Esercizi su risposta libera e impulsiva

1. Dato il seguente sistema a tempo continuo (LTI)

$$\frac{d^2v(t)}{dt^2} - 5\frac{dv(t)}{dt} - 6v(t) = \frac{du(t)}{dt} + 5u(t), \quad t \in \mathbb{R}_+,$$

e le seguenti condizioni inziali:

$$v(0^-) = 3 \qquad \frac{dv(0^-)}{dt} = 1.$$

(a) si calcoli la risposta libera nel tempo,

(b) si calcoli la risposta impulsica nel tempo.

a) Risposta libera

Equazione del sistema

$$v''(t) - 5v'(t) - 6v(t) = u'(t) + 5u(t), \quad t \in \mathbb{R}_+$$

Condizioni iniziali

$$\begin{cases} v(0^-) = 3 \\ v'(0^-) = 1 \end{cases}$$

Equazione omogenea del polinomio caratteristico

$$P(s) = s^2 - 5s - 6s = 0$$

Soluzioni

$$s_{1,2} = \frac{5 \pm \sqrt{25+24}}{2} = \frac{5 \pm 7}{2} = \begin{cases} \frac{12}{2} = 6 \\ -\frac{2}{2} = -1 \end{cases}$$

$\lambda_1 = -1$ $\qquad r = 2$ (numero di soluzioni)

$\lambda_2 = 6$ $\qquad \mu_{1,2} = 1$ (molteplicitá)

Risposta libera generica

$$v_l(t) = \sum_{i=1}^{r} \sum_{l=0}^{\mu_i - 1} c_{i,l} \cdot e^{\lambda_i t} \cdot \frac{t^l}{l!}$$

$$= c_1 \cdot e^{-t} \cdot \frac{t^0}{0!} + c_2 \cdot e^{6t} \cdot \frac{t^0}{0!}$$

$$= c_1 \cdot e^{-t} + c_2 \cdot e^{6t}$$

Derivata della risposta libera

$$v(t) = c_1 \cdot e^{-t} + c_2 \cdot e^{6t}$$

$$v'(t) = c_1 \cdot e^{-t} \cdot (-1) + c_2 \cdot e^{6t} \cdot 6$$

$$= -c_1 \cdot e^{-t} + 6 c_2 \cdot e^{6t}$$

Calcolo dei coefficienti c_1 e c_2

$$\begin{cases} v(0^-) = c_1 \cdot e^{-0^-} + c_2 \cdot e^{6 \cdot 0^-} \\ v'(0^-) = -c_1 \cdot e^{-0^-} + 6 c_2 \cdot e^{6 \cdot 0^-} \end{cases}$$

$$\begin{cases} v(0^-) = c_1 + c_2 \\ v'(0^-) = -c_1 + 6 c_2 \end{cases}$$

$$\begin{cases} v(0^-) = 3 \\ v'(0^-) = 1 \end{cases} \rightarrow \begin{cases} c_1 + c_2 = 3 \\ -c_1 + 6 c_2 = 1 \end{cases}$$

$$\begin{cases} c_1 = 3 - c_2 \\ -3 + c_2 + 6 c_2 = 1 \end{cases}$$

$$\begin{cases} c_1 = \frac{17}{7} \\ c_2 = \frac{4}{7} \end{cases}$$

Risposta libera specifica

$$v_l(t) = \frac{17}{7} e^{-t} + \frac{4}{7} e^{6t}$$

b) Risposta impulsiva

Equazione del sistema

$$v''(t) - 5 v'(t) - 6 v(t) = u'(t) + 5 u(t), \qquad t \in \mathbb{R}_+$$

Equazione omogenea del polinomio caratteristico

$$P(s) = s^2 - 5s - 6s = 0$$

Soluzioni

$\lambda_1 = -1$ $\quad r = 2$ (numero di soluzioni)

$\lambda_2 = 6$ $\quad \mu_{1,2} = 1$ (molteplicitá)

