Esame 1/3/2024

PARTE 1

Esercizio 1 (punti: ....../4)
Determinare la soluzione del seguente problema di Cauchy

$$\begin{cases} 2yy' - xy^2 = x \\ y(0) = 1 \end{cases}$$

e indicare il più ampio **intervallo** su cui tale soluzione è definita.

$$2yy' - xy^2 = x$$

$$2yy' = x + xy^2$$

$$2yy' = x(1+y^2)$$

$$\frac{2yy'}{1+y^2} = x$$

$$\int \frac{2y}{1+y^2}\,dy = \frac{x^2}{2} + c$$

$$1+y^2 = u$$

$$du = 2y\,dy$$

$$\int \frac{1}{u}\,du = \frac{x^2}{2} + c$$

$$\ln(1+y^2) = \frac{x^2}{2} + c$$

$$1+y^2 = e^{\frac{x^2}{2}+c}$$

$$y(t) = \pm\sqrt{e^{\frac{x^2}{2}+c} - 1}$$

$$y(0) = 1$$

$$\Downarrow$$

$$1 = \pm\sqrt{e^c - 1}$$

$$e^c - 1 = 1$$

$$e^c = 2$$

$$c = \ln 2$$

$$y(t) = \pm\sqrt{2e^{\frac{x^2}{2}} - 1}$$

$$y(0) = +\sqrt{2-1} = +\sqrt{1} = 1$$

Siccome $y(0)=1>0$ prendo solo la parte positiva di $y(t)$

$$y(t)=\sqrt{2e^{\frac{x^2}{2}}-1}$$

Il più ampio intervallo su cui questa funzione è definita è:

$$2e^{\frac{x^2}{2}}-1\geq 0 \qquad e^{\frac{x^2}{2}}\geq 1$$

$$\frac{x^2}{2}\geq \ln 1$$

$$2e^{\frac{x^2}{2}}\geq 1 \longleftarrow x^2\geq 0 \quad \forall x$$

$$\downarrow$$

$$\forall x\in\mathbb{R}$$

Esercizio 2 (punti: ....../4)
Si trovi la soluzione del seguente problema di Cauchy, definita per ogni $t\in\mathbb{R}$:

$$\begin{cases} y'''(t)-4y'(t)=3\sin t \\ y(0)=\frac{3}{5} \\ y'(0)=0 \\ y''(0)=\frac{2}{5} \end{cases}$$

Suggerimento: utilizzare la sostituzione di variabile $z(t)=y'(t)$ per portare l'equazione differenziale al secondo ordine. Quindi, prima trovare $z(t)$ e poi $y(t)$.

$$z(t)=y'(t)$$

$$y'''(t)-4y'(t)=0$$

$$\downarrow$$

$$z''(t)-4z(t)=0$$

$$s^2-4=0$$

$$s_1=-2$$

$$s_2=2$$

$$z_0=C_1e^{-2t}+C_2e^{2t}$$

$$z_F=a\sin t+b\cos t$$

$$z'_F=a\cos t-b\sin t$$

$$z''_F=-a\sin t-b\cos t$$

$$-a\sin t-b\cos t-4a\sin t-4b\cos t=3\sin t$$

$$\sin t(-a-4a)+\cos t(-b-4b)=3\sin t$$

$$\begin{cases} -5a=3 \\ -5b=0 \end{cases} \rightarrow \begin{cases} a=-\frac{3}{5} \\ b=0 \end{cases}$$

$$z_F=-\frac{3}{5}\sin t$$

$$z(t)=z_0+z_F=C_1e^{-2t}+C_2e^{2t}-\frac{3}{5}\sin t$$

$$z'(t)=-2C_1e^{-2t}+2C_2e^{2t}-\frac{3}{5}\cos t$$

$$\begin{cases} y'(0)=z(0)=0 \\ y''(0)=z'(0)=\frac{2}{5} \end{cases}$$

$$\downarrow$$

$$\begin{cases} C_1+C_2=0 \\ -2C_1+2C_2-\frac{3}{5}=\frac{2}{5} \end{cases} \rightarrow \begin{cases} C_1=-C_2 \\ 4C_2=1 \end{cases} \rightarrow \begin{cases} C_1=-\frac{1}{4} \\ C_2=\frac{1}{4} \end{cases}$$

