Esercizi presi dall'eserciziario su moodle

Equazioni differenziali di primo grado

✍ **Esercizio 1.1.1.** *Si determini la soluzione $y(t)$ del seguente problema di Cauchy*

$$\begin{cases} y' = \dfrac{y^2}{y^2+4}t \\ y(0) = 2 \end{cases}$$

Inoltre si determini il valore $\alpha > 0$ per cui $\dfrac{y(t)}{t^\alpha}$ tende a un numero finito e non nullo per $t \to +\infty$.

$$\frac{y^2+4}{y^2} \cdot y' = t$$

$$\int \frac{y^2+4}{y^2} \cdot y' \, dt = \int t \, dt + C$$

$u = y$

$du = y' dt$

$\downarrow u = y$

$$\int \frac{y^2+4}{y^2} \, dy = \frac{t^2}{2} + C$$

$$\int 1 \, dy + 4 \int y^{-2} \, dy = \frac{t^2}{2} + C$$

$$y - \frac{4}{y} = \frac{t^2}{2} + C$$

$$\frac{y^2-4}{y} = \frac{t^2}{2} + C$$

$y(0) = 2$

$\downarrow$

$$\frac{2^2-4}{2} = \frac{0}{2} + C$$

$$\frac{0}{2} = C$$

$$C = 0 \rightarrow \frac{y^2-4}{y} = \frac{t^2}{2} \rightarrow \frac{y^2 - 4 - \frac{t^2 y}{2}}{y} = 0$$

$$y(t) = \frac{\frac{t^2}{2} \pm \sqrt{\frac{t^4}{4} + 16}}{2} = \frac{t^2 \pm \sqrt{t^4+64}}{4}$$

$$\downarrow$$

$$y(0) = 2$$

$$\downarrow$$

$$\frac{0^2 \pm \sqrt{64}}{4} = \begin{cases} +2 \\ -2 \quad \text{(Non accettabile)} \to y(0)=2 \end{cases}$$

Soluzione 1:

$$y(t) = \frac{t^2 + \sqrt{t^4+64}}{4}$$

$$\lim_{t \to +\infty} \frac{y(t)}{t^\alpha} = \frac{t^2 + \sqrt{t^4+64}}{4t^\alpha} = \lim_{t\to+\infty} \frac{t^2 + t^2 \overbrace{\sqrt{1+\frac{64}{t^4}}}^{1+\frac{64}{+\infty}=1}}{4t^\alpha} =$$

$$= \lim_{t\to+\infty} \frac{\cancel{2}t^2}{\cancel{4}t^\alpha} = \lim_{t\to+\infty} \frac{t^2}{2t^\alpha}$$

L'unico modo per avere un numero finito è che il grado del numeratore e del denominatore sia uguale, quindi:

$$\alpha = 2 \to \lim_{t\to+\infty} \frac{t^2}{2t^2} = \lim_{t\to+\infty} \frac{1}{2} = \frac{1}{2}$$

✍ **Esercizio 1.1.2.** *Si determini la soluzione $y(t)$ del seguente problema di Cauchy*

$$\begin{cases} y' = \dfrac{t^2+t}{2e^{2y}+6e^y} \\ y(0) = 0 \end{cases}$$

$$y' \cdot (2e^{2y} + 6e^y) = t^2 + t$$

$$\int y' \cdot (2e^{2y} + 6e^y)\,dt = \int t^2 + t \, dt + C$$

$u = y$

$du = y'dt$

$\downarrow u = y$

$$2\int e^{2y}\,dy + 6\int e^{y}\,dy = \frac{t^3}{3} + \frac{t^2}{2} + C$$

$$\cancel{2}\cdot\frac{e^{2y}}{\cancel{2}} + 6e^{y} = \frac{t^3}{3} + \frac{t^2}{2} + C$$

$y(0)=0$

$$e^{2\cdot 0} + 6e^{0} = \frac{0}{3} + \frac{0}{2} + C$$

$1+6 = C$

$C = 7$

$\downarrow$

$$e^{2y} + 6e^{y} = \frac{t^3}{3} + \frac{t^2}{2} + 7$$

$x = e^{y} \rightarrow y = \ln(x)$

$$x^2 + 6x - \frac{t^3}{3} - \frac{t^2}{2} - 7 = 0$$

$$x_{1,2} = \frac{-\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - ac}}{a} = -3 \pm \sqrt{\frac{t^3}{3} + \frac{t^2}{2} + 16}$$

$$e^{y} = -3 \pm \sqrt{\frac{t^3}{3} + \frac{t^2}{2} + 16}$$

$y(0)=0$

$$e^{0} = -3 \pm \sqrt{\frac{0}{3} + \frac{0}{3} + 16} = -3 \pm \sqrt{16} = -3 \pm 4 = \begin{cases} -7 & \text{Non accettabile} \\ 1 & \end{cases}$$

