# Esercizi su risposta libera e impulsiva

1. Dato il seguente sistema a tempo continuo (LTI)

$$\frac{d^2v(t)}{dt^2} - 5\frac{dv(t)}{dt} - 6v(t) = \frac{du(t)}{dt} + 5u(t), \quad t \in \mathbb{R}_+,$$

e le seguenti condizioni inziali:

$$v(0^-) = 3 \qquad \frac{dv(0^-)}{dt} = 1.$$

(a) si calcoli la risposta libera nel tempo,

(b) si calcoli la risposta impulsica nel tempo.

## a) Risposta libera

Equazione del sistema

$$v''(t) - 5v'(t) - 6v(t) = u'(t) + 5u(t), \qquad t \in \mathbb{R}_+$$

Condizioni iniziali

$$\begin{cases} v(0^-) = 3 \\ v'(0^-) = 1 \end{cases}$$

Equazione omogenea del polinomio caratteristico

$$P(s) = s^2 - 5s - 6s = 0$$

Soluzioni

$$s_{1,2} = \frac{5 \pm \sqrt{25+24}}{2} = \frac{5 \pm 7}{2} = \begin{cases} \frac{12}{2} = 6 \\ -\frac{2}{2} = -1 \end{cases}$$

$\lambda_1 = -1 \qquad r = 2$ (numero di soluzioni)

$\lambda_2 = 6 \qquad \mu_{1,2} = 1$ (molteplicitá)

Risposta libera generica

$$v_l(t) = \sum_{i=1}^{r} \sum_{l=0}^{\mu_i - 1} c_{i,l} \cdot e^{\lambda_i t} \cdot \frac{t^l}{l!}$$

$$= c_1 \cdot e^{-t} \cdot \frac{t^0}{0!} + c_2 \cdot e^{6t} \cdot \frac{t^0}{0!}$$

$$= c_1 \cdot e^{-t} + c_2 \cdot e^{6t}$$

Derivata della risposta libera

$$v(t) = c_1 \cdot e^{-t} + c_2 \cdot e^{6t}$$

$$v'(t) = c_1 \cdot e^{-t} \cdot (-1) + c_2 \cdot e^{6t} \cdot 6$$

$$= -c_1 \cdot e^{-t} + 6 c_2 \cdot e^{6t}$$

Calcolo dei coefficienti $c_1$ e $c_2$

$$\begin{cases} v(0^-) = c_1 \cdot e^{-0^-} + c_2 \cdot e^{6 \cdot 0^-} \\ v'(0^-) = -c_1 \cdot e^{-0^-} + 6 c_2 \cdot e^{6 \cdot 0^-} \end{cases}$$

$$\begin{cases} v(0^-) = c_1 + c_2 \\ v'(0^-) = -c_1 + 6 c_2 \end{cases}$$

$$\begin{cases} v(0^-) = 3 \\ v'(0^-) = 1 \end{cases} \rightarrow \begin{cases} c_1 + c_2 = 3 \\ -c_1 + 6 c_2 = 1 \end{cases}$$

$$\begin{cases} c_1 = 3 - c_2 \\ -3 + c_2 + 6 c_2 = 1 \end{cases}$$

$$\begin{cases} c_1 = \frac{17}{7} \\ c_2 = \frac{4}{7} \end{cases}$$

Risposta libera specifica

$$v_l(t) = \frac{17}{7} e^{-t} + \frac{4}{7} e^{6t}$$

## b) Risposta impulsiva

Equazione del sistema

$$v''(t) - 5 v'(t) - 6 v(t) = u'(t) + 5 u(t), \qquad t \in \mathbb{R}_+$$

Equazione omogenea del polinomio caratteristico

$$P(s) = s^2 - 5s - 6s = 0$$

Soluzioni

$\lambda_1 = -1$ $\qquad r = 2$ (numero di soluzioni)

$\lambda_2 = 6$ $\qquad \mu_{1,2} = 1$ (molteplicità)

## Risposta impulsiva generica

$$h(t) = d_0 \cdot \delta(t) + \left( \sum_{i=1}^{r} \sum_{l=0}^{\mu_i - 1} d_{i,l} \cdot e^{\lambda_i t} \cdot \frac{t^l}{l!} \right) \cdot \delta_{-1}(t)$$

