ODS

Fabio Bühler, fabuehle@ethz.ch

Version: 19. März 2024

1 Math Preliminaries

1.1 Linear Algebra

Cauchy-Schwarz Inequality

$$|\mathbf{u}^\top \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\| \Leftrightarrow -1 \leq \underbrace{\frac{\mathbf{u}^\top \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}}_{\cos(\alpha)} \leq 1$$

Equality is obtained if $\alpha = 0$ or $\alpha = \pi$.
Also holds for other norms.

Spectral Norm

Let $\mathbf{A}$ be a $(m \times d)$-matrix. Then the spectral norm of $\mathbf{A}$ is defined as:

$$\|\mathbf{A}\| := \max_{\mathbf{v} \in \mathbb{R}^d, \mathbf{v} \neq 0} \frac{\|A\mathbf{v}\|}{\|\mathbf{v}\|} = \max_{\|\mathbf{v}\|=1} \|\mathbf{A}\mathbf{v}\| = \sqrt{\lambda_{\text{max}}(\mathbf{A}^\top \mathbf{A})}$$

From this it follows: $\|A\mathbf{v}\| \leq \|A\| \, \|\mathbf{v}\|$

Mean Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Fundemental Theorem of Calculus

Let $f : \mathbf{dom}(f) \to \mathbb{R}$ be a differentiable function on an open domain $\mathbf{dom}(f) \supset [a, b]$ and let f' be continuous on $[a, b]$. Then

$$f(b) - f(a) = \int_a^b f'(x) \, dx$$

Differentiability

Let $f : \mathbf{dom}(f) \to \mathbb{R}^m$ where $\mathbf{dom}(f) \subseteq \mathbb{R}^d$. The function f is differentiable at $\mathbf{x} \in \mathbf{dom}(f)$ if there exists a matrix $\mathbf{A} \in \mathbb{R}^{m \times d}$ and an error function $r : \mathbb{R}^d \to \mathbb{R}^m$ defined in some neighborhood of $\mathbf{0} \in \mathbb{R}^d$ such that for all $\mathbf{y}$ in some neighborhood of $\mathbf{x}$:

$$f(\mathbf{y}) = f(\mathbf{x}) + \mathbf{A}(\mathbf{y} - \mathbf{x}) + r(\mathbf{y} - \mathbf{x})$$

where

$$\lim_{\mathbf{v} \to \mathbf{0}} \frac{r(\mathbf{v})}{\|\mathbf{v}\|} = \mathbf{0}$$

From this it follows that the matrix $\mathbf{A}$ is unique and is called the differential or Jacobian of f at $\mathbf{x}$ and is denoted by $Df(\mathbf{x})$.
$Df(\mathbf{x})$ is the matrix of partial derivatives at the point $\mathbf{x}$:

$$Df(\mathbf{x})_{ij} = \frac{\partial f_i}{\partial x_j}(\mathbf{x})$$

f is called differentiable if f is differentiable at every point in $\mathbf{dom}(f)$ (implies that $\mathbf{dom}(f)$ is open).

Chain Rule

Let $f : \mathbf{dom}(f) \to \mathbb{R}^m$, $\mathbf{dom}(f) \subseteq \mathbb{R}^d$ and $g : \mathbf{dom}(g) \to \mathbb{R}^d$.
Suppose that g is differentiable at $\mathbf{x} \in \mathbf{dom}(g)$ and f is differentiable at $g(\mathbf{x}) \in \mathbf{dom}(f)$.
Then the composition $f \circ g$ is differentiable at $\mathbf{x}$ with differential:

$$D(f \circ g)(\mathbf{x}) = Df(g(\mathbf{x}))Dg(\mathbf{x})$$

2 Convex Optimization

2.1 Convex Sets

Convex Set

A set $C \subseteq \mathbb{R}^d$ is called *convex* if for any two points $\mathbf{x}, \mathbf{y} \in C$, the connecting line segment is contained in C.

$$\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in C, \quad \forall \mathbf{x}, \mathbf{y} \in C, \lambda \in [0, 1]$$

$C = \cap_{i \in I} C_i$ is convex if all C_i are convex.

Mean Value Inequality

Let $f : \mathbf{dom}(f) \to \mathbb{R}^m$ be differentiable, $X \subseteq \mathbf{dom}(f)$ a convex set, $B \in \mathbb{R}^+$. If $X \subseteq \mathbf{dom}(f)$ is nonempty and open, the following two statements are equivalent.

(i) f is B-Lipschitz, meaning that

$$||f(\mathbf{x}) - f(\mathbf{y})|| \leq B||\mathbf{x} - \mathbf{y}||, \quad \mathbf{x}, \mathbf{y} \in X$$

(ii) f has differentials bounded by B (in spectral norm)

$$||Df(\mathbf{x})|| \leq B, \quad \forall \mathbf{x} \in X.$$

Moreover, for every (not necessarily open) convex $X \subseteq \mathbf{dom}(f)$, (ii) implies (i), and this is the mean value inequality.

2.2 Convex Functions

We are considering real valued functions $f : \mathbf{dom}(f) \to \mathbb{R}, \mathbf{dom}(f) \subseteq \mathbb{R}^d$.

Convexity Definition

A function $f : \mathbf{dom}(f) \to \mathbb{R}$ is convex if

(i) $\mathbf{dom}(f)$ is convex, and

(ii) for all $\mathbf{x}, \mathbf{y} \in \mathbf{dom}(f)$ and all $\lambda \in [0, 1]$, we have

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y})$$

Epigraph

The epigraph of a function $f : \mathbf{dom}(f) \to \mathbb{R}$ is defined as

$$\text{epi}(f) = \{(\mathbf{x}, \alpha) \in \mathbf{R}^{d+1} : \mathbf{x} \in \mathbf{dom}(f), \alpha \geq f(\mathbf{x})\}$$

The function f is convex if and only if epi(f) is a convex set.

Jensens Inequality

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function, $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbf{dom}(f)$, and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}_+$ sucht that $\sum_{i=1}^m \lambda_i = 1$. Then

$$f\left(\sum_{i=1}^m \lambda_i \mathbf{x}_i\right) \leq \sum_{i=1}^m \lambda_i f(\mathbf{x}_i)$$

If f is concave, the inequality is reversed.