Esercizio 1 (40 punti)

Si consideri il modello ingresso/uscita a tempo continuo descritto dalla sequente equazione differenziale,

$$\frac{d^2v(t)}{dt^2} - 5\frac{dv(t)}{dt} - 6v(t) = \frac{d^2u(t)}{dt^2} - 7\frac{du(t)}{dt} + 10u(t), \quad t \in \mathbb{R}_+,$$

e le seguenti condizioni inziali:

$$v(0^-) = 1 \qquad \frac{dv(0^-)}{dt} = 0.$$

a) Si studi la stabilitá del sistema

$$s^2 - 5s - 6 = 0$$

$$\downarrow$$

$$(s-6)(s+1) = 0$$

$s_1 = -1$ $\quad \mu_{1,2} = 1$

$s_2 = 6$

Il sistema non è asintoticamente stabile perchè non tutte le radici hanno parte reale negativa. Per verificare se il sistema è BIBO stabile bisogna vedere se le radici che non hanno parte reale negativa si semplificano nella funzione di trasferimento data dal rapporto tra il polinomio caratteristico dell'ingresso e dell'uscita:

$$H(s) = \frac{s^2 - 7s + 10}{(s-6)(s+1)} = \frac{(s-5)(s-2)}{(s-6)(s+1)}$$

Il sistema non è BIBO stabile.

b) Calcolare la risposta totale usando Laplace considerando la seguente funzione in input

$$u(t) = 5t e^{2t} \delta_{-1}(t)$$

$$v''(t) - 5v'(t) - 6v(t) = u''(t) - 7u'(t) + 10u(t)$$

$$\downarrow \mathcal{L}$$

$$\mathcal{L}\left[v''(t) - 5v'(t) - 6v(t)\right](s) = \mathcal{L}\left[u''(t) - 7u'(t) + 10u(t)\right](s)$$

$$\mathcal{L}\left[v''(t)\right](s) = s^2V(s) - sv(0^-) - v'(0^-) = s^2V(s) - s$$

$$-5\mathcal{L}\left[v'(t)\right](s) = -5sV(s) + 5v(0^-) = -5sV(s) + 5$$

$$-6\mathcal{L}\left[v(t)\right](s) = -6V(s)$$

$$\mathcal{L}\left[u''(t) - 7u'(t) + 10u(t)\right](s) = (s^2 - 7s + 10)U(s) = (s-5)(s-2)U(s)$$

$$U(s) = \mathcal{L}\left[5t e^{2t} \delta_{-1}(t)\right](s) = \frac{5}{(s-2)^2}$$

$$s^2 V(s) - s - 5s\,V(s) + 5 - 6V(s) = (s-5)\cancel{(s-2)} \cdot \frac{5}{(s-2)^{\cancel{2}}}$$

$$V(s)\left(s^2 - 5s - 6\right) - s + 5 = \frac{5(s-5)}{s-2}$$

$$V(s)(s-6)(s+1) - s + 5 = \frac{5(s-5)}{s-2}$$

$$V(s) = \frac{5(s-5)}{(s-6)(s-2)(s+1)} + \frac{s-5}{(s-6)(s+1)}$$

Fratti semplici

$$V_L(s) = \frac{s-5}{(s-6)(s+1)} = \frac{A}{s-6} + \frac{B}{s+1} = \frac{As + A + Bs - 6B}{(s-6)(s+1)} = \frac{s(A+B) + A - 6B}{(s-6)(s+1)}$$

$$\begin{cases} A+B=1 \\ A-6B=-5 \end{cases} \rightarrow \begin{cases} A = 1 - B \\ -7B = -6 \end{cases} \rightarrow \begin{cases} A = \frac{1}{7} \\ B = \frac{6}{7} \end{cases}$$

$$V_L(s) = \frac{1}{7} \frac{1}{s-6} + \frac{6}{7} \cdot \frac{1}{s+1}$$

$$\downarrow \mathcal{L}^{-1}$$

$$v_L(t) = \left(\frac{1}{7} e^{6t} + \frac{6}{7} e^{-t}\right) \delta_{-1}(t)$$

$$V_F(s) = \frac{5(s-5)}{(s-6)(s-2)(s+1)} = \frac{A}{s-6} + \frac{B}{s-2} + \frac{C}{s+1}$$

$$A = \lim_{s \to 6} \frac{d^{1-0-1}}{ds^{1-0-1}} \cancel{(s-6)} \frac{5(s-5)}{\cancel{(s-6)}(s-2)(s+1)} = \frac{5}{28}$$

$$B = \lim_{s \to 2} \frac{d^{1-0-1}}{ds^{1-0-1}} \cancel{(s-2)} \frac{5(s-5)}{(s-6)\cancel{(s-2)}(s+1)} = \frac{-15}{-12} = \frac{5}{4}$$

$$C = \lim_{s \to -1} \frac{d^{1-0-1}}{ds^{1-0-1}} \cancel{(s+1)} \frac{5(s-5)}{(s-6)(s-2)\cancel{(s+1)}} = \frac{-30}{21} = -\frac{10}{7}$$

$$V_F(s) = \frac{5}{28}\cdot\frac{1}{s-6} + \frac{5}{4}\cdot\frac{1}{s-2} - \frac{10}{7}\cdot\frac{1}{s+1}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

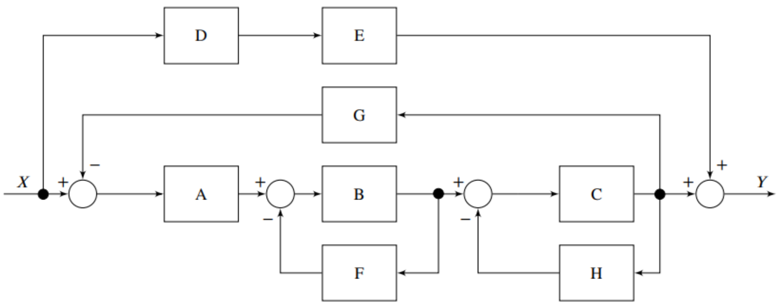
$$v_F(t) = \left(\frac{5}{28}e^{6t} + \frac{5}{4}e^{2t} - \frac{10}{7}e^{-t}\right)\delta_{-1}(t)$$

