Esame 19/06/2024

Esercizio 5 (punti: ....../4)

a) (2 punti) Determinare e classificare i punti stazionari della funzione $f(x, y) = x^3 + y^2 - 11x$.

$$F(x,y) = x^3 + y^2 - 11x$$

$$\nabla f(x,y) = \begin{pmatrix} 3x^2 - 11 \\ 2y \end{pmatrix}$$

$$\nabla F(x,y) = 0$$

$$\downarrow$$

$$\begin{cases} 3x^2 - 11 = 0 \\ 2y = 0 \end{cases} \rightarrow \begin{cases} x = \pm\sqrt{\frac{11}{3}} \\ y = 0 \end{cases}$$

$$A\left(\sqrt{\tfrac{11}{3}}, 0\right) \qquad B\left(-\sqrt{\tfrac{11}{3}}, 0\right)$$

$$H_F(x,y) = \begin{pmatrix} 6x & 0 \\ 0 & 2 \end{pmatrix}$$

Troviamo autovalori

$$\det(H_F - \lambda I) = \det\begin{pmatrix} 6x - \lambda & 0 \\ 0 & 2 - \lambda \end{pmatrix}$$

$$= (6x - \lambda)(2 - \lambda)$$

$$\lambda_1 = 6x$$

$$\lambda_2 = 2$$

$$H_F(A) = \begin{pmatrix} 6\sqrt{\frac{11}{3}} & 0 \\ 0 & 2 \end{pmatrix} \rightarrow \begin{matrix} \lambda_1 > 0 \\ \lambda_2 > 0 \end{matrix} \rightarrow A \text{ Minimo locale}$$

$$H_F(B) = \begin{pmatrix} -6\sqrt{\frac{11}{3}} & 0 \\ 0 & 2 \end{pmatrix} > 0 \rightarrow \begin{matrix} \lambda_1 < 0 \\ \lambda_2 > 0 \end{matrix} \rightarrow B \text{ Sella}$$

b) (2 punti) Si consideri l'insieme

$$E = \{(x, y) \in \mathbb{R}^2 : 4x^2 + y^2 = 1\}$$

La funzione f, ristretta a E, ha estremi assoluti? Motivare la risposta e, in caso affermativo, calcolarli.

$$E = \{(x,y) \in \mathbb{R}^2 : 4x^2 + y^2 = 1\} \qquad F(x,y) = x^3 + y^2 - 11x$$

$$\downarrow$$

$$(2x)^2 + y^2 = 1$$

Coord polari

$$\begin{cases} 2x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \rightarrow \begin{cases} x = \frac{\rho}{2}\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad \theta \in [0, 2\pi]$$

$\rho = 1$ (Raggio)

$$\downarrow$$

$$\begin{cases} x = \frac{1}{2}\cos\theta \\ y = \sin\theta \end{cases}$$

$$F(\theta) = \left(\tfrac{1}{2}\cos\theta\right)^3 + \sin^2\theta - \frac{11}{2}\cos\theta$$

$$= \frac{1}{8}\cos^3\theta + \sin^2\theta - \frac{11}{2}\cos\theta$$

$$F'(\theta) = -\frac{1}{8}\cdot 3\cos^2\theta\sin\theta + 2\sin\theta\cos\theta + \frac{11}{2}\sin\theta$$

$$= -\frac{1}{8}\cdot 3\cos^2\theta\sin\theta + \sin(2\theta) + \frac{11}{2}\sin\theta$$

$$F' = 0 \rightarrow -\frac{1}{8}\cdot 3\cos^2\theta\sin\theta + \sin(2\theta) + \frac{11}{2}\sin\theta = 0$$

$$\sin\theta\left(-\frac{1}{8}\cdot 3\cos^2\theta + 2\cos\theta + \frac{11}{2}\right) = 0$$

$$\sin\theta = 0 \ \vee \ -\frac{1}{8}\cdot 3\cos^2\theta + 2\cos\theta + \frac{11}{2} = 0$$

$$\theta = 0 \vee \theta = \pi \ \vee \ \underbrace{\cos\theta\left(-\frac{3}{8}\cos\theta + 2\right) = -\frac{11}{2}}_{\nexists\,\theta \in [0, 2\pi]}$$

