

A kinetic theory for quantum information transport

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We propose a novel theoretical framework to describe the transport of information in quantum systems. This is aimed at modeling and understanding how out of equilibrium open quantum systems move information across their quantum state space, using an approach inspired by transport theories. The main goal is to build new mathematical tools, together with physical intuition, to improve our understanding of complex (many-body and out-of-equilibrium) quantum systems. In particular, we are aiming at unraveling the interplay between dynamical properties and information-theoretic features. The main rationale here is to have a framework that can imitate, and potentially replicate, the decades-long history of empirical success of classical transport theories in modeling out-of-equilibrium phenomena.

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INTRODUCTION.

Understanding out-of-equilibrium phenomena in many-body systems is crucial to harness the dynamical properties of spatially extended systems. In this sense, transport theories are phenomenological descriptions of a detailed out-of-equilibrium behavior of a system. Their aim is to capture, at the phenomenological level, the way spatially extended system move quantities which, in principle, can be globally conserved by can be locally exchanged. Well-known examples are the theories of transport of charge, mass, momentum and heat, which underlie macroscopic phenomena like hydrodynamics. Most of them were originally formulated to understand Avogadro-scale phenomena, like the tendency of a system to transfer mass to minimize the concentration difference. Over time, they were put on more rigorous grounds using statistical mechanics and kinetic theory, and their use has been incredibly powerful in understanding and modeling emergent non-equilibrium properties of many-particles systems. This is true both scientifically and at the industrial level, where these established theories have now become vital parts of our industrial development.

Most notably, Boltzmann used this approach to describe the behavior of gases, and explained the rise of macroscopic irreversibility from the underlying time-symmetric classical mechanics. After Boltzmann's work, the core idea of treating motion and interactions in a statistical way has led to innumerable advances, both of fundamental and applied nature. With the rise of quantum theory, these techniques have also been adapted to include quantum fluctuations, leading to improved descriptions of transport phenomena at the nanoscale.

In this work, we explore the idea that these techniques can be used to understand how many-body quantum systems move quantum information across its quantum state space, thus laying a bridge between the field of quantum information processing in quantum systems and that of transport theories. The long-term goal we have in mind here is to have a comprehensive framework to understand the flows of quantum information in out-of-equilibrium many-body quantum systems. Thus, here we provide a self-consistent framework to track and model the transport of probability across a quantum state space. We do so by building a kinetic theory of quantum state evolution, both in closed and open configurations.

At the technical level, this is done by leveraging recent work [1–3] on Geometric Quantum Mechanics (GQM): a differential-geometric approach to quantum mechanics that gets rid of the phase redundancy of the Hilbert space and exposes the underlying phase-space-like structure of the space of quantum states, with probabilities and phases playing the role of canonically conjugated coordinates.

The paper is organized as follows. In Section I we give a brief summary of GQM, and of the tools introduced in [1, 2] to describe open quantum systems. In section II we provide our first result: a microscopic derivation of the continuity equation for the information transport in quantum systems, together with the appropriate microscopic definitions for phenomenological quantities such as the information flux and sink/sources. Section III deals with the kinetic aspects underlying the continuity equation and gives the second result: a general set of kinetic equations regulating the transport of information. In Sections II and IV we use the theoretical framework developed in the previous two sections to analyze a few concrete examples, supported by numerical analysis. Eventually, in Section V we draw some conclusions.

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I. GEOMETRIC QUANTUM MECHANICS

References [4–22] give a comprehensive introduction to GQM. Here, we briefly summarize only the elements we need, working with Hilbert spaces \mathcal{H} of finite dimension D . For the details of the derivations, we send the reader to the literature cited above.

Given an arbitrary basis $\{|e_n\rangle\}_{n=0}^{D-1}$, a pure state is parametrized by D complex homogeneous coordinates $Z = \{Z^n\}_n$, up to normalization and an overall phase:

$$|\psi\rangle = \sum_{n=0}^{D-1} Z^n |e_n\rangle .$$

Here, and throughout the paper, we will always use upper indices to identify different coordinates of the same point and lower indices to identify different points. The Hilbert space formulation is redundant since we can multiply Z by a complex number and get the same physical state. Therefore, $Z \in \mathbb{C}^D$, $Z \sim \lambda Z$, with $\lambda \in \mathbb{C}/\{0\}$. This equivalence relation means pure states of a quantum system are points in the complex projective space $\mathcal{P}(\mathcal{H}) \sim \mathbb{CP}^{D-1}$. We will often refer to $\mathcal{P}(\mathcal{H})$ as the *quantum state space*. One can always use probability-phase coordinates $Z^n = \sqrt{p_n} e^{i\phi_n}$. We will often use this set of coordinates as they play a particular useful role.

Observables and POVMs. In GQM, an *observable* is a function $\mathcal{O}(Z) \in \mathbb{R}$ that associates to each point $Z \in \mathcal{P}(\mathcal{H})$ the expectation value $\langle\psi|\mathcal{O}|\psi\rangle / \langle\psi|\psi\rangle$ of the corresponding operator \mathcal{O} on state $|\psi\rangle$ with coordinates Z :

$$\mathcal{O}(Z) = \frac{\sum_{\alpha,\beta} \mathcal{O}_{\alpha,\beta} Z^\alpha \bar{Z}^\beta}{\sum_\gamma |Z^\gamma|^2} , \quad (1)$$

where $\mathcal{O}_{\alpha\beta}$ is Hermitian $\mathcal{O}_{\beta,\alpha} = \overline{\mathcal{O}_{\alpha,\beta}}$.

Measurement outcome probabilities are determined by *positive operator-valued measurements* (POVMs) $\{E_j\}_{j=1}^D$ applied to a state [23, 24]. They are nonnegative operators $E_j \geq 0$, called *effects*, that sum up to the identity: $\sum_{j=1}^D E_j = \mathbb{I}$. In GQM they consist of nonnegative real functions $E_j(Z) \geq 0$ on $\mathcal{P}(\mathcal{H})$ whose sum is always unity:

$$E_j(Z) = \frac{\sum_{m,n} (E_j)_{m,n} Z^m \bar{Z}^n}{\sum_k |Z^k|^2} , \quad (2)$$

where $\sum_{j=1}^D E_j(Z) = 1$.

The quantum state space $\mathcal{P}(\mathcal{H})$ has a preferred metric g_{FS} —the *Fubini-Study metric* [14]—and an associated volume element dV_{FS} that is coordinate-independent and invariant under unitary transformations. The geometric derivation of dV_{FS} is beyond our immediate goals here. That said, it is sufficient to give its explicit form in the

probability-phase coordinate system $Z^n = \sqrt{p_n} e^{i\phi_n}$:

$$\begin{aligned} dV_{FS} &= \sqrt{\det g_{FS}} \prod_{n=0}^{D-1} dZ^n d\bar{Z}^n \\ &= \prod_{n=1}^{D-1} \frac{dp_n d\phi_n}{2} . \end{aligned}$$

Note how p_0 and ϕ_0 are not involved. This is due to $\mathcal{P}(\mathcal{H})$'s projective nature which guarantees that we can choose a coordinate patch in which $p_0 = 1 - \sum_{n=1}^{D-1} p_n$ and $\phi_0 = 0$. As we see now, the probability-phase coordinates play a particular role: they are canonically conjugated.

State-space structure. Beyond the Riemannian structures, it can be shown that $\mathcal{P}(\mathcal{H})$ also has another interesting geometric feature: a symplectic structure. This is the hallmark of classical state spaces and justifies the use of the term *quantum state space* for $\mathcal{P}(\mathcal{H})$. With a more common jargon, a symplectic structure is the geometric entity allowing us to define “Poisson Brackets” and the existence of canonically conjugated coordinates. In particular, using probabilities and phases coordinates $\{(p_n, \phi_n)\}$ one has that $\{p_n, \phi_m\} = \frac{1}{\hbar} \delta_{nm}$. Thus, for arbitrary functions A and B on $\mathcal{P}(\mathcal{H})$ we have

$$\{A, B\} := \frac{1}{\hbar} \sum_{n=1}^{D-1} \frac{\partial A}{\partial p_n} \frac{\partial B}{\partial \phi_n} - \frac{\partial B}{\partial p_n} \frac{\partial A}{\partial \phi_n} \quad (3)$$

Unitary evolution. In QM, an isolated quantum system evolves with a unitary propagator $U(t, t_0) = e^{-\frac{i}{\hbar} H(t-t_0)}$, where the generator H is the (time-independent) Hamiltonian of the system. Surprisingly, it can be shown [14] that this evolution in Hilbert space is equivalent to a classical Hamiltonian dynamics on $\mathcal{P}(\mathcal{H})$. In particular, using a vector notation $\{p_n, \phi_n\}_n = (\vec{p}, \vec{\phi})$, $E(\vec{p}, \vec{\phi}) = \langle \vec{p}, \vec{\phi} | H | \vec{p}, \vec{\phi} \rangle$ the expectation value of H on a generic state $|\vec{p}, \vec{\phi}\rangle = \sum_{n=0}^{D-1} \sqrt{p_n} e^{i\phi_n} |e_n\rangle$, the unitary evolution of a generic function A on $\mathcal{P}(\mathcal{H})$ is given by

$$\frac{\partial A}{\partial t} = \{A, E\} \quad (4)$$

Or, equivalently, a generic point $Z \sim (\vec{p}, \vec{\phi}) \in \mathcal{P}(\mathcal{H})$ will evolve according to Hamilton's equations of motion:

$$\frac{dp_n}{dt} = \frac{1}{\hbar} \frac{\partial E}{\partial \phi_n} \quad (5a)$$

$$\frac{d\phi_n}{dt} = -\frac{1}{\hbar} \frac{\partial E}{\partial p_n} \quad (5b)$$

Here the analogies with classical Hamiltonian mechanics are particularly evident. While quantum mechanics is clearly very different from its classical counterpart, at a certain descriptive level we can still use the intuition of classical mechanics to understand the geometry of the

dynamical flows in a quantum state space. We proceed in our summary of GQM by looking at situations in which the state of the system can not be characterized by a single point but needs a probability distribution over the whole state space.

