

Diffusion Models: DALL-E

Deep Learning and Neural Networks: Advanced Topics

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March 1, 2023

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TELECOMMUNICATIONS,
COMPUTER
ENGINEERING,
AND PHOTONICS
INSTITUTE



Sant'Anna
School of Advanced Studies – Pisa



Introduction

Diffusion Models

Broader Impacts

Introduction



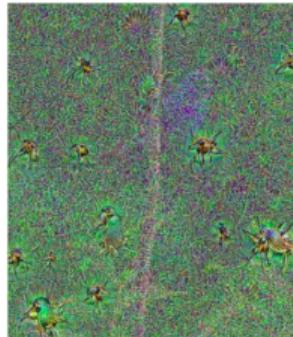
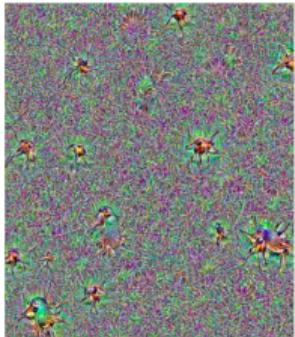
Diffusion Models



Overview

Diffusion models are generative models that aim at denoising data

Inverse Diffusion Process



Diffusion Process

Timeline

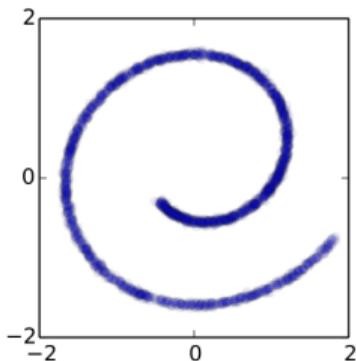
2015) ...*Non-equilibrium Thermodynamics*. Sohl-Dickstein et al. ICML

2020) *Denoising Diffusion Probabilistic Models*. Ho et al. NeurIPS.

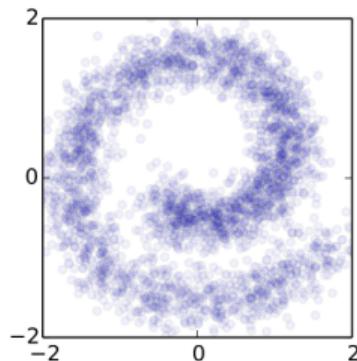
2021) *Score-Based Generative Modeling Through SDE*. Song et al. ICLR.

Deep Unsupervised Learning using Non-Equilibrium Thermodynamics

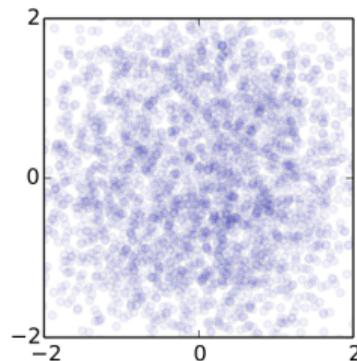
$t = 0$



$t = \frac{T}{2}$



$t = T$



Diffusion process as a **Markov Chain** with Continuous State Space and Discrete Time.¹

¹thermodynamic.

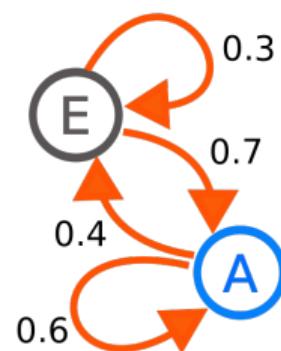
Reminder: Markov Chains with Discrete Time

Informal Definition

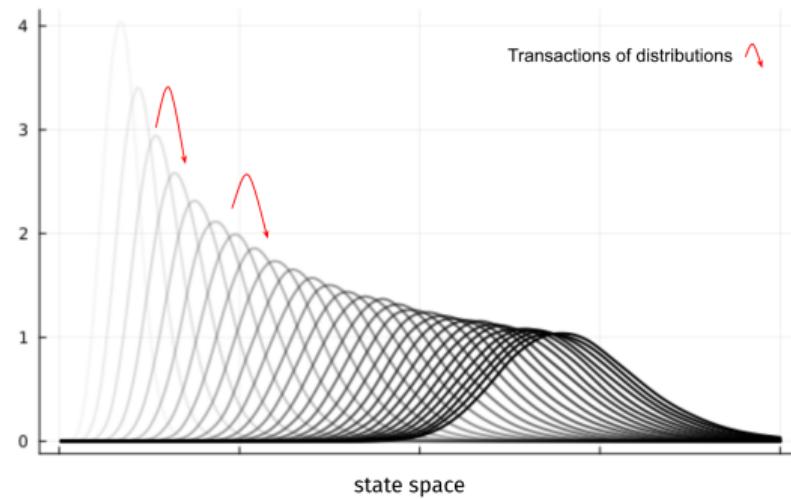
A sequence of random variables $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(t)}, \dots$, such that:

- $\mathbf{x}^{(t)} \in S$, where S State Space
- The future $\mathbf{x}^{(t+1)}$ depends on the present $\mathbf{x}^{(t)}$ but not on the past $\mathbf{x}^{(t-1)}$

Discrete State Space S



Continuous State Space S



Reminder: MCDT with Discrete State Space

Definition

A sequence $\{\mathbf{x}^{(t)}\}_{t \in \mathbb{N}} \subseteq S$, a matrix $P = (p_{ij})$.

