

Diffusion Models and an application to DALL-E2

Deep Learning and Neural Networks: Advanced Topics

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COMPUTER
ENGINEERING,
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INSTITUTE



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School of Advanced Studies – Pisa



Introduction

Diffusion Models

DALL-E2

Broader Impacts

Introduction



Generative Models

1D example:
we illustrate the
effet of G over
the entire
distribution

*Generative model
to be learned*

Simple 1D gaussian
distribution we know
how to sample from

Targeted complex 1D
distribution we don't know
how to sample from

$$G(\text{---}) = \text{---}$$

High dimension
example:
we illustrate the
effet of G over a
single sample

$$G(\text{---}) = \text{---}$$



*Generative model
to be learned*

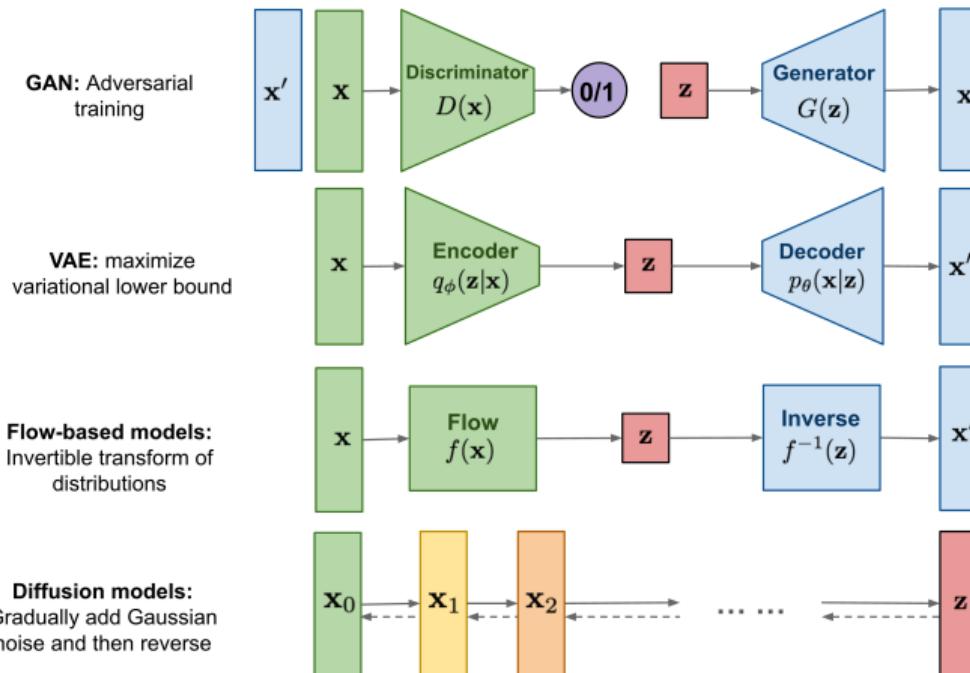
High dimension data
point from simple
noise distribution

High dimension data
point from complex
image distribution

Schema of generative models¹

¹Credits <https://towardsdatascience.com/understanding-diffusion-probabilistic-models-dpms-1940329d6048>

Generative Models



Overview of different generative models²

²Credits <https://lilianweng.github.io/posts/2021-07-11-diffusion-models/>

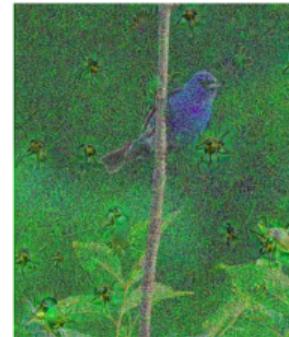
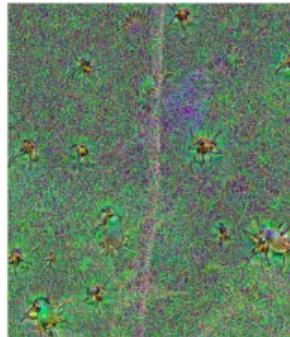
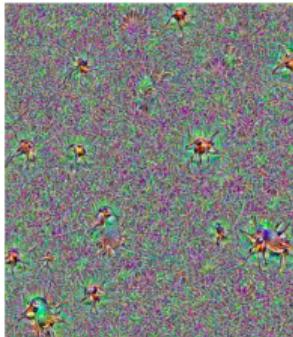
Diffusion Models



Overview

Diffusion models are generative models that aim at denoising data

Inverse Diffusion Process

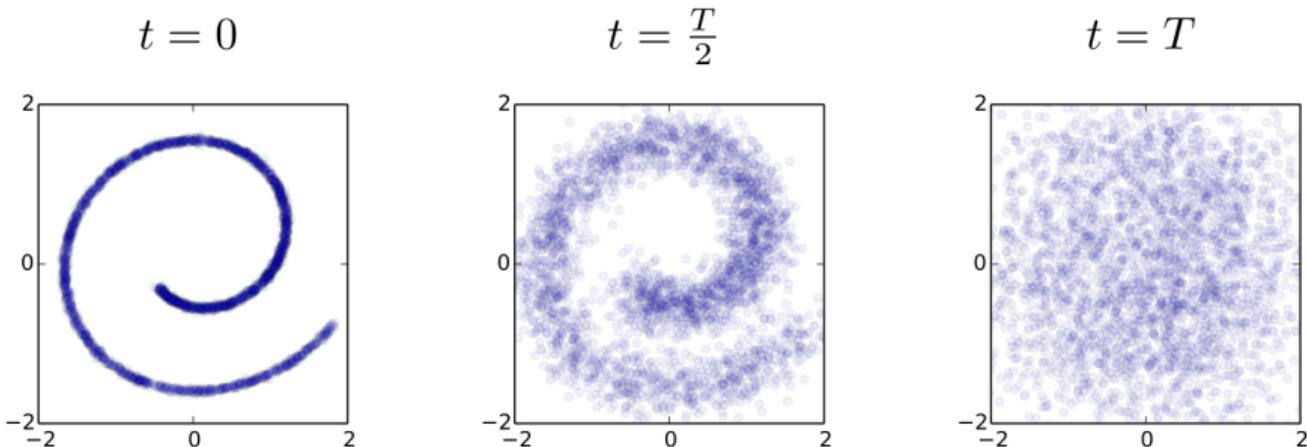


Diffusion Process

Timeline

- 2015) ...*Non-equilibrium Thermodynamics*. Sohl-Dickstein et al. ICML
- 2020) *Denoising Diffusion Probabilistic Models*. Ho et al. NeurIPS.
- 2021) *Score-Based Generative Modeling Through SDE*. Song et al. ICLR.

Deep Unsupervised Learning using Non-Equilibrium Thermodynamics



Diffusion process as a Markov Chain with Continuous State Space and Discrete Time.³

³Sohl-Dickstein et al., "Deep Unsupervised Learning using Nonequilibrium Thermodynamics".

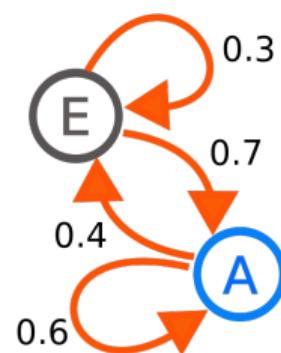
Reminder: Markov Chains with Discrete Time

Informal Definition

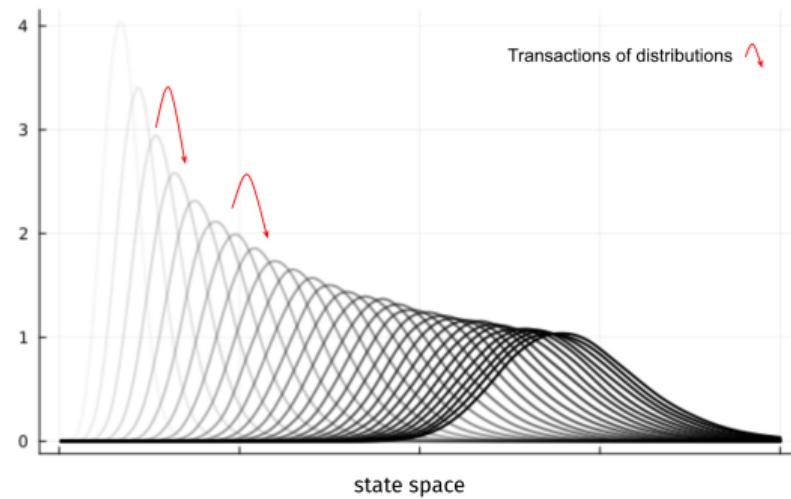
A sequence of random variables $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(t)}, \dots$, such that:

- $\mathbf{x}^{(t)} \in S$, where S State Space
- The future $\mathbf{x}^{(t+1)}$ depends on the present $\mathbf{x}^{(t)}$ but not on the past $\mathbf{x}^{(t-1)}$

Discrete State Space S



Continuous State Space S



Reminder: MCDT with Discrete State Space

Definition

A sequence $\{\mathbf{x}^{(t)}\}_{t \in \mathbb{N}} \subseteq S$, a matrix $P = (p_{ij})$.

