

# Algebra Lineare - Esercizi da consegnare

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## Scheda 1

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2° Semestre 2023/2024

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# 1 Scheda 1

## 1.1 Esercizio 1

Date le seguenti matrici a coefficienti complessi:

$$A = \begin{pmatrix} i & 0 \\ -1 & 1+i \end{pmatrix} \quad B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 \\ \frac{1}{2} & -1 \end{pmatrix}$$

calcolare

(a)  $(CD)A$

$$\begin{aligned} (CD)A &= \left( \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix} \begin{pmatrix} 2 & 0 \\ \frac{1}{2} & -1 \end{pmatrix} \right) \begin{pmatrix} i & 0 \\ -1 & 1+i \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{2} & -1 \\ 2 + \frac{1}{2}i & -i \end{pmatrix} \begin{pmatrix} i & 0 \\ -1 & 1+i \end{pmatrix} \\ &= \begin{pmatrix} 1 + \frac{5}{2}i & -1-i \\ -\frac{1}{2} + 3i & 1-i \end{pmatrix} \end{aligned}$$

(b)  $B^T B$

$$\begin{aligned} B^T &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\ B^T B &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

(c)  $3A(B + 4D^T)$

$$\begin{aligned} D^T &= \begin{pmatrix} 2 & \frac{1}{2} \\ 0 & -1 \end{pmatrix} \\ 3A(B + 4D^T) &= 3 \begin{pmatrix} i & 0 \\ -1 & 1+i \end{pmatrix} \left( \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + 4 \begin{pmatrix} 2 & \frac{1}{2} \\ 0 & -1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 3i & 0 \\ -3 & 3+3i \end{pmatrix} \left( \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 8 & 2 \\ 0 & -4 \end{pmatrix} \right) \\ &= \begin{pmatrix} 3i & 0 \\ -3 & 3+3i \end{pmatrix} \begin{pmatrix} 8 & 2-i \\ i & -4 \end{pmatrix} \\ &= \begin{pmatrix} 24i & 3+6i \\ -27+3i & -18-9i \end{pmatrix} \end{aligned}$$

(d)  $C^2 A^T$

$$\begin{aligned} A^T &= \begin{pmatrix} i & -1 \\ 0 & 1+i \end{pmatrix} \\ C^2 A^T &= \left( \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix} \right) \begin{pmatrix} i & -1 \\ 0 & 1+i \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1+i \\ 1+i & 0 \end{pmatrix} \begin{pmatrix} i & -1 \\ 0 & 1+i \end{pmatrix} \\ &= \begin{pmatrix} 2i & -2+2i \\ -1+i & -1-i \end{pmatrix} \end{aligned}$$

(e)  $\frac{1}{2} (B^2 - 3D^T C)$

$$D^T = \begin{pmatrix} 2 & \frac{1}{2} \\ 0 & -1 \end{pmatrix}$$

$$\begin{aligned} \frac{1}{2} (B^2 - 3D^T C) &= \frac{1}{2} \left( \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - 3 \begin{pmatrix} 2 & \frac{1}{2} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix} \right) \\ &= \frac{1}{2} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 6 & \frac{3}{2} \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix} \right) \\ &= \frac{1}{2} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{15}{2} & 6 + \frac{3}{2}i \\ -3 & -3i \end{pmatrix} \right) \\ &= \frac{1}{2} \begin{pmatrix} -\frac{13}{2} & -6 - \frac{3}{2}i \\ 3 & 1 + 3i \end{pmatrix} \\ &= \begin{pmatrix} -\frac{13}{4} & -3 - \frac{3}{4}i \\ \frac{3}{2} & \frac{1}{2} + \frac{3}{2}i \end{pmatrix} \end{aligned}$$

## 1.2 Esercizio 2

Date le seguenti matrici a coefficienti complessi:

$$A = \begin{pmatrix} 1 & -1 & -1 & 7 \\ 3+i & -3-i & -2-i & 11+7i \\ 3 & -3 & -2 & 11 \end{pmatrix} \quad B = \begin{pmatrix} 5 & -5 \\ i & -1 \\ 5 & -5+i \end{pmatrix}$$

$$C = \begin{pmatrix} i & 0 & -1 \\ 7 & i & 0 \\ 6 & 1 & -7+6i \end{pmatrix} \quad D = \begin{pmatrix} 1 & -2 & \frac{1}{2} & 4 & 0 \\ 1 & -1 & \frac{1}{2} & 4+i & 1 \\ 3 & -5 & \frac{3}{2} & 12+i & 1 \end{pmatrix}$$

