Algebra Lineare - Esercizi da consegnare

UniVR - Dipartimento di Informatica

Scheda 1

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1.1 Esercizio 1

Date le seguenti matrici a coefficienti complessi:

$$A = \begin{pmatrix} i & 0 \\ -1 & 1+i \end{pmatrix} \quad B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 \\ \frac{1}{2} & -1 \end{pmatrix}$$

calcolare

(a) $(CD)A$

$$\begin{aligned} (CD)A &= \left(\begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix} \begin{pmatrix} 2 & 0 \\ \frac{1}{2} & -1 \end{pmatrix} \right) \begin{pmatrix} i & 0 \\ -1 & 1+i \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{2} & -1 \\ 2+\frac{1}{2}i & -i \end{pmatrix} \begin{pmatrix} i & 0 \\ -1 & 1+i \end{pmatrix} \\ &= \begin{pmatrix} 1+\frac{5}{2}i & -1-i \\ -\frac{1}{2}+3i & 1-i \end{pmatrix} \end{aligned}$$

(b) $B^T B$

$$B^T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$B^T B = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

(c) $3A\left(B + 4D^T\right)$

$$D^T = \begin{pmatrix} 2 & \frac{1}{2} \\ 0 & -1 \end{pmatrix}$$

$$\begin{aligned} 3A\left(B + 4D^T\right) &= 3 \begin{pmatrix} i & 0 \\ -1 & 1+i \end{pmatrix} \left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + 4 \begin{pmatrix} 2 & \frac{1}{2} \\ 0 & -1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 3i & 0 \\ -3 & 3+3i \end{pmatrix} \left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 8 & 2 \\ 0 & -4 \end{pmatrix} \right) \\ &= \begin{pmatrix} 3i & 0 \\ -3 & 3+3i \end{pmatrix} \begin{pmatrix} 8 & 2-i \\ i & -4 \end{pmatrix} \\ &= \begin{pmatrix} 24i & 3+6i \\ -27+3i & -18-9i \end{pmatrix} \end{aligned}$$

(d) $C^2 A^T$

$$A^T = \begin{pmatrix} i & -1 \\ 0 & 1+i \end{pmatrix}$$

$$\begin{aligned} C^2 A^T &= \left(\begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix} \right) \begin{pmatrix} i & -1 \\ 0 & 1+i \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1+i \\ 1+i & 0 \end{pmatrix} \begin{pmatrix} i & -1 \\ 0 & 1+i \end{pmatrix} \\ &= \begin{pmatrix} 2i & -2+2i \\ -1+i & -1-i \end{pmatrix} \end{aligned}$$

(e) $\frac{1}{2}\left(B^2 - 3D^TC\right)$

$$D^T = \begin{pmatrix} 2 & \frac{1}{2} \\ 0 & -1 \end{pmatrix}$$

$$\begin{aligned} \frac{1}{2}\left(B^2 - 3D^TC\right) &= \frac{1}{2}\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - 3\begin{pmatrix} 2 & \frac{1}{2} \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix}\right) \\ &= \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 6 & \frac{3}{2} \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix}\right) \\ &= \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{15}{2} & 6 + \frac{3}{2}i \\ -3 & -3i \end{pmatrix}\right) \\ &= \frac{1}{2}\begin{pmatrix} -\frac{13}{2} & -6 - \frac{3}{2}i \\ 3 & 1 + 3i \end{pmatrix} \\ &= \begin{pmatrix} -\frac{13}{4} & -3 - \frac{3}{4}i \\ \frac{3}{2} & \frac{1}{2} + \frac{3}{2}i \end{pmatrix} \end{aligned}$$

1.2 Esercizio 2

Date le seguenti matrici a coefficienti complessi:

$$A = \begin{pmatrix} 1 & -1 & -1 & 7 \\ 3+i & -3-i & -2-i & 11+7i \\ 3 & -3 & -2 & 11 \end{pmatrix} \quad B = \begin{pmatrix} 5 & -5 \\ i & -1 \\ 5 & -5+i \end{pmatrix}$$

$$C = \begin{pmatrix} i & 0 & -1 \\ 7 & i & 0 \\ 6 & 1 & -7+6i \end{pmatrix} \quad D = \begin{pmatrix} 1 & -2 & \frac{1}{2} & 4 & 0 \\ 1 & -1 & \frac{1}{2} & 4+i & 1 \\ 3 & -5 & \frac{3}{2} & 12+i & 1 \end{pmatrix}$$