Risposta impulsiva generica

$$h(t) = d_0 \cdot \delta(t) + \left(\sum_{i=1}^{r} \sum_{l=0}^{\mu_i - 1} d_{i,l} \cdot e^{\lambda_i t} \cdot \frac{t^l}{l!} \right) \cdot \delta_{-1}(t)$$

$d_0 = 0$ perché il sistema non é proprio

$$= \left(d_1 \cdot e^{-t} \cdot \frac{t^0}{0!} + d_2 \cdot e^{6t} \cdot \frac{t^0}{0!} \right) \cdot \delta_{-1}(t)$$

$$= \left(d_1 \cdot e^{-t} + d_2 \cdot e^{6t} \right) \cdot \delta_{-1}(t)$$

Riscrivo l'equazione del sistema ponendo $v(t) = h(t)$ e $u(t) = \delta(t)$

$$v''(t) - 5v'(t) - 6v(t) = u'(t) + 5u(t)$$

$$\downarrow$$

$$h''(t) - 5h'(t) - 6h(t) = \delta'(t) + 5\delta(t)$$

Calcolo delle derivate di $h(t)$

$$h(t) = \left(d_1 \cdot e^{-t} + d_2 \cdot e^{6t} \right) \cdot \delta_{-1}(t)$$

$$h'(t) = \left(-d_1 \cdot e^{-t} + 6 d_2 e^{6t} \right) \cdot \delta_{-1}(t) + \left(d_1 \cdot e^{-t} + d_2 \cdot e^{6t} \right) \cdot \delta(t)$$

$$h''(t) = \left(d_1 \cdot e^{-t} + 36 d_2 e^{6t} \right) \cdot \delta_{-1}(t) + \left(-d_1 \cdot e^{-t} + 6 d_2 e^{6t} \right) \cdot \delta(t) +$$
$$\left(-d_1 \cdot e^{-t} + 6 d_2 \cdot e^{6t} \right) \cdot \delta(t) + \left(d_1 \cdot e^{-t} + d_2 \cdot e^{6t} \right) \cdot \delta_1(t)$$

Sostituisco nell' equazione del sistema

$$h''(t) - 5h'(t) - 6h(t) = \delta'(t) + 5\delta(t)$$

$$\downarrow$$

$$\overbrace{\Big[\left(d_1 \cdot e^{-t} + 36 d_2 e^{6t} \right) \cdot \delta_{-1}(t) + \left(-d_1 \cdot e^{-t} + 6 d_2 e^{6t} \right) \cdot \delta(t) +}^{h''}$$
$$\left(-d_1 \cdot e^{-t} + 6 d_2 \cdot e^{6t} \right) \cdot \delta(t) + \left(d_1 \cdot e^{-t} + d_2 \cdot e^{6t} \right) \cdot \delta_1(t)\Big]$$
$$-5\overbrace{\Big[\left(-d_1 \cdot e^{-t} + 6 d_2 e^{6t} \right) \cdot \delta_{-1}(t) + \left(d_1 \cdot e^{-t} + d_2 \cdot e^{6t} \right) \cdot \delta(t)\Big]}^{h'}$$
$$-6\overbrace{\Big[\left(d_1 \cdot e^{-t} + d_2 \cdot e^{6t} \right) \cdot \delta_{-1}(t)\Big]}^{h} = \delta'(t) + 5\delta(t)$$

Valuto le funzioni in $t=0^-$

$$\Big[\left(d_1 \cdot e^{-0^-} + 36 d_2 e^{6\cdot 0^-}\right)\cdot \delta_{-1}(0^-) + \left(-d_1 \cdot e^{-0^-} + 6 d_2 e^{6\cdot 0^-}\right)\cdot \delta(0^-) +$$
$$\left(-d_1 \cdot e^{-0^-} + 6 d_2 \cdot e^{6\cdot 0^-}\right)\cdot \delta(0^-) + \left(d_1 \cdot e^{-0^-} + d_2 \cdot e^{6\cdot 0^-}\right)\cdot \delta_1(0^-)\Big]$$
$$-5\Big[\left(-d_1 \cdot e^{-0^-} + 6 d_2 e^{6\cdot 0^-}\right)\cdot \delta_{-1}(0^-) + \left(d_1 \cdot e^{-0^-} + d_2 \cdot e^{6\cdot 0^-}\right)\cdot \delta(0^-)\Big]$$
$$-6\Big[\left(d_1 \cdot e^{-0^-} + d_2 \cdot e^{6\cdot 0^-}\right)\cdot \delta_{-1}(0^-)\Big] = \delta'(0^-) + 5\,\delta(0^-)$$

