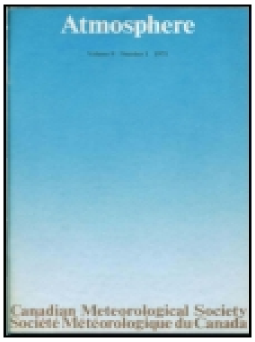

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Fourier series on spheres

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ABSTRACT

A spectral representation consisting of a two-dimensional Fourier series for use on a sphere is described.

The method is applied to the advection of a passive scalar field over the poles and is compared to the pseudo-spectral and grid-point representations.

The results show that the double Fourier series method compares favourably with both the pseudo-spectral and grid-point schemes.

1 Introduction

A method of representing scalar and vector component fields on the sphere by means of two-dimensional series has been developed and applied to the test problem of advection of a conical shape with a non-divergent flow field.

The method grew out of the pseudo-spectral approach of Merilees (1973) and resembles the method of Orszag (1974) although it was developed independently.

The results of the advection calculation using double Fourier series is compared with both the pseudo-spectral method and the "conservative" grid point scheme obtained using the second order Arakawa Jacobian (Arakawa, 1966).

2 Fourier Series Representation

The representation of a scalar or vector field in terms of an expansion in simple functions has a number of advantages for computational purposes. Among these are the analytic nature of the representation and the ability to calculate derivatives and integrals accurately. It may also be possible, given suitable expansion functions, to do unaliased calculations, to maintain accuracy in a least squares sense, and to maintain certain conservative properties inherent in the continuous physical system. For Fourier series, the existence of a fast transform routine is an important consideration.

It is common to represent a scalar field in terms of a truncated Fourier series expansion in longitude (the usual meteorological notation is used):

$$A(\lambda, \phi) = \sum_{|p| \leq \infty} A_p(\phi) e^{ip\lambda} \tag{2.1}$$

The least squares approximation to the function is obtained for

$$A_p = \frac{1}{2\pi} \int_0^{2\pi} A e^{-ip\lambda} d\lambda \tag{2.2}$$

The expression for A is often decomposed further by representing the $A_p(\phi)$ in terms of associated Legendre functions. The resulting spherical harmonic expansion is in terms of functions which are orthogonal with respect to area weighting over the sphere. The orthogonality of the expansion functions implies many desirable properties for theoretical and numerical treatment of atmospheric motions.

While spherical harmonics have desirable properties from several points of view, they are somewhat difficult to use in numerical computations. Fourier series, on the other hand, are particularly simple both analytically and numerically and are amenable to fast transform techniques. Complications arise, however, since a Fourier series representation is not orthogonal with respect to area weighting on the sphere.

Consider $A_p(\phi)$ in eqn. (2.1), with the zero of "latitude" taken at the south pole so that $\phi \in [0, \pi]$. $A_p(\phi)$ may be expanded in terms of a sine or cosine series in this interval. Such a representation, while it will converge to $A_p(\phi)$ in the interval if sufficient terms are retained in the expansion, may exhibit Gibb's phenomenon and need not converge when differentiated term by term.

It is possible, however, to expand $A_p(\phi)$ in a Fourier series which is properly convergent in $[0, \pi]$ and which may be differentiated term by term. A simple demonstration of this is as follows. Given $A(\lambda, \phi)$ defined in the "primary", region, $\lambda \in [0, 2\pi]$, $\phi \in [0, \pi]$, extend the definition of A to the region $\lambda \in [0, 2\pi]$ $\phi \in [0, 2\pi]$ thus:

$$A'(\lambda, \phi) = \begin{cases} A(\lambda, \phi) & , \quad \phi \in [0, \pi] \\ A(\lambda + \pi, 2\pi - \phi), & \quad \phi \in [\pi, 2\pi]. \end{cases} \tag{2.3}$$

Then A' is continuous (if A is) and periodic with period 2π in both λ and ϕ. Moreover, $A' = A$ in the primary region. This definition for A' is just that used by Merilees (*loc cit*) to obtain periodic functions in ϕ for pseudo-spectral calculations. For a given λ, $A'(\lambda, \phi)$ is the value of A which would be traced out following a meridian of longitude completely around the globe.

