Comparative study of integrable systems on spaces of polygons, matrices and bundles

Fabiola Valeria Cordero Uriona

PUC-Rio

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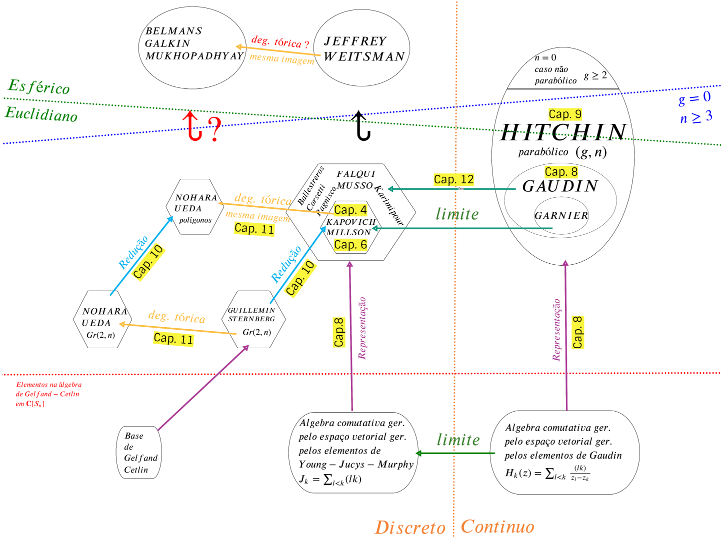
Relations between distinct integrable systems



Integrable Systems and Symplectic Reduction

Integrable Systems

Integrable System on symplectic variety (M, ω) is $n = \frac{\dim_{\mathbf{R}} M}{2}$ independent functions $f_1, \ldots, f_n : M \to \mathbf{R}$ *in involution*: $\{f_i, f_j\} = 0$.

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Then M_{red} has symplectic form ω_ε determined by $i^*\omega = \pi^*\omega_\varepsilon$.

Definition. Walker 84

$M(r) := \{(e_1, ..., e_n) \in \prod_{i=1}^n S^2_{r_i} : e_1 + ... + e_n = 0\} / \mathrm{SO}(3, \mathbf{R})$ for $r = (r_1, \ldots, r_n),\ r_i > 0.$

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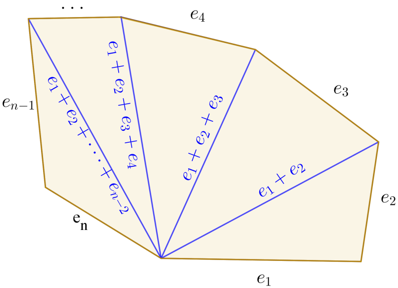
Kapovich–Millson's integrable system of bending flows



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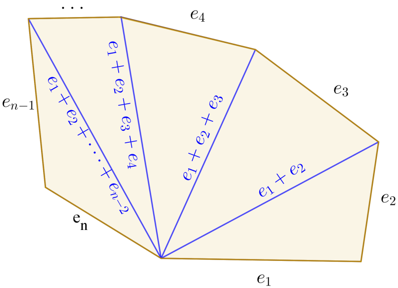

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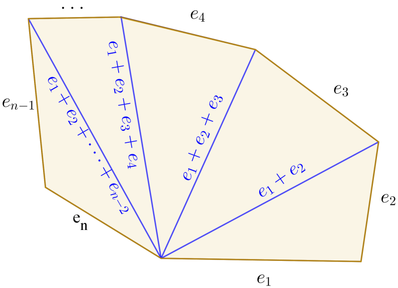

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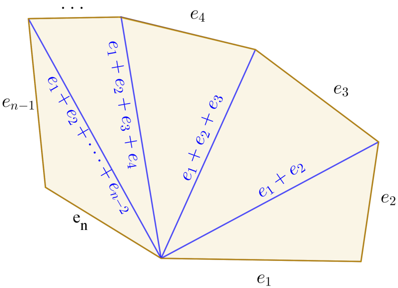