- Discrete state space: $S = \{s_0, \dots, s_n, \dots\}$
- Markov Property: $\mathbf{x}^{(t+1)}$ not dep. $\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(t-1)}$.
- Transaction Matrix: $\mathbb{P}(\mathbf{x}^{(t+1)} = s_j | \mathbf{x}^{(t)} = s_i) = p_{ij}$

Reminder: MCDT with Discrete State Space

Definition

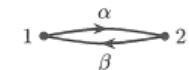
A sequence $\{\mathbf{x}^{(t)}\}_{t \in \mathbb{N}} \subseteq S$, a matrix $P = (p_{ij})$.

P is a stochastic matrix!

$$\forall i, \quad \sum_{j \in \mathbb{N}} p_{ij} = 1$$

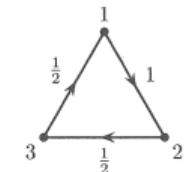
- Discrete state space: $S = \{s_0, \dots, s_n, \dots\}$

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$



- Markov Property: $\mathbf{x}^{(t+1)}$ not dep. $\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(t-1)}$.

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$



- Transaction Matrix: $\mathbb{P}(\mathbf{x}^{(t+1)} = s_j | \mathbf{x}^{(t)} = s_i) = p_{ij}$

Reminder: DTMC with Continuous State Space

Let assume $\mathbf{x}, \mathbf{y} \in S$ where S continuous state space (e.g. $S = \mathbb{R}^d$).

Joint Distribution $p(\mathbf{x}, \mathbf{y})$

$$\mathbb{P}(\mathbf{x} \in A \mid \mathbf{y} \in B) = \int_A \int_B p(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}$$

Transactional Kernel $p(\mathbf{x} \mid \mathbf{y})$

$$p(\mathbf{x}, \mathbf{y}) = p(\mathbf{x} \mid \mathbf{y}) p(\mathbf{y})$$

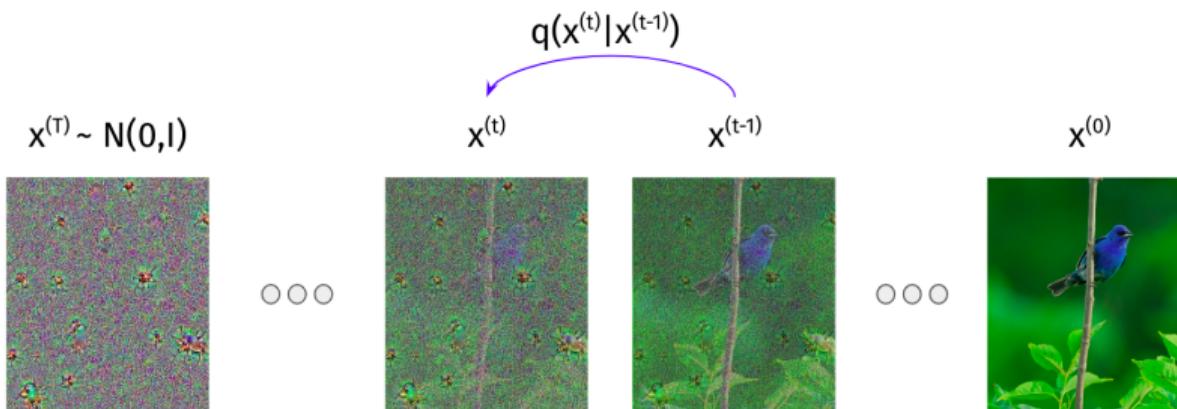
Marginal Distribution $p(\mathbf{x})$

$$p(\mathbf{x}) = \int_S p(\mathbf{x}, \mathbf{y}) d\mathbf{y} = \int_S p(\mathbf{x} \mid \mathbf{y}) p(\mathbf{y}) d\mathbf{y}$$

Forward Diffusion Process

“Adding noise to data...”

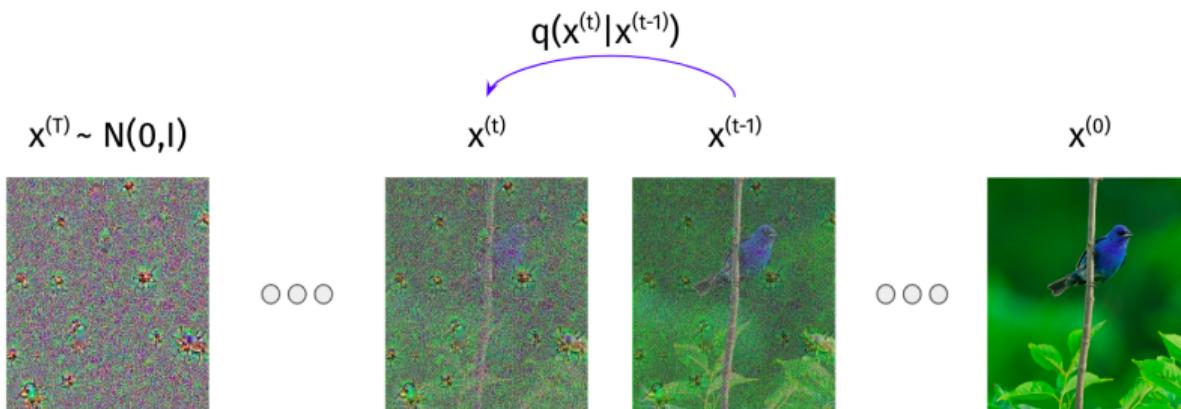
- Data Distribution: $\mathbf{x}^{(0)} \sim q$
- Transaction Kernel: $q(\mathbf{x}^{(t)} | \mathbf{x}^{(t-1)}) = \mathcal{N}(\mathbf{x}^{(t)}; \sqrt{1 - \beta_t} \mathbf{x}^{(t-1)}; \beta_t I)$
- Variance Scheduler: $\beta_1, \dots, \beta_T \in (0, 1]$



Forward Diffusion Process

“Adding noise to data...”

- Data Distribution: $\mathbf{x}^{(0)} \sim q$ Not Analytic!!
- Transaction Kernel: $q(\mathbf{x}^{(t)} | \mathbf{x}^{(t-1)}) = \mathcal{N}(\mathbf{x}^{(t)}; \sqrt{1 - \beta_t} \mathbf{x}^{(t-1)}; \beta_t I)$
- Variance Scheduler: $\beta_1, \dots, \beta_T \in (0, 1]$ \beta_T = 1



Forward Diffusion Process: Explicit Representation

$$\mathbf{x}^{(t)} = \sqrt{1 - \beta_t} \mathbf{x}^{(t-1)} + \sqrt{\beta_t} \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \sim \mathcal{N}(0, I)$$

Observation: Many small noisy steps \approx Large Noisy step

$$\mathbf{x}^{(t)} = \sqrt{1 - \alpha_t} \mathbf{x}^{(0)} + \sqrt{\alpha_t} \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim \mathcal{N}(0, I)$$

where

$$\alpha_t = 1 - \prod_{i=0}^t (1 - \beta_i)$$

Forward Diffusion Process: Distribution Representation

Markov property allows breaking up distributional Representation...

$$q(\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T)}) = q\left(\mathbf{x}^{(T)} \mid \mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T-1)}\right) q\left(\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T-1)}\right)$$

Forward Diffusion Process: Distribution Representation

Markov property allows breaking up distributional Representation...

