A phenomenological theory of information transport

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TBD

Introduction. Dynamical systems theory describes long-term recurrent behavior via a system's *attractors*: stable dynamically-invariant sets. Said simply there are regions of state space—points, curves, smooth manifolds, or fractals—the system repeatedly visits. These objects are implicitly determined by the underlying equations of motion and the probability distributions they support—Sinai-Bowen-Ruelle (SRB) measures on the system's state space—are interpreted as the analogs of thermodynamic macrostates [**? ?**]. Here lie the foundations of classical statistical mechanics.

Building on this, the following introduces tools aimed at studying analogously-important state-space structures for quantum systems. This requires developing a more fundamental concept of "state of a quantum system", essentially moving beyond the standard notion of density matrices; though they can be directly recovered. We call these objects the system's *geometric quantum states* and, paralleling SRB measures, they are specified by a probability distribution on the space of pure quantum states.

Quantum mechanics is firmly grounded in a vector formalism in which states $|\psi\rangle$ are elements of a complex Hilbert space $\mathcal{H}$. These are the system's *pure states*. To account for more general situations, one employs *density matrices* ρ. These are operators in $\mathcal{H}$ that are positive semi-definite $\rho \geq 0$, self-adjoint $\rho = \rho^\dagger$, and normalized $\operatorname{Tr} \rho = 1$.

The interpretation of a density matrix as a system's *probabilistic state* is given by *ensemble theory* [**? ?**]. Accordingly, since a density matrix always decomposes into eigenvalues λ_i and eigenvectors $|\lambda_i\rangle$:

$$\rho = \sum_i \lambda_i |\lambda_i\rangle \langle\lambda_i| \ , \tag{1}$$

one interprets ρ as an ensemble of pure states—the eigenvectors—in which λ_i is the probability of an observer interacting with state $|\lambda_i\rangle$.

However, this interpretation is problematic: It is not unique. One can write the same ρ using different decompositions, for example in terms of $\{|\psi_k\rangle\} \neq \{|\lambda_i\rangle\}$:

$$\rho = \sum_k p_k |\psi_k\rangle \langle\psi_k| \ .$$

Given the interpretation, all the decompositions identify the same quantum state ρ. While one often prefers Eq. (1)'s diagonal decomposition in terms of eigenvalues and eigenvectors, it is not the only one possible. More tellingly, in principle, there is no experimental reason to prefer it to others. This fact is often addressed by declaring density matrices with the same *barycenter* equal. A familiar example of this degeneracy is that the maximally mixed state ($\rho \propto \mathbb{I}$) has an infinite number of identical decompositions, each possibly representing a physically-distinct ensemble.

Moreover, it is rather straightforward to imagine systems that, despite being in obviously different physical situations, are described by the same density matrix. For example, consider two distinct state-preparation protocols. In one case, we prepare $\{|0\rangle, |1\rangle\}$ with classical probabilities $p_{0/1} = \frac{1\pm\lambda}{2}$; in the other, we prepare states

$$|\psi_0\rangle = \sqrt{\frac{1+\lambda}{2}} |0\rangle + \sqrt{\frac{1-\lambda}{2}} e^{i\chi} |1\rangle \ , \tag{2a}$$

$$|\psi_1\rangle = \sqrt{\frac{1+\lambda}{2}} |0\rangle + \sqrt{\frac{1-\lambda}{2}} e^{i(\chi+\pi)} |1\rangle \ , \tag{2b}$$

each with probability $1/2$. It is easy to check that they are described by the same ρ. It is also easy to see that the two state-preparation protocols simply traded a classical source of randomness, the bias $p_0 - p_1 = \lambda$, for one of quantum nature, the bias $|\langle\psi_0|\psi_1\rangle|^2 = \lambda$.

The following argues that an alternative—the *geometric formalism*—together with an appropriately adapted measure theory allows to overcome these ambiguities.

With this perspective in mind, first, we recall the basics of *geometric quantum mechanics* (GQM). Second, we discuss how it relates to density matrices. Then, we analyze two broad settings in which the geometric approach arises naturally. We consider a quantum system with finite-dimensional Hilbert space, interacting with another one with (1) infinite-dimensional Hilbert space and (2) finite-dimensional Hilbert space. These are relevant, respectively, for the contexts of quantum state manipulation [**?**] and quantum thermodynamics [**?**]. After discussing

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the results, we draw out several consequences.

Geometric quantum mechanics. References [**? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ?**] give a comprehensive introduction to GQM. Here, we briefly summarize only the elements we need, working with Hilbert spaces $\mathcal{H}$ of finite dimension D.

Pure states are points in the complex projective manifold $\mathcal{P}(\mathcal{H}) = \mathbb{CP}^{D-1}$. Therefore, given an arbitrary basis $\{|e_\alpha\rangle\}_{\alpha=0}^{D-1}$, a pure state is parametrized by D complex homogeneous coordinates $Z = \{Z^\alpha\}$, up to normalization and an overall phase:

$$|\psi\rangle = \sum_{\alpha=0}^{D-1} Z^\alpha |e_\alpha\rangle \ ,$$

where $Z \in \mathbb{C}^D$, $Z \sim \lambda Z$, and $\lambda \in \mathbb{C}/\{0\}$. If the system consists of a single qubit, for example, one can always use probability-phase coordinates $Z = (\sqrt{1-p}, \sqrt{p}e^{i\nu})$.

