Esame 19/06/2024

Esercizio 5 (punti: ....../4)

a) (2 punti) Determinare e classificare i punti stazionari della funzione $f(x, y) = x^3 + y^2 - 11x$.

$$F(x,y) = x^3 + y^2 - 11x$$

$$\nabla f(x,y) = \begin{pmatrix} 3x^2 - 11 \\ 2y \end{pmatrix}$$

$$\nabla F(x,y) = 0$$

$$\downarrow$$

$$\begin{cases} 3x^2 - 11 = 0 \\ 2y = 0 \end{cases} \rightarrow \begin{cases} x = \pm\sqrt{\frac{11}{3}} \\ y = 0 \end{cases}$$

$A\left(\sqrt{\frac{11}{3}}, 0\right)$ $\qquad$ $B\left(-\sqrt{\frac{11}{3}}, 0\right)$

$$H_F(x,y) = \begin{pmatrix} 6x & 0 \\ 0 & 2 \end{pmatrix}$$

Troviamo autovalori

$$\det(H_F - \lambda I) = \det\begin{pmatrix} 6x - \lambda & 0 \\ 0 & 2-\lambda \end{pmatrix}$$

$$= (6x - \lambda)(2 - \lambda)$$

$$\lambda_1 = 6x$$

$$\lambda_2 = 2$$

$$H_F(A) = \begin{pmatrix} 6\sqrt{\frac{11}{3}} & 0 \\ 0 & 2 \end{pmatrix} \rightarrow \begin{matrix} \lambda_1 > 0 \\ \lambda_2 > 0 \end{matrix} \rightarrow A \text{ Minimo locale}$$

$$H_F(B) = \begin{pmatrix} -6\sqrt{\frac{11}{3}} & 0 \\ 0 & 2 \end{pmatrix} > 0 \rightarrow \begin{matrix} \lambda_1 < 0 \\ \lambda_2 > 0 \end{matrix} \rightarrow B \text{ Sella}$$

b) (2 punti) Si consideri l'insieme

$$E = \{(x, y) \in \mathbb{R}^2 : 4x^2 + y^2 = 1\}$$

La funzione f, ristretta a E, ha estremi assoluti? Motivare la risposta e, in caso affermativo, calcolarli.

$E = \{(x,y) \in \mathbb{R}^2 : 4x^2+y^2=1\}$ $\qquad F(x,y) = x^3+y^2-11x$

$\downarrow$

$(2x)^2+y^2=1$

Coord polari

$$\begin{cases} 2x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \rightarrow \begin{cases} x = \frac{\rho}{2}\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$\theta \in [0, 2\pi]$

$\rho = 1$ (Raggio)

$\downarrow$

$$\begin{cases} x = \frac{1}{2}\cos\theta \\ y = \sin\theta \end{cases}$$

$$F(\theta) = \left(\tfrac{1}{2}\cos\theta\right)^3 + \sin^2\theta - \tfrac{11}{2}\cos\theta$$

$$= \frac{1}{8}\cos^3\theta + \sin^2\theta - \frac{11}{2}\cos\theta$$

$$F'(\theta) = -\frac{1}{8}\cdot 3\cos^2\theta\sin\theta + 2\sin\theta\cos\theta + \frac{11}{2}\sin\theta$$

$$= -\frac{1}{8}\cdot 3\cos^2\theta\sin\theta + \sin(2\theta) + \frac{11}{2}\sin\theta$$

$$F' = 0 \rightarrow -\frac{1}{8}\cdot 3\cos^2\theta\sin\theta + \sin(2\theta) + \frac{11}{2}\sin\theta = 0$$

$$\sin\theta\left(-\frac{1}{8}\cdot 3\cos^2\theta + 2\cos\theta + \frac{11}{2}\right) = 0$$

$$\sin\theta = 0 \ \vee\ -\frac{1}{8}\cdot 3\cos^2\theta + 2\cos\theta + \frac{11}{2} = 0$$

$$\theta = 0 \vee \theta = \pi \ \vee\ \underbrace{\cos\theta\left(-\frac{3}{8}\cos\theta + 2\right) = -\frac{11}{2}}_{\nexists\, \theta \in [0, 2\pi]}$$