- **Discrete state space:** $S = \{s_0, \dots, s_n, \dots\}$
- **Markov Property:** $\mathbf{x}^{(t+1)}$ not dep. $\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(t-1)}$.
- **Transaction Matrix:** $\mathbb{P}(\mathbf{x}^{(t+1)} = s_j | \mathbf{x}^{(t)} = s_i) = p_{ij}$

Reminder: MCDT with Discrete State Space

Definition

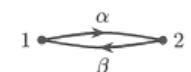
A sequence $\{\mathbf{x}^{(t)}\}_{t \in \mathbb{N}} \subseteq S$, a matrix $P = (p_{ij})$.

P is a stochastic matrix!

$$\forall i, \quad \sum_{j \in \mathbb{N}} p_{ij} = 1$$

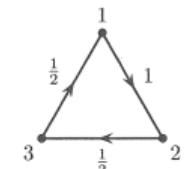
- Discrete state space: $S = \{s_0, \dots, s_n, \dots\}$

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$



- Markov Property: $\mathbf{x}^{(t+1)}$ not dep. $\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(t-1)}$.

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$



- Transaction Matrix: $\mathbb{P}(\mathbf{x}^{(t+1)} = s_j | \mathbf{x}^{(t)} = s_i) = p_{ij}$

Reminder: DTMC with Continuous State Space

Let assume $\mathbf{x}, \mathbf{y} \in S$ where S continuous state space (e.g. $S = \mathbb{R}^d$).

Joint Distribution $p(\mathbf{x}, \mathbf{y})$

$$\mathbb{P}(\mathbf{x} \in A \mid \mathbf{y} \in B) = \int_A \int_B p(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} \, d\mathbf{y}$$

Transactional Kernel $p(\mathbf{x} \mid \mathbf{y})$

$$p(\mathbf{x}, \mathbf{y}) = p(\mathbf{x} \mid \mathbf{y}) p(\mathbf{y})$$

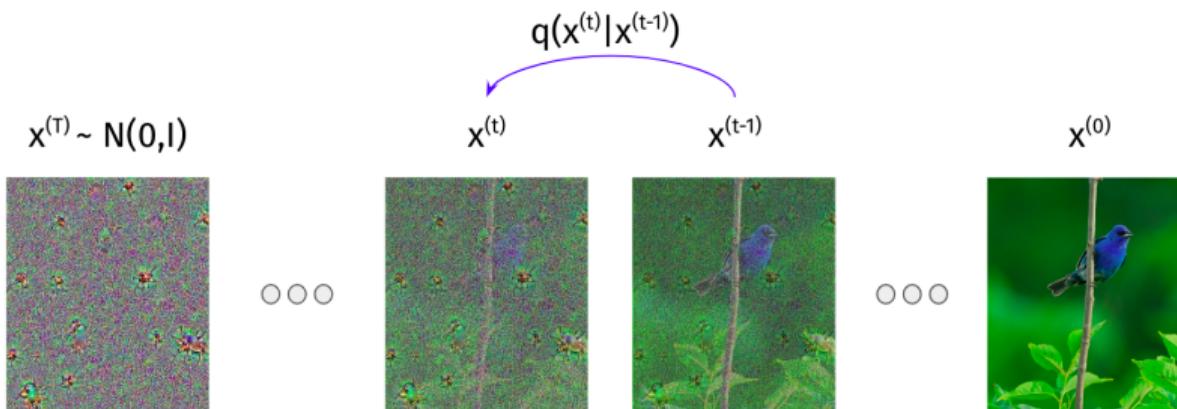
Marginal Distribution $p(\mathbf{x})$

$$p(\mathbf{x}) = \int_S p(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} = \int_S p(\mathbf{x} \mid \mathbf{y}) p(\mathbf{y}) \, d\mathbf{y}$$

Forward Diffusion Process

“Adding noise to data...”