Geometric quantum states. The geometric framework makes it very natural to view a quantum state as a functional encoding that associates expectation values to observables, paralleling the C^* -algebra formulation of quantum mechanics [25]. The idea is that one considers probability density functions on $\mathcal{P}(\mathcal{H})$, together with observable functions.

Geometric Quantum States are functionals $P_q[\mathcal{O}]$ from the algebra of observables \mathcal{A} to the real line:

$$P_\mu[\mathcal{O}] = \int_{\mathcal{P}(\mathcal{H})} \mathcal{O}(Z) \mu(dV_{FS}) , \quad (6)$$

where $\mathcal{O} \in \mathcal{A}$ and μ is a normalized probability measure on $\mathcal{P}(\mathcal{H})$ associated to the functional P_μ :

$$P_\mu[\mathbb{I}] = \int_{\mathcal{P}(\mathcal{H})} \mu(dV_{FS}) = 1 ,$$

and $P_\mu[\mathcal{O}] \in \mathbb{R}$. Here it is convenient to introduce De Finetti's notation for measures, $\mu(A) = \int_A \mu(dV_{FS})$ and think about the state of a quantum system as described by a *Random Quantum Variable* $\mathbf{Z} = (Z, \mu, \mathcal{P}(\mathcal{H}))$. This is a random variable, in the classical sense, in which Z are the realizations, which belong to the sampling space $\mathcal{P}(\mathcal{H})$, occurring with measure μ . If the measure is absolutely continuous with respect to the Fubini-Study measure dV_{FS} , then there is a continuous density $q(Z)$ such that $\mu(dV_{FS}) = q(Z)dV_{FS}$.

In this description, the quantumness of the system is taken care of by the non-trivial geometry of the sampling space, which is the state space of a quantum system.

Thus, a generic pure state $|\psi_0\rangle \sim Z_0$ is described by a Dirac measure δ_{Z_0} on $\mathcal{P}(\mathcal{H})$ such that:

$$\begin{aligned} P_{Z_0}[\mathcal{O}] &= \int_{\mathcal{P}(\mathcal{H})} \mathcal{O}(Z) \delta_{Z_0}(dV_{FS}) \\ &= \mathcal{O}(Z_0) = \langle \psi_0 | \mathcal{O} | \psi_0 \rangle . \end{aligned}$$

Extending by linearity, a generic ensemble $\{\lambda_j, |\lambda_j\rangle\}$ is described by a convex combinations of Dirac measures:

$$\mu_{\text{mix}} = \sum_j \lambda_j \delta_{Z_j} .$$

Equipped with these tools, we call both μ and q (when it exists) as the system's Geometric Quantum State.

Density matrix. The connection between geometric quantum states and density matrices is fairly straightforward. Since density matrices are a synthetic way of collecting probability outcomes about POVMs, which are functions of the form $A(Z) \propto \sum_{n,m} A_{nm} Z^n \bar{Z}^m$, given a

generic geometric quantum state μ the associated density matrix ρ^μ can be computed as follows:

$$\begin{aligned} \rho_{mn}^\mu &= P_\mu[Z^m \bar{Z}^n] \\ &= \int_{\mathcal{P}(\mathcal{H})} \mu(dV_{FS}) Z^m \bar{Z}^n . \end{aligned} \quad (7)$$

Owing to the specific form of POVMs on $\mathcal{P}(\mathcal{H})$, recall Eq. (2), they are sensitive to μ only via ρ^μ . Therefore, if two geometric quantum states μ_1 and μ_2 induce the same density matrix $\rho^{\mu_1} = \rho^{\mu_2}$, then all POVMs produce the same outcomes.

The GQS of an Open Quantum System. As shown above, if a system is in a pure state its GQS is a Dirac measure concentrated on a single point of the quantum state space. However, if the system is in contact with an environment, this is not always true, unless system and environment are in a product state. In general, contact with the environment causes the loss of information about which region of the state space the system inhabits. Hence, we need a probability distribution for the state of our system: a geometric quantum state. The explicit derivation of the generic geometric quantum state for an open quantum system can be found in Ref.[1]. Here we simply give the result, without discussing the derivation. Given a system and its environment, we call $\{|a_j\rangle\}_{j=0}^{d_S-1}$ a basis for the Hilbert space \mathcal{H}_S of the system (dimension d_S) and $\{|b_n\rangle\}_{n=0}^{d_E-1}$ a basis for the Hilbert space \mathcal{H}_E of the environment (dimension d_E). Assuming system and environment are in a pure state $|\psi_{SE}\rangle = \sum_{j,\alpha} \psi_{j\alpha} |a_j\rangle |b_\alpha\rangle$, the GQS of the system is

$$\mu = \sum_{\alpha=0}^{d_E-1} x_\alpha \delta_{Z(\chi_\alpha)} , \quad (8)$$

where

$$x_\alpha := \sum_j |\psi_{j\alpha}|^2 , \quad \text{with} \quad \sum_\alpha x_\alpha = 1 \quad (9a)$$

$$Z^j(\chi_\alpha) = \frac{\psi_{j\alpha}}{\sqrt{x_\alpha}} \quad \rightarrow \quad |\chi_\alpha\rangle = \sum_j \frac{\psi_{j\alpha}}{\sqrt{x_\alpha}} |a_j\rangle \quad (9b)$$

Here x_α is the probability that the environment is in state $|b_\alpha\rangle$ and $\{|\chi_\alpha\rangle\}$ is a set of d_E states of the system which constitutes the discrete support of the geometric quantum state. The GQS in Eq.(8) is in one-to-one correspondence with a specific decomposition of the reduced density matrix of the system $\rho^S(\psi_{SE}) = \sum_\alpha x_\alpha |\chi_\alpha\rangle \langle \chi_\alpha|$.

II. INFORMATION TRANSPORT: CONTINUITY EQUATION

So far, we have summarized previous results about GQM. We now build on them and, by bringing in the dynamics

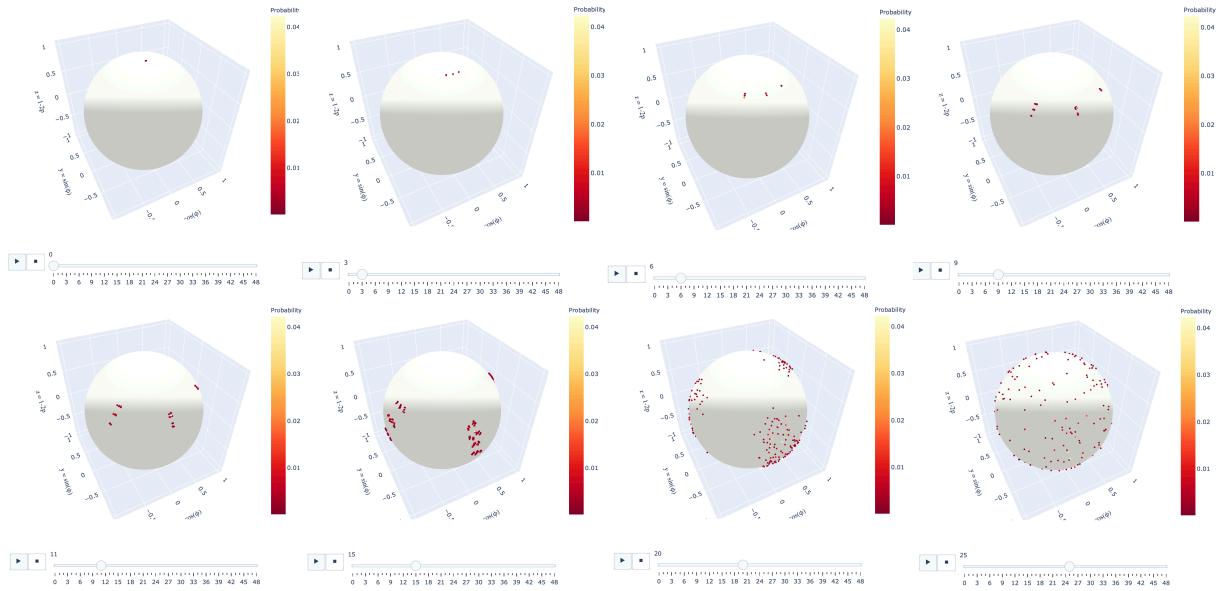


Figure 1. Evolution of the geometric quantum state of one qubit interacting with nine others via an Ising model with transverse field, visualized on the surface of the Bloch sphere. Each particle is $\Gamma_\alpha(t)$, represented in spherical coordinates $\Gamma_\alpha(t) = (\theta_\alpha(t), \phi_\alpha(t))$, with $|\psi(\theta, \phi)\rangle = \cos \theta / 2 |0\rangle + \sin \theta / 2 e^{i\phi} |1\rangle$. The color of the point encodes the probability $x_\alpha(t)$. Time increases left to right and top to bottom. Each particle carries a probability mass x_α , which is the probability to find the system in a state Γ_α . Thus, we think of it as an information carrier, moving the information about the state of a quantum system around the quantum state space. The position of each particle is determined by the state of the environment and, indeed, there are 2^9 particles. The interactive html file from which these snapshots were taken can be found at <http://csc.ucdavis.edu/~cmg/papers/GeoStateEvolution.html>

of the system, we derive a continuity equation which dictates how the geometric quantum state of an open quantum system out-of-equilibrium evolves, under very general assumptions. This is the fundamental kinetic equation governing how the probability of finding the state of the system in a region of $\mathcal{P}(\mathcal{H})$ changes as a result of its interaction with an environment. Throughout this section we will try to maintain a fairly general language, to emphasize how the treatment applies to quantum systems under very general assumptions. However, for concrete examples, both numerical and analytical, we will always refer to the case of a qubit.

