Esercizi presi dall'eserciziario su moodle

Equazioni differenziali di primo grado

✍ **Esercizio 1.1.1.** *Si determini la soluzione $y(t)$ del seguente problema di Cauchy*

$$\begin{cases} y' = \dfrac{y^2}{y^2+4}t \\ y(0) = 2 \end{cases}$$

Inoltre si determini il valore $\alpha > 0$ per cui $\dfrac{y(t)}{t^\alpha}$ tende a un numero finito e non nullo per $t \to +\infty$.

$$\frac{y^2+4}{y^2} \cdot y' = t$$

$$\int \frac{y^2+4}{y^2} \cdot y' \, dt = \int t \, dt + C$$

$u = y$

$du = y' dt$

$\downarrow u = y$

$$\int \frac{y^2+4}{y^2} \, dy = \frac{t^2}{2} + C$$

$$\int 1 \, dy + 4 \int y^{-2} \, dy = \frac{t^2}{2} + C$$

$$y - \frac{4}{y} = \frac{t^2}{2} + C$$

$$\frac{y^2-4}{y} = \frac{t^2}{2} + C$$

$y(0) = 2$

$\downarrow$

$$\frac{2^2-4}{2} = \frac{0}{2} + C$$

$$\frac{0}{2} = C$$

$$C = 0 \to \frac{y^2-4}{y} = \frac{t^2}{2} \to \frac{y^2 - 4 - \frac{t^2 y}{2}}{y} = 0$$

$$y(t) = \frac{\frac{t^2}{2} \pm \sqrt{\frac{t^4}{4} + 16}}{2} = \frac{t^2 \pm \sqrt{t^4+64}}{4}$$

$$\downarrow$$

$$y(0) = 2$$

$$\downarrow$$

$$\frac{0^2 \pm \sqrt{64}}{4} = \begin{cases} +2 \\ -2 \end{cases}$$

(Non accettabile) $\to y(0)=2$

Soluzione 1:

$$y(t) = \frac{t^2 + \sqrt{t^4+64}}{4}$$

$$\lim_{t \to +\infty} \frac{y(t)}{t^\alpha} = \frac{t^2 + \sqrt{t^4+64}}{4t^\alpha} = \lim_{t \to +\infty} \frac{t^2 + t^2\overbrace{\sqrt{1+\frac{64}{t^4}}}^{1+\frac{64}{+\infty}=1}}{4t^\alpha} =$$

$$= \lim_{t \to +\infty} \frac{\cancel{2}t^2}{\cancel{4}t^\alpha} = \lim_{t \to +\infty} \frac{t^2}{2t^\alpha}$$

L'unico modo per avere un numero finito è che il grado del numeratore e del denominatore sia uguale, quindi:

$$\alpha = 2 \to \lim_{t \to +\infty} \frac{t^2}{2t^2} = \lim_{t \to +\infty} \frac{1}{2} = \frac{1}{2}$$

✍ **Esercizio 1.1.2.** *Si determini la soluzione $y(t)$ del seguente problema di Cauchy*

$$\begin{cases} y' = \dfrac{t^2+t}{2e^{2y}+6e^y} \\ y(0) = 0 \end{cases}$$

$$y' \cdot (2e^{2y} + 6e^y) = t^2 + t$$

$$\int y' \cdot (2e^{2y} + 6e^y)\,dt = \int t^2 + t \, dt + C$$

$u = y$

$du = y'dt$

$\downarrow u = y$

$$2\int e^{2y}\,dy + 6\int e^{y}\,dy = \frac{t^3}{3} + \frac{t^2}{2} + C$$

$$\cancel{2}\cdot\frac{e^{2y}}{\cancel{2}} + 6e^{y} = \frac{t^3}{3} + \frac{t^2}{2} + C$$

$$y(0)=0$$

$$e^{2\cdot 0} + 6e^{0} = \frac{0}{3} + \frac{0}{2} + C$$

$$1+6=C$$

$$C=7$$

$$\downarrow$$

$$e^{2y} + 6e^{y} = \frac{t^3}{3} + \frac{t^2}{2} + 7$$

$$x = e^{y} \rightarrow y = \ln(x)$$

$$x^2 + 6x - \frac{t^3}{3} - \frac{t^2}{2} - 7 = 0$$

$$x_{1,2} = \frac{-\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - ac}}{a} = -3 \pm \sqrt{\frac{t^3}{3} + \frac{t^2}{2} + 16}$$

$$e^{y} = -3 \pm \sqrt{\frac{t^3}{3} + \frac{t^2}{2} + 16}$$

$$y(0)=0$$

$$e^{0} = -3 \pm \sqrt{\frac{0}{3} + \frac{0}{3} + 16} = -3 \pm \sqrt{16} = -3 \pm 4 = \begin{cases} -7 & \text{Non accettabile} \\ 1 \end{cases}$$

$$y(t) = \ln\left(-3 + \sqrt{\frac{t^3}{3} + \frac{t^2}{2} + 16}\right)$$

$$\downarrow$$

$$y(0) = \ln(1) = 0 \checkmark$$

✍ **Esercizio 1.1.4.** *Sia $y(t)$ la soluzione del problema di Cauchy*

$$\begin{cases} y' = \dfrac{e^{-x}\sqrt{y+1}}{e^{-x}+1} \\ y(0) = 1 \end{cases}$$

$$y' = \frac{e^{-x}\sqrt{y+1}}{e^{-x}+1}$$

$$\frac{y'}{\sqrt{y+1}} = \frac{e^{-x}}{e^{-x}+1}$$

$$\int \frac{y'}{\sqrt{y+1}}\,dy = \int \frac{e^{-x}}{e^{-x}+1}\,dx$$

$y = u$

$y'dy = du$

$u = y$

$$-\int (y+1)^{-\frac{1}{2}}\,dy = \int \frac{-e^{-x}}{e^{-x}+1}\,dx$$

$$\begin{cases} e^{-x}+1 = F \\ F' = -e^{-x} \end{cases}$$

$$C + \frac{-(y+1)^{\frac{1}{2}}}{\frac{1}{2}} = \int \frac{F'}{F}\,dF$$

$$-2\sqrt{y+1} = \int \frac{dF}{F}$$

$$-2\sqrt{y+1} = \ln(F) + C$$

$$\sqrt{y+1} = -\frac{1}{2}\ln(e^{-x}+1) + C$$

$$y = \left(-\frac{1}{2}\ln(e^{-x}+1) + C\right)^2 - 1$$

$$y(0) = 1$$

$$\downarrow$$

$$1 = \left(-\frac{1}{2}\ln(2) + C\right)^2 - 1$$

$$2 = \left(-\frac{1}{2}\ln(2) + C\right)^2$$

$$\sqrt{2} = -\frac{1}{2}\ln(2) + c$$

$$c = \sqrt{2} + \frac{1}{2}\ln(2)$$

$$y = \left(-\frac{1}{2}\ln\left(e^{-t}+1\right) + \sqrt{2} + \frac{1}{2}\ln(2)\right)^2 - 1$$

✍ **Esercizio 1.1.5.** *Si determini la soluzione $y(t)$ del seguente problema di Cauchy*

$$\begin{cases} y' = (e^{-3y}+1)(2x-1) \\ y(0) = -1 \end{cases}$$

$$y' = (e^{-3y}+1)(2x-1)$$

$$\frac{y'}{e^{-3y}+1} = 2x-1$$

$$\int \frac{y'}{e^{-3y}+1}\,dy = \int 2x-1\,dx$$

$$\int \frac{1}{e^{-3y}+1}\,dy = 2\int x\,dx - \int 1\,dx$$

$$\int \frac{1}{e^{-3y}+1}\,dy = x^2 - x + C$$

$$\int \frac{1}{e^{-3y}\left(1+\frac{1}{e^{-3y}}\right)}\,dy = x^2 - x + C$$

$$\int \frac{e^{3y}}{1+e^{3y}}\,dy = x^2 - x + C \qquad \frac{d\left(1+e^{3y}\right)}{dy} = 3e^{3y}$$

$$3\int \frac{e^{3y}}{1+e^{3y}}\,dy = 3\left(x^2 - x + C\right)$$

$$\int \frac{3e^{3y}}{1+e^{3y}}\,dy = 3x^2-3x+C$$

$F = 1+e^{3y}$

$F'\,dF = 3e^{3y}$

$$\int \frac{F'}{F}\,dF = 3x^2-3x+C$$

$$\int \frac{1}{F}\,dF = 3x^2-3x+C$$

$$\ln(F) = 3x^2-3x+C$$

$$\ln(1+e^{3y}) = 3x^2-3x+C$$

$$e^{\ln(1+e^{3y})} = e^{3x^2-3x+c}$$

$$e^{3y} = e^{3x^2-3x+c} - 1$$

$$\ln(e^{3y}) = \ln(e^{3x^2-3x+c}-1)$$

$$3y = \ln(e^{3x^2-3x+c}-1)$$

$$y = \frac{\ln(e^{3x^2-3x+c}-1)}{3}$$

$e^{\sim} \cdot e^{c}$

$y(0) = -1$

$$-1 = \frac{\ln(e^{0+c}-1)}{3}$$

$$-3 = \ln(e^{c}-1)$$

$$e^{-3} = e^{c}-1$$

$$e^{c} = e^{-3}+1$$

$$\ln(e^{c}) = \ln(e^{-3}+1)$$

$$c = \ln\left(e^{-3}+1\right)$$

$$y = \frac{\ln\left(e^{3x^2-3x} \cdot e^{\ln(e^{-3}+1)} - 1\right)}{3}$$

$$y = \frac{\ln\left(\left(1+e^{-3}\right) e^{3x^2-3x} - 1\right)}{3}$$

$$y = \frac{1}{3} \ln\left(\left(1+e^{-3}\right) e^{3x^2-3x} - 1\right)$$

✍ **Esercizio 1.1.6.** *Si determini la soluzione $y(t)$ del seguente problema di Cauchy*

$$\begin{cases} y' = \left(3+27y^2\right)\left(xe^{3x}-2x^2\right) \\ y(0)=0 \end{cases}$$

