Esame 20/02/2023

1. **(8 punti)**

   (a) Si consideri la seguente matrice:

$$A = \begin{pmatrix} \alpha & \alpha+3 & 2\alpha \\ \alpha & 2\alpha+2 & 3\alpha \\ 2\alpha & \alpha+7 & 4\alpha \end{pmatrix}$$

   Si studi $det(A)$, $rk(A)$ e invertibilità di A al variare di $\alpha \in \mathbb{R}$.

   (b) Si calcoli z^6 dove $z = \frac{2}{\sqrt{3}-i} + \frac{1}{i}$.

a)

$$\begin{pmatrix} \alpha & \alpha+3 & 2\alpha \\ \alpha & 2\alpha+2 & 3\alpha \\ 2\alpha & \alpha+7 & 4\alpha \end{pmatrix} \underset{\alpha \neq 0}{\overset{E_1\left(\frac{1}{\alpha}\right)}{\sim}} \begin{pmatrix} 1 & \frac{\alpha+3}{\alpha} & 2 \\ \alpha & 2\alpha+2 & 3\alpha \\ 2\alpha & \alpha+7 & 4\alpha \end{pmatrix} \underset{E_{31}(-2\alpha)}{\overset{E_{21}(-\alpha)}{\sim}} \begin{pmatrix} 1 & \frac{\alpha+3}{\alpha} & 2 \\ 0 & \alpha-1 & \alpha \\ 0 & -\alpha+1 & 0 \end{pmatrix} \underset{\alpha \neq 1}{\overset{E_2\left(\frac{1}{\alpha-1}\right)}{\sim}}$$

$$\begin{pmatrix} 1 & \frac{\alpha+3}{\alpha} & 2 \\ 0 & 1 & \frac{\alpha}{\alpha-1} \\ 0 & -\alpha+1 & 0 \end{pmatrix} \overset{E_{32}(\alpha-1)}{\sim} \begin{pmatrix} 1 & \frac{\alpha+3}{\alpha} & 2 \\ 0 & 1 & \frac{\alpha}{\alpha-1} \\ 0 & 0 & \alpha \end{pmatrix} \overset{E_3\left(\frac{1}{\alpha}\right)}{\sim} \begin{pmatrix} 1 & \frac{\alpha+3}{\alpha} & 2 \\ 0 & 1 & \frac{\alpha}{\alpha-1} \\ 0 & 0 & 1 \end{pmatrix}$$

$rk(A) = 3$ se $\alpha \neq 0 \wedge \alpha \neq 1$

$\alpha = 0$

$$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 7 & 0 \end{pmatrix} \overset{E_1\left(\frac{1}{3}\right)}{\sim} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 7 & 0 \end{pmatrix} \underset{E_{31}(-7)}{\overset{E_{21}(-2)}{\sim}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$rk(A) = 1$ se $\alpha = 0$

$\alpha = 1$

$$\begin{pmatrix} 1 & 4 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad rk(A) = 2 \text{ se } \alpha = 1$$

$$det\begin{pmatrix} \alpha & \alpha+3 & 2\alpha \\ \alpha & 2\alpha+2 & 3\alpha \\ 2\alpha & \alpha+7 & 4\alpha \end{pmatrix} = \alpha\, det\begin{pmatrix} 2\alpha+2 & 3\alpha \\ \alpha+7 & 4\alpha \end{pmatrix} - (\alpha+3)\, det\begin{pmatrix} \alpha & 3\alpha \\ 2\alpha & 4\alpha \end{pmatrix} + 2\alpha\, det\begin{pmatrix} \alpha & 2\alpha+2 \\ 2\alpha & \alpha+7 \end{pmatrix} =$$

$$= \alpha\left(8\alpha^2+8\alpha-(3\alpha^2+21\alpha)\right) - (\alpha+3)\left(4\alpha^2-6\alpha^2\right) + 2\alpha\left(\alpha^2+7\alpha-(4\alpha^2+4\alpha)\right) =$$

$$= \alpha(5\alpha^2-13\alpha) - (\alpha+3)(-2\alpha^2) + 2\alpha(-3\alpha^2+3\alpha) =$$

$= 5\alpha^3 - 13\alpha^2 + 2\alpha^3 + 6\alpha^2 - 6\alpha^3 + 6\alpha^2 =$

$= \alpha^3 - \alpha^2$

La matrice é invertibile se det $\neq 0$

$\alpha^3 - \alpha^2 \neq 0$

$\alpha^2(\alpha - 1) \neq 0$

$\alpha \neq 0$

$\alpha \neq 1$

La matrice é invertibile se $\alpha \neq 0 \wedge \alpha \neq 1$

b)

$$z = \frac{2}{\sqrt{3}-i} + \frac{1}{i} = \frac{\sqrt{3}+i}{\sqrt{3}+i} \cdot \frac{2}{\sqrt{3}-i} + \frac{1}{i} \cdot \frac{-i}{-i} = \frac{2\sqrt{3}+2i}{\cancel{4}\,2} - \frac{2i}{2} =$$

$$= \frac{\sqrt{3}+i-2i}{2} = \frac{\sqrt{3}-i}{2} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$z = r(\cos(\alpha) + i\sin(\alpha))$