General treatment

The following treatment, and its results, pertains quantum systems which are finite-dimensional, and interact with finite-dimensional environments but are, otherwise, arbitrary. A geometric quantum state is specified by two sets of quantities: $\{x_\alpha\}_{\alpha=0}^{d_E-1}$ and $\{\Gamma_\alpha\}_{\alpha=0}^{d_E-1}$ (see Eq. (9)) which is a short-hand notation for $\Gamma_\alpha = Z(|\chi_\alpha\rangle)$. The first one is a classical probability distribution resulting from measuring the environment on a generic eigenbasis $\{|b_\alpha\rangle\}$. For each of the x_α there is a corresponding pure state that the system inhabits: $\Gamma_\alpha \in \mathcal{P}(\mathcal{H}_S)$, which corresponds to the ket $|\chi_\alpha\rangle \in \mathcal{H}_S$. Since these $\Gamma_\alpha(t)$ are points moving in a state space, we can think of them as

particles on a classical phases space, which interact in a non-trivial way. These are the “*carriers of information*”, in the sense that each of these particles carries a probability mass $x_\alpha(t)$ that the system will be found in $\Gamma_\alpha(t)$. An example of how the geometric quantum state of a qubit evolves when the system interacts with an environment is given in Figure 1. Since the total amount of information has to be preserved, $\mu_t(\mathcal{P}(\mathcal{H})) = \sum_\alpha x_\alpha(t) = 1$, the geometric quantum state μ_t must satisfy a continuity equation. Conceptually, this is the starting point of virtually all transport theories, which deal with quantities that are globally conserved, but that are moved around as a result of the underlying microscopic dynamics.

Thus, in the dynamics of a geometric quantum state we identify two different terms, arising from the time-evolution of these two sets of quantities. Loosely speaking, by looking at Figure 1 we can see that there are two different ways in which the geometric quantum state changes in time. First, there is the color of each particle, which changes over time: $\dot{x}_\alpha \neq 0$. Second, there is the movements of each point: $\dot{\Gamma}_\alpha \neq 0$. By summing each term over all particles, and then summing the two terms, we get the following evolution equation for the geometric quantum state:

$$\dot{\mu}_t = \sum_\alpha \dot{x}_\alpha \delta_{\Gamma_\alpha} - x_\alpha \dot{\Gamma}_\alpha \delta'_{\Gamma_\alpha}. \quad (10)$$

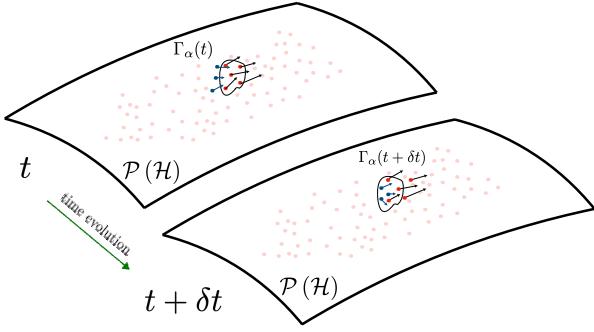


Figure 2. Kinetic interpretation of the Information flux J_t . If we look at a small region of the quantum state space, due to the underlying dynamics we see that information is locally conserved in the sense that there is a certain number of points which enters and leaves this region. As a result of this local process, probability is moved around and can concentrate in a certain region or get scrambled across the quantum state space.

Let's start by analyzing the second term. δ' is the distributional derivative of the Dirac measure δ_{Γ_α} and $\dot{\Gamma}_\alpha(t)$ is the velocity vector, in state space, of each point $\Gamma_\alpha(t)$. Thus, this term is the covariant divergence of a velocity field or, in other words, a flux term in a continuity equation. Indeed, since all the dependence on the set of which to evaluate the measure lies inside the δ' , the whole term can be explicitly written as the divergence¹

$$\sum_\alpha x_\alpha \delta'_{\Gamma_\alpha} \dot{\Gamma}_\alpha = \nabla \cdot J_t , \quad (11)$$

where we have identified the *Information Flux* J_t as the “single-particle” flux $J_\alpha(t)$, averaged with the probability mass it is carrying $x_\alpha(t)$:

$$J_t = \sum_\alpha x_\alpha(t) J_\alpha(t) , \quad (12a)$$

$$J_\alpha(t) = \delta_{\Gamma_\alpha(t)} \dot{\Gamma}_\alpha(t) . \quad (12b)$$

We now look at the first term in the right-hand side of Eq.(10): $\sum_\alpha \dot{x}_\alpha(t) \delta_{\Gamma_\alpha(t)}$. We recognize a source/sink term, which is independent on the underlying movement of the points in the quantum state space. This identifies the other functional term in our continuity equation:

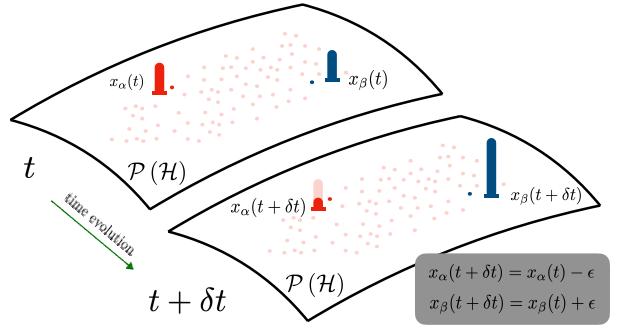


Figure 3. Kinetic interpretation of the sink/source term σ_t . Even if the points do not move around the quantum state space $\dot{\Gamma}_\alpha = 0$, the term σ_t allows the exchange of information between different regions. Note that this can be a non-local effect in state space, as depicted above. In the example above we have two states Γ_α and Γ_β fixed in time, but exchanging a certain amount ϵ of probability, thus moving information from one region of the state space to another one: $x_\beta(t + \delta t) = x_\beta(t) + \epsilon$, $x_\alpha(t + \delta t) = x_\alpha(t) - \epsilon$.

sinks and sources of information σ_t :

$$\sigma_t = \sum_{\alpha=0}^{d_E} \dot{x}_\alpha(t) \delta_{\Gamma_\alpha(t)} . \quad (13)$$

Eventually, we obtain the following continuity equation for the geometric quantum state of a finite-dimensional quantum system interacting with a finite-dimensional environment:

$$\dot{\mu}_t + \nabla \cdot J_t = \sigma_t \quad (14)$$

This is our first and most important result. It is the central tool which describes, in a phenomenologically meaningful way, how quantum information is spread around the state space, due to the underlying dynamics.

Before we move on and use the equation to study the flow of quantum information in some concrete systems, a few comments about general validity of the equation, and the physical interpretation of each of its terms are in order.

Discussion and interpretation

First, the derivation presented here pertains finite-dimensional quantum systems interacting with finite environments. However, generalization to countably-infinite or uncountably-infinite dimensional environment is straightforward. In the first case, a simple limit $d_E \rightarrow \infty$ suffices. This limit goes through the derivation, leaving it unaltered. In the second case, which pertains the case in which we have continuous variables or quantum fields as environment, only minor adjustments are

¹ With an arbitrary coordinate system we have $\text{div} \vec{f} = \frac{1}{\sqrt{g}} \partial_\alpha (\sqrt{g} f^\alpha)$, where $g = |\det g_{FS}|$ is the absolute value of the determinant of the metric which, in our case, is the Fubini-Study metric.

needed. However, the core aspects of the derivation hold, leaving the final form of the continuity equation unaltered. In summary, the only difference between the three cases of finite, countably-infinite or uncountably-infinite environment will be that the support of the geometric quantum state will go from being a set of dimension zero (finite number of points), to a more generic set which, in some cases, can even have fractal dimension. For a thorough analysis of these situations we send the interested reader to Ref.[26].

Second, an aspect that we deem quite interesting here is that the proposed theoretical framework allows us to talk about quantum information as a localized quantity, which is carried around in quantum state space. This is in analogy with all classical theories of transport of properties, such as mass or charge, which are carried around by particles, represented as points wondering in a classical phase-space.

Third, the derivation was performed using the generic measure μ_t . Thus, it holds also for the probability density $q_t(Z)$, when it exists.

Fourth, to strengthen the physical intuition about the theoretical derivation, we now argue that the physical interpretation of the flux J_t and source/sink σ_t terms are essentially the same as in other transport theories. Indeed, the property that is carried around is essentially probability mass or, equivalently, surprisal. We now describe them explicitly, using their kinetic interpretation, drawn in Figures 2 and 3.

J_t is an information flux, associated to an overall conserved quantity, probability mass, localized in its carriers Γ_α , which is moved around the quantum state space. To emphasize this point, we look at the situation in which $\dot{x}_\alpha = 0$. In this case, $\sigma_t = 0$ and the information carried around by the points in state space is a constant of motion ($x_\alpha(t) = x_\alpha(t_0)$). Thus, the analogy with a typical model of point particles carrying around a physical property, like charge or mass, becomes exact. As the carriers move around, the information is dispersed across the quantum state space in a way that guarantees its local preservation. This means the spread of quantum information can be described with a continuity equation, which equates the local time derivative of the probability distribution with the divergence of the local flux of information.

