alpha} \quad (5b)$$

Here the analogies with classical Hamiltonian mechanics are particularly evident. While quantum mechanics is clearly very different from its classical counterpart, at a certain descriptive level we can still use the intuition of classical mechanics to understand the unitary evolution of quantum systems. Thus, as we do for classical mechanics and statistical physics, we continue by introducing probability distributions on the quantum state space.

Geometric quantum states. The geometric framework makes it very natural to view a quantum state as a functional encoding that associates expectation values to observables, paralleling the C^* -algebra formulation of quantum mechanics [?]. The idea is that one considers probability density functions on $\mathcal{P}(\mathcal{H})$ (as $q(Z)$), together with observable functions (as $\mathcal{O}(Z)$). This was introduced in Ref. [?] and here we give a quick summary.

States are functionals $P[\mathcal{O}]$ from the algebra of observables \mathcal{A} to the real line:

$$P_q[\mathcal{O}] = \int_{\mathcal{P}(\mathcal{H})} q(Z) \mathcal{O}(Z) dV_{FS}, \quad (6)$$

where $\mathcal{O} \in \mathcal{A}$, $q(Z) \geq 0$ is the normalized distribution associated with functional P :

$$P_q[\mathbb{I}] = \int_{\mathcal{P}(\mathcal{H})} q(Z) dV_{FS} = 1,$$

and $P_q[\mathcal{O}] \in \mathbb{R}$. In this way, kets $|\psi_0\rangle$ are functionals with a Dirac-delta distribution $p_0(Z) = \tilde{\delta}[Z - Z_0]$:

$$\begin{aligned} P_0[\mathcal{O}] &= \int_{\mathcal{P}(\mathcal{H})} \tilde{\delta}(Z - Z_0) \mathcal{O}(Z) dV_{FS} \\ &= \mathcal{O}(Z_0) = \langle \psi_0 | \mathcal{O} | \psi_0 \rangle. \end{aligned}$$

$\tilde{\delta}(Z - Z_0)$ is shorthand for a coordinate-covariant Dirac-delta in arbitrary coordinates. In homogeneous coordi-

nates this reads:

$$\tilde{\delta}(Z - Z_0) := \frac{1}{\sqrt{\det g_{FS}}} \prod_{\alpha=0}^{D-1} \delta(X^\alpha - X_0^\alpha) \delta(Y^\alpha - Y_0^\alpha) ,$$

where $Z^\alpha = X^\alpha + iY^\alpha$. In (p_α, ν_α) coordinates this becomes simply:

$$\tilde{\delta}(Z - Z_0) = \prod_{\alpha=1}^{D-1} 2\delta(p_\alpha - p_\alpha^0) \delta(\nu_\alpha - \nu_\alpha^0) ,$$

where the coordinate-invariant nature of the functionals $P_q[\mathcal{O}]$ is now apparent. Extending by linearity, mixed states

$$\rho = \sum_j \lambda_j |\lambda_j\rangle \langle \lambda_j|$$

are convex combinations of these Dirac-delta functionals:

$$q_{\text{mix}}(Z) = \sum_j \lambda_j \tilde{\delta}(Z - Z_j) .$$

Thus, expressed as functionals from observables to the real line, mixed states are:

$$P_{\text{mix}}[\mathcal{O}] = \sum_j \lambda_j \langle \lambda_j | \mathcal{O} | \lambda_j \rangle . \quad (7)$$

Equipped with this tools, one identifies the distribution $q(Z)$ of Eq. (6) as a system's *geometric quantum state*. This is a generalized notion of quantum state.

A simple example of a geometric quantum state is the *geometric canonical ensemble*:

$$q(Z) = \frac{1}{Q_\beta} e^{-\beta h(Z)} ,$$

where:

$$Q_\beta = \int dV_{FS} e^{-\beta h(Z)} ,$$

$$h(Z) = \langle \psi(Z) | H | \psi(Z) \rangle ,$$

and H is the system's Hamiltonian operator. This was introduced in Refs. [?]. References [?] and [?] investigated its potential role in establishing a quantum foundation of thermodynamics that is an alternative to that based on Gibbs ensembles and von Neumann entropy.

