Notes

Fabio Brau

January 31, 2023

Contents

Chapter 1

Markov Chains

In the section we will introduce the model of a classical markov chain with discrete time for which each state belongs to some finite or contouble set of possible state. In the next section we extend the definition to states with a continuous states or in general a continuous density function.

1.1 Background

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. The class-function $\mathbb{P}$ is a finite measure over the sigma-algebra $\mathcal{F}$ such that $\mathbb{P}(\Omega) = 1$. A discrete random variable is represented by a measurable function $X : \Omega \to \mathbb{N}$. The variable is associated to a discrete distribution λ.

Definition 1 (Discrete Distribution)**.** A sequence $\lambda = (\lambda_0, \ldots, \lambda_n, \ldots)$ is a *discrete distribution* if and only if $\sum_{i \in \mathbb{N}} \lambda_i = 1$.

We will say that a random variable X has distribution λ if and only if for each possible outcome $i \in \mathbb{N}$,

$$\mathbb{P}(X = i) := \mathbb{P}(\{w \in \Omega \,:\, X(w) = i\}) = \lambda_i. \tag{1.1}$$

1.2 Countable States Markov Chains

Let $(X_t)_{t \in \mathbb{N}}$ a sequence of random variables. Let us assume that each instant $t \in \mathbb{N}$ the variable X_t takes values in a countable state space S.

Definition 2 (Markov Property)**.** The sequence $(X_t)_{t \in \mathbb{N}}$ satisfies the Markov property if for each time t and for each states $s, s_0, \ldots, s_n \in S$

$$\mathbb{P}(X_{n+1} = s \,|\, X_0 = s_0, \ldots, X_n = s_n) = \mathbb{P}(X_{n+1} = s \,|\, X_n = s_n). \tag{1.2}$$

That is, the state assumed at a certain instant t only depends on the previous state and not on the all history.

Observe that, since we are assuming that S is finite, then we are assuming that there exists an enumeration $S = \{s_1, \dots, s_n, \dots\}$. Hence, for the sake of simplicity, and without loss of generality, we can assume that $S = \mathbb{N}$, from which $X_t \in \mathbb{N}$ for each t.

Based on the latter assumption over S, a *Markov Chain* with discrete time and countable state set is represented by a tuple (λ, P) as stated in the following definition.

Definition 3. A transaction matrix $P = (p_{ij})$ is matrix with eventually infinite entries such that

$$\forall i \in \mathbb{N}, \quad \sum_{j \in \mathbb{N}} p_{ij} = 1 \tag{1.3}$$

Definition 4 (Markov Chain)**.** Let l<++>