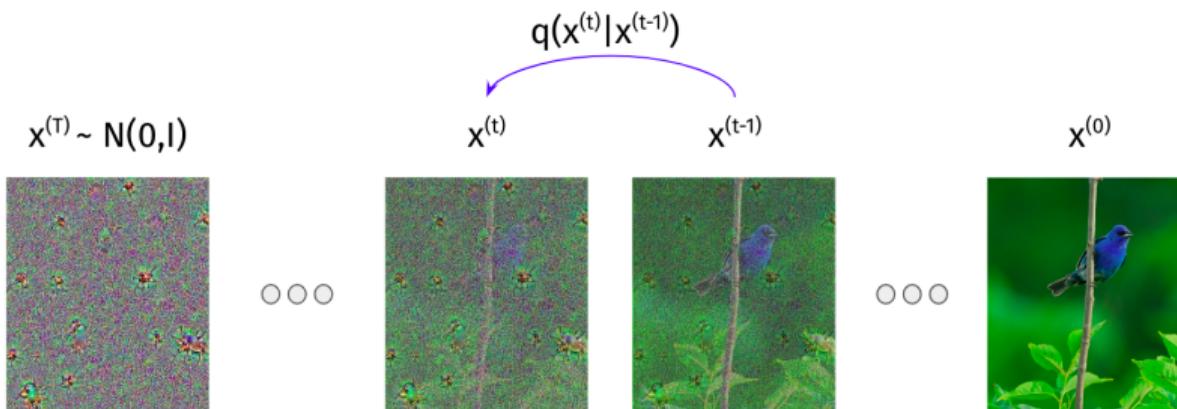
- Data Distribution: $\mathbf{x}^{(0)} \sim q$
- Transaction Kernel: $q(\mathbf{x}^{(t)} | \mathbf{x}^{(t-1)}) = \mathcal{N}(\mathbf{x}^{(t)}; \sqrt{1 - \beta_t} \mathbf{x}^{(t-1)}; \beta_t I)$
- Variance Scheduler: $\beta_1, \dots, \beta_T \in (0, 1]$



Forward Diffusion Process

“Adding noise to data...”

- Data Distribution: $\mathbf{x}^{(0)} \sim q$ Not Analytic!!
- Transaction Kernel: $q(\mathbf{x}^{(t)} | \mathbf{x}^{(t-1)}) = \mathcal{N}(\mathbf{x}^{(t)}; \sqrt{1 - \beta_t} \mathbf{x}^{(t-1)}; \beta_t I)$
- Variance Scheduler: $\beta_1, \dots, \beta_T \in (0, 1]$ \beta_T = 1



Forward Diffusion Process: Explicit Representation

$$\mathbf{x}^{(t)} = \sqrt{1 - \beta_t} \mathbf{x}^{(t-1)} + \sqrt{\beta_t} \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \sim \mathcal{N}(0, I)$$

Observation: Many small noisy steps \approx Large Noisy step

$$\mathbf{x}^{(t)} = \sqrt{1 - \alpha_t} \mathbf{x}^{(0)} + \sqrt{\alpha_t} \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim \mathcal{N}(0, I)$$

where

$$\alpha_t = 1 - \prod_{i=0}^t (1 - \beta_i)$$

Forward Diffusion Process: Distribution Representation

Markov property allows breaking up distributional Representation...

$$q(\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T)}) = q\left(\mathbf{x}^{(T)} \mid \mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T-1)}\right) q\left(\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T-1)}\right)$$

Forward Diffusion Process: Distribution Representation

Markov property allows breaking up distributional Representation...

$$\begin{aligned} q(\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T)}) &= q\left(\mathbf{x}^{(T)} \mid \mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T-1)}\right) q\left(\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T-1)}\right) \\ &= q\left(\mathbf{x}^{(T)} \mid \mathbf{x}^{(T-1)}\right) q\left(\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T-1)}\right) \\ &\vdots \end{aligned} \tag{1}$$

Forward Diffusion Process: Distribution Representation

Markov property allows breaking up distributional Representation...

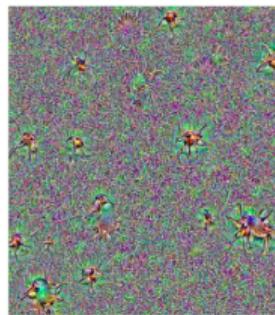
$$\begin{aligned} q(\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T)}) &= q\left(\mathbf{x}^{(T)} \mid \mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T-1)}\right) q\left(\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T-1)}\right) \\ &= q\left(\mathbf{x}^{(T)} \mid \mathbf{x}^{(T-1)}\right) q\left(\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T-1)}\right) \\ &\vdots \end{aligned} \tag{1}$$

Distributional Representation

$$q(\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T)}) = q(\mathbf{x}^{(0)}) \prod_{t=1}^T q\left(\mathbf{x}^{(t)} \mid \mathbf{x}^{(t-1)}\right)$$