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Forward Diffusion Process: Distribution Representation

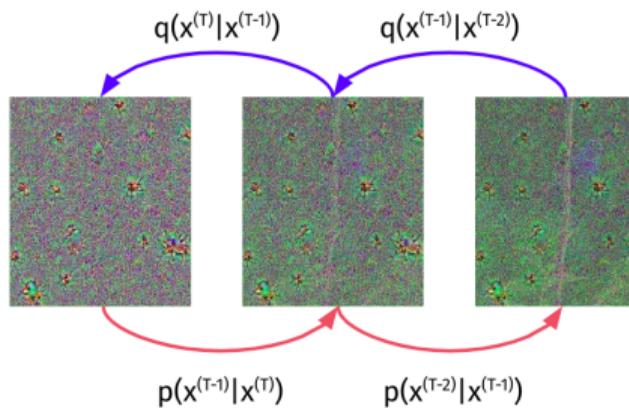
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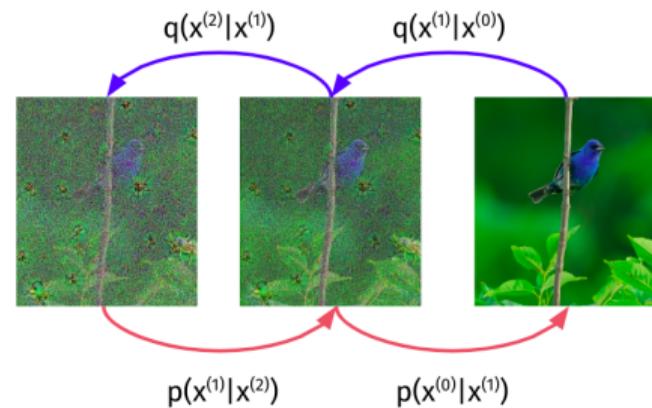
Distributional Representation

$$q(\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T)}) = q(\mathbf{x}^{(0)}) \prod_{t=1}^T q\left(\mathbf{x}^{(t)} \mid \mathbf{x}^{(t-1)}\right)$$

Reverse Diffusion Process



○ ○ ○



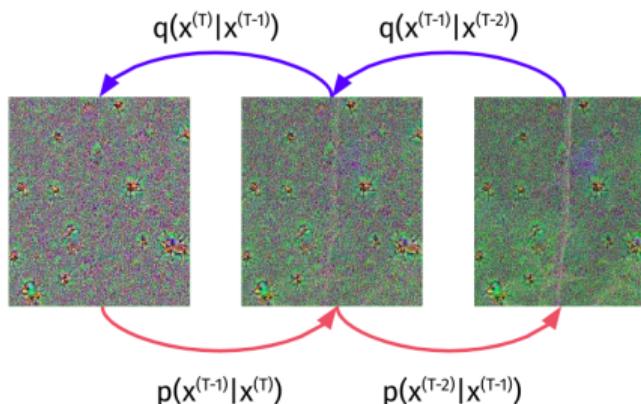
Learned Reverse Process

Reverse Diffusion Process

Fixed Forward Process

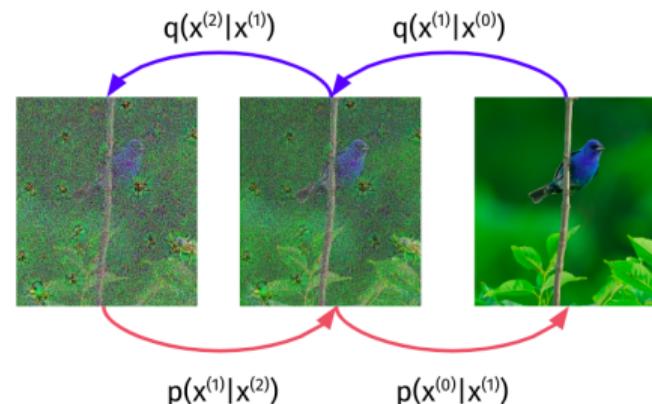
Initial Distribution

$$q(\mathbf{x}^{(0)})$$



Gaussian Transaction Kernel

$$q\left(\mathbf{x}^{(t)} | \mathbf{x}^{(t-1)}\right) = \mathcal{N}\left(\mathbf{x}^{(t)}; \sqrt{1 - \beta_t} \mathbf{x}^{(t-1)}; \beta_t I\right)$$



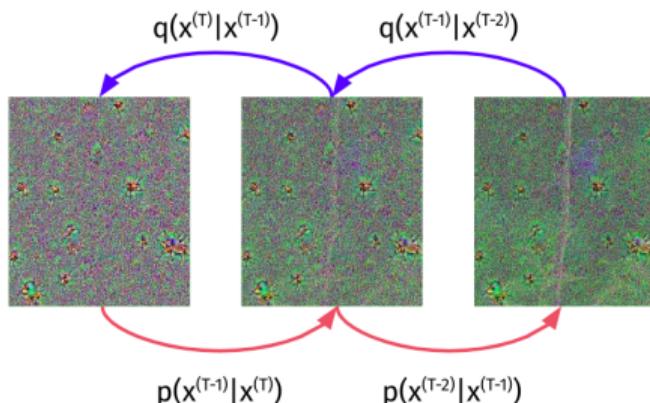
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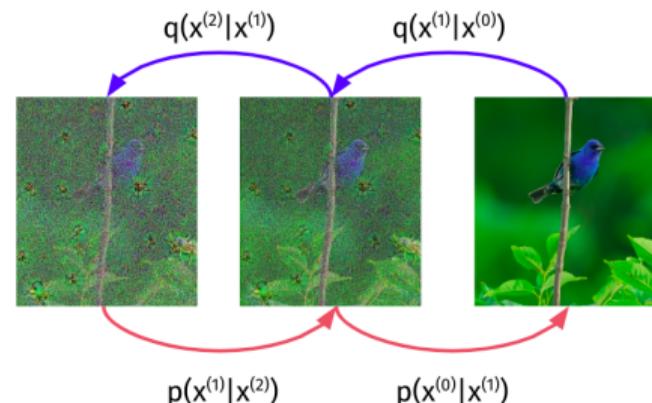
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Initial Distribution

$$p(\mathbf{x}^{(T)}) \sim \mathcal{N}(0, I)$$

Learned Reverse Process

Approximation of

$$q(\mathbf{x}^{(t-1)} \mid \mathbf{x}^{(t)})$$

Gaussian Kernel with parameters

$$p_{\theta}(\mathbf{x}^{(t-1)} \mid \mathbf{x}^{(t)}) = \mathcal{N}\left(\mathbf{x}^{(t-1)}; \boldsymbol{\mu}_{\theta}(\mathbf{x}^{(t)}, t), \boldsymbol{\Sigma}_{\theta}(\mathbf{x}^{(t)}, t)\right)$$

Reverse Diffusion Process

Forward Diffusion Process

$q(\mathbf{x}^{(0)})$ Data Distribution

$$q(\mathbf{x}^{(0\dots T)}) = q(\mathbf{x}^{(0)}) \prod_{t=1}^T q\left(\mathbf{x}^{(t)} \mid \mathbf{x}^{(t-1)}\right)$$

Reverse Diffusion Process

$q(x^{(T)}) = \mathcal{N}(0, I)$

$$q(\mathbf{x}^{(0\dots T)}) = q(\mathbf{x}^{(T)}) \prod_{t=1}^T q\left(\mathbf{x}^{(t-1)} \mid \mathbf{x}^{(t)}\right)$$

⁴Sohl-Dickstein et al., “Deep Unsupervised Learning using Nonequilibrium Thermodynamics”.

Reverse Diffusion Process

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$$q(\mathbf{x}^{(0\dots T)}) = q(\mathbf{x}^{(T)}) \prod_{t=1}^T q\left(\mathbf{x}^{(t-1)} \mid \mathbf{x}^{(t)}\right)$$

Theorem. Reverse of Gaussian DP is \approx Gaussian DP⁴

If $|\beta_i - \beta_{i+1}| \approx 0$, i.e. diffusion slow enough, then

$$q(\mathbf{x}^{(t-1)} \mid \mathbf{x}^{(t)}) \approx \mathcal{N}\left(\mathbf{x}^{(t-1)}; \boldsymbol{\mu}_\theta\left(\mathbf{x}^{(t)}, t\right), \boldsymbol{\Sigma}_\theta\left(\mathbf{x}^{(t)}, t\right)\right)$$

⁴Sohl-Dickstein et al., “Deep Unsupervised Learning using Nonequilibrium Thermodynamics”.

