

# Esercizi presi dall'eserciziario su moodle

## Equazioni differenziali di primo grado

▣ **Esercizio 1.1.1.** Si determini la soluzione  $y(t)$  del seguente problema di Cauchy

$$\begin{cases} y' = \frac{y^2}{y^2 + 4} t \\ y(0) = 2 \end{cases}$$

Inoltre si determini il valore  $\alpha > 0$  per cui  $\frac{y(t)}{t^\alpha}$  tende a un numero finito e non nullo per  $t \rightarrow +\infty$ .

$$\frac{y^2 + 4}{y^2} \cdot y' = t$$

$$\int \frac{y^2 + 4}{y^2} \cdot y' dt = \int t dt + C$$

$$\begin{aligned} u &= y \\ du &= y' dt \\ \downarrow u &= y \end{aligned}$$

$$\int \frac{y^2 + 4}{y^2} dy = \frac{t^2}{2} + C$$

$$\int 1 dy + 4 \int y^{-2} dy = \frac{t^2}{2} + C$$

$$y - \frac{4}{y} = \frac{t^2}{2} + C$$

$$\frac{y^2 - 4}{y} = \frac{t^2}{2} + C$$

$$y(0) = 2$$

$\downarrow$

$$\frac{2^2 - 4}{2} = \frac{0}{2} + C$$

$$\frac{0}{2} = C$$

$$C = 0 \rightarrow$$

$$\frac{y^2 - 4}{y} = \frac{t^2}{2} \rightarrow$$

$$\frac{y^2 - 4 - \frac{t^2 y}{2}}{y} = 0$$

$$y(t) = \frac{\frac{t^2}{2} \pm \sqrt{\frac{t^4}{4} + 16}}{2} = \frac{t^2 \pm \sqrt{t^4 + 64}}{4}$$

$$\downarrow$$

$$y(0) = 2$$

$$\downarrow$$

$$\frac{0^2 \pm \sqrt{64}}{4} = \begin{cases} +2 \\ -2 \end{cases} \quad (\text{Non accettabile}) \rightarrow y(0) = 2$$

Soluzione 1:

$$y(t) = \frac{t^2 + \sqrt{t^4 + 64}}{4}$$

$$\lim_{t \rightarrow +\infty} \frac{y(t)}{t^\alpha} = \frac{t^2 + \sqrt{t^4 + 64}}{4 t^\alpha} = \lim_{t \rightarrow +\infty} \frac{t^2 + t^2 \sqrt{1 + \frac{64}{t^4}}}{4 t^\alpha} =$$

$$= \lim_{t \rightarrow +\infty} \frac{x t^2}{4 t^\alpha} = \lim_{t \rightarrow +\infty} \frac{t^2}{2 t^\alpha}$$

$1 + \frac{64}{+\infty} = 1$

L'unico modo per avere un numero finito è che il grado del numeratore e del denominatore sia uguale, quindi:

$$\alpha = 2 \rightarrow \lim_{t \rightarrow +\infty} \frac{t^2}{2 t^2} = \lim_{t \rightarrow +\infty} \frac{1}{2} = \frac{1}{2}$$

▣ **Esercizio 1.1.2.** Si determini la soluzione  $y(t)$  del seguente problema di Cauchy

$$\begin{cases} y' = \frac{t^2 + t}{2e^{2y} + 6e^y} \\ y(0) = 0 \end{cases}$$

$$y' \cdot (2e^{2y} + 6e^y) = t^2 + t$$

$$\int y' \cdot (2e^{2y} + 6e^y) dt = \int t^2 + t dt + C$$

$$u = y$$

$$\downarrow u = y' dt$$

$$\downarrow u = y$$

$$2 \int e^{2y} dy + 6 \int e^y dy = \frac{t^3}{3} + \frac{t^2}{2} + C$$

$$2 \cdot \frac{e^{2y}}{2} + 6 e^y = \frac{t^3}{3} + \frac{t^2}{2} + C$$

$$y(0) = 0$$

$$e^{2 \cdot 0} + 6 e^0 = \frac{0}{3} + \frac{0}{2} + C$$

$$1 + 6 = C$$

$$C = 7$$

↓

$$e^{2y} + 6 e^y = \frac{t^3}{3} + \frac{t^2}{2} + 7$$

$$x = e^y \rightarrow y = \ln(x)$$

$$x^2 + 6x - \frac{t^3}{3} - \frac{t^2}{2} - 7 = 0$$

$$x_{1,2} = \frac{-\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - ac}}{a} = -3 \pm \sqrt{\frac{t^3}{3} + \frac{t^2}{2} + 16}$$

$$e^y = -3 \pm \sqrt{\frac{t^3}{3} + \frac{t^2}{2} + 16}$$

$$y(0) = 0$$

$$e^0 = -3 \pm \sqrt{\frac{0}{3} + \frac{0}{2} + 16} = -3 \pm \sqrt{16} = -3 \pm 4 = \begin{cases} -7 & \text{non accettabile} \\ 1 \end{cases}$$

$$y(t) = \ln \left( -3 + \sqrt{\frac{t^3}{3} + \frac{t^2}{2} + 16} \right)$$

↓

$$y(0) = \ln(1) = 0 \quad \checkmark$$

✎ Esercizio 1.1.4. Sia  $y(t)$  la soluzione del problema di Cauchy

$$\begin{cases} y' = \frac{e^{-x}\sqrt{y+1}}{e^{-x}+1} \\ y(0) = 1 \end{cases}$$

$$y' = \frac{e^{-x}\sqrt{y+1}}{e^{-x}+1}$$

$$\frac{y'}{\sqrt{y+1}} = \frac{e^{-x}}{e^{-x}+1}$$

$$\int \frac{y'}{\sqrt{y+1}} dy = \int \frac{e^{-x}}{e^{-x}+1} dx$$

$$y = v$$

$$- \int (y+1)^{-\frac{1}{2}} dy = \int \frac{-e^{-x}}{e^{-x}+1} dx$$

$$y' dy = dv$$

$$\leftarrow v = y$$

$$C + \frac{-(y+1)^{\frac{1}{2}}}{\frac{1}{2}} = \int \frac{F'}{F} dF$$

$$\leftarrow \begin{cases} e^{-x}+1 = F \\ F' = -e^{-x} \end{cases}$$

$$-2\sqrt{y+1} = \int \frac{dF}{F}$$

$$-2\sqrt{y+1} = \ln(F) + C$$

$$\sqrt{y+1} = -\frac{1}{2} \ln(e^{-x}+1) + C$$

$$y = \left( -\frac{1}{2} \ln(e^{-x}+1) + C \right)^2 - 1$$

$$y(0) = 1$$

↓

$$1 = \left( -\frac{1}{2} \ln(2) + C \right)^2 - 1$$

$$2 = \left( -\frac{1}{2} \ln(2) + C \right)^2$$

$$\sqrt{z} = -\frac{1}{2} \ln(z) + c$$

$$c = \sqrt{z} + \frac{1}{2} \ln(z)$$

$$y = \left( -\frac{1}{2} \ln(e^{-x} + 1) + \sqrt{z} + \frac{1}{2} \ln(z) \right)^2 - 1$$