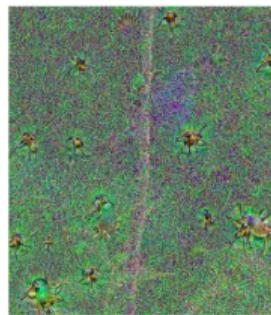
Reverse Diffusion Process

$$x^{(T)} \sim N(0, I)$$

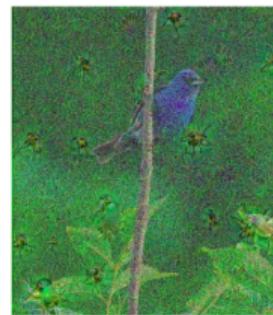


○ ○ ○

$$x^{(t)}$$



$$x^{(t-1)}$$

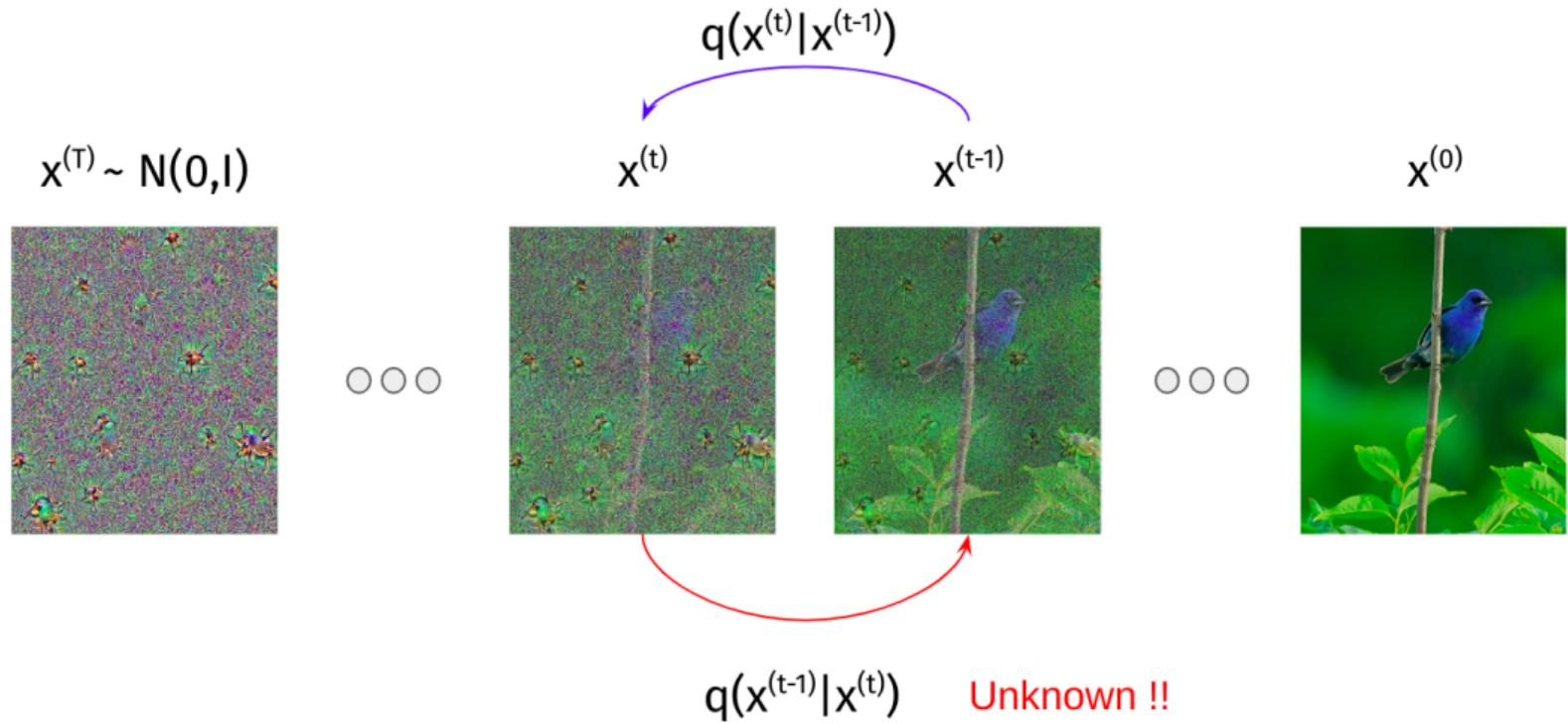


○ ○ ○

$$x^{(0)}$$



Reverse Diffusion Process



Reverse Diffusion Process

Forward Diffusion Process

$q(\mathbf{x}^{(0)})$ Data Distribution

$$q(\mathbf{x}^{(0\dots T)}) = q(\mathbf{x}^{(0)}) \prod_{t=1}^T q\left(\mathbf{x}^{(t)} \mid \mathbf{x}^{(t-1)}\right)$$

Reverse Diffusion Process

$q(x^{(T)}) = \mathcal{N}(0, I)$

$$q(\mathbf{x}^{(0\dots T)}) = q(\mathbf{x}^{(T)}) \prod_{t=1}^T q\left(\mathbf{x}^{(t-1)} \mid \mathbf{x}^{(t)}\right)$$

²thermodynamic.

Reverse Diffusion Process

Forward Diffusion Process

$q(\mathbf{x}^{(0)})$ Data Distribution

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Reverse Diffusion Process

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Theorem. Reverse of Gaussian DP is \approx Gaussian DP²

If $|\beta_i - \beta_{i+1}| \approx 0$, i.e. diffusion slow enough, then

$$q(\mathbf{x}^{(t-1)} \mid \mathbf{x}^{(t)}) \approx \mathcal{N}\left(\mathbf{x}^{(t-1)}; \boldsymbol{\mu}_\theta\left(\mathbf{x}^{(t)}, t\right), \boldsymbol{\Sigma}_\theta\left(\mathbf{x}^{(t)}, t\right)\right)$$

²thermodynamic.