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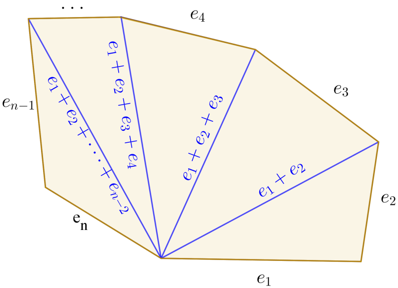

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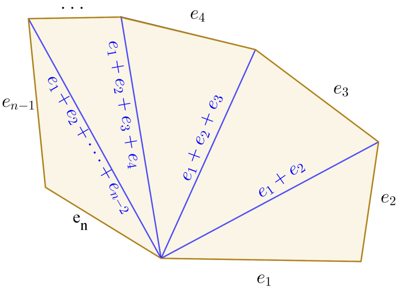

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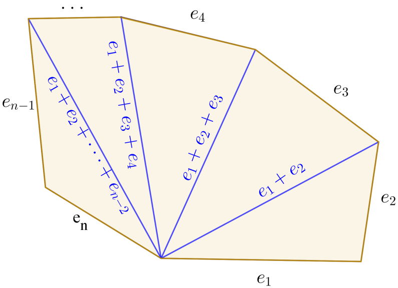

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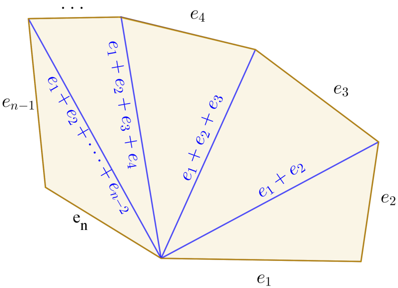

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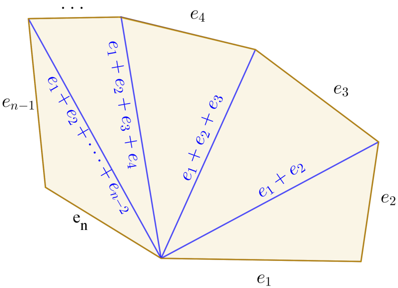

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$L(M(r)) \subset \mathbf{R}^{n-3}$ — polytope

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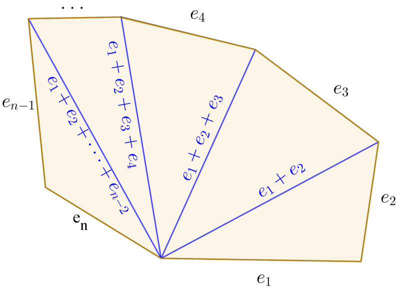

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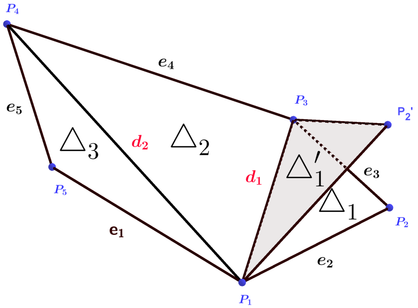
Action-angle coordinates



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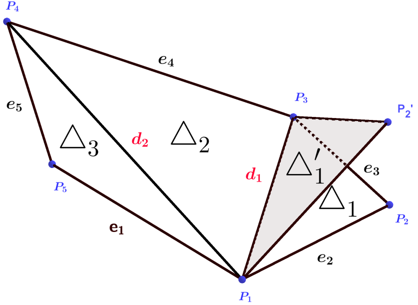

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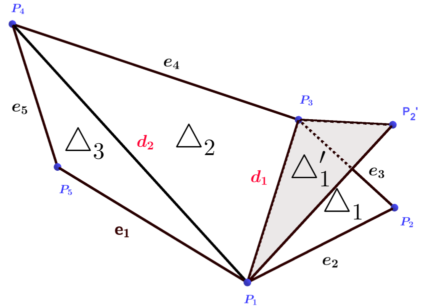