▮ **Esercizio 1.1.5.** Si determini la soluzione  $y(t)$  del seguente problema di Cauchy

$$\begin{cases} y' = (e^{-3y} + 1)(2x - 1) \\ y(0) = -1 \end{cases}$$

$$y' = (e^{-3y} + 1)(2x - 1)$$

$$\frac{y'}{e^{-3y} + 1} = 2x - 1$$

$$\int \frac{y'}{e^{-3y} + 1} dy = \int (2x - 1) dx$$

$$\int \frac{1}{e^{-3y} + 1} dy = 2 \int x dx - \int 1 dx$$

$$\int \frac{1}{e^{-3y} + 1} dy = x^2 - x + C$$

$$\int \frac{1}{e^{-3y} \left(1 + \frac{1}{e^{-3y}}\right)} dy = x^2 - x + C$$

$$\int \frac{e^{3y}}{1 + e^{3y}} dy = x^2 - x + C$$

$$\frac{d(1 + e^{3y})}{dy} = 3e^{3y}$$

$$3 \int \frac{e^{3y}}{1 + e^{3y}} dy = 3(x^2 - x + C)$$



$$c = \ln(e^{-3} + 1)$$

$$y = \frac{\ln(e^{3x^2-3x} \cdot e^{\ln(e^{-3}+1)} - 1)}{3}$$

$$y = \frac{\ln((1+e^{-3})e^{3x^2-3x} - 1)}{3}$$

$$y = \frac{1}{3} \ln((1+e^{-3})e^{3x^2-3x} - 1)$$

✎ **Esercizio 1.1.6.** Si determini la soluzione  $y(t)$  del seguente problema di Cauchy

$$\begin{cases} y' = (3 + 27y^2)(xe^{3x} - 2x^2) \\ y(0) = 0 \end{cases}$$

$$y' = (3 + 27y^2)(xe^{3x} - 2x^2)$$

$$\int \frac{y'}{3 + 27y^2} dy = \int (xe^{3x} - 2x^2) dx$$

$$\int \frac{1}{3 + 27y^2} dy = \int xe^{3x} dx - 2 \int x^2 dx$$

$$\frac{1}{3} \int \frac{1}{1 + 9y^2} dy = \underbrace{\int xe^{3x} dx}_{\downarrow} - 2 \frac{x^3}{3} + C$$

$$\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx$$

$$F: x \quad F': 1$$

$$g': e^{3x} \quad g: \frac{1}{3} e^{3x}$$

$$\frac{1}{3} \int \frac{1}{1 + 9y^2} dy = \frac{1}{3} x e^{3x} - \frac{1}{3} \int e^{3x} dx - 2 \frac{x^3}{3} + C$$

$$\frac{1}{3} \int \frac{1}{1+9y^2} dy = \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} - 2 \frac{x^3}{3} + C$$

$$\int \frac{1}{1+9y^2} dy = x e^{3x} - \frac{1}{3} e^{3x} - 2x^3 + C$$

↓

$$\frac{1}{x^2+a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right)$$

↓

$$\frac{1}{9} \int \frac{1}{y^2 + \frac{1}{9}} dy = x e^{3x} - \frac{1}{3} e^{3x} - 2x^3 + C$$

$$\frac{1}{3} \arctan(3y) = x e^{3x} - \frac{1}{3} e^{3x} - 2x^3 + C$$

$$\arctan(3y) = 3x e^{3x} - e^{3x} - 6x^3 + C$$

$$\arctan(x) = a$$

↓

$$x = \tan(a)$$

$$3y = \tan(3x e^{3x} - e^{3x} - 6x^3 + C)$$

$$y = \frac{1}{3} \tan(3x e^{3x} - e^{3x} - 6x^3 + C)$$

$$\tan(a) = 0$$

$$y(0) = 0$$

↓

$$a = \arctan(0)$$

$$0 = \frac{1}{3} \tan(0 - 1 - 0 + C)$$

$$\tan(-1 + C) = 0$$

$$-1 + C = \arctan(0)$$

$$C = 1$$

$$y = \frac{1}{3} \tan(3x e^{3x} - e^{3x} - 6x^3 + 1)$$



## Equazioni differenziali: lineari di secondo grado

▣ **Esercizio 1.2.1.** Si determini la soluzione  $y(t)$  del seguente problema di Cauchy

$$\begin{cases} y'' - 6y' + 9y = 3t + 2 \\ y(0) = -1 \\ y'(0) = 2 \end{cases}$$

$3t + 2$  è un polinomio di grado 1

$$q_1 = a_0 + a_1 x$$

Sostituisco  $\rightarrow$

$$-6a_1 + 9a_0 + 9a_1 x = 3x + 2$$

$$q_1' = a_1$$

$$-6a_1 + 9a_0 + 9a_1 x = 3x + 2$$

$$q_1'' = 0$$

$$\begin{cases} -6a_1 + 9a_0 = 2 \\ 9a_1 = 3 \end{cases}$$

$$\begin{cases} -2 + 9a_0 = 2 \\ a_1 = \frac{1}{3} \end{cases}$$

$$\begin{cases} a_0 = \frac{4}{9} \\ a_1 = \frac{1}{3} \end{cases}$$

$$\rightarrow y_1 = \frac{4}{9} + \frac{1}{3}t$$

Risolve l'equazione omogenea

$$r'' - 6r' + 9r = 0$$

$$(r-3)^2$$

$$r_{1,2} = 3 \rightarrow y_2 = C_1 e^{3t} + C_2 t e^{3t}$$

$$y(t) = \frac{4}{9} + \frac{1}{3}t + C_1 e^{3t} + C_2 t e^{3t}$$

Applico le condizioni di Cauchy

$$y(0) = \frac{4}{9} + 0 + C_1 + 0 = \frac{4}{9} + C_1 = -1 \rightarrow C_1 = -\frac{13}{9}$$

$$y'(t) = \frac{1}{3} + 3C_1 e^{3t} + C_2 e^{3t} + 3C_2 t e^{3t}$$

$$y'(0) = \frac{1}{3} + 3C_1 + C_2 = 2 \rightarrow \frac{1}{3} + 3 \cdot \left(-\frac{13}{9}\right) + C_2 = 2$$

$$\frac{1}{3} - \frac{13}{3} + C_2 = 2$$

$$1 - 13 + 3C_2 = 6$$

$$3C_2 = 6 + 12$$

$$C_2 = \frac{18}{3} = 6$$

$$C_1 = -\frac{13}{9} \quad C_2 = 6$$

↓

$$y(t) = \frac{4}{9} + \frac{1}{3}t + C_1 e^{3t} + C_2 t e^{3t} = \frac{4}{9} + \frac{1}{3}t - \frac{13}{9} e^{3t} + 6 t e^{3t}$$

▣ **Esercizio 1.2.2.** Sia  $y(t)$  la soluzione del problema di Cauchy

$$\begin{cases} y'' + 2y' - 3y = 0 \\ y(0) = 0 \\ y'(0) = 1. \end{cases}$$

Allora  $\lim_{t \rightarrow +\infty} y(t) =$

- ☐ 0;
- ☐ non esiste;
- ☐  $+\infty$ ;
- ☐  $-\infty$

Risolvere l'equazione caratteristica

$$r^2 + 2r - 3 = 0$$

$$(r+3)(r-1)$$

$$r_1 = -3 \quad r_2 = 1$$

$$y(t) = C_1 e^{-3t} + C_2 e^t$$

$$y'(t) = -3C_1 e^{-3t} + C_2 e^t$$

Impongo le condizioni di Cauchy

$$\begin{cases} y(0) = C_1 + C_2 = 0 \\ y'(0) = -3C_1 + C_2 = 1 \end{cases} \rightarrow \begin{cases} C_1 = -C_2 \\ 3C_2 + C_2 = 1 \end{cases} \rightarrow \begin{cases} C_1 = -\frac{1}{4} \\ C_2 = \frac{1}{4} \end{cases}$$

$$y(t) = -\frac{1}{4}e^{-3t} + \frac{1}{4}e^t$$

$$\lim_{t \rightarrow +\infty} y(t) = -\frac{1}{4}e^{-\infty} + \frac{1}{4}e^{\infty} = 0 + \infty = +\infty$$

La risposta corretta è la terza

✎ **Esercizio 1.2.3.** Si determini la soluzione  $y(t)$  del problema di Cauchy

$$\begin{cases} y'' - y' - 2y = \cos(2t) \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