$$y' = (3+27y^2)(xe^{3x}-2x^2)$$

$$\int \frac{y'}{3+27y^2}\,dy = \int xe^{3x} - 2x^2\,dx$$

$$\int \frac{1}{3+27y^2}\,dy = \int xe^{3x}\,dx - 2\int x^2\,dx$$

$$\frac{1}{3}\int \frac{1}{1+9y^2}\,dy = \underbrace{\int xe^{3x}\,dx} - 2\,\frac{x^3}{3} + C$$

$$\downarrow$$

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

$f: x \qquad f': 1$

$g': e^{3x} \qquad g: \frac{1}{3}e^{3x}$

$$\downarrow$$

$$\frac{1}{3}\int \frac{1}{1+9y^2}\,dy = \frac{1}{3}\,x\,e^{3x} - \frac{1}{3}\int e^{3x}\,dx \quad - 2\,\frac{x^3}{3} + C$$

$$\frac{1}{3}\int \frac{1}{1+9y^2}\,dy = \frac{1}{3}x\,e^{3x} - \frac{1}{9}e^{3x} - 2\frac{x^3}{3} + C$$

$$\int \underbrace{\frac{1}{1+9y^2}}\,dy = xe^{3x} - \frac{1}{3}e^{3x} - 2x^3 + C$$

$$\downarrow$$

$$\frac{1}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$$

$$\downarrow$$

$$\frac{1}{9}\int \frac{1}{y^2+\frac{1}{9}}\,dy = xe^{3x} - \frac{1}{3}e^{3x} - 2x^3 + C$$

$$\frac{1}{3}\arctan(3y) = xe^{3x} - \frac{1}{3}e^{3x} - 2x^3 + C$$

$$\arctan(3y) = 3xe^{3x} - e^{3x} - 6x^3 + C$$

$$\arctan(x) = a$$
$$\downarrow$$
$$x = \tan(a)$$

$$3y = \tan(3xe^{3x} - e^{3x} - 6x^3 + C)$$

$$y = \frac{1}{3}\tan(3xe^{3x} - e^{3x} - 6x^3 + C)$$

$$\tan(a) = 0$$

$$a = \arctan(0)$$

$$y(0) = 0$$

$$\downarrow$$

$$0 = \frac{1}{3}\tan(0 - 1 - 0 + C)$$

$$\tan(-1 + C) = 0$$

$$-1 + C = \arctan(0)$$

$$C = 1$$

$$y = \frac{1}{3}\tan(3xe^{3x} - e^{3x} - 6x^3 + 1)$$

Equazioni differenziali lineari di secondo grado

✍ **Esercizio 1.2.1.** *Si determini la soluzione $y(t)$ del seguente problema di Cauchy*

$$\begin{cases} y'' - 6y' + 9y = 3t + 2 \\ y(0) = -1 \\ y'(0) = 2 \end{cases}$$

$3t + 2$ è un polinomio di grado 1

$$y_1 = a_0 + a_1 x \xrightarrow{\text{Sostituisco}} -6a_1 + 9a_0 + 9a_1 x = 3x + 2$$

$$y_1' = a_1$$

$$-6a_1 + 9a_0 + 9a_1 x = 3x + 2$$

$$y'' = 0$$

$$\begin{cases} -6a_1 + 9a_0 = 2 \\ 9a_1 = 3 \end{cases} \quad \begin{cases} -2 + 9a_0 = 2 \\ a_1 = \frac{1}{3} \end{cases} \quad \begin{cases} a_0 = \frac{4}{9} \\ a_1 = \frac{1}{3} \end{cases} \rightarrow \quad y_1 = \frac{4}{9} + \frac{1}{3}t$$

Risolvo l'equazione omogenea

$$r'' - 6r' + 9r = 0$$

$$(r-3)^2$$

$$r_{1,2} = 3 \longrightarrow \quad y_2 = C_1 e^{3t} + C_2 t e^{3t}$$

$$y(t) = \frac{4}{9} + \frac{1}{3}t + C_1 e^{3t} + C_2 t e^{3t}$$

Applico le condizioni di Cauchy

$$y(0) = \frac{4}{9} + 0 + c_1 + 0 = \frac{4}{9} + c_1 = -1 \rightarrow \quad c_1 = -\frac{13}{9}$$

$$y'(t) = \frac{1}{3} + 3c_1 e^{3t} + c_2 e^{3t} + 3c_2 t e^{3t}$$

$$y'(0) = \frac{1}{3} + 3c_1 + c_2 = 2 \rightarrow \quad \frac{1}{3} + 3 \cdot \left(-\frac{13}{9}\right) + c_2 = 2$$

$$\frac{1}{3} - \frac{13}{3} + c_2 = 2$$

$$1 - 13 + 3c_2 = 6$$

$$3c_2 = 6 + 12$$

$$c_2 = \frac{18}{3} = 6$$

$$c_1 = -\frac{13}{9} \quad c_2 = 6$$

$$\downarrow$$

$$y(t) = \frac{4}{9} + \frac{1}{3}t + c_1 e^{3t} + c_2 t e^{3t} = \frac{4}{9} + \frac{1}{3}t - \frac{13}{9} e^{3t} + 6 t e^{3t}$$

✍ **Esercizio 1.2.2.** *Sia $y(t)$ la soluzione del problema di Cauchy*

$$\begin{cases} y'' + 2y' - 3y = 0 \\ y(0) = 0 \\ y'(0) = 1. \end{cases}$$

Allora $\lim_{t\to+\infty} y(t) =$

☐ 0;

☐ non esiste;

☐ $+\infty$;

☐ $-\infty$

Risolvo l'equazione caratteristica

$$r^2 + 2r - 3 = 0$$

$$(r+3)(r-1)$$

$$r_1 = -3 \quad r_2 = 1$$

$$y(t) = c_1 e^{-3t} + c_2 e^{t}$$

$$y'(t) = -3c_1 e^{-3t} + c_2 e^{t}$$

Impongo le condizioni di Cauchy

$$\begin{cases} y(0) = c_1 + c_2 = 0 \\ y'(0) = -3c_1 + c_2 = 1 \end{cases} \rightarrow \begin{cases} c_1 = -c_2 \\ 3c_2 + c_2 = 1 \end{cases} \quad \begin{cases} c_1 = -\frac{1}{4} \\ c_2 = \frac{1}{4} \end{cases}$$

$$y(t) = -\frac{1}{4}e^{-3t} + \frac{1}{4}e^{t}$$

$$\lim_{t \to +\infty} y(t) = -\frac{1}{4}e^{-\infty} + \frac{1}{4}e^{\infty} = 0 + \infty = +\infty$$

La risposta corretta è la terza

✍ **Esercizio 1.2.3.** *Si determini la soluzione $y(t)$ del problema di Cauchy*

$$\begin{cases} y'' - y' - 2y = \cos(2t) \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

Risolvo l'equazione omogenea associata

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1)$$

$$r_1 = -1 \quad r_2 = 2 \quad \to \quad z(t) = c_1 e^{-t} + c_2 e^{2t}$$

Con il metodo di somiglianza cerco una soluzione particolare dell'equazione non omogenea:

$$\bar{y}(t) = \alpha \sin(2t) + \beta \cos(2t)$$

$$\bar{y}'(t) = 2\alpha \cos(2t) - 2\beta \sin(2t)$$

$$\bar{y}''(t) = -4\alpha \sin(2t) - 4\beta \cos(2t)$$

$$y'' - y' - 2y = \cos(2t)$$

$$\downarrow$$

$$-4\alpha \sin(2t) - 4\beta\cos(2t) - 2\alpha\cos(2t) + 2\beta \sin(2t) - 2\alpha \sin(2t) - 2\beta\cos(2t) = \cos(2t)$$

$$\sin(2t)(-4\alpha + 2\beta - 2\alpha) + \cos(2t)(-4\beta - 2\alpha - 2\beta) = \cos(2t)$$

$$\sin(2t)(-6\alpha + 2\beta) + \cos(2t)(-6\beta - 2\alpha) = \cos(2t)$$

$$\begin{cases} -6\alpha + 2\beta = 0 \\ -6\beta - 2\alpha = 1 \end{cases} \qquad \begin{cases} \beta = 3\alpha \\ -18\alpha - 2\alpha = 1 \end{cases} \qquad \begin{cases} \beta = -\frac{3}{20} \\ \alpha = -\frac{1}{20} \end{cases}$$

$$\bar{y}(t) = -\frac{1}{20}\sin(2t) - \frac{3}{20}\cos(2t)$$

$$y(t) = z(t) + \bar{y}(t) = c_1 e^{-t} + c_2 e^{2t} - \frac{1}{20}\sin(2t) - \frac{3}{20}\cos(2t)$$

$$y'(t) = -c_1 e^{-t} + 2c_2 e^{2t} - \frac{1}{10}\cos(2t) + \frac{3}{10}\sin(2t)$$

Applico le condizioni di Cauchy

$$\begin{cases} y(0) = c_1 + c_2 - \frac{3}{20} = 1 \\ y'(0) = -c_1 + 2c_2 - \frac{1}{10} = 0 \end{cases} \rightarrow \begin{cases} c_1 = -c_2 + \frac{23}{20} \\ -c_1 + 2c_2 = \frac{1}{10} \end{cases} \quad \begin{cases} c_1 = -c_2 + \frac{23}{20} \\ c_2 - \frac{23}{20} + 2c_2 = \frac{1}{10} \end{cases}$$

$$\begin{cases} c_1 = -c_2 + \frac{23}{20} \\ 3c_2 = \frac{2}{20} + \frac{23}{20} \end{cases} \quad \begin{cases} c_1 = -\frac{5}{12} + \frac{23}{20} \\ c_2 = \frac{5}{12} \end{cases} \quad \begin{cases} c_1 = \frac{-25+69}{60} = \frac{44}{60} = \frac{11}{15} \\ c_2 = \frac{5}{12} \end{cases}$$

$$y(t) = \frac{11}{15} e^{-t} + \frac{5}{12} e^{2t} - \frac{1}{20}\sin(2t) - \frac{3}{20}\cos(2t)$$