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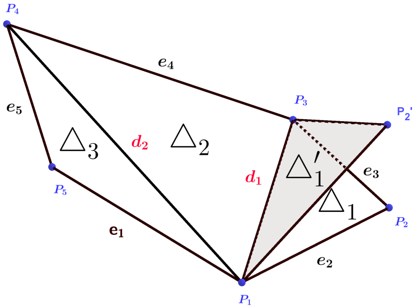

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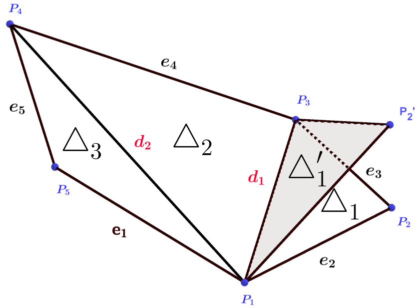

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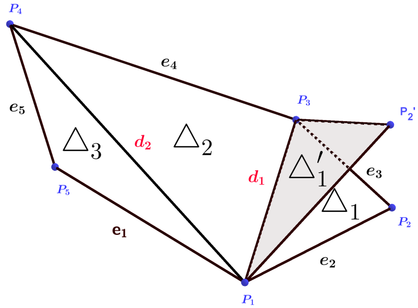

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Lie bracket $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ defines Lie–Poisson bracket $\{,\}$ on polynomials $S(\mathfrak{g}) = \mathbf{C}[\mathfrak{g}^*]$ and functions $C^\infty(\mathfrak{g}^*)$ by

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Coadjoint action $G:\mathfrak{g}^*$ is Poisson.

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$$\omega_\rho(ad^*_X R, ad^*_Y R):=\langle R,[X,Y]\rangle$$

for $X,Y\in\mathfrak{g}$, $R\in\mathfrak{g}^*$.

Symplectic structure on $\mathfrak{g}$-polygon spaces

Fix n given by invariants $\mathrm{P} := (\rho_1, ..., \rho_n)$.

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So

$$\begin{aligned} M(\mathrm{P}) &:= \mu^{-1}(0)/G \\ &= \{\vec{e} \in S_{\mathrm{P}} : e_1 + e_2 + ... + e_n = 0\}/G \end{aligned}$$

is also symplectic.

KM functions on the space of $\mathfrak{g}$-polygons

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Theorem

Euclidean ($\mathfrak{g} = \mathfrak{so}(3, \mathbf{R})$) and Minkowski ($\mathfrak{g} = \mathfrak{sl}(2, \mathbf{R})$) polygon spaces are immersed in complex version $M(\mathrm{P})$ ($\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$) under the following interpretation.

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Kohno–Drinfeld algebra

Recall

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i, j, k, l distinct.

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Definition. K85, D91

Kohno–Drinfeld Lie algebra $\mathfrak{t}_n$ is defined by generators $t_{ij} = t_{ji}$, $1 \leq i \neq j \leq n$ and relations

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From Kohno–Drinfeld to polynomials on polygon spaces

Proposition.

For any metric Lie algebra $(\mathfrak{g}, B)$ there is a unique Lie morphism

$$G_{\mathfrak{g}}^{B} : (\mathfrak{t}_n, [,]) \longrightarrow (\mathbf{C}[\mathfrak{g}^*]^{\otimes n}, \{,\})$$

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(Recall that Poisson bracket on RHS is given on linear generators by $\{X, Y\} = [X, Y]$.)

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$$G : \mathfrak{t}_n \longrightarrow \mathfrak{s}_n$$

s.t. $G(t_{ij}) = (ij) \in \mathfrak{s}_n$, where $(ij) \in S_n \subset \mathfrak{s}_n$ is a transposition.

