Notes

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Contents

Chapter 1

Markov Chains

In the section we will introduce the model of a classical Markov chain with discrete time for which each state belongs to some finite or countable set of possible state. In the next section we extend the definition to states with a continuous states or in general a continuous density function.

1.1 Background

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. The class-function $\mathbb{P}$ is a finite measure over the sigma-algebra $\mathcal{F}$ such that $\mathbb{P}(\Omega) = 1$. A discrete random variable is represented by a measurable function $X : \Omega \to S$, where $S = \{s_1, \ldots, s_n, \ldots\}$ equipped with an uniform. The variable is associated to a discrete distribution λ.

Definition 1 (Discrete Distribution)**.** A sequence $\lambda = (\lambda_0, \ldots, \lambda_n, \ldots)$ is a *discrete distribution* if and only if $\sum_{i \in \mathbb{N}} \lambda_i = 1$.

We will say that a random variable X has distribution λ if and only if for each possible outcome $i \in \mathbb{N}$,

$$\mathbb{P}(X = i) := \mathbb{P}(\{w \in \Omega \,:\, X(w) = i\}) = \lambda_i. \tag{1.1}$$

1.2 Countable States Markov Chains

Let $(X_t)_{t \in \mathbb{N}}$ a sequence of random variables. Let us assume that each instant $t \in \mathbb{N}$ the variable X_t takes values in a countable state space S.

Definition 2 (Markov Property)**.** The sequence $(X_t)_{t \in \mathbb{N}}$ satisfies the Markov property if for each time t and for each states $s, s_0, \ldots, s_n \in S$

$$\mathbb{P}(X_{n+1} = s \,|\, X_0 = s_0, \ldots, X_n = s_n) = \mathbb{P}(X_{n+1} = s \,|\, X_n = s_n). \tag{1.2}$$

That is, the state assumed at a certain instant t only depends on the previous state and not on the whole history.

Observe that, since we are assuming that S is finite, then we are assuming that there exists an enumeration $S = \{s_1, \dots, s_n, \dots\}$. Hence, for the sake of simplicity, and without loss of generality, we can assume that $S = \mathbb{N}$, from which $X_t \in \mathbb{N}$ for each t.

Based on the latter assumption over S, a *Markov Chain* with discrete time and countable state set is represented by a tuple (λ, P) as stated in the following definition.

Definition 3 (Transaction Matrix)**.** A transaction matrix $P = (p_{ij})$ is matrix with infinite entries such that

$$\forall i \in \mathbb{N}, \quad \sum_{j \in \mathbb{N}} p_{ij} = 1. \tag{1.3}$$

In other words, a matrix P is a transaction matrix if every row $P_{i:}$ is a discrete distribution.

Definition 4 (Markov Chain)**.** Let λ be a discrete distribution, and let P be a transaction matrix. A sequence of random variable $(x_i)_{i \in \mathbb{N}}$ is a Markov chain with initial distribution λ and transaction matrix P

- $\lambda_i = \mathbb{P}\left(X_0 = i\right)$;
- $p_{ij} = \mathbb{P}\left(X_n = j \mid X_{n-1} = i\right)$ for each state n.
- X_{n+1} is independent from $X_0, \dots, X_{n-1}$.

1.3 Markov Chain with Continuous State Space

Let assume $(X_t)_{t \in \mathbb{N}}$ a sequence of continuous random variables, and that the initial variable X_t as a continuous distribution $p(x)$.