(a) Usare l'algoritmo di Eliminazione di Gauss per determinare una forma ridotta di ognuna delle matrici:

(i) A

$$\begin{pmatrix} 1 & -1 & -1 & 7 \\ 3+i & -3-i & -2-i & 11+7i \\ 3 & -3 & -2 & 11 \end{pmatrix} \overset{R_2-(3+i)R_1}{\rightsquigarrow}$$

$$\begin{pmatrix} 1 & -1 & -1 & 7 \\ 0 & 0 & 1 & -10 \\ 3 & -3 & -2 & 11 \end{pmatrix} \overset{R_3-3R_1}{\rightsquigarrow}$$

$$\begin{pmatrix} 1 & -1 & -1 & 7 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 1 & -10 \end{pmatrix} \overset{R_3-R_2}{\rightsquigarrow}$$

$$\begin{pmatrix} 1 & -1 & -1 & 7 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(ii) B

$$\begin{pmatrix} 5 & -5 \\ i & -1 \\ 5 & -5+i \end{pmatrix} \overset{\frac{1}{5}R_1}{\rightsquigarrow} \begin{pmatrix} 1 & -1 \\ i & -1 \\ 5 & -5+i \end{pmatrix} \overset{R_2-iR_1}{\rightsquigarrow}$$

$$\begin{pmatrix} 1 & -1 \\ 0 & -1+i \\ 5 & -5+i \end{pmatrix} \overset{R_3-5R_1}{\rightsquigarrow} \begin{pmatrix} 1 & -1 \\ 0 & -1+i \\ 0 & i \end{pmatrix} \overset{\frac{1}{-1+i}R_2}{\rightsquigarrow}$$

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & i \end{pmatrix} \overset{R_3-iR_2}{\rightsquigarrow} \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(iii) C

$$\begin{pmatrix} i & 0 & -1 \\ 7 & i & 0 \\ 6 & 1 & -7+6i \end{pmatrix} \overset{-iR_1}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & i \\ 7 & i & 0 \\ 6 & 1 & -7+6i \end{pmatrix} \overset{R_2-7R_1}{\rightsquigarrow}$$

$$\begin{pmatrix} 1 & 0 & i \\ 0 & i & -7i \\ 6 & 1 & -7+6i \end{pmatrix} \overset{R_3-6R_1}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & i \\ 0 & i & -7i \\ 0 & 1 & -7 \end{pmatrix} \overset{-iR_2}{\rightsquigarrow}$$

$$\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & -7 \\ 0 & 1 & -7 \end{pmatrix} \overset{R_3-R_2}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{pmatrix}$$

(iv) D

$$\begin{pmatrix} 1 & -2 & \frac{1}{2} & 4 & 0 \\ 1 & -1 & \frac{1}{2} & 4+i & 1 \\ 3 & -5 & \frac{3}{2} & 12+i & 1 \end{pmatrix} \overset{R_2-R_1}{\rightsquigarrow} \begin{pmatrix} 1 & -2 & \frac{1}{2} & 4 & 0 \\ 0 & 1 & 0 & i & 1 \\ 3 & -5 & \frac{3}{2} & 12+i & 1 \end{pmatrix}$$

$$\overset{R_3-3R_1}{\rightsquigarrow} \begin{pmatrix} 1 & -2 & \frac{1}{2} & 4 & 0 \\ 0 & 1 & 0 & i & 1 \\ 0 & 1 & 0 & i & 1 \end{pmatrix} \overset{R_3-R_2}{\rightsquigarrow} \begin{pmatrix} 1 & -2 & \frac{1}{2} & 4 & 0 \\ 0 & 1 & 0 & i & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Calcolare il rango di ognuna delle matrici:

(i) $rk(A) = 2$

(ii) $rk(B) = 2$

(iii) $rk(C) = 2$

(iv) $rk(D) = 2$

(c) Scrivere i sistemi lineari per cui le matrici A, B, C, D sono le corrispondenti matrici aumentate ed usare il Teorema di Rouchè-Capelli per stabilire se tali sistemi lineari ammettono o non ammettono soluzioni

(i)

$$A = \begin{cases} x - y - z = 7 \\ z = -10 \end{cases} \qquad \text{Ha infinite soluzioni}$$

(ii)

$$B = \begin{cases} x = -1 \\ 0 = 1 \\ 0 = 0 \end{cases} \qquad \text{Non ha soluzioni}$$