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Corollary

If $y_1, \ldots, y_N \in \mathfrak{t}_n$ commute then $Gy_1, \ldots, Gy_N \in \mathfrak{s}_n$ also commute, and $G^B_{\mathfrak{g}} y_1, \ldots, G^B_{\mathfrak{g}} y_N \in \mathbf{C}[\mathfrak{g}^*]^{\otimes n}$ commute for every metric Lie algebra $(\mathfrak{g}, B)$.

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Observation (Enriquez–Roubtsov 95)

For $\mathfrak{g} = \mathfrak{sl}(2)$ this goes back to Garnier (1919).

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$G_{\mathfrak{g}}^B G^{-1} J_k = f_{k-1}$, so Young's/Jucys's/Murphy's Thm (3?/71/81) + Gaudin's method (76) $\implies$ KM/BCR/K Thm (96/96/98): $\{f_k, f_l\} = 0$.

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Hitchin system

Chevalley: $\mathbf{C}[\mathfrak{g}^*]^G = \mathbf{C}[P_1, \ldots, P_{\operatorname{rank}\mathfrak{g}}]$, e.g. $P_k(M) = \operatorname{Tr}(M^k)$ for $\mathfrak{gl}(n)$.

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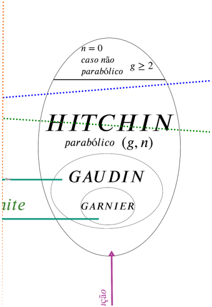
Theorem. Hitchin 87

H is an algebraic integrable system.

Gaudin como Hitchin



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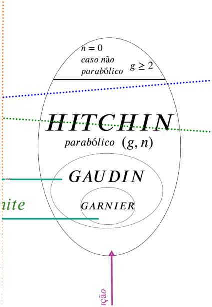

$C = \mathbf{CP}^1$ with coordinate z

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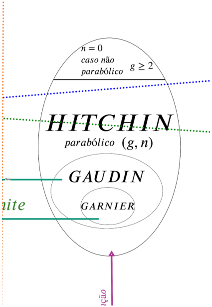

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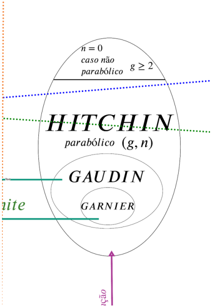

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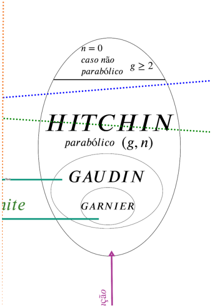

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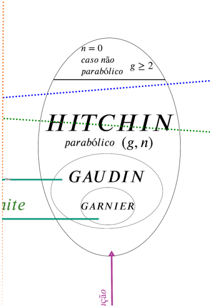

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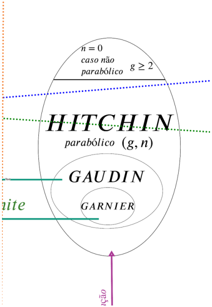

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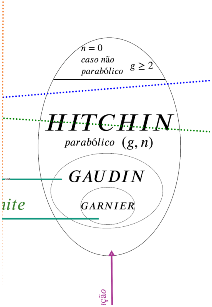

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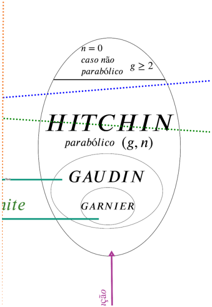