$$y'(t)=z(t)=-\frac{1}{4}e^{-2t}+\frac{1}{4}e^{2t}-\frac{3}{5}\sin t$$

$$y(t)=\int y'(t)dt=\int z(t)dt=-\frac{1}{4}\int e^{-2t}dt+\frac{1}{4}\int e^{2t}dt-\frac{3}{5}\int \sin t\, dt$$

$$=-\frac{1}{4}\cdot\frac{1}{-2}e^{-2t}+\frac{1}{4}\cdot\frac{1}{2}e^{2t}+\frac{3}{5}\cos t+c$$

$$y(0)=\frac{3}{5}$$

$$\downarrow$$

$$\frac{1}{8}+\frac{1}{8}\cancel{-\frac{3}{5}}+c=\cancel{\frac{3}{5}}$$

$$c=-\frac{1}{4}$$

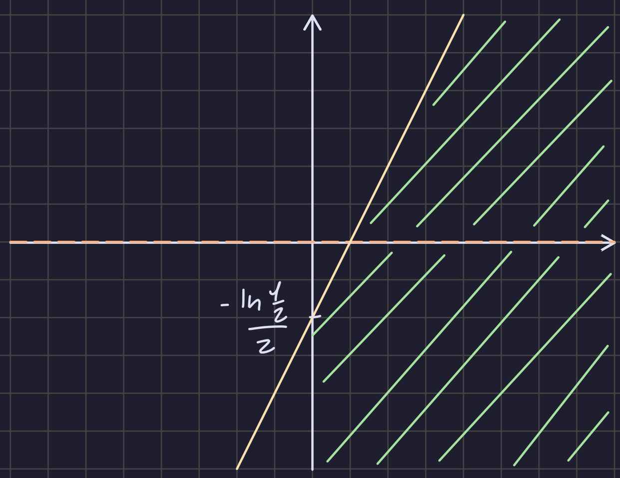
$$y(t)=\frac{1}{8}e^{-2t}+\frac{1}{8}e^{2t}+\frac{3}{5}\cos t-\frac{1}{4}$$

Esercizio 3 (punti: ....../4)
Data la funzione

$$f(x,y) = \sqrt{2e^{x-2y} - 1} + \frac{x}{y}$$

a) (2 punti) determinare analiticamente il suo dominio naturale D e poi rappresentarlo (con cura!) nel piano cartesiano. Stabilire se D è un insieme limitato/illimitato, aperto/chiuso, connesso/sconnesso (motivare le risposte!)

$$D = \begin{cases} 2e^{x-2y} - 1 \geq 0 \\ y \neq 0 \end{cases} = \begin{cases} x - 2y \geq \ln\left(\frac{1}{2}\right) \\ y \neq 0 \end{cases} = \begin{cases} y \leq \frac{1}{2}x - \frac{\ln\frac{1}{2}}{2} \\ y \neq 0 \end{cases}$$



L'insieme è illimitato, nè chiuso, nè aperto e sconnesso

b) (2 punti) scrivere l'equazione del piano tangente in $P(2,1,f(2,1))$ al grafico di f.

