# Esercitazione in classe sulle curve

✍ **Esercizio 2.1.1.** *Sia* $\gamma$ *la curva piana una cui parametrizzazione in coordinate polari è* $\rho(\vartheta) = \vartheta^2 + 1$, *on* $0 \leq \vartheta \leq 2\pi$. *Dopo aver disegnato sommariamente il sostegno di* $\gamma$, *determinare i versori tangente e normale al sostegno di* $\gamma$ *nel punto* $\gamma(\pi)$ *e scrivere un'equazione della retta tangente nello stesso punto.*

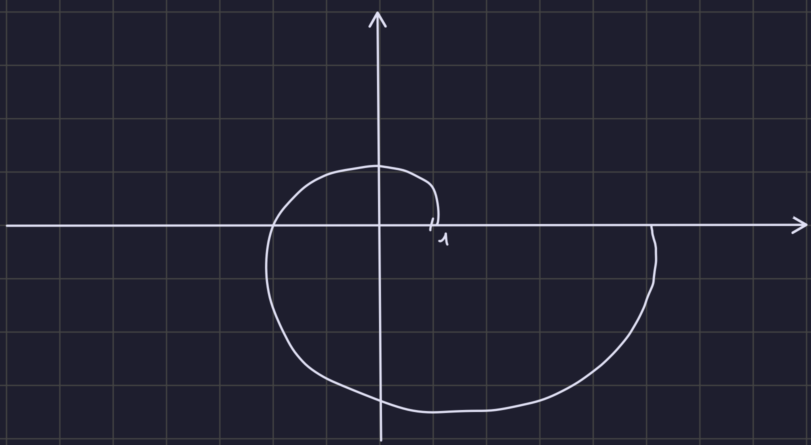
$$\rho(\theta) = \theta^2 + 1 \qquad \theta \in [0, 2\pi]$$

In coordinate cartesiane equivale a

$$\begin{cases} x(\theta) = \rho(\theta)\cos\theta = (\theta^2+1)\cos\theta \\ y(\theta) = \rho(\theta)\sin\theta = (\theta^2+1)\sin\theta \end{cases}$$

coordinate polari

Diamo valori a caso a theta e plottiamo il grafico a grandi linee




Rappresenta la curva

$$\gamma(\theta) = (x(\theta), y(\theta)) = \left((\theta^2+1)\cos\theta, (\theta^2+1)\sin\theta\right)$$

$$\gamma'(\theta) = \left(2\theta\cos\theta - (\theta^2+1)\sin\theta, \; 2\theta\sin\theta + (\theta^2+1)\cos\theta\right)$$

$$\|\gamma'(\theta)\| = \sqrt{4\theta^2\cos^2\theta \cancel{- 4\theta(\theta^2+1)\cos\theta\sin\theta} + (\theta^2+1)^2\sin\theta + 4\theta^2\sin^2\theta \cancel{+ 4\theta(\theta^2+1)\cos\theta\sin\theta} + (\theta^2+1)^2\cos^2\theta}$$

$$= \sqrt{4\theta^2 + (\theta^2+1)^2}$$

Tangente:

$$T(\theta) = \frac{\gamma'(\theta)}{\|\gamma'(\theta)\|} = \frac{\left(2\theta\cos\theta - (\theta^2+1)\sin\theta, \; 2\theta\sin\theta + (\theta^2+1)\cos\theta\right)}{\sqrt{4\theta^2 + (\theta^2+1)^2}}$$

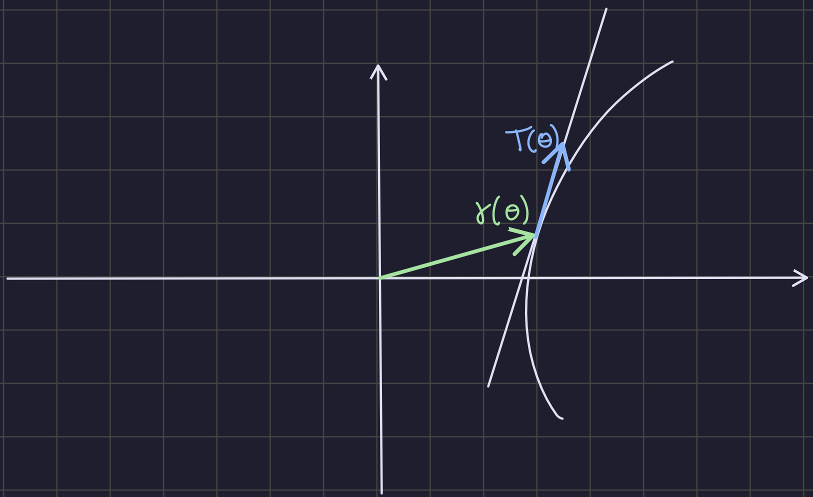
$$T(\pi) = \frac{\left(-2\pi,\ -(\pi^2+1)\right)}{\sqrt{4\pi^2 + (\pi^2+1)^2}}$$

Direzione della tangente in pi

In R² il versore normale è il versore tangente ruotato di 90°

$$N(\pi) = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} T(\pi)$$

Per trovare la retta tangente alla curva bisogna trovare quel vettore che sposta lo spazio vettoriale formato dal vettore tangente sopra la curva e quel vettore è proprio $\gamma(\theta)$




Quindi la retta tangente è:

$$\gamma(\pi) + t\gamma'(\pi)$$

Componente x del vettore tangente

$$\begin{cases} x(t) = -(\pi^2+1) - t\,2\pi \\ y(t) = 0 - t(\pi^2+1) \end{cases}$$

Componente y del vettore tangente

$$-\frac{x + (1+\pi^2)}{2\pi} = t$$

$$y = \frac{x + (1+\pi^2)}{2\pi}(\pi^2+1)$$

$$y = \frac{\pi^2+1}{2\pi}x + \frac{(\pi^2+1)^2}{2\pi}$$

✍ **Esercizio 2.2.4.** *Calcolare l'integrale (curvilineo) di*

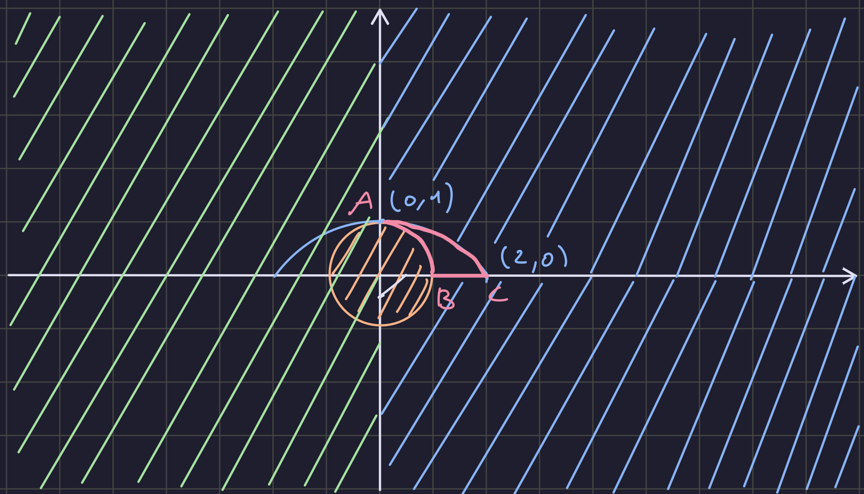
$$f(x,y) = \frac{xy}{\sqrt{4+x^2}}$$

*lungo la curva $\gamma$ il cui sostegno è il bordo $\partial E$ di*

$$E = \left\{(x,y) : x \geq 0, x^2+y^2 \geq 1, 0 \leq y \leq 1 - \frac{x^2}{4}\right\}$$

*e determinare la retta tangente a $\gamma$ nel punto* $\left(1, \frac{3}{4}\right)$.

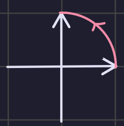
Disegnamo l'insieme E



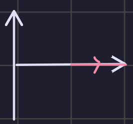

Consideriamo solo l'area rossa

Integriamo lungo le 3 curve parametrizzate:

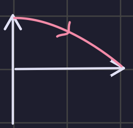
$\gamma_{BA}(t) = (\cos t, \sin t) \quad t \in \left[0, \frac{\pi}{2}\right]$



$\gamma_{BC}(t) = (t, 0) \quad t \in [1,2]$



$\gamma_{AC}(t) = \left(t, 1 - \frac{t^2}{4}\right)$



Integriamo

$$\int_{\gamma_{BA}} F\, ds = \int_0^{\frac{\pi}{2}} \frac{\cos t \sin t}{\sqrt{4+\cos^2 t}} \underbrace{\cancel{\sqrt{(-\sin t)^2 + (\cos t)^2}}}_{|\gamma'(\theta)| = 1} dt$$

$F(x,y) = \frac{xy}{\sqrt{4+x^2}}$

$w = \cos^2 t$

$dw = -2\cos t \sin t\, dt$

$$= -\frac{1}{2} \int \frac{-2\cos t \sin t\, dt}{\sqrt{4+\cos^2 t}}$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{4+w}}\,dw = -(4+w)^{\frac{1}{2}}$$

$$= \left[-\sqrt{4+\cos^2 t}\,\right]_0^{\frac{\pi}{2}} = -2+\sqrt{5}$$

Retta tangente a γ in (1,3/4) (γ(1))

$$\gamma(t)\begin{cases}\gamma_{BA} \xrightarrow{\times} \\ \gamma_{BC} \xrightarrow{\times} \\ \gamma_{AC} \xrightarrow{\checkmark}\end{cases} \left(1,\tfrac{3}{4}\right) \in \gamma_{AC}$$

$$\gamma_{AC}(t) = \left(t,\, 1-\frac{t^2}{4}\right) \rightarrow t \mid \gamma_{AC}(t) = \left(1, \tfrac{3}{4}\right) \Longrightarrow t = 1$$

$$\begin{cases} x(t) = 1 \\ y(t) = \frac{3}{4}\end{cases} \quad \dots$$

$$\gamma'_{AC}(t) = \left(1, -\frac{t}{2}\right)$$

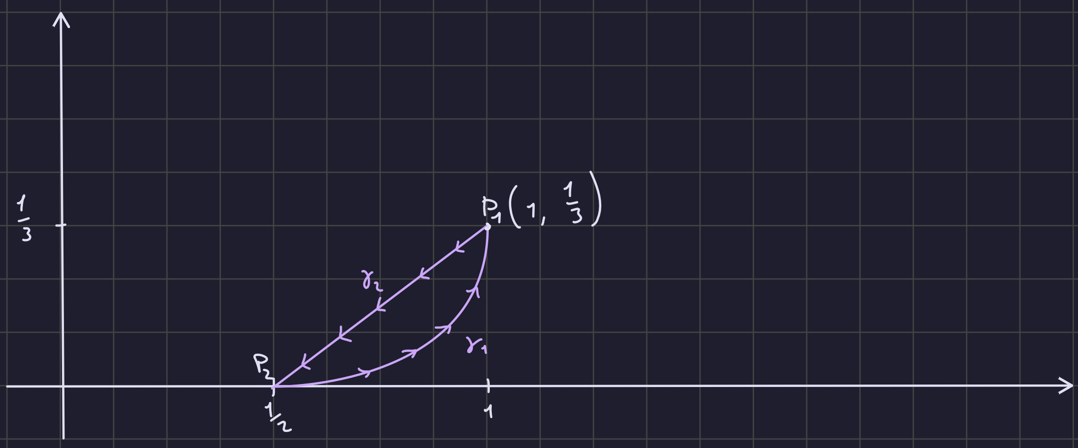
$$r_{TAN}(t) = \gamma_{AC}(t) + s\,\gamma'_{AC}(t) \qquad s \in \mathbb{R}$$

$$r_{TAN}(1) = \gamma_{AC}(1) + s\,\gamma'_{AC}(1) = r_{TAN}(s) = \begin{cases} x(s) = 1+s \\ y(s) = \frac{3}{4} - \frac{s}{2}\end{cases}$$

$$\begin{cases} x(s) = 1+s \\ y(s) = \frac{3}{4} - \frac{s}{2}\end{cases} \rightarrow \begin{cases} s = x-1 \\ y = \frac{3}{4} - \frac{x-1}{2}\end{cases} \rightarrow y = -\frac{x}{2} + \frac{5}{4} \quad \text{retta tangente}$$

✍ **Esercizio 2.1.2.** *Determinare una parametrizzazione della curva chiusa $\gamma$ che si ottiene percorrendo prima da sinistra verso destra il grafico di $f(x) = (1/3)(2x-1)^{3/2}$ per $1/2 \leq x \leq 1$ e poi da destra a sinistra il segmento congiungente gli estremi del grafico di $f$ stessa. Disegnare quindi il sostegno di $\gamma$ e calcolarne la lunghezza.*

$F(x) = \frac{1}{3}(2x-1)^{3/2}$ $\quad F'(x) = \frac{1}{3}(2x-1)^{\frac{1}{2}}$ $\quad F''(x) = \frac{1}{3}\frac{1}{(2x-1)^{\frac{1}{2}}} > 0$ concavità verso l'alto




$$\gamma_1(t) = \left(t, \frac{1}{3}(2t-1)^{3/2}\right) \quad t \in \left[\frac{1}{2}, 1\right]$$

$$\gamma_2(t) = (1-t)\begin{pmatrix}1\\ \frac{1}{3}\end{pmatrix} + t\begin{pmatrix}\frac{1}{2}\\ 0\end{pmatrix} \longleftarrow t \in [0,1] \quad (1-t)p_1 + tp_2$$

$$= \left((1-t) + \frac{t}{2}, \frac{1-t}{3}\right) \quad t \in [0,1]$$

Le due rette sono definite nello stesso intervallo, quindi in t = 1/2 si avrà un valore corrispondente a 2 rette contemporaneamente, e noi non vogliamo questo, ma vogliamo che gamma2 sia collegata a gamma1. Cambiamo di nuovo parametrizzazione

$$\gamma_1(t) \quad t \in \left[\frac{1}{2}, 1\right] \to \left[0, \frac{1}{2}\right] \to t = As + B \to \begin{cases}\frac{1}{2} = A \cdot 0 + B\\ 1 = A \cdot \frac{1}{2} + B\end{cases} \to \begin{cases}B = \frac{1}{2}\\ A = 1\end{cases}$$

$$\Downarrow \quad t = s + \frac{1}{2}$$

$$\gamma_1(s) = \left(s + \frac{1}{2}, \frac{1}{3}(2s+1-1)^{3/2}\right) \longleftarrow t = s + \frac{1}{2}$$

$$= \left(s + \frac{1}{2}, \frac{1}{3}(2s)^{3/2}\right) \longrightarrow \begin{bmatrix} s = 0 \to \left(\frac{1}{2}, 0\right)\\ s = \frac{1}{2} \to \left(1, \frac{1}{3}\right)\end{bmatrix} \quad s \in \left[0, \frac{1}{2}\right]$$