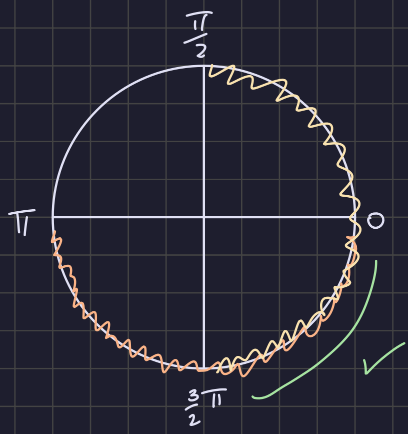
Punti stazionari: $F(0)$ $F(\pi)$

$F(\frac{1}{2}, 0)$ $F(-\frac{1}{2}, 0)$

$$F(0) = \frac{1}{8} - \frac{11}{2} = \frac{1-44}{8} = -\frac{43}{8} \qquad \text{Minimo assoluto}$$

$$F(\pi) = -\frac{1}{8} + \frac{11}{2} = \frac{-1+44}{8} = \frac{43}{8} \qquad \text{Massimo assoluto}$$

Esercizio 6 (punti: ....../4)
Sia $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le a^2, x \ge 0,\ y \le 0\}$.
Determinare per quale valore di $a \in \mathbb{R}^+$ si ha
$$\iint_D xy(x^2+y^2)^{\frac{3}{2}}\,dx\,dy = -\frac{1}{14}$$



$x^2+y^2=a^2$
$\downarrow$
Polari

$$\begin{cases} y \le 0 \Rightarrow \sin\theta \ge 0 \\ x \ge 0 \Rightarrow \cos\theta \ge 0 \end{cases} \rightarrow$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \quad \theta \in [\tfrac{3}{2}\pi, 2\pi] \quad \rho \in [0, a]$$

$$\iint_D \rho^2\cos\theta\sin\theta\,(\rho^2)^{\frac{3}{2}}\,\underbrace{\rho}_{\text{Jacobiano}}\,d\rho\,d\theta$$

$$\iint_D \rho^2\cos\theta\sin\theta\,\rho^3\,\rho\,d\rho\,d\theta$$

$$\iint_D \rho^6\cos\theta\sin\theta\,d\rho\,d\theta$$

$$\int_0^a \rho^6\,d\rho \cdot \int_{\frac{3}{2}\pi}^{2\pi} \cos\theta\sin\theta\,d\theta \qquad u=\sin\theta \quad du=\cos\theta\,d\theta$$

$$\left[\frac{\rho^7}{7}\right]_0^a \cdot \left[\frac{\sin^2\theta}{2}\right]_{\frac{3}{2}\pi}^{2\pi}$$

$$\frac{a^7}{7}\cdot\left(0-\left(\frac{(-1)^2}{2}\right)\right)$$

$$\frac{a^7}{7}\cdot\left(-\frac{1}{2}\right)$$

$$-\frac{a^7}{14} = -\frac{1}{4}$$

$$a = 1$$

Esercizio 7 (punti: ....../4)
Sia $\Omega = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D,\ 0 \le z \le x + 4y\}$, dove D è la regione finita nel piano xy limitata da $y = -2x$ e $y = x^2$. Calcolare