$r = \|z\| = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1 \qquad \alpha = \arctan\left(\cdot\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{3}}\right) + 2\pi = \frac{11}{6}\pi$

$$z^6 = 1^6\left(\cos\left(6 \cdot \frac{11}{6}\pi\right) + i\sin\left(6 \cdot \frac{11}{6}\pi\right)\right) = \cos(11\pi) + i\sin(11\pi) = -1$$

2. **(8 punti)** Si consideri la seguente matrice:

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

(a) Si calcolino tutti gli autovalori di B su $\mathbb{R}$ e si trovino delle basi dei loro autospazi.

(b) Si verifichi che la matrice B è diagonalizzabile e si trovino la matrice diagonale D e le matrici S, S^{-1} tali che $B = SDS^{-1}$.

a)

$$\det\begin{bmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{bmatrix} = (-\lambda)\det\begin{pmatrix} -\lambda & -1 \\ -1 & -\lambda \end{pmatrix} + \det\begin{pmatrix} 0 & -\lambda \\ 1 & -1 \end{pmatrix} =$$

$$= -\lambda(\lambda^2 - 1) + \lambda = -\lambda((\lambda^2-1)-1) = -\lambda(\lambda^2 - 2)$$

$\lambda_1 = 0$

$\lambda_2 = \sqrt{2}$

$\lambda_3 = -\sqrt{2}$

$$E(0) = N(B - 0I_3) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \overset{E_{31}}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \underset{E_{32}(-1)}{\overset{E_2(-1)}{\sim}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 - x_2 = 0 \\ x_3 = 0 \\ x_2 = t \end{cases} \quad \begin{cases} x_1 = t \\ x_2 = t \\ x_3 = 0 \end{cases} \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ é una base di } E(0)$$

$$E(\sqrt{2}) = N(B - \sqrt{2}I_3) = \begin{bmatrix} -\sqrt{2} & 0 & 1 \\ 0 & -\sqrt{2} & -1 \\ 1 & -1 & -\sqrt{2} \end{bmatrix} \overset{E_1(-\frac{1}{\sqrt{2}})}{\sim} \begin{bmatrix} 1 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -\sqrt{2} & -1 \\ 1 & -1 & -\sqrt{2} \end{bmatrix} \overset{E_{31}(-1)}{\sim} \begin{bmatrix} 1 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -\sqrt{2} & -1 \\ 0 & -1 & -\frac{1}{\sqrt{2}} \end{bmatrix} \overset{E_2(-\frac{1}{\sqrt{2}})}{\sim}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} \\ 0 & -1 & -\frac{1}{\sqrt{2}} \end{bmatrix} \overset{E_{32}(1)}{\sim} \begin{bmatrix} 1 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 - \frac{1}{\sqrt{2}}x_3 = 0 \\ x_2 + \frac{1}{\sqrt{2}}x_3 = 0 \\ x_3 = t \end{cases} \quad \begin{cases} x_1 = \frac{1}{\sqrt{2}}t \\ x_2 = -\frac{1}{\sqrt{2}}t \\ x_3 = t \end{cases} \quad \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 1 \end{pmatrix} \right\} \text{ é una base di } E(\sqrt{2})$$

$$E(-\sqrt{2}) = N(B + \sqrt{2}I_3) = \begin{bmatrix} \sqrt{2} & 0 & 1 \\ 0 & \sqrt{2} & -1 \\ 1 & -1 & \sqrt{2} \end{bmatrix} \overset{E_1(\frac{1}{\sqrt{2}})}{\sim} \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} & -1 \\ 1 & -1 & \sqrt{2} \end{bmatrix} \overset{E_{31}(-1)}{\sim} \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} & -1 \\ 0 & -1 & \frac{1}{\sqrt{2}} \end{bmatrix} \overset{E_2(\frac{1}{\sqrt{2}})}{\sim}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & \frac{1}{\sqrt{2}} \end{bmatrix} \overset{E_{32}(1)}{\sim} \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 + \frac{1}{\sqrt{2}}x_3 = 0 \\ x_2 - \frac{1}{\sqrt{2}}x_3 = 0 \\ x_3 = t \end{cases} \quad \begin{cases} x_1 = -\frac{1}{\sqrt{2}}t \\ x_2 = \frac{1}{\sqrt{2}}t \\ x_3 = t \end{cases} \quad \left\{ \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix} \right\} \text{ é una base di } E(-\sqrt{2})$$

b) La matrice é diagonalizzabile perché possiede 3 autovalori distinti

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{bmatrix} \qquad S = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & 1 \end{bmatrix}$$