σ_t is a proper source/sink term, with the usual interpretation. Indeed, first, we note that it can not be written as a divergence term. Second, we look at the simpler dynamics in which the position of the points in $\mathcal{P}(\mathcal{H})$ does not change: $\dot{\Gamma}_\alpha = 0$. In this case the support of the distribution is fixed and the particles don't move around $\Gamma_\alpha(t) = \Gamma_\alpha(t_0)$. However, there can still be a non-trivial dynamics, due to $\dot{x}_\alpha(t) \neq 0$. Depending on the actual support of the distribution, as depicted in Figure 3, the time-evolution generated by this term is generally non-local in the quantum state space. Thus, σ_t possesses all the hallmarks of the standard sinks and sources terms

in transport theories: it is not associated with particles moving around the state space and it can not be rewritten as a divergence term, while still affecting the time-derivative of the quantity that is being transported—probability mass.

EXAMPLE 1: ISOLATED QUANTUM SYSTEMS

As first use-case, here we describe the flow of quantum information when our system of interest is isolated. Indeed, in this case both J_t and σ_t assume a generic form, which can be characterized in full generality. To ease the exposition of the results, in this subsection we will assume that $\mu_t(dV_{FS}) = q_t(Z)dV_{FS}$, where q_t is an absolutely continuous probability density. The results do not depend on this choice.

The first task at hand is to find the kinetic equations which regulate the evolution of q_t , namely \dot{x}_α and $\dot{\Gamma}_\alpha$. To find these equations we can simply look at the unitary of the global $|\psi_{SE}\rangle$:

$$\begin{aligned} |\psi_{SE}(t)\rangle &= \sum_\alpha \sqrt{x_\alpha(t)} |\chi_\alpha(t)\rangle |e_\alpha\rangle \\ &= U_S(t) \otimes \mathbb{I}_E |\psi_{SE}(0)\rangle \\ &= U_S(t) \otimes \mathbb{I}_E \sum_\alpha \sqrt{x_\alpha(0)} |\chi_\alpha\rangle |e_\alpha\rangle \\ &= \sum_\alpha \sqrt{x_\alpha(0)} |\chi_\alpha(t)\rangle |e_\alpha\rangle \end{aligned} \quad (15)$$

Two features are relevant here. First, $x_\alpha(t) = x_\alpha(0)$, leading to $\dot{x}_\alpha = 0$. In turn, this leads to $\sigma_t = 0$: This supports the idea that $\sigma_t \neq 0$ is related to the loss or gain of information and, likely, to a dissipative dynamics resulting from the interaction of our system with an environment. Second, $|\chi_\alpha(t)\rangle = U_S(t) |\chi_\alpha(0)\rangle$. This means all elements $\Gamma_\alpha(t)$ of the support of q_t evolve rigidly, according to the same equation. Indeed, starting from the definition of the flux in Eq.(12), we can explicitly write the form of the flux J_t , using Hamilton's equations of motion. Writing Γ_α in canonical coordinates and with the following vector notation $\Gamma_\alpha = (\vec{p}(\Gamma_\alpha), \vec{\phi}(\Gamma_\alpha)) = (\vec{p}_\alpha, \vec{\phi}_\alpha)$, we have

$$\begin{aligned} \dot{\Gamma}_\alpha &= \left(\frac{d\vec{p}_\alpha}{dt}, \frac{d\vec{\phi}_\alpha}{dt} \right) \\ &= \left(\frac{1}{\hbar} \frac{\partial E}{\partial \vec{\phi}} \Big|_{\Gamma_\alpha}, - \frac{1}{\hbar} \frac{\partial E}{\partial \vec{p}} \Big|_{\Gamma_\alpha} \right) = v_H(\vec{p}_\alpha, \vec{\phi}_\alpha) \end{aligned} \quad (16)$$

Note how the flow, v_H , does not depend on the index α : all Γ_α evolve according to the same (Hamiltonian) flow. Inserting this into the definition of J_t we obtain

$$J_t = q_t v_H , \quad (17)$$

leading to the following continuity equation:

$$\frac{\partial q_t}{\partial t} = -v_H \cdot \nabla q_t - q_t \nabla \cdot v_H \quad (18)$$

This can be further simplified by remembering that the Hamiltonian vector field is divergence free, $\nabla \cdot v_H = 0$. For a discussion on this (and more) see Ref.[27]. This leads us to the final form of the continuity equation of an isolated quantum system:

$$\frac{\partial q_t}{\partial t} + v_H \cdot \nabla q_t = 0 \quad (19)$$

This is a convection equation, typically used to describe an incompressible fluid with zero mass diffusivity, as dilute gases at very low temperatures. We will come back to this point later, when discussing the physical interpretation of our results.

Liouville's theorem for GQM. As a further point of contact with the techniques of classical statistical mechanics and kinetic theory, we now show that a generic Hamiltonian dynamics for the geometric quantum state satisfies Liouville's theorem [28]. Indeed, by writing explicitly the total derivative of $q_t(Z)$ with respect to time, inserting Hamilton's equations of motion (Eq. (5)) and then using the continuity equation we get

$$\begin{aligned} \frac{dq_t}{dt} &= \sum_{\alpha} \frac{\partial q_t}{\partial \vec{p}_{\alpha}} \frac{d\vec{p}_{\alpha}}{dt} + \frac{\partial q_t}{\partial \vec{\phi}_{\alpha}} \frac{d\vec{\phi}_{\alpha}}{dt} + \frac{\partial q_t}{\partial t} \\ &= \{q_t, E\} - \nabla \cdot q_t v_H \\ &= 0 \end{aligned} \quad (20)$$

Before we turn to the general case of an open quantum system, we now discuss in detail the example of the evolution of the simplest isolated system: a qubit.

Isolated qubit

Before we begin our case studies, here we collect useful technical details about the geometric quantum mechanics of a qubit. Then, using as a reference basis an arbitrary basis $\{|0\rangle, |1\rangle\}$ the (probability,phase) coordinates are identified via the scalar product operation on the Hilbert space $\langle s|\psi\rangle = \sqrt{p_s} e^{i\phi_s}$, with $s = 0, 1$, giving $(p_0, \phi_0, p_1, \phi_1)$. However, the quantum state space gets rid of two fundamental redundancies in this description. First, we can always assume that $\phi_0 = 0$. Second, due to normalization we have $p_0 = 1 - p_1$. Thus, we only have a pair of independent coordinates: (p_1, ϕ_1) . We can therefore drop the index and say that a general point on the state space of a qubit is uniquely identified by $(p, \phi) \in [0, 1] \times [0, 2\pi[$, with an embedding on the Hilbert space defined by $|\psi(p, \phi)\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}e^{i\phi}|1\rangle$. In analogy with the Bloch sphere, we call the square representation of $\mathbb{C}P^1$ the “Bloch square”. Note that the

embedding is smooth everywhere, aside for two isolated points $|0\rangle$ and $|1\rangle$, where ϕ is not defined and which are represented only by $p = 0$ and $p = 1$, respectively. These coordinates (p, ϕ) are canonically conjugated: $\{p, \phi\}$ and the determinant of the Fubini-Study metric is simply $g = \frac{1}{2}$. Moreover, since the total volume of the state space is π [14], we renormalize the Fubini-Study volume element with the total volume, so that $dV = \frac{dp d\phi}{2\pi} = \frac{dV_{FS}}{\text{vol}(\mathbb{C}P^1)}$.

Analytical solution

Our first case study is an isolated qubit, with an arbitrary initial geometric quantum state $f_0(p, \phi)$ and Hamiltonian operator $H = \vec{B} \cdot \vec{m} = E_0 |E_0\rangle \langle E_0| + E_1 |E_1\rangle \langle E_1|$, with \vec{B} an arbitrary 3D magnetic vector field, $\vec{m} = \frac{\hbar}{2} \vec{\sigma}$ and $\vec{\sigma}$ are the Pauli matrices. Thus, the generator of the geometric evolution is:

$$\begin{aligned} \mathcal{E}(p, \phi) &= \langle \psi(p, \phi) | H | \psi(p, \phi) \rangle = (1-p)H_{00} + pH_{11} \\ &\quad + \sqrt{p(1-p)} (e^{i\phi} H_{01} + e^{-i\phi} H_{10}) \\ &= \vec{B} \cdot \vec{S}, \end{aligned} \quad (21)$$

where $\langle \vec{m} \rangle = \vec{S}(p, \phi)$:

$$\vec{S} = \frac{\hbar}{2} (2\sqrt{p(1-p)} \cos \phi, 2\sqrt{p(1-p)} \sin \phi, 1 - 2p)$$

Since the evolution is Hamiltonian $\sigma_t = 0$ and the continuity equation is

$$\frac{\partial f_t(p, \phi)}{\partial t} + \nabla \cdot J_t = 0. \quad (22)$$

The divergence operator can be computed, in arbitrary coordinates, using the Voss-Weyl formula:

$$\frac{\partial f_t(p, \phi)}{\partial t} + \frac{\partial J_t^p \sqrt{p(1-p)}}{\partial p} + \frac{1}{\sqrt{p(1-p)}} \frac{\partial J_t^\phi}{\partial \phi} = 0. \quad (23)$$