$$y(t) = \ln\left(-3 + \sqrt{\frac{t^3}{3} + \frac{t^2}{2} + 16}\right)$$

$\downarrow$

$y(0) = \ln(1) = 0$ ✓

✍ **Esercizio 1.1.4.** *Sia $y(t)$ la soluzione del problema di Cauchy*

$$\begin{cases} y' = \dfrac{e^{-x}\sqrt{y+1}}{e^{-x}+1} \\ y(0) = 1 \end{cases}$$

$$y' = \frac{e^{-x}\sqrt{y+1}}{e^{-x}+1}$$

$$\frac{y'}{\sqrt{y+1}} = \frac{e^{-x}}{e^{-x}+1}$$

$$\int \frac{y'}{\sqrt{y+1}}\,dy = \int \frac{e^{-x}}{e^{-x}+1}\,dx$$

$y = u$

$y'dy = du$

$$-\int (y+1)^{-\frac{1}{2}}\,dy = \int \frac{-e^{-x}}{e^{-x}+1}\,dx \longleftarrow u = y$$

$$c + \frac{-(y+1)^{\frac{1}{2}}}{\frac{1}{2}} = \int \frac{F'}{F}\,dF \longleftarrow \begin{cases} e^{-x}+1 = F \\ F' = -e^{-x} \end{cases}$$

$$-2\sqrt{y+1} = \int \frac{dF}{F}$$

$$-2\sqrt{y+1} = \ln(F) + c$$

$$\sqrt{y+1} = -\frac{1}{2}\ln(e^{-x}+1) + c$$

$$y = \left(-\frac{1}{2}\ln(e^{-x}+1) + c\right)^2 - 1$$

$$y(0) = 1$$
$$\downarrow$$

$$1 = \left(-\frac{1}{2}\ln(2) + c\right)^2 - 1$$

$$2 = \left(-\frac{1}{2}\ln(2) + c\right)^2$$

$$\sqrt{2} = -\frac{1}{2}\ln(2) + c$$

$$c = \sqrt{2} + \frac{1}{2}\ln(2)$$

$$y = \left(-\frac{1}{2}\ln(e^{-t}+1) + \sqrt{2} + \frac{1}{2}\ln(2)\right)^2 - 1$$

✍ **Esercizio 1.1.5.** *Si determini la soluzione $y(t)$ del seguente problema di Cauchy*

$$\begin{cases} y' = (e^{-3y}+1)(2x-1) \\ y(0) = -1 \end{cases}$$

$$y' = (e^{-3y}+1)(2x-1)$$

$$\frac{y'}{e^{-3y}+1} = 2x-1$$

$$\int \frac{y'}{e^{-3y}+1}\,dy = \int 2x-1\,dx$$

$$\int \frac{1}{e^{-3y}+1}\,dy = 2\int x\,dx - \int 1\,dx$$

$$\int \frac{1}{e^{-3y}+1}\,dy = x^2 - x + C$$

$$\int \frac{1}{e^{-3y}\left(1+\frac{1}{e^{-3y}}\right)}\,dy = x^2 - x + C$$

$$\int \frac{e^{3y}}{1+e^{3y}}\,dy = x^2 - x + C \qquad \frac{d\,(1+e^{3y})}{dy} = 3e^{3y}$$

$$3\int \frac{e^{3y}}{1+e^{3y}}\,dy = 3(x^2 - x + C)$$

$$\int \frac{3e^{3y}}{1+e^{3y}}\,dy = 3x^2-3x+C$$

$F = 1+e^{3y}$

$F'dF = 3e^{3y}$

$$\int \frac{F'}{F}\,dF = 3x^2-3x+C$$

$$\int \frac{1}{F}\,dF = 3x^2-3x+C$$

$$\ln(F) = 3x^2-3x+C$$

$$\ln(1+e^{3y}) = 3x^2-3x+C$$

$$e^{\ln(1+e^{3y})} = e^{3x^2-3x+c}$$

$$e^{3y} = e^{3x^2-3x+c} - 1$$

$$\ln(e^{3y}) = \ln(e^{3x^2-3x+c}-1)$$

$$3y = \ln(e^{3x^2-3x+c}-1)$$

$$y = \frac{\ln(e^{3x^2-3x+c}-1)}{3} \qquad e^{\sim} \cdot e^{c}$$

$$y(0) = -1$$

$$-1 = \frac{\ln(e^{0+c}-1)}{3}$$

$$-3 = \ln(e^{c}-1)$$

$$e^{-3} = e^{c}-1$$

$$e^{c} = e^{-3}+1$$

$$\ln(e^{c}) = \ln(e^{-3}+1)$$

$$c = \ln(e^{-3}+1)$$

$$y = \frac{\ln\left(e^{3x^2-3x}\cdot e^{\ln(e^{-3}+1)}-1\right)}{3}$$

$$y = \frac{\ln\left(\left(1+e^{-3}\right)e^{3x^2-3x}-1\right)}{3}$$

$$y = \frac{1}{3}\ln\left(\left(1+e^{-3}\right)e^{3x^2-3x}-1\right)$$

✍ **Esercizio 1.1.6.** *Si determini la soluzione* $y(t)$ *del seguente problema di Cauchy*