Reverse Diffusion Process

Forward Diffusion Process

$q(\mathbf{x}^{(0)})$ Data Distribution

$$q(\mathbf{x}^{(0\dots T)}) = q(\mathbf{x}^{(0)}) \prod_{t=1}^T q\left(\mathbf{x}^{(t)} | \mathbf{x}^{(t-1)}\right)$$

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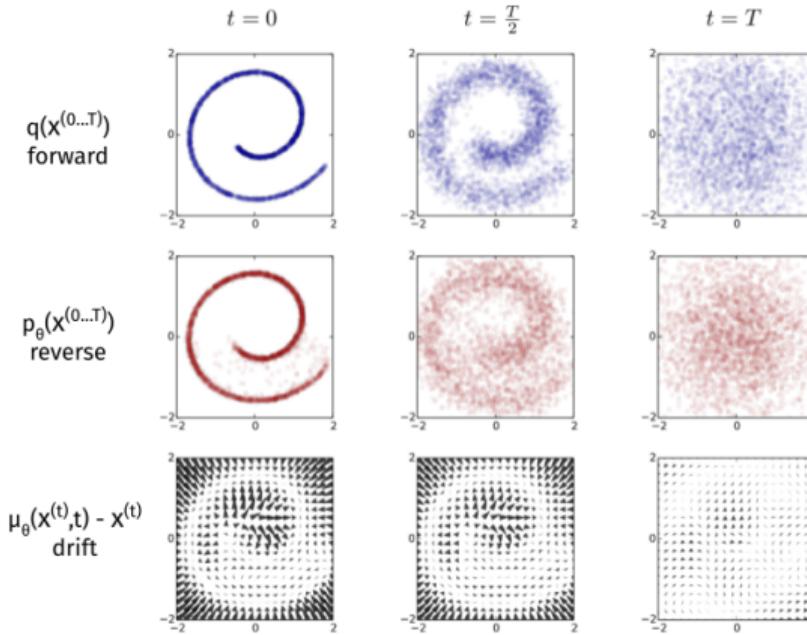
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$$q(\mathbf{x}^{(t-1)} | \mathbf{x}^{(t)}) \approx \mathcal{N}\left(\mathbf{x}^{(t-1)}; \mu_\theta(\mathbf{x}^{(t)}, t), \Sigma_\theta(\mathbf{x}^{(t)}, t)\right)$$

Mean μ_θ and covariance Σ_θ have to be learned!!

⁴Sohl-Dickstein et al., "Deep Unsupervised Learning using Nonequilibrium Thermodynamics".

Visualization of Diffusion Process: 2D dimensional case



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⁵Sohl-Dickstein et al., "Deep Unsupervised Learning using Nonequilibrium Thermodynamics".

Training of μ_θ and Σ_θ

Aim

Search for the best parameters θ

$$q(\mathbf{x}^{(0)}) \approx p_\theta(\mathbf{x}^{(0)})$$

where $\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T)}$ diffusion process

Estimated Reverse Process

$$p_\theta(\mathbf{x}^{(T)}) = \mathcal{N}(\mathbf{x}^{(T)}; 0, I)$$

$$p_\theta(\cdot | \mathbf{x}^{(t)}) = \mathcal{N}(\boldsymbol{\mu}_\theta(\mathbf{x}^{(t)}, t), \boldsymbol{\Sigma}_\theta(\mathbf{x}^{(t)}, t))$$

Training of μ_θ and Σ_θ

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Search for the best parameters θ

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where $\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T)}$ diffusion process

Method

Minimize the *Kullback–Leibler Divergence*

$$D_{KL}(q || p_\theta) := \int q(\mathbf{x}^{(0)}) \log \left(\frac{q(\mathbf{x}^{(0)})}{p_\theta(\mathbf{x}^{(0)})} \right) d\mathbf{x}^{(0)}$$

Estimated Reverse Process

$$p_\theta(\mathbf{x}^{(T)}) = \mathcal{N} \left(\mathbf{x}^{(T)}; 0, I \right)$$

$$p_\theta(\cdot | \mathbf{x}^{(t)}) = \mathcal{N} \left(\boldsymbol{\mu}_\theta \left(\mathbf{x}^{(t)}, t \right), \boldsymbol{\Sigma}_\theta \left(\mathbf{x}^{(t)}, t \right) \right)$$

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Easy??

Training of μ_θ and Σ_θ

Aim

Search for the best parameters θ

$$q(\mathbf{x}^{(0)}) \approx p_\theta(\mathbf{x}^{(0)})$$

where $\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(T)}$ diffusion process

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Method

Minimize the *Kullback–Leibler Divergence*

$$D_{KL}(q || p_\theta) := \int q(\mathbf{x}^{(0)}) \log \left(\frac{q(\mathbf{x}^{(0)})}{p_\theta(\mathbf{x}^{(0)})} \right) d\mathbf{x}^{(0)}$$

Easy??

No. $q(\mathbf{x}^{(0)})$ is analytically intractable!!



Training of μ_θ and Σ_θ

Aim: Deduce a tractable loss function

$$D_{KL}(q \parallel p_\theta) := \int q(\mathbf{x}^{(0)}) \log \left(\frac{q(\mathbf{x}^{(0)})}{p_\theta(\mathbf{x}^{(0)})} \right) d\mathbf{x}^{(0)}$$

Training of μ_θ and Σ_θ

Aim: Deduce a tractable loss function

$$D_{KL}(q \parallel p_\theta) := \int q(\mathbf{x}^{(0)}) \log \left(\frac{q(\mathbf{x}^{(0)})}{p_\theta(\mathbf{x}^{(0)})} \right) d\mathbf{x}^{(0)}$$

Simplification I: Minimize the Cross Entropy

$$D_{KL} \left(q(\mathbf{x}^{(0)}) \parallel p_\theta(\mathbf{x}^{(0)}) \right) = \int q(\mathbf{x}^{(0)}) \log(q(\mathbf{x}^{(0)})) d\mathbf{x}^{(0)} + \int -q(\mathbf{x}^{(0)}) \log(p_\theta(\mathbf{x}^{(0)})) d\mathbf{x}^{(0)}$$

Training of μ_θ and Σ_θ

Aim: Deduce a tractable loss function

$$D_{KL}(q \parallel p_\theta) := \int q(\mathbf{x}^{(0)}) \log \left(\frac{q(\mathbf{x}^{(0)})}{p_\theta(\mathbf{x}^{(0)})} \right) d\mathbf{x}^{(0)}$$

Simplification I: Minimize the Cross Entropy

$$D_{KL} \left(q(\mathbf{x}^{(0)}) \parallel p_\theta(\mathbf{x}^{(0)}) \right) = \underbrace{\int q(\mathbf{x}^{(0)}) \log(q(\mathbf{x}^{(0)})) d\mathbf{x}^{(0)}}_{-\mathbb{H}(q(\mathbf{x}^{(0)}))} + \underbrace{\int -q(\mathbf{x}^{(0)}) \log(p_\theta(\mathbf{x}^{(0)})) d\mathbf{x}^{(0)}}_{L_{CE}(p_\theta)}$$

Training of μ_θ and Σ_θ

Minimize the Cross Entropy Loss

$$L_{CE}(p_\theta(\mathbf{x}^{(0)})) := - \int q(\mathbf{x}^{(0)}) \log(p_\theta(\mathbf{x}^{(0)})) d\mathbf{x}^{(0)}$$

Training of μ_θ and Σ_θ

Minimize the Cross Entropy Loss

$$L_{CE}(p_\theta(\mathbf{x}^{(0)})) := - \int q(\mathbf{x}^{(0)}) \log(p_\theta(\mathbf{x}^{(0)})) d\mathbf{x}^{(0)}$$

Observation: Marginal Distribution

$$p_\theta(\mathbf{x}^{(0)}) = \int p_\theta(\mathbf{x}^{(0\dots T)}) d\mathbf{x}^{(1\dots T)}$$

Training of μ_θ and Σ_θ

Minimize the Cross Entropy Loss

$$L_{CE}(p_\theta(\mathbf{x}^{(0)})) := - \int q(\mathbf{x}^{(0)}) \log(p_\theta(\mathbf{x}^{(0)})) d\mathbf{x}^{(0)}$$

Observation: Marginal Distribution

$$p_\theta(\mathbf{x}^{(0)}) = \int p_\theta(\mathbf{x}^{(0\dots T)}) d\mathbf{x}^{(1\dots T)}$$

Simplification II: Jensen Inequality

$$L_{CE}(p_\theta) \leq -\mathbb{E}_{q(\mathbf{x}^{(0\dots T)})} \left[\log \frac{q(\mathbf{x}^{(1\dots T)} | \mathbf{x}^{(0)})}{p_\theta(\mathbf{x}^{(0\dots T)})} \right]$$

Training of μ_θ and Σ_θ

...after some algebraic steps :)