$$\iiint_\Omega x\,dx\,dy\,dz$$

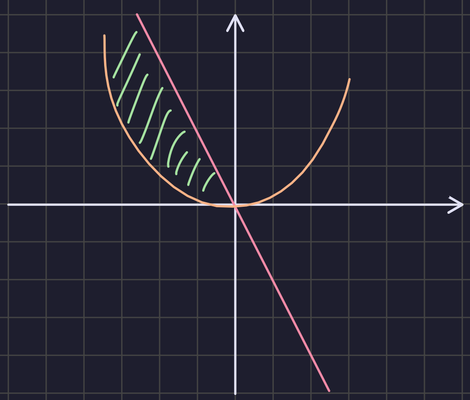
$y = -2x$

$y = x^2$

$-2x = x^2$

$x = -2$

$D \to$



$$x \int_{-2x}^{x^2} dy + \int_{-2x}^{x^2} 4y\,dy$$

$$\iiint_\Omega x\,dx\,dy\,dz =$$

$$= \iint_D \int_0^{x+4y} dz\,dx\,dy$$

$$= \int_{-2}^{0} \int_{-2x}^{x^2} x \int_0^{x+4y} dz\,dy\,dx$$

$$= \int_{-2}^{0} \int_{-2x}^{x^2} x(x+4y)\,dy\,dx$$

$$= \int_{-2}^{0} \int_{-2x}^{x^2} x^2 + 4xy\,dy\,dx$$

$$= \int_{-2}^{0} \left[x^2 y + 4x\frac{y^2}{2}\right]_{-2x}^{x^2} dx$$

$$= \int_{-2}^{0} \left(x^2 x^2 + 2x x^4 - x^2(-2x) + 2x(-2x)^2\right) dx$$

$$= \int_{-2}^{0} \left(x^4 + 2x^5 - (-2x^3 + 8x^3)\right) dx$$

$$= \int_{-2}^{0} x^4 + 2x^5 - 6x^3 \, dx$$

$$= \left[\frac{x^5}{5}\right]_{-2}^{0} + 2\left[\frac{x^6}{6}\right]_{-2}^{0} - 6 \left[\frac{x^4}{4}\right]_{-2}^{0}$$

$$= \frac{-2^5}{5} + 2 \frac{2^6}{6} - 6 \frac{2^4}{4}$$

$$= -\frac{2^5}{5} + \frac{2^7}{6} - \frac{2^5 \cdot 3}{4}$$

$$= \frac{-2^5(6\cdot 4) + 2^7(5\cdot 4) - 2^5 \cdot 3 \, (5 \cdot 6)}{5 \cdot 6 \cdot 4}$$

$$= \frac{-768 + 2560 - 2880}{120}$$

$$= -\frac{1088}{120}$$

$$= -\frac{136}{15}$$

Esercizio 8 (punti: . . . . . . /4)

a) (2 punti) Il campo vettoriale piano

$$\vec{F}(x, y) = (x^2 + y, \, -y^2 - 2x)$$

è somma dei campi $\vec{F_1}$, $\vec{F_2}$, dove $\vec{F_1}(x, y) = (0, \, -3x)$ e $\vec{F_2}(x, y) = (x^2 + y, \, -y^2 + x)$.
Verificare che $\vec{F_2}$ è conservativo e trovarne un potenziale.

$$\vec{F}(x,y) = (x^2 + y, -y^2 - 2x)$$

$$\vec{F}_1(x,y) = (0, -3x) \qquad \vec{F}_2(x,y) = (x^2 + y, -y^2 + x)$$

$$\vec{F} = \vec{F}_1 i + \vec{F}_2 j$$

$$\nabla \times \vec{F} = 0$$

$$\nabla \times \vec{F}_2 = \det \begin{pmatrix} i & j & k \\ \partial x & \partial y & 0 \\ F_{2x} & F_{2y} & 0 \end{pmatrix} = k \left(\frac{\partial \vec{F}_{2y}}{dx} - \frac{d \vec{F}_{2x}}{dy} \right) = k(1-1) = 0$$