Risolvero l'equazione omogenea associata

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1)$$

$$r_1 = -1 \quad r_2 = 2 \rightarrow z(t) = c_1 e^{-t} + c_2 e^{2t}$$

Con il metodo di somiglianza cerco una soluzione particolare dell'equazione non omogenea:

$$\bar{y}(t) = \alpha \sin(2t) + \beta \cos(2t)$$

$$\bar{y}'(t) = 2\alpha \cos(2t) - 2\beta \sin(2t)$$

$$\bar{y}''(t) = -4\alpha \sin(2t) - 4\beta \cos(2t)$$

$$y'' - y' - 2y = \cos(2t)$$

↓

$$-4\alpha \sin(2t) - 4\beta \cos(2t) - 2\alpha \cos(2t) + 2\beta \sin(2t) - 2\alpha \sin(2t) - 2\beta \cos(2t) = \cos(2t)$$

$$\sin(2t)(-4\alpha + 2\beta - 2\alpha) + \cos(2t)(-4\beta - 2\alpha - 2\beta) = \cos(2t)$$

$$\sin(2t)(-6\alpha + 2\beta) + \cos(2t)(-6\beta - 2\alpha) = \cos(2t)$$

$$\begin{cases} -6\alpha + 2\beta = 0 \\ -6\beta - 2\alpha = 1 \end{cases} \quad \begin{cases} \beta = 3\alpha \\ -18\alpha - 2\alpha = 1 \end{cases} \quad \begin{cases} \beta = -\frac{3}{20} \\ \alpha = -\frac{1}{20} \end{cases}$$

$$\bar{y}(t) = -\frac{1}{20} \sin(2t) - \frac{3}{20} \cos(2t)$$

$$y(t) = z(t) + \bar{y}(t) = C_1 e^{-t} + C_2 e^{2t} - \frac{1}{20} \sin(2t) - \frac{3}{20} \cos(2t)$$

$$y'(t) = -C_1 e^{-t} + 2C_2 e^{2t} - \frac{1}{10} \cos(2t) + \frac{3}{10} \sin(2t)$$

Applico le condizioni di Cauchy

$$\begin{cases} y(0) = C_1 + C_2 - \frac{3}{20} = 1 \\ y'(0) = -C_1 + 2C_2 - \frac{1}{10} = 0 \end{cases} \rightarrow \begin{cases} C_1 = -C_2 + \frac{23}{20} \\ -C_1 + 2C_2 = \frac{1}{10} \end{cases} \begin{cases} C_1 = -C_2 + \frac{23}{20} \\ C_2 - \frac{23}{20} + 2C_2 = \frac{1}{10} \end{cases}$$

$$\begin{cases} C_1 = -C_2 + \frac{23}{20} \\ 3C_2 = \frac{2}{20} + \frac{23}{20} \end{cases} \begin{cases} C_1 = -\frac{5}{12} + \frac{23}{20} \\ C_2 = \frac{5}{12} \end{cases} \begin{cases} C_1 = \frac{-25+69}{60} = \frac{44}{60} = \frac{11}{15} \\ C_2 = \frac{5}{12} \end{cases}$$

$$y(t) = \frac{11}{15} e^{-t} + \frac{5}{12} e^{2t} - \frac{1}{20} \sin(2t) - \frac{3}{20} \cos(2t)$$

▣ **Esercizio 1.2.4.** Si determini la soluzione  $y(t)$  del problema di Cauchy

$$\begin{cases} y'' - 4y' + 8y = e^{-2t} \\ y(0) = -1 \\ y'(0) = 0. \end{cases}$$

Risolvero l'equazione omogenea associata

$$r^2 - 4r + 8 = 0$$

$$r_{1,2} = \frac{4 \pm \sqrt{16 - 32}}{2} = \frac{4 \pm i\sqrt{16}}{2} = \frac{4 \pm 4i}{2} = 2 \pm 2i$$

$$r_1 = 2 - 2i \quad r_2 = 2 + 2i$$

$$z(t) = C_1 e^{2t} \sin(2t) + C_2 e^{2t} \cos(2t)$$

Bisogna trovare una soluzione particolare del tipo:

$$\bar{y}(t) = e^{-2t} \gamma(t) \quad \begin{matrix} A = 1 \\ \lambda = -2 \end{matrix}$$

$$\downarrow$$

$$\gamma'' + \gamma'(2(-2) - 4) + \gamma(4 + 8 + 8) = 1$$

$$y'' - 3y' + 20y = 1$$

$$\lambda^2 + \lambda a + b \neq 0 \rightarrow y(t) = \cos t + a_1 t + e = \frac{A}{\lambda^2 + \lambda a + b} = \frac{1}{20}$$

$$\bar{y}(t) = \frac{1}{20} e^{-2t}$$

$$y(t) = z(t) + \bar{y}(t) = c_1 e^{2t} \sin(2t) + c_2 e^{2t} \cos(2t) + \frac{1}{20} e^{-2t}$$

$$y'(t) = 2c_1 e^{2t} \sin(2t) + 2c_1 e^{2t} \cos(2t) + 2c_2 e^{2t} \cos(2t) - 2c_2 e^{2t} \sin(2t) - \frac{1}{10} e^{-2t}$$

$$\begin{cases} y(0) = c_2 + \frac{1}{20} = -1 \\ y'(0) = 2c_1 + 2c_2 - \frac{1}{10} = 0 \end{cases} \rightarrow \begin{cases} c_2 = -\frac{21}{20} \\ c_1 = \frac{11}{10} \end{cases}$$

$$y(t) = \frac{11}{10} e^{2t} \sin(2t) - \frac{21}{20} e^{2t} \cos(2t) + \frac{1}{20} e^{-2t}$$

✎ **Esercizio 1.2.5.** Si determini la soluzione  $y(t)$  del problema di Cauchy

$$\begin{cases} y'' - y' - 2y = \sin(2t) \\ y(0) = 0 \\ y'(0) = 1. \end{cases}$$

Risolvo l'equazione omogenea associata

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1)$$

$$r_1 = -1 \quad r_2 = 2$$

$$z(t) = c_1 e^{-t} + c_2 e^{2t}$$

Bisogna trovare una soluzione particolare del tipo:

$$\bar{y}(t) = \alpha \sin(2t) + \beta \cos(2t)$$

$$\bar{y}'(t) = 2\alpha \cos(2t) - 2\beta \sin(2t)$$

$$\bar{y}''(t) = -4\alpha \sin(2t) - 4\beta \cos(2t)$$

$$y'' - y' - 2y = \sin(2t)$$

↓

$$-4\alpha \sin(2t) - 4\beta \cos(2t) - 2\alpha \cos(2t) + 2\beta \sin(2t) - 2\alpha \sin(2t) - 2\beta \cos(2t) = \sin(2t)$$

$$\sin(2t) (-4\alpha + 2\beta - 2\alpha) + \cos(2t) (-4\beta - 2\alpha - 2\beta) = \sin(2t)$$

$$\sin(2t) (-6\alpha + 2\beta) + \cos(2t) (-6\beta - 2\alpha) = \sin(2t)$$

$$\begin{cases} -6\alpha + 2\beta = 1 \\ -6\beta - 2\alpha = 0 \end{cases} \quad \begin{cases} -6\alpha - \frac{2}{3}\alpha = 1 \\ \beta = -\frac{1}{3}\alpha \end{cases} \quad \begin{cases} -\frac{20}{3}\alpha = 1 \\ \beta = -\frac{1}{3}\alpha \end{cases} \quad \begin{cases} \alpha = -\frac{3}{20} \\ \beta = +\frac{1}{20} \end{cases}$$

$$\overline{y}(t) = -\frac{3}{20}\sin(2t) + \frac{1}{20}\cos(2t)$$

$$y(t) = c_1 e^{-t} + c_2 e^{2t} - \frac{3}{20}\sin(2t) + \frac{1}{20}\cos(2t)$$

$$y'(t) = -c_1 e^{-t} + 2c_2 e^{2t} - \frac{3}{10}\cos(2t) - \frac{1}{10}\sin(2t)$$

Impongo le condizioni di Cauchy

$$\begin{cases} y(0) = c_1 + c_2 + \frac{1}{20} = 0 \\ y'(0) = -c_1 + 2c_2 - \frac{3}{10} = 1 \end{cases} \rightarrow \begin{cases} c_1 = -c_2 - \frac{1}{20} \\ c_2 + \frac{1}{20} + 2c_2 - \frac{3}{10} = 1 \end{cases} \quad \begin{cases} c_1 = -c_2 - \frac{1}{20} \\ 3c_2 - \frac{5}{20} = 1 \end{cases}$$

$$\begin{cases} c_1 = -\frac{25}{60} - \frac{1}{20} = \frac{-25-3}{60} = -\frac{28}{60} = -\frac{7}{15} \\ c_2 = \frac{25}{60} = \frac{5}{12} \end{cases}$$

$$y(t) = -\frac{7}{15} e^{-t} + \frac{5}{12} e^{2t} - \frac{3}{20}\sin(2t) + \frac{1}{20}\cos(2t)$$

✎ **Esercizio 1.2.6.** Determinate la soluzione generale dell'equazione differenziale  $y'' - 4y' + 13y = 4x$ .