Reverse Diffusion Process

Forward Diffusion Process

$q(\mathbf{x}^{(0)})$ Data Distribution

$$q(\mathbf{x}^{(0\dots T)}) = q(\mathbf{x}^{(0)}) \prod_{t=1}^T q\left(\mathbf{x}^{(t)} | \mathbf{x}^{(t-1)}\right)$$

Reverse Diffusion Process

$q(x^{(T)}) = \mathcal{N}(0, I)$

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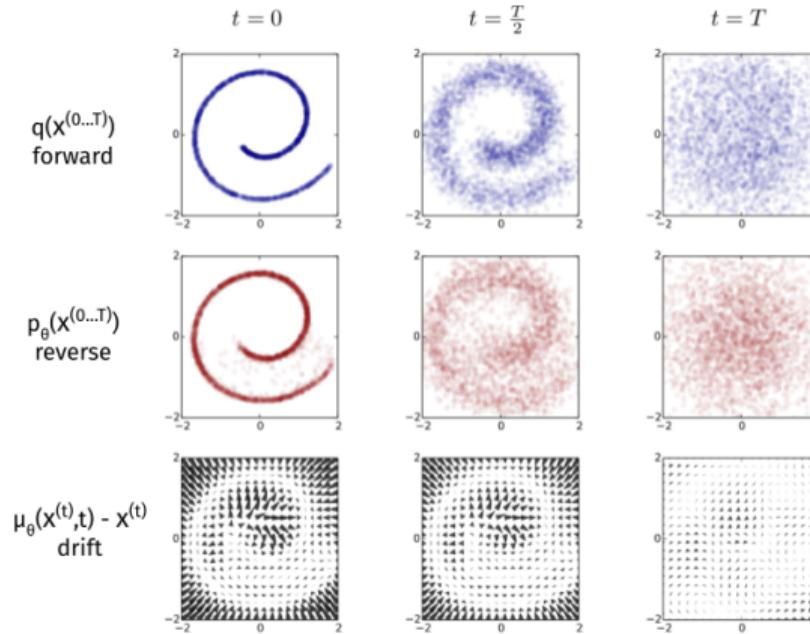
If $|\beta_i - \beta_{i+1}| \approx 0$, i.e. diffusion slow enough, then

$$q(\mathbf{x}^{(t-1)} | \mathbf{x}^{(t)}) \approx \mathcal{N}\left(\mathbf{x}^{(t-1)}; \mu_\theta(\mathbf{x}^{(t)}, t), \Sigma_\theta(\mathbf{x}^{(t)}, t)\right)$$

Mean μ_θ and covariance Σ_θ have to be learned!!

²thermodynamic.

Visualization of Diffusion Process: 2D dimensional case



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³thermodynamic.

Training of μ_θ and Σ_θ

Aim

Search for the best parameters θ

$$q(\mathbf{x}^{(0)}) \approx p_\theta(\mathbf{x}^{(0)})$$

where $\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T)}$ diffusion process

Estimated Reverse Process

$$p_\theta(\mathbf{x}^{(T)}) = \mathcal{N}\left(\mathbf{x}^{(T)}; 0, I\right)$$

$$p_\theta(\cdot | \mathbf{x}^{(t)}) = \mathcal{N}\left(\boldsymbol{\mu}_\theta\left(\mathbf{x}^{(t)}, t\right), \boldsymbol{\Sigma}_\theta\left(\mathbf{x}^{(t)}, t\right)\right)$$

Training of μ_θ and Σ_θ

Aim

Search for the best parameters θ

$$q(\mathbf{x}^{(0)}) \approx p_\theta(\mathbf{x}^{(0)})$$

where $\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T)}$ diffusion process

Method

Minimize the Kullback–Leibler Divergence

$$D_{KL}(q || p_\theta) := \int q(\mathbf{x}^{(0)}) \log \left(\frac{q(\mathbf{x}^{(0)})}{p_\theta(\mathbf{x}^{(0)})} \right) d\mathbf{x}^{(0)}$$

Estimated Reverse Process

$$p_\theta(\mathbf{x}^{(T)}) = \mathcal{N} \left(\mathbf{x}^{(T)}; 0, I \right)$$

$$p_\theta(\cdot | \mathbf{x}^{(t)}) = \mathcal{N} \left(\boldsymbol{\mu}_\theta \left(\mathbf{x}^{(t)}, t \right), \boldsymbol{\Sigma}_\theta \left(\mathbf{x}^{(t)}, t \right) \right)$$

Training of μ_θ and Σ_θ

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Estimated Reverse Process

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$$p_\theta(\cdot | \mathbf{x}^{(t)}) = \mathcal{N} \left(\boldsymbol{\mu}_\theta \left(\mathbf{x}^{(t)}, t \right), \boldsymbol{\Sigma}_\theta \left(\mathbf{x}^{(t)}, t \right) \right)$$

Easy??