La parametrizzazione è corretta

$$\gamma_2(t) \quad t \in [0,1] \to \left[\frac{1}{2}, 1\right] \to t = As + B \qquad t = 2s - 1$$

$$\begin{cases}0 = A \cdot \frac{1}{2} + B\\ 1 = A \cdot 1 + B\end{cases} \to \begin{cases}A = 2\\ B = -1\end{cases}$$

$$\gamma_2(t) = \left((1-2s+1) + \frac{2s-1}{2}, \frac{(1-2s+1)}{3}\right)$$

$$= \left(2 - 2s + s - \frac{1}{2}, \ \frac{2(1-s)}{3}\right)$$

$$= \left(\frac{3}{2} - s, \ \frac{2(1-s)}{3}\right) \longrightarrow \begin{bmatrix} s = \frac{1}{2} \to \left(1, \frac{1}{3}\right) \\ s = 1 \to \left(\frac{1}{2}, 0\right) \end{bmatrix} \quad s \in \left[\tfrac{1}{2}, 1\right]$$

$$\gamma(s) = \begin{cases} \left(s + \frac{1}{2}, \ \frac{1}{3}(2s)^{3/2}\right) & \text{se } s \in \left[0, \frac{1}{2}\right] \\ \left(\frac{3}{2} - s, \ \frac{2(1-s)}{3}\right) & \text{se } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

I passaggi per cambiare parametro sono:

$$\gamma : [a, b] \to \mathbb{R}^n$$

$[c, d] \nearrow C^1$, (Mappa monotona)

$$\downarrow$$

$t \in [a, b]$
$s \in [c, d]$

$$t = As + B \to \begin{cases} a = Ac + B \\ b = Ad + B \end{cases} \to \begin{cases} A = \cdots \\ B = \cdots \end{cases}$$

✍ **Esercizio 2.2.6.** *Si calcoli l'integrale curvilineo (rispetto all'ascissa curvilinea)* $\int_\alpha z\,ds$, *ove* $\alpha$ *è la curva di parametrizzazione* $\alpha(t) = (t\cos t, t\sin t, t)$, $t \in [0, 2\pi]$. *Si determini inoltre il piano normale ad* $\alpha$ *nel punto* $(-\pi, 0, \pi)$ *(ovvero il piano normale alla retta tangente in quel punto).*

$$\int_{\alpha(t)} F(x, y, z) \overbrace{ds}^{\text{spazio}} = \int_0^{2\pi} F(\alpha(t)) \, |\alpha'(t)| \, dt$$

$F(x, y, z) = z$

$$\alpha(t) = (t\cos t, \ t\sin t, \ t)$$

$$\alpha'(t) = (\cos t - t\sin t, \ \sin t + t\cos t, \ 1)$$

$z = t$

$$\int_0^{2\pi} F(\alpha(t)) \, |\alpha'(t)| \, dt = \int_0^{2\pi} t \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1} \ dt$$

$$= \int_0^{2\pi} t \sqrt{\cos^2 t - \cancel{2t\cos t}\sin t + t^2 \sin^2 t + \sin^2 t + \cancel{2t\sin t \cos t} + t^2 \cos^2 t + 1}\; dt$$

$$= \int_0^{2\pi} t \sqrt{t^2+2}\; dt = \left[ \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} \left(t^2+2\right)^{\frac{3}{2}} \right]_0^{2\pi} = \frac{\left(4\pi^2+2\right)^{3/2} - 2^{3/2}}{3}$$

Calcoliamo il piano normale in $(-\pi, 0, \pi)$

$$\alpha(t) = (t\cos t,\ t\sin t,\ t) \Rightarrow (-\pi, 0, \pi) \leftrightarrow t = \pi$$

$$\alpha'(t) = (\cos t - t\sin t,\ \sin t + t\cos t,\ 1)$$

Cerchiamo una direzione tangente alla curva in $\pi$

$$\alpha'(\pi) = (-1, -\pi, 1)$$

L'equazione della retta tangente alla curva nel punto $(-\pi, 0, \pi) \rightarrow r = (-\pi, 0, \pi) + s(-1, -\pi, 1)$

Bisogna trovare il piano perpendicolare alla retta tangente

Metodo 1:

$$\left\langle \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \begin{pmatrix} -1 \\ -\pi \\ 1 \end{pmatrix} \right\rangle = 0$$

$$-s_1 - \pi s_2 + s_3 = 0 \qquad s_1 = \underset{\substack{\uparrow \\ k}}{s_3} - \pi \underset{\substack{\uparrow \\ t}}{s_2}$$ Teorema rouche capelli

$$\begin{pmatrix} k - \pi t \\ t \\ k \end{pmatrix} \longrightarrow k \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -\pi \\ 1 \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} -\pi \\ 0 \\ \pi \end{pmatrix}}_{\text{Vettore che sposta il piano nel punto interessato}} + \underbrace{k \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -\pi \\ 1 \\ 0 \end{pmatrix}}_{\text{Piano perpendicolare alla retta tangente (piano normale)}} \rightarrow \begin{cases} x(k,t) = -\pi + k - \pi t \\ y(k,t) = t \\ z(k,t) = \pi + k \end{cases}$$

$$\begin{cases} x = -\pi + z - \pi - \pi y \\ t = y \\ k = z - \pi \end{cases} \quad \rightarrow \quad z - x - \pi y = 2\pi$$

Metodo 2:

$$\left\langle \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \begin{pmatrix} -1 \\ -\pi \\ 1 \end{pmatrix} \right\rangle = 0$$

Trasciniamo lo spazio affine nell'origine, così non bisogna calcolare il vettore che trasla lo spazio nel punto della retta tangente

$$\left\langle \begin{pmatrix} x - (-\pi) \\ y \\ z - \pi \end{pmatrix}, \begin{pmatrix} -1 \\ -\pi \\ 1 \end{pmatrix} \right\rangle = 0$$

$$-(x + \pi) - \pi y + z - \pi = 0$$

$$z - x - \pi y = 2\pi$$

✍ **Esercizio 3.6.2.** *Dire se la seguente funzione è differenziabile*

$$x^{3/2} + (xy)^{3/2}$$

$$F(x,y) = \sqrt{x^3} + \sqrt{(xy)^3}$$

Dominio

$$\begin{cases} xy \geq 0 \\ x \geq 0 \end{cases} \rightarrow \begin{cases} y \geq 0 \\ x \geq 0 \end{cases} \rightarrow \begin{matrix} x : [0, +\infty) \\ y : [0, +\infty) \end{matrix}$$

Derivata

$$\frac{\partial}{\partial x} f(x,y) = \frac{3}{2}\sqrt{x} + \frac{3}{2}\sqrt{xy}\; y$$

$$\frac{\partial}{\partial y} F(x,y) = \frac{3}{2}\sqrt{xy}\; x$$

Le derivate parziali esistono e sono continue, quindi la funzione è differenziabile

✍ **Esercizio 3.6.3.** *Dire se la seguente funzione è differenziabile*

$$f(x,y) = \begin{cases} \dfrac{x^2y^3}{x^4+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$(x,y) \neq 0 : \frac{\partial}{\partial x} F(x,y) = \frac{2x y^3 (x^4 + y^4) - x^2 y^3\, 4x^3}{(x^4+y^4)^2}$$

$$\frac{\partial}{\partial y} F(x,y) = \frac{3y^2 x^2 (x^4 + y^4) - 2x\, y^3\, 4y^3}{(x^4+y^4)^2}$$

Le derivate parziali sono continue per (x,y) ≠ 0

$$(x,y) = (0,0) : \frac{\partial}{\partial x} F(0,0) = \lim_{h \to 0} \frac{F(h,0) - F(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial}{\partial y} F(0,0) = \lim_{k \to 0} \frac{F(0,k) - F(0,0)}{k} = \lim_{k \to} \frac{0-0}{k} = 0$$

$$\lim_{(h,k)=(0,0)} \frac{f(0+h,\,0+k)-f(0,0)-\frac{\partial}{\partial x}f(0,0)\,h-\frac{\partial}{\partial y}f(0,0)\,k}{\sqrt{h^2+k^2}} =$$

$$= \lim_{(h,k)=(0,0)} \frac{\frac{h^2k^3}{h^4+k^4}-0-0-0}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)=(0,0)} \frac{h^2k^3}{(h^4+k^4)\sqrt{h^2+k^2}} = \text{Non esiste}$$

$$\downarrow h=k$$

$$\lim_{h\to 0} \frac{h^5}{2h^4\sqrt{2h^2}} = \lim_{h\to 0}\frac{h}{2\sqrt{2}\,|h|} = \pm\frac{1}{2\sqrt{2}} \neq 0$$

✍ **Esercizio 3.3.7.** *Calcolare*

$$\lim_{(x,y)\to(0,0)} \frac{xy^3 - 2\sin(x^2y)\cos(x+2y)}{x^2+y^2}$$

$$\text{se } (x,0) \to \lim_{x\to 0} \frac{0-0\cdot\cos(x)}{x^2} = 0$$

$$\text{se } (0,y) \to \lim_{y\to 0} \frac{0-0\,\cos(2y)}{y^2} = 0$$

Il limite, fissato x e fissato y esiste, quindi bisogna trovare un'altra sequenza che non tenda a 0

$$\underbrace{\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^2}}_{} - 2\underbrace{\lim_{(x,y)\to(0,0)} \frac{\sin(x^2y)\cos(x+2y)}{x^2+y^2}}_{}$$

$$y = x^{\frac{1}{3}}$$

$$\lim_{x\to 0} \frac{x^2}{x^2+x^{\frac{2}{3}}} = 0$$

$$\sin(x^2y) = x^2y + o(x^2y)$$

$$\cos(x+2y) = 1 + o(x+2y)$$

$$\downarrow$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y}$$

$$\downarrow\ f\to 0 \Leftrightarrow |f|\to 0$$

$$0 \le \lim_{(x,y)\to(0,0)} |y| \underbrace{\frac{x^2}{x^2+y^2}}_{\le 1} \le \lim_{y\to 0} |y| = 0$$

$$\downarrow$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y} = 0 \quad (\text{esiste})$$

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^2}$$

$$\Big\downarrow\ f\to 0 \Leftrightarrow |f|\to 0$$

$$0 \le \lim_{(x,y)\to(0,0)} \frac{|xy^3|}{x^2+y^2} = \lim_{(x,y)\to(0,0)} |xy| \underbrace{\frac{y^2}{x^2+y^2}}_{\le 1} \le \lim_{(x,y)\to(0,0)} |x,y| = 0 \quad (\text{esiste})$$

Quindi il limite esiste

Un altro modo è quello di usare le coordinate polari

$$\lim_{\rho\to 0} \frac{\rho^4\cos\theta\,\sin^3\theta - 2\sin(\rho^3\cos^2\theta\,\sin\theta)\cos(\rho\cos\theta + 2\rho\sin\theta)}{\rho^2}$$

$$\lim_{\rho\to 0} \rho^2 \underbrace{\cos\theta\,\sin^3\theta}_{\in[-1,1]} - 2\lim_{\rho\to 0} \frac{\sin(\rho^3\cos^2\theta\,\sin\theta)\cos(\rho\cos\theta + 2\rho\sin\theta)}{\rho^2}$$

$$\downarrow$$

$\circ$

$$-2 \lim_{\rho \to 0} \frac{\sin(\rho^3 \cos^2\theta \sin\theta) \cos(\rho\cos\theta + 2\rho\sin\theta)}{\rho^2}$$

$\downarrow$ Approssimo seno e coseno con l'argomento

$$\lim_{\rho \to 0} \rho \cos^2\theta \sin\theta = 0$$

$= 0$ quindi esiste

✍ **Esercizio 3.3.14.** *Si consideri la funzione*

$$f(x,y) = \frac{x^2\,(y-x)}{(x^2+y^2)^\alpha}, \qquad (x,y) \neq (0,0).$$

*Si determini se esiste (e in caso affermativo si calcoli)*

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

*quando* $\alpha = 1$ *e quando* $\alpha = 2$.