$$F(x,y) = \sqrt{2e^{x-2y} - 1} + \frac{x}{y}$$

$$\nabla F(x,y) = \begin{pmatrix} (2e^{x-2y} - 1)^{-\frac{1}{2}} \cdot e^{x-2y} + \frac{1}{y} \\ -(2e^{x-2y} - 1)^{-\frac{1}{2}} \cdot 2e^{x-2y} - \frac{x}{y^2} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{e^{x-2y}}{\sqrt{2e^{x-2y} - 1}} + \dfrac{1}{y} \\ \dfrac{-2e^{x-2y}}{\sqrt{2e^{x-2y} - 1}} - \dfrac{x}{y^2} \end{pmatrix}$$

$$\nabla F(2,1) = \begin{pmatrix} \dfrac{e^{2-2}}{\sqrt{2e^{2-2}-1}} + \dfrac{1}{1} \\ \\ \dfrac{-2e^{2-2}}{\sqrt{2e^{2-2}-1}} - 2 \end{pmatrix} = \begin{pmatrix} 2 \\ \\ -4 \end{pmatrix}$$

$$F(2,1) = 3$$

$$T(x,y) = F(2,1) + \nabla F(2,1) \begin{pmatrix} x-2 \\ y-1 \end{pmatrix}$$

$$= 3 + (2 \quad -4) \begin{pmatrix} x-2 \\ y-1 \end{pmatrix}$$

$$= 3 + 2(x-2) - 4(y-1)$$

$$= 2x - 4 - 4y + 4 + 3$$

$$= 2x - 4y + 3$$

$$\Downarrow$$

$$z = 2x - 4y + 3$$

In forma vettoriale

$$\begin{cases} z = 2t - 4s + 3 \\ x = t \\ y = s \end{cases}$$

$$\Downarrow$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ s \\ 2t - 4s + 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$$

a) (2 punti) Data la funzione

$$f(x,y) = \begin{cases} \dfrac{x^{\alpha-2}e^{x+y}}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

stabilire per quali valori reali di α f è continua in $\mathbb{R}^2$.

$$\lim_{(x,y)\to(0,0)} \frac{x^{\alpha-2} e^{x+y}}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^{\alpha-2}}{x^2+y^2} \cdot \lim_{(x,y)\to(0,0)} e^{x+y}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^{\alpha-2}}{x^2+y^2} \cdot 1$$

Coordinate polari

$$\frac{\rho^{\alpha-2}(\cos\theta)^{\alpha-2}}{\rho^2} = \rho^{\alpha-4}(\cos\theta)^{\alpha-2}$$

$$\lim_{\rho\to 0} \rho^{\alpha-4}(\cos\theta)^{\alpha-2} = \nexists \quad \text{se } \alpha \leq 4$$

$$0 \leq \left|\rho^{\alpha-4}(\cos\theta)^{\alpha-2}\right| \leq \rho^{\alpha-4}$$

$$\lim_{\rho\to 0} \rho^{\alpha-4} = 0 \quad \text{se } \alpha > 4$$

Quindi la funzione è continua per $\alpha > 4$

b) (2 punti) Calcolare la lunghezza dell'arco di curva parametrizzato da

$$\gamma(t) = (2t + 3, t^{\frac{3}{2}} + 1),\ t \in [0,\ 1]$$

$$\gamma(t) = (2t+3,\ t^{\frac{3}{2}}+1) \quad t \in [0,1]$$

$$\gamma'(t) = (2,\ \tfrac{3}{2} t^{\frac{1}{2}})$$

$$\|\gamma'(t)\| = \sqrt{2^2 + \left(\tfrac{3}{2}\sqrt{t}\right)^2} = \sqrt{2^2 + \frac{3^2}{2^2} t} = \sqrt{4 + \frac{9}{4} t}$$

$$L = \int_0^1 \|\gamma'(t)\|\, dt$$

$$= \int_0^1 \left(4 + \frac{9}{4} t\right)^{\frac{1}{2}} dt \qquad 4 + \frac{9}{4} t = u$$

$$= \frac{4}{9} \int_0^1 \sqrt{u}\, du \qquad du = \frac{9}{4} dt$$

$$= \frac{4}{9} \cdot \frac{2}{3} \left[\sqrt{\left(4 + \frac{9}{4} t\right)^3}\right]_0^1$$