We now come to writing explicitly the flux $J_t = (J_t^p, J_t^\phi)$. As argued above, with Hamiltonian evolution the flux is the product between the Hamiltonian vector field v_H and the geometric quantum state $f_t(p, \phi)$. Explicitly:

$$J_t = \frac{f_t(p, \phi)}{\hbar} \left(\frac{\partial \mathcal{E}(p, \phi)}{\partial \phi}, -\frac{\partial \mathcal{E}(p, \phi)}{\partial p} \right) \quad (24)$$

Plugging this into the continuity equation and remembering that a Hamiltonian vector field is divergence-free leads to a partial differential equation for f_t . Alternatively, using the symplectic two-form, we get

$$\frac{\partial f_t}{\partial t} + \{f_t, \mathcal{E}\} = 0 \quad (25)$$

Its solution is given by the propagator $P(t, t_0) =$

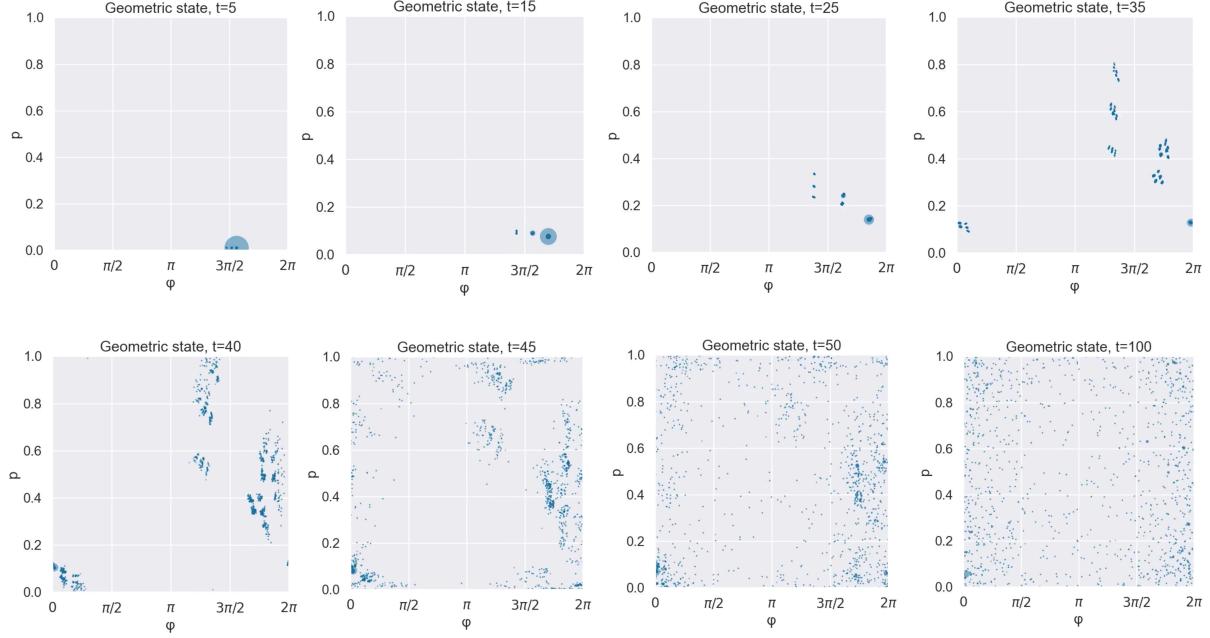


Figure 4. Evolution of the geometric quantum state of one qubit interacting with eleven others via an antiferromagnetic Ising model with both transverse and longitudinal field $\vec{B} = (1, 0, 0.5)$, visualized on the Bloch square. Each particle is $\Gamma_\alpha(t)$, represented in canonically conjugated coordinates $\Gamma_\alpha(t) = (p = p_\alpha(t), \phi = \phi_\alpha(t))$, with $|\psi(p, \phi)\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}e^{i\phi}|1\rangle$. The radius of the particle encodes its probability mass $x_\alpha(t)$. Time increases left to right and top to bottom. Each particle carries a probability mass x_α , which is the probability to find the system in a state Γ_α . Thus, we think of it as an information carrier, moving the information about the state of a quantum system around the quantum state space. After 60/70 time steps the distribution settles on an equilibrium distribution. A full video of the dynamical evolution is available at the following link: https://github.com/fabioanza/GeomQuantMech/blob/main/video_g1_h05_L12.mp4

$\text{Exp}[-(t-t_0)\{\mathcal{E}, \cdot\}]$. Since $\mathcal{E} = \vec{B} \cdot \vec{S}$ and, by using the algebra of poisson brackets, one can prove that $\vec{S} = \frac{\hbar}{2}(S_x(p, \phi), S_y(p, \phi), S_z(p, \phi))$ are the generators of $SO(3)$:

$$\{S_a, S_b\} = \epsilon_{abc} S_c, \quad (26)$$

here we recognize that the propagator $P(t, t_0)$ implements a 3D rotation of the generic expectation value $\vec{S}(t)$ around the axis \vec{B} , with angle $||\vec{B}||(t - t_0)$. In more physics-agnostic terms, this is a linear, first-order, partial differential equation, which can be solved in full generality by using the method of the characteristics. Here we simplify the treatment by switching to coordinates aligned to the convenient basis in which the Hamiltonian is diagonal. This leads to

$$\frac{\partial f_t}{\partial t} + \frac{\omega}{\sqrt{p(1-p)}} \frac{\partial f_t}{\partial \phi} = 0, \quad (27)$$

where $\omega = \frac{E_1 - E_0}{\hbar}$. This is very similar to an advection equation in which the speed of propagation along ϕ depends on p .

Equivalent to the transport equation for advection, or

heat equation, which leads to the analytical solution for the time-dependent geometric quantum state: $f_t(p, \phi) = f_0(p, \phi + \omega t)$, for any valid initial distribution $f_0(p, \phi)$.

Discussion

The evolution of a isolated qubit is periodic in time and this appears in the evolution of its geometric quantum state as $f_0(p, \phi + \omega t)$. Crucially, as the distribution changes over time with a Hamiltonian evolution, it does so in a rigid fashion, never changing its shape. This can, indeed, be seen explicitly. Say that, at $t = 0$, the geometric quantum state is made by a convex sum of dirac deltas concentrated on a certain number N of points $\Gamma_\alpha = (p_\alpha, \phi_\alpha)$: $q_0(p, \phi) = \sum_{\alpha=1}^N \lambda_\alpha 2\delta[p - p_\alpha] \delta[\phi - \phi_\alpha]$. The evolution of the geometric quantum state is therefore dictated by how the points $\Gamma_\alpha(t)$ change over time. Since, however, the evolution can be written as a linear change in the ϕ coordinate $\phi(t) = \phi(0) + \omega t$, this translates into $\phi_\alpha(t) = \phi_\alpha(0) - \omega t$, with $p_\alpha(t) = p_\alpha(0)$. As a result, the shape of the distribution never changes as the relative distance between the points $\Gamma_\alpha(t)$ does not depend on time. Calling $d_{FS}(\cdot, \cdot)$ the Fubini-Study dis-

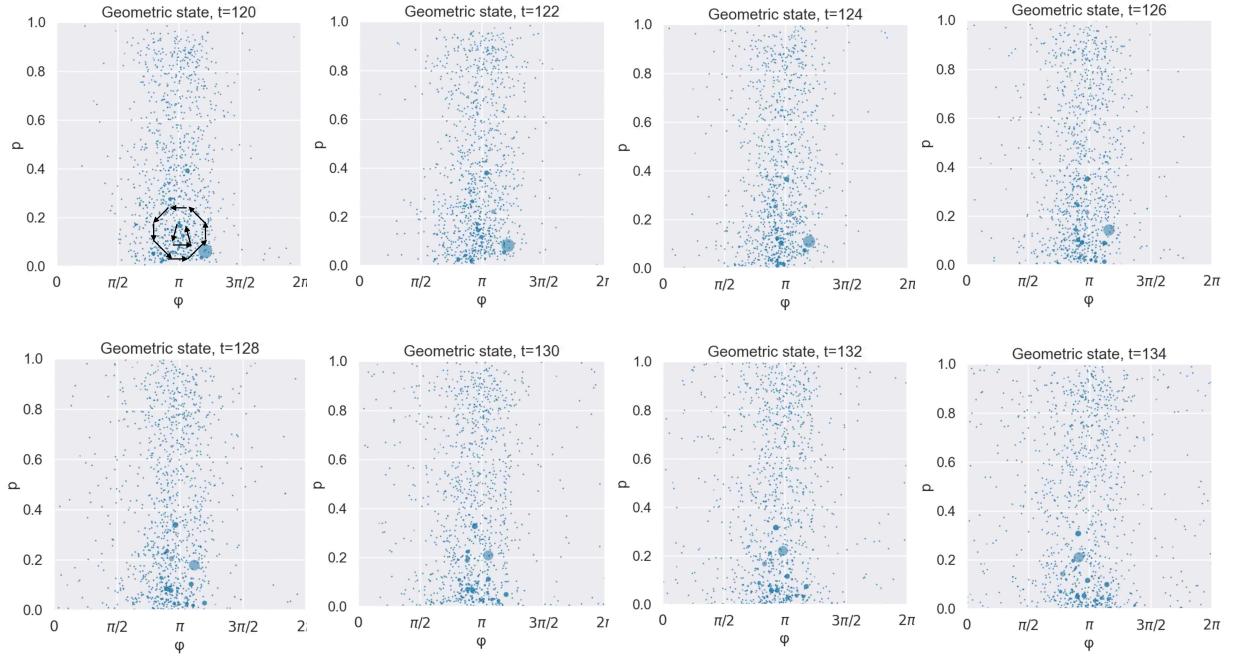


Figure 5. Evolution of the geometric quantum state of one qubit interacting with eleven others via a Ferromagnetic Ising model with longitudinal and transverse field, visualized on the Bloch square. Each particle is $\Gamma_\alpha(t)$, represented via canonically conjugated coordinates $\Gamma_\alpha(t) = (p = p_\alpha(t), \phi = \phi_\alpha(t))$, with $|\psi(p, \phi)\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}e^{i\phi}|1\rangle$. The radius of the particle encodes its probability mass $x_\alpha(t)$. Time increases left to right and top to bottom. Each particle carries a probability mass x_α , which is the probability to find the system in a state Γ_α . Here we show the detail of the evolution of a macroscopic coherent phenomenon that gives rise to sustained fluctuations and maintains the system out of thermal equilibrium. A full video of the dynamical evolution is available https://github.com/fabioanza/GeomQuantMech/blob/main/video_g1_h05_L12_ferro_long.mp4