Density matrix. The connection between geometric quantum states and density matrices is fairly straightforward. Since density matrices are a synthetic way of collecting measurement outcomes about POVMs, which functions of the form $A(Z) \propto \sum_{\alpha,\beta} A_{\alpha\beta} Z^\alpha \bar{Z}^\beta$, given a generic geometric quantum state $q(Z)$ the associated den-

sity matrix ρ^q can be simply computed as follows:

$$\rho_{\alpha\beta}^q = P_q[Z^\alpha \bar{Z}^\beta] = \int_{\mathcal{P}(\mathcal{H})} dV_{FS} q(Z) Z^\alpha \bar{Z}^\beta . \quad (8)$$

Owing to the specific form of POVMs on $\mathcal{P}(\mathcal{H})$, recall Eq. (2), they are sensitive to $q(Z)$ only via ρ^q . Therefore, if two geometric quantum states q_1 and q_2 induce the same density matrix $\rho^{q_1} = \rho^{q_2}$, then all POVMs produce the same outcomes.

The GQS of an Open Quantum System. As shown above, if a system is in a pure state its GQS is simply a Dirac delta concentrated on a single point. However, if the system is in contact with an environment, this is not true anymore, unless they are exactly in a product state. In general, contact with the environment causes the lost of information about which pure state the system inhabits. Hence, we need a probability distribution for the state of our system: a geometric quantum state. The explicit derivation of the generic geometric quantum state for an open quantum system can be found in Ref.[?]. Here we simply give the result, without discussing the derivation. Given a system and its environment, we call $\{|a_j\rangle\}_{j=0}^{d_S-1}$ a basis for the Hilbert space \mathcal{H}_S of the system (dimension d_S) and $\{|b_\alpha\rangle\}_{\alpha=0}^{d_E-1}$ a basis for the Hilbert space \mathcal{H}_E of the environment (dimension d_E). Assuming system and environment are in a pure state $|\psi_{SE}\rangle = \sum_{j,\alpha} \psi_{j\alpha} |a_j\rangle |b_\alpha\rangle$, the geometric quantum state of the system is

$$q(Z) = \sum_{\alpha=0}^{d_E-1} x_\alpha \tilde{\delta}(Z - Z(\chi_\alpha)) , \quad (9)$$

where $x_\alpha = \sum_j |\psi_{j\alpha}|^2$ is the probability that the environment is in state $|b_\alpha\rangle$ and $|\chi_\alpha\rangle = \sum_j \frac{\psi_{j\alpha}}{\sqrt{x_\alpha}} |a_j\rangle \in \mathcal{H}_S$ is a set of d_E states of the system which constitutes the discrete support of the geometric quantum state. The geometric quantum state in Eq.(9) provides the correct reduced density matrix of the system $\rho^S(\psi_{SE}) = \sum_\alpha x_\alpha |\chi_\alpha\rangle \langle \chi_\alpha|$.

II. INFORMATION TRANSPORT: CONTINUITY EQUATION

In the past section we have summarized previous results about GQM. In this section we build on them and, by bringing in the dynamics of the system, we derive a continuity equation which dictates how the geometric quantum state of a non-equilibrium open quantum system evolves, under very general assumptions. This is the fundamental kinetic equation that dictates how the information about the state of a quantum system changes as a result of its interaction with an environment. Throughout