Since A' is continuous and periodic it may be represented as a double Fourier series

$$A' = \sum_{|p| \leqslant N} \sum_{|q| \leqslant M} A'_{pq} e^{ip\lambda} e^{iq\phi} \tag{2.4}$$

$$A'_{pq} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} A' e^{-ip\lambda} e^{-iq\phi} d\lambda d\phi. \tag{2.5}$$

This series is everywhere well behaved, possesses derivatives and integrals which may be obtained by operating term by term on the series and, in particular, represents the scalar in the primary region.

It can be easily shown that the symmetry of A' in the extended region together with the specification that A' is real leads to the following relations among the coefficients:

$$A'_{p-q} = (-1)^p A'_{pq},$$
$$A'_{-pq} = (-1)^p A'^*_{pq},$$
$$A'_{-p-q} = A'^*_{pq}.$$

The Fourier series (2.4) may be written to apply to the primary region in the form

$$A(\lambda, \phi) = \sum_{|p| \leq N} \sum_{q=0}^{M} \delta A_{pq} e^{ip\lambda} F_p{}^q \tag{2.7}$$

where

$$F_p{}^q = e^{iq\phi} + (-1)^p e^{-iq\phi} \tag{2.8}$$

$$\delta = \begin{cases} \frac{1}{2}, q = 0 \\ 1, q \neq 0 \end{cases} \tag{2.9}$$

whence

$$A_{pq} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\pi} A e^{-ip\lambda} F_p{}^{-q} d\phi d\lambda. \tag{2.10}$$

If A is to be single valued at the poles, $\phi = 0, \pi$ it follows that

$$\begin{aligned} &\sum_{|p| \leq N} \sum_{q=0}^{M} \delta A_{pq}(1 + (-1)^p) e^{ip\lambda} = \text{const} \quad , \phi = 0 \\ &\sum_{|p| \leq N} \sum_{q=0}^{M} \delta A_{pq}(1 + (-1)^p)(-1)^q e^{ip\lambda} = \text{const}, \phi = \pi \end{aligned} \tag{2.11}$$

and therefore that

$$\sum_{q\text{even}} \delta A_{pq} = \sum_{q\text{odd}} A_{pq} = 0; p \neq 0, p \text{ even}$$

is a necessary condition on the coefficients. Equations (2.7–2.11) define the representation of a scalar on the sphere in terms of a two-dimensional Fourier series.

The Fourier series representation of a vector component is obtained in an analogous way. To obtain continuity in the extended region a change of sign is required, i.e.,

$$u'(\lambda, \phi) = \begin{cases} u(\lambda, \phi) & , \phi \in [0, \pi] \\ -u(\lambda + \pi, 2\pi - \phi), & \phi \in [\pi, 2\pi] \end{cases} \tag{2.12}$$

The resulting Fourier series is of the form

$$\begin{aligned} u &= \sum_{|p| \leq N} \sum_{q=0}^{M} \delta U_{pq} e^{ip\lambda} F_{p+1}^q \\ U_{pq} &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\pi} u e^{-ip\lambda} F_{p+1}^{-q} \, d\phi d\lambda \end{aligned} \tag{2.13}$$

and similarly for v. The necessary conditions at the poles may be obtained by representing u and v in terms of the polar sterographic components U_s and V_s as follows:

$$\begin{aligned} u &= -U_s \sin\lambda + V_s \cos\lambda \\ v &= -U_s \cos\lambda - V_s \sin\lambda \end{aligned}$$

where U_s, V_s behave as scalars in terms of spherical polar representation. At the poles

$$u = \sum_{|p|\leqslant N} \sum_{q=0}^{M} \delta U_{pq} e^{ip\lambda}(1 + (-1)^{p+1}) = \tfrac{1}{2}(We^{i\lambda} + W^* e^{-i\lambda}) \qquad, \phi = 0$$

$$v = \sum_{|p|\leqslant N} \sum_{q=0}^{M} \delta V_{pq} e^{ip\lambda}(1 + (-1)^{p+1}) = \frac{i}{2}(We^{i\lambda} - W^* e^{-i\lambda}) \qquad, \phi = 0$$

$$u = \sum_{|p|\leqslant N} \sum_{q=0}^{M} \delta U_{pq} e^{ip\lambda}(1 + (-1)^{p+1})(-1)^q = \tfrac{1}{2}(We^{i\lambda} + W^* e^{-i\lambda}), \ \phi = \pi$$

$$v = \sum_{|p|\leqslant N} \sum_{q=0}^{M} \delta V_{pq} e^{ip\lambda}(1 + (-1)^{p+1})(-1)^q = \frac{i}{2}(We^{i\lambda} - W^* e^{-i\lambda}), \ \phi = \pi$$

$$W = V_s + iU_s$$

so that we require

$$\left.\begin{aligned} \sum_{q\,\mathrm{even}} \delta U_{pq} &= \sum_{q\,\mathrm{odd}} U_{pq} = 0 \\ \sum_{q\,\mathrm{even}} \delta V_{pq} &= \sum_{q\,\mathrm{odd}} V_{pq} = 0 \end{aligned}\right\} \quad |p| \neq 1, p \text{ odd.} \tag{2.14}$$