$$\Big[\cancel{\left(d_1 \cdot e^{-0^-} + 36 d_2 e^{6\cdot 0^-}\right)\cdot \delta_{-1}(0^-)} + \left(-d_1 \cdot \cancel{e^{-0^-}} + 6 d_2 \cancel{e^{6\cdot 0^-}}\right)\cdot \delta(0^-) +$$
$$\left(-d_1 \cdot \cancel{e^{-0^-}} + 6 d_2 \cdot \cancel{e^{6\cdot 0^-}}\right)\cdot \delta(0^-) + \left(d_1 \cdot \cancel{e^{-0^-}} + d_2 \cdot \cancel{e^{6\cdot 0^-}}\right)\cdot \delta_1(0^-)\Big]$$
$$-5\Big[\cancel{\left(-d_1 \cdot e^{-0^-} + 6 d_2 e^{6\cdot 0^-}\right)\cdot \delta_{-1}(0^-)} + \left(d_1 \cdot \cancel{e^{-0^-}} + d_2 \cdot \cancel{e^{6\cdot 0^-}}\right)\cdot \delta(0^-)\Big]$$
$$-6\Big[\cancel{\left(d_1 \cdot e^{-0^-} + d_2 \cdot e^{6\cdot 0^-}\right)\cdot \delta_{-1}(0^-)}\Big] = \delta'(0^-) + 5\,\delta(0^-)$$

tolgo il gradino perché in $t=0^-$ fa 0

$$\Big[(-d_1 + 6 d_2)\cdot \delta(0^-) + (-d_1 + 6 d_2)\cdot \delta(0^-) + (d_1 + d_2)\cdot \delta_1(0^-)\Big]$$
$$-5\Big[(d_1 + d_2)\cdot \delta(0^-)\Big] = \delta'(0^-) + 5\,\delta(0^-)$$

$$(-d_1 + 6 d_2)\, 2\, \delta(0^-) + (d_1 + d_2)\cdot \delta_1(0^-)$$
$$-5(d_1 + d_2)\cdot \delta(0^-) = \delta'(0^-) + 5\,\delta(0^-)$$

Sposto tutto a sinistra

$$\underline{(-d_1 + 6 d_2)\, 2\, \delta(0^-)} + \underline{(d_1 + d_2)\cdot \delta_1(0^-)}$$
$$\underline{-5(d_1 + d_2)\cdot \delta(0^-)} \quad \underline{-\delta_1(0^-)}\, \underline{-5\,\delta(0^-)} = 0$$

Raccolgo per $\delta(0), \delta_1(0), \ldots, \delta_n(0)$ che sono linearmente indipendenti

$$\underline{\big(2(-d_1 + 6 d_2) - 5(d_1 + d_2) - 5\big)\ \delta(0^-)} + \underline{(d_1 + d_2 - 1)\cdot \delta_1(0^-)} = 0$$

$$(-2d_1+12d_2-5d_1-5d_2-5)\,\delta(0^-)+(d_1+d_2-1)\,\delta_1(0^-)=0$$

$$(-7d_1+7d_2-5)\,\delta(0^-)+(d_1+d_2-1)\,\delta_1(0^-)=0$$

Risolvo il sistema

$$\begin{cases}(-7d_1+7d_2-5)\,\cancel{\delta(0^-)}=0\\(d_1+d_2-1)\,\cancel{\delta_1(0^-)}=0\end{cases}$$

$$\begin{cases}-7d_1+7d_2=5\\d_1+d_2=1\end{cases}$$

$$\begin{cases}7d_2-7+7d_2=5\\d_1=-d_2+1\end{cases}$$

$$\begin{cases}d_2=\frac{12}{14}=\frac{6}{7}\\d_1=-\frac{6}{7}+\frac{7}{7}=\frac{1}{7}\end{cases}$$