An *observable* is a function $\mathcal{O}(Z) \in \mathbb{R}$ that associates to each point $Z \in \mathcal{P}(\mathcal{H})$ the expectation value $\langle\psi| \mathcal{O} |\psi\rangle / \langle\psi|\psi\rangle$ of the corresponding operator $\mathcal{O}$ on state $|\psi\rangle$ with coordinates Z:

$$\mathcal{O}(Z) = \frac{\sum_{\alpha,\beta} \mathcal{O}_{\alpha,\beta} Z^\alpha \overline{Z}^\beta}{\sum_\gamma |Z^\gamma|^2} \ , \tag{3}$$

where $\mathcal{O}_{\alpha\beta}$ is Hermitian $\mathcal{O}_{\beta,\alpha} = \overline{\mathcal{O}}_{\alpha,\beta}$.

Measurement outcome probabilities are determined by *positive operator-valued measurements* (POVMs) $\{E_j\}_{j=1}^n$ applied to a state [**? ?**]. They are nonnegative operators $E_j \geq 0$, called *effects*, that sum up to the identity: $\sum_{j=1}^n E_j = \mathbb{I}$. In GQM they consist of nonnegative real functions $E_j(Z) \geq 0$ on $\mathcal{P}(\mathcal{H})$ whose sum is always unity:

$$E_j(Z) = \frac{\sum_{\alpha,\beta} (E_j)_{\alpha,\beta} Z^\alpha \overline{Z}^\beta}{\sum_\gamma |Z^\gamma|^2} \ , \tag{4}$$

where $\sum_{j=1}^n E_j(Z) = 1$.

The projective Hilbert space of the pure states of a quantum system $\mathcal{P}(\mathcal{H})$ has a preferred metric g_{FS}—the *Fubini-Study metric* [**?**]—and an associated volume element dV_{FS} that is coordinate-independent and invariant under unitary transformations. The geometric derivation of dV_{FS} is beyond our immediate goals here. That said, it is sufficient to give its explicit form in the probability-phase coordinate system $Z^\alpha = \sqrt{p_\alpha}e^{i\nu_\alpha}$ that we use for explicit calculations:

$$dV_{FS} = \sqrt{\det g_{FS}} \prod_{\alpha=0}^{D-1} dZ^\alpha d\overline{Z}^\alpha$$
$$= \prod_{\alpha=1}^{D-1} \frac{dp_\alpha d\nu_\alpha}{2} \ .$$

Notice how p_0 and ν_0 are not involved. This is due to $\mathcal{P}(\mathcal{H})$'s projective nature which guarantees that we can choose a coordinate patch in which $p_0 = 1 - \sum_{\alpha=1}^{D-1} p_\alpha$ and $\nu_0 = 0$.

Geometric quantum states. This framework makes it very natural to view a quantum state as a functional encoding that associates expectation values to observables, paralleling the C^*-algebra formulation of quantum mechanics [**?**]. The idea is that one considers probability density functions on $\mathcal{P}(\mathcal{H})$ (as $q(Z)$), together with observable functions (as $\mathcal{O}(Z)$). This was introduced in Ref. [**?**] and here we give a quick summary.

States are functionals $P[\mathcal{O}]$ from the algebra of observables $\mathcal{A}$ to the real line:

$$P_q[\mathcal{O}] = \int_{\mathcal{P}(\mathcal{H})} q(Z)\mathcal{O}(Z)dV_{FS} \ , \tag{5}$$

where $\mathcal{O} \in \mathcal{A}$, $q(Z) \geq 0$ is the normalized distribution associated with functional P:

$$P_q[\mathbb{I}] = \int_{\mathcal{P}(\mathcal{H})} q(Z)dV_{FS} = 1 \ ,$$

and $P_q[\mathcal{O}] \in \mathbb{R}$. In this way, pure states $|\psi_0\rangle$ are functionals with a Dirac-delta distribution $p_0(Z) = \widetilde{\delta}[Z - Z_0]$:

$$P_0[\mathcal{O}] = \int_{\mathcal{P}(\mathcal{H})} \widetilde{\delta}(Z - Z_0)\mathcal{O}(Z)dV_{FS}$$
$$= \mathcal{O}(Z_0) = \langle\psi_0| \mathcal{O} |\psi_0\rangle \ .$$

$\widetilde{\delta}(Z - Z_0)$ is shorthand for a coordinate-covariant Dirac-delta in arbitrary coordinates. In homogeneous coordinates this reads:

$$\widetilde{\delta}(Z - Z_0) := \frac{1}{\sqrt{\det g_{FS}}} \prod_{\alpha=0}^{D-1} \delta(X^\alpha - X_0^\alpha)\delta(Y^\alpha - Y_0^\alpha) \ ,$$

where $Z^\alpha = X^\alpha + iY^\alpha$. In (p_α, ν_α) coordinates this becomes simply:

$$\widetilde{\delta}(Z - Z_0) = \prod_{\alpha=1}^{D-1} 2\delta(p_\alpha - p_\alpha^0)\delta(\nu_\alpha - \nu_\alpha^0) \ ,$$

where the coordinate-invariant nature of the functionals $P_q[\mathcal{O}]$ is now apparent.