For $|p| = 1$

$$\sum_{q=0}^{M} 2\delta U_{1q} = \frac{W}{2}, \ \sum_{q=0}^{M} 2\delta V_{1q} = \frac{iW}{2} \ (\text{For } \varphi = 0)$$

$$\sum_{q=0}^{M} 2\delta U_{1q}(-1)^q = \frac{W}{2}, \ \sum_{q=0}^{M} 2\delta V_{1q}(-1)^q = \frac{iW}{2} \ (\text{For } \varphi = \pi) \tag{2.15}$$

and

$$\sum_{q\,\mathrm{even}} \delta U_{1q} = i \sum_{q\,\mathrm{even}} \delta V_{1q}; \ \sum_{q\,\mathrm{odd}} U_{1q} = i \sum_{q\,\mathrm{odd}} V_{1q}.$$

The equations above give an expansion of a scalar or vector component field on the globe in terms of double Fourier series. The useful features of such an expansion have been alluded to at the beginning of the section. The drawback of this representation is that the expansion functions are not orthogonal with respect to area weighting on the sphere and, therefore, the expansion coefficients are not those appropriate to a least squares representation and do not exhibit the condition of finality. The coefficients are those appropriate to a least squares representation for a weighting function of unity (rather than the cosine of latitude as in area weighting). These disadvantages may be outweighed by the accuracy and ease of calculation with Fourier series using the FFT.

3 The simple advection case

The simplest non-trivial test of the method is that of the passive advection of a scalar field by a non-divergent flow field. The equation of motion is

$$\frac{\partial A}{\partial t} + \frac{u}{a \sin \phi} \frac{\partial A}{\partial \lambda} + \frac{v}{a} \frac{\partial A}{\partial \varphi} = 0 \tag{3.1}$$

or in terms of a stream function ψ

$$V = k \times \nabla \psi$$

$$\frac{\partial A}{\partial t} + \frac{1}{a^2 \sin \phi} \left(- \frac{\partial \psi}{\partial \phi} \frac{\partial A}{\partial \lambda} + \frac{\partial \psi}{\partial \lambda} \frac{\partial A}{\partial \phi} \right) = 0 \tag{3.2}$$

The flow field chosen was that of solid rotation about an axis lying in the equatorial plane of the spherical coordinate system. Such a flow field requires the computation method to account successfully for the convergence of the meridians and the behaviour at the pole during the advection of the passive scalar over the poles. The stream function chosen was

$$\psi = -a^2\omega \sin\phi \cos\lambda \tag{3.3}$$

whence

$$\begin{aligned} u &= -a\omega \cos\phi \cos\lambda \\ v &= a\omega \sin\lambda. \end{aligned} \tag{3.4}$$

The scalar field chosen was conical in shape with a base which formed a circle on the sphere. The defining expression is

$$A(\lambda, \phi) = \begin{cases} A_0(1 - R/R_0), & R \le R_0 \\ 0 & , R > R_0 \end{cases} \tag{3.5}$$

$$R = 2a[\sin^2\phi \sin^2(\lambda - \lambda_0)/2 + \sin^2(\phi - \phi_0)/2]^{1/2}$$

where A_0 is the height of the cone and R_0 is the radius of the circular base of the cone; ϕ_0 and λ_0 are the latitude and longitude of the central axis of the cone. Equation (3.1) was rewritten in the form

$$\frac{\partial A}{\partial t} + \frac{1}{a}\left(u\frac{\partial \hat{A}}{\partial \lambda} + v\frac{\partial A}{\partial \phi}\right) = 0 \tag{3.6}$$

where $\hat{A} = A/\sin\phi$. It may be shown that if A is represented in terms of a double Fourier series expansion (2.7) with the conditions (2.11), then $\hat{A}$ may be expanded in a series of the form

$$\hat{A} = \sum_{|p| \leqslant N} \sum_{q=0}^{M} \delta\hat{A}_{pq} e^{ip\lambda} F_{p+1}^{q} \tag{3.7}$$

where the $\hat{A}_{pq}$ are expressed in terms of the A_{pq} as shown in Appendix I.

