An introduction to PCA

Weekly AI pills

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SSSA, Emerging Digital Technologies, Pisa.

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DI TECNOLOGIE DELLA
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DELL'INFORMAZIONE
E DELLA
PERCEZIONE

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Sant'Anna

Retis
Real-Time Systems Laboratory

Summary

- The aim of Principal Component Analysis
- Derivation
    1. A Geometrical idea
    2. A statistical Derivation
    3. Singolar Value Decomposition
- PCA from Encoder Decoder NN
- Dummy examples

Geometrical Introduction

Let $X \in \mathbb{R}^{N \times n}$ be a dataset of N **observation** within n **variables**.

$$X = \begin{bmatrix} x_1^T \\ \vdots \\ x_N^T \end{bmatrix} = \begin{bmatrix} x^{(1)} & \Big| & \cdots & \Big| & x^{(n)} \end{bmatrix} \tag{1}$$

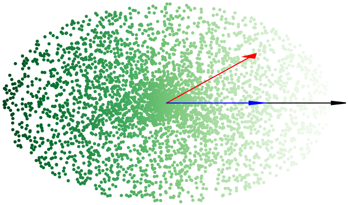
Notations:

- $x_i \in \mathbb{R}^n$ represents a single **observation**, i.e a **sample** in the feature space.
- $x^{(i)} \in \mathbb{R}^N$ represents the single **variable**, i.e a **column** of the dataset.
- The object $\mathbb{1}_n \in \mathbb{R}^n$ is the unitary columnar vector of length n $\mathbb{1}_n = [1, \cdots, 1]^T$.
- X is centered if $X^T \mathbb{1} = 0$

Geometrical Introduction: Finding a principal direction.

1. Scalar product measures the projection of x_j along the direction w.
2. We are only interested on module.
3. Summation over samples to get the global projection's contribute.
4. Searching for w which maximizes projection.
5. Adding constraint to avoid $w \to \infty$ solution.

$$w_1 \in \underset{\|w\|_2=1}{\operatorname{argmax}} \sum_{j=1}^{N} \left(w \cdot x_j\right)^2$$



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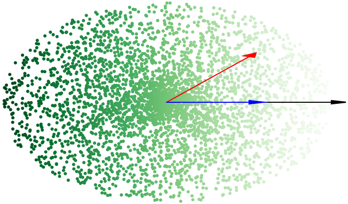

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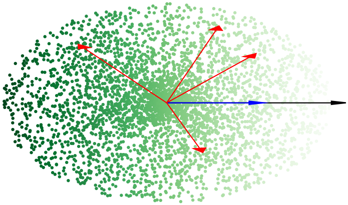

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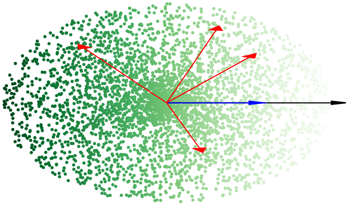

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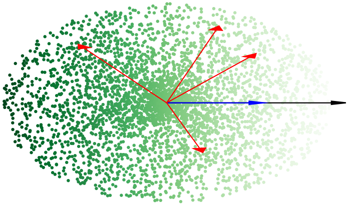

Geometrical Introduction: Finding other directions

We search for other orthogonal directions which maximize projections.

$$w_1 \in \underset{\|w\|_2=1}{\operatorname{argmax}} \sum_{j=1}^{N} (w \cdot x)^2$$

$$w_2 \in \underset{\|w\|_2=1}{\operatorname{argmax}} \sum_{j=1}^{N} (w \cdot x)^2 \quad \text{and} \quad w_2 \perp w_1$$

$$\vdots$$

$$w_n \in \underset{\|w\|_2=1}{\operatorname{argmax}} \sum_{j=1}^{N} (w \cdot x)^2 \quad \text{and} \quad w_2 \perp \{w_1, \ldots, w_{n-1}\}$$

Example

Geometrical Introduction: Direction Selection

$$V(w) = \sum_j (w \cdot x_j)^2 \qquad \text{momentum along } w$$

If w_1, w_2, w_3 orthogonal that maximizes V in the 3D example, then

1. $V(w_1) = 3181.20$ $\approx$82.5%
2. $V(w_2) = 646.25$ $\approx$17.0%
3. $V(w_3) = 19.23$ $\approx$0.5 %

What if we forget the last direction?