Passiamo in coordinate polari

$$\lim_{\rho \to 0} \frac{\rho^3 \cos^2\theta\,(\sin\theta - \cos\theta)}{\rho^{2\alpha}} = \lim_{\rho \to 0} \rho^{3-2\alpha} \cos^2\theta\,(\sin\theta - \cos\theta)$$

$$\begin{cases} \text{se } 3-2\alpha > 0 & \text{esiste e fa } 0 \\ \text{altrimenti} & \text{non esiste} \end{cases}$$

$\downarrow$

$$\begin{cases} \text{se } \alpha < \frac{3}{2} & \text{esiste e fa } 0 \\ \text{altrimenti} & \text{non esiste} \end{cases}$$

✍ **Esercizio 3.6.5.** *Si verifichi che la funzione*

$$f(x, y) = |x| \log(1 + y)$$

*è differenziabile in (0,0).*

$$\frac{\partial}{\partial x} f(x,y) = \lim_{h \to 0} \frac{|x+h| \log(1) - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$$\frac{\partial}{\partial x} f(0,0) = 0$$

$$\frac{\partial}{\partial y} f(x,y) = \frac{|x|}{1+y}$$

$$\frac{\partial}{\partial y} f(0,0) = 0$$

$$\lim_{(h,k)\to(0,0)} \frac{|h| \log(1+k) - \cancel{|0| \log(1)} - \cancel{0h} - 0k}{\sqrt{h^2+k^2}} =$$

Trasformo in coordinate polari

$$\lim_{\rho \to 0} \frac{\rho |\cos\theta| \log(1+\rho \sin\theta)}{\rho}$$

$$\downarrow \; F \to 0 \iff |F| \to 0$$

$$0 \le \lim_{\rho\to 0} |\cos\theta| \, |\log(1+\rho\sin\theta)| \le \lim_{\rho\to 0} |\log(1+\rho\sin\theta)| = 0$$

Metodo 2:

✍ **Esercizio 3.6.2.** *Dire se la seguente funzione è differenziabile*

$$x^{3/2} + (xy)^{3/2}$$

$x \geq 0$

$y \geq 0$

$x^{\frac{3}{2}} (1 + y^{\frac{3}{2}})$

$$\frac{\partial}{\partial x} F(x,y) = \frac{3}{2} x^{\frac{1}{2}} \cdot (1 - y^{\frac{3}{2}})$$

$$\frac{\partial}{\partial y} F(x,y) = x^{\frac{3}{2}} \cdot \left(1 - \frac{3}{2} y^{\frac{1}{2}}\right)$$

Le derivate parziali sono continue, quindi la funzione è differenziabile

✍ **Esercizio 3.1.6.** *Trovare l'insieme di definizione della funzione* $f(x,y) = \arcsin \dfrac{4xy}{x^2+y^2}$

$$f(x,y) = \arcsin\left(\frac{4xy}{x^2+y}\right)$$

$$\mathbb{D} = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; -1 \leq \frac{4xy}{x^2+y^2} \leq 1 \;,\; (x,y) \neq (0,0) \right\}$$

$$-1 \leq \frac{4xy}{x^2+y^2} \leq 1$$

$$-x^2 - y^2 \leq 4xy \leq x^2 + y^2$$

$$\downarrow$$

$$\begin{cases} x^2 + y^2 + 4xy \geq 0 \\ x^2 + y^2 - 4xy \geq 0 \end{cases}$$

Trasformo in coordinate polari

$$\begin{cases} \rho^2 + 4\rho^2 \cos\theta \sin\theta \geq 0 \\ \rho^2 - 4\rho^2 \cos\theta \sin\theta \geq 0 \end{cases}$$

$$\begin{cases} \rho^2(1+4\cos\theta\sin\theta) \geq 0 \\ \rho^2(1-4\cos\theta\sin\theta) \geq 0 \end{cases}$$

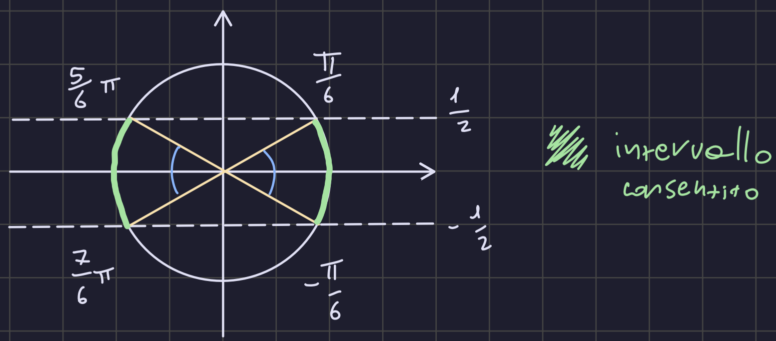
$\rho^2$ é sempre positivo

$$\begin{cases} 1+2\sin(2\theta) \geq 0 \\ 1-2\sin(2\theta) \geq 0 \end{cases}$$

$2\sin x\cos x = \sin 2x$

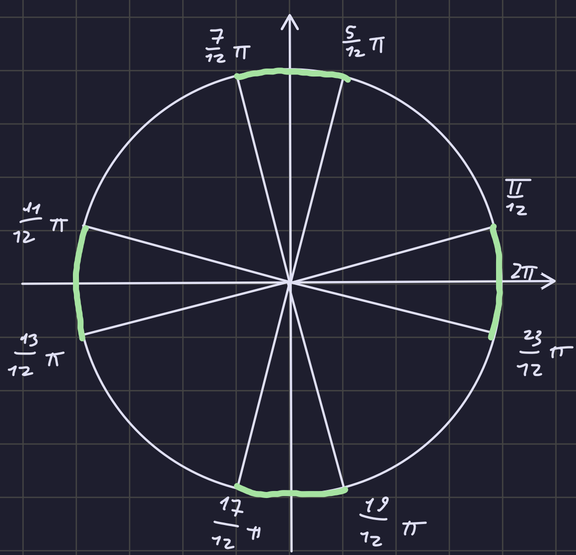
$$\begin{cases} \sin(2\theta) \geq -\frac{1}{2} \\ \sin(2\theta) \leq \frac{1}{2} \end{cases} \rightarrow \quad \sin(2\theta) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

$\downarrow$

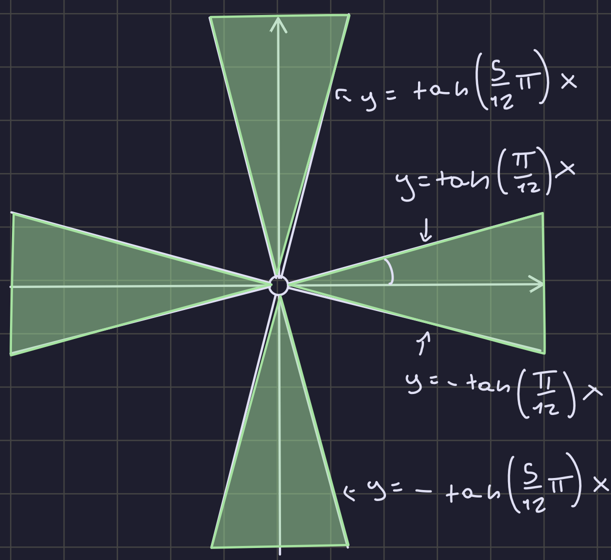



$$\begin{cases} -\frac{\pi}{6}+2k\pi \leq 2\theta \leq \frac{\pi}{6}+2k\pi \\ \frac{5}{6}\pi+2k\pi \leq 2\theta \leq \frac{7}{6}\pi+2k\pi \end{cases}$$

$$\begin{cases} -\frac{\pi}{12}+k\pi \leq \theta \leq \frac{\pi}{12}+k\pi \\ \frac{5}{12}\pi+k\pi \leq \theta \leq \frac{7}{12}\pi+k\pi \\ \theta \in [0, 2\pi] \end{cases}$$



Dominio:



$y = \tan\left(\frac{5}{12}\pi\right)x$
$y = \tan\left(\frac{\pi}{12}\right)x$
$y = -\tan\left(\frac{\pi}{12}\right)x$
$y = -\tan\left(\frac{5}{12}\pi\right)x$

✍ **Esercizio 3.8.2.** *Data la funzione*

$$f(x,y) = \sqrt[3]{x^2(y-1)} + 1$$

a) si verifichi che non è differenziabile in (0,1)

b) si calcolino tutte le derivate direzionali $D_v f(0,1)$ ($v$ versore di $\mathbb{R}^2$)

$$F(x,y) = \sqrt[3]{x^2(y-1)} + 1 = \left(x^2(y-1)\right)^{\frac{1}{3}} + 1 = \left(x^2y - x^2\right)^{\frac{1}{3}} + 1$$

$$F(0,1) = 1$$

$$\left.\begin{array}{l} F(x,1) = 1 \quad \forall x \in \mathbb{R} \\ \\ F(0,y) = 1 \quad \forall y \in \mathbb{R} \end{array}\right\} \to \vec{\nabla} F(0,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\lim_{(h,k)\to(0,0)} \frac{F(h,1+k) - F(0,1) - \langle \vec{\nabla} F(0,1), \begin{pmatrix} h \\ k \end{pmatrix} \rangle}{\left\| \begin{pmatrix} h \\ k \end{pmatrix} \right\|} =$$

$$= \lim_{(h,k)\to(0,0)} \frac{\sqrt[3]{h^2k} + 1 - 1 - 0}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{\sqrt[3]{h^2k}}{\sqrt{h^2+k^2}} \xrightarrow{h=k} \lim_{h\to 0} \frac{h}{\sqrt{2}\,|h|} = \pm \frac{1}{\sqrt{2}} \text{ (Non esiste)}$$

$$D_v F(0,1) = \lim_{t\to 0} \frac{F\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) - F\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}{t}$$

Non si può usare la formula del gradiente

$$v = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

Tutti i vettori di norma 1

$$= \lim_{t\to 0} \frac{F\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right) - F\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}{t}$$

$$= \lim_{t \to 0} \frac{F(t\cos\theta, 1+t\sin\theta) - 1}{t}$$

$$= \lim_{t \to 0} \frac{\sqrt[3]{t^2\cos^2\theta \; t\sin\theta} + 1 - 1}{t}$$

$$= \lim_{t \to 0} \frac{\sqrt[3]{t^3\cos^2\theta \sin\theta}}{t}$$

$$= \lim_{t \to 0} \frac{\cancel{t}\sqrt[3]{\cos^2\theta \sin\theta}}{\cancel{t}}$$

$$= \sqrt[3]{\cos^2\theta \sin\theta}$$

✍ **Esercizio 3.8.3.** *Data la funzione*

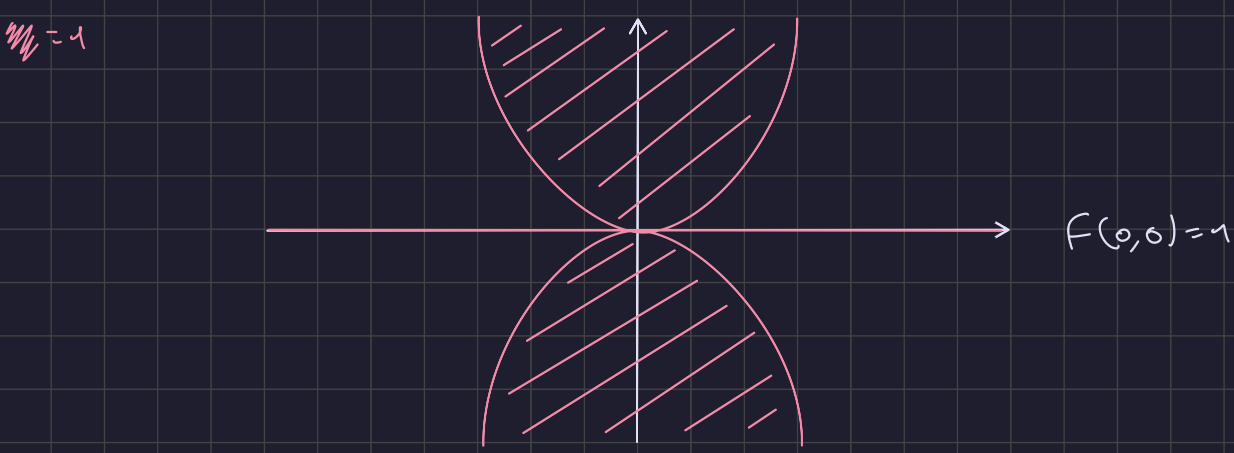
$$f(x,y) = \begin{cases} 1 & |y| > x^2 \vee y = 0 \\ 0 & \text{altrove} \end{cases}$$

*si calcoli* $D_v f(0,0)\ \forall v \in \mathbb{R}^2$ *versore e si verifichi che*

$$D_v f(0,0) = < \nabla f(0,0), v > .$$

*La funzione è differenziabile in (0,0)?*

$$F(x,y) = \begin{cases} 1 & y < -x^2 \wedge y > x^2 \vee y = 0 \\ 0 & \text{altrove} \end{cases}$$



$$D_v f(0,0) = \frac{\overbrace{f(t\cos\theta, t\sin\theta)}^{=1} - 1}{t} \longrightarrow (t, mt)\ \exists t \mid |mt| > t^2 ?$$

$$\Downarrow$$

$$\lim_{t \to 0} \frac{f(t, mt) - 1}{t} = 0$$

$$\lim_{x \to 0^+} f(x,0) = 1 \qquad \lim_{x \to 0^+} f(x, x^3) = 0$$

$$\Downarrow$$

Non continua

✍ **Esercizio 3.9.6.** *La temperatura nel punto* $(x, y)$ *in una regione del piano* $xy$ *è* $T$ *(misurata in gradi centigradi), dove*

$$T(x, y) = x^2\, e^{-y}.$$

*In quale direzione aumenta più rapidamente la temperatura nel punto* $(2, 1)$*? Con quale rapidità aumenta* $T$ *in quella direzione?*

$$T(x,y) = x^2 e^{-y}$$

$$T(2,1) = 4e^{-1}$$

$$\frac{\partial}{\partial x} = 2x e^{-y}$$

$$\frac{\partial}{\partial y} = -x^2 e^{-y}$$

$$\overline{\nabla} T(2,1) = (4e^{-1}, -4e^{-1}) = 4e^{-1}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Direzione: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Rapidità: $4e^{-1} \cdot \left\|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\| = 4e^{-1} \cdot \sqrt{2}$

✍ **Esercizio 3.9.3.** *Data la funzione*

$$f(x,y)=e^{x^2}(\alpha x - y^3) \qquad \alpha\in\mathbb{R};$$

*si determini $\alpha$ in modo che:*

a) la direzione di massima crescita in $(0,1)$ sia lungo la tangente alla parabola $y=(x+1)^2$ nel verso negativo dell'asse $x$;

b) il piano tangente in $(0,1)$ sia perpendicolare alla retta $\frac{x}{2}=\frac{y}{3}=z$.