The transformed equation (3.6) can be written most compactly in the form

$$\frac{d}{dt}A_{pq} + \frac{1}{a}\sum_{|\alpha||\beta| \leqslant N}\{i\alpha\hat{A}_{\alpha\beta}U_{p-\alpha q-\beta} + i\beta A_{\alpha\beta}V_{p-\alpha q-\beta}\} = 0 \tag{3.8}$$

Coefficients with negative values in the second subscript are defined as previously indicated, i.e. for a variable which behaves as a scalar

$$A_{\alpha-\beta} = (-1)^{\alpha} A_{\alpha\beta}$$

and for a variable which behaves as a vector component

$$V_{\alpha-\beta} = (-1)^{\alpha+1} V_{\alpha\beta}.$$

The equation was integrated using central differences in time and the convolution terms were calculated using the transform method of Orszag (1971a) (see also Orszag (1971b).

Linear stability analysis of the problem gives the stability restriction that, approximately,

$$\Delta t \leqslant |\tan\phi|/\omega p_{\max}.$$

In order to meet the stability requirement while retaining a reasonable time step, Fourier filtering in the λ direction was performed poleward of 60° latitude ($\phi_0 = 30°, 150°$). The maximum zonal wave number retained was, $p_{\max}$, where

$$p_{\max} \leq \left| \frac{\tan \phi}{\tan \phi_0} \right| N,\ \phi < 30°,\ \phi > 150°.$$

A value of $\omega = 2.40885 \times 10^{-6}\ \mathrm{sec}^{-1}$ was chosen to give an advection velocity which would move the apex of the cone one grid interval in an integral number of time steps of 1 hour. The stability condition for this value of ω and for $N = 32$ requires, approximately, $\Delta t \leqslant 1.4$ hours.

The Fourier filtering also automatically maintains the conditions (2.11). These conditions are imposed in the filtering procedure simply by the condition that $p_{\max} = 0$ at the poles, i.e. that the filtered variable is a constant at the poles.

The conservation of "total A" was tested by calculating

$$\int_0^{2\pi} \int_0^{\pi} A \sin \phi d\phi d\lambda = 2\pi \sum_{\substack{q=0 \\ q\,\mathrm{even}}}^{M} 4\delta A_{oq}/(1 - q_2).$$

It is not obvious that "total A" should be conserved in the truncated system (3.8) although it will be seen that is is, to good approximation, at least in the cases tested.

4 Comparison with other numerical methods

The same calculations were carried out using two other methods; a finite difference formulation of equation (3.2) using the Arakawa second order Jacobian (Arakawa, 1966) and a pseudo-spectral formulation of equation (3.1) after Merilees (1973).

The equation used in the finite difference case was

$$\begin{aligned}
&\frac{A_{nm}{}^{\tau+1} - A_{nm}{}^{\tau-1}}{2\Delta t} \\
&\quad + \frac{1}{a^2 \sin \phi_m (12\Delta^2)} \{\psi_{n+1m}(A_{nm-1} + A_{n+1m-1} - A_{nm+1} - A_{n+1m+1}) \\
&\quad + \psi_{n-1m}(A_{nm+1} + A_{n-1m+1} - A_{nm-1} - A_{n-1m-1}) \\
&\quad + \psi_{nm+1}(A_{n+1m} + A_{n+1m+1} - A_{n-1m} - A_{n+1m-1}) \\
&\quad + \psi_{nm-1}(A_{n-1m} + A_{n-1m-1} - A_{n+1m} - A_{n+1m-1}) \\
&\quad + \psi_{n+1m+1}(A_{n+1m} - A_{nm+1}) + \psi_{n+1m-1}(A_{nm-1} - A_{n+1m}) \\
&\quad + \psi_{n-1m+1}(A_{nm+1} - A_{n-1m}) + \psi_{n-1m-1}(A_{n-1m} - A_{nm-1})\} = 0
\end{aligned}$$

The pseudo-spectral equations used can be written in the form

$$\frac{A_{nm}{}^{\tau+1} - A_{nm}{}^{\tau-1}}{2\Delta t} + \frac{u_{nm}}{a \sin \phi_m} \left.\frac{\partial A}{\partial \lambda}\right|_{nm} + \frac{v_{nm}}{a} \left.\frac{\partial A}{\partial \phi}\right|_{nm} = 0$$

where the net of grid points is defined as

$$\lambda_n = \frac{\pi}{N} n \qquad , n = 0, 1, \ldots, 2N - 1$$

$$\phi_m = \frac{\pi}{M}(m + 1/2),\ m = 0, 1, \ldots, M - 1.$$

TABLE 1. Characteristics of different advection calculations.