Reformulated Loss Function

$$\mathcal{L} = \mathcal{L}_T + \sum_{t=1}^{T-1} \mathcal{L}_t + \mathcal{L}_0$$

where,

$$\mathcal{L}_T = \mathbb{E}_{q(\mathbf{x}^{(0\dots T)})} \left[D_{KL} \left(q(\mathbf{x}^{(T)} | \mathbf{x}^{(0)}) \parallel p_\theta(\mathbf{x}^{(T)}) \right) \right]$$

$$\mathcal{L}_t = \mathbb{E}_{q(\mathbf{x}^{(0\dots T)})} \left[D_{KL} \left(q(\mathbf{x}^{(t)} | \mathbf{x}^{(t+1)}, \mathbf{x}^{(0)}) \parallel p_\theta(\mathbf{x}^{(t)} | \mathbf{x}^{(t+1)}) \right) \right]$$

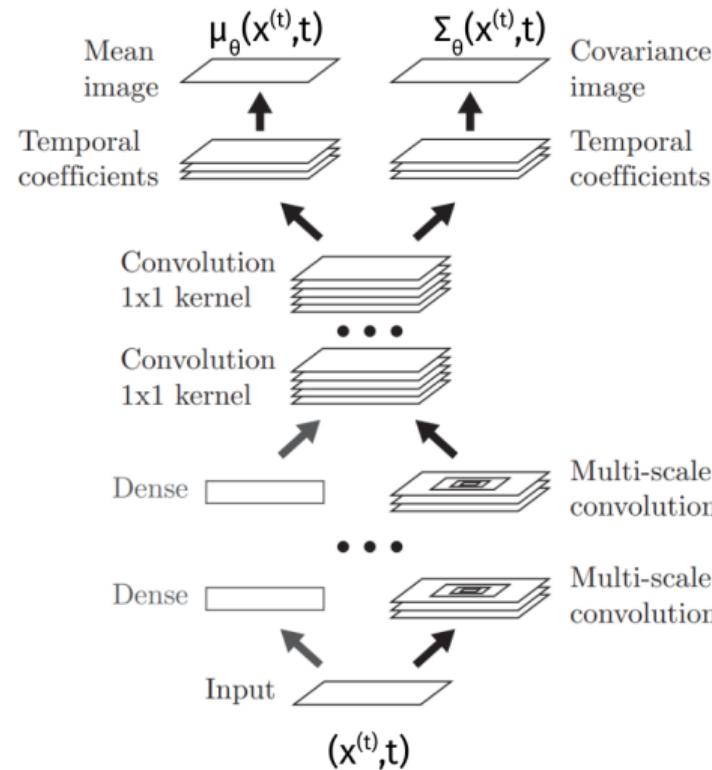
$$\mathcal{L}_0 = \mathbb{E}_{q(\mathbf{x}^{(0\dots T)})} \left[p_\theta(\mathbf{x}^{(0)} | \mathbf{x}^{(1)}) \right]$$

Note.

\mathcal{L}_T is constant.

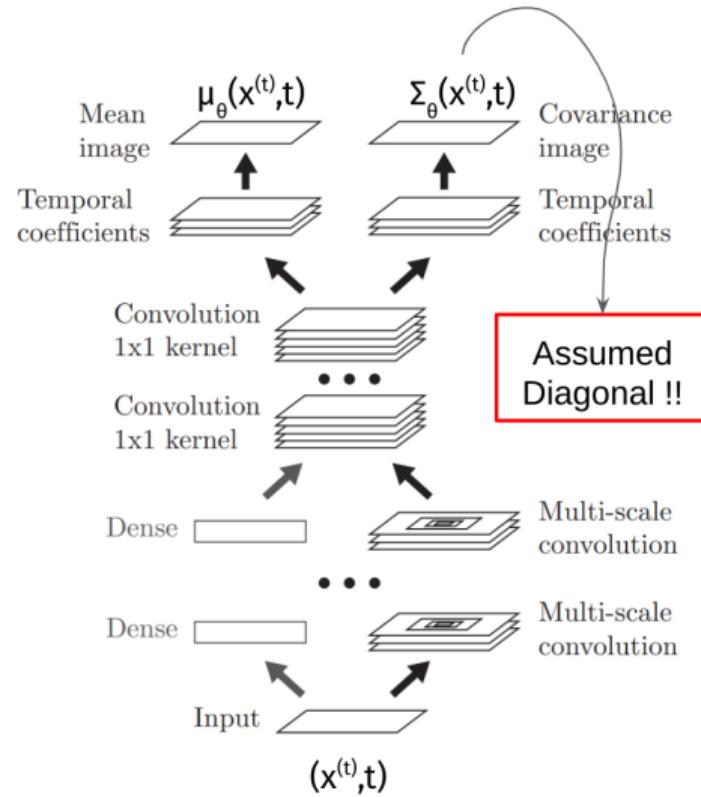
$\mathcal{L}_0, \mathcal{L}_t$ explicit.

Neural Network that estimate μ_θ and Σ_θ



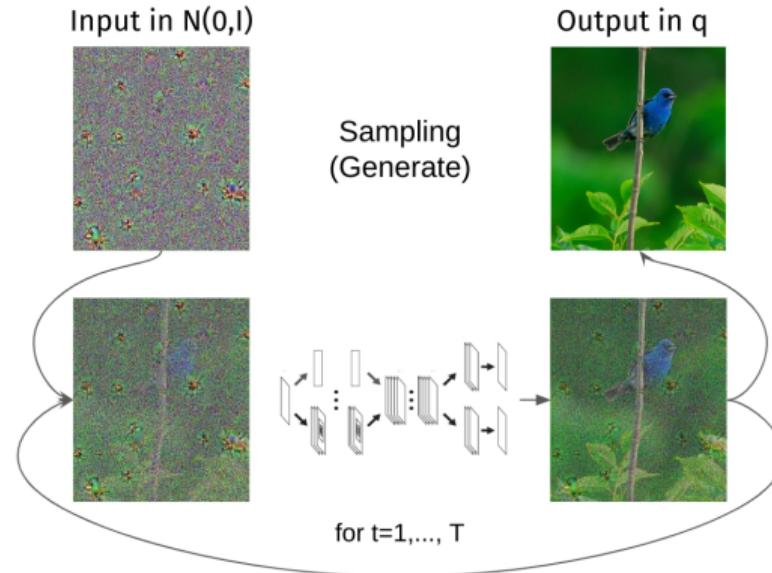
Prposed Neural Network for CIFAR10 image generation. T=1000

Neural Network that estimate μ_θ and Σ_θ

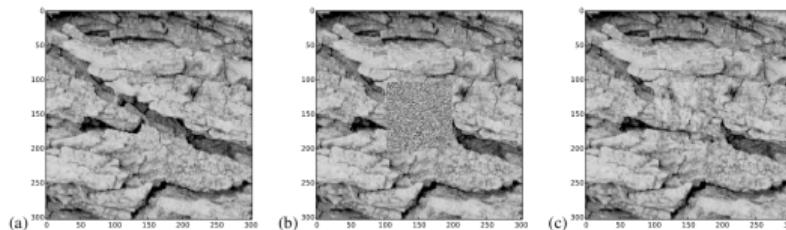


Proposed Neural Network for CIFAR10 image generation. T=1000

Sampling or Generative Stage



Experiments



MNIST

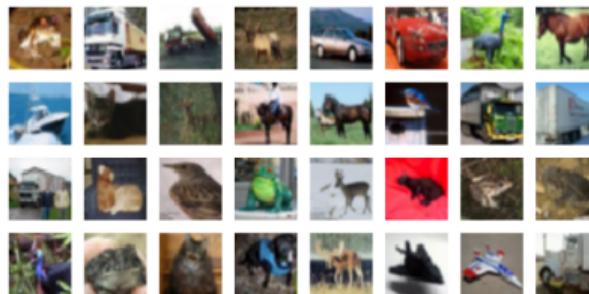


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⁶Sohl-Dickstein et al., “Deep Unsupervised Learning using Nonequilibrium Thermodynamics”.

Experiments

CIFAR10 (original)



CIFAR10 (generated)



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⁷Sohl-Dickstein et al., “Deep Unsupervised Learning using Nonequilibrium Thermodynamics”.