- (a) Usare l'algoritmo di Eliminazione di Gauss per determinare una forma ridotta di ognuna delle matrici:

(i)  $A$

$$\begin{aligned} &\begin{pmatrix} 1 & -1 & -1 & 7 \\ 3+i & -3-i & -2-i & 11+7i \\ 3 & -3 & -2 & 11 \end{pmatrix} \xrightarrow{R_2 - (3+i)R_1} \\ &\begin{pmatrix} 1 & -1 & -1 & 7 \\ 0 & 0 & 1 & -10 \\ 3 & -3 & -2 & 11 \end{pmatrix} \xrightarrow{R_3 - 3R_1} \\ &\begin{pmatrix} 1 & -1 & -1 & 7 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 1 & -10 \end{pmatrix} \xrightarrow{R_3 - R_2} \\ &\begin{pmatrix} 1 & -1 & -1 & 7 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

(ii)  $B$

$$\begin{pmatrix} 5 & -5 \\ i & -1 \\ 5 & -5+i \end{pmatrix} \xrightarrow[\sim]{\frac{1}{5}R_1} \begin{pmatrix} 1 & -1 \\ i & -1 \\ 5 & -5+i \end{pmatrix} \xrightarrow[\sim]{R_2-iR_1} \begin{pmatrix} 1 & -1 \\ 0 & -1+i \\ 0 & i \end{pmatrix} \xrightarrow[\sim]{\frac{1}{-1+i}R_2} \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & i \end{pmatrix} \xrightarrow[\sim]{R_3-iR_2} \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(iii)  $C$

$$\begin{pmatrix} i & 0 & -1 \\ 7 & i & 0 \\ 6 & 1 & -7+6i \end{pmatrix} \xrightarrow[\sim]{-iR_1} \begin{pmatrix} 1 & 0 & i \\ 7 & i & 0 \\ 6 & 1 & -7+6i \end{pmatrix} \xrightarrow[\sim]{R_2-7R_1} \begin{pmatrix} 1 & 0 & i \\ 0 & i & -7i \\ 0 & 1 & -7 \end{pmatrix} \xrightarrow[\sim]{R_3-6R_1} \begin{pmatrix} 1 & 0 & i \\ 0 & i & -7i \\ 0 & 1 & -7 \end{pmatrix} \xrightarrow[\sim]{-iR_2} \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & -7 \\ 0 & 1 & -7 \end{pmatrix} \xrightarrow[\sim]{R_3-R_2} \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{pmatrix}$$

(iv)  $D$

$$\begin{pmatrix} 1 & -2 & \frac{1}{2} & 4 & 0 \\ 1 & -1 & \frac{1}{2} & 4+i & 1 \\ 3 & -5 & \frac{3}{2} & 12+i & 1 \end{pmatrix} \xrightarrow[\sim]{R_2-R_1} \begin{pmatrix} 1 & -2 & \frac{1}{2} & 4 & 0 \\ 0 & 1 & 0 & i & 1 \\ 3 & -5 & \frac{3}{2} & 12+i & 1 \end{pmatrix} \xrightarrow[\sim]{R_3-3R_1} \begin{pmatrix} 1 & -2 & \frac{1}{2} & 4 & 0 \\ 0 & 1 & 0 & i & 1 \\ 0 & 1 & 0 & i & 1 \end{pmatrix} \xrightarrow[\sim]{R_3-R_2} \begin{pmatrix} 1 & -2 & \frac{1}{2} & 4 & 0 \\ 0 & 1 & 0 & i & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Calcolare il rango di ognuna delle matrici:

(i)  $rk(A) = 2$

(ii)  $rk(B) = 2$

(iii)  $rk(C) = 2$

(iv)  $rk(D) = 2$

(c) Scrivere i sistemi lineari per cui le matrici  $A, B, C, D$  sono le corrispondenti matrici aumentate ed usare il Teorema di Rouchè-Capelli per stabilire se tali sistemi lineari ammettono o non ammettono soluzioni