tance, we have

$$\begin{aligned} \cos d_{FS}(p_\alpha, \phi_\alpha; p_\beta, \phi_\beta) &:= \left(1 - (p_\alpha + p_\beta) + p_\alpha p_\beta \right. \\ &\quad \left. + \sqrt{p_\alpha(1-p_\alpha)} \sqrt{p_\beta(1-p_\beta)} \cos(\phi_\alpha - \phi_\beta)\right)^{1/2}, \end{aligned} \quad (28)$$

giving

$$d_{FS}(\Gamma_\alpha(t), \Gamma_\beta(t)) = d_{FS}(\Gamma_\alpha(0), \Gamma_\beta(0)) \quad \forall \alpha, \beta. \quad (29)$$

Since Hamiltonian evolutions can generically support folding and stretching in state space, causing chaotic behavior, we conclude that this is not the most generic Hamiltonian evolution a qubit state space can support. Note that this goes beyond the assumption of a time-independent Hamiltonian. Even with time-dependent Hamiltonian, this aspect of the isolated evolution of a qubit still holds true, thus confirming that a unitary evolution might not be the most general Hamiltonian evolution that can occur in a quantum state space.

III. KINETIC EQUATIONS OF INFORMATION TRANSPORT IN OPEN QUANTUM SYSTEMS

As we move to analyze how open quantum systems scramble information around the quantum state space, the goal of this section is to provide a microscopic approach to the transport of information: a kinetic theory of how the probability mass of finding our system in a region of its quantum state space changes as a result of its interaction with a structured, non-thermal, environment. The main outcome of this section is a concrete set of microscopic equations that determines the evolution of the geometric quantum state of an open quantum system.

Calling H_S and H_E the Hamiltonian operators of the system and environment, respectively, the total Hamiltonian of the joint system is $H = H_S + H_E + H_{int}$. Since H_{int} is the interaction term between system and environment, we can always put it in the form

$$H_{int} = \sum_{k=1}^M A^{(k)} \otimes B^{(k)}, \quad (30)$$

where $A^{(k)}$ and $B^{(k)}$ are operators with support on \mathcal{H}_S and \mathcal{H}_E , respectively. With a slight abuse of notation we will often conflate $A^{(k)}$ with $A^{(k)} \otimes \mathbb{I}_E$ and $B^{(k)}$ with

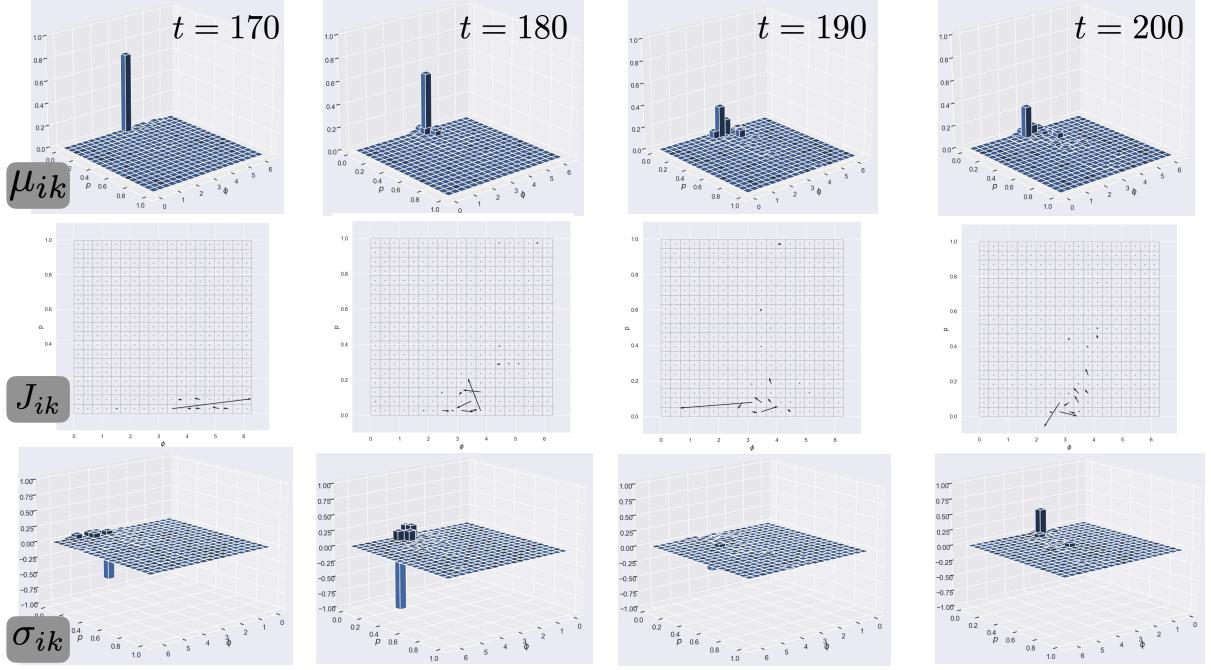


Figure 6. Evolution of the coarse-grained geometric quantum state (Top row), Information Flux (Middle row) and Information sink/source (Bottom row), for one qubit interacting with eleven other ones via a Ferromagnetic Ising mode with longitudinal and transverse field, visualized on the Bloch square. We have sliced the quantum state space in (p, ϕ) coordinates into a grid of 19×19 small uniform cells $\mathcal{C}_{ik} = \left[\frac{i}{20}, \frac{i+1}{20} \right] \times \left[2\pi \frac{k}{20}, 2\pi \frac{k+1}{20} \right]$ of equal volume and integrated the relevant quantities for the study of the continuity equation over \mathcal{C}_{ik} to obtain a time-dependent coarse-grained graphic representation: $\mu_{ik}(t)$, $J_{ik}(t)$ and $\sigma_{ik}(t)$.

$\mathbb{I}_S \otimes B^{(k)}$, where \mathbb{I}_S and \mathbb{I}_E are, respectively, the identity operator on \mathcal{H}_S and \mathcal{H}_E . Here we do not impose a specific choice for the basis of the system and environment. While a natural one is the bases that diagonalize the non-interacting Hamiltonians H_S and H_E , there are cases where a different choice is more appropriate. For example, in the case of a spin-1/2 chain one might be interested in using the computational basis. Thus, we keep things general and use a generic basis, with no particular properties with respect to the algebra of observables: $\{|a_j\rangle\}_{j=0}^{d_S}$ and $\{|b_\alpha\rangle\}_{\alpha=0}^{d_E-1}$.

Since the goal is to derive a dynamic equation for $\{x_\alpha(t)\}$ and $\{\Gamma_\alpha(t)\}$, we begin with the overall Schrödinger equation for the pure state $|\psi(t)\rangle$ of the joint system+environment, written in the generic tensor product basis defined above, with $\psi_{j\alpha}(t) = \langle a_j, e_\alpha | \psi(t) \rangle$:

$$i\hbar \frac{d\psi_{j\alpha}(t)}{dt} = \sum_{k,\beta} H_{j\alpha;k\beta} \psi_{k\beta}(t) , \quad (31)$$

where

$$H_{j\alpha;k\beta} := \langle a_j, b_\alpha | H | a_k, b_\beta \rangle \quad (32)$$

$$= (H_S)_{jk} \delta_{\alpha\beta} + \delta_{jk} (H_E)_{\alpha\beta} + \sum_{n=1}^M A_{jk}^{(n)} B_{\alpha\beta}^{(n)} .$$

We now collect the probability $x_\alpha(t)$ and the states $|\chi_\alpha(t)\rangle$ into a single quantity: a non-normalized ket $|\Phi_\alpha(t)\rangle := \sqrt{x_\alpha} |\chi_\alpha\rangle \in \mathcal{H}_S$. By plugging Schrödinger's equation into the time-derivative of $|\Phi_\alpha(t)\rangle$ we get the following set of d_E coupled linear equations:

$$i\hbar \frac{d|\Phi_\alpha\rangle}{dt} = H_S |\Phi_\alpha\rangle + \sum_\beta (H_E)_{\alpha\beta} |\Phi_\beta\rangle + \hat{M}_{\alpha\beta} |\Phi_\beta\rangle , \quad (33)$$

where $\hat{M}_{\alpha\beta}$ is a set of operators acting on the system, defined as

$$\hat{M}_{\alpha\beta} := \sum_{k=1}^M B_{\alpha\beta}^{(k)} A^{(k)} \quad (34)$$