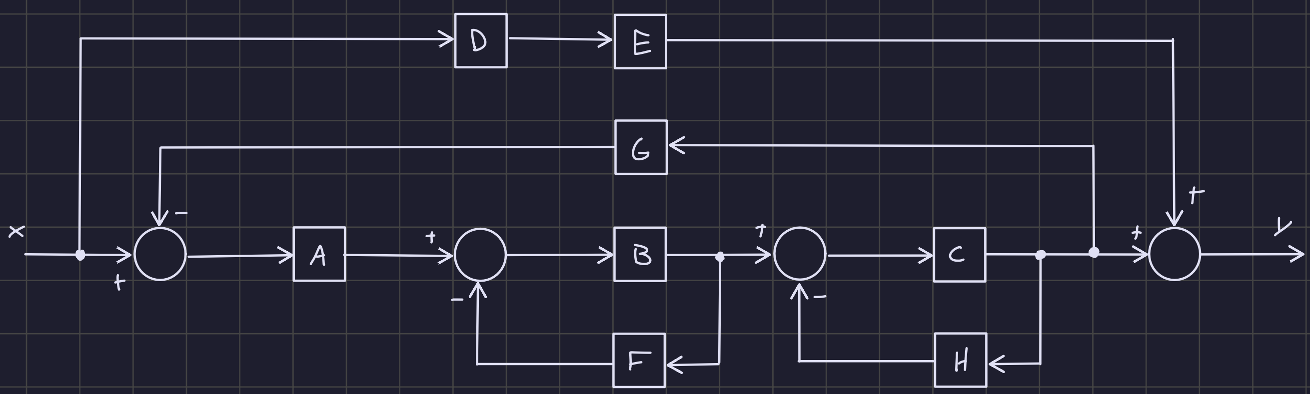
$$v_T(t) = v_L(t) + v_F(t) = \left(\frac{1}{7}e^{6t} + \frac{6}{7}e^{-t} + \frac{5}{28}e^{6t} + \frac{5}{4}e^{2t} - \frac{10}{7}e^{-t}\right)\delta_{-1}(t)$$

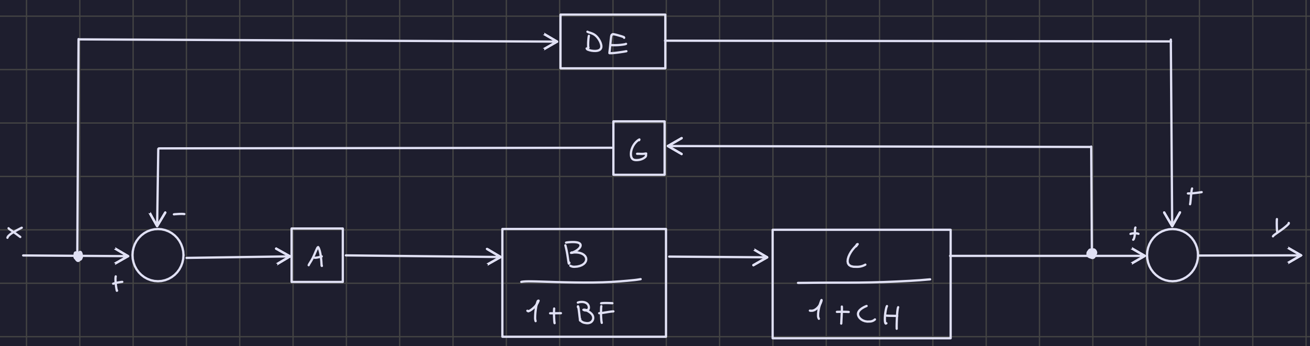
$$= \left(\frac{9}{28}e^{6t} - \frac{4}{7}e^{-t} + \frac{5}{4}e^{2t}\right)\delta_{-1}(t)$$

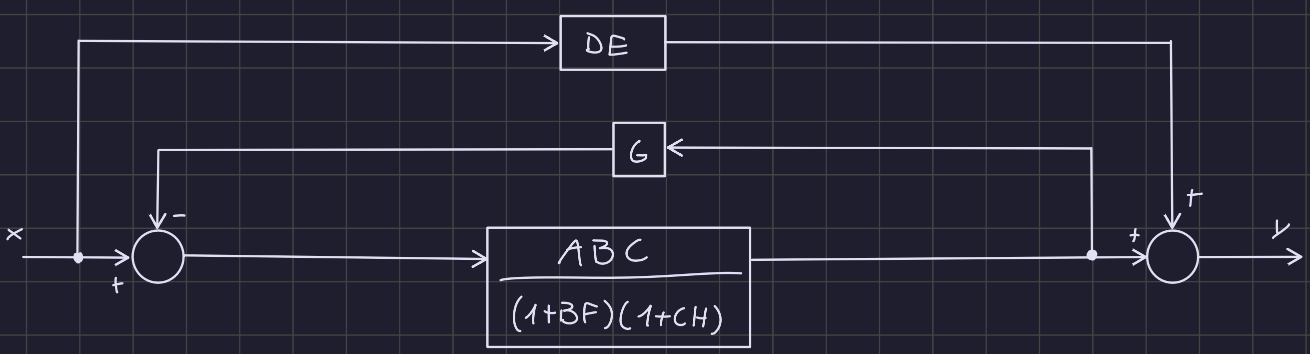
Esercizio 2 (20 punti)

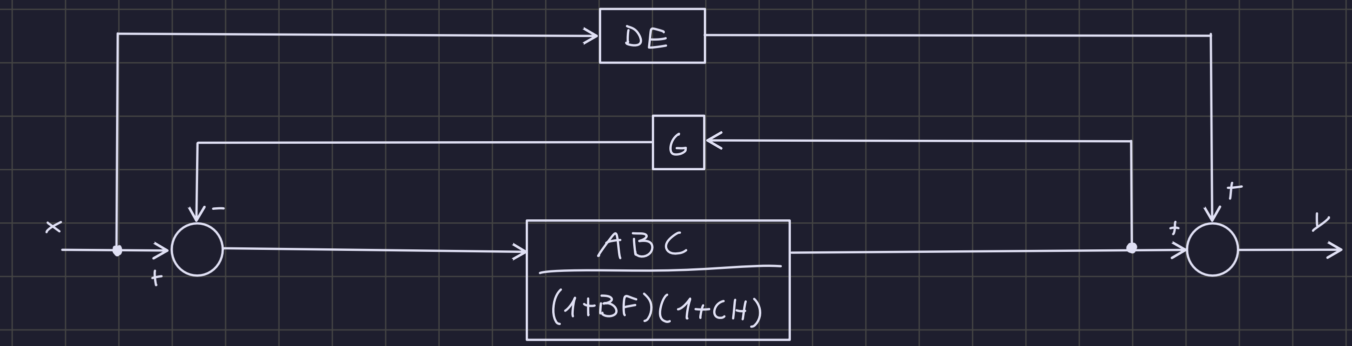
Calcolare la funzione di trasferimento del seguente schema a blocchi:





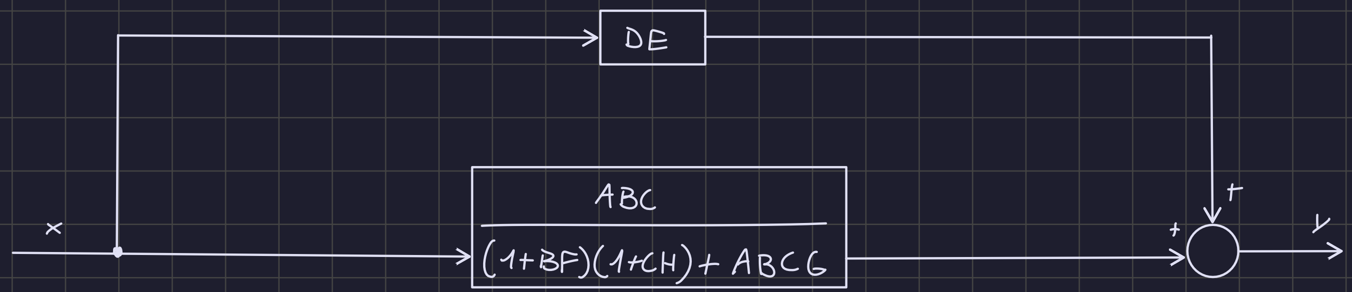






$$\frac{ABC}{\cancel{(1+BF)}(1+CH)} \cdot \frac{\cancel{(1+BF)}(1+CH)}{(1+BF)(1+CH)+ABCG}$$

$\downarrow$



$$\frac{ABC+DE\left((1+BF)(1+CH)+ABCG\right)}{(1+BF)(1+CH)+ABCG} = \frac{ABC+DE\left(1+CH+BF+BFCH+ABCG\right)}{1+CH+BF+BFCH+ABCG}$$

$$y = \frac{ABC+DE\left(1+CH+BF+BFCH+ABCG\right)}{1+CH+BF+BFCH+ABCG}$$

Esercizio 3 (30 punti)

Riportare la funzione in forma di Bode, disegnare i diagrammi di modulo e fase delle singole componenti elementari e il diagramma globale della seguente funzione di trasferimento:

$$G(s) = \frac{24(s^5+3s^4-10s^3)}{s^2+s+16}.$$

$$G(s) = 24 \cdot \frac{s^3(-10+3s+s^2)}{s^2+s+16} = \frac{3}{2}\,\frac{s^3(s+5)(s-2)}{1+\frac{1}{16}s+\frac{1}{16}s^2} = -15\,\frac{s^3\left(1+\frac{1}{5}s\right)\left(1-\frac{1}{2}s\right)}{1+\frac{1}{16}s+\frac{1}{16}s^2}$$