$$\begin{cases} y' = (3+27y^2)\left(xe^{3x}-2x^2\right) \\ y(0) = 0 \end{cases}$$

$$y' = (3+27y^2)(xe^{3x}-2x^2)$$

$$\int \frac{y'}{3+27y^2}\,dy = \int xe^{3x}-2x^2\,dx$$

$$\int \frac{1}{3+27y^2}\,dy = \int xe^{3x}\,dx - 2\int x^2\,dx$$

$$\frac{1}{3}\int \frac{1}{1+9y^2}\,dy = \underbrace{\int xe^{3x}\,dx}_{\downarrow} - 2\,\frac{x^3}{3} + C$$

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

$$f: x \qquad f': 1$$

$$g': e^{3x} \qquad g: \frac{1}{3}e^{3x}$$

$$\downarrow$$

$$\frac{1}{3}\int \frac{1}{1+9y^2}\,dy = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx - 2\,\frac{x^3}{3} + C$$

$$\frac{1}{3}\int \frac{1}{1+9y^2}\,dy = \frac{1}{3}x\,e^{3x} - \frac{1}{9}e^{3x} - 2\frac{x^3}{3} + C$$

$$\int \frac{1}{1+9y^2}\,dy = xe^{3x} - \frac{1}{3}e^{3x} - 2x^3 + C$$

$$\downarrow$$

$$\frac{1}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$$

$$\downarrow$$

$$\frac{1}{9}\int \frac{1}{y^2+\frac{1}{9}}\,dy = xe^{3x} - \frac{1}{3}e^{3x} - 2x^3 + C$$

$$\frac{1}{3}\arctan(3y) = xe^{3x} - \frac{1}{3}e^{3x} - 2x^3 + C$$

$$\arctan(3y) = 3xe^{3x} - e^{3x} - 6x^3 + C$$

$$\arctan(x) = a$$
$$\downarrow$$
$$x = \tan(a)$$

$$3y = \tan(3xe^{3x} - e^{3x} - 6x^3 + C)$$

$$y = \frac{1}{3}\tan(3xe^{3x} - e^{3x} - 6x^3 + C)$$

$$\tan(a) = 0$$

$$a = \arctan(0)$$

$$y(0) = 0$$

$$\downarrow$$

$$0 = \frac{1}{3}\tan(0 - 1 - 0 + C)$$

$$\tan(-1 + C) = 0$$

$$-1 + C = \arctan(0)$$

$$C = 1$$

$$y = \frac{1}{3}\tan(3xe^{3x} - e^{3x} - 6x^3 + 1)$$

Equazioni differenziali lineari di secondo grado

✍ **Esercizio 1.2.1.** *Si determini la soluzione $y(t)$ del seguente problema di Cauchy*

$$\begin{cases} y'' - 6y' + 9y = 3t + 2 \\ y(0) = -1 \\ y'(0) = 2 \end{cases}$$

$3t+2$ è un polinomio di grado 1

$y_1 = a_0 + a_1 x \xrightarrow{\text{Sostituisco}} -6a_1 + 9a_0 + 9a_1 x = 3x + 2$

$y_1' = a_1$

$-6a_1 + 9a_0 + 9a_1 x = 3x + 2$

$y'' = 0$

$$\begin{cases} -6a_1 + 9a_0 = 2 \\ 9a_1 = 3 \end{cases} \quad \begin{cases} -2 + 9a_0 = 2 \\ a_1 = \frac{1}{3} \end{cases} \quad \begin{cases} a_0 = \frac{4}{9} \\ a_1 = \frac{1}{3} \end{cases} \rightarrow y_1 = \frac{4}{9} + \frac{1}{3}t$$

Risolvo l'equazione omogenea

$r'' - 6r' + 9r = 0$

$(r-3)^2$

$r_{1,2} = 3 \longrightarrow y_2 = C_1 e^{3t} + C_2 t e^{3t}$

$$y(t) = \frac{4}{9} + \frac{1}{3}t + C_1 e^{3t} + C_2 t e^{3t}$$

Applico le condizioni di Cauchy

$y(0) = \frac{4}{9} + 0 + C_1 + 0 = \frac{4}{9} + C_1 = -1 \rightarrow C_1 = -\frac{13}{9}$