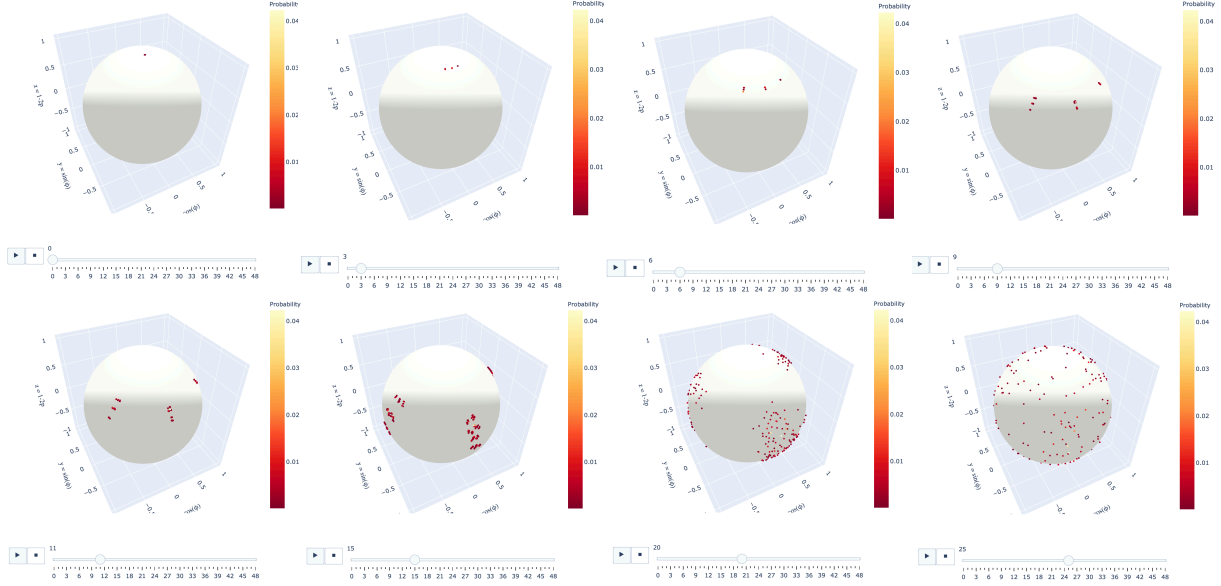


Figure 1: Evolution of the geometric quantum state of one qubit interacting with nine others via an Ising model with transverse field, visualized on the surface of the Bloch sphere. Each point is $\Gamma_\alpha(t)$ and the color of the point encodes the probability $x_\alpha(t)$. Time increases left to right and top to bottom. Each point indicates a possible state in which the system could be, and the color indicates the probability of finding it in that state. The position of each point is determined by the state of the environment and, indeed, there are 2^9 points.

this section we will try to maintain a fairly general language, to emphasize how the treatment applies to quantum systems under very general assumptions. However, for concrete examples we will always refer to the simple case of a qubit.

General treatment. The following treatment pertains quantum systems which are finite-dimensional, and interact with finite-dimensional environments. Other than that, we continuity equation holds for arbitrary quantum systems.

Given an initial geometric quantum state, this is specified by two sets of quantities: $\{x_\alpha\}_{\alpha=0}^{d_E-1}$ and $\{\Gamma_\alpha\}_{\alpha=0}^{d_E-1}$, which is a short-hand notation for $\Gamma_\alpha = Z(|\chi_\alpha\rangle)$. The first one is a classical probability distribution resulting from hypothetical measurements on the environment. For each of the x_α there is a corresponding pure state that the system inhabits: $\Gamma_\alpha \in \mathcal{P}(\mathcal{H}_S)$, which corresponds to the ket $|\chi_\alpha\rangle$. Since these are points moving in a state space $\Gamma_\alpha(t)$, we can think of them as particles on a classical phases space, which interact in a non-trivial way. These are the “carriers of information”, in the sense that each of these particles carries a probability mass $x_\alpha(t)$ that the system will be found on the state $\Gamma_\alpha(t)$. A clarifying example of how the geometric quantum state of a qubit evolves when the system interacts with an environment is given in Figure 1. Since overall information has to be preserved, $\int_{\mathcal{P}(\mathcal{H})} q_t(Z) dV_{FS} = \sum_\alpha x_\alpha(t) = 1$, the geometric quantum state $q_t(Z)$ must satisfy a continuity equation. Conceptually, this is the starting point of virtually all transport theories, which deal with quantities

that are globally conserved, but that are moved around as a result of the underlying dynamics.

Thus, the dynamics of a geometric quantum state has two different terms, arising from the time-evolution of these two sets of quantities. By looking at Figure 1 we can see that there are two different ways in which the geometric quantum state changes in time. First, there is the color of each particle, which changes over time: $\dot{x}_\alpha \neq 0$. Second, there is the pure movements of each point: $\dot{\Gamma}_\alpha \neq 0$. By summing each term over all particles, and then summing the two terms, we get the following evolution equation for the geometric quantum state:

$$\frac{\partial q_t(Z)}{\partial t} = \sum_\alpha \dot{x}_\alpha \tilde{\delta}(Z - \Gamma_\alpha) - x_\alpha \tilde{\delta}'(Z - \Gamma_\alpha) \dot{\Gamma}_\alpha. \quad (10)$$

Here δ' is the first distributional derivative of the dirac delta distribution. The second term in the right-hand side can be written as a covariant divergence¹, since every dependence on the coordinate Z is inside the delta distribution:

$$\sum_\alpha x_\alpha \tilde{\delta}'(Z - \Gamma_\alpha) \dot{\Gamma}_\alpha = (\nabla \cdot J_t)(Z) \quad (11)$$

¹ With an arbitrary coordinate system we have $\text{div} \vec{f} = \frac{1}{\sqrt{g}} \partial_\alpha (\sqrt{g} f^\alpha)$, where $g = |\det g_{FS}|$ is the absolute value of the determinant of the metric which, in our case, is the Fubini-Study metric.