$K_b = -15$

$Z_n = s^3$

$Z_{r_1} = 1+\frac{1}{5}s$

$Z_{r_2} = 1-\frac{1}{2}s$

$P_{cc} = \left(1+\frac{1}{16}s+\frac{1}{16}s^2\right)^{-1}$

$A = 20 \cdot \log_{10}(|15|) \approx 24$ $\quad K_b = -15$

$\phi = \begin{cases} 0 & K_b \geq 0 \\ -180 & K_b < 0 \end{cases}$

Diagramma di Bode

Ampiezza

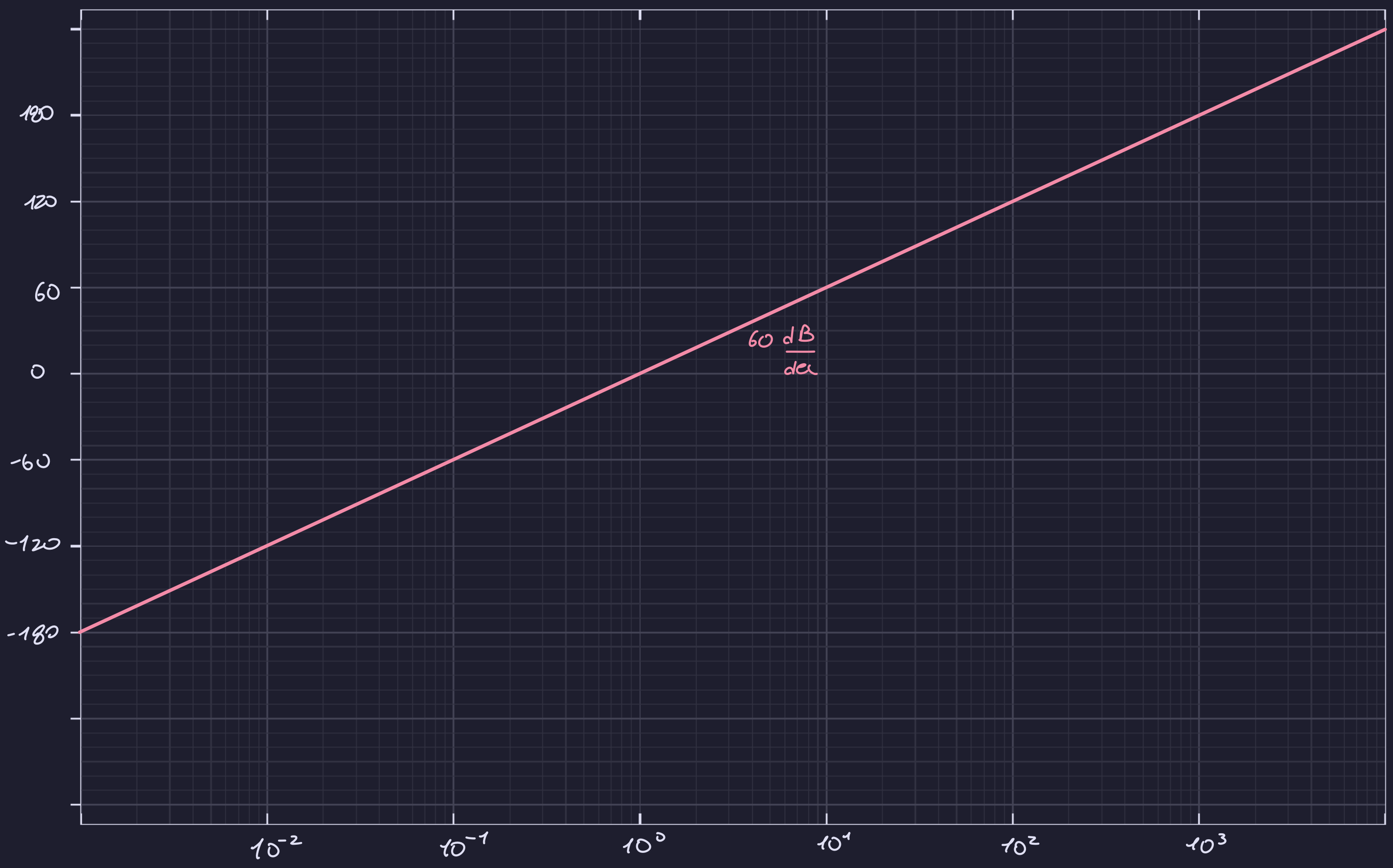
Fase

$A = 20 \cdot \mu \frac{dB}{dec} = 60 \frac{dB}{dec}$

$\phi = \mu \cdot 90 = 270°$

Diagramma di Bode s^3

Ampiezza



Fase

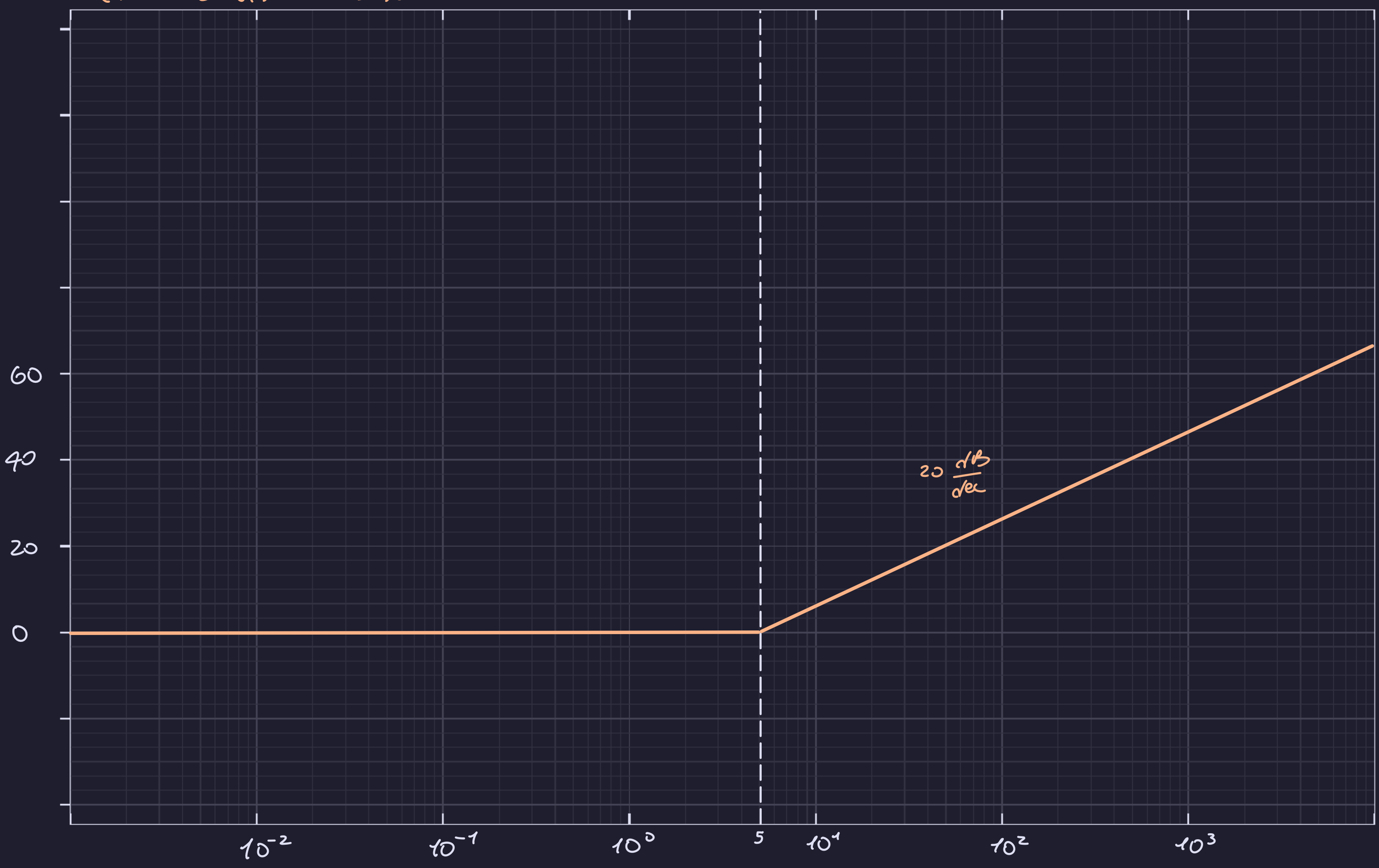
$$A = \begin{cases} 0 & \omega \le \omega_n \\ 20\,\mu \frac{dB}{dec} = 20 \frac{dB}{dec} \end{cases}$$

$$\phi = \begin{cases} 0 & \omega \le \omega_n \\ \mu \cdot 90 \cdot \text{segno}(\tau) & \omega > \omega_n = 90° \end{cases}$$

Diagramma di Bode

$1 + \frac{1}{5}s \qquad \tau = \frac{1}{5} \qquad \omega_n = \frac{1}{|\tau|} = 5$