$y'(t) = \frac{1}{3} + 3C_1 e^{3t} + C_2 e^{3t} + 3C_2 t e^{3t}$

$y'(0) = \frac{1}{3} + 3C_1 + C_2 = 2 \rightarrow \frac{1}{3} + 3 \cdot \left(-\frac{13}{9}\right) + C_2 = 2$

$$\frac{1}{3} - \frac{13}{3} + c_2 = 2$$

$$1 - 13 + 3c_2 = 6$$

$$3c_2 = 6 + 12$$

$$c_2 = \frac{18}{3} = 6$$

$$c_1 = -\frac{13}{9} \quad c_2 = 6$$

$$\downarrow$$

$$y(t) = \frac{4}{9} + \frac{1}{3}t + c_1 e^{3t} + c_2 t e^{3t} = \frac{4}{9} + \frac{1}{3}t - \frac{13}{9} e^{3t} + 6te^{3t}$$

✍ **Esercizio 1.2.2.** *Sia $y(t)$ la soluzione del problema di Cauchy*

$$\begin{cases} y'' + 2y' - 3y = 0 \\ y(0) = 0 \\ y'(0) = 1. \end{cases}$$

Allora $\lim_{t\to+\infty} y(t) =$

- □ 0;
- □ non esiste;
- □ $+\infty$;
- □ $-\infty$

Risolvo l'equazione caratteristica

$$r^2 + 2r - 3 = 0$$

$$(r+3)(r-1)$$

$$r_1 = -3 \quad r_2 = 1$$

$$y(t) = c_1 e^{-3t} + c_2 e^{t}$$

$$y'(t) = -3c_1 e^{-3t} + c_2 e^{t}$$

Impongo le condizioni di Cauchy

$$\begin{cases} y(0) = c_1 + c_2 = 0 \\ y'(0) = -3c_1 + c_2 = 1 \end{cases} \rightarrow \begin{cases} c_1 = -c_2 \\ 3c_2 + c_2 = 1 \end{cases} \quad \begin{cases} c_1 = -\frac{1}{4} \\ c_2 = \frac{1}{4} \end{cases}$$

$y(t) = -\frac{1}{4} e^{-3t} + \frac{1}{4} e^{t}$

$$\lim_{t \to +\infty} y(t) = -\frac{1}{4} e^{-\infty} + \frac{1}{4} e^{\infty} = 0 + \infty = +\infty$$

La risposta corretta è la terza

✍ **Esercizio 1.2.3.** *Si determini la soluzione $y(t)$ del problema di Cauchy*

$$\begin{cases} y'' - y' - 2y = \cos(2t) \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

Risolvo l'equazione omogenea associata

$r^2 - r - 2 = 0$

$(r-2)(r+1)$

$r_1 = -1 \quad r_2 = 2 \quad \to \quad z(t) = c_1 e^{-t} + c_2 e^{2t}$

Con il metodo di somiglianza cerco una soluzione particolare dell'equazione non omogenea:

$$\bar{y}(t) = \alpha \sin(2t) + \beta \cos(2t)$$

$$\bar{y}'(t) = 2\alpha \cos(2t) - 2\beta \sin(2t)$$

$$\bar{y}''(t) = -4\alpha \sin(2t) - 4\beta \cos(2t)$$

$$y'' - y' - 2y = \cos(2t)$$

$$\downarrow$$

$$-4\alpha \sin(2t) - 4\beta \cos(2t) - 2\alpha \cos(2t) + 2\beta \sin(2t) - 2\alpha \sin(2t) - 2\beta \cos(2t) = \cos(2t)$$

$$\sin(2t)(-4\alpha + 2\beta - 2\alpha) + \cos(2t)(-4\beta - 2\alpha - 2\beta) = \cos(2t)$$

$$\sin(2t)(-6\alpha + 2\beta) + \cos(2t)(-6\beta - 2\alpha) = \cos(2t)$$

$$\begin{cases} -6\alpha + 2\beta = 0 \\ -6\beta - 2\alpha = 1 \end{cases} \qquad \begin{cases} \beta = 3\alpha \\ -18\alpha - 2\alpha = 1 \end{cases} \qquad \begin{cases} \beta = -\frac{3}{20} \\ \alpha = -\frac{1}{20} \end{cases}$$

$$\bar{y}(t) = -\frac{1}{20}\sin(2t) - \frac{3}{20}\cos(2t)$$

$$y(t) = z(t) + \bar{y}(t) = c_1 e^{-t} + c_2 e^{2t} - \frac{1}{20}\sin(2t) - \frac{3}{20}\cos(2t)$$

$$y'(t) = -c_1 e^{-t} + 2c_2 e^{2t} - \frac{1}{10}\cos(2t) + \frac{3}{10}\sin(2t)$$