✍ **Esercizio 1.2.4.** *Si determini la soluzione $y(t)$ del problema di Cauchy*

$$\begin{cases} y'' - 4y' + 8y = e^{-2t} \\ y(0) = -1 \\ y'(0) = 0. \end{cases}$$

Risolvo l'equazione omogenea associata

$$r^2 - 4r + 8 = 0$$

$$r_{1,2} = \frac{4 \pm \sqrt{16-32}}{2} = \frac{4 \pm i\sqrt{16}}{2} = \frac{4 \pm 4i}{2} = 2 \pm 2i$$

$r_1 = 2 - 2i \qquad r_2 = 2 + 2i$

$$z(t) = c_1 e^{2t}\sin(2t) + c_2 e^{2t}\cos(2t)$$

Bisogna trovare una soluzione particolare del tipo:

$$A = 1$$
$$\lambda = -2$$

$$\bar{y}(t) = e^{-2t}\gamma(t)$$

$$\downarrow$$

$$\gamma'' + \gamma'(2(-2) - 4) + \gamma(4 + 8 + 8) = 1$$

$$y'' - 8y' + 20y = 1$$

$$\lambda^2 + \lambda a + b \neq 0 \rightarrow y(t) = \text{costante} = \frac{A}{\lambda^2 + \lambda a + b} = \frac{1}{20}$$

$$\bar{y}(t) = \frac{1}{20} e^{-2t}$$

$$y(t) = z(t) + \bar{y}(t) = c_1 e^{2t} \sin(2t) + c_2 e^{2t} \cos(2t) + \frac{1}{20} e^{-2t}$$

$$y'(t) = 2c_1 e^{2t} \sin(2t) + 2c_1 e^{2t} \cos(2t) + 2c_2 e^{2t} \cos(2t) - 2c_2 e^{2t} \sin(2t) - \frac{1}{10} e^{-2t}$$

$$\begin{cases} y(0) = c_2 + \frac{1}{20} = -1 \\ y'(0) = 2c_1 + 2c_2 - \frac{1}{10} = 0 \end{cases} \rightarrow \begin{cases} c_2 = -\frac{21}{20} \cdot 2 \\ c_1 = \frac{11}{10} \end{cases}$$

$$y(t) = \frac{11}{10} e^{2t} \sin(2t) - \frac{21}{20} e^{2t} \cos(2t) + \frac{1}{20} e^{-2t}$$

✍ **Esercizio 1.2.5.** *Si determini la soluzione $y(t)$ del problema di Cauchy*

$$\begin{cases} y'' - y' - 2y = \sin(2t) \\ y(0) = 0 \\ y'(0) = 1. \end{cases}$$

Risolvo l'equazione omogenea associata

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1)$$

$$r_1 = -1 \quad r_2 = 2$$

$$z(t) = c_1 e^{-t} + c_2 e^{2t}$$

Bisogna trovare una soluzione particolare del tipo:

$$\bar{y}(t) = \alpha \sin(2t) + \beta \cos(2t)$$

$$\bar{y}'(t) = 2\alpha \cos(2t) - 2\beta \sin(2t)$$

$$\bar{y}''(t) = -4\alpha \sin(2t) - 4\beta \cos(2t)$$

$$y'' - y' - 2y = \sin(2t)$$

$$\downarrow$$

$$-4\alpha \sin(2t) - 4\beta\cos(2t) - 2\alpha\cos(2t) + 2\beta\sin(2t) - 2\alpha\sin(2t) - 2\beta\cos(2t) = \sin(2t)$$

$$\sin(2t)\left(-4\alpha + 2\beta - 2\alpha\right) + \cos(2t)\left(-4\beta - 2\alpha - 2\beta\right) = \sin(2t)$$

$$\sin(2t)\left(-6\alpha + 2\beta\right) + \cos(2t)\left(-6\beta - 2\alpha\right) = \sin(2t)$$

$$\begin{cases} -6\alpha + 2\beta = 1 \\ -6\beta - 2\alpha = 0 \end{cases} \quad \begin{cases} -6\alpha - \frac{2}{3}\alpha = 1 \\ \beta = -\frac{1}{3}\alpha \end{cases} \quad \begin{cases} -\frac{20}{3}\alpha = 1 \\ \beta = -\frac{1}{3}\alpha \end{cases} \quad \begin{cases} \alpha = -\frac{3}{20} \\ \beta = +\frac{1}{20} \end{cases}$$

$$\bar{y}(t) = -\frac{3}{20}\sin(2t) + \frac{1}{20}\cos(2t)$$

$$y(t) = c_1 e^{-t} + c_2 e^{2t} - \frac{3}{20}\sin(2t) + \frac{1}{20}\cos(2t)$$

$$y'(t) = -c_1 e^{-t} + 2c_2 e^{2t} - \frac{3}{10}\cos(2t) - \frac{1}{10}\sin(2t)$$

Impongo le condizioni di Cauchy

$$\begin{cases} y(0) = c_1 + c_2 + \frac{1}{20} = 0 \\ y'(0) = -c_1 + 2c_2 - \frac{3}{10} = 1 \end{cases} \rightarrow \begin{cases} c_1 = -c_2 - \frac{1}{20} \\ c_2 + \frac{1}{20} + 2c_2 - \frac{3}{10} = 1 \end{cases} \quad \begin{cases} c_1 = -c_2 - \frac{1}{20} \\ 3c_2 - \frac{5}{20} = 1 \end{cases}$$

$$\begin{cases} c_1 = -\frac{25}{60} - \frac{1}{20} = \frac{-25-3}{60} = -\frac{28}{60} = -\frac{7}{15} \\ c_2 = \frac{25}{60} = \frac{5}{12} \end{cases}$$

$$y(t) = -\frac{7}{15}e^{-t} + \frac{5}{12}e^{2t} - \frac{3}{20}\sin(2t) + \frac{1}{20}\cos(2t)$$

✍ **Esercizio 1.2.6.** *Determinate la soluzione generale dell'equazione differenziale* $y'' - 4y' + 13y = 4x$.

Risolvo l'equazione omogenea associata

$$r^2 - 4r + 13 = 0$$

$$r_{1,2} = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm i\sqrt{24^2}}{2} = \frac{4 \pm 4i\sqrt{2}}{2} = 2 \pm 2\sqrt{2}\,i$$

$$z(t) = c_1 e^{2t} \cos(2\sqrt{2}t) + c_2 e^{2t} \sin(2\sqrt{2}t)$$

✍ **Esercizio 1.2.7.** *Determinare la soluzione generale dell'equazione differenziale*

$$2y'' + 3y' + 4y = 0.$$

$$2r^2 + 3r + 4 = 0$$

$$r_{1,2} = \frac{-3 \pm \sqrt{9-32}}{4} = \frac{-3 \pm i\sqrt{23}}{4} = \frac{-3 \pm 4\sqrt{7}\,i}{4} = -\frac{3}{4} \pm \sqrt{7}\,i$$

$$y(t) = c_1 e^{-\frac{3}{4}t} \cos(\sqrt{7}t) + c_2 e^{-\frac{3}{4}t} \sin(\sqrt{7}t)$$

✍ **Esercizio 1.2.8.** *Si risolva il seguente problema di Cauchy:*

$$y'' + 6y' + 8y = e^{4t} + t^2, \qquad y(1) = 2, \qquad y'(1) = 3.$$

Risolvo l'equazione omogenea associata

$$r^2 + 6r + 8 = 0$$

$$(r+4)(r+2)$$

$$r_1 = -4 \quad r_2 = -2$$

$$z(t) = c_1 e^{-4t} + c_2 e^{-2t}$$

Bisogna trovare una soluzione particolare:

$$\bar{y}_1(t) = e^{\lambda t}\, \gamma(t) \qquad \lambda = 4 \quad A = 1$$

$$\Downarrow$$

$$\lambda^2 + \lambda a + b = 16 + 4 + 6 = 26 \neq 0 \longrightarrow \gamma = \text{costante} = \frac{A}{\lambda^2 + \lambda a + b} = \frac{1}{26}$$

$$\bar{y}_1(t) = \frac{1}{26} e^{4t}$$

$$\bar{y}_2(t) = a_0 + a_1 x + a_2 x^2 = t^2$$

$$\bar{y}_2'(t) = a_1 + a_2 x \qquad \xrightarrow{\text{sostituisco}} \qquad y'' + 6y' + 8y = t^2$$

$$\bar{y}_2''(t) = a_2$$

$$a_2 + 6a_1 + 6a_2 x + 8a_0 + 8a_1 x + 8a_2 x^2 = t^2$$

$$a_2 + 6a_1 + 8a_0 + x(6a_2 + 8a_1) + 8a_2 x^2 = t^2$$

$$\begin{cases} a_2 + 6a_1 + 8a_0 = 0 \\ 6a_2 + 8a_1 = 0 \\ a_2 = \frac{1}{8} \end{cases} \qquad \begin{cases} \frac{1}{8} - \frac{18}{32} + 8a_0 = 0 \\ a_1 = -\frac{3}{32} \\ a_2 = \frac{1}{8} \end{cases} \qquad \begin{cases} a_0 = \frac{7}{128} \\ a_1 = -\frac{3}{32} \\ a_2 = \frac{1}{8} \end{cases}$$

$$\bar{y}_2(t) = \frac{7}{128} - \frac{3}{32} x + \frac{1}{8} x^2$$

$$\bar{y}(t) = \bar{y}_1 + \bar{y}_2 = \frac{1}{26} e^{4t} + \frac{7}{128} - \frac{3}{32} x + \frac{1}{8} x^2$$

