Esercizio 1 (40 punti)

Si consideri il modello ingresso/uscita a tempo continuo descritto dalla seguente equazione differenziale,

$$3\frac{d^2v(t)}{dt^2} + 6\frac{dv(t)}{dt} - 9v(t) = 2\frac{d^2u(t)}{dt^2} + 2(5+k)\frac{du(t)}{dt} + 10ku(t), \quad t \in \mathbb{R}_+,$$

e le seguenti condizioni inziali:

$$v(0^-) = 2 \qquad \frac{dv(0^-)}{dt} = 0.$$

a) Si studi la stabilitá al variare di k

$$3s^2 + 6s - 9 = 0$$

$$s_{1,2} = \frac{-6 \pm \sqrt{36+108}}{6} = \frac{-6 \pm 12}{6} = -1 \pm 2 = \begin{cases} -3 \\ 1 \end{cases}$$

$s_1 = -3$ $\qquad \mu_{1,2} = 1$

$s_2 = 1$

Il sistema non è asintoticamente stabile perchè non tutte le radici hanno parte reale negativa. Per controllare se il sistema è BIBO stabile bisogna verificare se le radici con parte reale positiva si semplificano all'interno della funzione di trasferimento, cioè il rapporto tra il polinomio caratteristico dell'entrata e quello dell'uscita:

$$H(s) = \frac{2s^2 + 2(5+k)s + 10k}{3(s+3)(s-1)} = \frac{2}{3}\frac{s^2 + (5+k)s + 5k}{(s+3)(s-1)}$$

$$s^2 + (5+k)s + 5k = s^2 + 5s + ks + 5k = s(s+5) + k(s+5) = (s+5)(s+k)$$

$$\frac{2}{3}\frac{(s+5)(s+k)}{(s+3)(s-1)} \xrightarrow{k=-1} \frac{2}{3}\frac{(s+5)\cancel{(s-1)}}{(s+3)\cancel{(s-1)}} = \frac{s+5}{s+3}$$

Il sistema è BIBO stabile solo se k = -1

b) Calcolare la risposta libera con Laplace

$$3v''(t) + 6v'(t) - 9v(t) = 0$$

$$\downarrow \mathcal{L}$$

$$3\mathcal{L}[v''(t)] = 3s^2V(s) - 3sv(0^-) - 3v'(0^-) = 3s^2V(s) - 6s$$

$$6\mathcal{L}[v'(t)] = 6sV(s) - 6v(0^-) = 6sV(s) - 12$$

$$-9\mathcal{L}[v(t)] = -9V(s)$$

$$3s^2V(s)-6s+6sV(s)-12-9V(s)=0$$

$$V(s)(3s^2+6s-9)=6s+12$$

$$V(s)=\frac{6(s+2)}{3s^2+6s-9}=\frac{2(s+2)}{(s+3)(s-1)}$$

Fratti semplici:

$$\frac{2s+4}{(s+3)(s-1)}=\frac{A}{s+3}+\frac{B}{s-1}=\frac{As-A+Bs+3B}{(s+3)(s-1)}=\frac{s(A+B)-A+3B}{(s+3)(s-1)}$$

$$\begin{cases}A+B=2\\-A+3B=4\end{cases}\rightarrow\begin{cases}A=2-B\\-2+B+3B=4\end{cases}\rightarrow\begin{cases}A=2-\frac{3}{2}\\B=\frac{3}{2}\end{cases}\rightarrow\begin{cases}A=\frac{1}{2}\\B=\frac{3}{2}\end{cases}$$

$$V(s)=\frac{1}{2}\cdot\frac{1}{s+3}+\frac{3}{2}\cdot\frac{1}{s-1}$$

$$\downarrow \mathcal{L}^{-1}$$

$$v(t)=\left(\frac{1}{2}e^{-3t}+\frac{3}{2}e^{t}\right)\delta_{-1}(t)$$

c) Dato $k=3$ calcolare la risposta forzata con Laplace considerando la seguente funzione in ingresso:

$$u(t)=3t\,e^{2t}\,\delta_{-1}(t)$$

$$V(s)(3s^2+6s-9)-6s-12=\mathcal{L}\left[2u''(t)+2(5+k)u'(t)+10k\,u(t)\right](s)$$

$$2\mathcal{L}\left[u''(t)\right]=2s^2U(s)$$

$$2(5+k)\mathcal{L}\left[u'(t)\right]=2(5+k)sU(s)=16sU(s)$$

$$10k\,\mathcal{L}\left[u(t)\right]=10ksU(s)=30sU(s)$$

$$V(s)(3s^2+6s-9)-6s-12=2s^2U(s)+16U(s)+30sU(s)$$

$$V(s)(3s^2+6s-9)-6s-12=U(s)(2s^2+30s+16)$$

$$V(s) = \frac{2(s+2)}{(s+3)(s-1)} + \frac{2s^2+30s+16}{3(s+3)(s-1)} U(s)$$

$$V_f(s) = \frac{2s^2+30s+16}{3(s+3)(s-1)} U(s) = \frac{2(s+5)\cancel{(s+3)}}{3\cancel{(s+3)}(s-1)} U(s) = \frac{2}{3} \frac{s+5}{s-1} U(s)$$

$$U(s) = \mathcal{L}\left[3te^{2t}\delta_{-1}(t)\right] = \frac{3}{(s-2)^2}$$

$$V_f(s) = \frac{2}{\cancel{3}} \frac{s+5}{s-1} \cdot \frac{\cancel{3}}{(s-2)^2} = \frac{2s+10}{(s-1)(s-2)}$$

Fratti semplici

$$\frac{2s+10}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$A = \lim_{s \to 1} \frac{d^{2-0-1}}{ds^{1-0-1}} \cancel{(s-1)} \frac{2(s+5)}{\cancel{(s-1)}(s-2)^2} = \frac{12}{1} = 12$$

$$B = \lim_{s \to 2} \frac{d^{2-0-1}}{ds^{1-0-1}} \cancel{(s-2)^2} \frac{2(s+5)}{(s-1)\cancel{(s-2)^2}} = \frac{d}{ds} \frac{2s+10}{s-1} = \frac{2(s-1)-(2s+10)}{(s-1)^2} = \frac{2-14}{1} = -12$$

$$C = \lim_{s \to 2} \frac{d^{2-1-1}}{ds^{2-1-1}} \cancel{(s-2)^2} \frac{2(s+5)}{(s-1)\cancel{(s-2)^2}} = 14$$

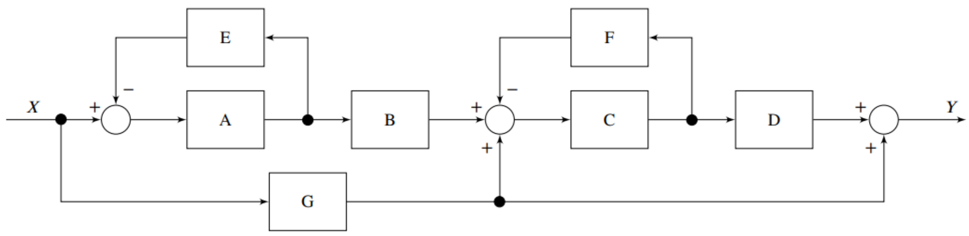
$$V_f(s) = \frac{12}{s-1} - \frac{12}{s-2} + \frac{14}{(s-2)^2}$$