Risposta impulsiva specifica

$$h(t)=\frac{1}{7}\cdot e^{-t}+\frac{6}{7}\cdot e^{6t}\cdot\delta_{-1}(t)$$

2. Dato il seguente sistema a tempo continuo (LTI)

$$2\frac{d^2v(t)}{dt^2}-3\frac{dv(t)}{dt}-2v(t)=2\frac{du(t)}{dt}+u(t),\quad t\in\mathbb{R}_+.$$

e le seguenti condizioni inziali:

$$v(0^-)=4\qquad\frac{dv(0^-)}{dt}=-2,$$

(a) si calcoli la risposta libera nel tempo,

(b) si calcoli la risposta impulsiva nel tempo,

a) Risposta libera

Equazione del sistema

$$2v''(t)-3v'(t)-2v(t)=2u'(t)+u(t)\qquad t\in\mathbb{R}_+$$

Condizioni iniziali

$$\begin{cases}v(0^-)=4\\v'(0^-)=-2\end{cases}$$

Equazione omogenea del polinomio caratteristico

$$P(s) = 2s^2 - 3s - 2 = 0$$

Soluzioni

$$s_{1,2} = \frac{3 \pm \sqrt{9+16}}{4} = \frac{3 \pm 5}{4} = \begin{cases} \frac{8}{4} = 2 \\ -\frac{2}{4} = -\frac{1}{2} \end{cases}$$

$\lambda_1 = -\frac{1}{2}$ $\quad r = 2$ (numero di soluzioni)

$\lambda_2 = 2$ $\quad \mu_{1,2} = 1$ (molteplicità)

Risposta libera generica

$$v_L(t) = \sum_{i=1}^{r} \sum_{l=0}^{\mu_i - 1} c_{i,l} \cdot e^{\lambda_i t} \cdot \frac{t^l}{l!}$$

$$= c_1 \cdot e^{-\frac{1}{2}t} \cdot \frac{t^0}{0!} + c_2 \cdot e^{2t} \cdot \frac{t^0}{0!}$$

$$= c_1 \cdot e^{-\frac{1}{2}t} + c_2 \cdot e^{2t}$$

Derivata risposta impulsiva

$$v(t) = c_1 \cdot e^{-\frac{1}{2}t} + c_2 \cdot e^{2t}$$

$$v'(t) = -\frac{1}{2}c_1 e^{-\frac{1}{2}t} + 2c_2 \cdot e^{2t}$$

Calcolo dei coefficienti c_1 e c_2

$$\begin{cases} v(0^-) = c_1 \cdot e^{-\frac{1}{2} \cdot 0^-} + c_2 \cdot e^{2 \cdot 0^-} \\ v'(0^-) = -\frac{1}{2}c_1 e^{-\frac{1}{2} \cdot 0^-} + 2c_2 \cdot e^{2 \cdot 0^-} \end{cases}$$

$$\begin{cases} v(0^-) = c_1 + c_2 = 4 \\ v'(0^-) = -\frac{1}{2}c_1 + 2c_2 = -2 \end{cases}$$

$$\begin{cases} c_1 = 4 - c_2 \\ -\frac{1}{2}(4 - c_2) + 2c_2 = -2 \end{cases}$$

$$\begin{cases} c_1 = 4 \\ c_2 = 0 \end{cases}$$

Risposta libera specifica

$$v(t) = 4 \cdot e^{-\frac{1}{2}t}$$

$$v(t) = c_1 \cdot e^{-\frac{1}{2}t} + c_2 \cdot e^{2t}$$

b) Risposta impulsiva nel tempo

Equazione del sistema

$$2v''(t) - 3v'(t) - 2v(t) = 2u'(t) + u(t) \qquad t \in \mathbb{R}_+$$

Equazione omogenea del polinomio caratteristico

$$P(s) = 2s^2 - 3s - 2 = 0$$

Soluzioni

$$s_{1,2} = \frac{3 \pm \sqrt{9+16}}{4} = \frac{3 \pm 5}{4} = \begin{cases} \frac{8}{4} = 2 \\ -\frac{2}{4} = -\frac{1}{2} \end{cases}$$