In this way, too, mixed states:

$$\rho = \sum_j \lambda_j |\lambda_j\rangle \langle\lambda_j|$$

are convex combinations of these Dirac-delta functionals:

$$q_{\text{mix}}(Z) = \sum_j \lambda_j \widetilde{\delta}(Z - Z_j) \ .$$

Thus, expressed as functionals from observables to the

real line, mixed states are:

$$P_{\text{mix}}[\mathcal{O}] = \sum_j \lambda_j \langle \lambda_j | \mathcal{O} | \lambda_j \rangle \ . \qquad (6)$$

Equipped with this tools, one identifies the distribution $q(Z)$ of Eq. (5) as a system's *geometric quantum state*. This is a generalized notion of quantum state.

A simple example of a geometric quantum state is the *geometric canonical ensemble*:

$$q(Z) = \frac{1}{Q_\beta} e^{-\beta h(Z)} \ ,$$

where:

$$Q_\beta = \int dV_{FS} e^{-\beta h(Z)} \ ,$$
$$h(Z) = \langle \psi(Z) | H | \psi(Z) \rangle \ ,$$

and H is the system's Hamiltonian operator. This was introduced in Refs. [**?**]. References [**?**] and [**?**] investigated its potential role in establishing a quantum foundation of thermodynamics that is an alternative to that based on Gibbs ensembles and von Neumann entropy.

Density matrix. The connection between geometric quantum states and density matrices is two-fold. On the one hand, when $q(Z)$ falls into one of the two aforementioned cases—Dirac-deltas or finite convex combinations of them—the present formalism is equivalent to the standard one. However, not all functionals fall into the Dirac-delta form. Given this, $q(Z)$ is clearly a more general notion of a quantum system's state than density matrices.

On the other hand, given an arbitrary $q(Z)$, there is a unique density matrix ρ^q associated to q:

$$\rho_{\alpha\beta}^q = P_q[Z^\alpha \overline{Z}^\beta]$$
$$= \int_{\mathcal{P}(\mathcal{H})} dV_{FS} \, q(Z) \, Z^\alpha \overline{Z}^\beta \ . \qquad (7)$$

Owing to the specific form of POVMs on $\mathcal{P}(\mathcal{H})$, recall Eq. (4), they are sensitive to $q(Z)$ only via ρ^q. Therefore, if two geometric quantum states q_1 and q_2 induce the same density matrix $\rho^{q_1} = \rho^{q_2}$, then all POVMs produce the same outcomes.

A well-known consequence of this fact is that two density matrices with the same barycenter are considered equal, even if they describe experiments with different physical configurations. In these cases, the statistics of POVM outcomes are described by the same density matrix. Note that this statement does not mean that the two physical configurations are the same. Rather, it means that there is no POVM on the system that distinguishes between q_1 and q_2.

To emphasize, consider the example of two geometric quantum states, q_1 and q_2, with very different charac-

teristics:

$$q_1(Z) = 0.864 \, \tilde{\delta}(Z - Z_+) + 0.136 \, \tilde{\delta}(Z - Z_-) \ ,$$
$$q_2(Z) = \frac{1}{Q} e^{-\frac{1}{2}\overline{Z}\rho^{-1}Z} \ ,$$

where $Q = \int_{\mathbb{C}P^1} dV_{FS} e^{-\frac{1}{2}Z\rho^{-1}\overline{Z}}$, $Z_+ = (0.657, 0.418 + i0.627)$, and $Z_- = (0.754, -0.364 - i0.546)$. However, states q_1 and q_2 have same density matrix ρ ($\rho_{00} = 0.45 = 1 - \rho_{11}$ and $\rho_{01} = 0.2 - i0.3 = \rho_{10}^*$) and so the same POVM outcomes. From Fig. 1 one appreciates the profound difference between q_1 and q_2, despite the equality of their POVM statistics.
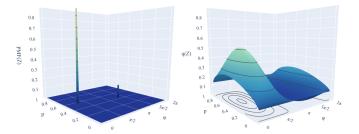

Figure 1. Geometric quantum states in *(probability,phase)* coordinates (p, ϕ) of $\mathbb{C}P^1$: (Left) Geometric quantum state q_1 is the convex sum of two Dirac delta-functions, centered on the eigenvectors $(p_+, \phi_+) = (0.568, 0.983)$ and $(p_-, \phi_-) = (0.432, 4.124)$ of density matrix ρ. (Right) Geometric quantum state q_2 differs markedly: A smooth distribution across the entire pure-state manifold $\mathbb{C}P^1$. However, q_1 and q_2 have the same density matrix $\rho_{q_1} = \rho_{q_2} = \rho$, where $\rho_{00} = 1 - \rho_{11} = 0.45$, $\rho_{01} = \rho_{10}^* = 0.2 - 0.3i$. $\rho_\pm$ are the eigenvalues of the density matrix: $\rho_+ = 0.864$ and $\rho_- = 0.136$. Thus, the marked difference in the structure of geometric quantum states q_1 and q_2 is not reflected in their shared density-matrix $\rho_1 = \rho_2 = \rho$.