$$\Downarrow$$

è conservativo

$$U_x = \frac{dU}{dx} \qquad \begin{cases} U_x = F_{2x} \\ U_y = F_{2y} \end{cases} \rightarrow \begin{cases} U_{xy} = \partial_y F_{2x} \\ U_{yx} = \partial_x F_{2y} \end{cases}$$

$$\frac{dU}{dx} = x^2+y \rightarrow U = \int x^2+y+c(y)\,dx = \frac{x^3}{3}+xy+C(y)$$

$$\frac{dU}{dy} = \frac{\partial\left(\frac{x^3}{3}+xy+C(y)\right)}{dy} = x + C'(y)$$

$$x + c'(y) = \vec{F}_{2y}$$

$$\cancel{x} + c'(y) = -y^2 + \cancel{x}$$

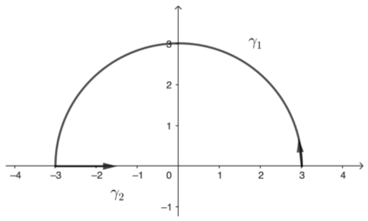
$$c'(y) = -y^2$$

$$c(y) = -\int y^2\,dy$$

$$C(y) = -\frac{y^3}{3} + k$$

$$U_{F_2} = \frac{x^3}{3} + xy - \frac{y^3}{3} + k$$

b) (2 punti) Calcolare il lavoro del campo $\vec{F}$ lungo il cammino chiuso orientato $\gamma_1 \cup \gamma_2$ in figura, che ha punto d'inizio $A = (3, 0)$.



$$\vec{F}(x,y) = (x^2+y,\ -y^2-2x)$$

$$\nabla \times \vec{F} = \det\begin{pmatrix} i & j & k \\ \partial x & \partial y & 0 \\ \vec{F}_x & \vec{F}_y & 0 \end{pmatrix} = k\left(\frac{\partial \vec{F}_y}{\partial x} - \frac{\partial \vec{F}_x}{\partial y}\right) = k(-2-1) = k(-3)$$

Non Conservativo

$L = \int_\gamma \vec{F}\, d\vec{\ell}$

$\vec{F}_2$ é conservativo $\rightarrow L = 0$

$= \int_\gamma \vec{F}_1\, d\vec{\ell} + \cancel{\int_\gamma \vec{F}_2\, d\vec{\ell}}$

$= \int_{\gamma_1} \vec{F}_1\, d\vec{\ell} + \cancel{\int_{\gamma_2} \vec{F}_1\, d\vec{\ell}}$

$\vec{F}_1$ é $\perp$ a $\gamma_2 \rightarrow L = 0$

$= \int_{\gamma_1} F_1\, d\vec{\ell}$

$= \int_0^\pi \vec{F}_1(\gamma_1(t)) \cdot \gamma'(t)\, dt$

$\gamma_1(t) = (3\cos t,\ 3\sin t) \quad t \in [0, \pi)$

$\gamma_1'(t) = (-3\sin t,\ 3\cos t)$

$= \int_0^\pi (0, -9\cos t) \cdot (-3\sin t, 3\cos t)\, dt$

$= \int_0^\pi -27\cos^2 t\, dt$

$= -27 \int_0^\pi \cos^2 t\, dt$

$\cos^2 t = \dfrac{1 + \cos(2t)}{2}$

$= -27 \int_0^\pi \dfrac{1 + \cos(2t)}{2}\, dt$

$= -\dfrac{27}{2} \int_0^\pi 1 + \cos(2t)\, dt$

$= -\dfrac{27}{2}\left(\int_0^\pi dt + \int_0^\pi \cos(2t)\, dt\right)$

$= -\dfrac{27}{2}\left(\pi + \int_0^\pi \cos(2t)\, dt\right)$

$u = 2t$

$du = 2\, dt$

$= -\dfrac{27}{2}\left(\pi + \dfrac{1}{2}\left[\sin(2t)\right]_0^\pi\right)$

$= -\dfrac{27}{2}\pi$