Type of calculation	Resolution	Δt (hrs.)	No. of Revolutions	Approx. CPU Time (sec.)	Apex Value
Double Fourier	64	0.5	1	1995	7.90
	64	1.0	1	998	7.60
			2		7.50
	32	1.0	1	173	7.85
	16	1.0	1	43	7.74
Pseudo-spectral	64	1.0	1	170	7.62
			2	becomes unstable	
(with Fourier chopping)	64	1.0	1		7.57
			2	333	7.40
Finite difference	64	1.0	1		6.89
			2	55	6.14
(no polar filtering)	64	0.1	1	127	7.05

and the derivatives are evaluated as described by Merilees (*loc cit*). An alternative pseudo-spectral formulation can be defined in terms of a series representation of the form (2.7) and (2.13) and will of course be identical to Merilees' scheme.

5 Discussion of results

Little effort was made to optimize the calculation using the Fourier series method other than the use of Orszag's (1971) convolution method for calculating the two non-linear products involved. The same FFT subroutine was used in both Fourier methods. Advective calculations with a number of resolutions and time steps were performed for the several methods.

Table 1 summarizes the results of several such calculations. The results are for 512 hours at which time the cone should have completed one revolution, and after 1024 hours at which time the cone should have completed two revolutions. The cone is initially situated at the equator for the Fourier series case and is displaced one-half a grid distance from the equator in the pseudo-spectral and finite difference cases. In the two Fourier methods the resolution listed in the table is $2N$ (equivalent to the number of grid points involved in a discrete transform). In all cases $M = N$. For the finite difference calculation, the resolution is the number of grid points around a latitude circle. The number of points in the north-south direction is one half this number.

In all cases the "mass" of the cone, i.e. the mean value of A was conserved to 10^{-11} in the spectral calculations and to 10^{-4} in the pseudo-spectral calculations.

Fig. 1 gives a north-south profile through the cone after 512 hours (one revolution) with $\Delta t = 1$ hr for the cases: (a) double Fourier series method, $2N = 64$; (b) double Fourier series method, $2N = 32$; (c) double Fourier series method, $2N = 16$; (d) pseudo-spectral method, $2N = 64$; and (e) second order Arakawa/Jacobian, $\Delta\lambda = \Delta\phi = 2\pi/64$. The cone has been advected over both poles and should have returned to its original position indicated by the light solid lines in the figure. It is apparent from the figures that:

(i) both the shape of the cone and its position are very well treated by the double Fourier series and pseudo-spectral methods while the second order finite difference calculation is clearly inferior.
(ii) even the low resolution double Fourier series calculation ($2N = 16$), where computation times become comparable to times for finite difference methods, gives very good results.
(iii) in the double Fourier and pseudo-spectral cases, the point of the cone suffers some distortion. This is due primarily to time truncation error. The inset of Fig. (1a) gives the shape of the cone apex with Δt reduced to 0.5 hr. In this case the tip of the cone is rather well represented.