Applico le condizioni di Cauchy

$$\begin{cases} y(0) = c_1 + c_2 - \frac{3}{20} = 1 \\ y'(0) = -c_1 + 2c_2 - \frac{1}{10} = 0 \end{cases} \rightarrow \begin{cases} c_1 = -c_2 + \frac{23}{20} \\ -c_1 + 2c_2 = \frac{1}{10} \end{cases} \quad \begin{cases} c_1 = -c_2 + \frac{23}{20} \\ c_2 - \frac{23}{20} + 2c_2 = \frac{1}{10} \end{cases}$$

$$\begin{cases} c_1 = -c_2 + \frac{23}{20} \\ 3c_2 = \frac{2}{20} + \frac{23}{20} \end{cases} \quad \begin{cases} c_1 = -\frac{5}{12} + \frac{23}{20} \\ c_2 = \frac{5}{12} \end{cases} \quad \begin{cases} c_1 = \frac{-25+69}{60} = \frac{44}{60} = \frac{11}{15} \\ c_2 = \frac{5}{12} \end{cases}$$

$$y(t) = \frac{11}{15}e^{-t} + \frac{5}{12}e^{2t} - \frac{1}{20}\sin(2t) - \frac{3}{20}\cos(2t)$$

✍ **Esercizio 1.2.4.** *Si determini la soluzione $y(t)$ del problema di Cauchy*

$$\begin{cases} y'' - 4y' + 8y = e^{-2t} \\ y(0) = -1 \\ y'(0) = 0. \end{cases}$$

Risolvo l'equazione omogenea associata

$$r^2 - 4r + 8 = 0$$

$$r_{1,2} = \frac{4 \pm \sqrt{16-32}}{2} = \frac{4 \pm i\sqrt{16}}{2} = \frac{4 \pm 4i}{2} = 2 \pm 2i$$

$r_1 = 2 - 2i \qquad r_2 = 2 + 2i$

$$z(t) = c_1 e^{2t}\sin(2t) + c_2 e^{2t}\cos(2t)$$

Bisogna trovare una soluzione particolare del tipo:

$$A = 1$$
$$\lambda = -2$$

$$\bar{y}(t) = e^{-2t}\gamma(t)$$
$$\downarrow$$
$$\gamma'' + \gamma'(2(-2) - 4) + \gamma(4 + 8 + 8) = 1$$

$$\gamma'' - 8\gamma' + 20\gamma = 1$$

$$\lambda^2 + \lambda a + b \neq 0 \rightarrow \gamma(t) = \text{costante} = \frac{A}{\lambda^2 + \lambda a + b} = \frac{1}{20}$$

$$\bar{y}(t) = \frac{1}{20} e^{-2t}$$

$$y(t) = z(t) + \bar{y}(t) = c_1 e^{2t} \sin(2t) + c_2 e^{2t} \cos(2t) + \frac{1}{20} e^{-2t}$$

$$y'(t) = 2c_1 e^{2t} \sin(2t) + 2c_1 e^{2t} \cos(2t) + 2c_2 e^{2t} \cos(2t) - 2c_2 e^{2t} \sin(2t) - \frac{1}{10} e^{-2t}$$

$$\begin{cases} y(0) = c_2 + \frac{1}{20} = -1 \\ y'(0) = 2c_1 + 2c_2 - \frac{1}{10} = 0 \end{cases} \rightarrow \begin{cases} c_2 = -\frac{21}{20} \cdot 2 \\ c_1 = \frac{11}{10} \end{cases}$$

$$y(t) = \frac{11}{10} e^{2t} \sin(2t) - \frac{21}{20} e^{2t} \cos(2t) + \frac{1}{20} e^{-2t}$$

✍ **Esercizio 1.2.5.** *Si determini la soluzione $y(t)$ del problema di Cauchy*

$$\begin{cases} y'' - y' - 2y = \sin(2t) \\ y(0) = 0 \\ y'(0) = 1. \end{cases}$$

Risolvo l'equazione omogenea associata

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1)$$

$$r_1 = -1 \quad r_2 = 2$$

$$z(t) = c_1 e^{-t} + c_2 e^{2t}$$

Bisogna trovare una soluzione particolare del tipo:

$$\bar{y}(t) = \alpha \sin(2t) + \beta \cos(2t)$$

$$\bar{y}'(t) = 2\alpha \cos(2t) - 2\beta \sin(2t)$$

$$\bar{y}''(t) = -4\alpha \sin(2t) - 4\beta \cos(2t)$$

$$y'' - y' - 2y = \sin(2t)$$

$$\downarrow$$

$$-4\alpha\sin(2t) - 4\beta\cos(2t) - 2\alpha\cos(2t) + 2\beta\sin(2t) - 2\alpha\sin(2t) - 2\beta\cos(2t) = \sin(2t)$$