$$= \frac{8}{27} \left(\sqrt{\left(4 + \frac{9}{4}\right)^3} - \sqrt{4^3}\right)$$

$$= \frac{8}{27} \left(\sqrt{\frac{25^3}{4^3}} - \sqrt{4^3}\right)$$

$$= \frac{8}{27} \left(\frac{25}{4}\sqrt{\frac{25}{4}} - 8\right)$$

$$= \frac{8}{27} \left(\frac{25}{4} \cdot \frac{5}{2} - 8\right)$$

$$= \frac{8}{27} \left(\frac{125}{8} - \frac{64}{8}\right)$$

$$= \frac{8}{27} \cdot \frac{61}{8}$$

$$= \frac{61}{27}$$

PARTE 2

Esercizio 5 (punti: ....../4)

a) (2 punti) Determinare e classificare i punti stazionari della funzione

$$f(x, y) = 2x^2 + 4xy - 3y^3$$

$F(x,y) = 2x^2 + 4xy - 3y^3$

$$\nabla F(x,y) = \begin{pmatrix} 4x + 4y \\ 4x - 9y^2 \end{pmatrix} = 0$$

$$\Downarrow$$

$$\begin{cases} 4x + 4y = 0 \\ 4x - 9y^2 = 0 \end{cases} \to \begin{cases} y = -x \\ 4x + 9x^2 = 0 \end{cases} \to \begin{cases} y = -x \\ x(4 + 9x) = 0 \end{cases} \to \begin{cases} y_1 = 0 \wedge y_2 = \frac{4}{9} \\ x_1 = 0 \wedge x_2 = -\frac{4}{9} \end{cases}$$

Punti critici

$A = (0, 0) \qquad B = \left(\frac{4}{9}, -\frac{4}{9}\right)$

$$HF(x,y) = \begin{pmatrix} 4 & 4 \\ 4 & -18y \end{pmatrix}$$

$$\det(HF(A) - \lambda I) = \det \begin{pmatrix} 4 - \lambda & 4 \\ 4 & 0 - \lambda \end{pmatrix}$$

$$= (4 - \lambda)(-\lambda) - 16$$

$$= -4\lambda + \lambda^2 - 16$$

$$= \lambda^2 - 4\lambda - 16$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 + 4 \cdot 16}}{2} = \frac{4 \pm 4\sqrt{5}}{2} = 2 \pm 2\sqrt{5}$$

$HF(A)$ indefinito $\to$ A sella

$$\det(HF(A)-\lambda I) = \det \begin{pmatrix} 4-\lambda & 4 \\ 4 & 8-\lambda \end{pmatrix}$$

$$= (4-\lambda)(8-\lambda) - 16$$

$$= \lambda^2 - 4\lambda - 8\lambda + 32$$

$$= \lambda^2 - 12\lambda + 32$$

$$\lambda_{1,2} = \frac{12 \pm \sqrt{144-128}}{2} = \frac{12 \pm 4}{2} = 6 \pm 2 \rightarrow \begin{matrix} \lambda_1 = 8 \\ \lambda_2 = 4 \end{matrix}$$

$HF(B)$ definita positiva $\rightarrow$ B minimo locale

b) (2 punti) Calcolare (se esistono) massimo e minimo assoluto della funzione $g(x, y) = f(x, y) - 4xy$ sulla circonferenza di centro l'origine e raggio 1.

$$g(x,y) = f(x,y) - 4xy = 2x^2 - 3y^3$$

$$D = x^2 + y^2 = 1$$

$$\downarrow$$

$$x^2 = 1 - y^2$$

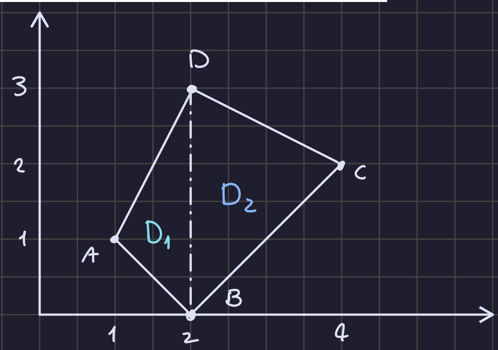
$g(y) = 2 - 2y^2 - 3y^3 \qquad y \in [-1, 1]$