Ampiezza



Fase

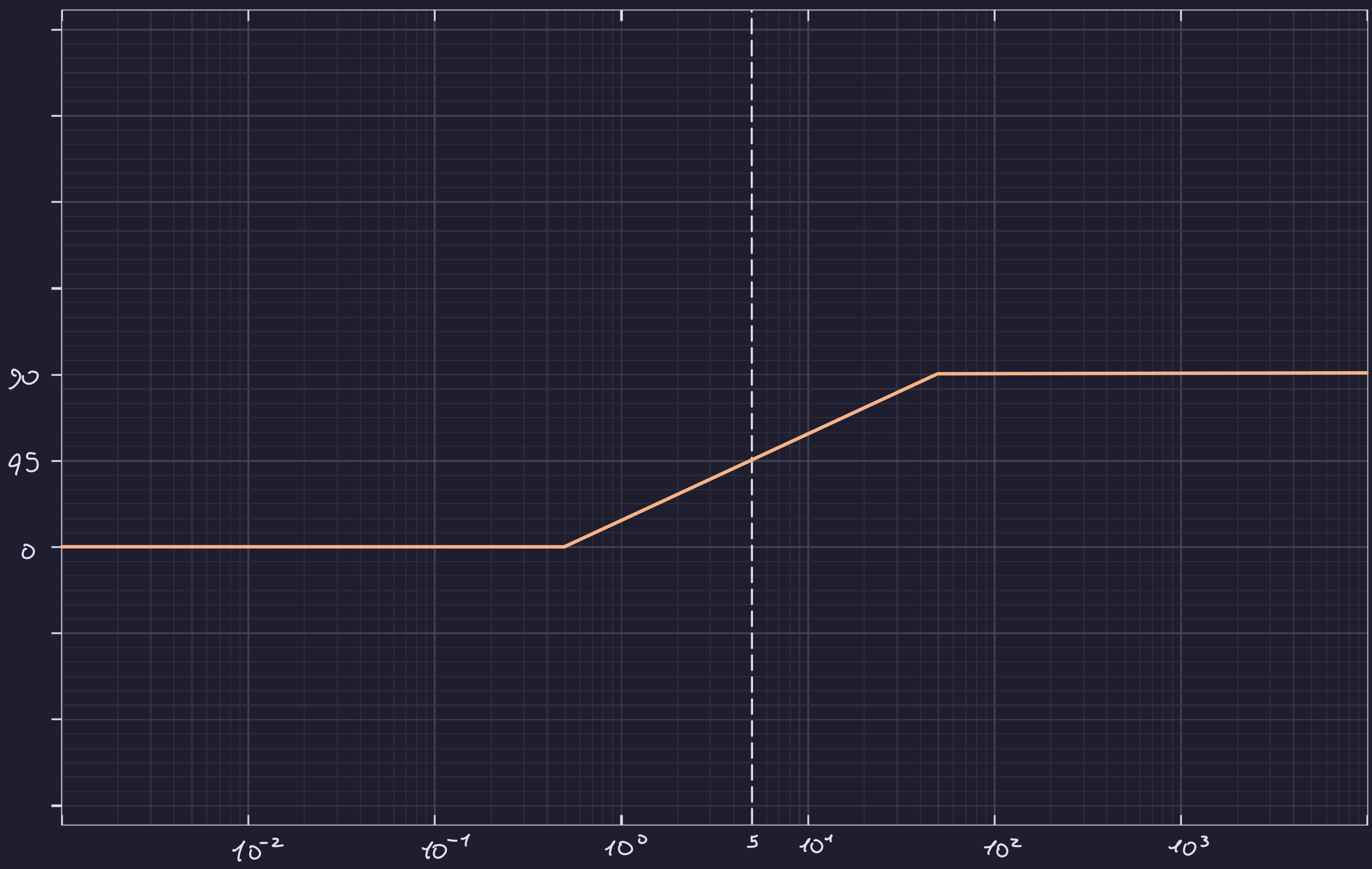


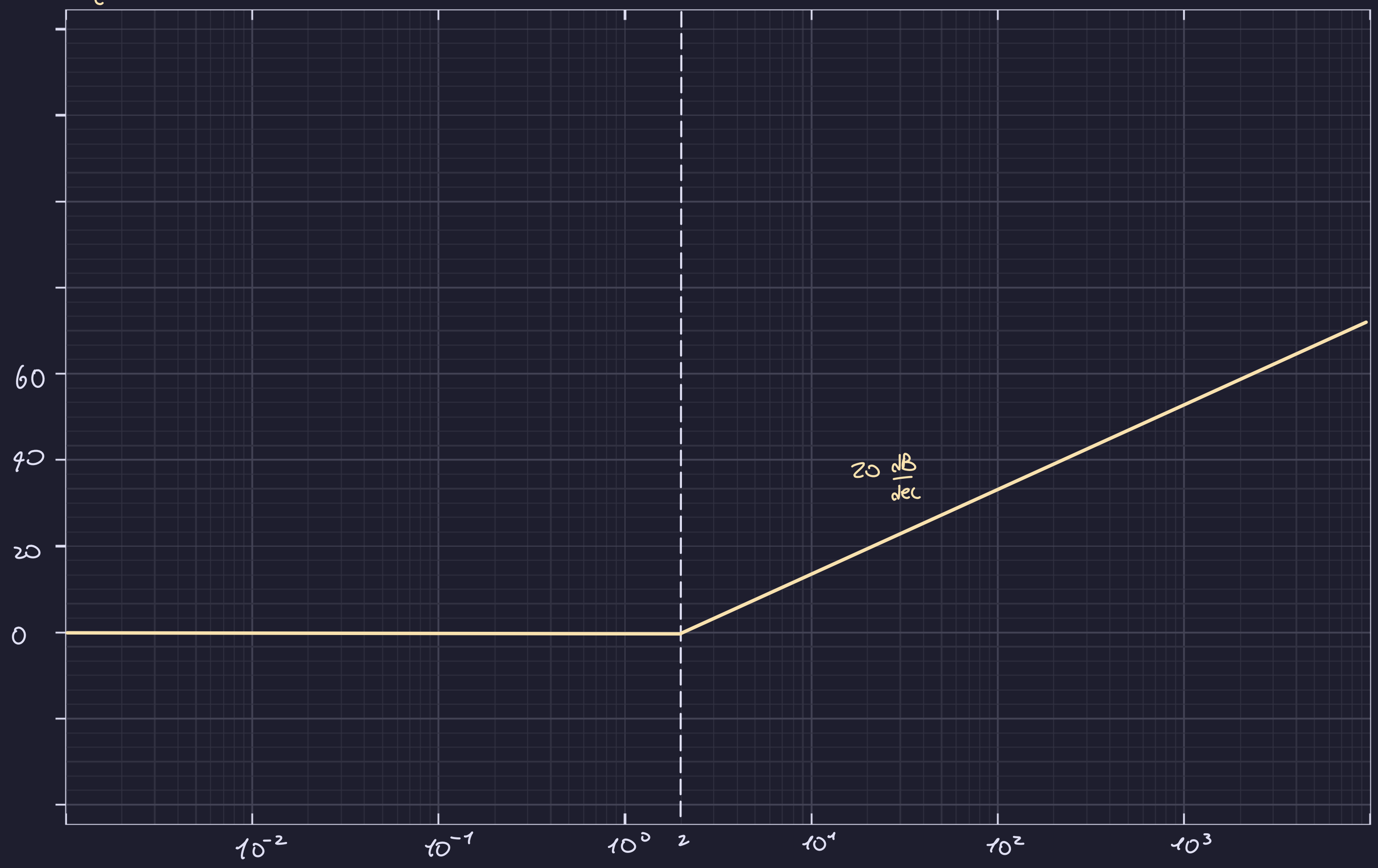
Diagramma di Bode

$$A = \begin{cases} 0 & \omega \le \omega_n \\ 20\,\mu \frac{dB}{dec} & \omega > \omega_n = 20 \frac{dB}{dec} \end{cases}$$

$$1 - \frac{1}{2}s \qquad \tau = -\frac{1}{2} \qquad \omega_n = \frac{1}{|\tau|} = 2$$

$$\phi = \begin{cases} 0 & \omega \le \omega_n \\ 90\,\mu\,\text{segno}(\tau) & \omega > \omega_n = -90^\circ \end{cases}$$

Ampiezza



Fase

$$A = \begin{cases} 0 & \omega \le \omega_n \\ 40\mu \frac{dB}{dec} & \omega > \omega_n = -40 \frac{dB}{dec} \end{cases}$$

$$\Phi = \begin{cases} 0 & \omega \le \omega_n \\ 180\mu \, \text{segno}(\xi) & \omega > \omega_n = -180^\circ \end{cases}$$

Diagramma di Bode $\left(1+\frac{1}{16}s+\frac{1}{16}s^2\right)^{-1}$

$\omega_n^2 = 4$

$\frac{2\xi}{\omega_n} = \frac{1}{\omega_n^2} \rightarrow \xi = \frac{1}{8} \rightarrow$ Picco

$\omega_r = \omega_n \cdot \sqrt{1-2\xi^2} \approx 3.94$

$M_r = 20\mu \cdot \log(2\xi\sqrt{1-\xi^2}) \approx 12.11$

Ampiezza

Fase

Ampiezze

	10^{-1}	10^{0}	2	4	5	10^{1}
k_b	24	24	24	24	24	24
z_n	-60	0	18	36	42	60
z_{r1}	0	0	0	0	0	6
z_{r2}	0	0	0	6	8	14
P_{cc}	0	0	0	0	-4	-16
Totale	-36	24	42	66	70	88

