Esercizi risposta libera

1. Calcolare la risposta libera del sistema a tempo continuo (LTI) descritto come:

$$\frac{d^2v(t)}{dt^2} + \frac{dv(t)}{dt} + v(t) = \frac{du(t)}{dt},$$

considerando le seguenti condizioni inziali:

$$v(0^-) = 2 \qquad \frac{dv(0^-)}{dt} = 0.$$

Equazione del sistema:

$$v''(t) + v'(t) + v(t) = u'(t)$$

Condizioni iniziali:

$$\begin{cases} v(0^-) = 2 \\ v'(0^-) = 0 \end{cases}$$

Equazione omogenea del polinomio caratteristico

$$P(s) = s^2 + s + 1 = 0$$

Soluzioni:

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\lambda_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \qquad r = 2 \quad (\text{num. soluzioni})$$

$$\lambda_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i \qquad \mu_{1,2} = 1 \quad (\text{molteplicità})$$

Risposta libera generica

$$v_L(t) = \sum_{i=1}^{r} \sum_{l=0}^{\mu_i - 1} c_{i,l} \cdot e^{\lambda_i t} \cdot \frac{t^l}{l!}$$

$$= \left(c_{1,0} \cdot e^{\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)t} \cdot \frac{t^0}{0!} \right) + \left(c_{2,0} \cdot e^{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)t} \cdot \frac{t^0}{0!} \right)$$

$$= c_{1,0} \cdot e^{\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)t} + c_{2,0} \cdot e^{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)t}$$

Derivata della risposta libera

$$v_L(t) = c_1 \cdot e^{\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)t} + c_2 \cdot e^{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)t}$$

$$v_L'(t) = c_1 \cdot e^{\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)t} \cdot \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) + c_2 \cdot e^{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)t} \cdot \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)$$

Calcolo dei coefficienti c_1 e c_2

$$\begin{cases} v(0^-) = c_1 \cdot e^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)0^-} + c_2 \cdot e^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)0^-} \\ v'(0^-) = c_1 \cdot e^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)0^-} \cdot \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) + c_2 \cdot e^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)0^-} \cdot \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) \end{cases}$$

$$\begin{cases} v(0^-) = c_1 \cdot e^{0^-} + c_2 \cdot e^{0^-} \\ v'(0^-) = \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) c_1 \cdot e^{0^-} + \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) c_2 \cdot e^{0^-} \end{cases}$$

$$\begin{cases} v(0^-) = c_1 + c_2 \\ v'(0^-) = \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) c_1 + \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) c_2 \end{cases}$$

$$\begin{cases} v(0^-) = 2 \\ v'(0^-) = 0 \end{cases} \rightarrow \begin{cases} c_1 + c_2 = 2 \\ \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) c_1 + \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) c_2 = 0 \end{cases}$$

$$\begin{cases} c_1 = 2 - c_2 \\ \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)(2-c_2) + \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) c_2 = 0 \end{cases}$$

$$\begin{cases} c_1 = 2 - c_2 \\ (-1+\sqrt{3}i) - \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) c_2 + \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) c_2 = 0 \end{cases}$$

$$\begin{cases} c_1 = 2 - c_2 \\ c_2 \left(\cancel{\frac{1}{2}} - \frac{\sqrt{3}}{2}i \cancel{-\frac{1}{2}} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}i \end{cases}$$

$$\begin{cases} c_1 = 2 - c_2 \\ -\sqrt{3}i\, c_2 = 1 - \sqrt{3}i \end{cases}$$

$$\begin{cases} c_1 = 2 - c_2 \\ c_2 = \frac{1-\sqrt{3}i}{-\sqrt{3}i} = -\frac{1}{\sqrt{3}i} + \frac{\sqrt{3}i}{\sqrt{3}i} = 1 - \frac{\sqrt{3}i}{-3} \end{cases}$$

$$\begin{cases} c_1 = 2 - 1 + \frac{\sqrt{3}i}{-3} = 1 + \frac{\sqrt{3}i}{-3} \\ c_2 = 1 - \frac{\sqrt{3}i}{-3} \end{cases}$$

$$\begin{cases} c_1 = 1 + \frac{\sqrt{3}i}{-3} \\ c_2 = 1 - \frac{\sqrt{3}i}{-3} \end{cases}$$

Risposta libera specifica:

$$v_c(t) = \left(1 + \frac{\sqrt{3}\,i}{-3}\right) \cdot e^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)t} + \left(1 - \frac{\sqrt{3}\,i}{-3}\right) \cdot e^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)t}$$