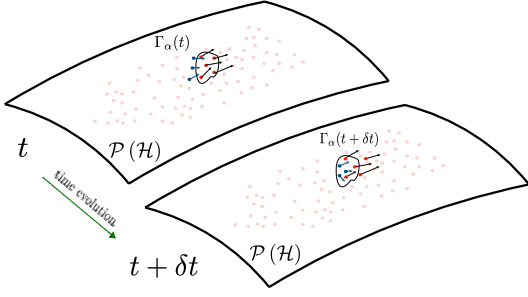


Figure 2: TBD

This allows us to identify the first functional term of the continuity equation, the *flux of information* $J(t)$, which is a weighted average of the “single-particle” flux $J_\alpha(t)$:

$$J_\alpha(t) = \tilde{\delta}(Z - \Gamma_\alpha(t)) \dot{\Gamma}_\alpha(t) \quad (12a)$$

$$J(t) = \sum_\alpha x_\alpha(t) J_\alpha(t) \quad (12b)$$

We now look at the first term in the right-hand: $\sum_\alpha \dot{x}_\alpha(t) \delta(Z - \Gamma_\alpha(t))$. This is identified to be a source/sink term, which is independent on the underlying movement of the points in the quantum state space. This identifies the second functional term of our continuity equation: *sinks and sources of information* $\sigma_t(Z)$:

$$\sigma_t(Z) = \sum_{\alpha=0}^{d_E} \dot{x}_\alpha(t) \tilde{\delta}(Z - \Gamma_\alpha(t)) . \quad (13)$$

Eventually, we obtain the following continuity equation for the geometric quantum state of a finite-dimensional quantum system interacting with a finite-dimensional environment:

$$\frac{\partial q_t}{\partial t} + \nabla \cdot J_t = \sigma_t \quad (14)$$

This is our first result. A crucial aspect here is that this theoretical framework allows us to talk about information as a physically localized (and overall conserved) quantity, which is carried around by points in quantum state space. This is in analogy with the classical theories of information, which deal with properties of particles such as mass or charge, which are carried around by particles, represented as points wondering in a classical phase-space.

To reinforce this point, the physical interpretation of the flux J_t and source/sink σ_t term is essentially the same as in other transport theories. We now describe them explicitly, together with their kinetic interpretation, drawn in Figures 2 and 3. First, J is an information flux, in the proper sense of flux. As one can appreciate in Figure 2, this flux is associated to an overall conserved quantity, localized in its carriers Γ_α , which move around and distribute it across the quantum state space. To em-

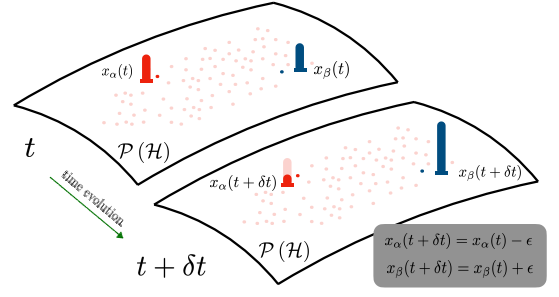


Figure 3: TBD

phasize this point we look at the simpler situation in which $\dot{x}_\alpha = 0$. In this case, the information carried around by the points in state space is a constant of motion ($x_\alpha(t) = x_\alpha(t_0)$). However, as the carriers move around, the information is still dispersed across the quantum state space.

To understand this, first we note that this can not be written as a divergence term. Then, we look at the simpler case in which the position of the points in $\mathcal{P}(\mathcal{H})$ does not change: $\dot{\Gamma}_\alpha = 0$. In this case the support of the distribution is fixed and the particles don't move $\Gamma_\alpha(t) = \Gamma_\alpha(t_0)$. However, even in this case there can still be a non-trivial dynamics, due to $\dot{x}_\alpha(t) \neq 0$. Therefore, this term possesses the hallmarks of sinks and sources terms in general transport theories: it is not associated with particles moving around the state space, and it can not be rewritten as a divergence term.