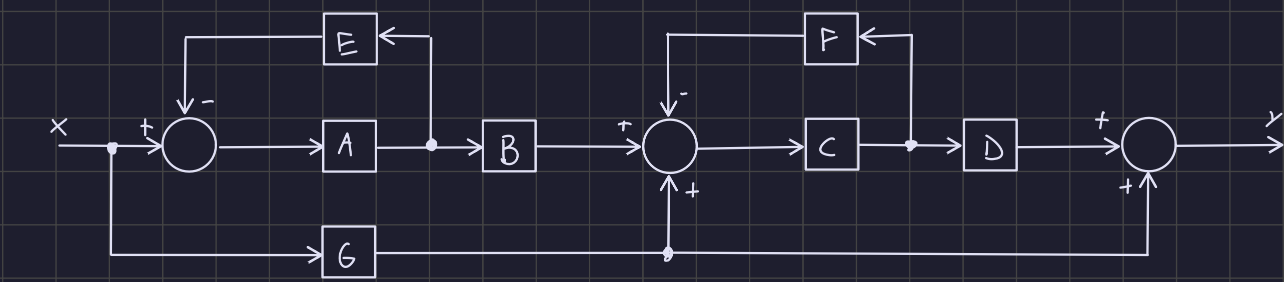
$$\downarrow \mathcal{L}^{-1}$$

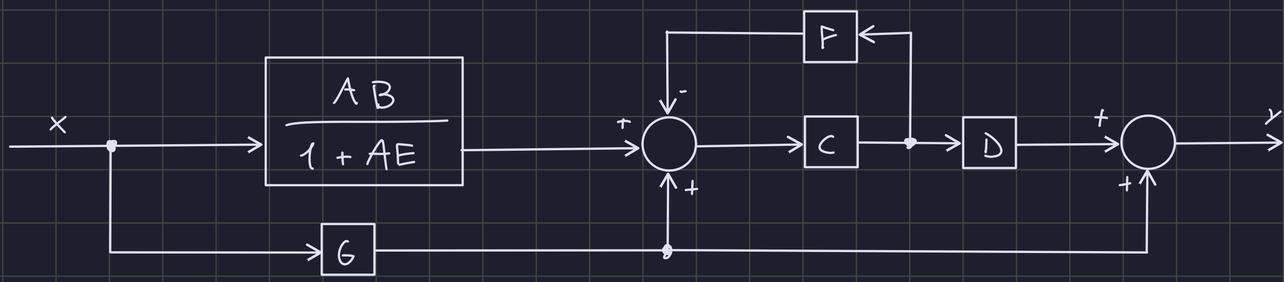
$$v_f(t) = \left(12e^{t} - 12e^{2t} + 14te^{2t}\right)\delta_{-1}(t)$$

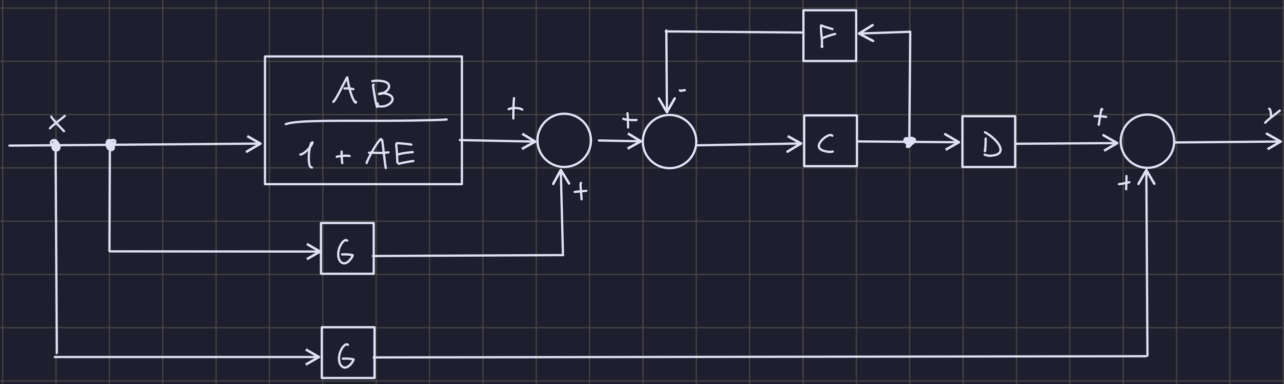
Esercizio 2 (20 punti)

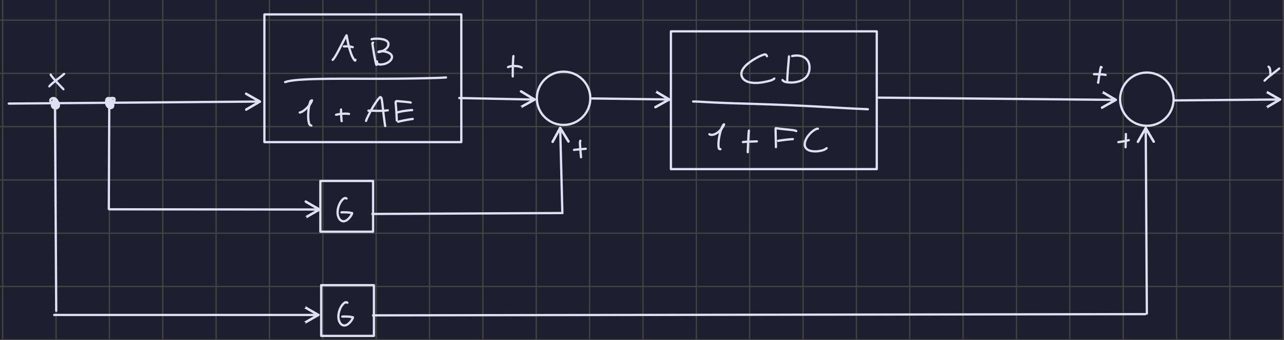
Calcolare la funzione di trasferimento del seguente schema a blocchi:

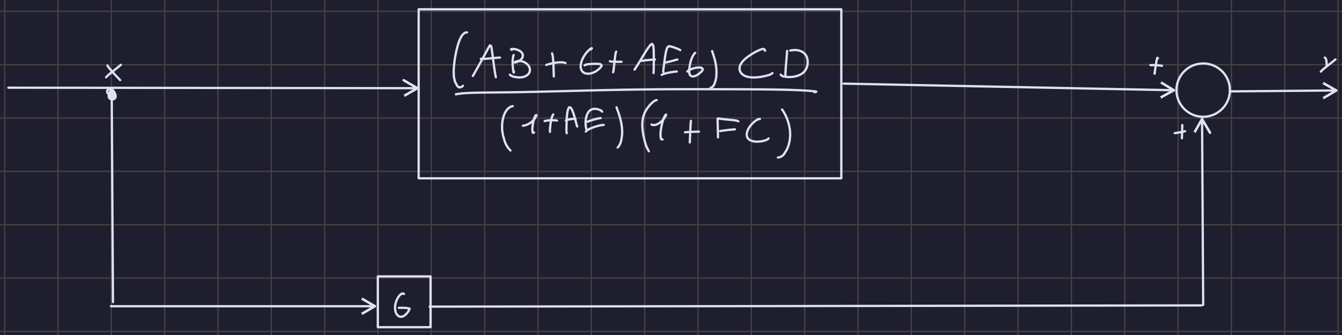


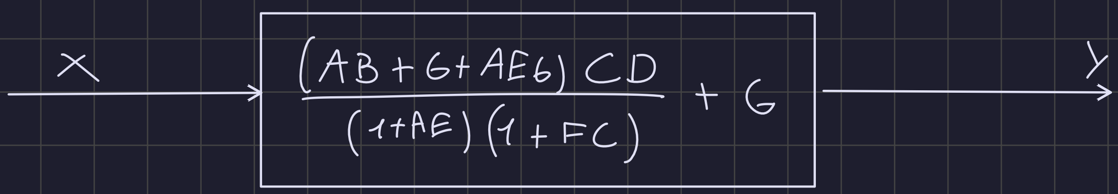












Esercizio 3 (25 punti)