✍ **Esercizio 1.2.10.** *Si determini la soluzione $y(t)$ del problema di Cauchy*

$$\begin{cases} y'' + y' - 2y = -e^x \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Risolviamo l'equazione omogenea associata

$$r^2 + r - 2 = 0$$

$$(r-1)(r+2)$$

$$r_1 = -2 \qquad r_2 = 1 \qquad\qquad z(x) = c_1 e^{-2t} + c_2 e^{t}$$

Bisogna trovare una soluzione particolare

$$\bar{y}(t) = A e^{\lambda t} \gamma(t) \qquad \lambda = 1 \quad A = -1$$

$$\downarrow$$

$$\gamma'' + \gamma'(2\lambda + a) + \gamma(\lambda^2 + \lambda a + b) = A$$

$$\gamma'' + \gamma' \underset{\neq 0}{(3)} + \gamma \underset{= 0}{(0)} = A$$

$$\downarrow$$

$$\gamma(t) = \frac{At}{2\lambda + a} = \frac{-t}{2+1} = -\frac{1}{3} t$$

$$\bar{y}(t) = + e^t \frac{1}{3} t = t \frac{e^t}{3}$$

$$y(t) = c_1 e^{-2t} + c_2 e^t + t \frac{e^t}{3}$$

$$y'(t) = -2c_1 e^{-2t} + c_2 e^t + \frac{e^t}{3} + t \frac{e^t}{3}$$

$$\begin{cases} y(0) = c_1 + c_2 = 0 \\ y'(0) = -2c_1 + c_2 + \frac{1}{3} = 0 \end{cases} \rightarrow \begin{cases} c_1 = -c_2 \\ 2c_2 + c_2 = -\frac{1}{3} \end{cases} \quad \begin{cases} c_1 = \frac{1}{9} \\ c_2 = -\frac{1}{9} \end{cases}$$

$$y(t) = \frac{1}{9} e^{-2t} - \frac{1}{9} e^t + t \frac{e^t}{3}$$

Calcolo infinitesimale per le curve

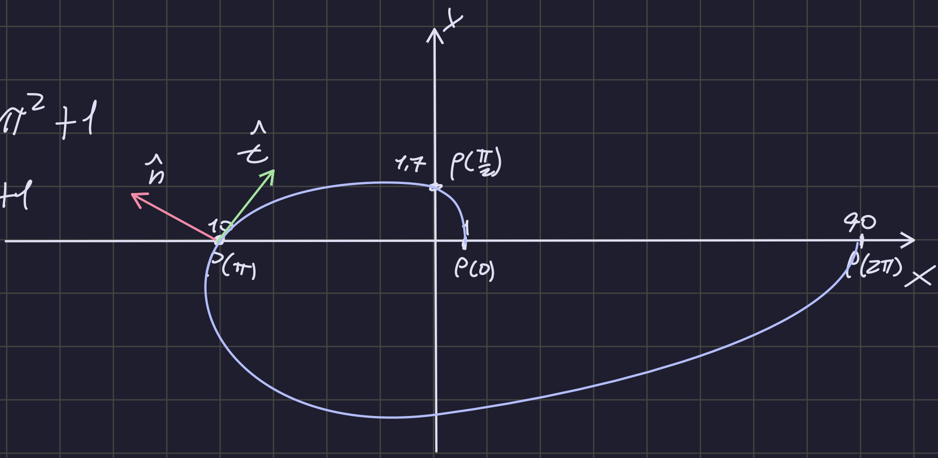
✍ **Esercizio 2.1.1.** *Sia γ la curva piana una cui parametrizzazione in coordinate polari è $\rho(\vartheta) = \vartheta^2 + 1$, on $0 \le \vartheta \le 2\pi$. Dopo aver disegnato sommariamente il sostegno di γ, determinare i versori tangente e normale al sostegno di γ nel punto $\gamma(\pi)$ e scrivere un'equazione della retta tangente nello stesso punto.*

$\rho(\theta) = \theta^2 + 1 \qquad \theta \in [0, 2\pi]$

$\rho(0) = 1$

$\rho(2\pi) = 4\pi^2 + 1$

$\rho(\pi) = \pi^2 + 1$



Trasformiamo in coordinate cartesiane

$$\begin{cases} x(\theta) = \rho(\theta)\cos\theta = (\theta^2+1)\cos\theta \\ y(\theta) = \rho(\theta)\sin\theta = (\theta^2+1)\sin\theta \end{cases}$$

$$\gamma(\theta) = (x(\theta), y(\theta))$$

$$\hat{t} = \frac{\gamma'(\theta)}{\|\gamma'(\theta)\|}$$

$$\gamma'(\theta) = \left(2\theta\cos\theta - (\theta^2+1)\sin\theta,\ 2\theta\sin\theta + (\theta^2+1)\cos\theta\right)$$

$$\|\gamma'(\theta)\| = \sqrt{x'(\theta)^2 + y'(\theta)^2}$$

$$= \sqrt{(2\theta\cos\theta - (\theta^2+1)\sin\theta)^2 + (2\theta\sin\theta + (\theta^2+1)\cos\theta)^2}$$

$$= \sqrt{4\theta^2\cos^2\theta + (\theta^2+1)^2\sin^2\theta - \cancel{4\theta\cos\theta(\theta^2+1)\sin\theta}}$$

$$\overline{+ 4\theta^2\sin^2\theta + (\theta^2+1)^2\cos^2\theta + \cancel{4\theta\sin\theta(\theta^2+1)\cos\theta}}$$

$$= \sqrt{4\theta^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} + (\theta^2+1)^2\underbrace{(\sin^2\theta + \cos^2\theta)}_{=1}}$$

$$= \sqrt{4\theta^2 + (\theta^2+1)^2}$$

$$\hat{t}(\theta) = \frac{\gamma'(\theta)}{\|\gamma'(\theta)\|} = \frac{(2\theta\cos\theta - (\theta^2+1)\sin\theta \,,\; 2\theta\sin\theta + (\theta^2+1)\cos\theta)}{\sqrt{4\theta^2+(\theta^2+1)^2}}$$

$$\hat{t}(\pi) = \frac{\gamma'(\pi)}{\|\gamma'(\pi)\|} = \frac{(2\pi\cos\pi - (\pi^2+1)\sin\pi \,,\; 2\pi\sin\pi + (\pi^2+1)\cos\pi)}{\sqrt{4\pi^2+(\pi^2+1)^2}}$$

$$= \frac{(-2\pi \,,\, -(\pi^2+1))}{\sqrt{4\pi^2+(\pi^2+1)^2}}$$

La normale è semplicemente la tangente ruotata di 90°, e ciò equivale alla moltiplicazione della tangente con una matrice di rotazione:

$$\hat{n}(\theta) = \hat{t}(\theta) \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\parallel$$

$$\begin{bmatrix} \cos(\theta) & -\sin\theta \\ \sin(\theta) & \cos\theta \end{bmatrix}$$

Retta tangente:

$$r_T = \gamma(\pi) + \gamma'(\pi)\, t$$

$$\begin{cases} x(t) = -(\pi^2+1) - t\, 2\pi = -t\,2\pi - (\pi^2+1) \\ y(t) = 0 \quad -t(\pi^2+1) = -t(\pi^2+1) \end{cases}$$

$$y = -t(\pi^2+1) \qquad x = -t\,2\pi - (\pi^2+1)$$

$$-t\,2\pi = x + (\pi^2+1)$$

$$t = \frac{x+(\pi^2+1)}{-2\pi}$$

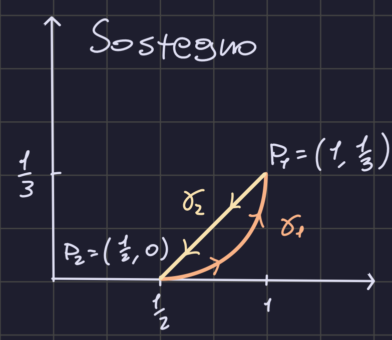
$$y = \frac{x + (\pi^2+1)}{2\pi}(\pi^2+1)$$

$$y = \frac{\pi^2+1}{2\pi}x + \frac{(\pi^2+1)^2}{2\pi}$$

✍ **Esercizio 2.1.2.** *Determinare una parametrizzazione della curva chiusa γ che si ottiene percorrendo prima da sinistra verso destra il grafico di $f(x) = (1/3)(2x-1)^{3/2}$ per $1/2 \leq x \leq 1$ e poi da destra a sinistra il segmento congiungente gli estremi del grafico di f stessa. Disegnare quindi il sostegno di γ e calcolarne la lunghezza.*

$\gamma_1 = \frac{1}{3}(2x-1)^{\frac{3}{2}}$ $\quad \gamma_1' = (2x-1)^{\frac{1}{2}}$ $\quad \gamma_1'' = (2x-1)^{-\frac{1}{2}} = \frac{1}{\sqrt{2x-1}} > 0$ concavità verso l'alto

$\gamma_1(\frac{1}{2}) = 0 \quad \gamma_1(1) = \frac{1}{3}$



$$\gamma_1(t) = \left(t, \frac{1}{3}(2t-1)^{\frac{3}{2}}\right) \quad t \in [\tfrac{1}{2}, 1]$$

$$\gamma_2(t) = (1-t)P_1 + tP_2$$
$$= (1-t)\begin{pmatrix}1\\ 1/3\end{pmatrix} + t\begin{pmatrix}\frac{1}{2}\\ 0\end{pmatrix}$$
$$= \left(1-t+\frac{1}{2}t, \frac{1}{3}(1-t)\right)$$
$$= \left(1-\frac{1}{2}t, \frac{1}{3}(1-t)\right) \quad t \in [0,1]$$

La parametrizzazione per t non è coerente, quindi cerco una parametrizzazione che vada da 0 a 1 per rappresentare entrambi i pezzi della curva come una sola curva, quindi da [0,1/2) per la prima e da [1/2,1] per la seconda

$$\gamma_1(t) = \left(t, \frac{1}{3}(2t-1)^{\frac{3}{2}}\right) \quad t \in [\tfrac{1}{2}, 1]$$
$$\downarrow$$
$$t = As + B$$

$$\begin{cases} \frac{1}{2} = A \cdot 0 + B \\ 1 = A \cdot \frac{1}{2} + B \end{cases} \quad \begin{cases} B = \frac{1}{2} \\ \frac{1}{2}A = 1 - \frac{1}{2} \end{cases} \quad \begin{cases} B = \frac{1}{2} \\ A = 1 \end{cases} \quad t = s + \frac{1}{2}$$

$$\gamma_1(s) = \left(s+\frac{1}{2}, \frac{1}{3}\left(2\left(s+\frac{1}{2}\right)-1\right)^{\frac{3}{2}} \right)$$

$$= \left(s+\frac{1}{2}, \frac{1}{3}(2s)^{\frac{3}{2}} \right) \quad s \in \left[0, \frac{1}{2}\right] \xrightarrow{\text{controllo}} \begin{cases} \gamma_1(0) = \left(\frac{1}{2}, \frac{1}{3}\left(2\cdot\frac{1}{2}\right)^{\frac{3}{2}} \right) \\ \quad = \left(\frac{1}{2}, 0 \right) \\ \gamma_1\left(\frac{1}{2}\right) = \left(\frac{1}{2}+\frac{1}{2}, \frac{1}{3}\left(2\cdot\frac{1}{2}\right)^{\frac{3}{2}} \right) \\ \quad = \left(1, \frac{1}{3} \right) \end{cases}$$