$20\,\mu\,\log_{10}(\omega)$

$\left.\right\}\, 20\,\mu\,\log_{10}(\omega|\tau|)$

$40\,\mu\,\log_{10}\left(\frac{\omega}{\omega_n}\right)$

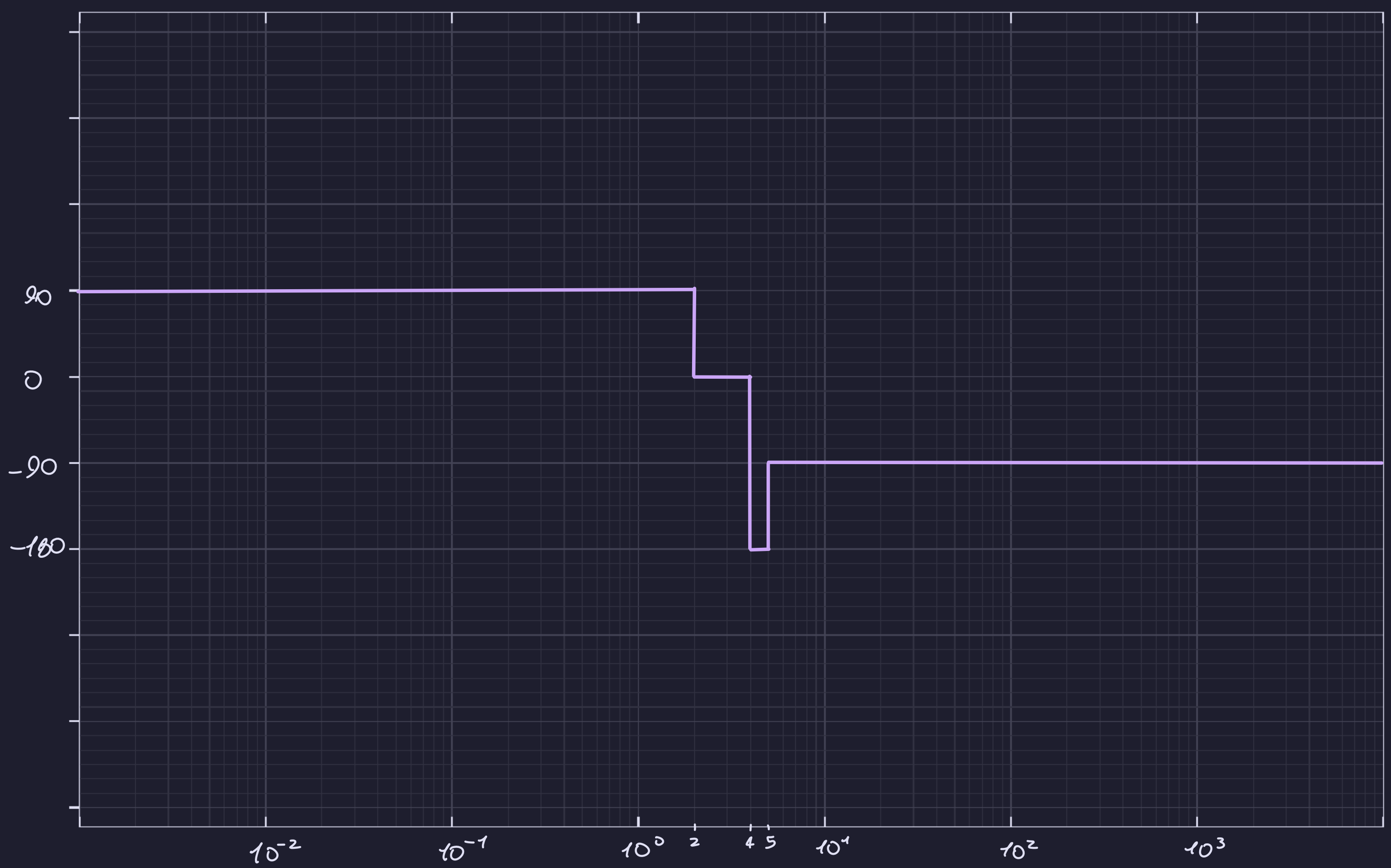
Fasi

	10^{-1}	10^{0}	2	4	5	10^{1}
k_b	-180	-180	-180	-180	-180	-180
z_n	270	270	270	270	270	270
z_{r1}	0	0	0	0	0	90
z_{r2}	0	0	0	-90	-90	-90
P_{cc}	0	0	0	0	-180	-180
Totale	90	90	90	0	-180	-90

Diagramma di Bode

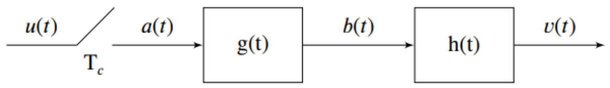
Ampiezza

Fase



Esercizio 4 (35 punti)

Dato il seguente schema a blocchi,



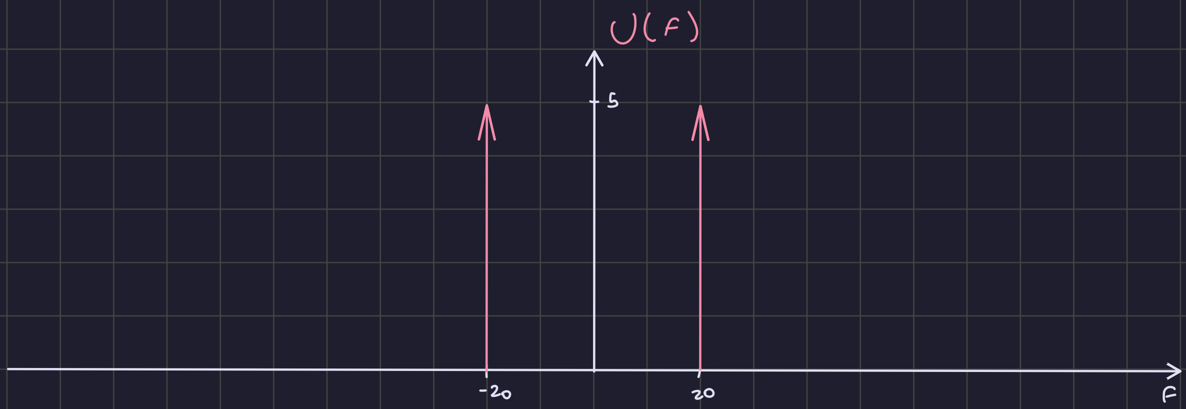
$$u(t) = 10\cos(40\pi t) \qquad g(t) = -15\,\mathrm{sinc}(150t) \qquad h(t) = 3\,\mathrm{sinc}^2(30t) \qquad T_c = \frac{1}{30}\,s$$

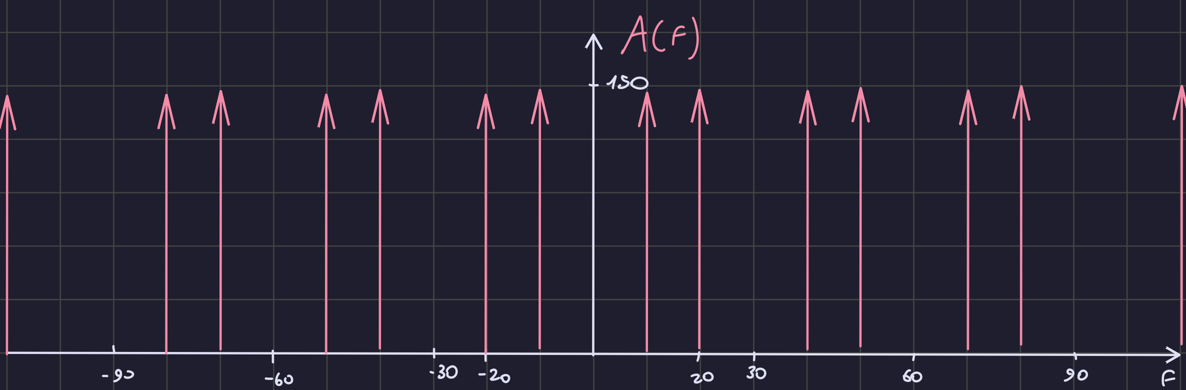
$$u(t) = 10\cos(2\cdot\pi\cdot 20t) \xrightarrow{F} U(f) = 5\,\delta(f-20) + 5\,\delta(f+20)$$