$g(-1) = 2 - 2 + 3 = 3$ Massimo assoluto

$g(1) = 2 - 2 - 3 = -3$ Minimo assoluto

Esercizio 6 (punti: ....../4)
Una lamina non omogenea, con densità variabile data da $f(x, y) = e^{x+y}$, occupa la regione piana delimitata dal quadrilatero di vertici $A(1, 1)$, $B(2, 0)$, $C(4, 2)$ e $D(2, 3)$. Calcolare la massa della lamina.

$$F(x,y) = e^{x+y}$$



$A_B(t) = (1-t)A + tB$

$= (1-t)\begin{pmatrix}1\\1\end{pmatrix} + t\begin{pmatrix}2\\0\end{pmatrix}$

$= (1+t, 1-t) = \begin{cases} x = 1+t \\ y = 1-t \end{cases} \to \begin{cases} t = x-1 \\ y = 1-x+1 \end{cases} \to y = -x+2 \quad x \in [1,2]$

$B_C(t) = (1-t)B + tC$

$= (1-t)\begin{pmatrix}2\\0\end{pmatrix} + t\begin{pmatrix}4\\2\end{pmatrix}$

$= (2+2t, 2t) = \begin{cases} x = 2+2t \\ y = 2t \end{cases} \to \begin{cases} t = \frac{x-2}{2} \\ y = x-2 \end{cases} \to y = x-2 \quad x \in [2,4]$

$C_D(t) = (1-t)C + tD$

$2t = \frac{4-x}{2} = 2 - \frac{x}{2}$

$= (1-t)\begin{pmatrix}4\\2\end{pmatrix} + t\begin{pmatrix}2\\3\end{pmatrix}$

$= (4-2t, 2+t) = \begin{cases} x = 4-2t \\ y = 2+t \end{cases} \to \begin{cases} t = \frac{4-x}{2} = 2 - \frac{x}{2} \\ y = 4 - \frac{x}{2} \end{cases} \to y = 4 - \frac{x}{2} \quad x \in [2,4]$

$D_A(t) = (1-t)D + tA$

$y = 3 - 2(2-x)$

$= 3 - 4 + 2x$

$x = 2-t$

$= (1-t)\begin{pmatrix}2\\3\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix}$ $\quad x - 2 = -t \quad t = 2-x$

$= (2-t, 3-2t) = \begin{cases} x = 2-t \\ y = 3-2t \end{cases} \to \begin{cases} t = 2-x \\ y = -1+2x \end{cases} \to y = -1+2x \quad x \in [1,2]$

$D_1 = \{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq 2, A_B \leq y \leq D_A\}$

$= \{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq 2, -x+2 \leq y \leq -1+2x\}$

$D_2 = \{(x,y) \in \mathbb{R}^2 : 2 \leq x \leq 4, B_C \geq y \leq C_D\}$

$= \{(x,y) \in \mathbb{R}^2 : 2 \leq x \leq 4, x-2 \geq y \leq 4 - \frac{x}{2}\}$