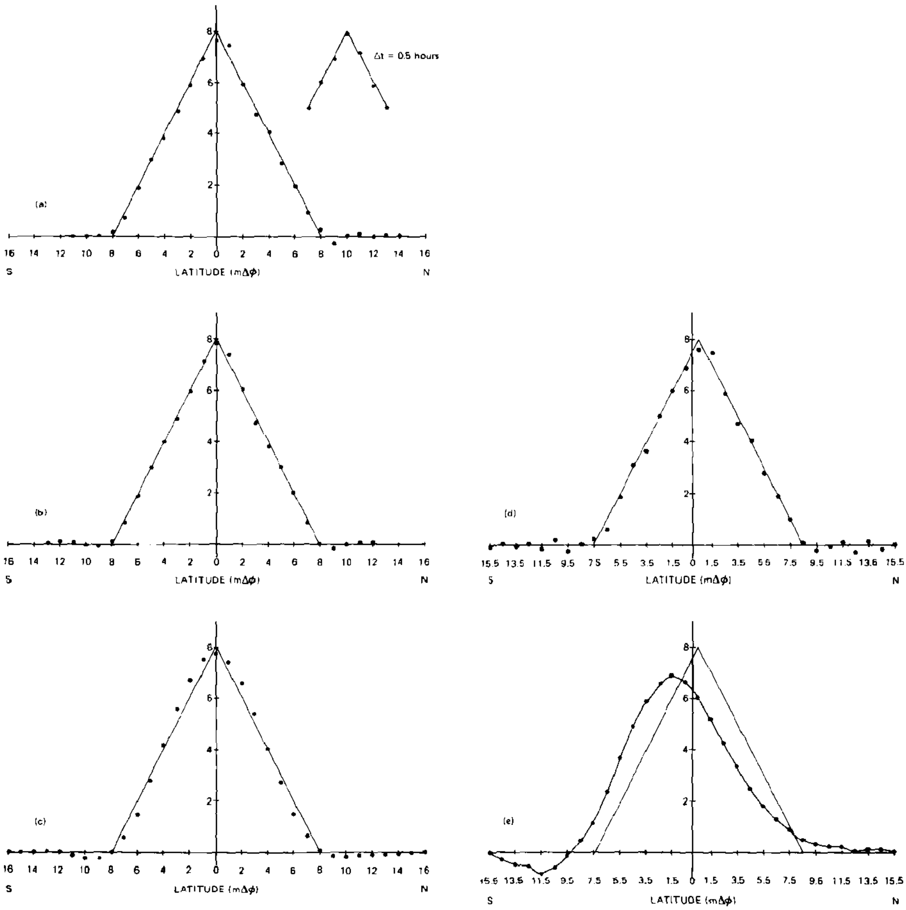
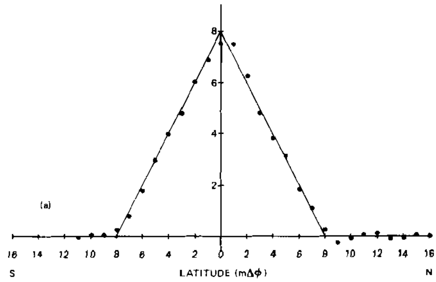
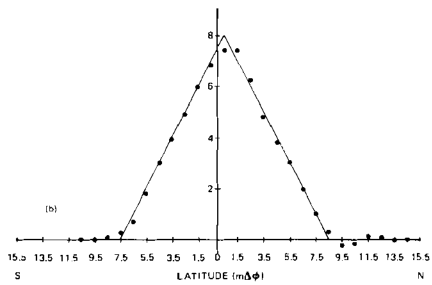


Fig. 1 North-South profile through the cone after one complete revolution about the sphere with $\Delta t = 1$ hr. (a) double Fourier method with $N = 32$, (b) double Fourier method with $N = 16$, (c) double Fourier method with $N = 8$, (d) Pseudo-spectral method $N = 32$ (e) second order finite difference method with $\Delta\lambda = \Delta\phi = 2\pi/64$. Note that the initial position of the cone is displaced one half a grid length from the equator in the pseudo-spectral and grid point methods.





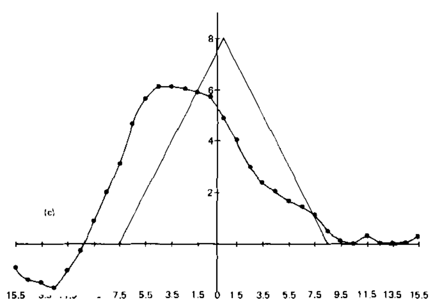


Fig. 2 North-South profile through the cone after two complete revolutions about the sphere with $\Delta t = 1$ hour: (a) double Fourier method with $N = 32$, (b) Pseudo-spectral method with $N = 32$ and Fourier chopping, (c) second order finite difference method with $\Delta\lambda = \Delta\phi = 2\pi/64$.

Fig. 2 gives a north-south profile through the cone after 1024 hrs. (2 revolutions again with $\Delta t = 1$ hr. for the double Fourier series and pseudo-spectral cases ($2N = 64$) and the second order grid point method. The shape and position of the cone has deteriorated very little from the previous case for the double-Fourier and pseudo-spectral calculations. The grid point calculations displays increased distortion.