La parametrizzazione è corretta

$$\gamma_2(t) = \left(1-\frac{1}{2}t, \frac{1}{3}(1-t) \right) \quad t \in [0,1]$$

$$\downarrow$$

$$t = As+B$$

$$\begin{cases} 0 = A\frac{1}{2}+B \\ 1 = A\cdot 1+B \end{cases} \quad \begin{cases} B = -\frac{1}{2}A \\ A-\frac{1}{2}A = 1 \end{cases} \quad \begin{cases} B = -\frac{1}{2}A \\ \frac{1}{2}A = 1 \end{cases} \quad \begin{cases} B = -1 \\ A = 2 \end{cases} \qquad t = 2s-1$$

$$\gamma_2(s) = \left(1-\frac{1}{2}(2s-1), \frac{1}{3}(1-(2s-1)) \right)$$

$$= \left(\frac{3}{2}-s, \frac{1}{3}(-2s+2) \right) \quad s \in \left[\frac{1}{2}, 1\right] \xrightarrow{\text{controllo}} \begin{cases} \gamma_2\left(\frac{1}{2}\right) = \left(\frac{3}{2}-\frac{1}{2}, \frac{1}{3}\left(-2\cdot\frac{1}{2}+2\right) \right) \\ \quad = \left(1, \frac{1}{3} \right) \\ \gamma_2(1) = \left(\frac{3}{2}-1, \frac{1}{3}(-2+2) \right) \\ \quad = \left(\frac{1}{2}, 0 \right) \end{cases}$$

La parametrizzazione è corretta

$$\gamma(s) = \begin{cases} \gamma_1 & \text{se } s \in \left[0, \frac{1}{2}\right] \\ \gamma_2 & \text{se } s \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} \left(s+\frac{1}{2}, \frac{1}{3}(2s)^{\frac{3}{2}} \right) & s \in \left[0, \frac{1}{2}\right] \\ \left(\frac{3}{2}-s, \frac{1}{3}(-2s+2) \right) & s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

https://www.desmos.com/calculator/4wdnbtbbdt

La lunghezza di gamma è calcolata come:

$$\ell(\gamma(s)) = \int_0^1 \|\gamma'(s)\|\,ds$$

Perchè una curva composta
da più curve rettificabili
che soddisfano la condizione
di raccordo è rettificabile

$$= \ell(\gamma_1(s)) + \ell(\gamma_2(s)) = \int_0^{\frac{1}{2}} \|\gamma_1'(s)\|\,ds + \int_{\frac{1}{2}}^1 \|\gamma_2'(s)\|\,ds$$

$$\gamma_1'(s) = \left(1, \frac{1}{3}\cdot\frac{3}{2}(2s)^{\frac{1}{2}}\cdot 2\right) = \left(1, (2s)^{\frac{1}{2}}\right)$$

$$\gamma_2'(s) = \left(-1, -\frac{2}{3}\right)$$

$$\ell(\gamma_1(s)) = \int_0^{\frac{1}{2}} \|\gamma_1'(s)\|\, ds =$$

$$= \int_0^{\frac{1}{2}} \sqrt{1^2 + (2s)^{\frac{1}{2}\cdot 2}}\; ds$$

$$= \int_0^{\frac{1}{2}} (1+2s)^{\frac{1}{2}}\, ds$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{1}{2}} = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^{\frac{1}{2}}$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}} + 0$$

$$= \frac{2}{3}\sqrt[3]{\frac{1}{4}}$$

$$\ell(\gamma_2(s) = \int_{\frac{1}{2}}^{1} \|\gamma_2'(s)\|\, ds$$

$$= \int_{\frac{1}{2}}^{1} \sqrt{(-1)^2 + \left(-\frac{2}{3}\right)^2}\; ds$$

$$= \sqrt{1 + \frac{4}{9}}$$

$$= \sqrt{\frac{13}{9}}$$

$$= \frac{\sqrt{13}}{3}$$

$$\ell(\gamma(s)) = \ell(\gamma_1(s) + \ell(\gamma_2(s)) = \frac{2}{3}\sqrt[3]{\frac{1}{4}} + \frac{\sqrt{13}}{3}$$

✍ **Esercizio 2.1.3.** *Data la curva γ avente equazione in coordinate polari $\rho = 2\theta^2$ con $-\pi \leq \theta \leq \pi$, determinate la lughezza di γ; determinate poi un versore tangente alla curva nel punto corrispondente a $\theta = \varepsilon$ e calcolate il limite per $\varepsilon \to 0^+$ di questo versore.*

$$\rho(\theta) = 2\theta^2 \qquad \theta \in [-\pi, \pi]$$

$$L(\rho(\theta)) = \int_{-\pi}^{\pi} \sqrt{\rho'(\theta)^2 + \rho(\theta)^2}\, d\theta$$

$$= \int_{-\pi}^{\pi} \sqrt{(4\theta)^2 + (2\theta^2)^2}\, d\theta$$

$$= \int_{-\pi}^{\pi} \sqrt{16\theta^2 + 4\theta^4}\, d\theta$$

$$= \int_{-\pi}^{\pi} \sqrt{4\theta^2(4+\theta^2)}\, d\theta$$

$$= \int_{-\pi}^{\pi} 2|\theta|(4+\theta^2)^{\frac{1}{2}}\, d\theta \qquad t = 4+\theta^2$$

$$dt = 2\theta\, d\theta$$

$$= 2\int_{0}^{\pi} 2\theta\, (4+\theta^2)^{\frac{1}{2}}\, d\theta$$

$$= 2\int (t)^{\frac{1}{2}}\, dt$$

$$= \frac{4}{3} t^{\frac{3}{2}} = \left[\frac{4}{3}(4+\theta^2)^{\frac{3}{2}}\right]_0^{\pi}$$

$$= \frac{4}{3}(4+\pi^2)^{\frac{3}{2}} - \left(\frac{4}{3}(4)^{\frac{3}{2}}\right)$$

$$= \frac{4}{3}(4+\pi^2)^{\frac{3}{2}} - \frac{4}{3}(4)^{\frac{3}{2}}$$

$$= \frac{4}{3}(4+\pi^2)^{\frac{3}{2}} - \frac{4}{3}\sqrt{4^2 \cdot 4}$$

$$= \frac{4}{3}(4+\pi^2)^{\frac{3}{2}} - \frac{4}{3}\, 4 \cdot 2$$

$$= \frac{4}{3}(4+\pi^2)^{\frac{3}{2}} - \frac{32}{3}$$

Per trovare il versore tangente bisogna calcolare la derivata nel punto ε in coordinate cartesiane

$$\rho(t) = (2t^2\cos t,\ 2t^2\sin t)$$

$$\rho'(t) = (4t\cos t - 2t^2\sin t,\ 4t\sin t + 2t^2\cos t)$$

$$\rho'(\varepsilon) = (4\varepsilon\cos\varepsilon - 2\varepsilon^2\sin\varepsilon,\ 4\varepsilon\sin\varepsilon + 2\varepsilon^2\cos\varepsilon)$$

Per ottenere il versore bisogna normalizzare

$$\hat{\rho}'(\varepsilon) = \frac{\rho'(\varepsilon)}{\|\rho'(\varepsilon)\|} = \frac{(4\varepsilon\cos\varepsilon - 2\varepsilon^2\sin\varepsilon,\ 4\varepsilon\sin\varepsilon + 2\varepsilon^2\cos\varepsilon)}{\sqrt{16\varepsilon^2 + 4\varepsilon^4}}$$

$$= \frac{(4\varepsilon\cos\varepsilon - 2\varepsilon^2\sin\varepsilon,\ 4\varepsilon\sin\varepsilon + 2\varepsilon^2\cos\varepsilon)}{2|\varepsilon|(4+\varepsilon^2)^{\frac{1}{2}}}$$

$$= \left(\frac{4\varepsilon\cos\varepsilon - 2\varepsilon^2\sin\varepsilon}{2|\varepsilon|(4+\varepsilon^2)^{\frac{1}{2}}},\ \frac{4\varepsilon\sin\varepsilon + 2\varepsilon^2\cos\varepsilon}{2|\varepsilon|(4+\varepsilon^2)^{\frac{1}{2}}}\right)$$

se $\varepsilon > 0$

$$= \left(\frac{\cancel{4\varepsilon}\cos\varepsilon - 2\varepsilon^{\cancel{2}}\sin\varepsilon}{\cancel{2\varepsilon}(4+\varepsilon^2)^{\frac{1}{2}}},\ \frac{\cancel{4\varepsilon}\sin\varepsilon + 2\varepsilon^{\cancel{2}}\cos\varepsilon}{\cancel{2\varepsilon}(4+\varepsilon^2)^{\frac{1}{2}}}\right)$$

$$= \left(\frac{2\cos\varepsilon - 2\varepsilon\sin\varepsilon}{(4+\varepsilon^2)^{\frac{1}{2}}},\ \frac{2\sin\varepsilon + 2\varepsilon\cos\varepsilon}{(4+\varepsilon^2)^{\frac{1}{2}}}\right)$$