$D = D_1 \cup D_2$

$$M = \iint_D e^{x+y}\, dx\, dy$$

$$= \iint_{D_1} e^{x+y}\, dx\, dy + \iint_{D_2} e^{x+y}\, dx\, dy$$

$$= \int_1^2 \int_{-x+2}^{2x-1} e^{x+y}\, dy\, dx + \int_2^4 \int_{x-2}^{-\frac{x}{2}+4} e^{x+y}\, dy\, dx$$

$$= \int_1^2 e^x \left(\int_{-x+2}^{2x-1} e^y\, dy \right) dx + \int_2^4 e^x \left(\int_{x-2}^{-\frac{x}{2}+4} e^y\, dy \right) dx$$

$$= \int_1^2 e^x \left[e^y \right]_{-x+2}^{2x-1} dx + \int_2^4 e^x \left[e^y \right]_{x-2}^{-\frac{x}{2}+4} dx$$

$$= \int_1^2 e^x \left(e^{2x-1} - e^{-x+2} \right) dx + \int_2^4 e^x \left(e^{-\frac{x}{2}+4} - e^{x-2} \right) dx$$

$$= e^{-1} \int_1^2 e^{3x}\, dx - e^2 \int_1^2 dx + e^4 \int_2^4 e^{\frac{1}{2}x}\, dx - e^{-2} \int_2^4 e^{2x}\, dx$$

$$= \frac{e^{-1}}{3} \left[e^{3x} \right]_1^2 - e^2 + 2e^4 \left[e^{\frac{1}{2}x} \right]_2^4 - \frac{e^{-2}}{2} \left[e^{2x} \right]_2^4$$

$$= \frac{e^{-1}}{3} (e^6 - e^3) - e^2 + 2e^4 (e^2 - e) - \frac{e^{-2}}{2} (e^8 - e^4)$$

$$= \frac{e^5}{3} - \frac{e^2}{3} - e^2 + 2e^6 - 2e^5 - \frac{e^6}{2} + \frac{e^2}{2}$$

$$= -\frac{5}{3} e^5 + \frac{3}{2} e^6 - \frac{5}{6} e^2$$

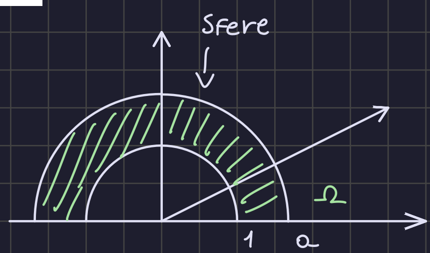
Esercizio 7 (punti: ....../4)
Si dica per quale valore di $a \in \mathbb{R}^+$ si ha:

$$\iiint_\Omega \frac{3}{\sqrt{x^2+y^2+z^2}}\, dx\, dy\, dz = \pi$$

dove

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : 1 \le x^2+y^2+z^2 \le a^2, y \ge 0, z \ge 0\}$$

$$\Omega = \begin{cases} x^2+y^2+z^2 \ge 1 \\ x^2+y^2+z^2 \le a^2 \\ y \ge 0 \\ z \ge 0 \end{cases}$$



Coordinate sferiche

$$\begin{cases} x = \rho \sin\varphi \cos\theta & \rho \in [1, a] \\ y = \rho \sin\varphi \sin\theta & \theta \in [0, \pi] \\ z = \rho \cos\varphi & \varphi \in [0, \frac{\pi}{2}] \end{cases}$$

$$\iiint_\Omega \frac{3}{\sqrt{\rho^2(\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi)}} \overbrace{\rho^2 \sin\varphi}^{\text{Jacobiano}} \, d\rho \, d\varphi \, d\theta$$

$$= \iiint_\Omega \frac{3}{\rho} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \iiint_\Omega 3\rho \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= 3 \int_1^a \rho \int_0^{\frac{\pi}{2}} \sin\varphi \int_0^\pi d\theta \, d\varphi \, d\rho$$

$$= 3 \int_1^a \rho \int_0^{\frac{\pi}{2}} \pi \sin\varphi \, d\varphi \, d\rho$$

$$= 3\pi \int_1^a \rho (1 - 0) \, d\rho$$

$$= 3\pi \left[\frac{\rho^2}{2}\right]_1^a$$

$$= 3\pi \left(\frac{a^2}{2} - \frac{1}{2}\right)$$

$$\frac{3}{2}\pi a^2 - \frac{3}{2}\pi = \pi$$

$$\frac{3}{2}\pi a^2 = \frac{5}{2}\pi$$

$$a^2 = \frac{5}{\cancel{2}}\cancel{\pi} \cdot \frac{\cancel{2}}{3\cancel{\pi}}$$