If the system is closed, upon unitary evolution the distribution remains a Dirac delta, but the point such state is concentrated on changes over time. For example, in the case of a qubit we have coordinates $Z^0 = \sqrt{1-p}$ and $Z^1 = \sqrt{p}e^{i\phi}$. Hence we only have a pair of independent canonically conjugated coordinates (p, ϕ) . If the qubit is isolated, with Hamiltonian $H = E_0 |E_0\rangle \langle E_0| + E_1 |E_1\rangle \langle E_1|$, we can choose as a basis the Hamiltonian eigenstates and its evolution is simply written. Indeed, with this choice of coordinates we have $E(p, \phi) = E_0 + (E_1 - E_0)p = E(p)$. Using Hamilton equations of motion (see Eq.5) we can see that p is preserved, $\dot{p} = \frac{1}{\hbar} \frac{\partial E}{\partial \phi} = 0$ and that ϕ is a cyclical coordinate $\dot{\phi} = -\frac{1}{\hbar} \frac{\partial E}{\partial p} = -\frac{E_1 - E_0}{\hbar}$, giving $\phi(t) = \phi(t_0) - \frac{E_1 - E_0}{\hbar} (t - t_0)$. This is easily generalized to the case of a generic Hilbert space, in which p_α are preserved and $\phi_\alpha(t) = \phi_\alpha(t_0) - \frac{E_\alpha - E_0}{\hbar} (t - t_0)$. In a general coordinate system this means $q_t(Z) = \tilde{\delta}(Z - Z_t)$.

Unitary evolution do not spread information around the state space, but keep it localized, while moving around support of the distribution.

III. DYNAMICS

IV. EXAMPLES

V. CONCLUSIONS

Standard quantum mechanics' concept of state is the density matrix. However, while density matrices provide a complete account of POVM statistics, they are not in one-to-one correspondence with the ensembles that generated them. This is a well-known fact that underlies the freedom in writing a decomposition of the density matrix in terms of probabilities and pure states. All such decompositions yield the same POVM statistics, but they are not physically equivalent since they are realized in physically different ways. The abiding question then becomes: Which of these physically different ways exhibit observable or, even perhaps, manipulable phenomena?

From a purification perspective [?], the physical information about an ensemble's realization can always be thought of as coming from a larger system that is in a pure state. While the additional information about how the ensemble is realized is not relevant for the measurement statistics on our system, it does provide a much richer description. It preserves part (if not all) of the structural information about how the system's POVM statistics result from interactions with its surroundings.

Geometric quantum mechanics and its concept of geometric quantum state provide a framework that allows retaining such information. This yields a richer description of the system—one that goes beyond the system's POVM statistics, taking into account the physical manner in which an ensemble has been realized. The geometric formalism's benefits emerge in at least two important cases: (i) Hybrid continuous-discrete systems, e.g., electrons or other particles with spin or other discrete degrees of freedom and (ii) the thermodynamic setting of a system in contact with a large environment.

The geometric formalism directly handles the continuous nature of hybrid systems and the large number of degrees of freedom in thermodynamics. And, it does so in a fairly direct way. Thus, the geometric quantum state retains the structural information about how an ensemble is generated. While the two applications considered are similar, a crucial difference appears.

If we assume a finite environment, knowledge of the geometric quantum state of our system is sufficient to recover the globally pure joint state of system and environment. This does not occur for a hybrid discrete-continuous system, where knowledge of the geometric quantum state does not allow inferring the phase $\theta_0(\vec{x})$ of $f(\vec{x})$. Notably, fully recovering the overall pure state, whose physical relevance can be argued on the ground of continuity with the finite-dimensional case, effectively translates into a $U(1)$ gauge principle on the overall system. The requirement

that wave-functions differing from a local phase are physically equivalent— $\psi_s(\vec{x}) \sim e^{i\varphi(\vec{x})}\psi_s(\vec{x})$ —turns into a sufficient condition for recovering the global pure state from the local geometric quantum state since, in this case, one can always choose $f(\vec{x}) \in \mathbb{R}$. Effectively, this is the principle behind gauge symmetries and gauge theories. We must leave exploring the connection between recovering the global pure state from a local geometric quantum state and a gauge principle for a future investigation.