This is particularly important for quantum information processing where one encounters long-range and long-lived correlational and mechanistic demands. Quantum computing immediately comes to mind. There, one is not only interested in measurement outcomes, but also in predicting and understanding how a quantum system evolves under repeated external manipulations imposed by complex control protocols.

State manipulation. The following shows that the geometric formalism arises quite naturally when a discrete quantum system interacts and develops entanglement with a continuous one. Imagine a protocol controlling a system's continuous degrees of freedom to manipulate discrete ones that store a computation's result. As a physical reference, consider quantum particles with a given number of discrete degrees of freedom (e.g., spin), confined to a region $\mathcal{R} \subseteq \mathbb{R}^3$. The results we derive do not depend on this choice, since the technical methods straightforwardly extend to other systems where continuous and discrete degrees of freedom are mixed. A helpful illustration is intra-particle entanglement [**?**] that couples position and spin degrees of freedom to create

entangled states. In this way, one manipulates the spin by only acting on the positional degrees of freedom, possibly via a potential.

Consider a hybrid quantum system comprised of N continuous degrees of freedom and M q-dits that are discrete. The entire system's Hilbert space is:

$$\mathcal{H} = \mathcal{H}_N^c \otimes \mathcal{H}_M^d ,$$

where $\mathcal{H}_N^c \coloneqq \mathcal{H}_c^{\otimes N}$ hosts N one-dimensional continuous degrees of freedom and has infinite dimension, while $\mathcal{H}_M^d \coloneqq \mathcal{H}_d^{\otimes M}$ hosts M discrete degrees of freedom, each with dimension d, and therefore has dimension d^M. A basis for $\mathcal{H}_N^c$ is provided by $\{|\vec{x}\rangle\}$, where $\vec{x} \in \mathcal{R} \subseteq \mathbb{R}^N$ and a basis for $\mathcal{H}_M^d$ is $\{|s\rangle\}_{s=0}^{d^M-1}$. Thus, a generic pure state is:

$$|\psi\rangle = \int_{\mathcal{R}} d\vec{x} \sum_s \psi_s(\vec{x}) |\vec{x}\rangle |s\rangle ,$$

where $\vec{x}$ is a dimensionless counterpart of the physical continuous degrees of freedom, achieved by multiplying its value by appropriate physical quantities. So, the measure $d\vec{x}$ has no physical dimension. For an electron in a box, for example, this is achieved by renormalizing with the box's total volume.

The following theorem establishes that this can be done constructively.

Theorem 1. *Any $|\psi\rangle \in \mathcal{H}$ can be written as:*

$$|\psi\rangle = \int_{\mathcal{R}} d\vec{x} f(\vec{x}) |x\rangle |q(\vec{x})\rangle , \qquad (8)$$

where $f(\vec{x})$ is such that $\int_{\mathcal{R}} d\vec{x} |f(\vec{x})|^2 = 1$ and $|q(\vec{x})\rangle$ is a parametrized state of the discrete degrees of freedom:

$$|q(\vec{x})\rangle = \sum_{s=0}^{d^M-1} \sqrt{p_s(\vec{x})} e^{i\phi_s(\vec{x})} |s\rangle ,$$

where $\{p_s(\vec{x}), \phi_s(\vec{x})\}_s$ is a set of $2(d^M - 1)$ real functions such that $\sum_{s=0}^{d^M-1} p_s(\vec{x}) = 1$, $\phi_s(\vec{x}) \in [0, 2\pi]$, and $\{|s\rangle\}_{s=0}^{d^M-1}$ is a basis on $\mathcal{H}_M^d$.

(The Supplementary Material gives the proof.) Equation (8)'s pure-state parametrization preserves key information about the continuous degrees of freedom, namely $|f(\vec{x})|^2$, when working with the discrete degrees of freedom. Indeed, the partial trace over the continuous degrees of freedom yields:

$$\rho = \int_{\mathcal{R}} d\vec{x} |f(\vec{x})|^2 |q(\vec{x})\rangle\langle q(\vec{x})| .$$