$$a^2 = \frac{5}{3}$$

$$a = \sqrt{\frac{5}{3}}$$

Esercizio 8 (punti: ....../4)

a) (2 punti) Stabilire se il campo vettoriale

$$\vec{F}(x,y,z) = (3x^2y,\, x^3,\, -\tfrac{1}{z})$$

definito su $D = \{(x,y,z) \in \mathbb{R}^3 : z > 0\}$ è conservativo e, in caso affermativo, determinare un suo potenziale.

$\vec{F}(x,y,z) = (3x^2y,\, x^3,\, -\frac{1}{z}) \qquad z>0$

Condizione necessaria e sufficiente

Derivate in croce:

$F_1 \quad F_2 \quad F_3$

$\partial x \quad \partial y \quad \partial z$

$$\frac{\partial F_2}{\partial y} = 0 = \frac{\partial F_1}{\partial x}$$

Verifico se esiste un potenziale scalare

$$\frac{\partial U}{\partial x} = 3x^2y \rightarrow U = \int 3x^2y\,dx = yx^3 + c_1(y,z)$$

$$\frac{\partial U}{\partial y} = x^3 \rightarrow U = \int x^3 dy = x^3y + c_2(x,z)$$

$$\frac{\partial U}{\partial z} = -\frac{1}{z} \rightarrow U = -\int \frac{1}{z}\,dz = -\ln z + c_3(x,y)$$

$$\begin{cases} \cancel{y}\cancel{x^3} + c_1(y,z) = \cancel{x^3}\cancel{y} + c_2(x,z) \\ yx^3 + c_1(y,z) = -\ln z + c_3(x,y) \end{cases}$$

$$c_1(y,z) = c_2(x,z) = -\ln z$$

$$c_3(x,y) = yx^3$$

$U(x,y,z) = x^3y - \ln z + K$ Quindi il campo é conservativo

b) (2 punti) Calcolare in due modi diversi il lavoro di $\vec{F}$ lungo il segmento che congiunge nell'ordine i punti $A(0,0,2)$ e $B(3,2,1)$.

$\gamma(t) = (1-t)A + tB \qquad t \in [0,1]$

$$= (1-t)\begin{pmatrix}0\\0\\2\end{pmatrix} + t\begin{pmatrix}3\\2\\1\end{pmatrix}$$

$$= (3t, 2t, 2-t)$$

$\gamma'(t) = (3, 2, -1)$

$L = U(B) - U(A)$

$= U(3,2,1) - U(0,0,2)$

$= 3^2 \cdot 2 - \ln 1 - \ln 2$

$= 18 - \ln 2$

$$L = \int_0^1 F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$= \int_0^1 F(3t, 2t, 2-t) \cdot \begin{pmatrix}3\\2\\-1\end{pmatrix} dt$$

$$= \int_0^1 \left(3(3t)^2(2t), (3t)^3, -\frac{1}{2-t}\right) \cdot \begin{pmatrix}3\\2\\-1\end{pmatrix} dt$$

$$= \int_0^1 \left(54t^3, 27t^3, -\frac{1}{2-t}\right) \cdot \begin{pmatrix}3\\2\\-1\end{pmatrix} dt$$

$$= \int_0^1 54t^3 \cdot 3 + 27t^3 \cdot 2 + \frac{1}{2-t}\, dt$$

$$= \int_0^1 216t^3 + \frac{1}{2-t}\, dt$$

$$= 216\left[\frac{t^4}{4}\right]_0^1 - \left[\ln|2-t|\right]_0^1$$

$$= 216\ \frac{1}{4} + \ln 2$$

$$= 54 + \ln 2$$