Riportare la funzione in forma di Bode, disegnare i diagrammi di modulo e fase delle singole componenti elementari e il diagramma globale della seguente funzione di trasferimento:

$$G(s) = \frac{320(s^4 - 4s^3)}{(4s^2 + 2s + 1)(s - 4)^3}.$$

$$G(s) = \frac{-\cancel{4}\cdot 320\, s^3 \cancel{\left(1 - \frac{1}{4}s\right)}}{(4s^2+2s+1)\left(-4^3\left(1-\frac{1}{4}s\right)^{\cancel{3}}\right)} = 20 \cdot \frac{s^3}{(4s^2+2s+1)\left(1-\frac{1}{4}s\right)^2}$$

$K_B = 20$

$P_{cc} = (1 + 2s + 4s^2)^{-1}$

$Z_h = s^3$

$P_r = \left(1 - \frac{1}{4}s\right)^{-2}$

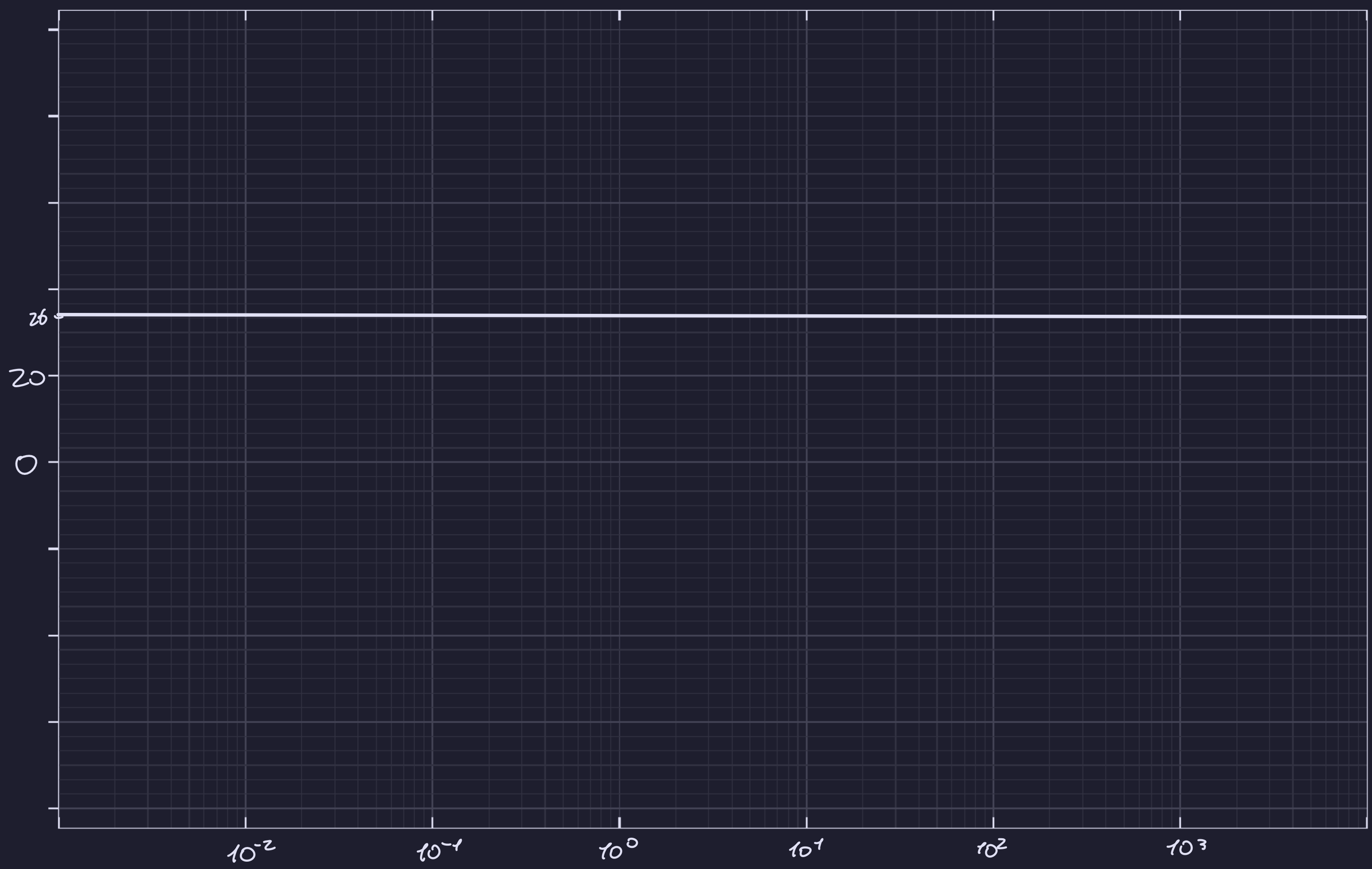