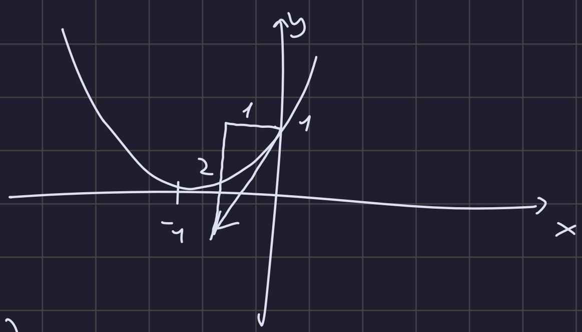
a)

$$F(x,y)=e^{x^2}(\alpha x - y^3)$$

$$\frac{d}{dx} = 2x e^{x^2}(\alpha x - y^3) + \alpha e^{x^2} = e^{x^2}(\alpha + 2x(\alpha x - y^3))$$

$$\frac{d}{dy} = -3y^2 e^{x^2}$$

$$\bar\nabla F(x,y) = e^{x^2}\begin{pmatrix}\alpha(1+2x^2)-2xy^3\\ -3y^2\end{pmatrix}$$




$$P(x)=(x+1)^2 \longrightarrow p(t)=(t,(t+1)^2)$$

$$\downarrow$$

$$p'(t)=\begin{pmatrix}1\\2t+2\end{pmatrix}$$

$$p'(0)=\begin{pmatrix}1\\2\end{pmatrix}$$

$$v_T = -p'(0) = \begin{pmatrix}-1\\-2\end{pmatrix}$$

$\uparrow$

direzione negativa asse x

$$r_T = \begin{pmatrix}0\\1\end{pmatrix}+t\begin{pmatrix}-1\\-2\end{pmatrix}$$

$$\downarrow$$

$$\begin{cases}x=-t\\ y=1-2t=1+2x\end{cases}$$

$$t\begin{pmatrix}-1\\-2\end{pmatrix}=\bar\nabla F(0,1)=\begin{pmatrix}\alpha\\-3\end{pmatrix} \longrightarrow \begin{cases}-t=\alpha\\ -2t=-3\end{cases} \quad \begin{cases}\alpha=-t\\ t=\frac{3}{2}\end{cases} \quad \alpha=-\frac{3}{2}$$

$t>0$

b) $\frac{x}{2} = \frac{y}{3} = z \rightarrow \begin{cases} \frac{x}{2} = \frac{y}{3} \\ \frac{y}{3} = z \\ \frac{x}{2} = z \end{cases} \rightarrow \begin{cases} 3x - 2y = 0 \\ y - 3z = 0 \\ x - 2z = 0 \end{cases}$

$$\left(\begin{array}{ccc|c} 3 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right) \overset{R_1 - 3R_3}{\sim} \left(\begin{array}{ccc|c} 0 & -2 & 6 & 0 \\ 0 & 1 & -3 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right) \overset{R_1 + 2R_2}{\sim} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right) \sim$$

$$\sim \begin{array}{c} \overbrace{\qquad\quad}^{\text{Pivot}} \\ \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \\ x \quad y \quad z = t \end{array} \rightarrow \begin{cases} x = 2t \\ y = 3t \\ z = t \end{cases} \rightarrow t \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$T(x,y) = F(0,1) + \left\langle \vec{\nabla} F(0,1), \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

$$= -1 + \alpha(x - 0) - 3(y - 1)$$

$$= \alpha x - 3y + 2$$

$$\downarrow$$

$$\{ \alpha x - 3y - z = 2 \longrightarrow \left(\begin{array}{ccc|c} \alpha & -3 & -1 & -2 \end{array}\right)$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{Pivot} & t & s \end{array}$$

$$\begin{cases} x = \frac{3}{\alpha} t + \frac{s}{\alpha} - \frac{2}{\alpha} \\ y = t \\ z = s \end{cases}$$

$$\downarrow$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{2}{\alpha} \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{3}{\alpha} \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} \frac{3}{\alpha} \\ 0 \\ 1 \end{pmatrix}$$

Il vettore della retta deve essere perpendicolare alla base del piano tangente della funzione in (0,1)

$$\begin{cases} \left\langle \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3/\alpha \\ 1 \\ 0 \end{pmatrix} \right\rangle \\ \left\langle \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1/\alpha \\ 0 \\ 1 \end{pmatrix} \right\rangle \end{cases} \rightarrow \begin{cases} \frac{6}{\alpha} + 3 = 0 \\ \frac{2}{\alpha} + 1 = 0 \end{cases} \rightarrow \left\{ \frac{2}{\alpha} + 1 = 0 \right. \qquad \alpha = -2$$

Burato 1-03-2024 es 3

Data la funzione

$$F(x,y)=\sqrt{2e^{x-2y}-1}+\frac{x}{y}$$

a) Determinare analiticamente il suo dominio naturale D e poi rappresentarlo (con cura!) nel piano cartesiano. Stabilire se D è un insieme limitato/illimitato, aperto/chiuso, connesso/sconnesso (motivare le risposte!)

b) Scrivere l'equazione del piano tangente in P(2,1,f(2,1)) al grafico di f

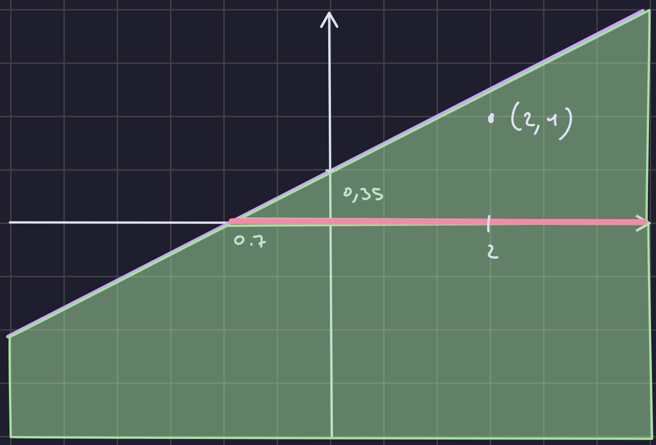
a) $\{(x,y)\in\mathbb{R}^2 \mid 2e^{x-2y}-1\geq 0 \vee y\neq 0\}$

$$e^{x-2y}\geq \frac{1}{2}$$

$$x-2y\geq \ln\left(\frac{1}{2}\right)$$

$$y\leq \frac{x}{2}-\frac{1}{2}\ln\left(\frac{1}{2}\right) \qquad \ln\left(\frac{1}{2}\right)= -0.693147181 \simeq 0{,}7$$

$$\{(x,y)\in\mathbb{R}^2 \mid y\leq \frac{x}{2}+0{,}35 \vee y\neq 0\}$$




L'insieme è illimitato, sconnesso (perchè y ≠ 0) e nè aperto nè chiuso

b)

$$P(2,1,F(2,1)) \qquad F(x,y)=\sqrt{2e^{x-2y}-1}+\frac{x}{y}$$

$$F(2,1)=\sqrt{2e^0-1}+2=3 \rightarrow P(2,1,3)$$

$$\nabla F(x,y)=\left(\frac{1}{2}\frac{2e^{x-2y}}{\sqrt{2e^{x-2y}-1}}+\frac{1}{y}, \frac{1}{2}\frac{-4e^{x-2y}}{\sqrt{2e^{x-2y}-1}}+\frac{x}{y^2}\right)$$

$$\nabla F(2,1)=(2,-4)$$

$$T(x,y)=3+2(x-2)-4(y-1) \rightarrow z=3+2x-\cancel{4}-4y+\cancel{4}$$

$$2x-4y-z=-3$$

$$x-2y-\frac{z}{2}=-\frac{3}{2}$$

$$\downarrow y=t,\ z=s$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2t + \frac{s}{2} - \frac{3}{2} \\ t \\ s \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{2} \\ 0 \\ 0 \end{pmatrix} t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

check

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -\frac{3}{2} \\ 0 \\ 0 \end{pmatrix} \rightarrow \left(\begin{array}{cc|c} 2 & 1 & \frac{1}{2} \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{array}\right)$$

Burato 1-03-2024 es 4

a) Data la funzione

$$f(x,y) = \begin{cases} \dfrac{x^{\alpha-2} e^{x+y}}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

stabilire per quali valori reali di alfa f è continua in R^2

b) Calcolare la lunghezza dell'arco di curva parametrizzato da

$$\gamma(t) = (2t+3, t^{\frac{3}{2}}+1) \quad t \in [0,1]$$

a) $$\lim_{\rho \to 0} \frac{\rho^{\alpha-2} \cos^{\alpha-2}\theta \, e^{\rho(\cos\theta + \sin\theta)}}{\rho^2 (\underbrace{\cos^2\theta + \sin^2\theta})} =$$

$$= \lim_{\rho \to 0} \frac{\rho^{\alpha-2} \cos^{\alpha-2}\theta \, e^{\rho(\cos\theta+\sin\theta)}}{\rho^2}$$

$$= \lim_{\rho \to 0} \rho^{\alpha-4} \cos^{\alpha-2}\theta \, \underbrace{e^{\rho(\cos\theta+\sin\theta)}}_{1}$$

$$f \to 0 \leftarrow |f| \to 0$$

Per $\alpha \leq 4$ il limite non esiste

Per $\alpha > 4$

$$\lim_{\rho \to 0} \rho^{\alpha-4} |\cos^{\alpha-2}\theta| \leq \lim_{\rho \to 0} \rho^{\alpha-4} = 0$$

Data la funzione

$$f(x,y)=\sqrt{x}\,\ln(x^2-y^2-1)-\pi$$

a) Determinare analiticamente il suo dominio naturale D e poi rappresentarlo (con cura!) nel piano cartesiano.
Stabilire se D è un insieme limitato/illimitato, aperto/chiuso, connesso/sconnesso (motivare le risposte!)

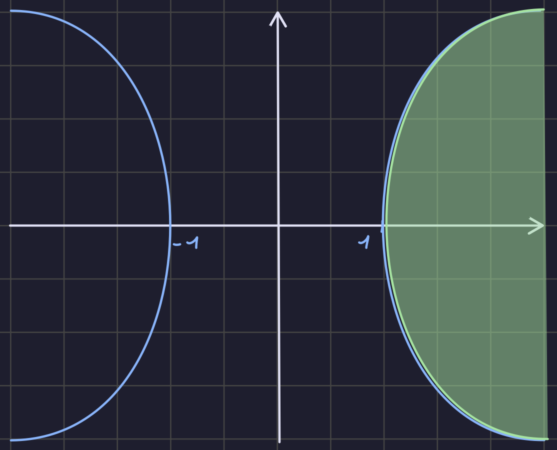
$$D=\{(x,y)\in\mathbb{R}^2 \mid x\geq 0 \ \wedge\ x^2-y^2-1>0\}$$

$x^2-y^2>1$

$x^2-y^2=1$

$[0^2-0^2>1$ ×

$\begin{cases} y=0 \\ x^2=1 \to x=\pm 1 \end{cases}$




Trovare una parametrizzazione per l'ellisse di equazione

$$g(x,y)=4x^2+y^2-8x-4y+4=0$$

Se (x_0,y_0) è un punto dell'ellisse, quale proprietà ha il vettore $\nabla g(x_0,y_0)$?

$$\text{Ellisse: } \left(\frac{x-\alpha}{a}\right)^2+\left(\frac{y-\beta}{b}\right)=1$$

$$4x^2+y^2-8x-4y+4=0$$

$$4(x-1)^2+y^2-4y=0$$

$$4(x-1)^2+(y^2-2)^2=4$$

$$(2(x-1))^2+(y-2)^2=4 \qquad \text{(cerchio)}$$

$$\begin{cases} 2(x-1)=t \\ y-2=s \end{cases}$$

$$t^2+s^2=4$$

$(2\cos\theta,\ 2\sin\theta)$ Param del cerchio

$$\begin{cases} 2(x-1)=2\cos\theta \\ y-2=2\sin\theta \end{cases} \qquad \begin{cases} x=\cos\theta+1 \\ y=2\sin\theta+2 \end{cases}$$

$$\left(\cos\theta + 4,\ 2\sin\theta + 2\right)$$

✍ **Esercizio 3.6.1.** *Dire se la seguente funzione è differenziabile*

$$f(x,y) = \begin{cases} x + \frac{1}{2}x^2 y & y \geq 0 \\ \frac{e^{xy}-1}{y} & y < 0 \end{cases}$$

Derivata rispetto a $x$

$$\begin{cases} \frac{\partial}{\partial x} F(x,y) = 1 + xy \\ y > 0 \end{cases} \qquad \begin{cases} \frac{\partial}{\partial x} F(x,y) = e^{xy} \\ y < 0 \end{cases}$$

$$\lim_{y\to 0^+} 1 + xy = 1 \qquad \lim_{y \to 0^-} e^{xy} = 1$$

($x$ qualsiasi)

Derivata rispetto a $y$

$$\begin{cases} \frac{\partial}{\partial y} F(x,y) = \frac{1}{2}x^2 \\ y > 0 \end{cases} \qquad \begin{cases} \frac{\partial}{\partial y} F(x,y) = \frac{(xy-1)e^{xy}+1}{y^2} \\ y < 0 \end{cases}$$

$$\lim_{y\to 0^+} \frac{1}{2}x^2 = \frac{1}{2}x^2 \qquad \lim_{y\to 0^-} \frac{(xy-1)e^{xy}+1}{y^2} = \frac{x^2}{2} \quad *$$

Da sopra e da sotto tutte e quattro le funzioni sembrano essere d'accordo (a coppie) sul valore da assumere.
Verifichiamo se si guasta tutto quando y=0

$$\frac{\partial}{\partial x} F(x,0) = 1 + x\cdot 0 = 1$$

$$\frac{\partial}{\partial y} F(x,0) = \lim_{k\to 0} \frac{F(x,k) - F(x,0)}{k} \longrightarrow \begin{cases} \lim_{k\to 0^+} \frac{F(x,k)-F(x,0)}{k} = \lim_{k\to 0^+} \frac{x + \frac{x^2}{2}k - x}{k} = \frac{x^2}{2} \\ \lim_{k\to 0^-} \frac{F(x,k)-F(x,0)}{k} = \lim_{k\to 0^-} \frac{\frac{e^{xk}-1}{k} - x}{k} = \frac{x^2}{2} \quad \times \end{cases}$$

Quindi

$$\frac{\partial}{\partial x} = \begin{cases} 1 + xy & y > 0 \\ 1 & y = 0 \\ e^{xy} & y < 0 \end{cases} \qquad \frac{\partial}{\partial y} = \begin{cases} \frac{1}{2}x^2 & y > 0 \\ \frac{x^2}{2} & y = 0 \\ \frac{(xy-1)e^{xy}+1}{y^2} & y > 0 \end{cases}$$

$$* \lim_{y\to 0^-} \frac{(xy-1)e^{xy}+1}{y^2} = \lim_{y\to 0^-} \frac{(xy-1)\left(1+xy+\frac{x^2y^2}{2}+o(y^2)\right)+1}{y^2} \quad \text{(Taylor)}$$

$$= \lim_{y\to 0^-} \frac{\cancel{xy}+x^2y^2+\frac{x^3y^3}{2}+xy\,o(y^2)\cancel{-1}-\cancel{xy}-\frac{x^2y^2}{2}-o(y^2)\cancel{+1}}{y^2}$$

$$= \lim_{y\to 0^-} \frac{x^2y^2+\frac{x^3y^3}{2}-\frac{x^2y^2}{2}\overbrace{o(y^2)(xy-1)}^{=0}}{y^2}$$

$$= \lim_{y\to 0^-} \frac{-\frac{x^2y^2}{2}}{y^2} = -\frac{x^2}{2}$$

$$* \lim_{k\to 0^-} \frac{\frac{e^{xk}-1}{k}-x}{x} = \lim_{k\to 0^-} \frac{e^{xk}-kx-1}{k^2}$$

$$= \lim_{k\to 0^-} \frac{\cancel{1+kx}+\frac{k^2x^2}{2}+o(x^2)\cancel{-kx-1}}{k^2} = \frac{x^2}{2}$$