Timeline

2015) ...*Non-equilibrium Thermodynamics*. Sohl-Dickstein et al. ICML

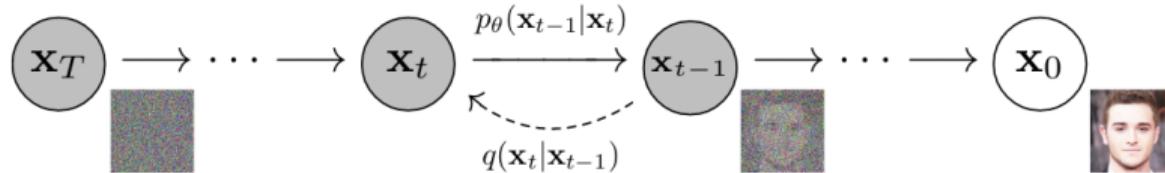


2020) *Denoising Diffusion Probabilistic Models*. Ho et al. NeurIPS.

2021) *Score-Based Generative Modeling Through SDE*. Song et al. ICLR.

Denoising Diffusion Probabilistic Model

Small technical improvements highly impact the performance...⁸



Simplification I: Diagonal Uniform Covariance Matrix

$$p_\theta(\mathbf{x}^{(t-1)} | \mathbf{x}^{(t)}) = \mathcal{N}\left(\mathbf{x}^{(t-1)}; \boldsymbol{\mu}_\theta\left(\mathbf{x}^{(t)}, t\right), \boldsymbol{\Sigma}_\theta\left(\mathbf{x}^{(t)}, t\right)\right)$$

where

$$\boldsymbol{\Sigma}_\theta\left(\mathbf{x}^{(t)}, t\right) = \sigma_t^2 I$$

⁸Ho, Jain, and Abbeel, "Denoising Diffusion Probabilistic Models".

Denoising Diffusion Probabilistic Model

Previous work aim at estimating $\mu_\theta(\mathbf{x}^{(t)}, t)$ and $\Sigma_\theta(\mathbf{x}^{(t)}, t)$.

Simplification II: Estimating the commited error

$$\mu_\theta(\mathbf{x}^{(t)}) = \frac{1}{\sqrt{1 - \beta_t}} \left(\mathbf{x}^{(t)} - \frac{1 - \beta_t}{\sqrt{\alpha_t}} \varepsilon_\theta(\mathbf{x}^{(t)}, t) \right)$$

Denoising Diffusion Probabilistic Model

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Simplification III: Training on random instants t

$$\mathcal{L}_{simple} = \mathbb{E}_{t, \mathbf{x}^{(0)}, \varepsilon} \left[\left\| \varepsilon - \varepsilon_\theta \left(\sqrt{1 - \alpha_t} \mathbf{x}^{(0)} + \sqrt{\alpha_t} \varepsilon, t \right) \right\| \right]$$

where

$$t \sim \mathcal{U}\{1, \dots, T\}, \quad \mathbf{x}^{(0)} \sim q, \quad \varepsilon \sim \mathcal{N}(0, I)$$

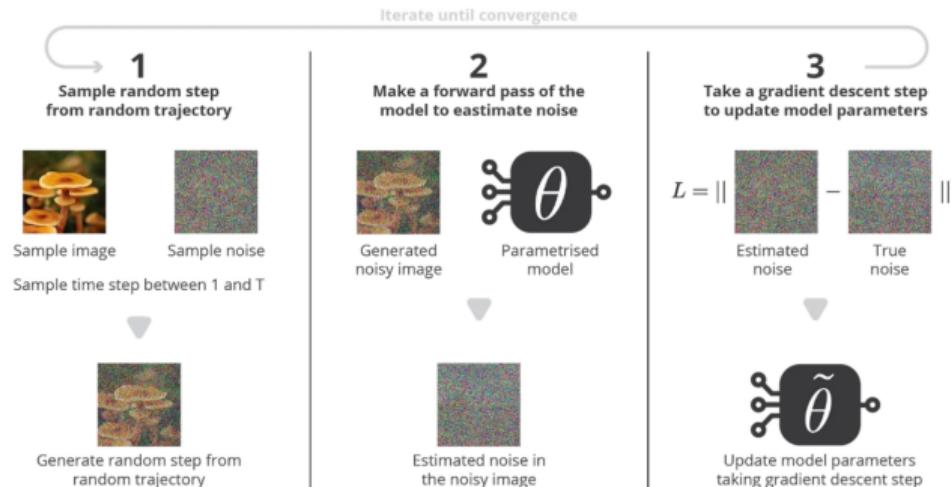
Training and Sampling Procedure

Algorithm 1 Training

```
1: repeat
2:    $\mathbf{x}_0 \sim q(\mathbf{x}_0)$ 
3:    $t \sim \text{Uniform}(\{1, \dots, T\})$ 
4:    $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ 
5:   Take gradient descent step on
        $\nabla_{\theta} \|\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_{\theta}(\sqrt{\bar{\alpha}_t} \mathbf{x}_0 + \sqrt{1 - \bar{\alpha}_t} \boldsymbol{\epsilon}, t)\|^2$ 
6: until converged
```

Algorithm 2 Sampling

```
1:  $\mathbf{x}_T \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ 
2: for  $t = T, \dots, 1$  do
3:    $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$  if  $t > 1$ , else  $\mathbf{z} = \mathbf{0}$ 
4:    $\mathbf{x}_{t-1} = \frac{1}{\sqrt{\alpha_t}} \left( \mathbf{x}_t - \frac{1 - \alpha_t}{\sqrt{1 - \alpha_t}} \boldsymbol{\epsilon}_{\theta}(\mathbf{x}_t, t) \right) + \sigma_t \mathbf{z}$ 
5: end for
6: return  $\mathbf{x}_0$ 
```



Experiments: Sample Quality

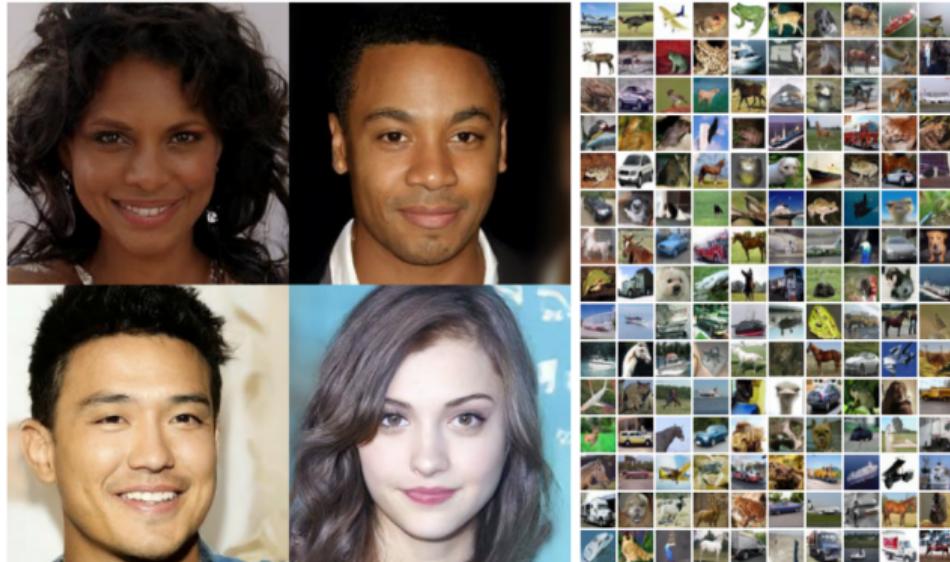


Figure 1: Generated samples on CelebA-HQ 256×256 (left) and unconditional CIFAR10 (right)

Objective	IS	FID
$\tilde{\mu}$ prediction (baseline)		
L , learned diagonal Σ	7.28 ± 0.10	23.69
L , fixed isotropic Σ	8.06 ± 0.09	13.22
$\ \tilde{\mu} - \tilde{\mu}_\theta\ ^2$	-	-
ϵ prediction (ours)		
L , learned diagonal Σ	-	-
L , fixed isotropic Σ	7.67 ± 0.13	13.51
$\ \tilde{\epsilon} - \epsilon_\theta\ ^2 (L_{\text{simple}})$	9.46 ± 0.11	3.17

Metrics for CIFAR10

Note.