Observation

- $x_j = \alpha_{1j} w_1 + \alpha_{2j} w_2 + \alpha_{3j} w_3$ (where $\alpha_{ij} = w_i \cdot x_j$).
- $\tilde{x}_j = \alpha_{1j} w_1 + \alpha_{2j} w_2$.

$$\frac{1}{N} \sum_j \|x_j - \tilde{x}_j\|^2 = \frac{V(w_3)}{N} \approx 4.8\,10^{-3} \qquad \text{(MSE)}$$

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Geometrical Introduction: Conclusion

- Given a set of data $X \in \mathbb{R}^{N \times n}$
- We can find $w_1, \cdots, w_n$ principal (orthonormal) directions the maximize their momentum.
- $V(w_1) > V(w_2) > \cdots > V(w_n)$
- Approximating X with $\tilde{X}$ by taking only the first k directions we are getting an error that is $V(w_{k+1})/N$

What's the catch?

$$\begin{aligned} \max_{w \in \mathbb{R}^n} \quad & \sum_{j=1}^{N} (w \cdot x_j)^2 \\ \text{s.t} \quad & w_i \cdot w = 0, \ \forall i < k \\ & w \cdot w = 1 \end{aligned} \tag{MP}$$

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Classical Derivation

Classical Derivation: Notations

$\mathcal{V}$ random variable, $V = (v_1, \dots, v_N)$ N observations of the variable.

- Expected Value $\mathbb{E}[\mathcal{V}] = \frac{1}{N}\sum_{j=1}^{N} v_j$
- Variance $Var(\mathcal{V}) = \mathbb{E}[(\mathcal{V} - \mathbb{E}[\mathcal{V}])^2]$
- Covariance $Cov(\mathcal{U}, \mathcal{V}) = \mathbb{E}[(\mathcal{U} - \mathbb{E}[\mathcal{U}])(\mathcal{V} - \mathbb{E}[\mathcal{V}])]$

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Under the assumption $\mathbb{E}[\mathcal{U}] = \mathbb{E}[\mathcal{V}] = 0$

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Classical Derivation: An Eigenvalue Problem

$$\max_{\|w\|=1} V(w) = \max_{\|w\|=1} \sum_j (w^T x_j)^2 = \max_{w^T w=1} w^T (X^T X) w \qquad \text{(MP)}$$

Lagrange Multipliers Technique

Let consider the Lagrangian Function of MP

$$\mathcal{L}(w, \lambda) = V(w) - \lambda(w^T w - 1), \quad \forall w \in \mathbb{R}^n, \lambda \in \mathbb{R}$$

Claim

If w^* is a solution of MP then there exists λ^* such that

$$\nabla \mathcal{L}(w^*, \lambda^*) = 0, \quad i.e \quad (X^T X) w^* - \lambda^* w^* = 0 \qquad (2)$$

w Principal Direction $\implies$ w Eigenvector of $X^T X$

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Why switching to an eigen-pair problem?

$X^T X \quad + \quad \longrightarrow$

- $w_1, \cdots, w_n$ eigenvectors
- $w_i^T X^T X w_i = V(w_i)$ eigenvalues
- $V(w_1) > \cdots > V(w_n)$

Why switching to an eigen-pair problem?

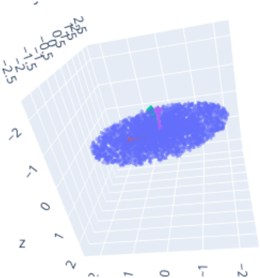
$X^T X$ + MATLAB $\longrightarrow$

- $w_1, \cdots, w_n$ eigenvectors
- $w_i^T X^T X w_i = V(w_i)$ eigenvalues
- $V(w_1) > \cdots > V(w_n)$

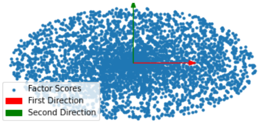
Dimensionality Reduction

The matrix $W = [w_1|\cdots|w_n]$ can be used to reduce the dimensionality

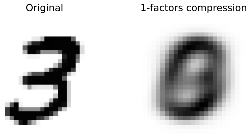
$$F = \left[f^{(1)} \quad |\cdots| \quad f^{(n)}\right] = X\,W \quad \text{(factors scores)}$$



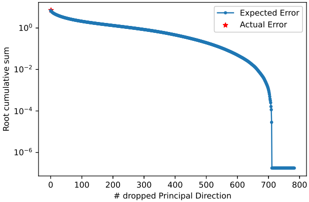
Feature space



Factor scores restricted to the first two principal directions.