Risolvo l'equazione omogenea associata

$$r^2 - 4r + 13 = 0$$

$$r_{1,2} = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm i\sqrt{24}}{2} = \frac{4 \pm 4i\sqrt{3}}{2} = 2 \pm 2i\sqrt{3}$$

$$z(t) = C_1 e^{2t} \cos(2\sqrt{3}t) + C_2 e^{2t} \sin(2\sqrt{3}t)$$

✎ **Esercizio 1.2.7.** Determinare la soluzione generale dell'equazione differenziale

$$2y'' + 3y' + 4y = 0.$$

$$2r^2 + 3r + 4 = 0$$

$$r_{1,2} = \frac{-3 \pm \sqrt{9 - 32}}{4} = \frac{-3 \pm i\sqrt{23}}{4} = \frac{-3 \pm 4i\sqrt{7}}{4} = -\frac{3}{4} \pm i\sqrt{7}$$

$$y(t) = C_1 e^{-\frac{3}{4}t} \cos(\sqrt{7}t) + C_2 e^{-\frac{3}{4}t} \sin(\sqrt{7}t)$$

✎ **Esercizio 1.2.8.** Si risolva il seguente problema di Cauchy:

$$y'' + 6y' + 8y = e^{4t} + t^2, \quad y(1) = 2, \quad y'(1) = 3.$$

Risolvo l'equazione omogenea associata

$$r^2 + 6r + 8 = 0$$

$$(r+4)(r+2)$$

$$r_1 = -4 \quad r_2 = -2$$

$$z(t) = C_1 e^{-4t} + C_2 e^{-2t}$$

Bisogna trovare una soluzione particolare:

$$\bar{y}_1(t) = e^{\lambda t} \gamma(t) \quad \lambda = 4 \quad A = 1$$

↓

$$\lambda^2 + \lambda a + b = 16 + 4 + 8 = 28 \neq 0 \rightarrow \gamma = \text{costante} = \frac{A}{\lambda^2 + \lambda a + b} = \frac{1}{28}$$

$$\bar{y}_1(t) = \frac{1}{26} e^{4t}$$

$$\bar{y}_2(t) = a_0 + a_1 x + a_2 x^2 = t^2$$

$$\bar{y}'_2(t) = a_1 + a_2 x$$

$$\bar{y}''_2(t) = a_2$$

Sostituisco

$$y'' + 6y' + 8y = t^2$$

$$a_2 + 6a_1 + 8a_2 x + 8a_0 + 8a_1 x + 8a_2 x^2 = t^2$$

$$a_2 + 6a_1 + 8a_0 + x(6a_2 + 8a_1) + 8a_2 x^2 = t^2$$

$$\begin{cases} a_2 + 6a_1 + 8a_0 = 0 \\ 6a_2 + 8a_1 = 0 \\ a_2 = \frac{1}{8} \end{cases} \quad \begin{cases} \frac{1}{8} - \frac{18}{32} + 8a_0 = 0 \\ a_1 = -\frac{3}{32} \\ a_2 = \frac{1}{8} \end{cases} \quad \begin{cases} a_0 = \frac{7}{128} \\ a_1 = -\frac{3}{32} \\ a_2 = \frac{1}{8} \end{cases}$$

$$\bar{y}_2(t) = \frac{7}{128} - \frac{3}{32} x + \frac{1}{8} x^2$$

$$\bar{y}(t) = \bar{y}_1 + \bar{y}_2 = \frac{1}{26} e^{4t} + \frac{7}{128} - \frac{3}{32} x + \frac{1}{8} x^2$$

▣ **Esercizio 1.2.10.** Si determini la soluzione  $y(t)$  del problema di Cauchy

$$\begin{cases} y'' + y' - 2y = -e^x \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Risolviamo l'equazione omogenea associata

$$r^2 + r - 2 = 0$$

$$(r-1)(r+2)$$

$$r_1 = -2 \quad r_2 = 1$$

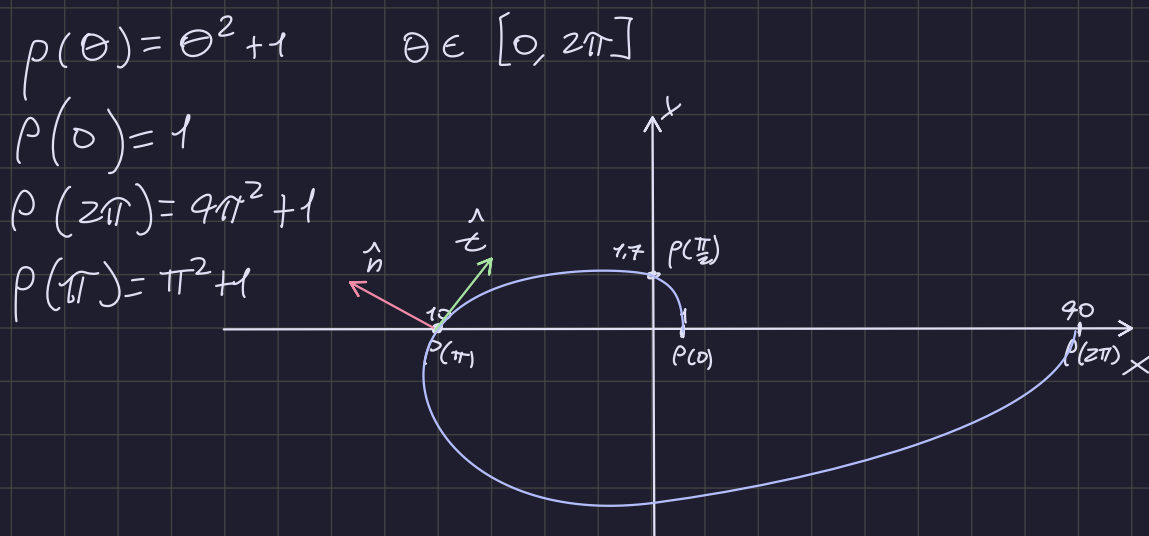
$$z(x) = c_1 e^{-2x} + c_2 e^x$$





## Calcolo infinitesimale per le curve

▣ **Esercizio 2.1.1.** Sia  $\gamma$  la curva piana la cui parametrizzazione in coordinate polari è  $\rho(\vartheta) = \vartheta^2 + 1$ , on  $0 \leq \vartheta \leq 2\pi$ . Dopo aver disegnato sommariamente il sostegno di  $\gamma$ , determinare i versori tangente e normale al sostegno di  $\gamma$  nel punto  $\gamma(\pi)$  e scrivere un'equazione della retta tangente nello stesso punto.



Trasformiamo in coordinate cartesiane

$$\begin{cases} x(\theta) = \rho(\theta) \cos \theta = (\theta^2 + 1) \cos \theta \\ y(\theta) = \rho(\theta) \sin \theta = (\theta^2 + 1) \sin \theta \end{cases}$$

$$\gamma(\theta) = (x(\theta), y(\theta))$$

$$\frac{1}{t} = \frac{\gamma'(\theta)}{\|\gamma'(\theta)\|}$$

$$\gamma'(\theta) = (2\theta \cos \theta - (\theta^2 + 1) \sin \theta, 2\theta \sin \theta + (\theta^2 + 1) \cos \theta)$$

$$\|\gamma'(\theta)\| = \sqrt{x'(\theta)^2 + y'(\theta)^2}$$

$$= \sqrt{(2\theta \cos \theta - (\theta^2 + 1) \sin \theta)^2 + (2\theta \sin \theta + (\theta^2 + 1) \cos \theta)^2}$$

$$= \sqrt{4\theta^2 \cos^2 \theta + (\theta^2 + 1)^2 \sin^2 \theta - 4\theta \cos \theta (\theta^2 + 1) \sin \theta}$$

$$+ 4\theta^2 \sin^2 \theta + (\theta^2 + 1)^2 \cos^2 \theta + 4\theta \sin \theta (\theta^2 + 1) \cos \theta$$

$$= \sqrt{4\theta^2 (\underbrace{\cos^2 \theta + \sin^2 \theta}_{=1}) + (\theta^2 + 1)^2 (\underbrace{\sin^2 \theta + \cos^2 \theta}_{=1})}$$

$$= \sqrt{4\theta^2 + (\theta^2 + 1)^2}$$

$$\hat{t}(\theta) = \frac{\gamma'(\theta)}{\|\gamma'(\theta)\|} = \frac{(2\theta \cos \theta - (\theta^2 + 1) \sin \theta, 2\theta \sin \theta + (\theta^2 + 1) \cos \theta)}{\sqrt{4\theta^2 + (\theta^2 + 1)^2}}$$

$$\begin{aligned} \hat{t}(\pi) &= \frac{\gamma'(\pi)}{\|\gamma'(\pi)\|} = \frac{(2\pi \cos \pi - (\pi^2 + 1) \sin \pi, 2\pi \sin \pi + (\pi^2 + 1) \cos \pi)}{\sqrt{4\pi^2 + (\pi^2 + 1)^2}} \\ &= \frac{(-2\pi, -(\pi^2 + 1))}{\sqrt{4\pi^2 + (\pi^2 + 1)^2}} \end{aligned}$$