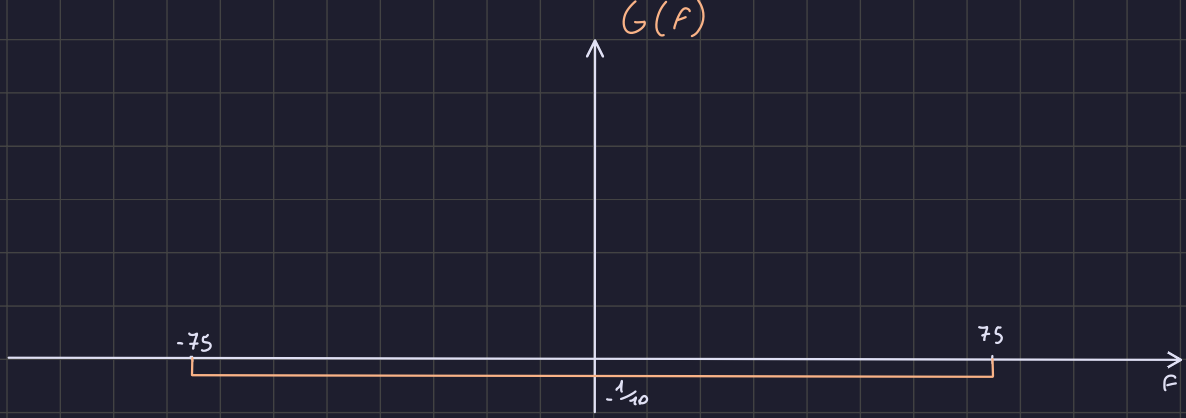
$$g(t) = -\frac{1}{10}\cdot 150\cdot \mathrm{sinc}(150t) \xrightarrow{F} G(f) = -\frac{1}{10}\,\Pi\left(\frac{f}{150}\right)$$

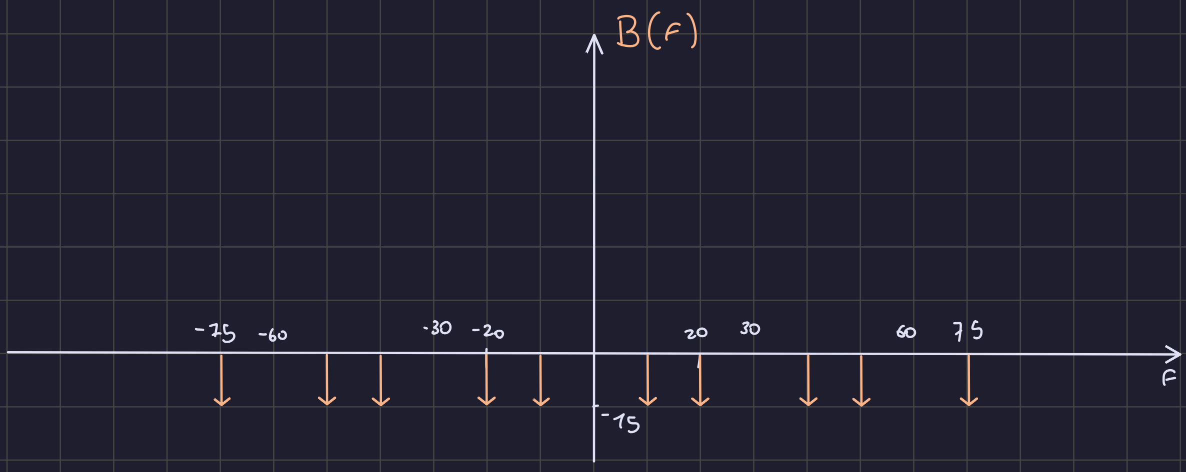
$$h(t) = \frac{1}{10}\cdot 30\,\mathrm{sinc}^2(30t) \xrightarrow{F} H(f) = \frac{1}{10}\cdot\Lambda\left(\frac{f}{30}\right)$$

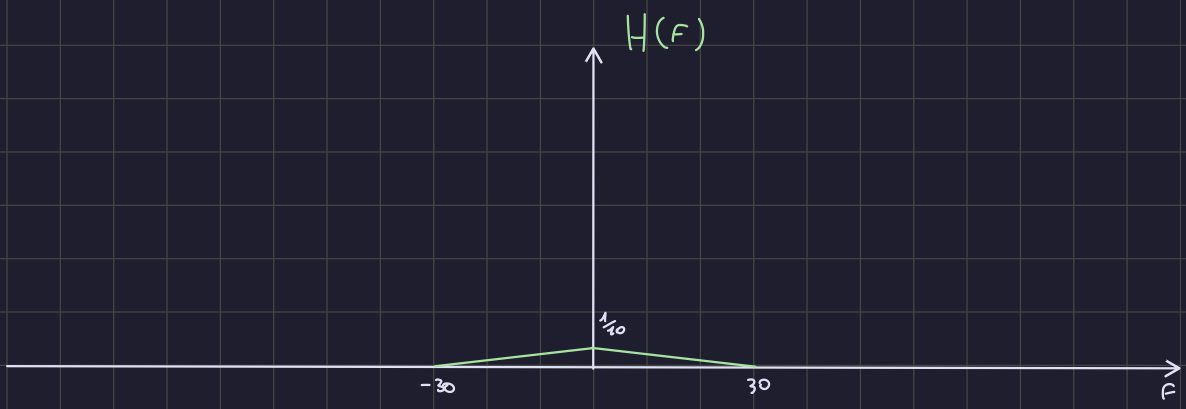
$$T_c = \frac{1}{30}\,s \rightarrow f_c = \frac{1}{T_c} = 30\,Hz$$











$$mx + \frac{1}{10} = 0$$

$$x = -\frac{1}{10m}$$

$$-\frac{1}{10m_1} = 30 \qquad -\frac{1}{10m_2} = -30$$

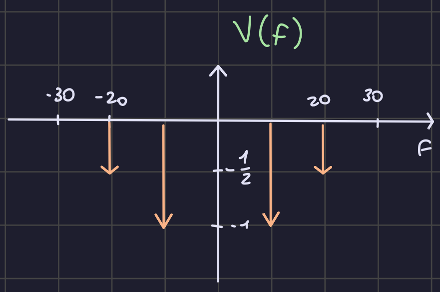
$$-\frac{1}{10} = 30m_1 \qquad -\frac{1}{10} = -30m_2$$

$$m_1 = -\frac{1}{300} \qquad m_2 = \frac{1}{300}$$

$$\rightarrow$$

$$f = 10 \rightarrow -15\left(-\frac{10}{300} + \frac{1}{10}\right) = -15 \cdot \frac{2}{30} = -1$$

$$f = 20 \rightarrow -15\left(-\frac{20}{300} + \frac{1}{10}\right) = -15 \cdot \frac{1}{30} = -\frac{1}{2}$$



Il segnale u(t) non può essere ricostruito dal segnale a(t) perchè si presenta aliasing siccome $f_c \leq 2B$, dove $B = 20$.