Conclusion. Geometric quantum mechanics is an alternative to the standard vector-based formalism. We introduced and then explored the concept of *geometric quantum state* $p(Z)$ as a probability distribution on the manifold of pure states, inspired by the thermodynamic formalism of chaotic attractors from the theory of dynamical systems or, more appropriately, its Sinai-Bowen-Ruelle measures [?]. This characterization accounts for the fact that singling out the density matrix as the sole descriptor of a quantum system's state entails ignoring how an ensemble is physically realized. While this does not have observable consequences if one is restricted to POVM statistics, in concrete situations the information about the ensemble realization can be key to accurate modeling. Reference [?] gives an example. That said, the geometric setting is always sufficiently general that density matrices are readily computed as quadratic averages from $p(Z)$ via Eq. (8).

We explored the physical relevance of geometric quantum states via an open quantum system in which a (finite) system under study is in contact with a larger environment and their joint state is assumed to be pure. In this thermodynamic setting, portions of the structural information about the joint pure state is directly preserved in the geometric quantum state of the smaller system under study. The result is a markedly richer picture of the system's state—a picture that goes substantially beyond the density matrix and its POVM statistics.

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Supplementary Materials

A kinetic theory of quantum information

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Appendix A: The Search for Quantum States

In those domains of the physical sciences that concern the organization and evolution of systems, a common first task is to determine a system's distinct configurations or *effective states*. Ultimately, this turns on what questions there are to answer. One goal is prediction—of properties or behaviors. And, in this, quantum mechanics stands out as a particularly telling arena in which to define effective states.

The very early history of its development can be construed partially as attempts to answer this question, from de Broglie's *phase-waves* [?] and Schrodinger's *wave functions* [?] to von Neumann's *statistical operators* in Refs. [?] and [?, Chap. IV], later labeled *density matrices* by Dirac [? ? ?]. And, these were paralleled by Heisenberg's “operational” *matrix mechanics* that focused on experimentally accessible observables and so avoided imputing internal, hidden structure [?].

The abiding challenge is that effective states are almost always inferred indirectly and through much trial and error. Quantum mechanics heightens the challenge greatly due to its foundational axiom that the detailed, microscopic, and fundamental degrees of freedom cannot be directly and completely measured *in principle*. The main text revisits this perennial question, What is a quantum state?

Appendix B: Theorem ??: Proof

This Appendix gives proves in detail Theorem ??. Let's first restate its setup.

Consider a hybrid quantum system comprised of N continuous degrees of freedom and M qudits that are the discrete ones. The entire system's Hilbert space is:

$$\mathcal{H} = \mathcal{H}_N^c \otimes \mathcal{H}_M^d ,$$

where \mathcal{H}_N^c hosts the continuous degrees of freedom and has infinite dimension, while \mathcal{H}_M^d hosts the discrete ones and has dimension d^M . A basis for \mathcal{H}_N^c is provided by $\{|\vec{x}\rangle\}$, where $\vec{x} \in \mathcal{R} \subseteq \mathbb{R}^N$ and a basis for \mathcal{H}_M^d is $\{|s\rangle\}_{s=0}^{d^M-1}$. Thus, a generic state is:

$$|\psi\rangle = \int_{\mathcal{R}} d\vec{x} \sum_s \psi_s(\vec{x}) |\vec{x}\rangle |s\rangle , \quad (\text{B1})$$

where \vec{x} is a dimensionless counterpart of the physical continuous degrees of freedom, achieved by multiplying its value by appropriate physical quantities. So, the measure $d\vec{x}$ has no physical dimension.

Theorem ??. *Any state $|\psi\rangle \in \mathcal{H}$ can be written as:*

$$|\psi\rangle = \int_{\mathcal{R}} d\vec{x} f(\vec{x}) |x\rangle |q(\vec{x})\rangle ,$$

where $f(\vec{x})$ is such that $\int_{\mathcal{R}} d\vec{x} |f(\vec{x})|^2 = 1$ and $|q(\vec{x})\rangle$ is a parametrized state of the discrete degrees of freedom:

$$|q(\vec{x})\rangle = \sum_{s=0}^{d^M-1} \sqrt{p_s(\vec{x})} e^{i\phi_s(\vec{x})} |s\rangle ,$$

where $\{p_s(\vec{x}), \phi_s(\vec{x})\}_s$ is a set of $2(d^M - 1)$ real functions such that $\sum_{s=0}^{d^M-1} p_s(\vec{x}) = 1$, $\phi_s(\vec{x}) \in [0, 2\pi]$, and $\{|s\rangle\}_{s=0}^{d^M-1}$ is a basis for \mathcal{H}_M^d .