Kb

$A = 20 \cdot \log_{10}(|Kb|) \approx 26$

$\phi = \begin{cases} 0 & Kb > 0 \\ -180 & Kb < 0 \end{cases}$

Diagramma di Bode

Ampiezza



Fase

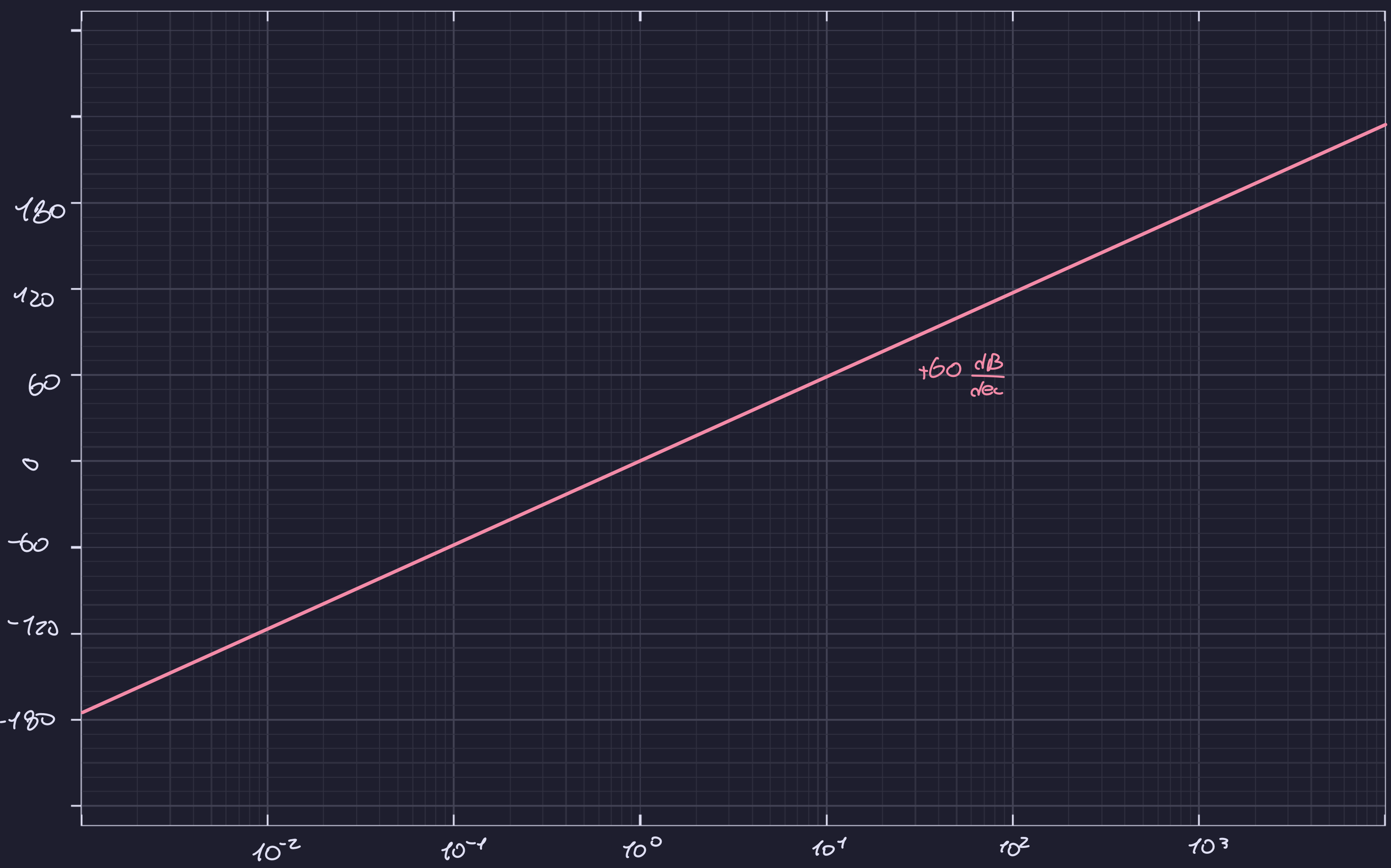
s^3

$A = 20\,\mu \frac{dB}{Dec} = 60$

$\Phi = 90 \cdot \mu = 270$

Diagramma di Bode

Ampiezza



Fase

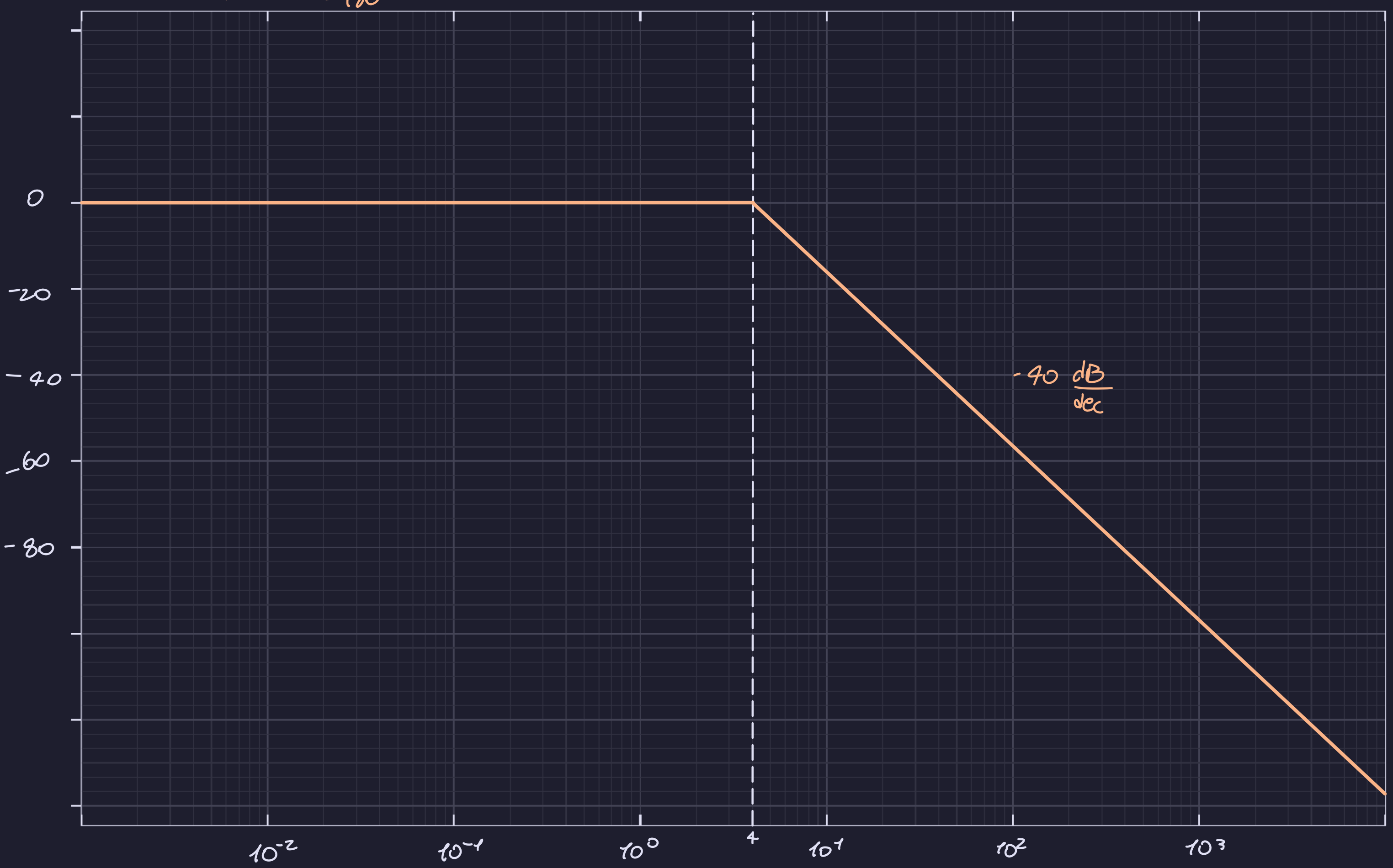
$A = \begin{cases} 0 & \omega \le \omega_n \\ 20 \cdot \mu & \omega > \omega_n = -40 \end{cases}$ $\quad \omega_n = \frac{1}{|\tau|} = 4$

Diagramma di Bode

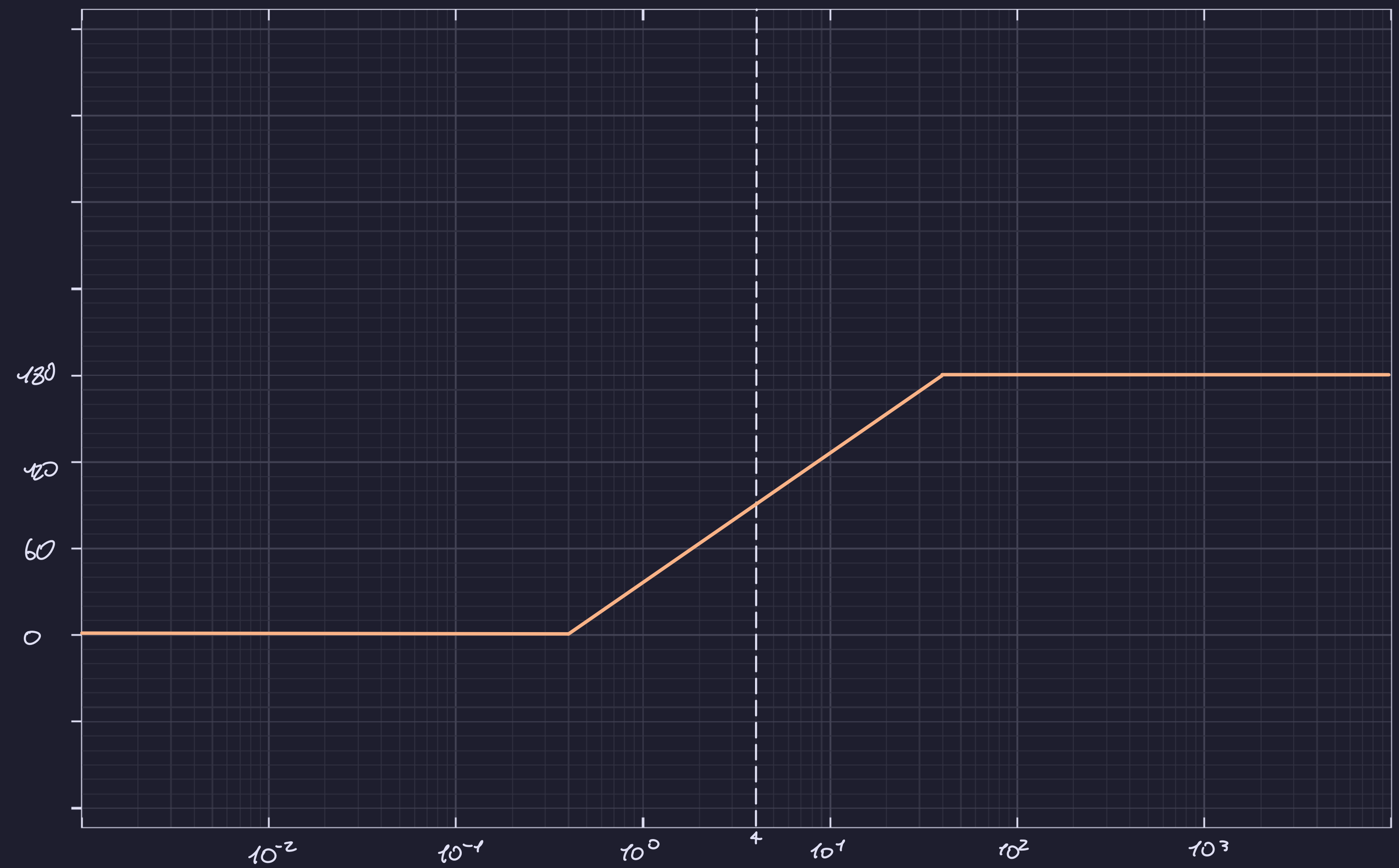
$\left(1 - \frac{1}{4}s\right)^{-2} \quad \tau = -\frac{1}{4}$