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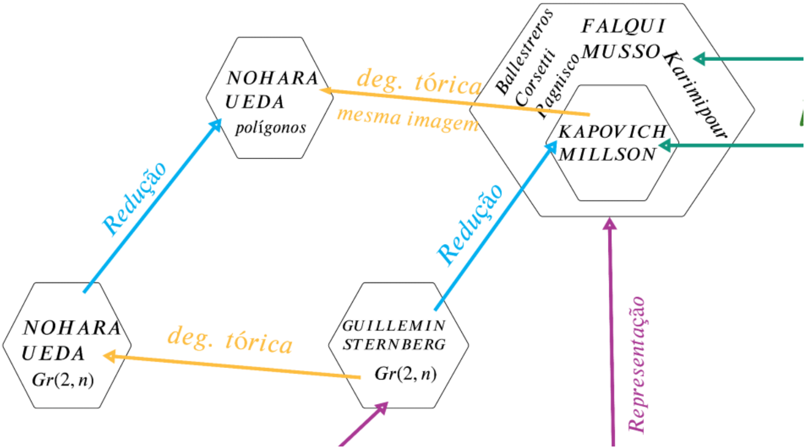
Toric degenerations



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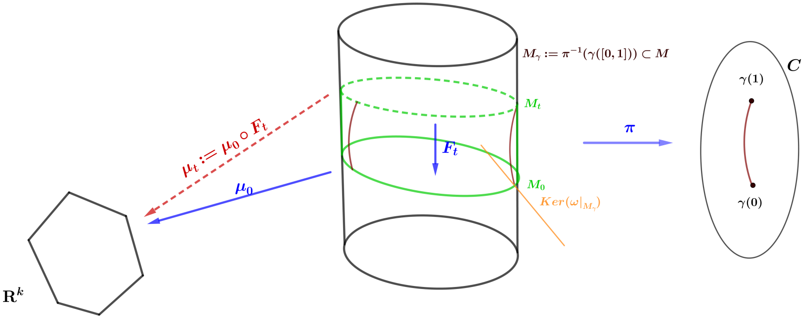
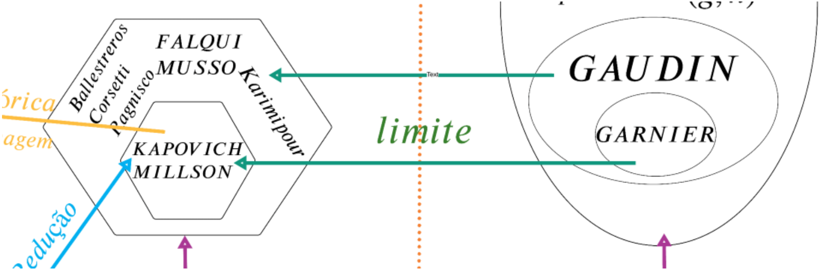


Kapovich–Millson systems.

YJM/Kapovich–Millson/BCR/K as a limit of Garnier/Gaudin/Hitchin



Common setup for KM and Gaudin–Hitchin

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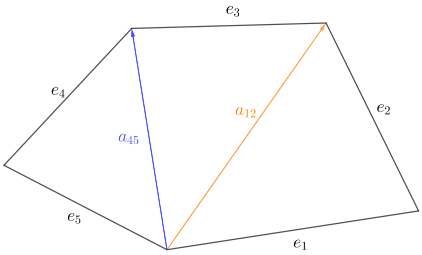
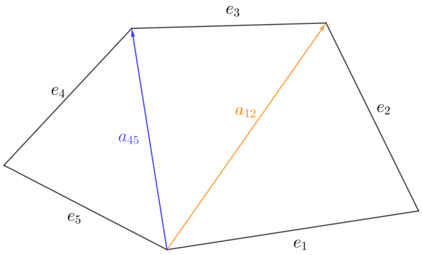
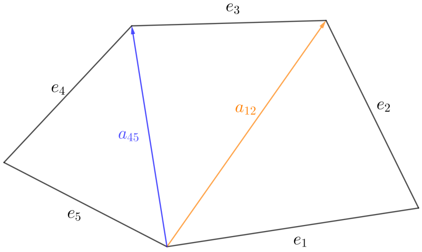

e_3
e_4
a_{45}
a_{12}
e_2
e_5
e_1



Pentagons

$\{a_{14}, a_{24}, a_{15}, a_{25}, a_{35}\}$ — basis of $\mathcal{U}/\mathcal{A}$.