Si trovi una parametrizzazione dell'arco di ellisse di equazione

$$\frac{(x-3)^2}{4} + y^2 = 1$$

Che congiunge (nell'ordine) i punti A(1,0) e B(3,-1) e si scriva poi l'equazione della tangente alla curva in P(2, -sqrt(3)/2)

$$\left(\frac{x-3}{2}\right)^2 + y^2 = 1 \to \begin{cases} \frac{x-3}{2} = t \\ y = s \end{cases} \to t^2 + s^2 = 1$$

$$\downarrow$$

$$\begin{cases} t = \rho\cos\theta \\ s = \rho\sin\theta \end{cases}$$

$$\downarrow$$

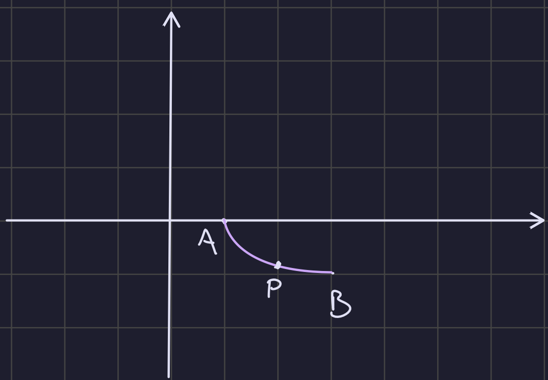
$$(\rho\cos\theta, \rho\sin\theta) \to (\sqrt{1}\cos\theta, \sqrt{1}\sin\theta)$$

$$\downarrow$$

$$(\cos\theta, \sin\theta)$$

$$\begin{cases} t = \cos\theta \\ s = \sin\theta \end{cases} \to \begin{cases} \frac{x-3}{2} = \cos\theta \\ y = \sin\theta \end{cases} \to \begin{cases} x = 2\cos\theta + 3 \\ y = \sin\theta \end{cases}$$

$$\gamma(\theta) = (2\cos\theta + 3, \sin\theta)$$




Bisogna trovare theta tale che contemporaneamente la curva sia nel punto A

$$\begin{cases} 2\cos\theta + 3 = 1 \\ \sin\theta = 0 \end{cases} \to \begin{cases} \cos\theta = -1 \\ \sin\theta = 0 \end{cases} \to \theta = \pi$$

Bisogna trovare theta tale che contemporaneamente la curva sia nel punto B

$$\begin{cases} 2\cos\theta + 3 = 3 \\ \sin\theta = -1 \end{cases} \to \begin{cases} \cos\theta = 0 \\ \sin\theta = -1 \end{cases} \to \theta = \frac{3}{2}\pi$$

La parametrizzazione finale è:

$$\gamma(\theta) = (2\cos\theta + 3,\ \sin\theta) \qquad \theta \in \left[\pi, \tfrac{3}{2}\pi\right]$$

$$P\left(2, -\frac{\sqrt{3}}{2}\right)$$

Bisogna trovare theta tale che contemporaneamente la curva sia nel punto P

$$\begin{cases} 2\cos\theta + 3 = 2 \\ \sin\theta = -\frac{\sqrt{3}}{2} \end{cases} \rightarrow \begin{cases} \cos\theta = -\frac{1}{2} \\ \sin\theta = -\frac{\sqrt{3}}{2} \end{cases} \rightarrow \theta = \frac{4}{3}\pi$$

$$\gamma'(\theta) = (-2\sin\theta,\ \cos\theta)$$

$$\gamma'\left(\tfrac{4}{3}\pi\right) = \left(-2\sin\left(\tfrac{4}{3}\pi\right),\ \cos\left(\tfrac{4}{3}\pi\right)\right) = ($$

$$r_T = \gamma\left(\tfrac{4}{3}\pi\right) + t\,\gamma'\left(\tfrac{4}{3}\pi\right)$$

$$= \begin{pmatrix} 2 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} + t \begin{pmatrix} \sqrt{3} \\ -\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} + t \begin{pmatrix} 2\sqrt{3} \\ -1 \end{pmatrix}$$

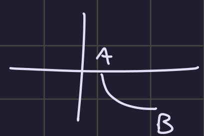
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} + t \begin{pmatrix} 2\sqrt{3} \\ -1 \end{pmatrix} \rightarrow \begin{cases} x = 2 + 2\sqrt{3}\,t \\ y = -\frac{\sqrt{3}}{2} - t \end{cases} \qquad \begin{cases} t = \frac{x-2}{2\sqrt{3}} \\ y = -\frac{\sqrt{3}}{2} + \frac{2-x}{2\sqrt{3}} \end{cases}$$

## Metodo 2

$$\left(\frac{x-3}{2}\right)^2 + y^2 = 1$$

$$y^2 = 1 - \left(\frac{x-3}{2}\right)^2$$

$$y = \pm\sqrt{1 - \left(\frac{x-3}{2}\right)^2} \rightarrow$$




Considero il $-$

$\downarrow$

$$y = -\sqrt{1 - \left(\frac{x-3}{2}\right)^2}$$

$$\phi(t) = \left(t, -\sqrt{1-\left(\frac{t-3}{2}\right)^2}\right) \quad t \in [1,3]$$

$$P\left(2, -\frac{\sqrt{3}}{2}\right)$$

$$\begin{cases} t = 2 \\ -\sqrt{1-\left(\frac{t-3}{2}\right)^2} = -\frac{\sqrt{3}}{2} \end{cases} \longrightarrow \quad t = 2$$

$$\phi'(t) = \left(1, -\frac{1}{2}\left(1-\left(\frac{t-3}{2}\right)^2\right)^{-\frac{1}{2}} \cdot \left(-\cancel{2}\left(\frac{t-3}{2}\right) \cdot \frac{1}{\cancel{2}}\right)\right) = \left(1, \frac{1}{2} \frac{\frac{t-3}{2}}{\sqrt{1-\left(\frac{t-3}{2}\right)^2}}\right)$$

$$r_T = \phi(2) + s\,\phi'(2) = \begin{pmatrix} 2 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} + s \begin{pmatrix} 1 \\ \frac{-\frac{1}{2} \cdot \frac{1}{2}}{\sqrt{1-\frac{1}{4}}} \end{pmatrix} = \begin{pmatrix} 2 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} - s \begin{pmatrix} 1 \\ -\frac{1}{2\sqrt{3}} \end{pmatrix}$$

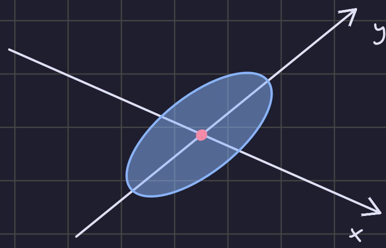
$$= \begin{pmatrix} 2 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} + s \begin{pmatrix} 2\sqrt{3} \\ -1 \end{pmatrix}$$

# Ottimizzazione vincolata

Ottimizzazione $\xrightarrow{\text{libera}}$ Gradiente + Punti critici $\longrightarrow$ Hessiana $\xrightarrow{\text{segno}}$
- $+$ Min
- $-$ Max
- $+/-$ Sella
- $0$ Inconcludente $\longrightarrow$ Studio dell'incremento $f(x,y) - f(0,0) > 0$

Ottimizzazione $\xrightarrow{\text{vincolata}}$ Vincolo esplicitabile $\longrightarrow$ Ottimizzazione libera su sottoinsieme

2) Trovare il valore massimo e il valore minimo di $f(x,y) = x^3 + y^2$ su $E = \{(x,y) \in \mathbb{R}^2 : \underbrace{4x^2 + y^2 \leq 1}_{\text{Esplicitabile}}\}$

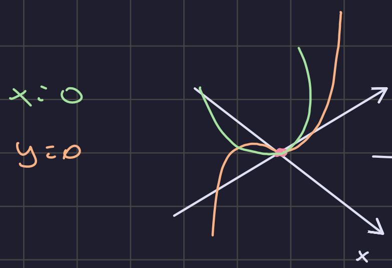



Facciamo ottimizzazione libera in tutto R e poi verifico se i punti sono nell'insieme E

$$\bar{\nabla} f(x,y) = \begin{pmatrix} 3x^2 \\ 2y \end{pmatrix} \qquad \text{Punto critico} \quad \bar{\nabla} f(x,y) = 0 \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$H_f(0,0) = \begin{pmatrix} 6x & 0 \\ 0 & 2 \end{pmatrix} \qquad H_f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

Inconcludente




$\longrightarrow$ Sella

## Metodo 1 Lagrange

$$\mathcal{L}(x,y,\lambda) = x^3 + y^2 - \lambda(4x^2 + y^2 - 1)$$

$$\bar{\nabla} \mathcal{L}(x,y,\lambda) = \begin{pmatrix} 3x^2 - 8\lambda x \\ 2y - 2\lambda y \\ -(4x^2 + y^2 - 1) \end{pmatrix}$$

$$\bar{\nabla} \mathcal{L}(x,y,\lambda) = 0$$

$\downarrow$

$$\begin{pmatrix}3x^2-8\lambda x\\ 2y-2\lambda y\\ -(4x^2+y^2-1)\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\to\begin{cases}3x^2-8\lambda x=0\\ 2y-2\lambda y=0\\ -(4x^2+y^2-1)=0\end{cases}$$

$$\begin{cases}3x^2-8\lambda x=0\\ 2y(1-\lambda)=0 \to y=0 \vee \lambda=1\\ -(4x^2+y^2-1)=0\end{cases}$$

$\lambda=1$

$$\begin{cases}x(3x-8)=0 \nearrow x=0 \vee x=\frac{8}{3} \to \text{Non accet.}\\ \lambda=1\\ -(4x^2+y^2-1)=0\end{cases}$$

$x=0 \to y=\pm1$

A $(0,1,1)$
B $(0,-1,1)$

$y=0$

$$\begin{cases}\frac{3}{4}-4\lambda=0\\ y=0\\ 4x^2=1 \to x=\pm\frac{1}{2}\end{cases}$$

$x=\frac{1}{2}\to\lambda=\frac{3}{16}$ → C $\left(\frac{1}{2},0,\frac{3}{16}\right)$

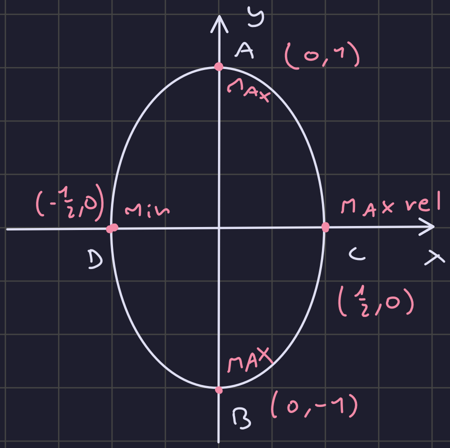
$x=-\frac{1}{2}\to\lambda=-\frac{3}{16}$ → D $\left(-\frac{1}{2},0,-\frac{3}{16}\right)$

$F(A)=1$

$F(B)=1$

$F(C)=\frac{1}{8}$

$F(D)=-\frac{1}{8}$




## Metodo 2 Esplicitazione del vincolo

Il vincolo è di disuguaglianza, ma visto che abbiamo già fatto ottimizzazione libera e abbiamo visto che all'interno c'è una sella possiamo considerare solo il bordo e quindi diventa un'uguaglianza

$4x^2+y^2=1$

$(2x)^2+y^2=1 \to$ Parametrizzazione $\begin{cases}2x=\cos\theta\\ y=\sin\theta\end{cases}\to\begin{cases}x=\frac{\cos\theta}{2}\\ y=\sin\theta\end{cases}$

$$F(x,y)=x^3+y^2 \to F(\theta)=\frac{\cos^3\theta}{8}+\sin^2\theta$$

La funzione è diventata ad una sola variabile e si può ottimizzare come in analisi 1

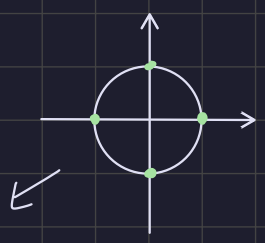
$$F'(\theta)=0 \to F'(\theta)=\frac{3}{8}\cos^2\theta(-\sin\theta)+2\sin\theta\cos\theta$$

$\downarrow$

$$\sin\theta\cos\theta\left(-\frac{3}{8}\cos\theta+2\right)=0$$

$$\Updownarrow$$

$$\sin\theta\cos\theta=0 \longrightarrow$$



$$\theta=k\frac{\pi}{2} \qquad k=0,1,2,3$$

$$\begin{cases} x_k=\dfrac{\cos\left(k\frac{\pi}{2}\right)}{2} \\ y_k=\sin\left(k\frac{\pi}{2}\right)\end{cases} \longrightarrow$$

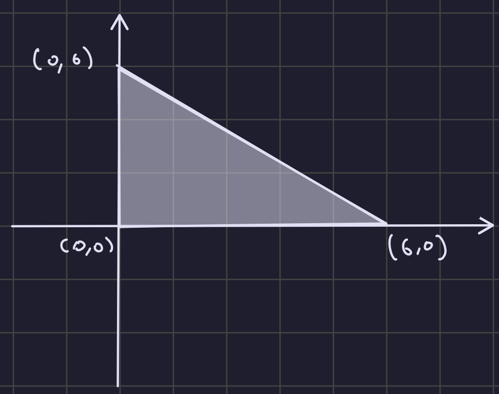
$\left(\frac{1}{2},0\right)=C \qquad k=0$