Proof: The proof is constructive. Given an arbitrary $\{\psi_s(\vec{x})\}_s$, we can always find the set of functions $f(\vec{x})$, $p_s(\vec{x})$, and $\phi_s(\vec{x})$. The converse holds trivially: Given these functions one can always compute the $\{\psi_s(\vec{x})\}_s$. First, we define

$\theta_s(\vec{x})$ as the phase of $\psi_s(\vec{x})$:

$$\theta_s(\vec{x}) \in [0, 2\pi[\quad \text{such that} \quad \psi_s(\vec{x}) = |\psi_s(\vec{x})| e^{i\theta_s(\vec{x})} ,$$

and define the functions

$$\phi_s(\vec{x}) := \theta_s(\vec{x}) - \theta_0(\vec{x}) ,$$

Starting from Eq.(B1) this gives

$$\begin{aligned} |\psi\rangle &= \int_{\mathcal{R}} d\vec{x} \sum_s \psi_s(\vec{x}) |\vec{x}\rangle |s\rangle = \int_{\mathcal{R}} d\vec{x} \sum_s |\psi_s(\vec{x})| e^{i\theta_s(\vec{x})} |\vec{x}\rangle |s\rangle = \int_{\mathcal{R}} d\vec{x} \sum_s |\psi_s(\vec{x})| e^{i\phi_s(\vec{x}) + i\theta_0(\vec{x})} |\vec{x}\rangle |s\rangle \\ &= \int_{\mathcal{R}} d\vec{x} e^{i\theta_0(\vec{x})} \sum_s |\psi_s(\vec{x})| e^{i\phi_s(\vec{x})} |\vec{x}\rangle |s\rangle \end{aligned}$$

We now define $f(\vec{x})$ and $p_s(\vec{x})$ as follows:

$$\begin{aligned} f(\vec{x}) &:= \sqrt{\sum_{s=0}^{d^M-1} |\psi_s(\vec{x})|^2} e^{i\theta_0(\vec{x})} \text{ and} \\ p_s(\vec{x}) &:= \frac{|\psi_s(\vec{x})|^2}{\sum_{l=0}^{d^M-1} |\psi_l(\vec{x})|^2} , \end{aligned}$$

It is easy to see how normalization of $|f(\vec{x})|^2$ and of $p_s(\vec{x})$ emerges from the definitions:

$$\begin{aligned} \int_{\mathcal{R}} d\vec{x} |f(\vec{x})|^2 &= \int_{\mathcal{R}} d\vec{x} \sum_{s=0}^{d^M-1} |\psi_s(\vec{x})|^2 \\ &= 1 , \\ \sum_{s=0}^{d^M-1} p_s(\vec{x}) &= \sum_{s=0}^{d^M-1} \frac{|\psi_s(\vec{x})|^2}{\sum_{l=0}^{d^M-1} |\psi_l(\vec{x})|^2} \\ &= 1 , \text{ and} \\ |e^{i\phi_s(\vec{x})}|^2 &= \frac{|\psi_s(\vec{x})|}{|\psi_s(\vec{x})|} \\ &= 1 . \end{aligned}$$

The latter gives $\phi_s(\vec{x}) \in [0, 2\pi]$.

With these definitions we obtain:

$$\begin{aligned} e^{i\phi_s(\vec{x})} f(\vec{x}) \sqrt{p_s(\vec{x})} &= \sqrt{|\psi_s(\vec{x})|^2} e^{i\theta_s(\vec{x})} \\ &= \psi_s(\vec{x}) . \end{aligned}$$

This in turn gives the desired result:

$$\begin{aligned} |\psi\rangle &= \int_{\mathcal{R}} d\vec{x} \sum_s \psi_s(\vec{x}) |\vec{x}\rangle |s\rangle \\ &= \int_{\mathcal{R}} d\vec{x} f(\vec{x}) |\vec{x}\rangle \sum_s e^{i\phi_s(\vec{x})} \sqrt{p_s(\vec{x})} |s\rangle \\ &= \int_{\mathcal{R}} d\vec{x} f(\vec{x}) |x\rangle |q(\vec{x})\rangle . \end{aligned}$$