Punti stazionari: $F(0)$ $F(\pi)$ — $F(\frac{1}{2}, 0)$ $F(-\frac{1}{2}, 0)$

$$F(0) = \frac{1}{8} - \frac{11}{2} = \frac{1-44}{8} = -\frac{43}{8} \qquad \text{Minimo assoluto}$$

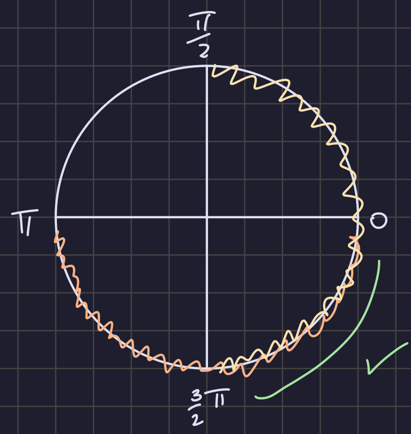
$$F(\pi) = -\frac{1}{8} + \frac{11}{2} = \frac{-1+44}{8} = \frac{43}{8} \qquad \text{Massimo assoluto}$$

Esercizio 6 (punti: ....../4)
Sia $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq a^2, x \geq 0,\ y \leq 0\}$.
Determinare per quale valore di $a \in \mathbb{R}^+$ si ha

$$\iint_D xy(x^2+y^2)^{\frac{3}{2}}\,dx\,dy = -\frac{1}{14}$$

$x^2+y^2=a^2$
$\downarrow$
Polari

$$\begin{cases} y \leq 0 \Rightarrow \sin\theta \geq 0 \\ x \geq 0 \Rightarrow \cos\theta \geq 0 \end{cases} \rightarrow$$



$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad \begin{matrix} \theta \in [\frac{3}{2}\pi, 2\pi] \\ \rho \in [0, a] \end{matrix}$$

$$\iint_D \rho^2\cos\theta\sin\theta\,(\rho^2)^{\frac{3}{2}}\,\overbrace{\rho}^{\text{Jacobiano}}\,d\rho\,d\theta$$

$$\iint_D \rho^2\cos\theta\sin\theta\,\rho^3\,\rho\,d\rho\,d\theta$$

$$\iint_D \rho^6\cos\theta\sin\theta\,d\rho\,d\theta$$

$$\int_0^a \rho^6\,d\rho \cdot \int_{\frac{3}{2}\pi}^{2\pi} \cos\theta\sin\theta\,d\theta \qquad \begin{matrix} u = \sin\theta \\ du = \cos\theta\,d\theta \end{matrix}$$

$$\left[\frac{\rho^7}{7}\right]_0^a \cdot \left[\frac{\sin^2\theta}{2}\right]_{\frac{3}{2}\pi}^{2\pi}$$

$$\frac{a^7}{7} \cdot \left(0 - \left(\frac{(-1)^2}{2}\right)\right)$$

$$\frac{a^7}{7} \cdot \left(-\frac{1}{2}\right)$$

$$-\frac{a^7}{14} = -\frac{1}{4}$$

$$a = 1$$

Esercizio 7 (punti: ....../4)
Sia $\Omega = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D,\ 0 \le z \le x+4y\}$, dove D è la regione finita nel piano xy limitata da $y=-2x$ e $y=x^2$. Calcolare