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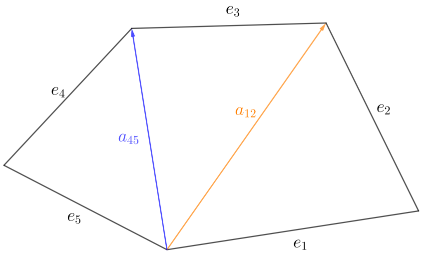
$$\begin{pmatrix} \frac{(x_1-x_2)(x_1-x_3)}{x_2-x_3} H_1 \\ \frac{(x_3-x_4)(x_4-x_5)}{x_3-x_5} H_4 \end{pmatrix}$$



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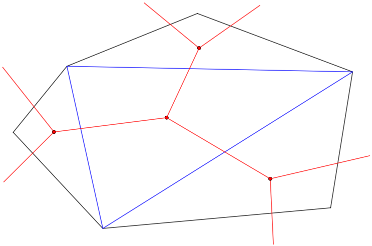
Hexagons, symmetric triangulation



$$H_j = \sum_{i \neq j} \frac{a_{ij}}{x_i - x_j}, j \in \{1, \ldots, 6\}$$

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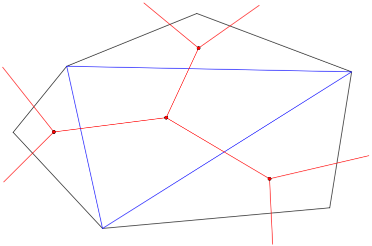

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If $k \neq j$:
$(x_k - x_j)H_j = \sum_{i \neq j} \frac{x_k - x_j}{x_i - x_j} a_{ij}$.

$$\begin{pmatrix} (x_2 - x_1)H_1 \\ (x_4 - x_3)H_3 \\ (x_6 - x_5)H_5 \end{pmatrix} \to \begin{pmatrix} a_{12} \\ a_{34} \\ a_{56} \end{pmatrix}$$

when $x_6 \to x_5$, $x_2 \to x_1$, $x_4 \to x_3$

Fan triangulation / caterpillar graph

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Chervov–Falqui–Rybnikov and Aguirre–Felder–Veselov

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where e — an edge of the tree, $c_k(e)$ — explicit functions of x parametrized by edges of the tree.

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For a pair of leaves $i \neq j$ the limit $\lim_{x_i \to x_j}(x_i - x_j)H_i = \lim_{x_i \to x_j}(x_j - x_i)H_j = a_{ij}$ is a Kapovich–Millson function

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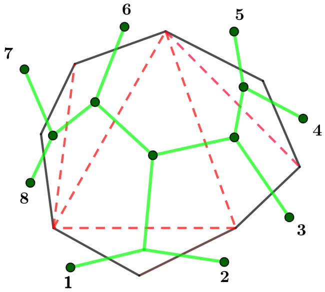
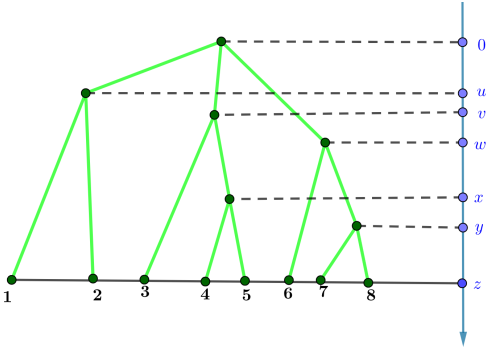
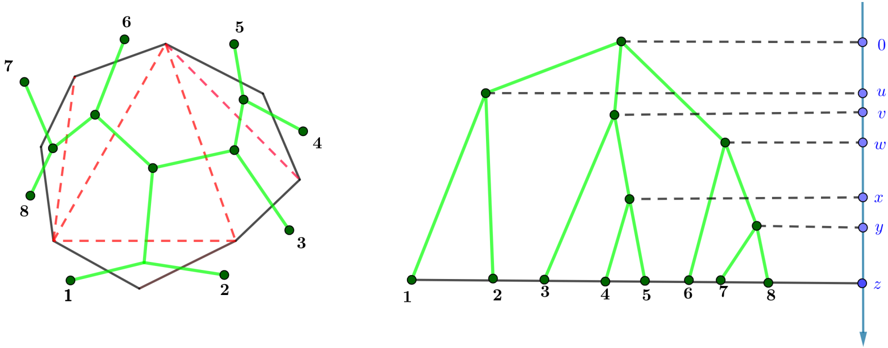
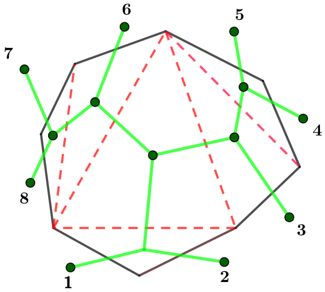