$$(S \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 & 0 & 0 \\ 1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \overset{E_{21}(-1)}{\sim} \left(\begin{array}{ccc|ccc} 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 & 0 & 0 \\ 0 & -\frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \overset{E_2\left(-\frac{\sqrt{2}}{2}\right)}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \overset{E_{32}(-1)}{\sim} \left(\begin{array}{ccc|ccc} 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 2 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 \end{array}\right) \overset{E_3\left(\frac{1}{2}\right)}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{2} \end{array}\right) \overset{E_{23}(1)}{\sim} \left(\begin{array}{ccc|ccc} 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{2} \end{array}\right) \overset{E_{13}\left(\frac{1}{\sqrt{2}}\right)}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{\sqrt{2}} & 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{2\sqrt{2}} \\ 0 & 1 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{2} \end{array}\right) \overset{E_{12}\left(-\frac{1}{2}\right)}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{6-\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} & \frac{2-\sqrt{2}}{4} \\ 0 & 1 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{2} \end{array}\right) = (I_3 \mid S^{-1})$$

$$S^{-1} = \begin{pmatrix} \frac{6-\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} & \frac{2-\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{2} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{2} \end{pmatrix}$$

3. **(8 punti)** Si considerino le seguenti matrici:

$$C = \begin{pmatrix} 3 & 0 \\ 2 & 13 \\ 0 & 8 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 2 & 0 \\ 1 & -2 & 3 \end{pmatrix}$$

(a) Si trova una base di ciascuno dei seguenti sottospazi di $\mathbb{R}^3$:
   i. Il sottospazio $C(C)$ generato dalle colonne di C.
   ii. Lo spazio nullo $N(D)$ di D.
   iii. La somma $C(C)+N(D)$ dei sottospazi $C(C)$ e $N(D)$.

(b) Si calcoli la dimensione dell'intersezione $C(C) \cap N(D)$ dei sottospazi $C(C)$ e $N(D)$.

a)

i) $C(C) = \left\langle \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 13 \\ 8 \end{pmatrix} \right\rangle$

$$\begin{bmatrix} 3 & 0 \\ 2 & 13 \\ 0 & 8 \end{bmatrix} \overset{E_1\left(\frac{1}{3}\right)}{\sim} \begin{bmatrix} 1 & 0 \\ 2 & 13 \\ 0 & 8 \end{bmatrix} \overset{E_{21}(-2)}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 13 \\ 0 & 8 \end{bmatrix} \overset{E_2\left(\frac{1}{13}\right)}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 8 \end{bmatrix} \overset{E_{32}(-8)}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\left\{ \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 13 \\ 8 \end{pmatrix} \right\}$ é una base di $C(C)$

ii) $\begin{pmatrix} 3 & 2 & 0 \\ 1 & -2 & 3 \end{pmatrix} \overset{E_{12}}{\sim} \begin{pmatrix} 1 & -2 & 3 \\ 3 & 2 & 0 \end{pmatrix} \overset{E_{21}(-3)}{\sim} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 8 & -9 \end{pmatrix} \overset{E_2(\frac{1}{8})}{\sim} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -\frac{9}{8} \end{pmatrix}$

$\begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ x_2 - \frac{9}{8}x_3 = 0 \\ x_3 = t \end{cases}$ $\begin{cases} x_1 = \frac{9}{4}t + 3t = \frac{21}{4}t \\ x_2 = \frac{9}{8}t \\ x_3 = t \end{cases}$ $\left\{ \begin{pmatrix} \frac{21}{4} \\ 9/8 \\ 1 \end{pmatrix} \right\}$ é una base di $N(D)$

iii) $\left\{ \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 13 \\ 8 \end{pmatrix}, \begin{pmatrix} \frac{21}{4} \\ 9/8 \\ 1 \end{pmatrix} \right\}$

b) $\dim(C(C) \cup N(D)) = \dim(C(C)) + \dim(N(D)) = 3$

4. **(6 punti)** Si considerino la matrice $P = \begin{pmatrix} 3 & i \\ -1 & 0 \end{pmatrix}$ Vero o falso? Si giustifichi la risposta!

(a) La matrice P è hermitiana, ovvero $P = P^H$.

(b) La matrice P è invertibile.

(c) Il vettore $c_{\mathcal{B}}(Pv)$ è uguale a $\begin{pmatrix} -1 \\ 4+i \end{pmatrix}$ dove $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ e $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

a)

$P = P^H$

$\Downarrow$

$\begin{pmatrix} 3 & i \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -i & 0 \end{pmatrix}$ FALSO

b) $\det \begin{pmatrix} 3 & i \\ -1 & 0 \end{pmatrix} = i$ VERO perché $\det(P) \neq 0$

c) $C_B(Pv) = C_B \begin{pmatrix} 3+i \\ -1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (4+i) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1+4+i \\ -1 \end{pmatrix} = \begin{pmatrix} 3+i \\ -1 \end{pmatrix}$

Vero