$$\iiint_\Omega x\,dx\,dy\,dz$$

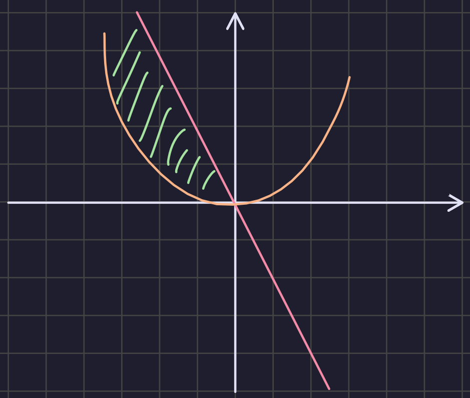
$y = -2x$

$y = x^2$

$-2x = x^2$

$x = -2$

$D \rightarrow$



$$\iiint_\Omega x\,dx\,dy\,dz =$$

$$= \iint_D \int_0^{x+4y} dz\ dx\,dy$$

$$= \int_{-2}^{0} \int_{-2x}^{x^2} x \int_0^{x+4y} dz\,dy\,dx$$

$$= \int_{-2}^{0} \int_{-2x}^{x^2} x\,(x+4y)\ dy\,dx$$

$$= \int_{-2}^{0} \int_{-2x}^{x^2} x^2 + 4xy\ dy\,dx$$

$$= \int_{-2}^{0} \left[x^2 y + 4x\,\frac{y^2}{2}\right]_{-2x}^{x^2} dx$$

$$= \int_{-2}^{0} \left(x^2 x^2 + 2x\,x^4 - x^2(-2x) + 2x(-2x)^2\right) dx$$

$$= \int_{-2}^{0} \left(x^4 + 2x^5 - \left(-2x^3 + 8x^3\right)\right) dx$$

$$x\int_{-2x}^{x^2} dy + \int_{-2x}^{x^2} 4y\,dy$$

$$= -\int_{-2}^{0} x^4 + 2x^5 - 6x^3 \, dx$$

$$= -\left(\left[\frac{x^5}{5}\right]_{-2}^{0} + 2\left[\frac{x^6}{6}\right]_{-2}^{0} - 6\left[\frac{x^4}{4}\right]_{-2}^{0} \right)$$

$$= -\left(\frac{2^5}{5} - 2\,\frac{2^6}{6} + 6\,\frac{2^4}{4} \right)$$

$$= -\left(\frac{2^5}{5} - \frac{2^7}{6} + \frac{2^5 \cdot 3}{4} \right)$$

$$= -\frac{2^5(6\cdot 4) - 2^7(5\cdot 4) + 2^5 \cdot 3\,(5\cdot 6)}{5\cdot 6\cdot 4}$$

$$= -\frac{768 - 2560 + 2880}{120}$$

$$= -\frac{1088}{120}$$

$$= -\frac{136}{15}$$

Esercizio 8 (punti: ....../4)

a) (2 punti) Il campo vettoriale piano

$$\vec{F}(x, y) = (x^2 + y,\ -y^2 - 2x)$$

è somma dei campi $\vec{F_1}$, $\vec{F_2}$, dove $\vec{F_1}(x, y) = (0,\ -3x)$ e $\vec{F_2}(x, y) = (x^2 + y,\ -y^2 + x)$.
Verificare che $\vec{F_2}$ è conservativo e trovarne un potenziale.

$$\vec{F}(x,y) = (x^2 + y,\ -y^2 - 2x)$$

$$\vec{F}_1(x,y) = (0,\ -3x) \qquad \vec{F}_2(x,y) = (x^2 + y,\ -y^2 + x)$$

$$\vec{F} = \vec{F}_1 i + \vec{F}_2 j$$

$$\nabla \times \vec{F} = 0$$

$$\nabla \times \vec{F}_2 = \det \begin{pmatrix} i & j & k \\ \partial x & \partial y & 0 \\ F_{2x} & F_{2y} & 0 \end{pmatrix} = k\left(\frac{\partial \vec{F}_{2y}}{dx} - \frac{d\vec{F}_{2x}}{dy} \right) = k(1-1) = 0$$

$$\Downarrow$$

è conservativo

$$U_x = \frac{dU}{dx} \qquad \begin{cases} U_x = F_{2x} \\ U_y = F_{2y} \end{cases} \rightarrow \begin{cases} U_{xy} = \partial_y F_{2x} \\ U_{yx} = \partial_x F_{2y} \end{cases}$$