$$\lim_{\varepsilon\to 0^+} \rho'(\varepsilon) = \left(\frac{2\cos 0 - 2\cdot 0\sin 0}{(4+0^2)^{\frac{1}{2}}},\ \frac{2\sin 0 + 2\cdot 0\cos 0}{(4+0^2)^{\frac{1}{2}}}\right)$$

$$= \left(\frac{2\cdot 1}{2},\ \frac{0}{(4+0^2)^{\frac{1}{2}}}\right) = (1, 0)$$

✍ **Esercizio 2.1.4.** *Data la curva γ parametrizzata da $(e^t\cos t, e^t\sin t)$ con $-2\pi \le t \le 2\pi$, determinate la lunghezza di γ; determinate poi la retta tangente alla curva nel punto corrispondente a $t = 0$.*

https://www.desmos.com/3d/nc44fxps9g

$$\gamma(t) = (e^t\cos t, e^t \sin t) \quad t \in [-2\pi, 2\pi]$$

$$\gamma'(t) = (e^t\cos t - e^t \sin t, e^t\sin t + e^t\cos t)$$

$$= (e^t(\cos t - \sin t), e^t(\sin t + \cos t))$$

$$L(\gamma(t)) = \int_{-2\pi}^{2\pi} \|\gamma'(t)\| \, dt$$

$$= \int \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2} \, dt$$

$$= \int \sqrt{e^{2t}(\underbrace{\cos^2 t + \sin^2 t}_{=1} - 2\cos t \sin t) + e^{2t}(\underbrace{\sin^2 t + \cos^2 t}_{=1} + 2\cos t \sin t)} \, dt$$

$$= \int \sqrt{e^{2t}(-\sin(2x) + \sin(2x) + 2)} \, dt$$

$$= \int_{-2\pi}^{2\pi} e^t\sqrt{2} \, dt$$

$$= \sqrt{2}\int_{-2\pi}^{2\pi} e^t \, dt$$

$$= \sqrt{2}\left[e^t\right]_{-2\pi}^{2\pi}$$

$$= \sqrt{2}\left(e^{2\pi} - e^{-2\pi}\right)$$

Trovo il vettore tangente nel punto 0

$$\gamma'(0) = (e^0(\cos 0 - \sin 0), e^0(\sin 0 + \cos 0)) = (1, 1)$$

La retta tangente è quella retta traslata nel punto $\gamma(0)$ e scalata per t volte il vettore tangente

$$r_t = \gamma(0) + t\gamma'(0)$$

$$\gamma(0) = (e^0 \cos 0, e^0 \sin 0) = (1, 0)$$

$$r_t = (1,0) + t(1,1)$$

$$= (1+t, t) \qquad t \in (-\infty, +\infty)$$

oppure

$$\begin{cases} x = 1+t \\ y = t \end{cases} \qquad t = x-1 \rightarrow y = x-1$$

✍ **Esercizio 2.1.5.** *Data la curva la cui equazione in coordinate polari è $\rho = 2\theta$, determinare un vettore tangente alla curva nel punto che corrisponde a $\theta = \frac{\pi}{2}$ e scrivere l'equazione cartesiana della retta tangente nello stesso punto.*

https://www.desmos.com/3d/z0do8lrwth

$$\rho(\theta) = 2\theta$$

$$\rho(\theta) = (2\theta \cos\theta, 2\theta \sin\theta)$$

$$\rho'(\theta) = (2\cos\theta - 2\theta \sin\theta, 2\sin\theta + 2\theta\cos\theta)$$

$$\rho'\left(\frac{\pi}{2}\right) = \left(2\cos\frac{\pi}{2} - 2\frac{\pi}{2}\sin\frac{\pi}{2}, 2\sin\frac{\pi}{2} + 2\frac{\pi}{2}\cos\frac{\pi}{2}\right)$$

$$= (-\pi, 2)$$

$$\rho\left(\frac{\pi}{2}\right) = \left(2\frac{\pi}{2}\cos\frac{\pi}{2}, 2\frac{\pi}{2}\sin\frac{\pi}{2}\right)$$

$$= (0, \pi)$$

$$r_t(t) = \rho\left(\frac{\pi}{2}\right) + t\rho'\left(\frac{\pi}{2}\right)$$

$$= (0, \pi) + t(-\pi, 2)$$

$$= (-\pi t, \pi + 2t)$$

$$= \begin{cases} x = -\pi t \\ y = \pi + 2t \end{cases}$$

$$\downarrow$$

$$t = -\frac{1}{\pi}x \rightarrow y = \pi - \frac{2}{\pi}x$$

✍ **Esercizio 2.1.6.** *Si calcoli la lunghezza l_γ della curva*

$$\underline{\gamma}(t) = \left(\frac{2+3t}{8t}, 2t-1, \ln(t)\right), \qquad \frac{1}{2} \le t \le 2.$$

Si calcolino inoltre le equazioni della retta r tangente alla curva nel punto $\underline{\gamma}(1)$ e del piano π perpendicolare alla curva nello stesso punto.

https://www.desmos.com/3d/8hrqnxhkqm

$$\gamma(t) = \left(\frac{2+3t}{8t}, 2t-1, \ln(t)\right) \quad t \in \left[\tfrac{1}{2}, 2\right]$$

$$\gamma'(t) = \left(\frac{3\cdot 8t - 8\cdot(2+3t)}{(8t)^2}, 2, \frac{1}{t}\right)$$

$$= \left(\frac{2\cancel{4t} - 16 - 2\cancel{4t}}{64t^2}, 2, \frac{1}{t}\right)$$

$$= \left(-\frac{1}{4t^2}, 2, \frac{1}{t}\right)$$

$$\gamma'(1) = \left(-\frac{1}{4}, 2, 1\right)$$

$$\gamma(1) = \left(\frac{2+3}{8}, 2-1, \ln(e^0)\right)$$

$$= \left(\frac{5}{8}, 1, 0\right)$$

$$r_t = \gamma(1) + t\,\gamma'(1)$$

$$= \begin{pmatrix} 5/8 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1/4 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5/8 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/4\, t \\ 2t \\ 1t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{8} - \frac{1}{4}t \\ 1 + 2t \\ t \end{pmatrix}$$

Calcoliamo il piano normale alla tangente nel punto 1

$$\pi = \left\langle \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \gamma'(1) \right\rangle = \left\langle \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \begin{pmatrix} -\frac{1}{4} \\ 2 \\ 1 \end{pmatrix} \right\rangle = 0$$

$$-\frac{1}{4}s_1 + 2s_2 + s_3 = 0$$

$$s_1 = 8s_2 + 4s_3$$

$$s_2 = u$$
$$s_3 = v$$

$$\begin{pmatrix} 8u+4v \\ u \\ v \end{pmatrix} = u\begin{pmatrix} 8 \\ 1 \\ 0 \end{pmatrix} + v\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

$$\pi = \gamma(1) + k\begin{pmatrix} 0 \\ 1 \\ \frac{1}{4} \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{5}{8} \\ 1 \\ 0 \end{pmatrix} + k\begin{pmatrix} 0 \\ 1 \\ \frac{1}{4} \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

✍ **Esercizio 2.2.1.** *Parametrizzate il tratto del grafico della funzione e^x compreso tra $x = 0$ e $x = 1$; detta γ tale curva, calcolate l'integrale su γ di $f(x, y) = ye^x$; calcolate infine la lunghezza di γ.*

$\gamma(t) = (t, e^t) \quad t \in [0, 1] \qquad f(x,y) = ye^x$

$$\int_\gamma f(x,y)\,ds = \int_0^1 f(t, e^t)\,\|\gamma'(t)\|\,dt$$

$\gamma'(t) = (1, e^t) \qquad \|\gamma'(t)\| = \sqrt{1+e^{2t}}$

$$= \int_0^1 e^t e^t \sqrt{1+e^{2t}}\,dt$$

$$= \int_0^1 e^{2t}\sqrt{1+e^{2t}}\,dt \qquad e^{2t} = u$$

$du = 2e^{2t}dt$

$$= \frac{1}{2}\int_0^1 \sqrt{1+u}\,du$$

$$= \frac{1}{2}\int (1+u)^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\left[\frac{2}{3}(1+u)^{\frac{3}{2}}\right]$$

$$= \left[\frac{1}{3}(1+e^{2t})^{\frac{3}{2}}\right]_0^1$$

$$= \frac{1}{3}(1+e^2)^{\frac{3}{2}} - \frac{1}{3}(2)^{\frac{3}{2}}$$

$$= \frac{1}{3}\left(\sqrt{(1+e^2)^2} - \sqrt{8}\right)$$

$$L(\gamma(t)) = \int_0^1 \|\gamma'(t)\|\,dt$$

$$= \int_0^1 \sqrt{1+e^{2t}}\, dt$$

$$= \int_0^1 (1+e^{2t})^{\frac{1}{2}}\, dt \qquad (1+e^{2t})^{\frac{1}{2}} = u$$

$$= \int u \cdot \left(\frac{1}{u^{-1}} \cdot \frac{1}{e^{\ln(u^2-1)}} \right) du \qquad 1+e^{2t} = u^2$$

$$e^{2t} = u^2 - 1$$

$$= \int u \cdot \frac{u}{u^2-1}\, du \qquad 2t = \ln(u^2-1)$$

$$du = (1+e^{2t})^{-\frac{1}{2}} e^{2t}\, dt$$

$$= \int \frac{u^2}{u^2-1}\, du$$

$$= \int \frac{u^2-1+1}{u^2-1}\, du$$

$$= \int 1 + \frac{1}{u^2-1}\, du$$

$$= u + \int \frac{1}{u^2-1}\, du$$

$$* = u + \int \frac{1}{(u+1)(u-1)}\, du$$

$$\frac{1}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} = \frac{Au - A + Bu + B}{(u+1)(u-1)} = \frac{u(A+B) - A + B}{(u+1)(u-1)}$$

$$\begin{cases} A+B=0 \\ -A+B=1 \end{cases} \qquad \begin{cases} A=-B \\ 2B=1 \end{cases} \qquad \begin{cases} A=-\frac{1}{2} \\ B=\frac{1}{2} \end{cases}$$

$$\frac{1}{u^2-1} = -\frac{1}{2}\frac{1}{u+1} + \frac{1}{2}\frac{B}{u-1}$$

$$* = u - \frac{1}{2}\int \frac{1}{u+1}\, du + \frac{1}{2}\int \frac{1}{u-1}\, du$$

$$= u - \frac{1}{2}\ln(u+1) + \frac{1}{2}\ln(u-1)$$

$$= u + \frac{1}{2}\ln\left(\frac{u-1}{u+1}\right)$$

$$= \left[\sqrt{1+e^{2t}} + \frac{1}{2}\ln\left(\frac{\sqrt{1+e^{2t}}-1}{\sqrt{1+e^{2t}}+1}\right)\right]_0^1$$

$$= \sqrt{1+e^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}\right) - \left(\sqrt{2} + \frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right)$$

$$= \sqrt{1+e^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}\right) - \sqrt{2} - \frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$$