$d_0 = 0$ perché il sistema non é proprio

$$= \left( d_1 \cdot e^{-t} \cdot \frac{t^0}{0!} + d_2 \cdot e^{6t} \cdot \frac{t^0}{0!} \right) \cdot \delta_{-1}(t)$$

$$= \left( d_1 \cdot e^{-t} + d_2 \cdot e^{6t} \right) \cdot \delta_{-1}(t)$$

## Riscrivo l'equazione del sistema ponendo $v(t) = h(t)$ e $u(t) = \delta(t)$

$$v''(t) - 5v'(t) - 6v(t) = u'(t) + 5u(t)$$

$$\downarrow$$

$$h''(t) - 5h'(t) - 6h(t) = \delta'(t) + 5\delta(t)$$

## Calcolo delle derivate di $h(t)$

$$h(t) = \left( d_1 \cdot e^{-t} + d_2 \cdot e^{6t} \right) \cdot \delta_{-1}(t)$$

$$h'(t) = \left( -d_1 \cdot e^{-t} + 6 d_2 e^{6t} \right) \cdot \delta_{-1}(t) + \left( d_1 \cdot e^{-t} + d_2 \cdot e^{6t} \right) \cdot \delta(t)$$

$$h''(t) = \left( d_1 \cdot e^{-t} + 36 d_2 e^{6t} \right) \cdot \delta_{-1}(t) + \left( -d_1 \cdot e^{-t} + 6 d_2 e^{6t} \right) \cdot \delta(t) +$$
$$\left( -d_1 \cdot e^{-t} + 6 d_2 \cdot e^{6t} \right) \cdot \delta(t) + \left( d_1 \cdot e^{-t} + d_2 \cdot e^{6t} \right) \cdot \delta_1(t)$$

## Sostituisco nell'equazione del sistema

$$h''(t) - 5h'(t) - 6h(t) = \delta'(t) + 5\delta(t)$$

$$\downarrow \quad h''$$

$$\Big[ \left( d_1 \cdot e^{-t} + 36 d_2 e^{6t} \right) \cdot \delta_{-1}(t) + \left( -d_1 \cdot e^{-t} + 6 d_2 e^{6t} \right) \cdot \delta(t) +$$
$$\left( -d_1 \cdot e^{-t} + 6 d_2 \cdot e^{6t} \right) \cdot \delta(t) + \left( d_1 \cdot e^{-t} + d_2 \cdot e^{6t} \right) \cdot \delta_1(t) \Big]$$

$$-5 \Big[ \left( -d_1 \cdot e^{-t} + 6 d_2 e^{6t} \right) \cdot \delta_{-1}(t) \overset{h'}{+} \left( d_1 \cdot e^{-t} + d_2 \cdot e^{6t} \right) \cdot \delta(t) \Big]$$

$$-6 \Big[ \left( d_1 \cdot e^{-t} + d_2 \overset{h}{\cdot} e^{6t} \right) \cdot \delta_{-1}(t) \Big] = \delta'(t) + 5\delta(t)$$

Valuto le funzioni in $t=0^-$

$$\Big[\big(d_1 \cdot e^{-0^-} + 36\, d_2\, e^{6\cdot 0^-}\big)\cdot \delta_{-1}(0^-) + \big(-d_1 \cdot e^{-0^-} + 6\, d_2\, e^{6\cdot 0^-}\big)\cdot \delta(0^-) +$$
$$\big(-d_1 \cdot e^{-0^-} + 6 d_2 \cdot e^{6\cdot 0^-}\big)\cdot \delta(0^-) + \big(d_1 \cdot e^{-0^-} + d_2 \cdot e^{6\cdot 0^-}\big)\cdot \delta_1(0^-)\Big]$$
$$-5\Big[\big(-d_1 \cdot e^{-0^-} + 6 d_2\, e^{6\cdot 0^-}\big)\cdot \delta_{-1}(0^-) + \big(d_1 \cdot e^{-0^-} + d_2 \cdot e^{6\cdot 0^-}\big)\cdot \delta(0^-)\Big]$$
$$-6\Big[\big(d_1 \cdot e^{-0^-} + d_2 \cdot e^{6\cdot 0^-}\big)\cdot \delta_{-1}(0^-)\Big] = \delta'(0^-) + 5\,\delta(0^-)$$