$$\frac{dU}{dx} = x^2 + y \rightarrow U = \int x^2 + y + c(y)\, dx = \frac{x^3}{3} + xy + C(y)$$

$$\frac{dU}{dy} = \frac{\partial\left(\frac{x^3}{3} + xy + C(y)\right)}{dy} = x + C'(y)$$

$$x + c'(y) = \vec{F}_{2y}$$

$$\cancel{x} + c'(y) = -y^2 + \cancel{x}$$

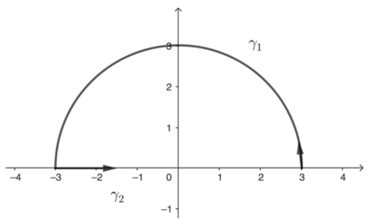
$$c'(y) = -y^2$$

$$c(y) = -\int y^2\, dy$$

$$C(y) = -\frac{y^3}{3} + k$$

$$U_{F_2} = \frac{x^3}{3} + xy - \frac{y^3}{3} + k$$

b) (2 punti) Calcolare il lavoro del campo $\vec{F}$ lungo il cammino chiuso orientato $\gamma_1 \cup \gamma_2$ in figura, che ha punto d'inizio $A = (3, 0)$.



$$\vec{F}(x,y) = (x^2 + y, -y^2 - 2x)$$

$$\nabla \times \vec{F} = \det\begin{pmatrix} i & j & k \\ \partial x & \partial y & 0 \\ \vec{F}_x & \vec{F}_y & 0 \end{pmatrix} = k\left(\frac{\partial \vec{F}_y}{\partial x} - \frac{\partial \vec{F}_x}{\partial y}\right) = k(-2-1) = k(-3)$$

Non Conservativo

$$L = \int_\gamma \vec{F} \, d\vec{l}$$

$\vec{F}_2$ é conservativo $\rightarrow L = 0$

$$= \int_\gamma \vec{F}_1 \, d\vec{l} + \cancel{\int_\gamma \vec{F}_2 \, d\vec{l}}$$

$$= \int_{\gamma_1} \vec{F}_1 \, d\vec{l} + \cancel{\int_{\gamma_2} \vec{F}_1 \, d\vec{l}}$$

$\vec{F}_1$ é $\perp$ a γ_2 $\rightarrow L = 0$

$$= \int_{\gamma_1} F_1 \, d\vec{l}$$

$$= \int_0^\pi \vec{F}_1(\gamma_1(t)) \cdot \gamma'(t) \, dt$$

$\gamma_1(t) = (3\cos t, 3\sin t) \quad t \in [0, \pi)$

$\gamma_1'(t) = (-3\sin t, 3\cos t)$

$$= \int_0^\pi (0, -9\cos t) \cdot (-3\sin t, 3\cos t) \, dt$$

$$= \int_0^\pi -27\cos^2 t \, dt$$

$$= -27 \int_0^\pi \cos^2 t \, dt$$

$$\cos^2 t = \frac{1 + \cos(2t)}{2}$$

$$= -27 \int_0^\pi \frac{1 + \cos(2t)}{2} \, dt$$

$$= -\frac{27}{2} \int_0^\pi 1 + \cos(2t) \, dt$$

$$= -\frac{27}{2} \left(\int_0^\pi dt + \int_0^\pi \cos(2t) \, dt \right)$$

$$= -\frac{27}{2} \left(\pi + \int_0^\pi \cos(2t) \, dt \right)$$

$u = 2t$

$du = 2\,dt$

$$= -\frac{27}{2} \left(\pi + \frac{1}{2} \Big[\sin(2t) \Big]_0^\pi \right)$$

$$= -\frac{27}{2} \pi$$

Esercizio 1 (punti: ....../4)
Tra tutte le soluzione della seguente equazione differenziale