La normale è semplicemente la tangente ruotata di  $90^\circ$ , e ciò equivale alla moltiplicazione della tangente con una matrice di rotazione:

$$\begin{aligned} \hat{n}(\theta) &= \hat{t}(\theta) \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &\parallel \\ &\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \end{aligned}$$

Retta tangente:

$$r_T = \gamma(\pi) + \gamma'(\pi) t$$

$$\begin{cases} x(t) = -(\pi^2 + 1) - t \cdot 2\pi = -t \cdot 2\pi - (\pi^2 + 1) \\ y(t) = 0 \quad -t(\pi^2 + 1) = -t(\pi^2 + 1) \end{cases}$$

$$y = -t(\pi^2 + 1)$$

$$x = -t \cdot 2\pi - (\pi^2 + 1)$$

$$-t \cdot 2\pi = x + (\pi^2 + 1)$$

$$t = \frac{x + (\pi^2 + 1)}{-2\pi}$$

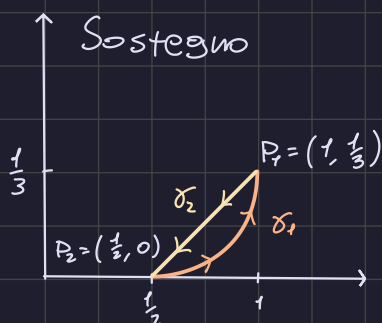
$$y = \frac{x + (\pi^2 + 1)}{2\pi} (\pi^2 + 1)$$

$$y = \frac{\pi^2 + 1}{2\pi} x + \frac{(\pi^2 + 1)^2}{2\pi}$$

▮ **Esercizio 2.1.2.** Determinare una parametrizzazione della curva chiusa  $\gamma$  che si ottiene percorrendo prima da sinistra verso destra il grafico di  $f(x) = (1/3)(2x - 1)^{3/2}$  per  $1/2 \leq x \leq 1$  e poi da destra a sinistra il segmento congiungente gli estremi del grafico di  $f$  stessa. Disegnare quindi il sostegno di  $\gamma$  e calcolarne la lunghezza.

$$\gamma_1 = \frac{1}{3} (2x - 1)^{\frac{3}{2}} \quad \gamma_1' = (2x - 1)^{\frac{1}{2}} \quad \gamma_1'' = (2x - 1)^{-\frac{1}{2}} = \frac{1}{\sqrt{2x - 1}} > 0 \quad \text{concavo verso l'alto}$$

$$\gamma_1\left(\frac{1}{2}\right) = 0 \quad \gamma_1(1) = \frac{1}{3}$$



$$\gamma_1(t) = \left( t, \frac{1}{3} (2t - 1)^{\frac{3}{2}} \right) \quad t \in \left[ \frac{1}{2}, 1 \right]$$

$$\begin{aligned} \gamma_2(t) &= (1-t)P_1 + tP_2 \\ &= (1-t) \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix} + t \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \\ &= \left( 1-t + \frac{1}{2}t, \frac{1}{3}(1-t) \right) \\ &= \left( 1 - \frac{1}{2}t, \frac{1}{3}(1-t) \right) \quad t \in [0, 1] \end{aligned}$$

La parametrizzazione per  $t$  non è coerente, quindi cerco una parametrizzazione che vada da 0 a 1 per rappresentare entrambi i pezzi della curva come una sola curva, quindi da  $[0, 1/2)$  per la prima e da  $[1/2, 1]$  per la seconda

$$\gamma_1(t) = \left( t, \frac{1}{3} (2t - 1)^{\frac{3}{2}} \right) \quad t \in \left[ \frac{1}{2}, 1 \right]$$

$$\downarrow \\ t = A s + B$$



$$\begin{cases} \frac{1}{2} = A \cdot 0 + B \\ 1 = A \cdot \frac{1}{2} + B \end{cases}$$

$$\begin{cases} B = \frac{1}{2} \\ \frac{1}{2}A = 1 - \frac{1}{2} \end{cases}$$

$$\begin{cases} B = \frac{1}{2} \\ A = 1 \end{cases}$$

$$t = s + \frac{1}{2}$$

$$\gamma_1(s) = \left( s + \frac{1}{2}, \frac{1}{3} \left( 2 \left( s + \frac{1}{2} \right) - 1 \right)^{\frac{3}{2}} \right)$$

$$= \left( s + \frac{1}{2}, \frac{1}{3} (2s)^{\frac{3}{2}} \right) \quad s \in \left[ 0, \frac{1}{2} \right] \xrightarrow{\text{controllo}}$$

$$\left\{ \begin{aligned} \gamma_1(0) &= \left( \frac{1}{2}, \frac{1}{3} \left( 2 \cdot \frac{1}{2} \right)^{\frac{3}{2}} \right) \\ &= \left( \frac{1}{2}, 0 \right) \\ \gamma_1\left(\frac{1}{2}\right) &= \left( \frac{1}{2} + \frac{1}{2}, \frac{1}{3} \left( 2 \cdot \frac{1}{2} \right)^{\frac{3}{2}} \right) \\ &= \left( 1, \frac{1}{3} \right) \end{aligned} \right.$$

La parametrizzazione è corretta

$$\gamma_2(t) = \left( 1 - \frac{1}{2}t, \frac{1}{3}(1-t) \right) \quad t \in [0, 1]$$

$$\downarrow$$

$$t = As + B$$

$$\begin{cases} 0 = A \cdot \frac{1}{2} + B \\ 1 = A \cdot 1 + B \end{cases} \quad \begin{cases} B = -\frac{1}{2}A \\ A - \frac{1}{2}A = 1 \end{cases} \quad \begin{cases} B = -\frac{1}{2}A \\ \frac{1}{2}A = 1 \end{cases} \quad \begin{cases} B = -1 \\ A = 2 \end{cases} \quad t = 2s - 1$$

$$\gamma_2(s) = \left( 1 - \frac{1}{2}(2s-1), \frac{1}{3}(1-(2s-1)) \right)$$

$$= \left( \frac{3}{2} - s, \frac{1}{3}(-2s+2) \right) \quad s \in \left[ \frac{1}{2}, 1 \right] \xrightarrow{\text{controllo}}$$

$$\left\{ \begin{aligned} \gamma_2\left(\frac{1}{2}\right) &= \left( \frac{3}{2} - \frac{1}{2}, \frac{1}{3} \left( -2 \cdot \frac{1}{2} + 2 \right) \right) \\ &= \left( 1, \frac{1}{3} \right) \\ \gamma_2(1) &= \left( \frac{3}{2} - 1, \frac{1}{3}(-2+2) \right) \\ &= \left( \frac{1}{2}, 0 \right) \end{aligned} \right.$$

La parametrizzazione è corretta

$$\gamma(s) = \begin{cases} \gamma_1 & \text{se } s \in \left[ 0, \frac{1}{2} \right] \\ \gamma_2 & \text{se } s \in \left[ \frac{1}{2}, 1 \right] \end{cases} = \begin{cases} \left( s + \frac{1}{2}, \frac{1}{3} (2s)^{\frac{3}{2}} \right) & s \in \left[ 0, \frac{1}{2} \right] \\ \left( \frac{3}{2} - s, \frac{1}{3} (-2s+2) \right) & s \in \left[ \frac{1}{2}, 1 \right] \end{cases}$$