$$\sin(2t)(-4\alpha + 2\beta - 2\alpha) + \cos(2t)(-4\beta - 2\alpha - 2\beta) = \sin(2t)$$

$$\sin(2t)(-6\alpha + 2\beta) + \cos(2t)(-6\beta - 2\alpha) = \sin(2t)$$

$$\begin{cases} -6\alpha + 2\beta = 1 \\ -6\beta - 2\alpha = 0 \end{cases} \quad \begin{cases} -6\alpha - \frac{2}{3}\alpha = 1 \\ \beta = -\frac{1}{3}\alpha \end{cases} \quad \begin{cases} -\frac{20}{3}\alpha = 1 \\ \beta = -\frac{1}{3}\alpha \end{cases} \quad \begin{cases} \alpha = -\frac{3}{20} \\ \beta = +\frac{1}{20} \end{cases}$$

$$\bar{y}(t) = -\frac{3}{20}\sin(2t) + \frac{1}{20}\cos(2t)$$

$$y(t) = c_1 e^{-t} + c_2 e^{2t} - \frac{3}{20}\sin(2t) + \frac{1}{20}\cos(2t)$$

$$y'(t) = -c_1 e^{-t} + 2c_2 e^{2t} - \frac{3}{10}\cos(2t) - \frac{1}{10}\sin(2t)$$

Impongo le condizioni di Cauchy

$$\begin{cases} y(0) = c_1 + c_2 + \frac{1}{20} = 0 \\ y'(0) = -c_1 + 2c_2 - \frac{3}{10} = 1 \end{cases} \rightarrow \begin{cases} c_1 = -c_2 - \frac{1}{20} \\ c_2 + \frac{1}{20} + 2c_2 - \frac{3}{10} = 1 \end{cases} \quad \begin{cases} c_1 = -c_2 - \frac{1}{20} \\ 3c_2 - \frac{5}{20} = 1 \end{cases}$$

$$\begin{cases} c_1 = -\frac{25}{60} - \frac{1}{20} = \frac{-25-3}{60} = -\frac{28}{60} = -\frac{7}{15} \\ c_2 = \frac{25}{60} = \frac{5}{12} \end{cases}$$

$$y(t) = -\frac{7}{15}e^{-t} + \frac{5}{12}e^{2t} - \frac{3}{20}\sin(2t) + \frac{1}{20}\cos(2t)$$

Esercizio 1.2.6. *Determinate la soluzione generale dell'equazione differenziale* $y'' - 4y' + 13y = 4x$.

Risolvo l'equazione omogenea associata

$r^2 - 4r + 13 = 0$

$$r_{1,2} = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm i\sqrt{24^2}}{2} = \frac{4 \pm 4i\sqrt{2}}{2} = 2 \pm 2\sqrt{2}\,i$$

$$z(t) = c_1 e^{2t} \cos(2\sqrt{2}t) + c_2 e^{2t} \sin(2\sqrt{2}t)$$

Esercizio 1.2.7. *Determinare la soluzione generale dell'equazione differenziale*

$$2y'' + 3y' + 4y = 0.$$

$2r^2 + 3r + 4 = 0$

$$r_{1,2} = \frac{-3 \pm \sqrt{9-32}}{4} = \frac{-3 \pm i\sqrt{23}}{4} = \frac{-3 \pm 4\sqrt{7}\,i}{4} = -\frac{3}{4} \pm \sqrt{7}\,i$$

$$y(t) = c_1 e^{-\frac{3}{4}t} \cos(\sqrt{7}t) + c_2 e^{-\frac{3}{4}t} \sin(\sqrt{7}t)$$

Esercizio 1.2.8. *Si risolva il seguente problema di Cauchy:*

$$y'' + 6y' + 8y = e^{4t} + t^2, \qquad y(1) = 2, \qquad y'(1) = 3.$$

Risolvo l'equazione omogenea associata

$r^2 + 6r + 8 = 0$

$(r+4)(r+2)$

$r_1 = -4 \quad r_2 = -2$

$$z(t) = c_1 e^{-4t} + c_2 e^{-2t}$$

Bisogna trovare una soluzione particolare:

$\bar{y}_1(t) = e^{\lambda t}\,\gamma(t) \qquad \lambda = 4 \quad A = 1$

$\Downarrow$

$$\lambda^2 + \lambda a + b = 16 + 4 + 6 = 26 \neq 0 \longrightarrow \gamma = \text{costante} = \frac{A}{\lambda^2 + \lambda a + b} = \frac{1}{26}$$

$$\bar{y}_1(t) = \frac{1}{26} e^{4t}$$

$$\bar{y}_2(t) = a_0 + a_1 x + a_2 x^2 = t^2$$

$$\bar{y}_2'(t) = a_1 + a_2 x$$

$$\bar{y}_2''(t) = a_2$$

$\xrightarrow{\text{sostituisco}}$ $y'' + 6y' + 8y = t^2$

$$a_2 + 6a_1 + 6a_2 x + 8a_0 + 8a_1 x + 8a_2 x^2 = t^2$$

$$a_2 + 6a_1 + 8a_0 + x(6a_2 + 8a_1) + 8a_2 x^2 = t^2$$

$$\begin{cases} a_2 + 6a_1 + 8a_0 = 0 \\ 6a_2 + 8a_1 = 0 \\ a_2 = \frac{1}{8} \end{cases} \quad \begin{cases} \frac{1}{8} - \frac{18}{32} + 8a_0 = 0 \\ a_1 = -\frac{3}{32} \\ a_2 = \frac{1}{8} \end{cases} \quad \begin{cases} a_0 = \frac{7}{128} \\ a_1 = -\frac{3}{32} \\ a_2 = \frac{1}{8} \end{cases}$$

$$\bar{y}_2(t) = \frac{7}{128} - \frac{3}{32} x + \frac{1}{8} x^2$$

$$\bar{y}(t) = \bar{y}_1 + \bar{y}_2 = \frac{1}{26} e^{4t} + \frac{7}{128} - \frac{3}{32} x + \frac{1}{8} x^2$$

✍ **Esercizio 1.2.10.** *Si determini la soluzione $y(t)$ del problema di Cauchy*

$$\begin{cases} y'' + y' - 2y = -e^x \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Risolviamo l'equazione omogenea associata

$$r^2 + r - 2 = 0$$

$$(r-1)(r+2)$$

$r_1 = -2 \quad r_2 = 1 \qquad Z(x) = C_1 e^{-2t} + C_2 e^t$

Bisogna trovare una soluzione particolare

$$\bar{y}(t) = A e^{\lambda t} \gamma(t) \qquad \lambda = 1 \quad A = -1$$

$$\downarrow$$

$$\gamma'' + \gamma'(2\lambda + a) + \gamma(\lambda^2 + \lambda a + b) = A$$

$$\gamma'' + \gamma'(\underset{\neq 0}{3}) + \gamma(\underset{=0}{0}) = A$$

$$\downarrow$$

$$\gamma(t) = \frac{At}{2\lambda + a} = \frac{-t}{2+1} = -\frac{1}{3}t$$

$$\bar{y}(t) = + e^t \frac{1}{3} t = t \frac{e^t}{3}$$

$$y(t) = C_1 e^{-2t} + C_2 e^t + t \frac{e^t}{3}$$

$$y'(t) = -2C_1 e^{-2t} + C_2 e^t + \frac{e^t}{3} + t \frac{e^t}{3}$$

$$\begin{cases} y(0) = C_1 + C_2 = 0 \\ y'(0) = -2C_1 + C_2 + \frac{1}{3} = 0 \end{cases} \rightarrow \begin{cases} C_1 = -C_2 \\ 2C_2 + C_2 = -\frac{1}{3} \end{cases} \quad \begin{cases} C_1 = \frac{1}{9} \\ C_2 = -\frac{1}{9} \end{cases}$$

$$y(t) = \frac{1}{9} e^{-2t} - \frac{1}{9} e^t + t \frac{e^t}{3}$$

Calcolo infinitesimale per le curve

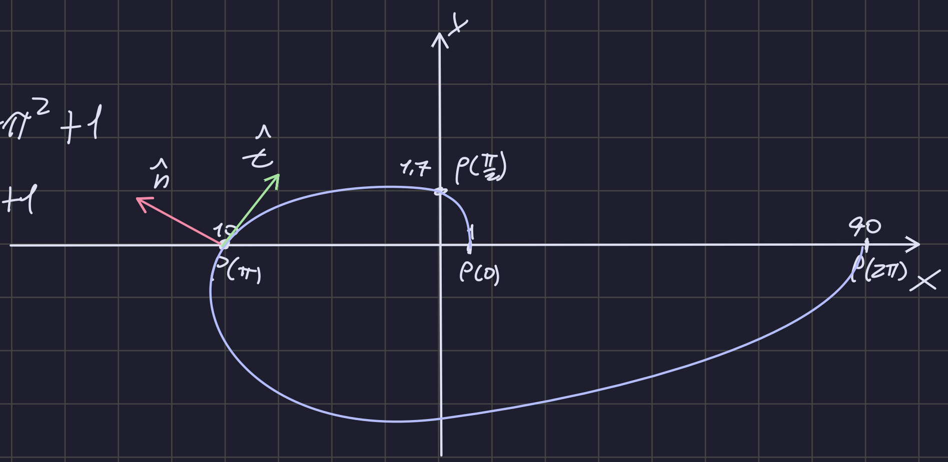
✍ **Esercizio 2.1.1.** *Sia γ la curva piana una cui parametrizzazione in coordinate polari è $\rho(\vartheta) = \vartheta^2 + 1$, on $0 \le \vartheta \le 2\pi$. Dopo aver disegnato sommariamente il sostegno di γ, determinare i versori tangente e normale al sostegno di γ nel punto $\gamma(\pi)$ e scrivere un'equazione della retta tangente nello stesso punto.*