(i)

$$A = \begin{cases} x - y - z = 7 \\ z = -10 \end{cases} \quad \text{Ha infinite soluzioni}$$

(ii)

$$B = \begin{cases} x = -1 \\ 0 = 1 \\ 0 = 0 \end{cases} \quad \text{Non ha soluzioni}$$

(iii)

$$C = \begin{cases} x = i \\ y = -7 \\ 0 = 0 \end{cases} \quad \text{Ha una sola soluzione}$$

(iv)

$$D = \begin{cases} x - 2y + \frac{1}{2}z + 4w = 0 \\ y + iw = 1 \\ 0 = 0 \end{cases} \quad \text{Ha infinite soluzioni}$$

(d) Trovare tutte le soluzioni del sistema lineare  $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 - i \\ -1 \end{pmatrix}$

$$\begin{cases} x_1 - x_2 - x_3 + 7x_4 = -1 \\ x_3 - 10x_4 = -1 - i \\ 0 = -1 \end{cases}$$

Il sistema lineare non ha soluzioni

(e) Trovare tutte le soluzioni del sistema lineare omogeneo associato alla matrice  $B$

$$\begin{cases} x_1 - x_2 = 0 \\ x_2 = 0 \end{cases}$$
$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

### 1.3 Esercizio 3

Si consideri la matrice

$$A_t = \begin{pmatrix} t-1 & 2t-2 & t-1 \\ t+1 & 2t+2 & t+1 \\ 1 & 2 & t+1 \end{pmatrix} \quad \text{con } t \in \mathbb{R}$$

(a) Calcolare il rango  $rk(A_t)$  per ogni valore di  $t \in \mathbb{R}$

$$\begin{pmatrix} t-1 & 2t-2 & t-1 \\ t+1 & 2t+2 & t+1 \\ 1 & 2 & t+1 \end{pmatrix} \xrightarrow[\sim]{\frac{1}{t-1}R_1} \begin{pmatrix} 1 & 2 & 1 \\ t+1 & 2t+2 & t+1 \\ 1 & 2 & t+1 \end{pmatrix}$$
$$\xrightarrow[\sim]{R_2 - (t+1)R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & t+1 \end{pmatrix} \xrightarrow[\sim]{R_3 - R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & t \end{pmatrix}$$

$$R_3 \xleftrightarrow{\sim} R_2 \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow[\sim]{\frac{1}{t}R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$rk(U) = rk(A_t) = 2 \quad \forall t \in \mathbb{R} - \{0, 1\}$$

- (b) Se  $A_t$  è la matrice aumentata di un sistema lineare, per quali valori di  $t \in \mathbb{R}$  tale sistema ammette soluzioni?

$$A_t = \begin{cases} x_1 + 2x_2 = 1 \\ 0 = 1 \end{cases}$$

Il sistema lineare non ammette soluzioni per nessun valore di  $t \in \mathbb{R}$

#### 1.4 Esercizio 4

Trovare tutte le soluzioni complesse di  $x^3 - 1 = 0$

$$x^3 - 1 = 0$$

$$x^3 = 1$$

$$x = \sqrt[3]{1}$$

$$z = a + bi \quad a = 1, \quad b = 0$$

$$r = \sqrt{a^2 + b^2} = 1$$

$$\alpha = \arctan \frac{b}{a} = \arctan \frac{0}{1} = 0$$

$$\sqrt[n]{z} = \sqrt[n]{r} \left( \cos \frac{\alpha + 2k\pi}{n} + i \sin \frac{\alpha + 2k\pi}{n} \right)$$

$$x = \sqrt[3]{1} = \cos \left( \frac{0 + 2k\pi}{3} \right) + i \sin \left( \frac{0 + 2k\pi}{3} \right) \quad k = 0, 1, 2$$

$$x_0 = \cos \left( \frac{0}{3} \right) + i \sin \left( \frac{0}{3} \right) = 1$$

$$x_1 = \cos \left( \frac{2\pi}{3} \right) + i \sin \left( \frac{2\pi}{3} \right) = -\frac{1}{2} + i \frac{\sqrt{3}}{2}$$

$$x_2 = \cos \left( \frac{4\pi}{3} \right) + i \sin \left( \frac{4\pi}{3} \right) = -\frac{1}{2} - i \frac{\sqrt{3}}{2}$$