Continuing, given an observable $\mathcal{O}$ with support only on $\mathcal{H}_M^d$, we have:

$$\langle \mathcal{O} \rangle = \mathrm{Tr}\, \rho \mathcal{O}$$
$$= \int_{\mathcal{R}} d\vec{x} |f(\vec{x})|^2 \mathcal{O}(q(\vec{x})) ,$$

where $\mathcal{O}(q(\vec{x})) = \langle q(\vec{x})| \mathcal{O} |q(\vec{x})\rangle$. Comparing with Eq. (5) one realizes that the functions $\{p_s(\vec{x}), \phi_s(\vec{x})\}$ provide an $\vec{x}$-dependent embedding of $\mathcal{R} \subseteq \mathbb{R}^N$ onto $\mathbb{C}P^n$, with $n = d^M - 1$, or a submanifold, via:

$$\Phi : \vec{x} \rightarrow \Phi(\vec{x}) = Z(\vec{x}) ,$$

where:

$$Z = (Z^0, \ldots, Z^n) ,$$

with $Z^\alpha(\vec{x}) = \sqrt{p_\alpha(\vec{x})} e^{i\phi_\alpha(\vec{x})}$. Thus, letting $\mathcal{R}^* = \Phi(\mathcal{R})$, we obtain:

$$\int_{\mathcal{R}} d\vec{x} |f(\vec{x})|^2 \mathcal{O}(q(\vec{x})) = \int_{\mathcal{R}^*} dV_{FS}\, q(Z)\mathcal{O}(Z) ,$$

where:

$$q(Z) = \frac{|\det D\Phi(Z)|}{\sqrt{\det g_{FS}}} \left| f(\Phi^{-1}(Z)) \right|^2 .$$

Here, $D\Phi$ denotes the Jacobian of the transformation Φ and g_{FS} is the Fubini-Study metric tensor and we assume the transformation is invertible. Generalizing to cases in which Φ^{-1} is not invertible, due to the fact that different $\vec{x}$ might yield the same $(p_s(\vec{x}), \phi_s(\vec{x}))$, is left to future efforts.

Let's illustrate with a familiar system: an electron in a 2D rectangular box $\mathcal{R} = [x_0, x_1] \times [y_0, y_1]$. In this case, $M = 1$ and $d = 2$ so that the system is described by $f(x, y)$ and $\{p_s(x, y), \phi_s(x, y)\}_{s=0,1}$. With this we have:

$$\langle \mathcal{O} \rangle = \int_{x_0}^{x_1} dx \int_{y_0}^{y_1} dy |f(x, y)|^2 \mathcal{O}(q(x, y))$$
$$= \frac{1}{2} \int_0^1 dp \int_0^{2\pi} d\phi\, q(Z(p, \phi))\, O(Z(p, \phi)) ,$$

where, for example, $p_0(x, y) = 1 - p_1(x, y)$, $p_1(x, y) = (x - x_0)/(x_1 - x_0)$, $\phi_0(x, y) = 0$, and $\phi_1(x, y) = 2\pi(y - y_0)/(y_1 - y_0)$.

In short, a generic ket $|\psi\rangle$ of the whole system uniquely defines a distribution $q(Z)$ on the manifold of pure states $\mathcal{P}(\mathcal{H}_M^d) = \mathbb{C}P^{d^M-1}$. The correspondence is not one-to-one as knowing $q(Z)$ does not allow recovering the entire state. The missing part is $\theta_0(\vec{x})$—$f(\vec{x})$'s phase. However, $q(Z)$ does circumscribe the possible realizations as it fixes the shape of the continuous variables' probability distribution $|f(\vec{x})|^2$.

Note how the embedding functions $p_s(\vec{x})$ and $\phi_s(\vec{x})$ play a key role in determining whether we can cover all of $\mathbb{C}P^n$ or only a submanifold. Consider the conditions that

guarantee the two extreme cases are captured: covering all of $\mathbb{C}P^n$ and covering tensor product sub-manifold $\mathbb{C}P^{d-1} \otimes \ldots \otimes \mathbb{C}P^{d-1}$ only. In the first case, $\mathbb{C}P^n$ is a complex manifold that requires $2n$ independent real coordinates to be completely covered. For M qudits this means:

$$M \leq M_{\max}^{\text{Full}}$$
$$= \frac{\log\left(N/2 + 1\right)}{\log d} .$$

Instead, if we need to cover only the submanifold of tensor product kets, the number of qudits we can control with N continuous degrees of freedom is much larger:

$$M \leq M_{\max}^{\text{Prod}}$$
$$= \frac{N}{2(d-1)} .$$

Most cases fall in between. And so, the number of qudits controllable with N continuous variables is $M \in \left[M_{\max}^{\text{Full}}, M_{\max}^{\text{Prod}}\right]$.

Thermodynamic framework. Another setting in which the geometric formalism arises naturally is quantum thermodynamics. There, one is often interested in modeling the behavior of a small system interacting with a larger environment. For modest-sized environments one can treat the system and environment as isolated and then simulate their evolution. As the environment's size grows, though, this quickly becomes infeasible. Nonetheless, as we now show, the geometric formalism allows appropriately writing the system's reduced density matrix in a way that retains much of the information about the environment. Theorem 1 guarantees this can be done.