Training of μ_θ and Σ_θ

Aim

Search for the best parameters θ

$$q(\mathbf{x}^{(0)}) \approx p_\theta(\mathbf{x}^{(0)})$$

where $\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T)}$ diffusion process

Method

Minimize the Kullback–Leibler Divergence

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Estimated Reverse Process

$$p_\theta(\mathbf{x}^{(T)}) = \mathcal{N} \left(\mathbf{x}^{(T)}; 0, I \right)$$

$$p_\theta(\cdot | \mathbf{x}^{(t)}) = \mathcal{N} \left(\boldsymbol{\mu}_\theta \left(\mathbf{x}^{(t)}, t \right), \boldsymbol{\Sigma}_\theta \left(\mathbf{x}^{(t)}, t \right) \right)$$

Easy??

No. $q(\mathbf{x}^{(0)})$ is analytically intractable!!



Training of μ_θ and Σ_θ

Aim: Deduce a tractable loss function

$$D_{KL}(q \parallel p_\theta) := \int q(\mathbf{x}^{(0)}) \log \left(\frac{q(\mathbf{x}^{(0)})}{p_\theta(\mathbf{x}^{(0)})} \right) d\mathbf{x}^{(0)}$$

Training of μ_θ and Σ_θ

Aim: Deduce a tractable loss function

$$D_{KL}(q \parallel p_\theta) := \int q(\mathbf{x}^{(0)}) \log \left(\frac{q(\mathbf{x}^{(0)})}{p_\theta(\mathbf{x}^{(0)})} \right) d\mathbf{x}^{(0)}$$

Simplification I: Minimize the Cross Entropy

$$D_{KL} \left(q(\mathbf{x}^{(0)}) \parallel p_\theta(\mathbf{x}^{(0)}) \right) = \int q(\mathbf{x}^{(0)}) \log(q(\mathbf{x}^{(0)})) d\mathbf{x}^{(0)} + \int -q(\mathbf{x}^{(0)}) \log(p_\theta(\mathbf{x}^{(0)})) d\mathbf{x}^{(0)}$$

Training of μ_θ and Σ_θ

Aim: Deduce a tractable loss function

$$D_{KL}(q \parallel p_\theta) := \int q(\mathbf{x}^{(0)}) \log \left(\frac{q(\mathbf{x}^{(0)})}{p_\theta(\mathbf{x}^{(0)})} \right) d\mathbf{x}^{(0)}$$

Simplification I: Minimize the Cross Entropy

$$D_{KL} \left(q(\mathbf{x}^{(0)}) \parallel p_\theta(\mathbf{x}^{(0)}) \right) = \underbrace{\int q(\mathbf{x}^{(0)}) \log(q(\mathbf{x}^{(0)})) d\mathbf{x}^{(0)}}_{-\mathbb{H}(q(\mathbf{x}^{(0)}))} + \underbrace{\int -q(\mathbf{x}^{(0)}) \log(p_\theta(\mathbf{x}^{(0)})) d\mathbf{x}^{(0)}}_{L_{CE}(p_\theta)}$$

Training of μ_θ and Σ_θ

Minimize the Cross Entropy Loss

$$L_{CE}(p_\theta(\mathbf{x}^{(0)})) := - \int q(\mathbf{x}^{(0)}) \log(p_\theta(\mathbf{x}^{(0)})) d\mathbf{x}^{(0)}$$

Training of μ_θ and Σ_θ

Minimize the Cross Entropy Loss

$$L_{CE}(p_\theta(\mathbf{x}^{(0)})) := - \int q(\mathbf{x}^{(0)}) \log(p_\theta(\mathbf{x}^{(0)})) d\mathbf{x}^{(0)}$$

Observation: Marginal Distribution

$$p_\theta(\mathbf{x}^{(0)}) = \int p_\theta(\mathbf{x}^{(0\dots T)}) d\mathbf{x}^{(1\dots T)}$$

Training of μ_θ and Σ_θ

Minimize the Cross Entropy Loss

$$L_{CE}(p_\theta(\mathbf{x}^{(0)})) := - \int q(\mathbf{x}^{(0)}) \log(p_\theta(\mathbf{x}^{(0)})) d\mathbf{x}^{(0)}$$

Observation: Marginal Distribution

$$p_\theta(\mathbf{x}^{(0)}) = \int p_\theta(\mathbf{x}^{(0\dots T)}) d\mathbf{x}^{(1\dots T)}$$

Simplification II: Jensen Inequality

$$L_{CE}(p_\theta) \leq -\mathbb{E}_{q(\mathbf{x}^{(0\dots T)})} \left[\log \frac{q(\mathbf{x}^{(1\dots T)} | \mathbf{x}^{(0)})}{p_\theta(\mathbf{x}^{(0\dots T)})} \right]$$

Training of μ_θ and Σ_θ

...after some algebraic steps :)