$\rho(\theta) = \theta^2 + 1 \qquad \theta \in [0, 2\pi]$

$\rho(0) = 1$

$\rho(2\pi) = 4\pi^2 + 1$

$\rho(\pi) = \pi^2 + 1$



Trasformiamo in coordinate cartesiane

$$\begin{cases} x(\theta) = \rho(\theta)\cos\theta = (\theta^2+1)\cos\theta \\ y(\theta) = \rho(\theta)\sin\theta = (\theta^2+1)\sin\theta \end{cases}$$

$$\gamma(\theta) = (x(\theta), y(\theta))$$

$$\hat{t} = \frac{\gamma'(\theta)}{\|\gamma'(\theta)\|}$$

$$\gamma'(\theta) = \left(2\theta\cos\theta - (\theta^2+1)\sin\theta,\ 2\theta\sin\theta + (\theta^2+1)\cos\theta\right)$$

$$\|\gamma'(\theta)\| = \sqrt{x'(\theta)^2 + y'(\theta)^2}$$

$$= \sqrt{(2\theta\cos\theta - (\theta^2+1)\sin\theta)^2 + (2\theta\sin\theta + (\theta^2+1)\cos\theta)^2}$$

$$= \sqrt{4\theta^2\cos^2\theta + (\theta^2+1)^2\sin^2\theta - \cancel{4\theta\cos\theta(\theta^2+1)\sin\theta}}$$
$$\overline{+ 4\theta^2\sin^2\theta + (\theta^2+1)^2\cos^2\theta + \cancel{4\theta\sin\theta(\theta^2+1)\cos\theta}}$$

$$= \sqrt{4\theta^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} + (\theta^2+1)^2\underbrace{(\sin^2\theta + \cos^2\theta)}_{=1}}$$

$$= \sqrt{4\theta^2 + (\theta^2+1)^2}$$

$$\hat{t}(\theta) = \frac{\gamma'(\theta)}{\|\gamma'(\theta)\|} = \frac{(2\theta\cos\theta - (\theta^2+1)\sin\theta,\ 2\theta\sin\theta + (\theta^2+1)\cos\theta)}{\sqrt{4\theta^2 + (\theta^2+1)^2}}$$

$$\hat{t}(\pi) = \frac{\gamma'(\pi)}{\|\gamma'(\pi)\|} = \frac{(2\pi\cos\pi - (\pi^2+1)\sin\pi,\ 2\pi\sin\pi + (\pi^2+1)\cos\pi)}{\sqrt{4\pi^2 + (\pi^2+1)^2}}$$

$$= \frac{(-2\pi,\ -(\pi^2+1))}{\sqrt{4\pi^2 + (\pi^2+1)^2}}$$

La normale è semplicemente la tangente ruotata di 90°, e ciò equivale alla moltiplicazione della tangente con una matrice di rotazione:

$$\hat{n}(\theta) = \hat{t}(\theta) \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\| $$

$$\begin{bmatrix} \cos(\theta) & -\sin\theta \\ \sin(\theta) & \cos\theta \end{bmatrix}$$

Retta tangente:

$$r_T = \gamma(\pi) + \gamma'(\pi)\, t$$

$$\begin{cases} x(t) = -(\pi^2+1) - t\ 2\pi = -t\,2\pi - (\pi^2+1) \\ y(t) = 0 \quad -t(\pi^2+1) = -t(\pi^2+1) \end{cases}$$

$$y = -t(\pi^2+1) \qquad x = -t\,2\pi - (\pi^2+1)$$

$$-t\,2\pi = x + (\pi^2+1)$$

$$t = \frac{x + (\pi^2+1)}{-2\pi}$$

$$y = \frac{x + (\pi^2 + 1)}{2\pi} (\pi^2 + 1)$$

$$y = \frac{\pi^2 + 1}{2\pi} x + \frac{(\pi^2 + 1)^2}{2\pi}$$