$(0,1)=A \qquad k=1$

$\left(-\frac{1}{2},0\right)=D \qquad k=2$

$(0,-1)=B \qquad k=3$

3) Trovare il valore massimo e il valore minimo di $f(x,y)=4xy^2-x^2y^2-xy^3$ sulla regione triangolare chiusa nel piano xy di vertici (0,0), (0,6) e (6,0)




$$f(x,y)=xy^2(4-x-y)$$

$$\bar{\nabla}f(x,y)=\begin{pmatrix}4y^2-2xy^2-y^3\\ 8xy-2x^2y-3xy^2\end{pmatrix}$$

$$=\begin{pmatrix}y^2(4-2x-y)\\ xy(8-2x-3y)\end{pmatrix}$$

$$\begin{cases} y^2(4-2x-y)=0\\ xy(8-2x-3y)=0\end{cases}$$

$$\downarrow$$

se $y=0$ : $x=$ qualunque

Y lo abbiamo già esaminato, quindi lo togliamo

$$\begin{cases} 4-2x-y=0 \\ x(8-2x-3y)=0 \end{cases}$$

$$\downarrow$$

se $x=0$ : $y=4$

X lo abbiamo già esaminato, quindi lo togliamo

$$\begin{cases} 4-2x-y=0 \\ 8-2x-3y=0 \end{cases} \to 4-2y=0 \to x=1,\ y=2$$

I punti critici sono:

$(x,0)$ $x \in [0,6]$

$(0,4)$

$(1,2)$

$$H_f(x,y)=\begin{pmatrix} 2y^2 & 8y-4xy-3y^2 \\ 8y-4xy-3y^2 & 8x-2x^2-6y^2 \end{pmatrix}$$

$$H_f(0,4)=\begin{pmatrix} 32 & -16 \\ -16 & 0 \end{pmatrix}$$ $\longrightarrow$ Autovalori discordi (det = prodotto di autovalori)

$$H_f(1,2)=\begin{pmatrix} 8 & -4 \\ -4 & -6 \end{pmatrix}$$ $\longrightarrow$ Autovalori discordi (det = prodotto di autovalori)

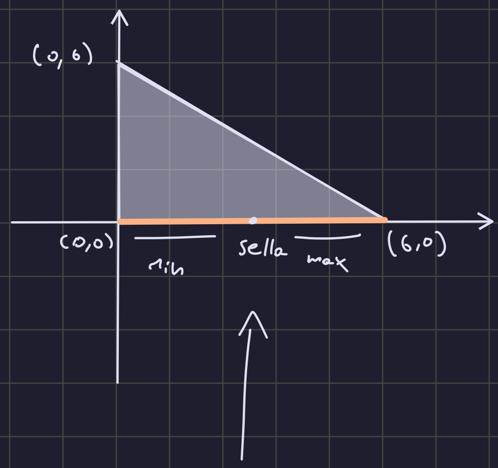
$$H_f(x,0)=\begin{pmatrix} 0 & 0 \\ 0 & 8x-2x^2 \end{pmatrix}$$ Inconcludente

Analizziamo l'incremento

$$f(x,y)-f(x,0)=xy^2(4-x-y)$$

Studiamo il segno

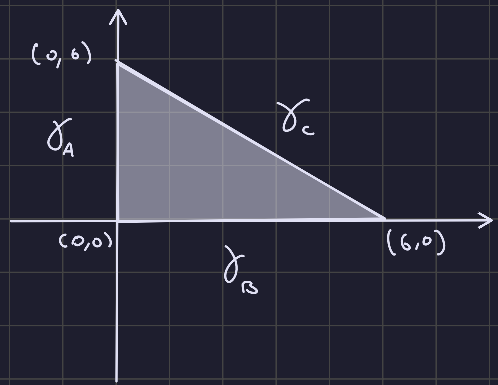
$(x \in [0,6] \quad y \geq 0)$




$$\underbrace{x}_{SP}\,\underbrace{y^2}_{SP}\,\underbrace{(4-x-y)}\geq 0 \longrightarrow y \leq 4-x$$

Da 0 a 4 l'incremento darà positivo
Da 4 a 6 sarà negativo

Studiamo i punti critici lungo il bordo




$$\gamma_A(t) = (0,t) \quad t \in [0,6]$$

$$\gamma_B(t) = (t,0) \quad t \in [0,6]$$

$$\gamma_C(t) = \begin{pmatrix} 0 \\ 6 \end{pmatrix} t + (1-t)\begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad t \in [0,1]$$

$$= ((1-t)6, 6t) \quad t \in [0,1]$$

$$F(\gamma_A(t)) = F(0,t) = 0$$

$$F(\gamma_B(t)) = F(t,0) = 0$$

Tutti massimi o tutti minimi

$$F(\gamma_C(t)) = F(6(1-t), 6t) = 6(1-t)36t^2(4-6(1-t)-6t)$$

$$= 432(1-t)t^2(2-3(1-t)-3t)$$

$$= 432(1-t)t^2(2-3+\cancel{3t}-\cancel{3t})$$

$$= -432(1-t)t^2$$

$$F'(\gamma_c(t)) = -864(1-t)t + 432t^2$$

$$= -864t + 864t^2 + 432t^2$$

$$= 432(-2t + 2t^2 + t^2)$$

$$= 432(-2t + 3t^2)$$

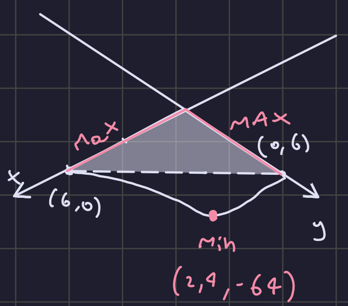
$$= 432\,t(3t-2)$$

$$t = 0 \quad \vee \quad t = \frac{2}{3}$$

↑
Già trovato

$$\gamma_c\left(\frac{2}{3}\right) = (2, 4)$$

$$F(2,4) = 32(4-2-4) = -64$$




10) Sia $F(x,y) = e^{\frac{x^2}{y^2+1}}$

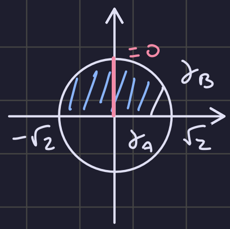
a) Calcolare i punti critici e gli estremi locali di f, specificando se sono globali

b) Trovare massimo e minimo assoluti di f su $E = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 2,\ y \geq 0\}$

a)

$$\nabla F(x,y) = \begin{pmatrix} e^{\frac{x^2}{y^2+1}} \dfrac{2x}{y^2+1} \\ e^{\frac{x^2}{y^2+1}} \dfrac{-2x^2 y}{(y^2+1)^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} e^{\frac{x^2}{y^2+1}} \dfrac{2x}{y^2+1} = 0 \\ e^{\frac{x^2}{y^2+1}} \dfrac{-2x^2y}{(y^2+1)^2} = 0 \end{cases} \longleftrightarrow \begin{cases} 2x = 0 \\ -2x^2 y = 0 \end{cases} \to x = 0 \quad y \in \mathbb{R}$$




$x^2 + y^2 \leq 2$

$y \leq +\sqrt{2-x^2}$

Consideriamo solo
la parte positiva

$\gamma_A(t) = (t, 0) \quad t \in [-\sqrt{2}, \sqrt{2}]$

$\gamma_B(t) = (t, \sqrt{2-t^2}) \quad t \in [-\sqrt{2}, \sqrt{2}]$

$F(\gamma_A(t)) = e^{t^2}$

$F(\gamma_B(t)) = e^{\frac{t^2}{2-t^2+1}} = e^{\frac{t^2}{3-t^2}}$

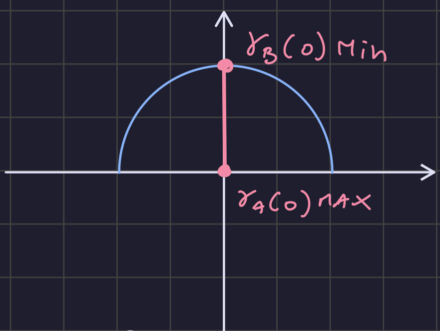
$F'(\gamma_A(t)) = 2t e^{t^2} = 0 \to t = 0 \to \gamma_A(0)$

$$F'(\gamma_B(t)) = \frac{2t(3-t^2) - t^2(-2t)}{(3-t)^2} e^{\frac{t^2}{3-t^2}} = 0$$

$$\updownarrow$$

$$2t(3-t^2) - t^2(-2t) = 0$$

$$6t - \cancel{2t^3} + \cancel{2t^3} = 0 \to t = 0 \to \gamma_B(0)$$




$F(\gamma_A(0)) = f(0,0) = e^{\frac{0^2}{0^2+1}} = 1$

$F(\gamma_B(0)) = f(0, \sqrt{2}) = e^{\frac{0}{2+1}} = 1$

Non abbiamo trovato i punti di massimo perchè il semicerchio forma un angolo retto con l'asse x e Lagrange non si applica più perchè non è un punto regolare, quindi troviamo una nuova parametrizzazione per il semicerchio

$$\gamma_c(\theta) = \left(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta\right)$$

$$F(\gamma_c(\theta)) = e^{\frac{2\cos^2\theta}{2\sin^2\theta+1}}$$

$$F'(\gamma_c(\theta)) = e^{\frac{2\cos^2\theta}{2\sin^2\theta+1}} \cdot \frac{-4\cos\theta\sin\theta(2\sin\theta+1) - 2\cos^2\theta\, 4\sin\theta\cos\theta}{(2\sin^2\theta+1)^2} = 0$$

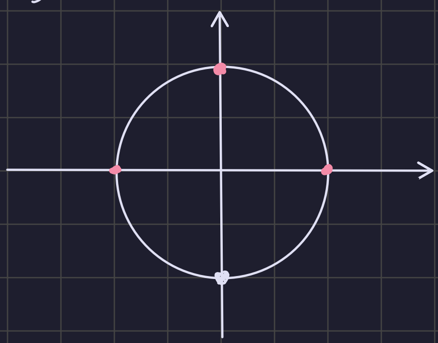
$$\Updownarrow$$

$$-4\cos\theta\sin\theta(2\sin\theta+1) - 2\cos^2\theta\, 4\sin\theta\cos\theta = 0$$

$$2\sin\theta\cos\theta\left(-2(2\sin^2\theta+1) - 4\cos^2\theta\right) = 0$$

$$2\sin\theta\cos\theta\left(-4\sin^2\theta - 4\cos^2\theta - 2\right) = 0$$

$$-12\sin\theta\cos\theta = 0$$



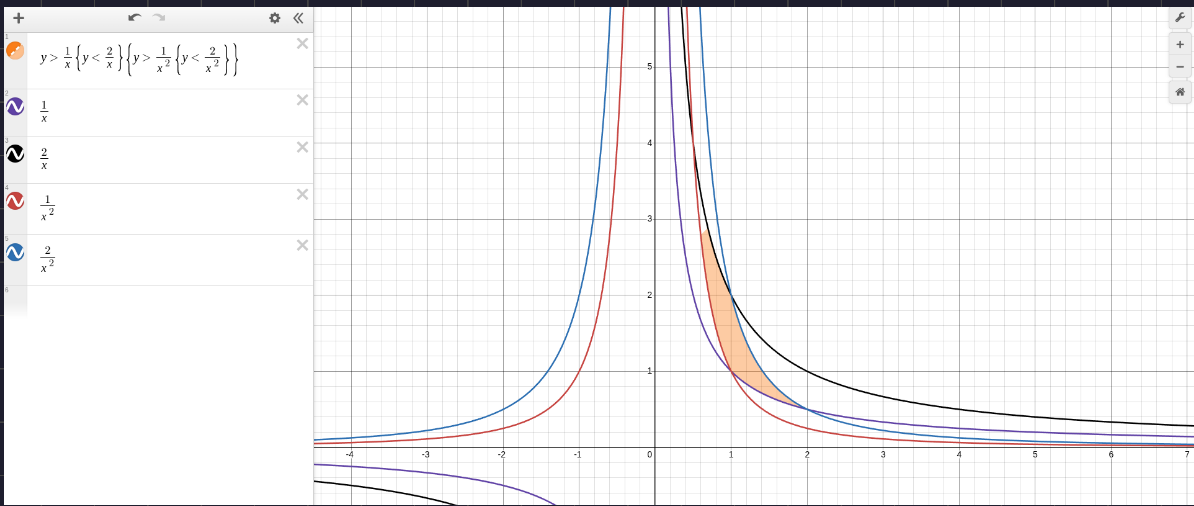
Ora si possono trovare i punti di massimo

# Integrali doppi e tripli

$$\iint_E xy\,dy\,dx \qquad E = \left\{ (x,y) \in \mathbb{R}^2 \mid 1 < xy < 2,\ 1 < x^2 y < 2 \right\}$$

$$\downarrow$$

$$E = \begin{cases} y > \frac{1}{x} \\ y < \frac{2}{x} \\ y > \frac{1}{x^2} \\ y < \frac{2}{x^2} \end{cases}$$




$$\begin{cases} xy = u \\ x^2 y = v \end{cases} \rightarrow \begin{cases} x = \frac{v}{u} \\ y = \frac{u^2}{v} \end{cases}$$

$$J_T(u,v) = \begin{pmatrix} -\frac{v}{u^2} & \frac{1}{u} \\ \frac{2u}{v} & -\frac{u^2}{v^2} \end{pmatrix}$$

$$|\det J_T| = \left| \frac{\cancel{v}}{\cancel{u^2}} \cdot \frac{\cancel{u^2}}{v^2} - \frac{2\cancel{u}}{v} \cdot \frac{1}{\cancel{u}} \right| = \left| \frac{1}{v} - \frac{2}{v} \right| = \left| -\frac{1}{v} \right| = \frac{1}{v}$$

$$\iint_E xy\,dy\,dx = \int_1^2\int_1^2 \frac{u}{v}\,du\,dv = \int_1^2 \frac{1}{v}\,dv \cdot \int_1^2 u\,du =$$

$$= \Big[\ln|v|\Big]_1^2 \cdot \left[\frac{u^2}{2}\right]_1^2 = \left(2 - \frac{1}{2}\right)(\ln 2 - 0) = \frac{3}{2}\ln 2$$

Calcolare il volume della regione:

$$V = \left\{ (x,y) \in \mathbb{R}^3 \mid x^2+y^2 \le 1,\ 0 < z \le 3\sqrt{x^2+y^2} \right\}$$

$$\iiint_V 1 \, dx\,dy\,dz = \iint_{\underset{\text{Cerchio}}{C}} \int_0^{3\sqrt{x^2+y^2}} 1 \, dz \, dx\,dy \qquad C = \left\{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1 \right\}$$