1. High FID (Frechet Implicit Distance) \Rightarrow high quality
2. Training improved

Experiments: Diffusion vs GAN/VAE

“Diffusion models get comparable result to Generative Adversarial Networks and Variational Autoencoders”

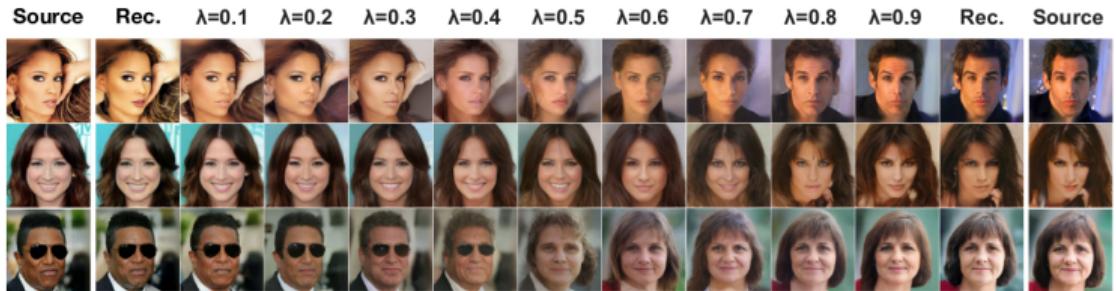
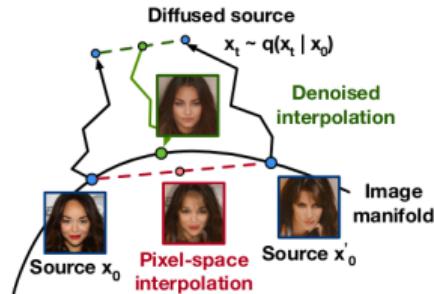
Table 1: CIFAR10 results. NLL measured in bits/dim.

Model	IS	FID	NLL Test (Train)
Conditional			
EBM [4]	8.30	37.9	
JEM [4]	8.76	38.4	
BigGAN [3]	9.22	14.73	
StyleGAN2 + ADA (v1) [20]	10.06	2.67	
Unconditional			
Diffusion (original) [53]			≤ 5.40
Gated PixelCNN [50]	4.60	65.93	3.03 (2.90)
Sparse Transformer [7]			2.80
PixelIQN [44]	5.29	49.46	
EBM [4]	6.78	38.2	
NCSNv2 [54]			31.75
NCSN [53]	8.87 ± 0.12	25.32	
SNGAN [29]	8.22 ± 0.05	21.7	
SNGAN-DDLS [4]	9.09 ± 0.10	15.42	
StyleGAN2 + ADA (v1) [20]	9.74 ± 0.05	3.26	
Ours (L , fixed isotropic Σ)	7.67 ± 0.13	13.51	≤ 3.70 (3.69)
Ours (L_{simple})	9.46 ± 0.11	3.17	≤ 3.75 (3.72)

CIFAR10 results. NLL measured in bits/dim¹⁰

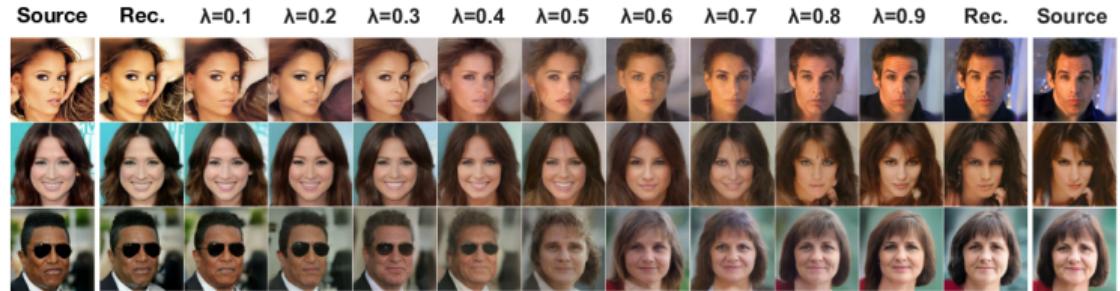
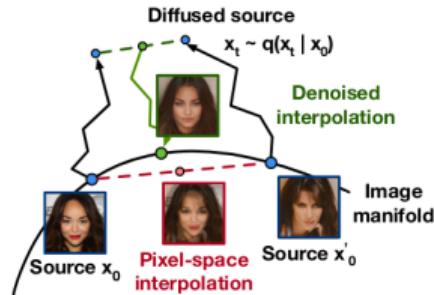
¹⁰Ho, Jain, and Abbeel, “Denoising Diffusion Probabilistic Models”.

Experiments: Images Interpolation



$$\mathbf{x}_\lambda^{(T)} := \lambda \mathbf{x}_{\text{source}_r}^{(T)} + (1 - \lambda) \mathbf{x}_{\text{source}_l}^{(T)}, \quad \lambda \in [0, 1]$$

Experiments: Images Interpolation



$$\mathbf{x}_\lambda^{(T)} := \lambda \mathbf{x}_{\text{source}_r}^{(T)} + (1 - \lambda) \mathbf{x}_{\text{source}_l}^{(T)}, \quad \lambda \in [0, 1]$$

$$\mathbf{x}_\lambda^{(0)} \sim p_\theta(\mathbf{x}^{(T)}), \quad \text{by diffusion}$$

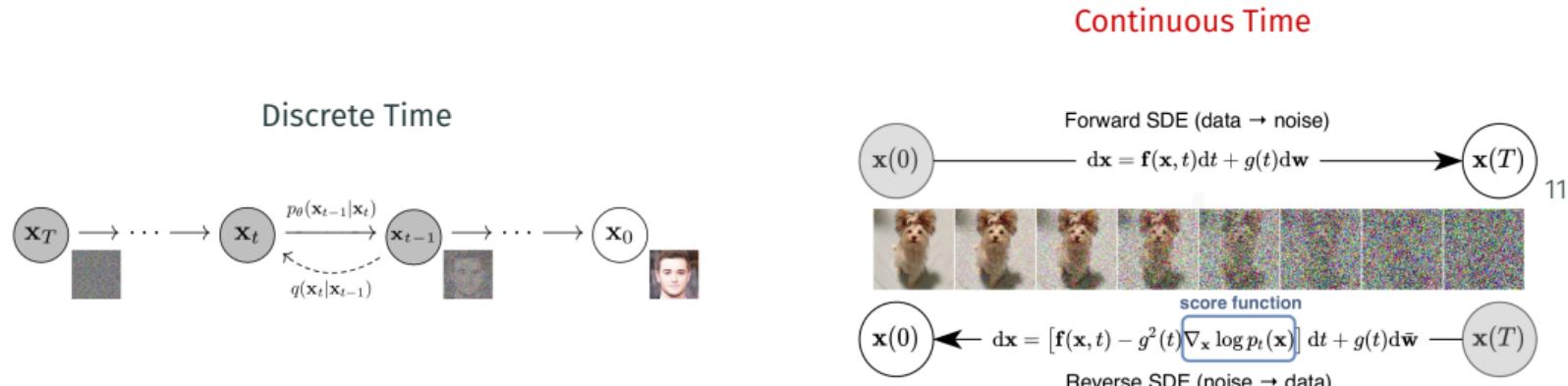
Timeline

2015) ...*Non-equilibrium Thermodynamics*. Sohl-Dickstein et al. ICML. ✓

2020) *Denoising Diffusion Probabilistic Models*. Ho et al. NeurIPS. ✓

2021) *Score-Based Generative Modeling Through SDE*. Song et al. ICLR.

Score-Based Generative Modeling Through SDE (no details)



Stochastic Process

Continuous sequence of \mathbf{x} indexed by $t \in [0, T]$, where $\mathbf{x}(t)$ has distribution $p_t(\mathbf{x})$;

¹¹Song et al., “Score-based generative modeling through stochastic differential equations”.

Continuous Diffusion Process described by SDE

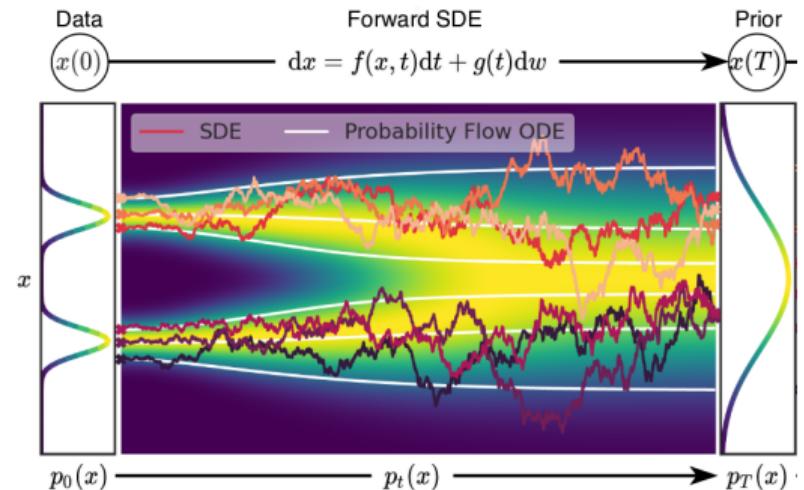
Continuous Diffusion Process

$$dx(t) = f(x, t)dt + g(t) d\mathbf{w}(t)$$

1. $f(\mathbf{x}, t)$ drift coefficient.
2. $g(t)$ diffusion coefficient.
3. $\mathbf{w}(t)$, Weiner Process