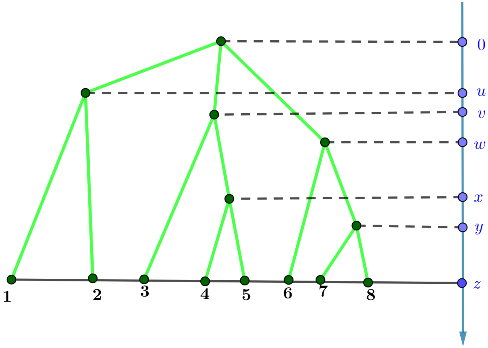


Idea — for a phylogenetic tree of n species evolution



Idea — for a phylogenetic tree of n species evolution, take $x_i(t) = \sum_{a \in \text{ancestors}(i)} evol(i; T_a) \cdot t^{T_a}$ — a generating function





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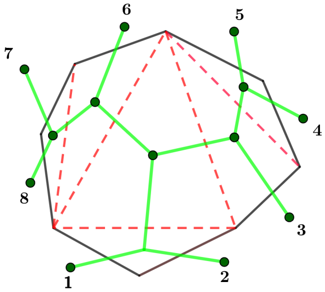

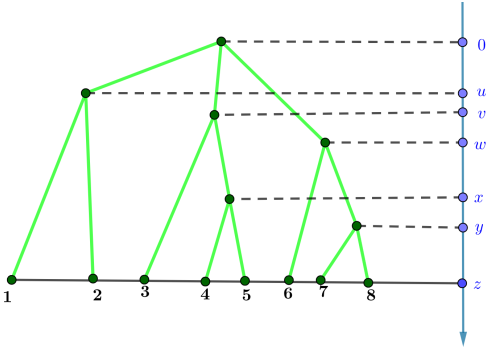

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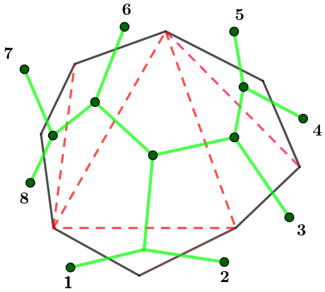