$$y' - \frac{y}{\sqrt{x}} = xy$$

determinare quella per cui

$$\lim_{x \to 0^+} y(x) = \sqrt{e}$$

$$y' - \frac{y}{\sqrt{x}} = xy$$

$$y' = xy + \frac{y}{\sqrt{x}}$$

$$y' = y\left(x + \frac{1}{\sqrt{x}}\right)$$

$$\frac{y'}{y} = x + \frac{1}{\sqrt{x}}$$

$$\int \frac{y'}{y}\, dy = \int x\, dx + \int x^{-\frac{1}{2}}\, dx \qquad -\frac{1}{2} + \frac{2}{2}$$

$$\ln|y| = \frac{x^2}{2} + 2\sqrt{x} + c$$

$$y = e^{\frac{x^2}{2} + 2\sqrt{x} + c}$$

$$\lim_{x \to 0^+} y(x) = \sqrt{e} \rightarrow y(0) = \sqrt{e}$$

$$\sqrt{e} = e^{\frac{0^2}{2} + 2\sqrt{0} + c}$$

$$\sqrt{e} = e^{c}$$

$$c = \frac{1}{2}$$

$$y(x) = e^{\frac{x^2}{2} + 2\sqrt{x} + \frac{1}{2}}$$

$$y(0) = \sqrt{e}$$

Esercizio 2 (punti: ....../4)
Le funzioni $x(t)$, $y(t)$, definite per ogni $t \in \mathbb{R}$, verificano il seguente sistema di equazioni differenziali del primo ordine:

$$(*) \begin{cases} x' = 2x + 3y + 9t^2 - 8e^t \\ y' = x \end{cases}$$

Sapendo che $(*)$ è equivalente a $y'' - 2y' - 3y = 9t^2 - 8e^t$, risolverlo con le condizioni iniziali $x(0) = 0$, $y(0) = -\frac{8}{3}$

$s^2 - 2s - 3 = 0$

$(s-3)(s+1) = 0$

$s_1 = -1$

$s_2 = 3$

$z = c_1 e^{-t} + c_2 e^{3t}$

$\bar{y} = a + bt + ct^2 + de^t$

$\bar{y}' = b + 2ct + de^t$

$\bar{y}'' = 2c + de^t$

$2c + de^t - 2(b + 2ct + de^t) - 3(a + bt + ct^2 + de^t) = 9t^2 - 8e^t$

$2c + de^t - 2b - 4ct - 2de^t - 3a - 3bt - 3ct^2 - 3de^t = 9t^2 - 8e^t$

$2c - 2b - 3a + e^t(d - 2d - 3d) + t(-4c - 3b) + t^2(-3c) = 9t^2 - 8e^t$

$$\begin{cases} 2c - 2b - 3a = 0 \\ d - 2d - 3d = -8 \\ -4c - 3b = 0 \\ -3c = 9 \end{cases} \rightarrow \begin{cases} 2c - 2b - 3a = 0 \\ d = 2 \\ -4c - 3b = 0 \\ -3c = 9 \end{cases} \rightarrow \begin{cases} -6 - 8 - 3a = 0 \\ d = 2 \\ b = 4 \\ c = -3 \end{cases} \rightarrow \begin{cases} a = -\frac{14}{3} \\ d = 2 \\ b = 4 \\ c = -3 \end{cases}$$

$\bar{y} = a + bt + ct^2 + de^t = -\frac{14}{3} + 4t - 3t^2 + 2e^t$

$y(t) = z + \bar{y} = c_1 e^{-t} + c_2 e^{3t} - \frac{14}{3} + 4t - 3t^2 + 2e^t$

$y'(t) = -c_1 e^{-t} + 3c_2 e^{3t} + 4 - 6t + 2e^t$

$y(0) = -\frac{8}{3}$

$y' = x \Rightarrow y'(0) = x(0) = 0$

$$\begin{cases} y(0) = -\frac{8}{3} \\ y'(0) = 0 \end{cases}$$

$\downarrow$

$$\begin{cases} C_1 + C_2 - \frac{14}{3} + 2 = -\frac{8}{3} \\ -C_1 + 3C_2 + 4 + 2 = 0 \end{cases} \rightarrow \begin{cases} C_1 + C_2 = 0 \\ -C_1 + 3C_2 = -6 \end{cases} \rightarrow \begin{cases} C_1 = -C_2 \\ C_2 + 3C_2 = -6 \end{cases} \rightarrow \begin{cases} C_1 = \frac{3}{2} \\ C_2 = -\frac{3}{2} \end{cases}$$