Coordinate polari

$$\begin{cases} x = \rho\cos\theta & \theta \in [0, 2\pi] \\ y = \rho\sin\theta & \rho \in [0,1] \end{cases}$$

$$= \iint_C 3\sqrt{x^2+y^2} \, dx\,dy$$

$$= \int_0^{2\pi} \int_0^1 3\rho^2 \, d\rho\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 3\rho^2 \, d\rho$$

$$= 2\pi \cdot 1$$

Coordinate cilindriche

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = t \end{cases} \rightarrow J_T(\rho,\theta,t) = \begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|\det J_T| = \rho$$

$$\downarrow$$

$$\int_0^{2\pi} \int_0^1 \int_0^{3\rho} \rho \, dt\, d\rho\, d\theta = 2\pi \left[\rho^3\right]_0^3 = 2\pi$$

✍ **Esercizio 6.1.4.** *Si calcoli l'integrale doppio*

$$\int\int_D (x^2+1)\,dx\,dy,$$

*ove $D$ è la parte dell'ellisse $\{(x,y)\in\mathbb{R}^2 : x^2+4y^2\le 1\}$ contenuta nel primo quadrante.*

$$\iint_D x^2+1 \, dx\,dy \qquad D=\{(x,y)\in\mathbb{R}^2 \mid x^2+(2y)^2\le 1\}$$

$$\begin{cases} x=\rho\cos\theta \\ 2y=\rho\sin\theta \end{cases} \qquad \begin{cases} x=\rho\cos\theta \\ y=\frac{\rho}{2}\sin\theta \end{cases} \qquad \begin{matrix} \theta\in[0,2\pi] \\ \rho\in[0,1] \end{matrix}$$

$$J_T(\rho,\theta)=\begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \frac{\sin\theta}{2} & \frac{\rho}{2}\cos\theta \end{pmatrix}$$

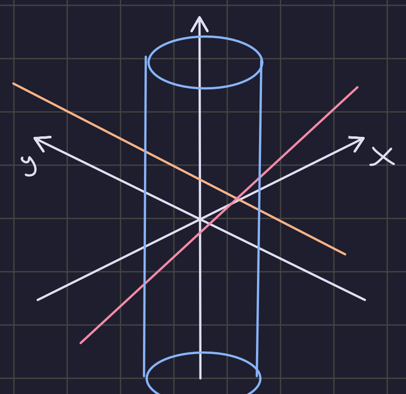
$$|\det J_T| = \cos\theta\cdot\frac{\rho}{2}\cos\theta+\rho\sin\theta\cdot\frac{\sin\theta}{2}=\frac{\rho}{2}\cos^2\theta+\frac{\rho}{2}\sin^2\theta=\frac{\rho}{2}$$

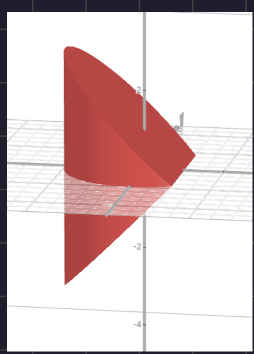
$$\int_0^{2\pi}\int_0^1 (\rho^2\cos^2\theta)\,\frac{\rho}{2}\,d\rho\,d\theta =$$

$$=\int_0^1 \frac{1}{2}\rho^3\,d\rho\cdot\int_0^{2\pi}\cos^2\theta\,d\theta+\int_0^1\frac{\rho}{2}\,d\rho\cdot\int_0^{2\pi}d\theta$$

✍ **Esercizio 6.3.1.** *Calcolate il volume della regione interna al cilindro di equazione $x^2+y^2\le 4$ e compresa tra i piani $z=x-1$ e $z=1-x$.*

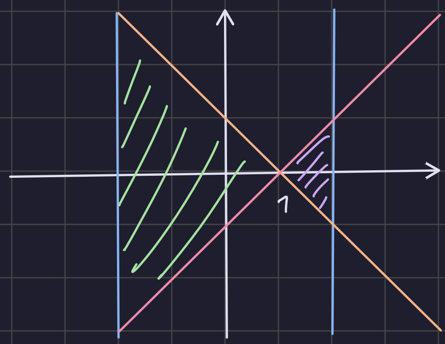
$$\begin{cases} x^2+y^2\le 4 \\ x-1 < z < 1-x \end{cases}$$



$$A = \{ (x,y) \mid x^2 + y^2 \leq 4, \ x \leq 1 \}$$

$$B = \{ (x,y) \mid x^2 + y^2 \leq 4, \ x \geq 1 \}$$




$$\iint_A \int_{x-1}^{1-x} dz\,dx\,dy + \iint_B \int_{1-x}^{x-1} dz\,dx\,dy$$

$$2\iint_A (1-x)\,dx\,dy + 2\iint_B (x-1)\,dx\,dy \qquad (\text{A caso})$$

✍ **Esercizio 6.3.3.** *Assegnati il paraboloide di equazione* $z = x^2 + y^2$ *ed il piano di equazione* $z = 4x - 12y$ *si calcoli il volume racchiuso dalle due superifici.*

Insieme z semplice $\rightarrow z \in [x^2 + y^2, \ 4x - 12y]$

$$\iint_A \int_{x^2+y^2}^{4x-12y} 1\,dz\,dx\,dy = \iint_A 4x - 12y - x^2 - y^2\,dx\,dy$$

$$-\left((x-2)^2 + (y+6)^2\right) = -\begin{pmatrix} x^2 - 4x + 4 \\ y^2 + 12y + 36 \end{pmatrix} = 4x - 12y - x^2 - y^2 - 40$$

$$\iint_A 4x - 12y - x^2 - y^2 \overbrace{- 40 + 40}^{=0}\,dx\,dy = \iint_A \underbrace{-(x-2)^2 - (y+6)^2 + 40}_{\text{Circonferenza di raggio } \sqrt{40}}\,dx\,dy$$

Coordinate polari

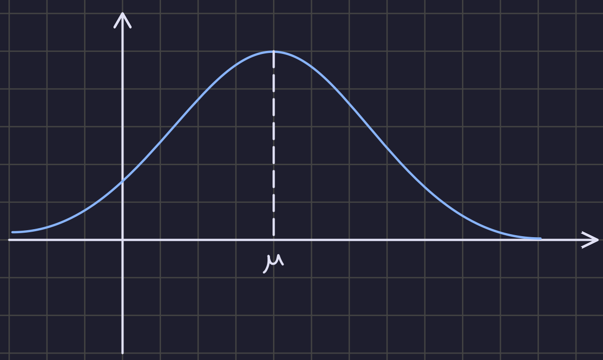
$$\begin{cases} x - 2 = \rho\cos\theta & \rho \in [0, \sqrt{40}] \\ y + 6 = \rho\sin\theta & \theta \in [0, 2\pi] \end{cases}$$

$$\int_0^{2\pi} \int_0^{\sqrt{40}} \left(-\rho^2 \cos^2\theta + \rho^2 \sin^2\theta + 40\right) \rho\,d\rho\,d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{40}} (-\rho^2 + 40)\rho \, d\rho \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{40}} -\rho^3 + 40\rho \, d\rho \, d\theta$$

$$= \int_0^{2\pi} d\theta \quad \left[ 20\rho^2 - \frac{\rho^4}{4} \right]_0^{\sqrt{40}} = 2\pi \cdot (20 \cdot 40 - 400) = 800\pi$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{(Campana di Gauss)}$$




$$g(x) = e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Si vuole capire quanto vale $C = \int_{-\infty}^{+\infty} g(x)\,dx$

$$\downarrow$$

$$f(x) = \frac{1}{C}\, g(x)\,dx$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} \frac{1}{C} g(x)\,dx = \frac{1}{C}\int_{-\infty}^{+\infty} g(x)\,dx = \frac{1}{C}\cdot C = 1$$

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{+\infty} e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2} dx$$

$$\downarrow$$

$$t = \frac{x-\mu}{\sqrt{2}\sigma}$$

$$\sqrt{2}\sigma t + \mu = x$$

$$\sqrt{2}\sigma\, dt = dx$$

$$\downarrow$$

$$\sqrt{2}\sigma \underbrace{\int_{-\infty}^{+\infty} e^{-t^2} dt} = C$$

$$\downarrow$$

$$0 \leq \int_{-\infty}^{+\infty} e^{-t^2} dt = \left(\left(\int_{-\infty}^{+\infty} e^{-t^2} dt\right)^2\right)^{\frac{1}{2}}$$

$$= \left(\int_{-\infty}^{+\infty} e^{-t^2} dt \cdot \int_{-\infty}^{+\infty} e^{-t^2} dt\right)^{\frac{1}{2}}$$

$$= \left(\int_{-\infty}^{+\infty} e^{-t^2} dt \cdot \int_{-\infty}^{+\infty} e^{-s^2} ds\right)^{\frac{1}{2}} \quad (t = s)$$

$$\downarrow$$

$$\left[\iint_{\Omega} f(x)\,g(y)\,dy\,dx \longleftrightarrow \int_a^b f(x)\,dx \cdot \int_c^d g(y)\,dy\right]$$

$$\downarrow$$

$$= \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-t^2} \cdot e^{-s^2}\,dt\,ds\right)^{\frac{1}{2}}$$

$$= \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(t^2+s^2)}\,dt\,ds\right)^{\frac{1}{2}}$$

Trasformo in coordinate polari

$$\begin{cases} t = \rho\cos\theta \\ s = \rho\sin\theta \end{cases}$$

$$= \left(\int_0^{2\pi}\int_0^{+\infty} e^{-\rho^2}\,\rho\,d\rho\,d\theta\right)^{\frac{1}{2}}$$

$$= \left(\int_0^{2\pi} 1\,d\theta \cdot \int_0^{+\infty} e^{-\rho^2}\,\rho\,d\rho\right)^{\frac{1}{2}}$$

$$= \left(2\pi\left[\frac{-e^{-\rho^2}}{2}\right]_0^{+\infty}\right)^{\frac{1}{2}}$$

$$= \left(2\pi \cdot \left(\lim_{k\to+\infty}\frac{-e^{-k^2}}{2} + \frac{e^{-0^2}}{2}\right)\right)^{\frac{1}{2}}$$

$$= \sqrt{\pi}$$

$$\swarrow$$

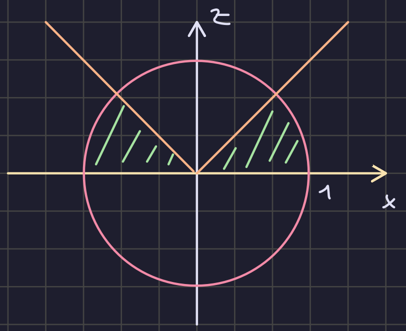
$$C = \sqrt{2}\,\sigma\,\sqrt{\pi}$$
$$= \sigma\sqrt{2\pi}$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Questo è il metodo per calcolare la normalizzazione della funzione campana e serve per far si che l'integrale della campana sia sempre 1.

$$\iiint_S \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$$

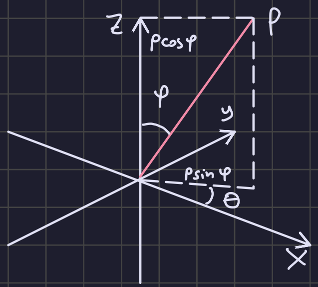
$$S = \Big\{ (x,y,z) \in \mathbb{R}^3 \;\Big|\; \underbrace{x^2+y^2+z^2 \leq 1}_{\substack{\text{Sfera piena} \\ r=1 \\ c=\text{origine}}} \wedge \underbrace{z^2-x^2-y^2 \leq 0}_{\substack{\text{Cerchio di} \\ \text{raggio } z \\ \text{(cono)}}} \wedge \underbrace{z \geq 0}_{\substack{\text{Solo} \\ \text{emisfero} \\ \text{nord}}} \Big\}$$




Vogliamo esprimerlo in coordinate sferiche

$$\begin{cases} x = \rho \cos\theta \sin\varphi \\ y = \rho \sin\theta \cos\varphi \\ z = \rho \cos\varphi \end{cases}$$

Proiezione sul piano (x,y)




Bisogna trovare la jacobiana

$$J(\rho,\theta,\varphi) = \begin{pmatrix} \cos\theta \sin\theta & -\rho \sin\theta \sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta \sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{pmatrix}$$

$$|\det J| = \Bigg| (-1)^{3+1} \cos\varphi \cdot \det\begin{pmatrix} -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \end{pmatrix} + 0 + (-1)^{3+3}(-\rho\sin\varphi)\cdot\det\begin{pmatrix} \cos\theta\sin\theta & -\rho\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi \end{pmatrix} \Bigg|$$

$$\left[ \det(c\cdot A) = c^n \cdot \det(A) \right]$$

$$= \Bigg| \cos\varphi\cdot\rho^2\cdot\det\begin{pmatrix} -\sin\theta\sin\varphi & \cos\theta\cos\varphi \\ \cos\theta\sin\varphi & \sin\theta\cos\varphi \end{pmatrix} - \rho\sin^3\varphi\cdot\det\begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix} \Bigg|$$

$$= \left| \rho^2 \cos\varphi \left( -\sin^2\Theta \underbrace{\sin\varphi\cos\varphi} - \cos^2\Theta \underbrace{\sin\varphi\cos\varphi} \right) - \rho^2 \sin^3\varphi \right|$$

$$= \left| -\rho^2 \cos^2\varphi \sin\varphi - \rho^2 \sin^3\varphi \right|$$

$$= \left| -\rho^2 \sin\varphi \overbrace{\left( \cos^2\varphi + \sin^2\varphi \right)}^{=1} \right|$$

$$= \left| -\rho^2 \sin\varphi \right| = \rho^2 \sin\varphi$$

$$\iiint_S \underbrace{\sqrt{x^2+y^2+z^2}}_{\rho} \, dx\,dy\,dz = \iiint \rho \quad \rho^2 \sin\varphi \; d\rho\, d\Theta\, d\varphi \qquad \begin{aligned} &\rho \in [0,1] \\ &\Theta \in [0, 2\pi] \\ &\varphi \in \left[\tfrac{\pi}{4}, \tfrac{\pi}{2}\right] \end{aligned}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho \; \rho^2 \sin\varphi \, d\rho \, d\Theta \, d\varphi$$

$$= \int_0^{2\pi} 1 \, d\Theta \cdot \int_0^1 \rho^3 \, d\rho \cdot \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\varphi \, d\varphi$$