(iii)

$$C = \begin{cases} x = i \\ y = -7 \\ 0 = 0 \end{cases} \qquad \text{Ha una sola soluzione}$$

(iv)

$$D = \begin{cases} x - 2y + \frac{1}{2}z + 4w = 0 \\ y + iw = 1 \\ 0 = 0 \end{cases} \qquad \text{Ha infinite soluzioni}$$

(d) Trovare tutte le soluzioni del sistema lineare $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1-i \\ -1 \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 7 & -1 \\ 3+i & -3-i & -2-i & 11+7i & -1-i \\ 3 & -3 & -2 & 11 & -1 \end{array}\right) \underset{\rightsquigarrow}{R_2-(3+i)R_1}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 7 & -1 \\ 0 & 0 & 1 & -10 & 2 \\ 3 & -3 & -2 & 11 & -1 \end{array}\right) \underset{\rightsquigarrow}{R_3-3R_1}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 7 & -1 \\ 0 & 0 & 1 & -10 & 2 \\ 0 & 0 & 1 & -10 & 2 \end{array}\right) \underset{\rightsquigarrow}{R_3-R_2}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 7 & -1 \\ 0 & 0 & 1 & -10 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 - x_2 - x_3 + 7x_4 = -1 \\ x_3 - 10x_4 = 2 \\ 0 = 0 \end{cases}$$

$$\begin{cases} x_1 = x_2 + x_3 - 7x_4 - 1 \\ x_3 = 10x_4 + 2 \end{cases}$$

Assegno dei parametri a x_2 e x_4

$$x_2 = t \quad x_4 = s$$

$$\begin{cases} x_1 = t + 3s + 1 \\ x_2 = t \\ x_3 = 10s + 2 \\ x_4 = s \end{cases}$$

Le soluzioni del sistema lineare sono:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} t + 3s + 1 \\ t \\ 10s + 2 \\ s \end{pmatrix}$$

(e) Trovare tutte le soluzioni del sistema lineare omogeneo associato alla matrice B

$$\begin{cases} x_1 - x_2 = 0 \\ x_2 = 0 \end{cases}$$

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

1.3 Esercizio 3

Si consideri la matrice

$$A_t = \begin{pmatrix} t-1 & 2t-2 & t-1 \\ t+1 & 2t+2 & t+1 \\ 1 & 2 & t+1 \end{pmatrix} \qquad \text{con } t \in \mathbb{R}$$

(a) Calcolare il rango $rk(A_t)$ per ogni valore di $t \in \mathbb{R}$

$$\begin{pmatrix} t-1 & 2t-2 & t-1 \\ t+1 & 2t+2 & t+1 \\ 1 & 2 & t+1 \end{pmatrix} \overset{\frac{1}{t-1}R_1}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 1 \\ t+1 & 2t+2 & t+1 \\ 1 & 2 & t+1 \end{pmatrix}$$

$$\overset{R_2-(t+1)R_1}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & t+1 \end{pmatrix} \overset{R_3-R_1}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & t \end{pmatrix}$$

$$\overset{R_3 \leftrightarrow R_2}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \overset{\frac{1}{t}R_2}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$rk(U) = rk(A_t) = 2 \quad \forall t \in \mathbb{R} - \{0, 1\}$$

(b) Se A_t è la matrice aumentata di un sistema lineare, per quali valori di $t \in \mathbb{R}$ tale sistema ammette soluzioni?

$$A_t = \begin{cases} x_1 + 2x_2 = 1 \\ 0 = 1 \end{cases}$$

Il sistema lineare non ammette soluzioni per nessun valore di $t \in \mathbb{R}$

1.4 Esercizio 4

Trovare tutte le soluzioni complesse di $x^3 - 1 = 0$

$$x^3 - 1 = 0$$
$$x^3 = 1$$
$$x = \sqrt[3]{1}$$

$$z = a + bi \quad a = 1,\ b = 0$$
$$r = \sqrt{a^2 + b^2} = 1$$
$$\alpha = \arctan\frac{b}{a} = \arctan\frac{0}{1} = 0$$

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\alpha + 2k\pi}{n} + i\sin\frac{\alpha + 2k\pi}{n}\right)$$
$$x = \sqrt[3]{1} = \cos\left(\frac{0 + 2k\pi}{3}\right) + i\sin\left(\frac{0 + 2k\pi}{3}\right) \qquad k = 0, 1, 2$$

$$x_0 = \cos\left(\frac{0}{3}\right) + i\sin\left(\frac{0}{3}\right) = 1$$
$$x_1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$x_2 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$