<https://www.desmos.com/calculator/4wdnbtbbdt>

La lunghezza di gamma è calcolata come:

$$\mathcal{L}(\gamma(s)) = \int_0^1 \|\gamma'(s)\| ds$$

Perché una curva composta da più curve rettificabili che soddisfano la condizione di raccordo è rettificabile

$$= \mathcal{L}(\gamma_1(s)) + \mathcal{L}(\gamma_2(s)) = \int_0^{\frac{1}{2}} \|\gamma_1'(s)\| ds + \int_{\frac{1}{2}}^1 \|\gamma_2'(s)\| ds$$

$$\gamma_1'(s) = \left( 1, \frac{1}{3} \cdot \frac{3}{2} (2s)^{\frac{1}{2}} \cdot 2 \right) = \left( 1, (2s)^{\frac{1}{2}} \right)$$

$$\gamma_2'(s) = \left( -1, -\frac{2}{3} \right)$$

$$L(\gamma_1(s)) = \int_0^{\frac{1}{2}} \|\gamma_1'(s)\| ds =$$

$$= \int_0^{\frac{1}{2}} \sqrt{1^2 + (2s)^{\frac{1}{2} \cdot 2}} ds$$

$$= \int_0^{\frac{1}{2}} (1+2s)^{\frac{1}{2}} ds$$

$$= \left[ \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^{\frac{1}{2}} = \left[ \frac{2}{3} x^{\frac{3}{2}} \right]_0^{\frac{1}{2}}$$

$$= \frac{2}{3} \left( \frac{1}{2} \right)^{\frac{3}{2}} + 0$$

$$= \frac{2}{3} \sqrt[3]{\frac{1}{4}}$$

$$L(\gamma_2(s)) = \int_{\frac{1}{2}}^1 \|\gamma_2'(s)\| ds$$

$$= \int_{\frac{1}{2}}^1 \sqrt{(-1)^2 + \left(-\frac{2}{3}\right)^2} ds$$

$$= \sqrt{1 + \frac{4}{9}}$$

$$= \sqrt{\frac{13}{9}}$$

$$= \frac{\sqrt{13}}{3}$$

$$L(\gamma(s)) = L(\gamma_1(s)) + L(\gamma_2(s)) = \frac{2}{3} \sqrt[3]{\frac{1}{4}} + \frac{\sqrt{13}}{3}$$

✎ **Esercizio 2.1.3.** Data la curva  $\gamma$  avente equazione in coordinate polari  $\rho = 2\theta^2$  con  $-\pi \leq \theta \leq \pi$ , determinate la lunghezza di  $\gamma$ ; determinate poi un versore tangente alla curva nel punto corrispondente a  $\theta = \varepsilon$  e calcolate il limite per  $\varepsilon \rightarrow 0^+$  di questo versore.

$$\rho(\theta) = 2\theta^2 \quad \theta \in [-\pi, \pi]$$

$$L(\rho(\theta)) = \int_{-\pi}^{\pi} \sqrt{\rho'(\theta)^2 + \rho(\theta)^2} \, d\theta$$

$$= \int_{-\pi}^{\pi} \sqrt{(4\theta)^2 + (2\theta^2)^2} \, d\theta$$

$$= \int_{-\pi}^{\pi} \sqrt{16\theta^2 + 4\theta^4} \, d\theta$$

$$= \int_{-\pi}^{\pi} \sqrt{4\theta^2(4 + \theta^2)} \, d\theta$$

$$= \int_{-\pi}^{\pi} 2|\theta|(4 + \theta^2)^{\frac{1}{2}} \, d\theta$$

$$t = 4 + \theta^2$$

$$dt = 2\theta \, d\theta$$

$$= 2 \int_0^{\pi} 2\theta (4 + \theta^2)^{\frac{1}{2}} \, d\theta$$

$$= 2 \int (t)^{\frac{1}{2}} \, dt$$

$$= \frac{4}{3} t^{\frac{3}{2}} = \left[ \frac{4}{3} (4 + \theta^2)^{\frac{3}{2}} \right]_0^{\pi}$$

$$= \frac{4}{3} (4 + \pi^2)^{\frac{3}{2}} - \left( \frac{4}{3} (4)^{\frac{3}{2}} \right)$$

$$= \frac{4}{3} (4 + \pi^2)^{\frac{3}{2}} - \frac{4}{3} (4)^{\frac{3}{2}}$$

$$= \frac{4}{3} (4 + \pi^2)^{\frac{3}{2}} - \frac{4}{3} \sqrt{4^2 \cdot 4}$$

$$= \frac{4}{3} (4 + \pi^2)^{\frac{3}{2}} - \frac{4}{3} 4 \cdot 2$$

$$= \frac{4}{3} (4 + \pi^2)^{\frac{3}{2}} - \frac{32}{3}$$

Per trovare il versore tangente bisogna calcolare la derivata nel punto  $\varepsilon$  in coordinate cartesiane

$$\rho(t) = (2t^2 \cos t, 2t^2 \sin t)$$

$$\rho'(t) = (4t \cos t - 2t^2 \sin t, 4t \sin t + 2t^2 \cos t)$$

$$\rho'(\varepsilon) = (4\varepsilon \cos \varepsilon - 2\varepsilon^2 \sin \varepsilon, 4\varepsilon \sin \varepsilon + 2\varepsilon^2 \cos \varepsilon)$$

Per ottenere il versore bisogna normalizzare

$$\hat{\rho}'(\varepsilon) = \frac{\rho'(\varepsilon)}{\|\rho'(\varepsilon)\|} = \frac{(4\varepsilon \cos \varepsilon - 2\varepsilon^2 \sin \varepsilon, 4\varepsilon \sin \varepsilon + 2\varepsilon^2 \cos \varepsilon)}{\sqrt{16\varepsilon^2 + 4\varepsilon^4}}$$

$$= \frac{(4\varepsilon \cos \varepsilon - 2\varepsilon^2 \sin \varepsilon, 4\varepsilon \sin \varepsilon + 2\varepsilon^2 \cos \varepsilon)}{2|\varepsilon|(4 + \varepsilon^2)^{\frac{1}{2}}}$$

$$= \left( \frac{4\varepsilon \cos \varepsilon - 2\varepsilon^2 \sin \varepsilon}{2|\varepsilon|(4 + \varepsilon^2)^{\frac{1}{2}}}, \frac{4\varepsilon \sin \varepsilon + 2\varepsilon^2 \cos \varepsilon}{2|\varepsilon|(4 + \varepsilon^2)^{\frac{1}{2}}} \right)$$

se  $\varepsilon > 0$

$$= \left( \frac{\cancel{4}\varepsilon \cos \varepsilon - 2\varepsilon^2 \sin \varepsilon}{\cancel{2}\varepsilon (4 + \varepsilon^2)^{\frac{1}{2}}}, \frac{\cancel{4}\varepsilon \sin \varepsilon + 2\varepsilon^2 \cos \varepsilon}{\cancel{2}\varepsilon (4 + \varepsilon^2)^{\frac{1}{2}}} \right)$$

$$= \left( \frac{2 \cos \varepsilon - 2\varepsilon \sin \varepsilon}{(4 + \varepsilon^2)^{\frac{1}{2}}}, \frac{2 \sin \varepsilon + 2\varepsilon \cos \varepsilon}{(4 + \varepsilon^2)^{\frac{1}{2}}} \right)$$

$$\lim_{\varepsilon \rightarrow 0^+} \hat{\rho}'(\varepsilon) = \left( \frac{2 \cos 0 - 2 \cdot 0 \sin 0}{(4 + 0^2)^{\frac{1}{2}}}, \frac{2 \sin 0 + 2 \cdot 0 \cos 0}{(4 + 0^2)^{\frac{1}{2}}} \right)$$

$$= \left( \frac{2 \cdot 1}{2}, \frac{0}{(4 + 0^2)^{\frac{1}{2}}} \right) = (1, 0)$$



🔗 **Esercizio 2.1.4.** Data la curva  $\gamma$  parametrizzata da  $(e^t \cos t, e^t \sin t)$  con  $-2\pi \leq t \leq 2\pi$ , determinate la lunghezza di  $\gamma$ ; determinate poi la retta tangente alla curva nel punto corrispondente a  $t = 0$ .