Reformulated Loss Function

$$\mathcal{L} = \mathcal{L}_T + \sum_{t=1}^{T-1} \mathcal{L}_t + \mathcal{L}_0$$

where,

$$\mathcal{L}_T = \mathbb{E}_{q(\mathbf{x}^{(0\dots T)})} \left[D_{KL} \left(q(\mathbf{x}^{(T)} | \mathbf{x}^{(0)}) \parallel p_\theta(\mathbf{x}^{(T)}) \right) \right]$$

$$\mathcal{L}_t = \mathbb{E}_{q(\mathbf{x}^{(0\dots T)})} \left[D_{KL} \left(q(\mathbf{x}^{(t)} | \mathbf{x}^{(t+1)}, \mathbf{x}^{(0)}) \parallel p_\theta(\mathbf{x}^{(t)} | \mathbf{x}^{(t+1)}) \right) \right]$$

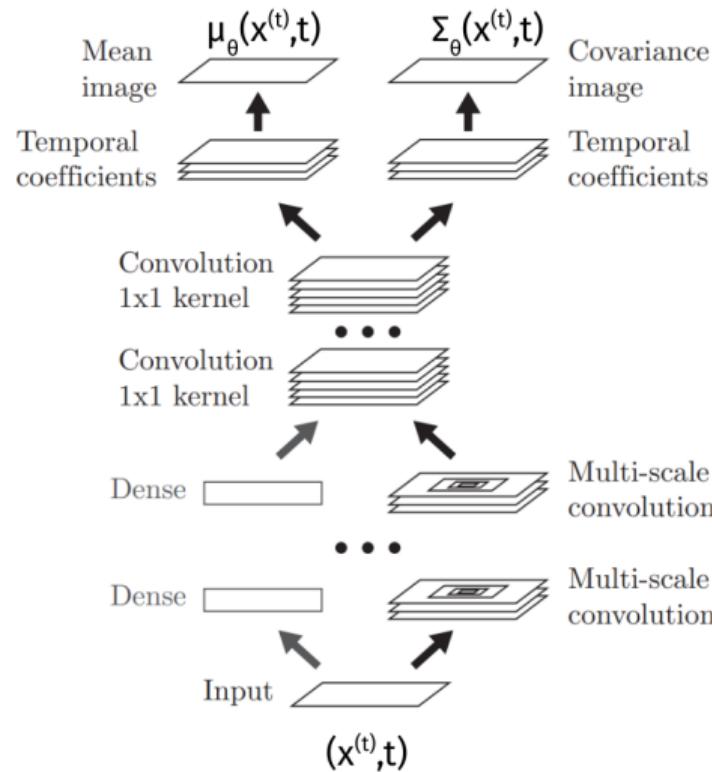
$$\mathcal{L}_0 = \mathbb{E}_{q(\mathbf{x}^{(0\dots T)})} \left[p_\theta(\mathbf{x}^{(0)} | \mathbf{x}^{(1)}) \right]$$

Note.

\mathcal{L}_T is constant.

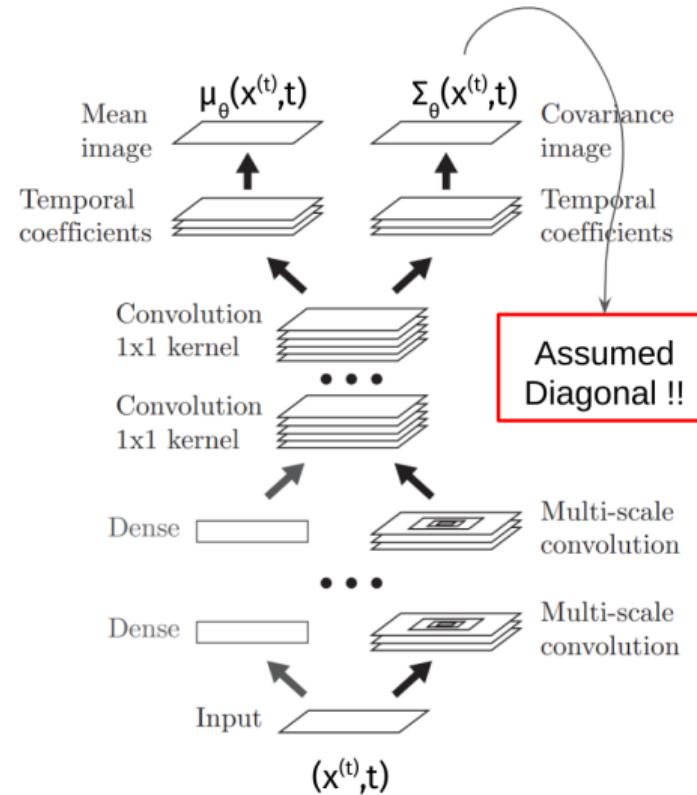
$\mathcal{L}_0, \mathcal{L}_t$ explicit.