$\phi = \begin{cases} 0 & \omega \le \omega_n \\ 90 \cdot \mu \cdot \text{segno}(\tau) & \omega > \omega_n = 180 \end{cases}$

Ampiezza



Fase



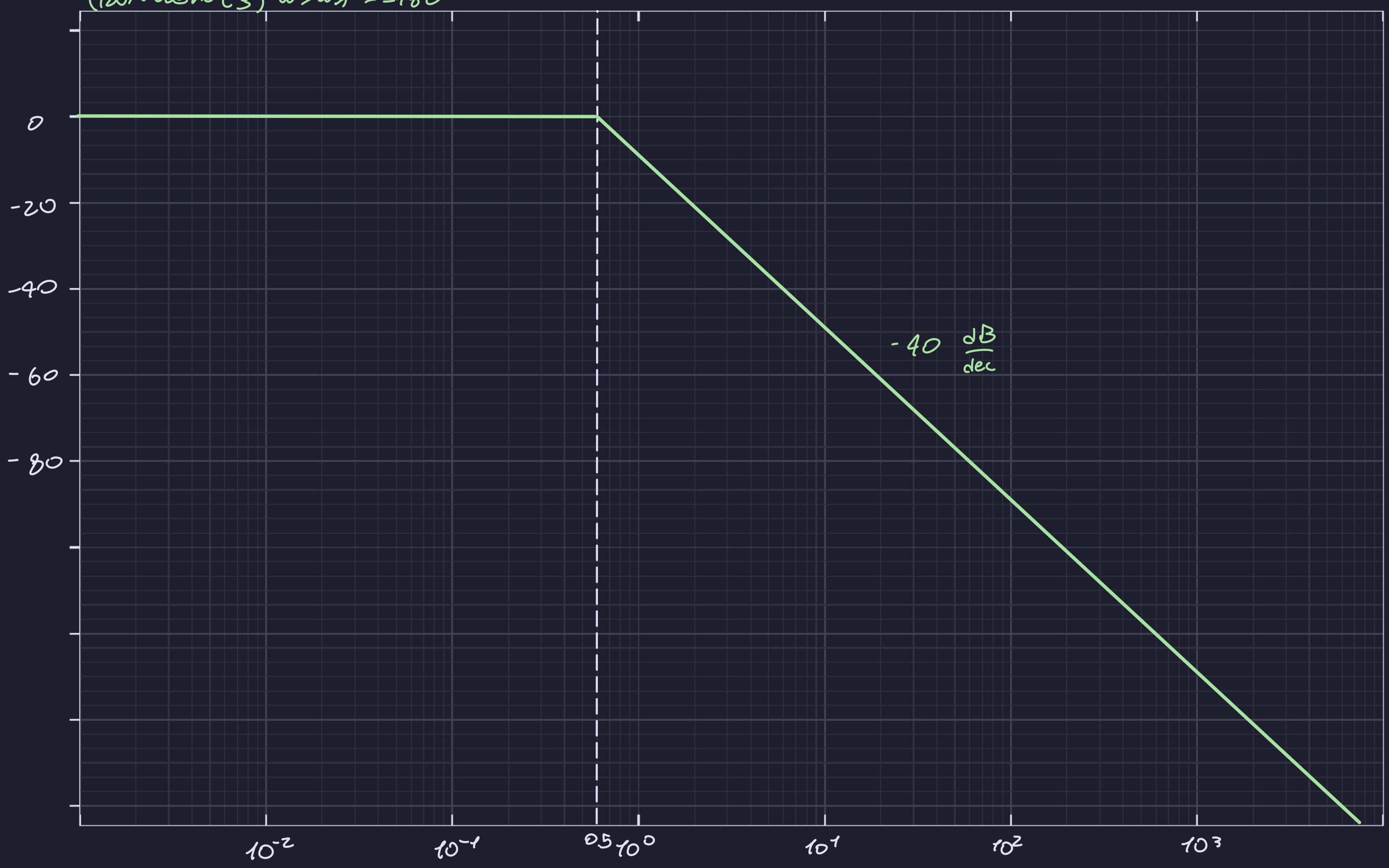
$$A = \begin{cases} 0 & w \le w_n \\ 40\mu \frac{dB}{dec} & w > w_n = -40 \frac{dB}{dec} \end{cases}$$

$$\Phi = \begin{cases} 0 & w \le w_n \\ 180\mu \ \text{segno}(\xi) & w > w_n = -180 \end{cases}$$

$(1+2s+4s^2)^{-1}$ $\quad w_n = \frac{1}{2}$ $\quad \xi = 1$ $\quad \left(\frac{2\xi}{w_n} = \frac{1}{w_n^2} \right)$

Diagramma di Bode

Ampiezza



Fase

Ampiezza

	10^{-1}	0.5	10^{0}	4	10^{1}
Kb	26	26	26	26	26
Zn	-60	-18	0	36	60
Pr	0	0	0	0	-16
Pcc	0	0	-12	-36	-52
Totale	-34	8	14	26	18