<https://www.desmos.com/3d/nc44fxps9g>

$$\gamma(t) = (e^t \cos t, e^t \sin t) \quad t \in [-2\pi, 2\pi]$$

$$\gamma'(t) = (e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t)$$

$$= (e^t (\cos t - \sin t), e^t (\sin t + \cos t))$$

$$L(\gamma(t)) = \int_{-2\pi}^{2\pi} \|\gamma'(t)\| dt$$

$$= \int \sqrt{e^{2t} (\cos t - \sin t)^2 + e^{2t} (\sin t + \cos t)^2} dt$$

$$= \int \sqrt{e^{2t} (\underbrace{\cos^2 t + \sin^2 t}_{=1} - 2 \cos t \sin t) + e^{2t} (\underbrace{\sin^2 t + \cos^2 t}_{=1} + 2 \cos t \sin t)} dt$$

$$= \int \sqrt{e^{2t} (-\sin(2x) + \sin(2x) + 2)} dt$$

$$= \int_{-2\pi}^{2\pi} e^t \sqrt{2} dt$$

$$= \sqrt{2} \int_{-2\pi}^{2\pi} e^t dt$$

$$= \sqrt{2} [e^t]_{-2\pi}^{2\pi}$$

$$= \sqrt{2} (e^{2\pi} - e^{-2\pi})$$

Trovo il vettore tangente nel punto 0

$$\gamma'(0) = (e^0 (\cos 0 - \sin 0), e^0 (\sin 0 + \cos 0)) = (1, 1)$$

La retta tangente è quella retta traslata nel punto  $\gamma(0)$  e scalata per  $t$  volte il vettore tangente

$$r_t = \gamma(0) + t \gamma'(0)$$

$$\gamma(0) = (e^0 \cos 0, e^0 \sin 0) = (1, 0)$$

$$r_t = (1, 0) + t(1, 1)$$

$$= (1+t, t) \quad t \in (-\infty, +\infty)$$

oppure

$$\begin{cases} x = 1+t \\ y = t \end{cases} \quad t = x-1 \rightarrow y = x-1$$

**Esercizio 2.1.5.** Data la curva la cui equazione in coordinate polari è  $\rho = 2\theta$ , determinare un vettore tangente alla curva nel punto che corrisponde a  $\theta = \frac{\pi}{2}$  e scrivere l'equazione cartesiana della retta tangente nello stesso punto.

<https://www.desmos.com/3d/z0do8Lrwth>

$$\rho(\theta) = 2\theta$$

$$\rho(\theta) = (2\theta \cos \theta, 2\theta \sin \theta)$$

$$\rho'(\theta) = (2 \cos \theta - 2\theta \sin \theta, 2 \sin \theta + 2\theta \cos \theta)$$

$$\rho'\left(\frac{\pi}{2}\right) = \left(2 \cos \frac{\pi}{2} - 2 \frac{\pi}{2} \sin \frac{\pi}{2}, 2 \sin \frac{\pi}{2} + 2 \frac{\pi}{2} \cos \frac{\pi}{2}\right)$$

$$= (-\pi, 2)$$

$$\rho\left(\frac{\pi}{2}\right) = \left(2 \frac{\pi}{2} \cos \frac{\pi}{2}, 2 \frac{\pi}{2} \sin \frac{\pi}{2}\right)$$

$$= (0, \pi)$$

$$r_t(t) = \rho\left(\frac{\pi}{2}\right) + t \rho'\left(\frac{\pi}{2}\right)$$

$$= (0, \pi) + t(-\pi, 2)$$

$$= (-\pi t, \pi + 2t)$$

$$= \begin{cases} x = -\pi t \\ y = \pi + 2t \end{cases}$$

$$\downarrow \\ t = -\frac{1}{\pi}x \rightarrow y = \pi - \frac{2}{\pi}x$$

✎ **Esercizio 2.1.6.** Si calcoli la lunghezza  $l_\gamma$  della curva

$$\underline{\gamma}(t) = \left( \frac{2+3t}{8t}, 2t-1, \ln(t) \right), \quad \frac{1}{2} \leq t \leq 2.$$

Si calcolino inoltre le equazioni della retta  $r$  tangente alla curva nel punto  $\underline{\gamma}(1)$  e del piano  $\pi$  perpendicolare alla curva nello stesso punto.

<https://www.desmos.com/3d/8hrqnxhkqm>

$$\gamma(t) = \left( \frac{2+3t}{8t}, 2t-1, \ln(t) \right) \quad t \in \left[ \frac{1}{2}, 2 \right]$$

$$\gamma'(t) = \left( \frac{3 \cdot 8t - 8 \cdot (2+3t)}{(8t)^2}, 2, \frac{1}{t} \right)$$

$$\gamma(1) = \left( \frac{2+3}{8}, 2-1, \ln(e^0) \right)$$

$$= \left( \frac{24t - 16 - 24t}{64t^2}, 2, \frac{1}{t} \right)$$

$$= \left( \frac{5}{8}, 1, 0 \right)$$

$$= \left( -\frac{1}{4t^2}, 2, \frac{1}{t} \right)$$

$$\gamma'(1) = \left( -\frac{1}{4}, 2, 1 \right)$$

$$r_t = \gamma(1) + t \gamma'(1)$$

$$= \begin{pmatrix} \frac{5}{8} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{4} \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{8} \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{4}t \\ 2t \\ 1t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{8} - \frac{1}{4}t \\ 1 + 2t \\ t \end{pmatrix}$$

Calcoliamo il piano normale alla tangente nel punto 1

$$\Pi = \left\langle \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \gamma'(1) \right\rangle = \left\langle \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \begin{pmatrix} -\frac{1}{4} \\ 2 \\ 1 \end{pmatrix} \right\rangle = 0$$

$$-\frac{1}{4}s_1 + 2s_2 + s_3 = 0$$

$$s_1 = 8s_2 + 4s_3$$

$$s_2 = U$$

$$s_3 = V$$

$$\begin{pmatrix} 8U+4V \\ U \\ V \end{pmatrix} = U \begin{pmatrix} 8 \\ 1 \\ 0 \end{pmatrix} + V \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

$$\Pi = \gamma(1) + K \begin{pmatrix} 0 \\ 1 \\ \frac{1}{4} \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{5}{8} \\ 1 \\ 0 \end{pmatrix} + K \begin{pmatrix} 0 \\ 1 \\ \frac{1}{4} \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

✎ **Esercizio 2.2.1.** Parametrize il tratto del grafico della funzione  $e^x$  compreso tra  $x = 0$  e  $x = 1$ ; detta  $\gamma$  tale curva, calcolate l'integrale su  $\gamma$  di  $f(x, y) = ye^x$ ; calcolate infine la lunghezza di  $\gamma$ .

$$\gamma(t) = (t, e^t) \quad t \in [0, 1] \quad f(x, y) = ye^x$$

$$\int_{\gamma} f(x, y) ds = \int_0^1 f(t, e^t) \|\gamma'(t)\| dt$$

$$\gamma'(t) = (1, e^t) \quad \|\gamma'(t)\| = \sqrt{1 + e^{2t}}$$

$$= \int_0^1 e^t e^t \sqrt{1 + e^{2t}} dt$$

$$= \int_0^1 e^{2t} \sqrt{1 + e^{2t}} dt$$

$$e^{2t} = u$$

$$du = 2e^{2t} dt$$

$$= \frac{1}{2} \int_0^1 \sqrt{1+u} du$$

$$= \frac{1}{2} \int (1+u)^{\frac{1}{2}} du$$

$$= \frac{1}{2} \left[ \frac{2}{3} (1+u)^{\frac{3}{2}} \right]$$

$$= \left[ \frac{1}{3} (1+e^{2t})^{\frac{3}{2}} \right]_0^1$$

$$= \frac{1}{3} (1+e^2)^{\frac{3}{2}} - \frac{1}{3} (2)^{\frac{3}{2}}$$

$$= \frac{1}{3} \left( \sqrt{(1+e^2)^2} - \sqrt{8} \right)$$

$$L(\gamma(t)) = \int_0^1 \|\gamma'(t)\| dt$$

$$= \int_0^1 \sqrt{1+e^{2t}} dt$$

$$= \int_0^1 (1+e^{2t})^{\frac{1}{2}} dt$$

$$(1+e^{2t})^{\frac{1}{2}} = u$$

$$= \int \downarrow u \cdot \left( \frac{1}{u-1} \cdot \frac{1}{e^{\ln(u^2-1)}} \right) du$$