Consider a large quantum system consisting of M qudits split in two asymmetric parts. Call the small part with N_S qudits the "system" and let the rest be the "environment" with $N_E = M - N_S$ qudits. A generic ket of the entire system $\mathcal{H}_S \otimes \mathcal{H}_E$ is:

$$|\psi_{SE}\rangle = \sum_{k=0}^{d_S-1} \sum_{\alpha=0}^{d_E-1} \psi_{k\alpha} |s_k\rangle |e_\alpha\rangle ,$$

where $\{|s_k\rangle\}_k$ and $\{|e_\alpha\rangle\}_k$ are bases for $\mathcal{H}_S$ and $\mathcal{H}_E$, respectively, d_E is the dimension of the environment's Hilbert space, and d_S that of the system under study.

Given $|\psi_{SE}\rangle$, it is not too hard to see that the system's density matrix is:

$$\rho^S = \sum_{\alpha=1}^{d_E} p_\alpha^S |\chi_\alpha^S\rangle \langle \chi_\alpha^S| , \qquad (9)$$

where:

$$p_\alpha^S = \sum_{k=0}^{d_S-1} |\psi_{k\alpha}|^2 ,$$

and

$$|\chi_\alpha^S\rangle = \frac{1}{\sqrt{p_\alpha^S}} \sum_{k=0}^{d_S-1} \psi_{k\alpha} |s_k\rangle .$$

In numerical analysis one often retains only the $d_S \times d_S$ matrix elements of ρ^S in a certain basis. However, this erases the functional information about the environment. Instead, the latter can be recovered from $\{p_\alpha^S, |\chi_\alpha^S\rangle\}$ as:

$$\left(\rho^E\right)_{\alpha\beta} = \sqrt{p_\alpha^S p_\beta^S} \langle \chi_\alpha^S | \chi_\beta^S \rangle .$$

As d_E grows, retaining this information as a set of probabilities and kets becomes quickly unrealistic.

However, the same information can be effectively encoded by switching to a geometric description. Indeed, at finite d_E, ρ^S becomes:

$$p_{d_E}^S(Z) = \sum_{\alpha=1}^{d_E} p_\alpha^E \tilde{\delta}\left[Z - Z(\chi_\alpha^S)\right]$$

and the thermodynamic limit is conveniently handled with $p_\infty(Z) = \lim_{d_E \to \infty} p_{d_E}^S(Z)$. Here, as usual, the limit is performed keeping finite the average energy density $\lim_{d_E \to \infty} \langle H \rangle / (N_S + N_E) = \epsilon$.

In this way, the geometric formalism emerges naturally in a quantum thermodynamics setting. In the limit of large environments, one simply cannot track exactly how an environment generates the ensemble of our system under study and so, per force, switches to a probabilistic description. The geometric formalism handles this with the continuous distribution $p_\infty(Z)$, or its approximations, that result from the thermodynamic limit procedure. See also Ref. [**?**] for an expanded exploration of the geometric formalism in quantum thermodynamics.

Before proceeding, though, let's highlight an interesting discrepancy between the two applications presented. In the thermodynamic setting, knowledge of the ensemble $\{p_\alpha^S, |\chi_\alpha^S\rangle\}$ leads to $|\psi_{SE}\rangle$. Indeed, it is easy to see that:

$$\psi_{k\alpha} = \sqrt{p_\alpha^S} \langle s_k | \chi_\alpha^S \rangle .$$

Substituting this into the ket $|\psi_{SE}\rangle$, we obtain a Schmidt-like decomposition in which the common label runs over the dimensions of the environment's Hilbert space:

$$|\psi_{SE}\rangle = \sum_{\alpha=0}^{d_E-1} \sqrt{p_\alpha^S} |\chi_\alpha^S\rangle |e_\alpha\rangle .$$

The price paid for the decomposition is that the kets $|\chi_\alpha^S\rangle$ are not orthogonal. However, we gain a more detailed description of our system's state. As we can see, here the challenge of recovering $|\psi_{SE}\rangle$ disappears thanks to $p_\alpha^S \in \mathbb{R}$. We comment on this discrepancy with the other case shortly.

Discussion. Standard quantum mechanics' concept of state is the density matrix. However, while density matrices provide a complete account of POVM statistics, they are not in one-to-one correspondence with the ensembles that generated them. This is a well-known fact that underlies the freedom in writing a decomposition of the density matrix in terms of probabilities and pure states. All such decompositions yield the same POVM statistics, but they are not physically equivalent since they are realized in physically different ways. The abiding question then becomes: Which of these physically different ways exhibit observable or, even perhaps, manipulable phenomena?

From a purification perspective [**?**], the physical information about an ensemble's realization can always be thought of as coming from a larger system that is in a pure state. While the additional information about how the ensemble is realized is not relevant for the measurement statistics on our system, it does provide a much richer description. It preserves part (if not all) of the structural information about how the system's POVM statistics result from interactions with its surroundings.

Geometric quantum mechanics and its concept of geometric quantum state provide a framework that allows retaining such information. This yields a richer description of the system—one that goes beyond the system's POVM statistics, taking into account the physical manner in which an ensemble has been realized. The geometric formalism's benefits emerge in at least two important cases: (i) Hybrid continuous-discrete systems, e.g., electrons or other particles with spin or other discrete degrees of freedom and (ii) the thermodynamic setting of a system in contact with a large environment.