Neural Network that estimate μ_θ and Σ_θ



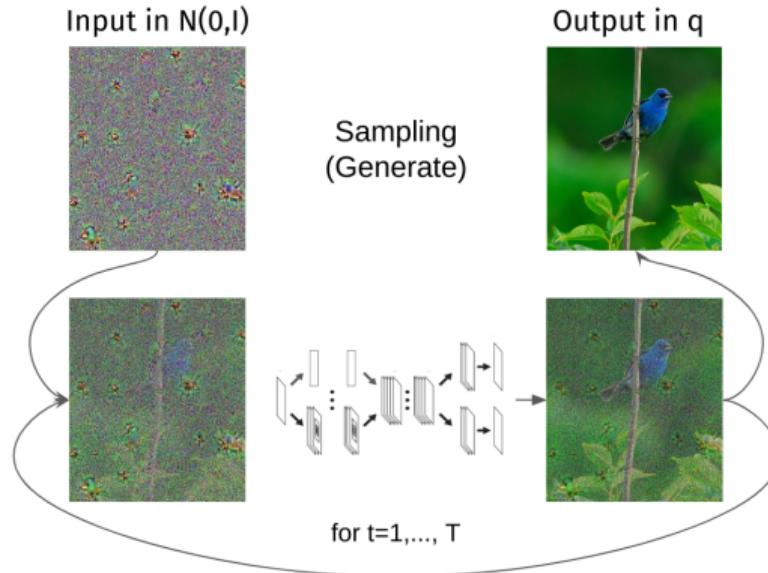
Proposed Neural Network for CIFAR10 image generation. T=1000

Neural Network that estimate μ_θ and Σ_θ

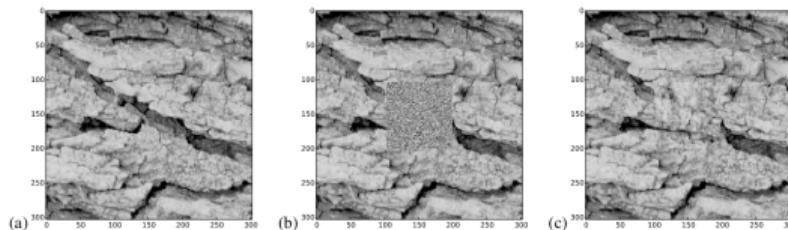


Proposed Neural Network for CIFAR10 image generation. T=1000

Sampling or Generative Stage



Experiments



MNIST

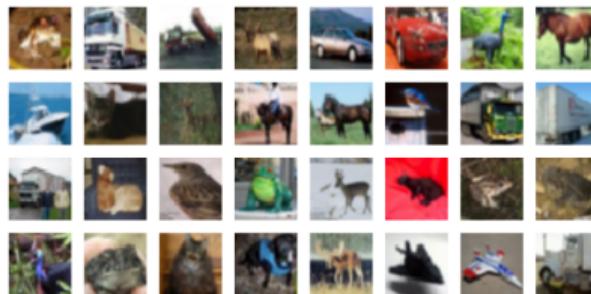


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thermodynamic.

Experiments

CIFAR10 (original)



CIFAR10 (generated)



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⁵thermodynamic.

Timeline

2015) ...*Non-equilibrium Thermodynamics*. Sohl-Dickstein et al. ICML



2020) *Denoising Diffusion Probabilistic Models*. Ho et al. NeurIPS.

2021) *Score-Based Generative Modeling Through SDE*. Song et al. ICLR.

Broader Impacts



"We also found discrepancies across gender and race for people categorized into the 'crime' and 'non-human' categories..."⁶

⁶clip.

Thanks for the attention

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ROBOTICS & AI



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Real-Time Systems Laboratory

Proof Details



Proof of Explicit Representation of Forward Diffusion Process

Let us proceed by induction by assuming $\mathbf{x}^{(t)} = \sqrt{1 - \alpha_t} \mathbf{x}^{(0)} + \sqrt{\alpha_t} \boldsymbol{\varepsilon}$ where $\boldsymbol{\varepsilon} \sim \mathcal{N}(0, I)$ and where $\alpha_t = 1 - \prod_{i=0}^t (1 - \beta_i)$.

$$\begin{aligned}\mathbf{x}^{(t+1)} &= \sqrt{1 - \beta_{t+1}} \mathbf{x}^{(t)} + \sqrt{\beta_{t+1}} \boldsymbol{\varepsilon}_{t+1} \\ &= \sqrt{1 - \beta_{t+1}} \left(\sqrt{1 - \alpha_t} \mathbf{x}^{(0)} + \sqrt{\alpha_t} \boldsymbol{\varepsilon} \right) + \sqrt{\beta_{t+1}} \boldsymbol{\varepsilon}_{t+1} \\ &= \sqrt{\left(\prod_{i=0}^{t+1} (1 - \beta_i) \right)} \mathbf{x}^{(0)} + \sqrt{(1 - \beta_{t+1})\alpha_t + \beta_{t+1}} \tilde{\boldsymbol{\varepsilon}}\end{aligned}\tag{2}$$

where the last term of the summation is obtained by observing that, since $\sqrt{(1 - \beta_{t+1})\alpha_t} \boldsymbol{\varepsilon}$ and $\sqrt{\beta_{t+1}} \boldsymbol{\varepsilon}_{t+1}$ are independent, then the variance of their sum (that still has a gaussian distribution) is given by $(1 - \beta_{t+1})\alpha_t + \beta_{t+1}$.