In the case of a cone advected by such a simple flow field one might expect the pseudo-spectral scheme to give results comparable to the double Fourier scheme. After one revolution this is essentially the case as indicated in Fig. 1(d). For longer times, however, aliasing effects begin to dominate the solution. Fig. 3

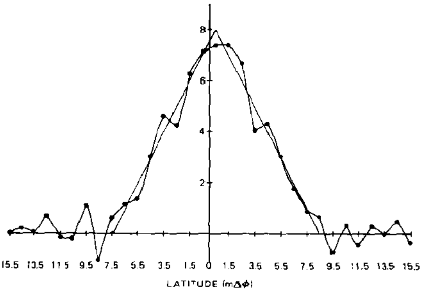


Fig. 3

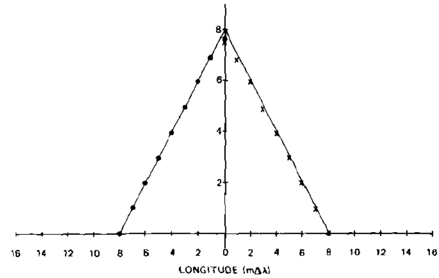


Fig. 4

Fig. 3 North-South profile through the cone after one and one-half revolutions about the sphere with $\Delta t = 1$ hour. Pseudo-spectral method with $N = 32$ but no Fourier chopping.

Fig. 4 East-West profile through the cone after one complete revolution (dots) and after two complete revolutions (crosses) about the sphere with $\Delta t = 1$ hr. Double Fourier method with $N = 32$.

indicates the result after 768 hours (one and one-half revolutions). The solution continues to deteriorate rapidly thereafter. In this case at least, it is easy to control the aliasing by Fourier chopping as proposed by Merilees (*loc. cit.*). The result of Fig. 2(b) has been obtained by chopping the high wave number one-third of the spectrum at 64 hour intervals.

The east-west profile of the cone is symmetric for the double Fourier method and very accurately maintained as indicated in Fig. 4 where the points plotted on the 'left' half of the cone are for one revolution while those on the 'right' are for two revolutions.

Finally, it should be pointed out that the apparent difference in efficiency between pseudo-spectral and double Fourier series methods as indicated in the table can be reduced considerably. The program used for calculating cross- product series in the double Fourier calculation was rather general in form. The calculations of $u(\partial\hat{A}/\partial\lambda)$ and $v(\partial A/\partial\phi)$ were done independently and the fact that U_{pq} and V_{pq} were constant in this case was not used to speed up the calculations.

6 Summary

A method of representing scalar and vector components on the sphere in terms of two-dimensional Fourier series has been used in the simple test calculation of passive advection of a conical shape. The cone is advected several times around the sphere in a meridional direction so as to pass over both poles. The results are compared with similar calculations using pseudo-spectral and grid point methods. The double Fourier series method gives good results, even at low resolution, and is not complicated by the aliasing problems inherent in the pseudo-spectral calculation or by the inaccuracies inherent in the grid-point scheme.

It should be noted that in the absence of proof of the conservation of quadratic quantities in this numerical scheme, it is not possible to claim that nonlinear instabilities must necessarily be absent in the calculation for this or for more nonlinear "meteorological" flow situations. The two-dimensional Fourier series calculation exhibited no non-linear instability behaviour, however while the pseudo-spectral calculation became unstable. This suggests at least, that the method is less prone to aliasing errors, as might be expected in view of the way in which the nonlinear terms are calculated. It is conceivable that some conservation statements may be possible of proof which, while not guaranteeing exact conservation of quadratic quantities with area weighting on the sphere, will guarantee the boundedness of these quantities and hence the absence of nonlinear instabilities. Such proofs however are not obvious.

While little effort was made to optimize the double Fourier calculations with respect to computer time, the accuracy of the method at low resolution compared to grid point schemes at "greater" resolution suggests that for similar accuracy the double Fourier series method may well be as efficient. In the case where storage is more of a constraint than computation time, the double Fourier method could be very attractive compared to grid point methods. In more complicated cases, the shallow water equations for instance, there appears to be no reason why semi-implicit time differencing could not be employed to yield a significant increase in efficiency.

For all schemes used here, polar filtering was used to control the time step and to maintain computational stability. The desirability of polar filtering in more complicated cases is by no means clear. It may prove to be necessary and/or desirable to maintain stability by some other method such as implicit time differencing.