Euclidean Distance : 7.23e+00
RMSE : 2.58e-01



Theoretical Error Estimation

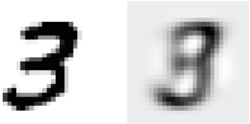
1. $x \in \mathbb{R}^n$ original sample.
2. $f = W^T x \in \mathbb{R}^n$ coordinates in factor-scores space.
3. $\tilde{f} = [f_1, \cdots, f_{n-k}] \in \mathbb{R}^{n-k}$ dropping last k coordinates.
4. $\tilde{x} = [w_1|\cdots|w_{n-k}]\tilde{f} \in \mathbb{R}^n$, approximation of x.

$$\|x - \tilde{x}\| \approx \frac{1}{\sqrt{N}}\|X - X_k\| = \frac{1}{\sqrt{N}}\sqrt{V(n-k+1) + \cdots + V(n)}$$

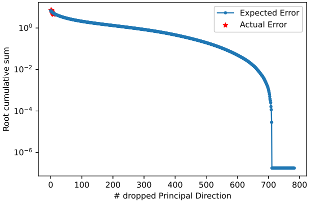
Dimensionality Reduction: A concrete example

Original

10-factors compression



Euclidean Distance : 4.57e+00
RMSE : 1.63e-01



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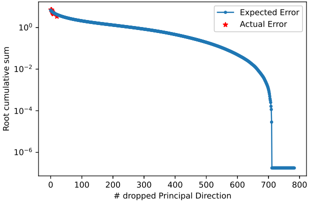
Dimensionality Reduction: A concrete example

Euclidean Distance : 3.46e+00
RMSE : 1.23e-01



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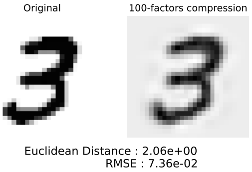

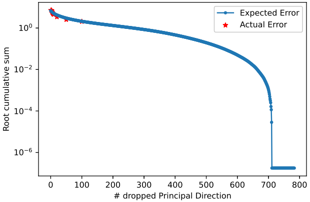

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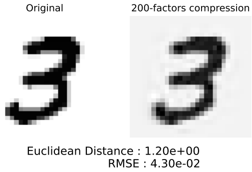
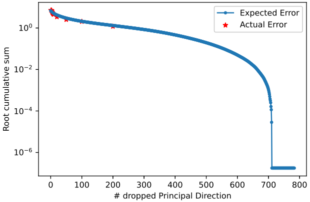
Dimensionality Reduction: A concrete example





Theoretical Error Estimation

1. $x \in \mathbb{R}^n$ original sample.
2. $f = W^T x \in \mathbb{R}^n$ coordinates in factor-scores space.
3. $\tilde{f} = [f_1, \cdots, f_{n-k}] \in \mathbb{R}^{n-k}$ dropping last k coordinates.
4. $\tilde{x} = [w_1 | \cdots | w_{n-k}] \tilde{f} \in \mathbb{R}^n$, approximation of x.

$$\|x - \tilde{x}\| \approx \frac{1}{\sqrt{N}} \|X - X_k\| = \frac{1}{\sqrt{N}} \sqrt{V(n-k+1) + \cdots + V(n)}$$

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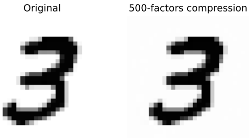
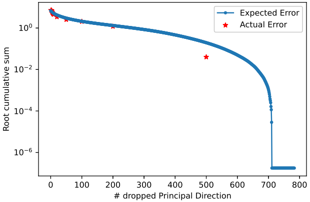


Euclidean Distance : 4.00e-02
RMSE : 1.43e-03



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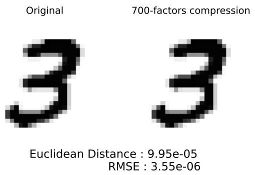

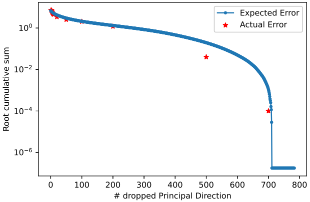

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