$\lambda_1 = -\frac{1}{2}$ $\qquad r = 2$ (numero di soluzioni)

$\lambda_2 = 2$ $\qquad \mu_{1,2} = 1$ (molteplicitá)

Risposta impulsiva generica

$$h(t) = d_0 \delta(t) + \left(\sum_{i=1}^{r} \sum_{l=0}^{\mu_i - 1} d_{i,l} \cdot e^{\lambda_i t} \cdot \frac{t^l}{l!} \right) \cdot \delta_{-1}(t)$$

$d_0 = 0$ perché il sistema non é proprio ($n \neq m$)

$$= \left(d_1 \cdot e^{-\frac{1}{2}t} \cdot \frac{t^0}{0!} + d_2 \cdot e^{2t} \cdot \frac{t^0}{0!} \right) \cdot \delta_{-1}(t)$$

$$= \left(d_1 \cdot e^{-\frac{1}{2}t} + d_2 \cdot e^{2t} \right) \delta_{-1}(t)$$

Riscrivo l'equazione del sistema ponendo $v(t)=h(t)$ e $u(t)=\delta(t)$

$$2v''(t)-3v'(t)-2v(t)=2u'(t)+u(t)$$

$$\Downarrow$$

$$2h''(t)-3h'(t)-2h(t)=2\delta_1(t)+\delta(t)$$

Calcolo le derivate di $h(t)$

$$h(t)=\left(d_1\cdot e^{-\frac{1}{2}t}+d_2\cdot e^{2t}\right)\delta_{-1}(t)$$

$$h'(t)=\left(-\tfrac{1}{2}d_1\cdot e^{-\frac{1}{2}t}+2d_2\cdot e^{2t}\right)\cdot\delta_{-1}(t)+\left(d_1\cdot e^{-\frac{1}{2}t}+d_2\cdot e^{2t}\right)\delta(t)$$

$$h''(t)=\left(\tfrac{1}{4}d_1\cdot e^{-\frac{1}{2}t}+4d_2\cdot e^{2t}\right)\cdot\delta_{-1}(t)+\left(-\tfrac{1}{2}d_1\cdot e^{-\frac{1}{2}t}+2d_2\cdot e^{2t}\right)\cdot\delta(t)$$
$$+\left(-\tfrac{1}{2}d_1\cdot e^{-\frac{1}{2}t}+2d_2\cdot e^{2t}\right)\delta(t)+\left(d_1\cdot e^{-\frac{1}{2}t}+d_2\cdot e^{2t}\right)\delta_1(t)$$

Sostituisco nell'equazione del sistema

$$2h''(t)-3h'(t)-2h(t)=2\delta_1(t)+\delta(t)$$

$$\Downarrow$$

$$2\Big[\overbrace{\left(\tfrac{1}{4}d_1\cdot e^{-\frac{1}{2}t}+4d_2\cdot e^{2t}\right)\cdot\delta_{-1}(t)+\left(-\tfrac{1}{2}d_1\cdot e^{-\frac{1}{2}t}+2d_2\cdot e^{2t}\right)\cdot\delta(t)}^{h''}$$
$$+\left(-\tfrac{1}{2}d_1\cdot e^{-\frac{1}{2}t}+2d_2\cdot e^{2t}\right)\delta(t)+\left(d_1\cdot e^{-\frac{1}{2}t}+d_2\cdot e^{2t}\right)\delta_1(t)\Big]$$
$$-3\Big[\overbrace{\left(-\tfrac{1}{2}d_1\cdot e^{-\frac{1}{2}t}+2d_2\cdot e^{2t}\right)\cdot\delta_{-1}(t)+\left(d_1\cdot e^{-\frac{1}{2}t}+d_2\cdot e^{2t}\right)\delta(t)}^{h'}\Big]$$
$$-2\Big[\overbrace{\left(d_1\cdot e^{-\frac{1}{2}t}+d_2\cdot e^{2t}\right)\delta_{-1}(t)}^{h}\Big]=2\delta_1(t)+\delta(t)$$