This can be further manipulated to separate the non-interacting part, acting on each vector $|\Phi_\alpha\rangle$, from the interacting part, acting on $|\Phi_\beta\rangle \neq |\Phi_\alpha\rangle$. The final form of our kinetic equation is

$$i\hbar \frac{d|\Phi_\alpha\rangle}{dt} = \hat{H}_\alpha |\Phi_\alpha\rangle + \sum_{\beta \neq \alpha} \hat{V}_{\alpha\beta} |\Phi_\beta\rangle , \quad (35)$$

where the single-particle Hamiltonian is \hat{H}_α is

$$\hat{H}_\alpha := H_S + (H_E)_{\alpha\alpha} \mathbb{I}_S + \sum_{k=1}^M B_{\alpha\alpha}^{(k)} A^{(k)}, \quad (36)$$

and the interaction between particles is vehiculated by the set of operators $\hat{V}_{\alpha\beta}$

$$\hat{V}_{\alpha\beta} := (H_E)_{\alpha\beta} \mathbb{I}_S + \sum_{k=1}^M B_{\alpha\beta}^{(k)} A^{(k)}. \quad (37)$$

The remaining step is to connect this with the quantities that determine the geometric quantum state of the system: x_α and Γ_α . This is easily done by using the original definition $|\Phi_\alpha\rangle = \sqrt{x_\alpha} |\chi_\alpha\rangle$. Since $\Gamma_\alpha(t) \leftrightarrow |\chi_\alpha\rangle$, we have $\Gamma_\alpha^j(t) = \frac{\langle a_j | \Phi_\alpha(t) \rangle}{\sqrt{x_\alpha(t)}}$ and $x_\alpha(t) = \langle \Phi_\alpha(t) | \Phi_\alpha(t) \rangle$.

Thus, Equation (35) provides a way to describe the dynamics of an open quantum system as a set of d_E linear, coupled, differential equations, for the geometric quantum state μ_t : they provide evolution equations for x_α and Γ_α which, by definition, satisfy the continuity equation in Eq.(14). By numerically simulating the globally unitary evolution, we can therefore access and study both phenomenological terms in the continuity equation: the information flux J_t and the sink/source term σ_t .

It is worth noting that the emergence of this set of coupled equations reinforces the interpretation of $\Gamma_\alpha(t)$ as behaving as “particle-like”, with their own local notion of energy, represented by H_α and their pair-wise interaction $\hat{V}_{\alpha\beta}$. While not treated here, by using the approach developed in Ref.[1] these results can be extended to include finite-systems interacting with infinite-dimensional environments.

As a self-consistency check, we note that the preservation of information within the quantum state space of the system is inherited by the unitarity of the global evolution. This is manifested in the relations $\hat{H}_\alpha^\dagger = \hat{H}_\alpha$ and $\hat{V}_{\alpha\beta}^\dagger = \hat{V}_{\beta\alpha}$, which can be easily proven from the definitions above, implying that probability is globally conserved: $\int_{\mathcal{P}(\mathcal{H})} \dot{\mu}_t = 0$.

here

So far, we have derived the microscopic, kinetic, equations regulating the information transport across the quantum state space. In the next few sections, using both analytical and numerical approaches, we explore the theoretical framework developed so far to look at the phenomenology of information transport in concrete physical systems. In particular, in our examples we look at the information transport in the state space of a qubit, both in isolated and open configurations. This choice is made to support the theoretical calculations with visual representations, both of which are simpler in the case of a qubit.

IV. SYSTEM 2: QUBIT IN ISING ENVIRONMENT

We now look at the case of a single qubit which interacts with an environment made by a 1D chain of qubits evolving with an Ising Hamiltonian with periodic boundary conditions $\vec{\sigma}_{k+L} = \vec{\sigma}_k$

$$H_{\text{Ising}} = \sum_{k=0}^L J_z \sigma_k^z \sigma_{k+1}^z + \vec{B} \cdot \vec{\sigma}_k, \quad (38)$$

with \vec{B} a homogeneous magnetic field. Here, since the number of points in quantum state space is $2^{L_{\text{env}}}$, where L_{env} is the size of the environment, solving the equations of motion analytically becomes unfeasible, and we resort to a numerical approach, in which we can directly evaluate physical quantities of interest, such as the information flux, the sink/source term and the entropies of the various distribution involved. In our numerical analysis, at $t = 0$ the system and the environment begin in a product state, in which the state of the system is $|1\rangle$. Hence, the geometric quantum state is a dirac delta distribution, concentrated on the north pole. However, as time goes by, the overall pure state is not a product anymore and we end up with a non-trivial geometric quantum state, whose evolution is dictated by Eq.(35) or, at the phenomenological level, by the continuity equation. We can map the evolution of the geometric quantum state on the surface of the Bloch sphere, as done in Figure 1 for example, or using (p, ϕ) coordinates, on the Bloch square.

Anti-Ferromagnetic example

In Figure 4 we show the evolution of the geometric quantum state of a qubit with an environment of 12 other qubits, all interacting with an Ising Hamiltonian with $J_z = 1$, $\vec{B} = (1, 0, 0.5)$ and with an initial uniform product state $|\uparrow \dots \uparrow\rangle$. Here we can see that the geometric quantum state, while initially localized, quickly spreads around to reach an equilibrium distribution. It is, however, interesting to see that it reaches equilibrium with a highly structured fashion. Here we can see that for $t \approx 0$ the distribution of points $\Gamma_\alpha(t)$ is organized in three clusters of states which are quite close to each other. As time goes by, each of these clusters *tri-furcates* into smaller clusters of less points. After this splitting occurs a certain number of time we loose track of the clustering and the distribution reaches an equilibrium configuration. The origin of this behavior has to be tracked to the structure of the operators $\hat{V}_{\alpha\beta}$. In particular, we believe this the tri-furcation to be due to the fact that there are two nearest neighbors directly interacting with our system. Increasing this number would result in a more detailed splitting, increasing the number of sub-clusters. To confirm this, we performed the same exact simulation, with a

next-to-nearest neighbor Ising model, observing the formation of 5 sub-clusters at each splitting. A phenomenon we can call ‘‘penta-furcation’’. This suggests the number of new subclusters n_{sc} to be equal to the number n_{int} of qubits our system interacts directly with: $n_{sc} = n_{int}$. While we do not have a definitive explanation for this phenomenon, it has a distinct, and unexpected, fractal flavor which deserves further investigation.

After a transient of 60/70 time-steps, with $\delta t = 0.005$, we can see that the distribution settles on an equilibrium configuration that is almost flat. This can be seen by looking directly at the distribution of information contained in a certain region of the state space. By discretizing the Bloch square, and therefore \mathbb{CP}^1 , via its canonical coordinates, we can integrate the geometric quantum state on the small cells, track their evolution and extract a phenomenological evolution equation. Given a certain number $N_p N_\phi$ of desired cells $\mathcal{C}_{ik} = \left[\frac{i}{N_p}, \frac{i+1}{N_p} \right] \times \left[\frac{2\pi k}{N_\phi}, \frac{2\pi(k+1)}{N_\phi} \right]$, we have $\bigcup_{i=1}^{N_p} \bigcup_{k=1}^{N_\phi} \mathcal{C}_{ik} = \mathcal{P}(\mathcal{H})$, resolution $\delta_p = \frac{1}{N_p}$ and $\delta_\phi = \frac{2\pi}{N_\phi}$. Given a geometric quantum state $q_t(Z)$, the probability mass in each cell is $\mu_{ik}(t) = \int_{\mathcal{C}_{ik}} q_t(Z) dV_{FS}$. By integrating the continuity equation (Eq.(14)) over a cell \mathcal{C}_{ik} we get the following equation:

$$\frac{d\mu_{ik}}{dt} + F_{ik} = \sigma_{ik} , \quad (39)$$

where

$$\sigma_{ik}(t) = \int_{\mathcal{C}_{ik}} dV_{FS} \sigma_t(Z) \quad (40)$$

$$F_{ik} = \int_{\mathcal{C}_{ik}} dV_{FS} \nabla \cdot J_t . \quad (41)$$

Using the divergence theorem, we can turn F_{ik} into an integral over the surface $\partial\mathcal{C}_{ik}$ of \mathcal{C}_{ik} :

$$F_{ik} = \int_{\partial\mathcal{C}_{ik}} J_t \cdot n dS_{FS} = \Phi_{ik}(J_t) , \quad (42)$$

where $n = (n_p, n_\phi)$ is the vector orthogonal to the boundary of \mathcal{C}_{ik} and dS_{FS} is the surface element. This quantity, $F_{ik} = \Phi_{ik}(J)$ is the flow of probability mass across the boundary of each cell \mathcal{C}_{ik} due to the underlying movement of the points $\Gamma_\alpha(t)$. On the other hand, $\sigma_{ik}(t)$ quantifies the probability mass that is produced or sinked in \mathcal{C}_{ik} . Beyond the initial transient, as the distribution settles on a more regular behavior, we expect a more regular evolution equation to emerge. In particular, we expect two emergent behavior. First, we assume the sink/sources term to be zero after a short transient $\sigma_{ik} \approx 0$. This assumption is motivated by two physical reasons. Since the $x_\alpha(t)$ are the probabilities of finding the environment in one of the elements of its eigenbasis $|b_\alpha\rangle$, for a large environment (the number of points is exponential in the size of the system) it is reasonable to expect that after a short

transient these probabilities will settle on ‘‘equilibrium’’ values (as the ones given by max entropy) and, possibly, oscillate around them. Second, we are interested in looking at the system on longer time-scales, where these oscillations can be averaged out. Together, these arguments supports the assumptions for $\sigma_{ik} \approx 0$. This implies that the evolution is guided only by the flux term, leading to a local conservation of probability mass. Thus, we can now build a microscopic, particle-like, model for $\Phi_{ik}(J_t)$. Since probability mass is locally conserved, the only way \mathcal{C}_{ik} can loose or gain probability mass is by exchanging particles with its immediate neighbors: $\mathcal{C}_{i\pm 1,k}$ and $\mathcal{C}_{i,k\pm 1}$. Thus, as first approximation, we assume there is a fixed rate of gain and loss of particles, in each direction (p, ϕ) : γ_p and γ_ϕ . This leads to the following phenomenological model for the evolution of $\mu_{ik}(t)$:

$$\frac{d\mu_{ik}}{dt} = \gamma_p (\mu_{i+1,k} + \mu_{i-1,k}) + \gamma_\phi (\mu_{i,k+1} + \mu_{i,k-1}) - \gamma_{loss} \mu_{ik} . \quad (43)$$