$$y(t) = \frac{3}{2} e^{-t} - \frac{3}{2} e^{3t} - \frac{14}{3} + 4t - 3t^2 + 2e^t$$

$$\begin{cases} x' = y''(t) = 2x + 3y + 9t^2 - 8e^t \\ y'(t) = -\frac{3}{2} e^{-t} - \frac{9}{2} e^{3t} + 4 - 6t + 2e^t = x \end{cases}$$

$$\begin{cases} y''(t) = -3e^{-t} - 9e^{3t} + 8 - 12t + 4e^t + 3\left(\frac{3}{2} e^{-t} - \frac{3}{2} e^{3t} - \frac{14}{3} + 4t - 3t^2 + 2e^t\right) + 9t^2 - 8e^t \\ y'(t) = -\frac{3}{2} e^{-t} - \frac{9}{2} e^{3t} + 4 - 6t + 2e^t = x \end{cases}$$

Esercizio 3 (punti: ....../4)
Si consideri la funzione

$$f(x, y) = e^{x + \ln(x^2 - 2x + y^2)} - xy^2$$

a) (2 punti) Indicato con D il dominio di f, rappresentare graficamente l'insieme

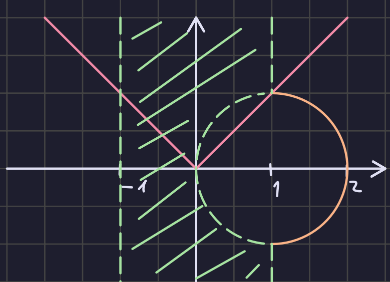
$$D \cap \{(x, y) \in \mathbb{R}^2 : |x| < 1\}$$

e dire se tale insieme è limitato/illimitato, aperto/chiuso, connesso/sconnesso (motivare le risposte!)

$D = x^2 - 2x + y^2 > 0$

$|x| < 1$

$D \cap \{(x, y) \in \mathbb{R}^2 : |x| < 1\}$



L'insieme è aperto, connesso per archi (perchè una linea che collega due punti qualsiasi potrebbe non avere tutti i suoi punti all'interno dell'insieme) e illimitato

b) (2 punti) Calcolare il valore di $\frac{\partial f}{\partial \vec{v}}(0,2)$, dove $\vec{v} = (\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}})$.

$$F(x,y) = e^{x+\ln(x^2-2x+y^2)} - xy^2 = e^x(x^2-2x+y^2) - xy^2$$

$$\nabla F(x,y) = \begin{pmatrix} e^x(x^2-2x+y^2) + e^x(2x-2) - y^2 \\ 2e^x y - 2xy \end{pmatrix}$$

$$\nabla F(0,2) = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$\frac{\partial F(0,2)}{\partial \vec{v}} = \nabla F(0,2) \cdot \vec{v} = (-2 \quad 4) \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix} = -\frac{2}{\sqrt{10}} + \frac{12}{\sqrt{10}} = \frac{10}{\sqrt{10}} = 10^{1-\frac{1}{2}} = \sqrt{10}$$

Esercizio 4 (punti: ....../4)

a) (2 punti) Quale valore (se esiste) bisogna assegnare alla funzione

$$f(x,y) = \frac{x^2+y^2-x^2y^2}{2x^2+2y^2}$$

nel punto $(0,0)$ affinché f sia definita e continua in $\mathbb{R}^2$?