Valuto in $t=0^-$

$$2\Big[\cancel{\left(\tfrac{1}{4}d_1\cdot e^{-\frac{1}{2}\cdot0^-}+4d_2\cdot e^{2\cdot0^-}\right)\cdot\delta_{-1}(0^-)}+\left(-\tfrac{1}{2}d_1\cdot \cancel{e^{-\frac{1}{2}0^-}}+2d_2\cdot\cancel{e^{2\cdot0^-}}\right)\cdot\delta(0^-)$$
$$+\left(-\tfrac{1}{2}d_1\cdot\cancel{e^{-\frac{1}{2}0^-}}+2d_2\cdot\cancel{e^{2\cdot0^-}}\right)\delta(0^-)+\left(d_1\cdot\cancel{e^{-\frac{1}{2}0^-}}+d_2\cdot\cancel{e^{2\cdot0^-}}\right)\delta_1(0^-)\Big]$$
$$-3\Big[\cancel{\left(-\tfrac{1}{2}d_1\cdot e^{-\frac{1}{2}\cdot0^-}+2d_2\cdot e^{2\cdot0^-}\right)\cdot\delta_{-1}(0^-)}+\left(d_1\cdot\cancel{e^{-\frac{1}{2}0^-}}+d_2\cdot\cancel{e^{2\cdot0^-}}\right)\delta(0^-)\Big]$$
$$-2\Big[\cancel{\left(d_1\cdot e^{-\frac{1}{2}\cdot0^-}+d_2\cdot e^{2\cdot0^-}\right)\delta_{-1}(0^-)}\Big]=2\delta_1(0^-)+\delta(0^-)$$

$$2\left[\left(-\frac{1}{2}d_1+2d_2\right)\cdot\delta(0^-)+\left(-\frac{1}{2}d_1+2d_2\right)\delta(0^-)+(d_1+d_2)\delta_1(0^-)\right]$$

$$-3\left[(d_1+d_2)\delta(0^-)\right]=2\delta_1(0^-)+\delta(0^-)$$

$$(-d_1+4d_2)\cdot\delta(0^-)+(-d_1+4d_2)\delta(0^-)+(2d_1+2d_2)\delta_1(0^-)+$$

$$(-3d_1-3d_2)\delta(0^-)=2\delta_1(0^-)+\delta(0^-)$$

$$2(-d_1+4d_2)\delta(0^-)+(-3d_1-3d_2)\delta(0^-)+(2d_1+2d_2)\delta_1(0^-)-2\delta_1(0^-)-\delta(0^-)=0$$

$$\left[(-2d_1+8d_2)+(-3d_1-3d_2)-1\right]\delta(0^-)+\left[(2d_1+2d_2)-2\right]\delta_1(0^-)=0$$

Risoluo il sistema

$$\begin{cases}(-2d_1+8d_2-3d_1-3d_2-1)\cancel{\delta(0^-)}=0\\(2d_1+2d_2-2)\cancel{\delta_1(0^-)}=0\end{cases}$$

$$\begin{cases}-5d_1+5d_2=1\\2d_1+2d_2=2\end{cases}$$

$$\begin{cases}-5+5d_2+5d_2=1\\d_1=1-d_2\end{cases}$$

$$\begin{cases}d_2=\frac{3}{5}\\d_1=\frac{2}{5}\end{cases}$$

Risposta impulsiva generica

$$h(t)=\left(\frac{2}{5}\cdot e^{-\frac{1}{2}t}+\frac{3}{5}\cdot e^{2t}\right)\delta_{-1}(t)$$