The geometric formalism directly handles the continuous nature of hybrid systems and the large number of degrees of freedom in thermodynamics. And, it does so in a fairly direct way. Thus, the geometric quantum state retains the structural information about how an ensemble is generated. While the two applications considered are similar, a crucial difference appears.

If we assume a finite environment, knowledge of the geometric quantum state of our system is sufficient to recover the globally pure joint state of system and environment. This does not occur for a hybrid discrete-continuous system, where knowledge of the geometric quantum state does not allow inferring the phase $\theta_0(\vec{x})$ of $f(\vec{x})$. Notably, fully recovering the overall pure state, whose physical relevance can be argued on the ground of continuity with the finite-dimensional case, effectively translates into a $U(1)$ gauge principle on the overall system. The requirement that wave-functions differing from a local phase are physically equivalent—$\psi_s(\vec{x}) \sim e^{i\varphi(\vec{x})}\psi_s(\vec{x})$—turns into a sufficient condition for recovering the global pure state from the local geometric quantum state since, in this case, one can always choose $f(\vec{x}) \in \mathbb{R}$. Effectively, this is the principle behind gauge symmetries and gauge theories. We must leave exploring the connection between recovering the global pure state from a local geometric quantum state and a gauge principle for a future investigation.

Conclusion. Geometric quantum mechanics is an alternative to the standard vector-based formalism. We introduced and then explored the concept of *geometric quantum state* $p(Z)$ as a probability distribution on the manifold of pure states, inspired by the thermodynamic formalism of chaotic attractors from the theory of dynamical systems or, more appropriately, its Sinai-Bowen-Ruelle measures [**?**]. This characterization accounts for the fact that singling out the density matrix as the sole descriptor of a quantum system's state entails ignoring how an ensemble is physically realized. While this does not have observable consequences if one is restricted to POVM statistics, in concrete situations the information about the ensemble realization can be key to accurate modeling. Reference [**?**] gives an example. That said, the geometric setting is always sufficiently general that density matrices are readily computed as quadratic averages from $p(Z)$ via Eq. (7).

We explored the physical relevance of geometric quantum states via an open quantum system in which a (finite) system under study is in contact with a larger environment and their joint state is assumed to be pure. In this thermodynamic setting, portions of the structural information about the joint pure state is directly preserved in the geometric quantum state of the smaller system under study. The result is a markedly richer picture of the system's state—a picture that goes substantially beyond the density matrix and its POVM statistics.

A phenomenological theory of information transport

Fabio Anza and James P. Crutchfield

Appendix A: The Search for Quantum States

In those domains of the physical sciences that concern the organization and evolution of systems, a common first task is to determine a system's distinct configurations or *effective states.* Ultimately, this turns on what questions there are to answer. One goal is prediction—of properties or behaviors. And, in this, quantum mechanics stands out as a particularly telling arena in which to define effective states.

The very early history of its development can be construed partially as attempts to answer this question, from de Broglie's *phase-waves* [**?**] and Schrodinger's *wave functions* [**?**] to von Neumann's *statistical operators* in Refs. [**?**] and [**?** , Chap. IV], later labeled *density matrices* by Dirac [**? ? ?**]. And, these were paralleled by Heisenberg's "operational" *matrix mechanics* that focused on experimentally accessible observables and so avoided imputing internal, hidden structure [**?**].

The abiding challenge is that effective states are almost always inferred indirectly and through much trial and error. Quantum mechanics heightens the challenge greatly due to its foundational axiom that the detailed, microscopic, and fundamental degrees of freedom cannot be directly and completely measured *in principle.* The main text revisits this perennial question, What is a quantum state?

Appendix B: Theorem 1: Proof

This Appendix gives proves in detail Theorem 1. Let's first restate its setup.

Consider a hybrid quantum system comprised of N continuous degrees of freedom and M qudits that are the discrete ones. The entire system's Hilbert space is:

$$\mathcal{H} = \mathcal{H}_N^c \otimes \mathcal{H}_M^d \ ,$$

where $\mathcal{H}_N^c$ hosts the continuous degrees of freedom and has infinite dimension, while $\mathcal{H}_M^d$ hosts the discrete ones and has dimension d^M. A basis for $\mathcal{H}_N^c$ is provided by $\{|\vec{x}\rangle\}$, where $\vec{x} \in \mathcal{R} \subseteq \mathbb{R}^N$ and a basis for $\mathcal{H}_M^d$ is $\{|s\rangle\}_{s=0}^{d^M-1}$. Thus, a generic state is:

$$|\psi\rangle = \int_{\mathcal{R}} d\vec{x} \sum_s \psi_s(\vec{x}) |\vec{x}\rangle |s\rangle \ , \tag{B1}$$

where $\vec{x}$ is a dimensionless counterpart of the physical continuous degrees of freedom, achieved by multiplying its value by appropriate physical quantities. So, the measure $d\vec{x}$ has no physical dimension.