The first two terms represent the gain due to a certain number of particles entering from each of the four sides of a cell \mathcal{C}_{ik} , either from p or ϕ direction. The last term represents the loss due to a certain number of particles leaving \mathcal{C}_{ik} for one of its four adjacent neighbors. By assuming that there are no special cells, unless otherwise specified, this leads to $\gamma_{loss} = 2(\gamma_p + \gamma_\phi)$. This model leads to a simple diffusive behavior that, while not entirely accurate, clearly captures the essence of the evolution beyond the transient. While finite-size fluctuations in time are to be expected, as we increase the size of the Ising environment we expect them to be less relevant, and our model more accurate. A full analysis of the phenomenology of this model will be carried elsewhere. Here we are interested in proving that this theoretical framework can lead to tractable phenomenological models. Indeed, in the limit of an infinitely fine-grained discretization this model leads to a Fick’s law of information diffusion $\mathcal{P}(\mathcal{H})$, as $J_t^{\text{diff}} = (-\gamma_p \frac{\partial q_t}{\partial p}, -\gamma_\phi \frac{\partial q_t}{\partial \phi}) = -\gamma \cdot \nabla q_t$ and therefore to a diffusion equation in $\mathcal{P}(\mathcal{H})$:

$$\frac{\partial q_t}{\partial t} = \gamma \cdot \nabla^2 q_t , \quad (44)$$

where $\gamma = (\gamma_p, \gamma_\phi)$ are direction-dependent diffusion coefficients.

Ferromagnetic example

In figure 5 we show the evolution of the geometric quantum state for a qubit in a ferromagnetic Ising environment. We do not show the short-time transient, as it is qualitatively similar to the antiferromagnetic case, still exhibiting the tri-furcation dynamical pattern. Rather, we focus on two aspects of this non-equilibrium evolution, which were absent from the antiferromag-

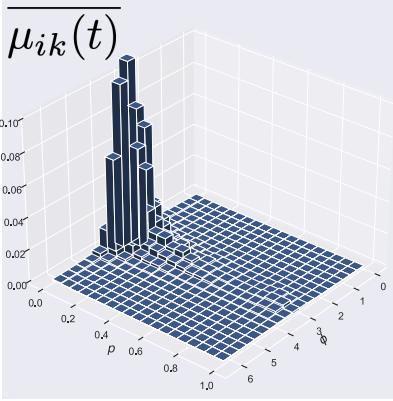


Figure 7. Time-average of the coarse-grained geometric quantum state, $\mu_{ik}(t)$, which conveys the information about how much information about the system is contained in a small cell C_{ik} of state space. In the case of a ferromagnetic Ising environment the non-equilibrium behavior of the quantum system maintains forces the information to be localized in a certain region of the state space, loosely localized around $Z_{center} = (p = 0.1, \phi = \pi)$. Here this is shown by plotting the time-average of $\mu_{ik}(t)$, after the first 100 time-steps, to avoid including transient behavior. A localized distribution is evident.

netic example: *sustained fluctuations*. We have highlighted them with the black arrow superimposed to the top left plot, and are evident in the video, which can be found at https://github.com/fabioanza/GeomQuantMech/blob/main/video_g1_h05_L12_ferro_long.mp4. At its core, this is an emergent phenomenon in which the probability mass, instead of diffusing across the whole state space as in the antiferromagnetic example, appears to be rotating around a point loosely localized as $Z_{center} \approx (p = 0.1, \phi = \pi)$. As a result of this, over time the distribution $\mu_{ik}(t)$ appears to remain localized in a small region around Z_{center} . This can be seen by plotting the time-averaged information distribution $\mu_{ik}(t) = \frac{1}{400} \sum_{n=101}^{500} \mu_{ik}(t_n)$, in which the time-average has been performed by excluding the first 100 time-steps, to exclude the transient behavior. The result is shown in the two bottom rows of Figure 6. We now use the approach developed in the previous section to understand how this occurs. Indeed, by plotting the coarse-grained flux of information $J_{ik}(t) = \int_{C_{ik}} J_t(Z) dV_{FS}$ and the coarse-grained information production term, in Figure 6 we see that the localization of the information around Z_{center} , and its stability over time, arises from the competition between a sink term, drawing up information and delocalizing it across the state space, and a rotating vector field $J_{ik}(t)$, moving the points away from the sink but maintaining them close to Z_{center} , with an overall rotation effect. Over time, these two effects balance each other out maintaining the information localized around Z_{center} , as it can be seen from the instantaneous distribution $\mu_{ik}(t)$, plotted in the top row of Figure 6.

While a fully analytical description of this phenomenon

requires an in-depth investigation of the microscopic details, the analysis given in term of information flux and information sink and source term provides a clear connection between the microscopic, kinetic, evolution and the macroscopic behavior provided by expectation values of observables like $\langle \sigma_z \rangle$.

V. CONCLUSIONS

We have developed a consistent framework to study how quantum systems move information around their state space. Our main results are the continuity equation (Eq.(14)) and the underlying kinetic equations (Eq.(35)). Together, they regulate how information can be transported around the quantum state space as a result of the interactions between the system and its environment. The kinetic equation represents the microscopic, detailed, model while the goal of the continuity equation formulation is to extract emergent, simplified, phenomenological descriptions. As an example of this logic, in Section II and IV we have looked at the transport of information in the state space of a qubit. In both cases we have seen the emergence of consistent behavior that can be described with sufficiently simple information fluxes. In the case of an isolated qubit, we were able to analytically extract the information flux. The case of an antiferromagnetic Ising environment with both longitudinal and transverse field led us to see how a simple diffusion model can reproduce the macroscopic feature of the kinetic evolution.

While here we focused only on the theoretical framework, we believe we have shown that the study of concrete kinetic models of information transport is phenomenologically quite rich, and deserves to be investigated in depth. The underlying reason behind our efforts in this kind of geometric-oriented research direction is that we believe these new tools will allow us to (1) gather a more accurate understanding and (2) build improved models for the nonequilibrium behavior of quantum systems. In particular, we believe these tools will reveal to be appropriate to get a more accurate picture of the interplay between the non-equilibrium physical properties of open quantum systems and the information-theoretic features of their dynamics, which emerge from their complex stochastic nature.

Future work in this direction will be focused on extracting more general, approximated, analytical models and developing more powerful numerical techniques, to investigate the behavior of quantum systems with larger environments. An interesting concrete issue is to understand how quantum information moves around the quantum state space when our system is part of a quantum computer performing quantum computation. Indeed, we believe this framework can help better understand how to improve and optimize quantum information processing for future quantum technologies.

We conclude with a general comment, clarifying our main

goal behind the paper, and with a series of future directions opened by our work. While our main goal here was to develop the theoretical framework and build the intuition to support it, we believe the general analyses presented in Sections II and IV revealed four features of interest, which deserve attention on their own. First, the dynamical emergence of the fractal pattern in the transient behavior of the antiferromagnetic model was unexpected and one wonders if similar dynamical patterns are typical of quantum models or not. This is clearly related to the Second point: the more general question of describing stationary states, with non-trivial non-equilibrium behavior. A natural route starts with imposing $\frac{\partial q_t}{\partial t} = 0$ in the continuity equation, which in turn leads to a balance equation between information flux and information sinks and sources. While one can expect that $J_t = \sigma_t = 0$ characterizes thermal equilibrium, this will not be true in general, and the continuity equation can be used to characterize stationary states. Third, in the Discussion part of Section II (see Eq. (29)), we have seen that a unitary dynamics is a very rigid evolution in which there is a unique velocity field (a divergence-free one) and all points in the support of any geometric quantum state have to move in the same way. It is well known from the theory of dynamical systems that this is not a feature of all Hamiltonian evolutions, which are able to support chaotic phenomenology via stretching and folding. Therefore, a quantum state space should be able to support more general Hamiltonian evolutions, exhibit-

ing richer phenomenology as chaos, of quantum nature. Thus, we believe the geometric formalism and the framework developed here to be a concrete route to improve our understanding of quantum chaos. Fourth, in Section II we have proven a novel form of Liouville's theorem for quantum systems. This poses a conundrum as it is in stark contrast with all other phase space formulation of quantum mechanics, such as Wigner's [29] or variations of it such as the one based on the Husimi function [30], both of which are known to violate Liouville's theorem [31]. The validity of Liouville's theorem is not just a mathematical fact, but a truly important property of the dynamical evolution, which can help us characterize useful properties of the dynamics, such as the emergence of chaos. Here, again, the geometric formalism and the theoretical framework developed here is bringing to light interesting new insights into the nonequilibrium behavior of quantum systems.

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