$$1+e^{2t} = u^2$$

$$e^{2t} = u^2 - 1$$

$$= \int u \cdot \frac{u}{u^2-1} du$$

$$2t = \ln(u^2-1)$$

$$= \int \frac{u^2}{u^2-1} du$$

$$du = (1+e^{2t})^{-\frac{1}{2}} e^{2t} dt$$

$$= \int \frac{u^2-1+1}{u^2-1} du$$

$$= \int 1 + \frac{1}{u^2-1} du$$

$$= u + \int \frac{1}{u^2-1} du$$

$$* = u + \int \frac{1}{(u+1)(u-1)} du$$

$$\frac{1}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} = \frac{Au-A+Bu+B}{(u+1)(u-1)} = \frac{u(A+B)-A+B}{(u+1)(u-1)}$$

$$\begin{cases} A+B=0 \\ -A+B=1 \end{cases}$$

$$\begin{cases} A=-B \\ 2B=1 \end{cases}$$

$$\begin{cases} A=-\frac{1}{2} \\ B=\frac{1}{2} \end{cases}$$

$$\frac{1}{u^2-1} = -\frac{1}{2} \frac{1}{u+1} + \frac{1}{2} \frac{1}{u-1}$$

$$* = u - \frac{1}{2} \int \frac{1}{u+1} du + \frac{1}{2} \int \frac{1}{u-1} du$$

$$\begin{aligned}
& \approx u - \frac{1}{2} \ln(u+1) + \frac{1}{2} \ln(u-1) \\
& = u + \frac{1}{2} \ln\left(\frac{u-1}{u+1}\right) \\
& = \left[ \sqrt{1+e^{2t}} + \frac{1}{2} \ln\left(\frac{\sqrt{1+e^{2t}}-1}{\sqrt{1+e^{2t}}+1}\right) \right]_0^1 \\
& = \sqrt{1+e^2} + \frac{1}{2} \ln\left(\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}\right) - \left( \sqrt{2} + \frac{1}{2} \ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) \right) \\
& = \sqrt{1+e^2} + \frac{1}{2} \ln\left(\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}\right) - \sqrt{2} - \frac{1}{2} \ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)
\end{aligned}$$

▮ **Esercizio 2.2.2.** Data la curva  $\gamma$  parametrizzata da  $\Phi(t) = (t \cos 2t, -t \sin 2t)$ , determinate la retta tangente alla curva nel punto che corrisponde a  $t = 0$  e calcolate l'integrale della funzione  $f(x, y) = \sqrt{x^2 + y^2}$  sulla parte di curva con  $-1 \leq t \leq 1$ .

<https://www.desmos.com/3d/w0f1mliadk>

$$\Phi(t) = (t \cos(2t), -t \sin(2t))$$

$$\Phi'(t) = (\cos(2t) - 2t \sin(2t), -\sin(2t) - 2t \cos(2t))$$

$$\Phi'(0) = (1, -2)$$

$$\|\Phi'(t)\| = \sqrt{(\cos(2t) - 2t \sin(2t))^2 + (-\sin(2t) - 2t \cos(2t))^2}$$

$$= \sqrt{\cos^2(2t) + 4t^2 \sin^2(2t) - 4t \cos(2t) \sin(2t) + \sin^2(2t) + 4t^2 \cos^2(2t) + 4t \sin(2t) \cos(2t)}$$

$$= \sqrt{\cos^2(2t) + 4t^2 \sin^2(2t) + \sin^2(2t) + 4t^2 \cos^2(2t)}$$

$$= \sqrt{4t^2 (\underbrace{\sin^2(2t) + \cos^2(2t)}_{=1}) + \underbrace{\cos^2(2t) + \sin^2(2t)}_{=1}}$$

$$= \sqrt{4t^2 + 1}$$

$$\begin{aligned}
\int_{\gamma} F(x, y) ds &= \int_{-1}^1 F(\Phi(t)) \|\Phi'(t)\| dt \\
&= \int_{-1}^1 \sqrt{t^2 \cos^2(2t) + t^2 \sin^2(2t)} \cdot \sqrt{4t^2 + 1} dt \\
&= \int_{-1}^1 |t| \sqrt{\cos^2(2t) + \sin^2(2t)} \cdot \sqrt{4t^2 + 1} dt \\
&= 2 \int_0^1 t \cdot \sqrt{4t^2 + 1} dt && 4t^2 + 1 = u \\
&= \frac{1}{4} \int u^{\frac{1}{2}} du && du = 8t \\
&= \frac{1}{4} \cdot \left[ \frac{2}{3} u^{\frac{3}{2}} \right] \\
&= \left[ \frac{1}{6} \sqrt{(4t^2 + 1)^3} \right]_0^1 \\
&= \frac{1}{6} \sqrt{5^3} - \left( \frac{1}{12} \right) \\
&= \frac{1}{6} (5\sqrt{5} - 1)
\end{aligned}$$

$$(t \cos(2t), -t \sin(2t))$$

$$\begin{cases} x = t \cos(2t) \\ y = -t \sin(2t) \end{cases}$$

$$z = \sqrt{x^2 + y^2} = \sqrt{t^2 \cos^2(2t)}$$



▣ **Esercizio 2.2.3.** Considerate la curva  $\gamma$  parametrizzata da  $(\sin t, t, 1)$  con  $0 \leq t \leq 2\pi$ ; determinare il vettore tangente a  $\gamma$  in ciascuno dei punti corrispondenti a  $t = 0$ ,  $t = \frac{\pi}{2}$ ,  $t = \pi$ , disegnate accuratamente  $\gamma$  e calcolate l'integrale su  $\gamma$  della funzione  $xyz\sqrt{1+\cos^2 y}$ .

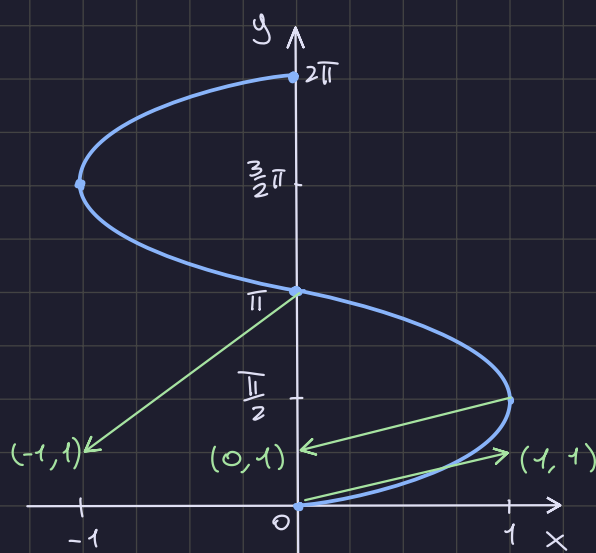
<https://www.desmos.com/3d/2mg7et4np5>

$$\gamma(t) = (\sin t, t, 1) \quad t \in [0, 2\pi]$$

$$\gamma'(t) = (\cos t, 1, 0) \quad \gamma'(0) = (1, 1, 0)$$

$$\gamma'\left(\frac{\pi}{2}\right) = (0, 1, 0)$$

$$\gamma'(\pi) = (-1, 1, 0)$$



$$F(x, y, z) = xyz\sqrt{1+\cos^2 y}$$

$$\|\gamma'(t)\| = \sqrt{\cos^2 t + 1}$$

$$\int_{\gamma} F(x, y, z) \, ds = \int_0^{2\pi} F(\gamma(t)) \|\gamma'(t)\| \, dt$$

$$= \int_0^{2\pi} t \sin t \sqrt{1+\cos^2 t} \sqrt{1+\cos^2 t} \, dt$$

$$= \int_0^{2\pi} t \sin t (1+\cos^2 t) \, dt$$

$$= \int_0^{2\pi} t \sin t + t \sin t \cos^2 t \, dt$$