tolgo il gradino perché in $t=0^-$ fa 0

$$\Big[\cancel{\big(d_1 \cdot e^{-0^-} + 36\, d_2\, e^{6\cdot 0^-}\big)\cdot \delta_{-1}(0^-)} + \big(-d_1 \cdot \cancel{e^{-0^-}} + 6\, d_2\, \cancel{e^{6\cdot 0^-}}\big)\cdot \delta(0^-) +$$
$$\big(-d_1 \cdot \cancel{e^{-0^-}} + 6 d_2 \cdot \cancel{e^{6\cdot 0^-}}\big)\cdot \delta(0^-) + \big(d_1 \cdot \cancel{e^{-0^-}} + d_2 \cdot \cancel{e^{6\cdot 0^-}}\big)\cdot \delta_1(0^-)\Big]$$
$$-5\Big[\cancel{\big(-d_1 \cdot e^{-0^-} + 6 d_2\, e^{6\cdot 0^-}\big)\cdot \delta_{-1}(0^-)} + \big(d_1 \cdot \cancel{e^{-0^-}} + d_2 \cdot \cancel{e^{6\cdot 0^-}}\big)\cdot \delta(0^-)\Big]$$
$$-6\Big[\cancel{\big(d_1 \cdot e^{-0^-} + d_2 \cdot e^{6\cdot 0^-}\big)\cdot \delta_{-1}(0^-)}\Big] = \delta'(0^-) + 5\,\delta(0^-)$$

$$\Big[\big(-d_1 + 6 d_2\big)\cdot \delta(0^-) + \big(-d_1 + 6 d_2\big)\cdot \delta(0^-) + \big(d_1 + d_2\big)\cdot \delta_1(0^-)\Big]$$
$$-5\Big[\big(d_1 + d_2\big)\cdot \delta(0^-)\Big] = \delta'(0^-) + 5\,\delta(0^-)$$

$$\big(-d_1 + 6 d_2\big)\, 2\, \delta(0^-) + \big(d_1 + d_2\big)\cdot \delta_1(0^-)$$
$$-5\big(d_1 + d_2\big)\cdot \delta(0^-) = \delta'(0^-) + 5\,\delta(0^-)$$

Sposto tutto a sinistra

$$\underline{\big(-d_1 + 6 d_2\big)\, 2\, \delta(0^-)} + \underline{\big(d_1 + d_2\big)\cdot \delta_1(0^-)}$$
$$\underline{-5\big(d_1 + d_2\big)\cdot \delta(0^-)} \; \underline{-\delta_1(0^-)} \; \underline{-5\,\delta(0^-)} = 0$$

Raccolgo per $\delta(0), \delta_1(0), \dots, \delta_n(0)$ che sono linearmente indipendenti

$$\underline{\big(2(-d_1 + 6 d_2) - 5(d_1 + d_2) - 5\big)\; \delta(0^-)} + \underline{\big(d_1 + d_2 - 1\big)\cdot \delta_1(0^-)} = 0$$

$$(-2d_1+12d_2-5d_1-5d_2-5)\,\delta(0^-)+(d_1+d_2-1)\,\delta_1(0^-)=0$$

$$(-7d_1+7d_2-5)\,\delta(0^-)+(d_1+d_2-1)\,\delta_1(0^-)=0$$

Risolvo il sistema

$$\begin{cases}(-7d_1+7d_2-5)\,\cancel{\delta(0^-)}=0\\(d_1+d_2-1)\,\cancel{\delta_1(0^-)}=0\end{cases}$$

$$\begin{cases}-7d_1+7d_2=5\\d_1+d_2=1\end{cases}$$

$$\begin{cases}7d_2-7+7d_2=5\\d_1=-d_2+1\end{cases}$$

$$\begin{cases}d_2=\frac{12}{14}=\frac{6}{7}\\d_1=-\frac{6}{7}+\frac{7}{7}=\frac{1}{7}\end{cases}$$

Risposta impulsiva specifica

$$h(t)=\frac{1}{7}\cdot e^{-t}+\frac{6}{7}\cdot e^{6t}\cdot\delta_{-1}(t)$$