Theorem 1. *Any state $|\psi\rangle \in \mathcal{H}$ can be written as:*

$$|\psi\rangle = \int_{\mathcal{R}} d\vec{x} f(\vec{x}) |x\rangle |q(\vec{x})\rangle \ ,$$

where $f(\vec{x})$ is such that $\int_{\mathcal{R}} d\vec{x}|f(\vec{x})|^2 = 1$ and $|q(\vec{x})\rangle$ is a parametrized state of the discrete degrees of freedom:

$$|q(\vec{x})\rangle = \sum_{s=0}^{d^M-1} \sqrt{p_s(\vec{x})} e^{i\phi_s(\vec{x})} |s\rangle \ ,$$

where $\{p_s(\vec{x}), \phi_s(\vec{x})\}_s$ is a set of $2(d^M - 1)$ real functions such that $\sum_{s=0}^{d^M-1} p_s(\vec{x}) = 1$, $\phi_s(\vec{x}) \in [0, 2\pi]$, and $\{|s\rangle\}_{s=0}^{d^M-1}$ is a basis for $\mathcal{H}_M^d$.

Proof: The proof is constructive. Given an arbitrary $\{\psi_s(\vec{x})\}_s$, we can always find the set of functions $f(\vec{x})$, $p_s(\vec{x})$, and $\phi_s(\vec{x})$. The converse holds trivially: Given these functions one can always compute the $\{\psi_s(\vec{x})\}_s$. First, we define

$\theta_s(\vec{x})$ as the phase of $\psi_s(\vec{x})$:

$$\theta_s(\vec{x}) \in [0, 2\pi[\quad \text{such that} \quad \psi_s(\vec{x}) = |\psi_s(\vec{x})|e^{i\theta_s(\vec{x})} \ ,$$

and define the functions

$$\phi_s(\vec{x}) := \theta_s(\vec{x}) - \theta_0(\vec{x}) \ ,$$

Starting from Eq.(B1) this gives

$$|\psi\rangle = \int_{\mathcal{R}} d\vec{x} \sum_s \psi_s(\vec{x}) |\vec{x}\rangle |s\rangle = \int_{\mathcal{R}} d\vec{x} \sum_s |\psi_s(\vec{x})|e^{i\theta_s(\vec{x})} |\vec{x}\rangle |s\rangle = \int_{\mathcal{R}} d\vec{x} \sum_s |\psi_s(\vec{x})|e^{i\phi_s(\vec{x})+i\theta_0(\vec{x})} |\vec{x}\rangle |s\rangle$$

$$= \int_{\mathcal{R}} d\vec{x} e^{i\theta_0(\vec{x})} \sum_s |\psi_s(\vec{x})|e^{i\phi_s(\vec{x})} |\vec{x}\rangle |s\rangle$$

We now define $f(\vec{x})$ and $p_s(\vec{x})$ as follows:

$$f(\vec{x}) := \sqrt{\sum_{s=0}^{d^M-1} |\psi_s(\vec{x})|^2 \ e^{i\theta_0(\vec{x})}} \text{ and}$$

$$p_s(\vec{x}) := \frac{|\psi_s(\vec{x})|^2}{\sum_{l=0}^{d^M-1} |\psi_l(\vec{x})|^2} \ ,$$

It is easy to see how normalization of $|f(\vec{x})|^2$ and of $p_s(\vec{x})$ emerges from the definitions:

$$\int_{\mathcal{R}} d\vec{x} |f(\vec{x})|^2 = \int_{\mathcal{R}} d\vec{x} \sum_{s=0}^{d^M-1} |\psi_s(\vec{x})|^2$$

$$= 1 \ ,$$

$$\sum_{s=0}^{d^M-1} p_s(\vec{x}) = \sum_{s=0}^{d^M-1} \frac{|\psi_s(\vec{x})|^2}{\sum_{l=0}^{d^M-1} |\psi_l(\vec{x})|^2}$$

$$= 1 \ , \text{ and}$$

$$|e^{i\phi_s(\vec{x})}|^2 = \frac{|\psi_s(\vec{s})|}{|\psi_s(\vec{x})|}$$

$$= 1 \ .$$

The latter gives $\phi_s(\vec{x}) \in [0, 2\pi]$.
With these definitions we obtain:

$$e^{i\phi_s(\vec{x})} f(\vec{x}) \sqrt{p_s(\vec{x})} = \sqrt{|\psi_s(\vec{x})|^2}e^{i\theta_s(\vec{x})}$$

$$= \psi_s(\vec{x}) \ .$$

This in turn gives the desired result:

$$|\psi\rangle = \int_{\mathcal{R}} d\vec{x} \sum_s \psi_s(\vec{x}) |\vec{x}\rangle |s\rangle$$

$$= \int_{\mathcal{R}} d\vec{x} f(\vec{x}) |\vec{x}\rangle \sum_s e^{i\phi_s(\vec{x})} \sqrt{p_s(\vec{x})} |s\rangle$$

$$= \int_{\mathcal{R}} d\vec{x} f(\vec{x}) |x\rangle |q(\vec{x})\rangle \ .$$