$20 \cdot \log_{10}(|Kb|)$

$20 \cdot \mu \cdot \log(\omega)$

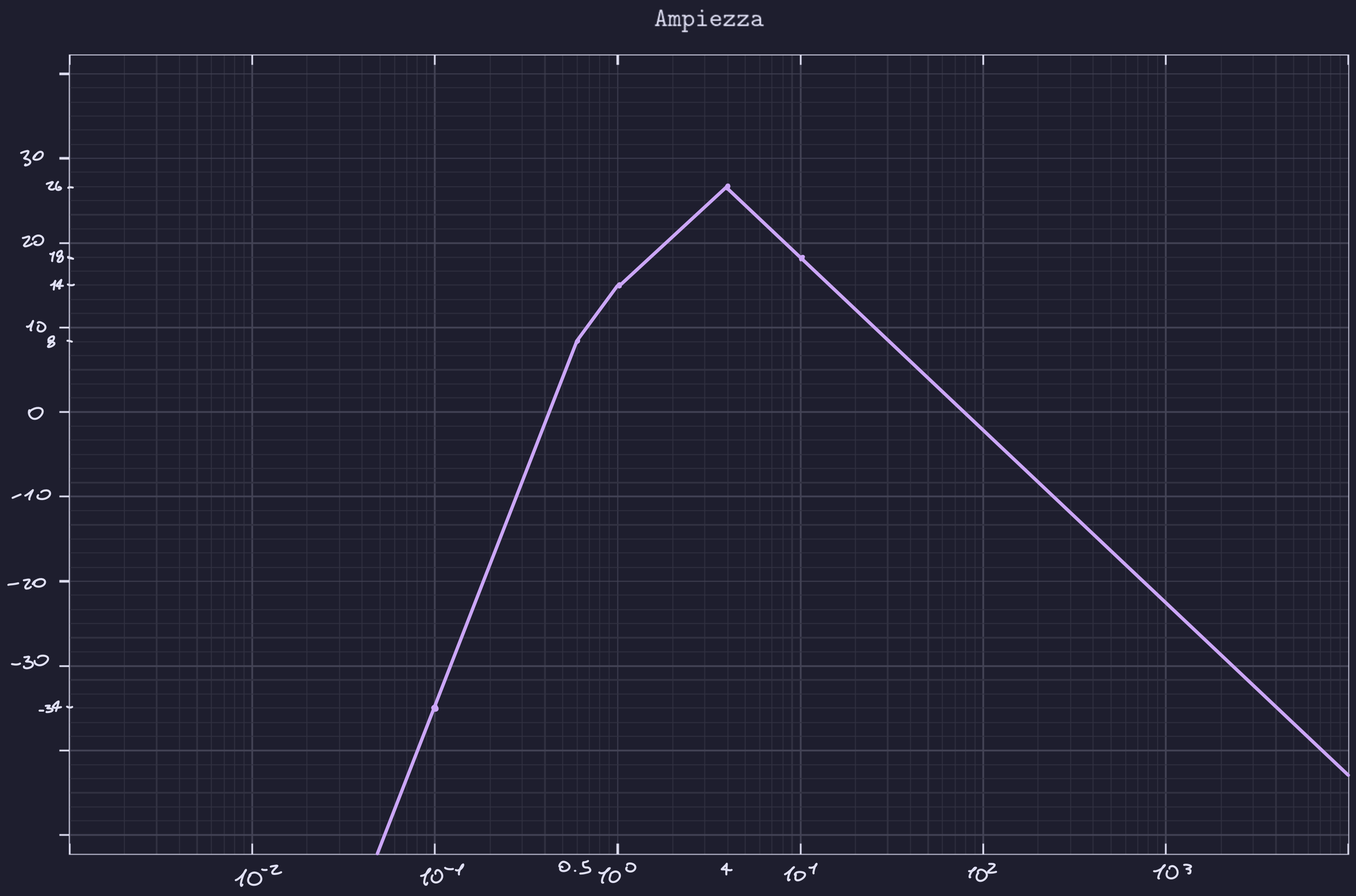
$20 \cdot \mu \cdot \log(\omega |\tau|)$

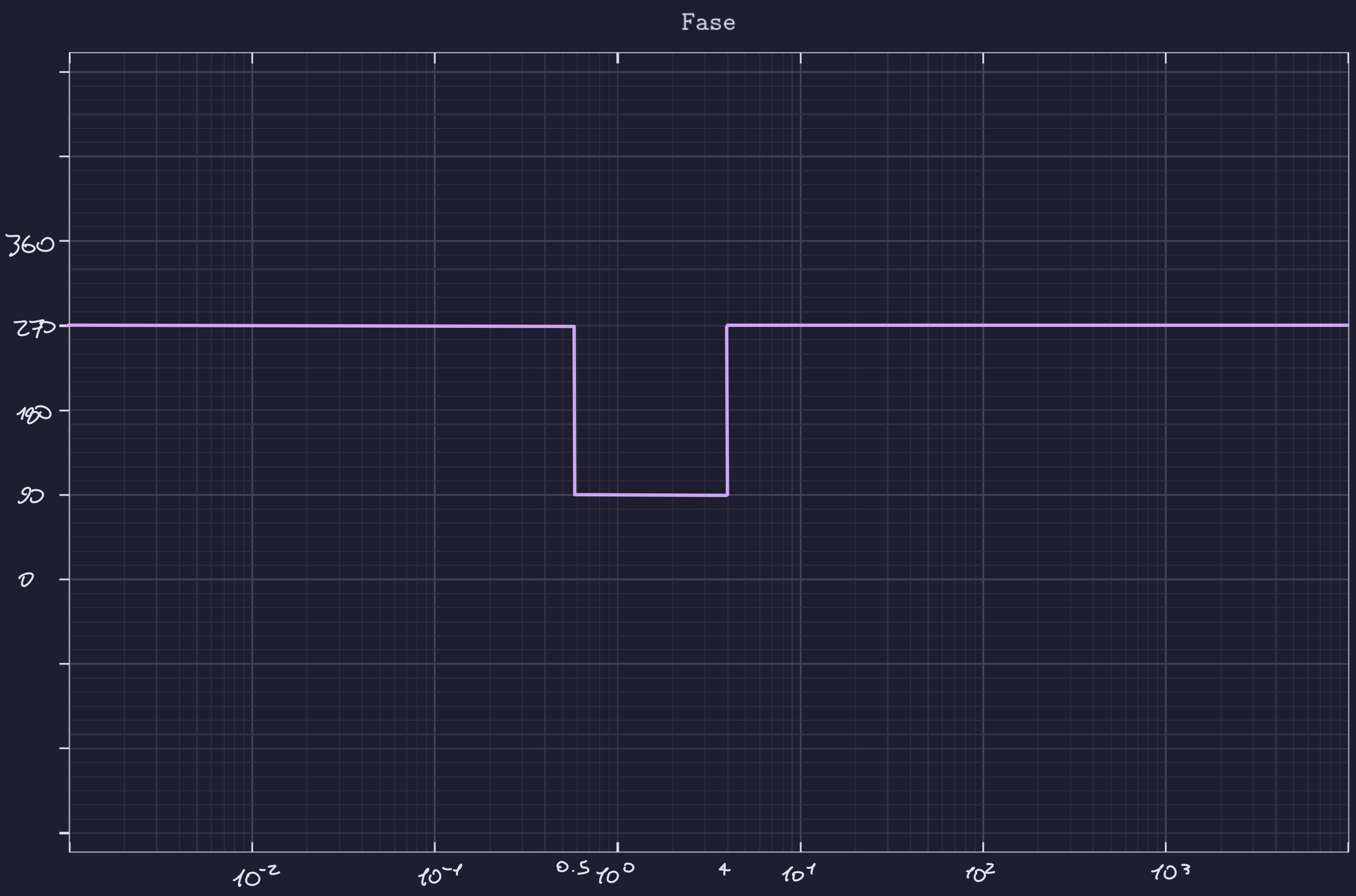
$40 \cdot \mu \cdot \log\left(\frac{\omega}{\omega_n}\right)$

Fase

	10^{-1}	0.5	10^{0}	4	10^{1}
Kb	0	0	0	0	0
Zn	270	270	270	270	270
Pr	0	0	0	0	180
Pcc	0	0	-180	-180	-180
Totale	270	270	90	90	270

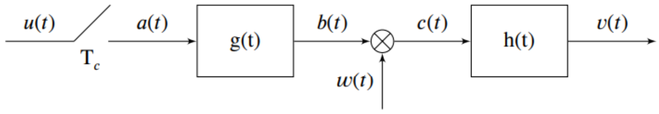
Diagramma di Bode





Esercizio 4 (35 punti)

Dato il seguente schema a blocchi,



$u(t) = 40\cos(40\pi t)$ $\quad g(t) = 9\ \mathrm{sinc}(90t) - 7\ \mathrm{sinc}(70t)$ $\quad T_c = 0.1\ s$