$$F(x,y) = \frac{x^2+y^2-x^2y^2}{2x^2+2y^2} = \frac{\cancel{x^2+y^2}}{2(\cancel{x^2+y^2})} - \frac{x^2y^2}{2(x^2+y^2)} = \frac{1}{2} - \frac{x^2y^2}{2(x^2+y^2)}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2(x^2+y^2)} = \lim_{\rho\to 0} \frac{\rho^4 \cos^2\theta \sin^2\theta}{2\rho^2} = \lim_{\rho\to 0} \frac{1}{2}\rho^2 \underbrace{\cos^2\theta \sin^2\theta}_{\in [0,\frac{1}{4}]} = 0$$

oppure

$$0 \le \frac{x^2y^2}{2(x^2+y^2)} \le \underbrace{\frac{x^2}{x^2+y^2}}_{\le 1} \cdot \frac{y^2}{2} \le \frac{y^2}{2} \qquad \lim_{y\to 0} \frac{y^2}{2} = 0 \rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2(x^2+y^2)} = 0$$

Quindi

$$F(x,y) = \frac{1}{2} - \frac{x^2y^2}{2(x^2+y^2)} \qquad \lim_{(x,y)\to(0,0)} F(x,y) = \frac{1}{2} - 0 = \frac{1}{2}$$

b) (2 punti) Un arco di curva γ ha equazioni parametriche $\begin{cases} x(\theta) = \theta^2 \cos\theta \\ y(\theta) = \theta^2 \sin\theta \end{cases}$ con $\theta \in [0,\, 2\pi]$.

Calcolare la lunghezza di γ.

$$\gamma(\theta) = (\theta^2 \cos\theta\,,\ \theta^2 \sin\theta)$$

$$\gamma'(\theta) = (2\theta\cos\theta - \theta^2\sin\theta,\ 2\theta\sin\theta + \theta^2\cos\theta)$$

$$\|\gamma'(\theta)\| = \sqrt{(2\theta\cos\theta - \theta^2\sin\theta)^2 + (2\theta\sin\theta + \theta^2\cos\theta)^2}$$

$$= \sqrt{4\theta^2\cos^2\theta + \theta^4\sin^2\theta - \cancel{4\theta^3\cos\theta\sin\theta} + 4\theta^2\sin^2\theta + \theta^4\cos^2\theta + \cancel{4\theta^3\cos\theta\sin\theta}}$$

$$= \sqrt{4\theta^2\cos^2\theta + \theta^4\sin^2\theta + 4\theta^2\sin^2\theta + \theta^4\cos^2\theta}$$

$$= \theta\sqrt{4\cos^2\theta + \theta^2\sin^2\theta + 4\sin^2\theta + \theta^2\cos^2\theta}$$

$$= \theta\sqrt{\cos^2\theta(4+\theta^2) + \sin^2\theta(4+\theta^2)}$$

$$= \theta\sqrt{(4+\theta^2)(\cos^2\theta + \sin^2\theta)}$$

$$= \theta\sqrt{4+\theta^2}$$

$$= \theta\sqrt{4+\theta^2}$$

$$L_\gamma = \int_0^{2\pi} \|\gamma'(\theta)\|\, d\theta$$

$$= \int_0^{2\pi} \theta\sqrt{4+\theta^2}\, d\theta \qquad u = 4+\theta^2$$

$$du = 2\theta\, d\theta$$

$$= \frac{1}{2}\int \sqrt{u}\, du$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}(4+\theta^2)^{\frac{3}{2}}}{3}\right]_0^{2\pi}$$

$$= \frac{1}{3}\left((4+4\pi^2)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\right)$$

$$= \frac{1}{3}\left(\left(4\left(1+\pi^2\right)\right)^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)$$

$$= \frac{1}{3}\left(4^{\frac{3}{2}}\left(1+\pi^2\right)^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)$$

$$= \frac{1}{3}\left(4^{\frac{3}{2}}\left(\left(1+\pi^2\right)^{\frac{3}{2}} - 1\right)\right)$$

$$= \frac{4^{\frac{3}{2}}}{3}\left(\sqrt{\left(1+\pi^2\right)^3} - 1\right)$$

$$= \frac{8}{3}\left(\sqrt{\left(1+\pi^2\right)^3} - 1\right)$$