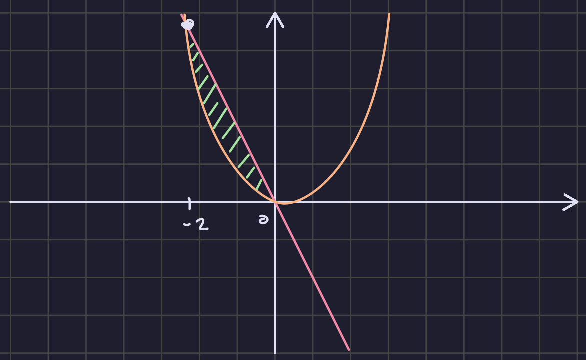
$$= 2\pi \cdot \frac{1}{4} \left[ -\cos\varphi \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \sqrt{2} \, \frac{\pi}{4}$$

Esame furato 19 Giugno 2024

$$\iiint_{\Omega} x \, dx\, dy\, dz \qquad \Omega = \{ (x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D,\ 0 \le z \le x+4y \}$$

D è la regione definita nel piano xy limitata da y = −2x e y = x^2

$$\iint_D \left( \int_0^{x+4y} x \, dz \right) dx\, dy =$$




$$= \iint_D x(x+4y)\, dx\, dy$$

$$= \int_{-2}^{0} \int_{x^2}^{-2x} x(x+4y)\, dx\, dy$$

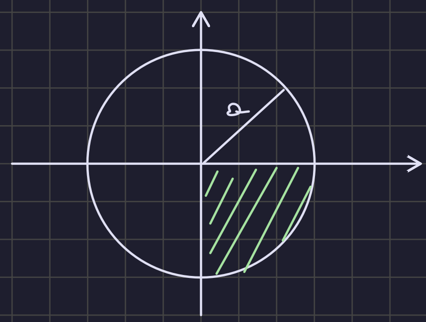
Intersezione $x^2 = -2x \quad x^2 + 2x = 0 \quad x(x+2) = 0$

= Per casa

Esame Burato 19 Giugno 2024

$$\iint_D xy\,(x^2+y^2)^{\frac{3}{2}}\,dx\,dy = -\frac{1}{14}$$

$D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le a^2,\ x \ge 0,\ y \le 0\}$




Trasformo in coordinate polari

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$\int_{\frac{3}{2}\pi}^{2\pi}\int_0^a \rho^6 \cos\theta\sin\theta\,d\rho\,d\theta =$$

$$= \int_0^a \rho^6\,d\rho \cdot \int_{\frac{3}{2}\pi}^{2\pi} \cos\theta\sin\theta\,d\theta$$

$$= \left[\frac{\rho^7}{7}\right]_0^a \cdot \left[\frac{\sin^2\theta}{2}\right]_{\frac{3}{2}\pi}^{2\pi}$$

$$= \frac{a^7}{7} \cdot -\frac{1}{2} = -\frac{a^7}{14}$$

$$-\frac{a^7}{14} = -\frac{1}{14}$$

$$\downarrow$$

$$a = 1$$

$$\int_0^1 dx \int_0^1 \frac{x+y}{1+(x-y)^2}\,dy = \int_0^1\int_0^1 \frac{x+y}{1+(x-y)^2}\,dx\,dy$$

$$\int_0^1\int_0^1 \frac{x+y}{1+(x-y)^2}\,dx\,dy \longrightarrow \quad \begin{cases} u = x+y \\ v = x-y \end{cases} \to \quad u+v = 2x \quad x = \frac{u+v}{2}$$

$$u-v = 2y \quad y = \frac{u-v}{2}$$

$$J(u,v) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$|\det J| = -\frac{1}{2}$$

Cambio degli
estremi di
integrazione

Calcolare $\int_\gamma G \, d\gamma$

$$G(x,y) = (-\cos y, x \sin y) e^{x(\cos y)} \qquad \gamma : [0,1] \to \mathbb{R}^2 \qquad t \mapsto (\cos(t\pi) + t^2, 1+t^2)$$

Controlliamo se il campo vettoriale è conservativo

$$D_x\left(x \sin y \, e^{x(\cos y)}\right) = \sin y \, e^{x(\cos x)} + x \sin y \, e^{x(\cos y)} \cos y$$

$$D_y\left(-\cos y \, e^{x(\cos y)}\right) = \sin y \, e^{x(\cos y)} + \cos y \, e^{x(\cos y)} x \sin y$$

$$D_x = D_y$$

$$\sin y \, \cancel{e^{x(\cos x)}} + x \sin y \, \cancel{e^{x(\cos y)}} \cos y =$$

$$\sin y \, \cancel{e^{x(\cos y)}} + \cos y \, \cancel{e^{x(\cos y)}} x \sin y$$

$$\sin y + x \sin y \cos y = \sin y + x \cos y \sin y \qquad \checkmark$$

Borato 6/09/23 B

**Esercizio 5 (punti: ....../4)**

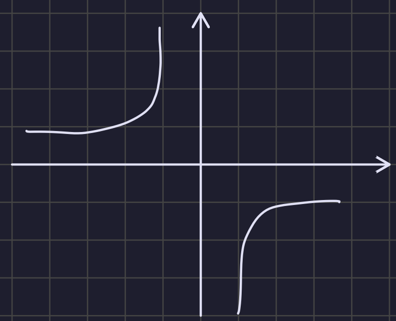
a) (2 punti) Determinare e classificare i punti stazionari della funzione $f(x, y) = xy^2 - 4x + y$.

b) (2 punti) Calcolare (se esistono) massimo e minimo assoluto della funzione $f$ su

$$E = \{(x, y) \in \mathbb{R}^2 : xy + 3 = 0\}$$

b) $F(x,y) = xy^2 - 4x + y$

$xy + 3 = 0 \to x = -\frac{3}{y}$



$g(x) = f\left(x, -\frac{3}{x}\right)$

$= x\frac{9}{x^2} - 4x - \frac{3}{x}$

$= \frac{6 - 4x^2}{x}$

$g'(x) = \frac{-8x^2 - (6 - 4x^2)}{x^2}$

$= \frac{-4x^2 - 6}{x^2} \to 4x^2 = -6$ $\nexists x \in \mathbb{R} \to$ Non ci sono punti critici

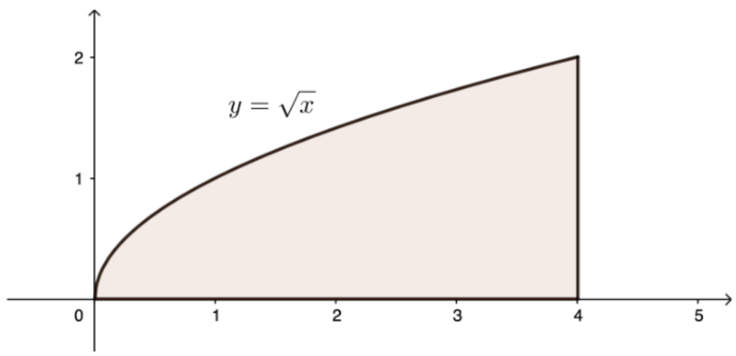
Metodo 2 (Lagrange)

$\mathcal{L}(x,y) = xy^2 - 4x + y + \lambda(xy + 3)$

$$\nabla \mathcal{L}(x,y) = \begin{pmatrix} y^2 - 4 + \lambda y \\ 2xy + 1 + \lambda x \\ xy + 3 \end{pmatrix} = 0 \quad \begin{cases} y_{1,2} = \frac{-\lambda \pm \sqrt{\lambda^2 + 16}}{2} \\ -6 + 1 + \lambda x = 0 \to \underbrace{\lambda x = 5} \\ y = -\frac{3}{x} \end{cases}$$

$\frac{9}{x^2} - 4 - \frac{3\lambda}{x} = \frac{9 - 4x^2 - 3\lambda x}{x^2} = 0 = \frac{-6 - 4x^2}{x^2}$

**Esercizio 6 (punti: ....../4)**
Calcolare il baricentro della regione piana rappresentata in figura.




$$x_B = \frac{\iint_D x \, dx\, dy}{\iint_D dx\, dy} \quad ] \text{ massa}$$

$$y_B = \frac{\iint_D y \, dx\, dy}{\iint_D dx\, dy}$$

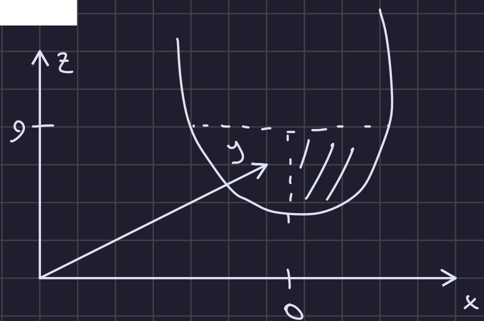
$$M = \int_0^4 \int_0^{\sqrt{x}} dy\, dx = \int_0^4 \sqrt{x}\, dx = \left[\frac{2}{3} x^{\frac{3}{2}}\right]_0^4 = \frac{16}{3}$$

**Esercizio 7 (punti: ....../4)**
Calcolare

$$\iiint_\Omega \frac{4}{3} x \sqrt{x^2+y^2}\, dx\, dy\, dz$$

dove $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \le z \le 9,\ x \ge 0\}$

$$\Omega = \begin{cases} z \ge x^2+y^2 \\ z \le 9 \\ x \ge 0 \end{cases}$$




Coordinate cilindriche

$$\begin{cases} x = \rho \cos\theta \\ y = -\rho \sin\theta \\ z = z \end{cases} \qquad \begin{array}{l} \theta \in [-\frac{\pi}{2}, \frac{\pi}{2} \\ \rho \in [0, \sqrt{z}] \\ z \in [0, 9] \end{array}$$

$$\iint_D \int_{x^2+x^2}^{9} \frac{4}{3} x \sqrt{x^2+y^2}\, dz\, dx\, dy$$

$$= \iint_D \frac{4}{3} x \sqrt{x^2+y^2} \left( 9 - x^2 + y^2 \right) dx\, dy$$

$$\begin{cases} x = \rho\cos\theta & \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ y = \rho\sin\theta & \rho \in [0,3] \end{cases}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 \frac{4}{3} \rho\cos\theta\, \rho\, (9-\rho^2)\, \rho\, d\rho\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta \cdot \int_0^3 \frac{4}{3} \rho^3 (9-\rho^2)\, d\rho$$

$$= \left[\sin\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 12\rho^3 - \frac{4}{3}\rho^5\, d\rho$$

$$= 2 \left[3\rho^4 - \frac{4}{18}\rho^6\right]_0^3$$

$$= \left[6\rho^4 - \frac{4}{9}\rho^6\right]$$

$$= 6 \cdot 81 - 4 \cdot 81$$

$$= 162$$

Metodo 2

$$\int_0^9 \int_0^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4}{3}\rho^3\cos\theta\, d\theta\, d\rho\, dz$$

$$= \frac{4}{3} \int_0^9 \left( \int_0^{\sqrt{z}} \rho^3\, d\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta \right) dz$$

$$= \frac{8}{3} \int_0^9 \left[\frac{\rho^4}{4}\right] dz$$

$$= \frac{8}{3} \int_0^9 \frac{z^2}{4}\, dz$$

$$= \frac{8}{3}\left[\frac{z^3}{12}\right]_0^9$$

$$= \frac{8\cdot 3^6}{3^2\cdot 4} = 2\cdot 3^4 = 162$$

**Esercizio 8 (punti: ....../4)**

a) (2 punti) Verificare che il campo vettoriale

$$\vec{F}(x,y,z) = (ze^{zx} + \alpha y,\ \alpha x - \beta + z,\ xe^{zx} + y)$$

è conservativo per ogni $\alpha, \beta \in \mathbb{R}$ e determinare un suo potenziale nel caso $\alpha = 2, \beta = 0$.

b) (2 punti) Con $\alpha = 2$, $\beta = 0$, calcolare $\int_\gamma \vec{F}\cdot d\vec{r}$ e $\int_\gamma F_x\, ds$, dove $\gamma(t) = (0, 3\cos t, 3\sin t)$, $t \in [0, \frac{\pi}{4}]$.

Nota: $F_x$ è la prima componente del campo vettoriale $\vec{F}$.

$$J_F = \begin{pmatrix} 0 & \alpha & e^{zx} + xze^{zx} \\ \alpha & 0 & 1 \\ e^{zx} + xze^{zx} & 1 & 0 \end{pmatrix}$$

É simmetrica quindi é conservativo

$\alpha = 2 \quad \beta = 0$

$\downarrow$

$$\vec{F}(x,y,z) = \begin{pmatrix} ze^{zx} + 2y \\ 2x + z \\ xe^{zx} + y \end{pmatrix}$$

$$\int ze^{zx} + 2y\, dx = \underline{e^{zx}} + \underline{2xy} + c(y,z)$$

$$\int 2x + z\, dy = \underline{2xy} + \underline{yz} + c(x,z)$$

$$\int xe^{zx} + y\, dz = \underline{e^{zx}} + \underline{yz} + c(x,y)$$

Potenziale: $U(x,y,z) = e^{xz} + yz + 2xy + K \qquad K \in \mathbb{R}$

$$\int_\gamma \vec{F}\cdot d\vec{r}$$

$\downarrow$

$$U(\gamma(\tfrac{\pi}{4})) - U(\gamma(0))$$

$$= U\left(0, \frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right) - U(0,3,0) = 1 + \frac{9}{2} - 1 = \frac{9}{2}$$

$$\int_\gamma F_x\, ds = \int_0^{\frac{\pi}{4}} F_x(\gamma(t))\cdot|\gamma'(t)|\, dt$$

$\gamma(t) = (0, 3\cos t, 3\sin t)$

$\gamma'(t) = (0, -3\sin t, 3\cos t)$

$|\gamma'(t)| = 3$

$$= \int_0^{\frac{\pi}{4}} 3\sin t + 6\cos t \cdot 3\, dt$$

$$= 9\int_0^{\pi/4} \sin t + 2\cos t \, dt$$

$$= 9\left[2\sin t - \cos t\right]_0^{\pi/4}$$

$$= 9\frac{\sqrt{2}}{2} + 9$$