$w(t) = 200\ \mathrm{sinc}^2(20t)$ $\quad h(t) = -28\ \mathrm{sinc}(140t)$

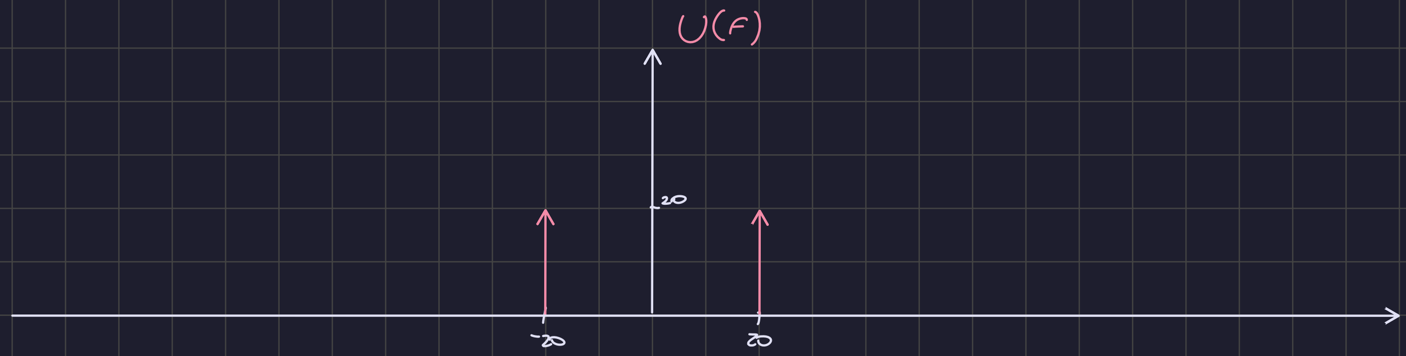
$$u(t) = 40\cos(2\pi\cdot 20t) \xrightarrow{F} U(f) = 20\,\delta(f-20) + 20\,\delta(f+20)$$

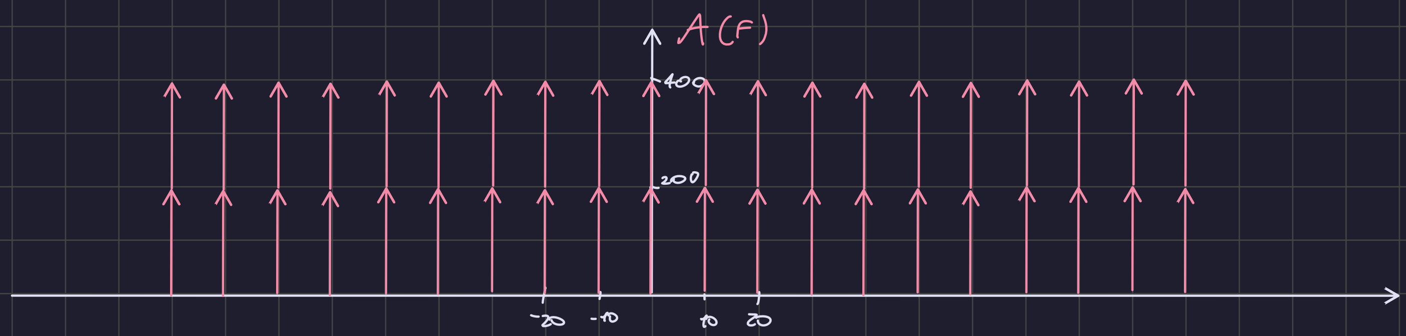
$$w(t) = 10\cdot 20\ \mathrm{sinc}^2(20t) \xrightarrow{F} W(f) = 10\,\Lambda\left(\frac{f}{20}\right)$$

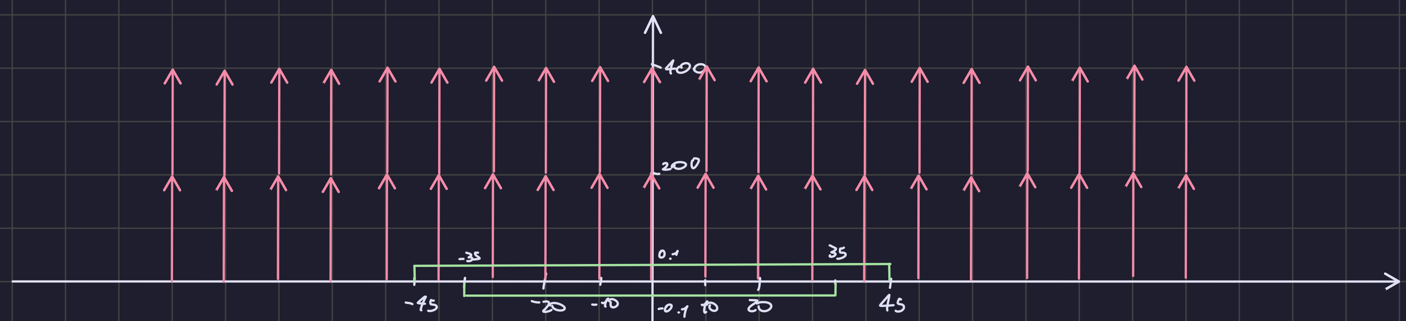
$$g(t) = \frac{1}{10}\cdot 90\cdot \mathrm{sinc}(90t) - \frac{1}{10}70\ \mathrm{sinc}(70t) \xrightarrow{F} G(f) = \frac{1}{10}\Pi\left(\frac{f}{90}\right) - \frac{1}{10}\Pi\left(\frac{f}{70}\right)$$

$$h(t) = 140\cdot\left(-\frac{1}{5}\right)\mathrm{sinc}(140t) \xrightarrow{F} H(f) = -\frac{1}{5}\Pi\left(\frac{f}{140}\right)$$

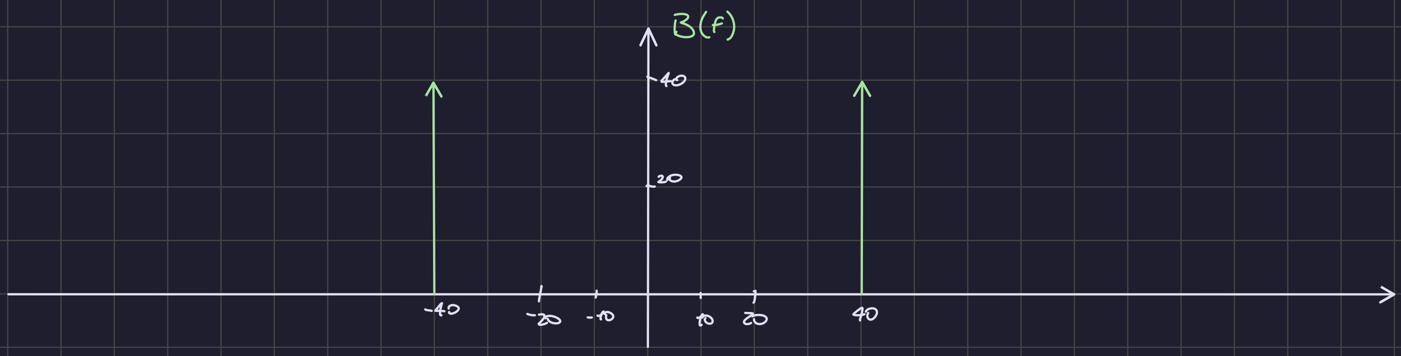
$$T_c = 0.1s \Rightarrow f_c = \frac{1}{T_c} = 10\ Hz$$