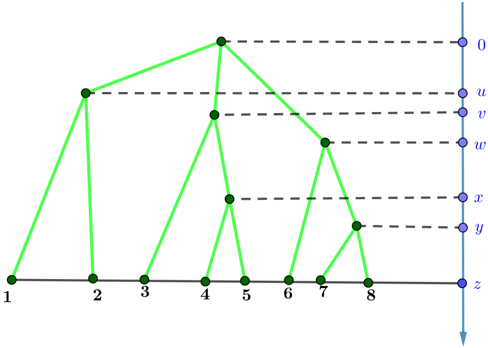

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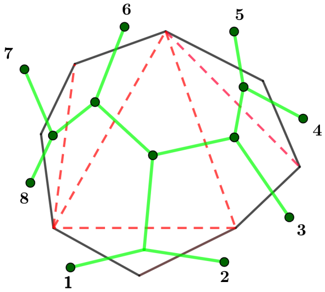
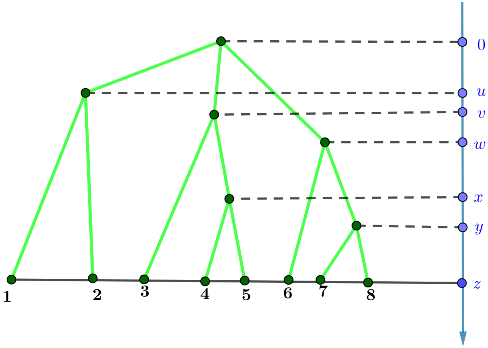
Trees and ultrametrics are used in *tropicalizations*





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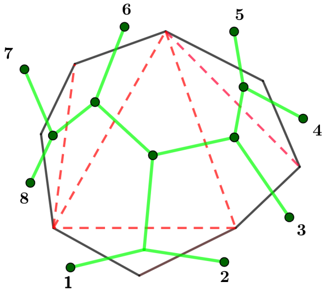

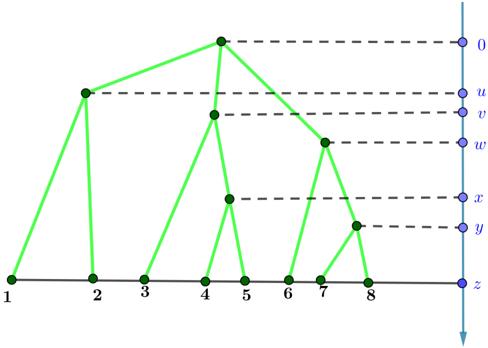

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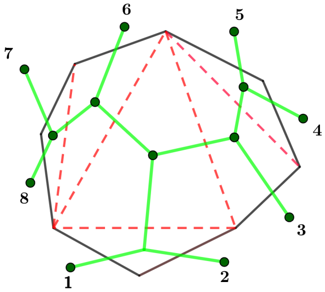

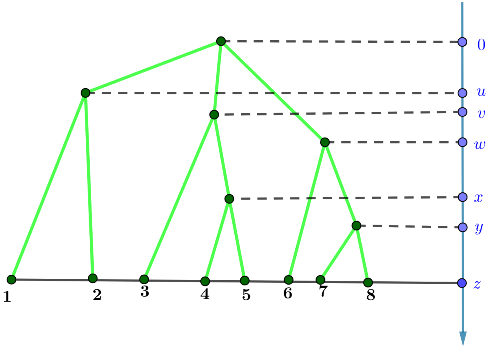

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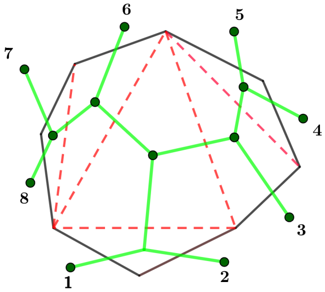

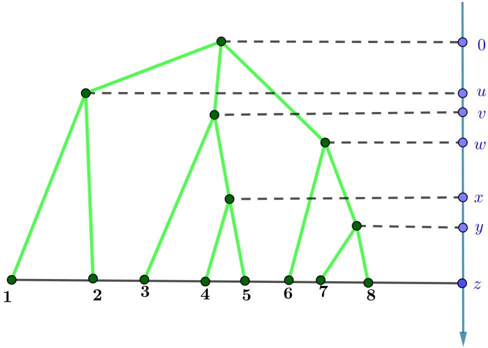

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Trees and ultrametrics are used in *tropicalizations* of $\overline{M}_{0,n}$ (Mikhalkin, Rau), $Gr(2,n)$ (Speyer), and in Nohara–Ueda.

More work in progress, and related problems

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