✍ **Esercizio 2.2.2.** *Data la curva γ parametrizzata da $\Phi(t) = (t\cos 2t, -t\sin 2t)$, determinate la retta tangente alla curva nel punto che corrisponde a $t = 0$ e calcolate l'integrale della funzione $f(x, y) = \sqrt{x^2 + y^2}$ sulla parte di curva con $-1 \le t \le 1$.*

https://www.desmos.com/3d/w0f1mliadk

$$\Phi(t) = (t\cos(2t), -t\sin(2t))$$

$$\Phi'(t) = (\cos(2t) - 2t\sin(2t), -\sin(2t) - 2t\cos(2t))$$

$$\Phi'(0) = (1, 2)$$

$$\|\Phi'(t)\| = \sqrt{(\cos(2t) - 2t\sin(2t))^2 + (-\sin(2t) - 2t\cos(2t))^2}$$

$$= \sqrt{\cos^2(2t) + 4t^2\sin^2(2t) - \cancel{4t\cos(2t)\sin(2t)} + \sin^2(2t) + 4t^2\cos^2(2t) + \cancel{4t\sin(2t)\cos(2t)}}$$

$$= \sqrt{4t^2(\underbrace{\sin^2(2t) + \cos^2(2t)}_{=1}) + \underbrace{\cos^2(2t) + \sin^2(2t)}_{=1}}$$

$$= \sqrt{4t^2+1}$$

$$\int_\gamma F(x,y)\,ds = \int_{-1}^{1} F(\Phi(t))\,\|\Phi'(t)\|\,dt$$

$$= \int_{-1}^{1} \sqrt{t^2\cos^2(2t) + t^2\sin^2(2t)} \cdot \sqrt{4t^2+1}\,dt$$

$$= \int_{-1}^{1} |t| \sqrt{\cos^2(2t)+\sin^2(2t)} \cdot \sqrt{4t^2+1}\,dt$$

$$= 2\int_0^1 t \cdot \sqrt{4t^2+1}\,dt \qquad 4t^2+1 = u$$

$$= \frac{1}{4}\int u^{\frac{1}{2}}\,du \qquad du = 8t$$

$$= \frac{1}{4} \cdot \left[\frac{2}{3}u^{\frac{3}{2}}\right]$$

$$= \left[\frac{1}{6}\sqrt{(4t^2+1)^3}\right]_0^1$$

$$= \frac{1}{6}\sqrt{5^2 \cdot 5} - \left(\frac{1}{12}\right)$$

$$= \frac{1}{6}\left(5\sqrt{5} - 1\right)$$

$$(t\cos(2t), -t\sin(2t))$$

$$\begin{cases} x = t\cos(2t) \\ y = -t\sin(2t) \\ z = \sqrt{x^2+y^2} = \sqrt{t^2\cos^2(2t)} \end{cases}$$

✍ **Esercizio 2.2.3.** *Considerate la curva γ parametrizzata da $(\sin t, t, 1)$ con $0 \le t \le 2\pi$; determinare il vettore tangente a γ in ciascuno dei punti corrispondenti a $t=0$, $t=\frac{\pi}{2}$, $t=\pi$, disegnate accuratamente γ e calcolate l'integrale su γ della funzione $xyz\sqrt{1+\cos^2 y}$.*

https://www.desmos.com/3d/2mg7et4np5

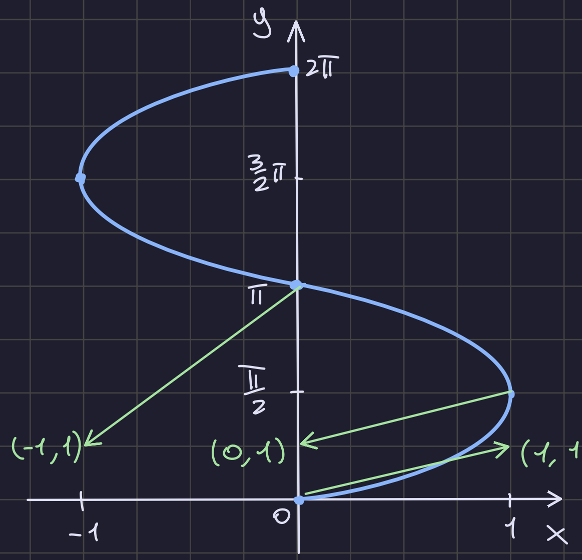
$\gamma(t) = (\sin t, t, 1) \qquad t \in [0, 2\pi]$

$\gamma'(t) = (\cos t, 1, 0)$

$\gamma'(0) = (1, 1, 0)$

$\gamma'(\frac{\pi}{2}) = (0, 1, 0)$

$\gamma'(\pi) = (-1, 1, 0)$



$F(x,y,z) = xyz\sqrt{1+\cos^2 y} \qquad \gamma(t) = (\sin t, t, 1)$

$\|\gamma'(t)\| = \sqrt{\cos^2 t + 1}$

$$\int_\gamma F(x,y,z)\, ds = \int_0^{2\pi} F(\gamma(t))\, \|\gamma'(t)\|\, dt$$

$$= \int t \sin t \sqrt{1+\cos^2 t}\,\sqrt{1+\cos^2 t}\; dt$$

$$= \int t \sin t\,(1+\cos^2 t)\, dt$$

$$= \int t\sin t + t \sin t \cos^2 t\, dt$$

$$= \int t\sin t\, dt + \int t\sin t\cos^2 t\, dt$$

$f: t \quad f' \, 1$

$g': \sin t \quad g: -\cos t$

$fg - \int f'g$

$$= \int t\sin t\, dt + \int t\sin t(1-\sin^2 t)dt$$

$\int t\sin t\, dt = -t\cos t + \sin t + C$

$$* = 2\int t\sin t dt - \int t\sin^3 t\, dt$$

$F: t \quad f': 1$

$g': \sin^3 t \quad g: ?$

$$\int \sin^3 t\, dt = \int \sin^2 t \sin t\, dt = \int (1-\cos^2 t) \sin t\, dt = \dots$$

$U = \cos t$

$dU = -\sin t\, dt$

$dt = \frac{dU}{-\sin t}$

$$\dots = \int (1-U^2) \cancel{\sin t} \cdot \frac{1}{\cancel{-\sin t}} dU = -\int 1-U^2\, dU = -U + \int U^2\, dU$$

$$= -U + \frac{U^3}{3} + C = -\cos t + \frac{\cos^3 t}{3} + C$$

$$g: -\cos t + \frac{\cos^3 t}{3}$$

$$* = 2\int t\sin t dt - \left(-t\cos t + \frac{t\cos^3 t}{3} - \int -\cos t + \frac{\cos^3 t}{3}\, dt\right)$$

$$= 2\int t\sin t dt + t\cos t - \frac{t\cos^3 t}{3} + \int -\cos t + \frac{\cos^3 t}{3}\, dt$$

$$= 2\int t\sin t dt + t\cos t - \frac{t\cos^3 t}{3} - \int \cos t\, dt + \frac{1}{3}\int \cos^3 t\, dt$$

$$= 2\int t\sin t dt + t\cos t - \frac{t\cos^3 t}{3} - \sin t + \frac{1}{3}\int \cos^3 t\, dt$$

$$= 2\int t\sin t dt + t\cos t - \frac{t\cos^3 t}{3} - \sin t + \frac{1}{3}\int \cos^2 t\cos t\, dt$$

$$= 2\int t\sin t\,dt + t\cos t - \frac{t\cos^3 t}{3} - \sin t + \frac{1}{3}\int (1-\sin^2 t)\cos t\,dt$$

$$u = \sin t$$
$$du = \cos t\,dt$$
$$dt = \frac{du}{\cos t}$$

$$= 2\int t\sin t\,dt + t\cos t - \frac{t\cos^3 t}{3} - \sin t + \frac{1}{3}\int (1-u^2)\cancel{\cos t}\cdot\frac{1}{\cancel{\cos t}}\,du$$

$$= 2\int t\sin t\,dt + t\cos t - \frac{t\cos^3 t}{3} - \sin t + \frac{1}{3}\int 1-u^2\,du$$

$$= 2\int t\sin t\,dt + t\cos t - \frac{t\cos^3 t}{3} - \sin t + \frac{1}{3}\left(u - \frac{u^3}{3}\right)$$

$$= 2\int t\sin t\,dt + t\cos t - \frac{t\cos^3 t}{3} - \sin t + \frac{\sin t}{3} - \frac{\sin^3 t}{9}$$

$$= 2\left(-t\cos t + \sin t\right) + t\cos t - \frac{t\cos^3 t}{3} - \sin t + \frac{\sin t}{3} - \frac{\sin^3 t}{9}$$

$$= -2t\cos t + 2\sin t + t\cos t - \frac{t\cos^3 t}{3} - \sin t + \frac{\sin t}{3} - \frac{\sin^3 t}{9}$$

$$= \left[-t\cos t - \frac{t\cos^3 t}{3} + \frac{4}{3}\sin t - \frac{\sin^3 t}{9}\right]_0^{2\pi}$$

$$= -2\pi - \frac{2}{3}\pi - \left(-2\pi - \frac{2}{3}\pi\right) = 0$$

✍ **Esercizio 2.2.4.** *Calcolare l'integrale (curvilineo) di*

$$f(x,y) = \frac{xy}{\sqrt{4+x^2}}$$

lungo la curva γ il cui sostegno è il bordo ∂E di

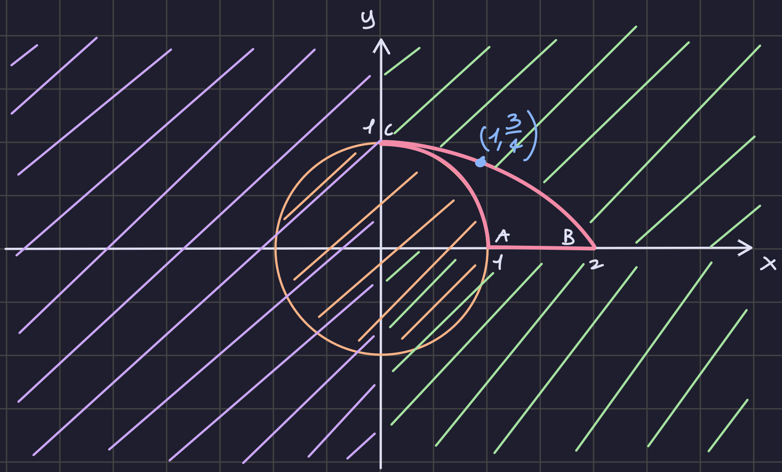
$$E = \left\{(x,y) : x \geq 0, x^2+y^2 \geq 1, 0 \leq y \leq 1 - \frac{x^2}{4}\right\}$$