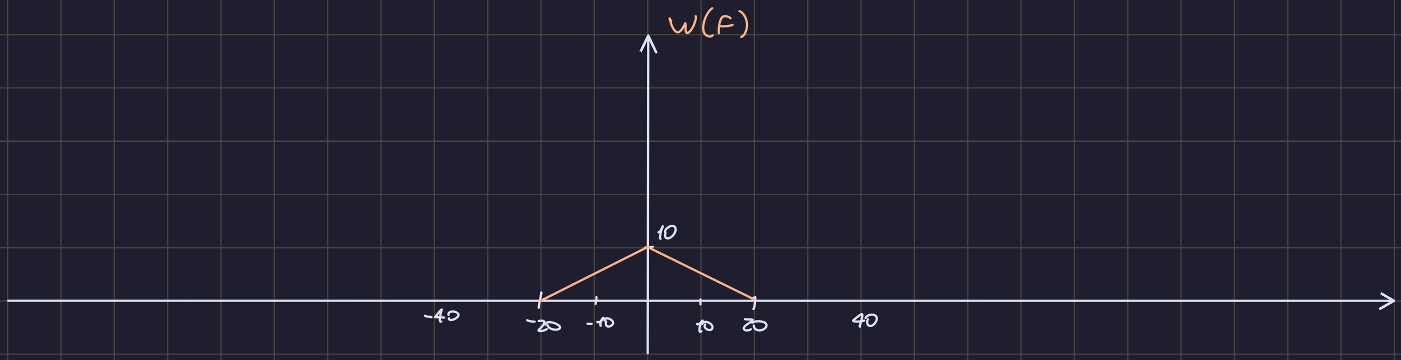


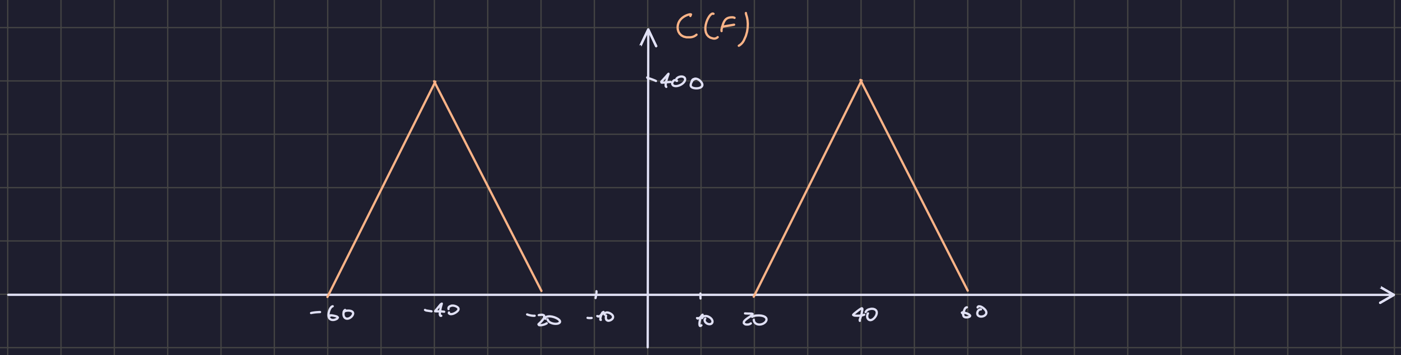


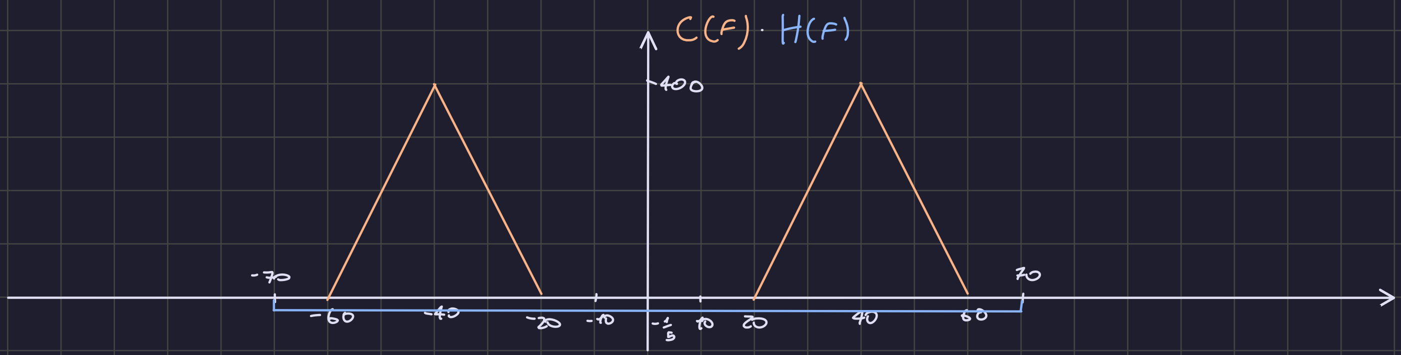


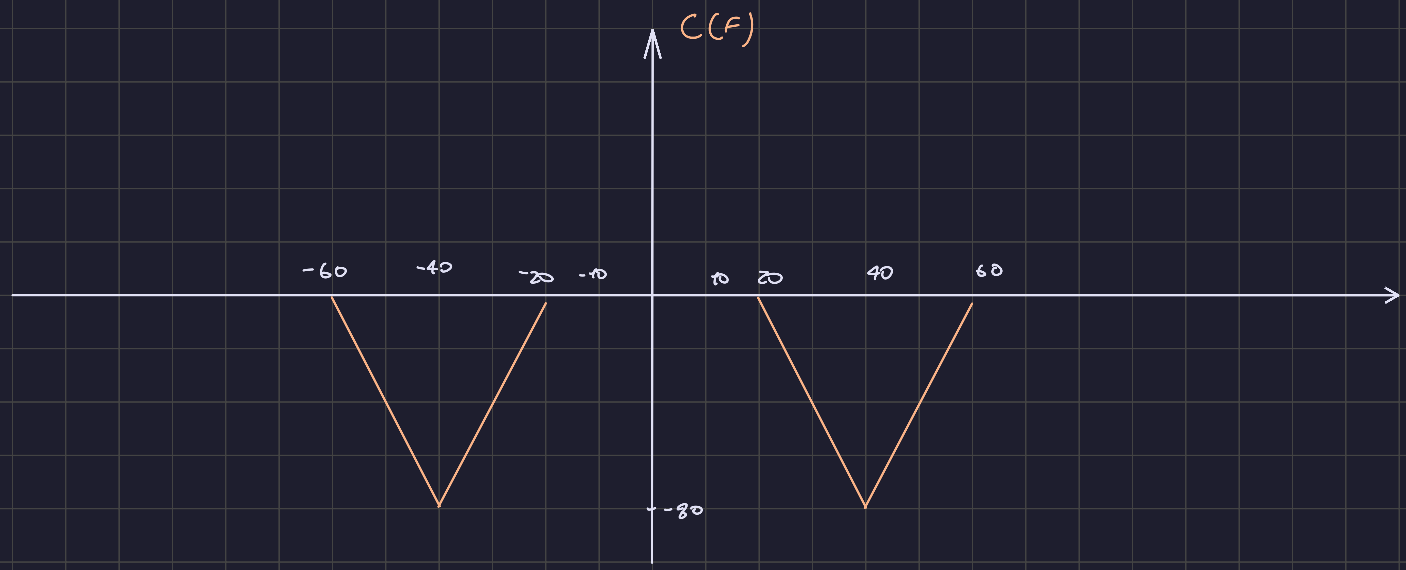

400
200
0.1
-0.1
-35
35
-45
-20
-10
10
20
45


B(f)
40
20
-40
-20
-10
10
20
40


W(f)
10
-40
-20
-10
10
20
40


C(f)
400
-60
-40
-20
-10
10
20
40
60


C(f) · H(f)
400
-70
70
-60
-40
-20
-10
-1/5
10
20
40
60



$$V(f) = \left(\delta(f-40) + \delta(f+40)\right) * -80 \Lambda\left(\frac{f}{20}\right)$$

$$\downarrow \mathcal{F}^{-1}$$

$$v(t) = 2\cos(2\pi \cdot 40t) \cdot \left(-80 \cdot 20 \,\mathrm{sinc}^2(20t)\right)$$

No perchè c'è aliasing. Questo accade perchè abbiamo che $f_c \leq 2B$ dove $B = 20$.