The double Fourier series method applied to the sphere has a number of desirable features compared to pseudo-spectral and grid point methods as mentioned above. Compared to the more usual spherical harmonic representation the ease of calculation using the FFT must be weighed against the "pole problem" and the fact that the expansion functions are not orthogonal with respect to area weighting on the sphere. A more definitive comparison of the various methods must await more meteorologically realistic calculations.

Appendix

Fourier series representation of $\hat{A} = A/\sin\phi$.

Given a scalar

$$A = \sum_{|p|\leqslant N} \sum_{q=0}^{M} \delta A_{pq} e^{ip\lambda} F_p{}^q$$

$$\sum_{q\,\text{even}} \delta A_{pq} = \sum_{q\,\text{odd}} A_{pq} = 0,\ p \text{ even},\ p \neq 0, \tag{A1}$$

we wish to determine the Fourier series representation of $\hat{A} = A/\sin\phi$. It is apparent that $\hat{A}$ must have a formal expansion like that of a vector component.

$$\hat{A} = \sum_{|p|\leqslant N} \sum_{q=0}^{M} \delta \hat{A}_{pq} e^{ip\lambda} F^q_{p+1}$$

and

$$\hat{A}_{pq} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\pi} \frac{A}{\sin\phi} e^{-ip\lambda} F^{-q}_{p+1}\, d\phi d\lambda$$

$$= \frac{1}{4\pi^2} \sum_{|\alpha| \leqslant N} \sum_{\beta=0}^{M} \delta A_{\alpha\beta} \int_0^{2\pi} e^{i(\alpha-p)\lambda} d\lambda \int_0^{\pi} \frac{F_\alpha{}^\beta F_{p+1}^{-q}}{\sin\phi} d\phi$$

$$= \frac{1}{2\pi} \sum_{\beta=0}^{M} \delta A_{p\beta} \int_0^{\pi} \frac{F_p{}^\beta F_{p+1}^{-q}}{\sin\phi} d\phi$$

for

$$\int_0^{\pi} \frac{F_p{}^\beta F_{p+1}^{-q}}{\sin\phi} d\phi = 2i \int_0^{\pi} \frac{\sin(\beta - q)\phi}{\sin\phi} d\phi + 2i(-1)^{p+1} \int_0^{\pi} \frac{\sin(\beta + q)\phi}{\sin\phi} d\phi$$

$$= 2i \begin{Bmatrix} 0; \beta = q, \beta - q \text{ even} \\ \pi; \beta - q \text{ odd}, >0 \\ -\pi; \beta - q \text{ odd}, <0 \end{Bmatrix} + 2i(-1)^{p+1} \begin{Bmatrix} 0; \beta + q \text{ even} \\ \pi; \beta + q \text{ odd} \end{Bmatrix}$$

we have

$$\hat{A}_{pq} = \begin{cases} -2i \sum\limits_{\substack{\beta=1 \\ \beta \text{odd}}}^{q-1} A_{p\beta}; p \text{ even}, q \text{ even} \ : 2i \sum\limits_{\substack{\beta=q+1 \\ \beta \text{odd}}}^{M-1} A_{p\beta}; p \text{ odd}, q \text{ even} \\ -2i \sum\limits_{\substack{\beta=0 \\ \beta \text{even}}}^{q-1} \delta A_{p\beta}; p \text{ even}, q \text{ odd}: 2i \sum\limits_{\substack{\beta=q+1 \\ \beta \text{even}}}^{M} A_{q\beta}; p \text{ odd}, q \text{ odd} \end{cases}$$

The condition (A1) on A_{pq} implies that

$$\hat{A}_{pq} = 0, q \geqq M$$

so that $\hat{A}$ is expressible in a double Fourier series with the same number of terms as A.

This formal derivation gives relations for $\hat{A}_{pq}$ which are also those obtained by equating terms in the series for $\hat{A}$ and $A \sin\phi$ and solving the resulting sets of equations subject to the conditions (A1). The case for $p = 0$ however is anomalous. The series derivation, or the consideration that $A/\sin\phi$ is singular for $\phi \to 0, \pi$ shows that the values for A_{0q} are indeterminant. This is not a problem in the transformed equation of motion in the case treated here since A enters as a derivative with respect to λ, i.e., as $ip\hat{A}_{pq}$, so that the term is well defined for $p = 0$.

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