Weiner Process

$$\mathbf{w}(t) - \mathbf{w}(s) \sim \mathcal{N}(0, (t-s)I)$$



Continuous Diffusion Process described by SDE

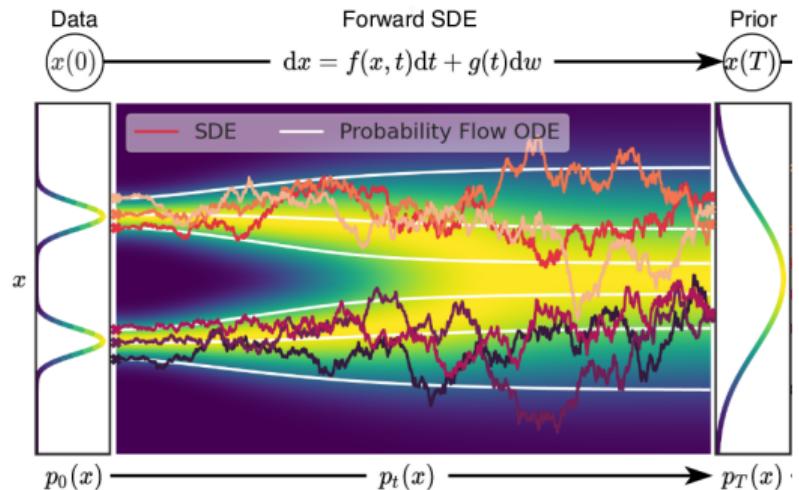
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Reverse Process explicity given by $\nabla_x p_t(\mathbf{x})$

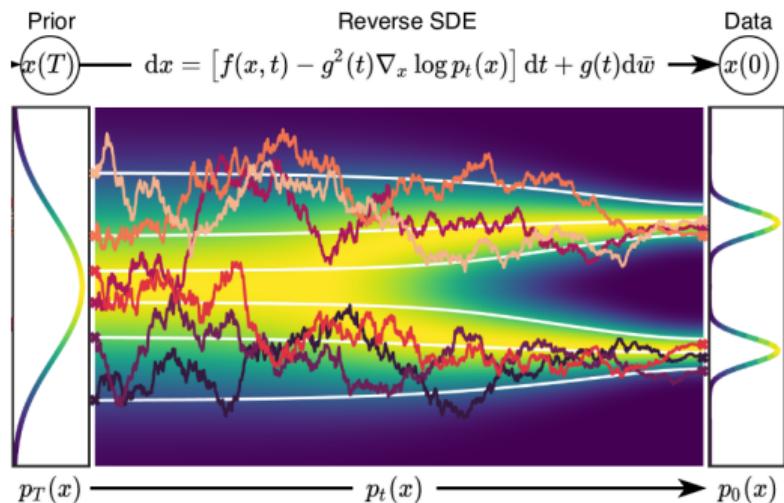
Theorem

Reverse Process is still a diffusion process

$$dx(t) = \bar{f}(\mathbf{x}, t) dt + g(t) d\bar{w}(t)$$

where

- $\bar{f}(\mathbf{x}, t) = f(\mathbf{x}, t) - g(t)^2 \nabla_{\mathbf{x}} \log(p_t(\mathbf{x}))$
- $\bar{w}(t) = w(T - t)$, reverse Weiner Process

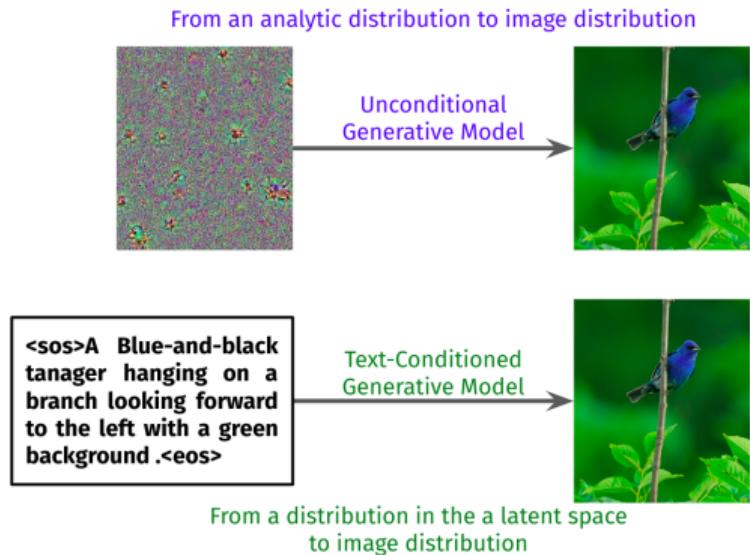


DALL-E2



Introduction

A Ramesh et al. (Open AI). “*Hierarchical Text-Conditional Image Generation with CLIP Latents*.” February 2022.



Introduction

DALL-E2: a new AI system that can create realistic images and art from a description in natural language

An astronaut **Teddy bears** A bowl of soup

mixing sparkling chemicals as mad scientists shopping for groceries working on new AI research

as a 1990s Saturday morning cartoon as digital art in a steampunk style



12

¹²Credits: DALL-E2 website

Introduction

DALL-E2: a new AI system that can create realistic images and art from a description in natural language

An astronaut Teddy bears A bowl of soup

riding a horse lounging in a tropical resort in space playing basketball with cats in space

in a photorealistic style in the style of Andy Warhol as a pencil drawing



Broader Impacts



Image Generation

1. Diffusion models, as generative models, can be used for malicious porpose. Fake images can become less detectable.
2. Diffusion models reflect the biases in the dataset in which they are trianed. Hence, using generated images for training other models can produce a **fade in** effect.

¹³Ho, Jain, and Abbeel, “Denoising Diffusion Probabilistic Models”.

Image Generation

1. Diffusion models, as generative models, can be used for malicious porpose. Fake images can become less detectable.
2. Diffusion models reflect the biases in the dataset in which they are trianed. Hence, using generated images for training other models can produce a **fade in effect**.

“If samples from generative models trained on these datasets proliferate throughout the internet, then these biases will only be reinforced further.¹³”

¹³Ho, Jain, and Abbeel, “Denoising Diffusion Probabilistic Models”.

CLIP Model

“We also found discrepancies across gender and race for people categorized into the ‘crime’ and ‘non-human’ categories...”¹⁴

¹⁴Radford et al., “Learning Transferable Visual Models From Natural Language Supervision”.

Thanks for the attention

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Proof Details



Proof of Explicit Representation of Forward Diffusion Process

Let us proceed by induction by assuming $\mathbf{x}^{(t)} = \sqrt{1 - \alpha_t} \mathbf{x}^{(0)} + \sqrt{\alpha_t} \boldsymbol{\varepsilon}$ where $\boldsymbol{\varepsilon} \sim \mathcal{N}(0, I)$ and where $\alpha_t = 1 - \prod_{i=0}^t (1 - \beta_i)$.

$$\begin{aligned}\mathbf{x}^{(t+1)} &= \sqrt{1 - \beta_{t+1}} \mathbf{x}^{(t)} + \sqrt{\beta_{t+1}} \boldsymbol{\varepsilon}_{t+1} \\ &= \sqrt{1 - \beta_{t+1}} \left(\sqrt{1 - \alpha_t} \mathbf{x}^{(0)} + \sqrt{\alpha_t} \boldsymbol{\varepsilon} \right) + \sqrt{\beta_{t+1}} \boldsymbol{\varepsilon}_{t+1} \\ &= \sqrt{\left(\prod_{i=0}^{t+1} (1 - \beta_i) \right)} \mathbf{x}^{(0)} + \sqrt{(1 - \beta_{t+1})\alpha_t + \beta_{t+1}} \tilde{\boldsymbol{\varepsilon}}\end{aligned}\tag{2}$$

where the last term of the summation is obtained by observing that, since $\sqrt{(1 - \beta_{t+1})\alpha_t} \boldsymbol{\varepsilon}$ and $\sqrt{\beta_{t+1}} \boldsymbol{\varepsilon}_{t+1}$ are independent, then the variance of their sum (that still has a gaussian distribution) is given by $(1 - \beta_{t+1})\alpha_t + \beta_{t+1}$.