$y \leq -\frac{1}{4}x^2 + 1$

e determinare la retta tangente a γ nel punto $\left(1, \frac{3}{4}\right)$.

$x=0 \Rightarrow y=1$

$y=0 \Rightarrow x=2$



$\gamma_{AB}(t) = (t, 0) \quad t \in [1,2] \qquad \|\gamma'_{AB}(t)\| = 1$

$\gamma_{AC}(t) = (\cos t, \sin t) \quad t \in [0, \frac{\pi}{2}] \qquad \|\gamma'_{AC}(t)\| = \sqrt{\cos^2 t + \sin^2 t} = 1$

$\gamma_{CB}(t) = (t, 1 - \frac{t^2}{4}) \quad t \in [0,2] \qquad \|\gamma_{CB}(t)\| = \sqrt{1 + (-\frac{1}{2}t)^2} = \sqrt{1 + \frac{1}{4}t^2}$

$$F(x,y) = \frac{xy}{\sqrt{4+x^2}}$$

$$\int_{\gamma_{AB}} F(x,y)\,ds = \int_1^2 F(\gamma_{AB}(t))\,\|\gamma'_{AB}(t)\|\,dt$$

$$= \int_1^2 \frac{t \cdot 0}{\sqrt{4+t}} \cdot 1\,dt = \int_1^2 0\,dt = 0$$

$$\int_{\gamma_{AC}} F(x,y)\,ds = \int_0^{\frac{\pi}{2}} F(\gamma_{AC}(t))\,\|\gamma'_{AC}(t)\|\,dt$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos t \sin t}{\sqrt{4+\cos^2 t}}\,dt \qquad u = \cos^2 t$$

$$du = -2\cos t \cdot \sin t\,dt$$

$$= -\frac{1}{2}\int \frac{-2\cos t \sin t}{\sqrt{4+u}}\,dt$$

$$= -\frac{1}{2} \int \frac{1}{\sqrt{4+u}} \, du$$

$$= -\frac{1}{2} \int (4+u)^{-\frac{1}{2}} \, du$$

$$= -\frac{1}{2} \left[2\sqrt{4+u} \right]$$

$$= -\left[\sqrt{4+\cos^2 t} \right]_0^{\frac{\pi}{2}}$$

$$= -\left(\sqrt{4} - \sqrt{5} \right) = -\sqrt{4} + \sqrt{5} = -2 + \sqrt{5}$$

$$\int_{\gamma_{CB}} F(x,y) \, ds = \int_0^2 F(\gamma_{CB}(t)) \, \|\gamma'_{CB}(t)\| \, dt$$

$$= \int_0^2 \frac{t - \frac{t^3}{4}}{\sqrt{4+t^2}} \sqrt{1+\frac{1}{4}t^2} \, dt$$

$$= \int \frac{t - \frac{t^3}{4}}{\sqrt{4(1+\frac{1}{4}t^2)}} \sqrt{1+\frac{1}{4}t^2} \, dt$$

$$= \frac{1}{2} \int \frac{t - \frac{t^3}{4}}{\cancel{\sqrt{1+\frac{1}{4}t^2}}} \cancel{\sqrt{1+\frac{1}{4}t^2}} \, dt$$

$$= \frac{1}{2} \int t - \frac{t^3}{4} \, dt$$

$$= \frac{1}{2} \left(\int t \, dt - \frac{1}{4} \int t^3 \, dt \right)$$

$$= \frac{1}{2} \left(\left[\frac{t^2}{2} \right]_0^2 - \frac{1}{4} \left[\frac{t^4}{4} \right]_0^2 \right)$$

$$= \frac{1}{2} \left(2 - \frac{1}{\cancel{4}} \cdot \cancel{4} \right) = \frac{1}{2}(2-1) = \frac{1}{2}$$

$$\int_{\gamma} F(x,y) \, ds = \int_{\gamma_{AB}} F(x,y) \, ds + \int_{\gamma_{AC}} F(x,y) \, ds + \int_{\gamma_{CB}} F(x,y) \, ds$$

$$= 0 + -2 + \sqrt{5} + \frac{1}{2} = -\frac{3}{2} + \sqrt{5}$$

$$\left(1, \tfrac{3}{4}\right) \in \gamma_{CB} \rightarrow \gamma_{CB}(1) = \left(1, \tfrac{3}{4}\right) \qquad \gamma_{CB}(t) = \left(t, 1 - \tfrac{t^2}{4}\right) \quad t \in [0, 2]$$

$$\gamma_{CB}'(t) = \left(1, -\tfrac{1}{2}t\right)$$

$$r_T(t) = \gamma_{CB}(1) + t\,\gamma_{CB}'(1) = \begin{pmatrix} 1 \\ \frac{3}{4} \end{pmatrix} + t \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

$$\downarrow$$

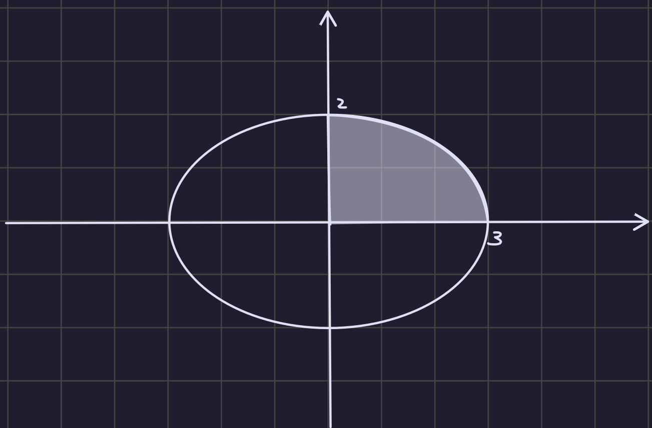
$$\begin{cases} x = 1 + t \\ y = \frac{3}{4} - \frac{1}{2}t \end{cases} \qquad \begin{cases} t = x - 1 \\ y = -\frac{1}{2}x + \frac{1}{2} + \frac{3}{4} = -\frac{1}{2}x + \frac{5}{4} \end{cases}$$

$$r_T = \begin{pmatrix} 1 \\ \frac{3}{4} \end{pmatrix} + t \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \iff y = -\frac{1}{2}x + \frac{5}{4}$$

✍ **Esercizio 2.2.5.** *Si calcoli l'integrale curvilineo di prima specie della funzione* $f(x, y) = x\,y$ *sulla parte dell'ellisse*

$$\left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} = 1\right\}$$

contenuta nel primo quadrante. [Sugerimento: nel corso del procedimento potrebbe venire utile un cambiamento di variabile del tipo $s = \sin t \ldots$*]*



$$f(x, y) = xy$$

$$\gamma(t) = (3\cos t, 2\sin t) \quad t \in \left[0, \tfrac{\pi}{2}\right]$$

$$\gamma'(t) = (-3\sin t, 2\cos t) \qquad \|\gamma'(t)\| = \sqrt{9\sin^2 t + 4\cos^2 t}$$

$$\int_\gamma f(x, y)\, ds = \int_0^{\frac{\pi}{2}} f(\gamma(t))\, \|\gamma'(t)\|\, dt$$

$$= \int_0^{\frac{\pi}{2}} 6\cos t \sin t \sqrt{9\sin^2 t + 4\cos^2 t}\, dt$$

$$= \int 6\cos t \sin t \sqrt{5\sin^2 t + 4\sin^2 t + 4\cos^2 t}\ dt$$

$$= \int 6\cos t \sin t \sqrt{5\sin^2 t + 4}\ dt$$

$$u = 5\sin^2 t + 4$$

$$du = 10\sin t \cdot \cos t$$

$$= \frac{3}{5}\int 10\cos t \sin t \sqrt{5\sin^2 t + 4}\ dt$$

$$= \frac{3}{5}\int u^{\frac{1}{2}}\ du$$

$$= \left[\frac{3}{5} \cdot \frac{2}{3} u^{\frac{3}{2}}\right]$$

$$= \left[\frac{2}{5} \cdot \left(5\sin^2 t + 4\right)^{\frac{3}{2}}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{5}\sqrt{9^3} - \frac{2}{5}\sqrt{4^3}$$

$$= \frac{2}{5} \cdot \left(\sqrt{9^3} - \sqrt{4^3}\right)$$

$$= \frac{2}{5}\left(9\sqrt{9} - 4\sqrt{4}\right)$$

$$= \frac{2}{5}\left(9 \cdot 3 - 4 \cdot 2\right)$$

$$= \frac{2}{5}\left(27 - 8\right) = \frac{2}{5} \cdot 19 = \frac{38}{5}$$

Calcolo differenziale funzioni a piú variabili