IN-STK5000 - Assignment 1

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Contents

1 Probability theory

1.1 EXERCISE 1

If A,B are mutually exclusive events i.e. $A \cap B = \emptyset$, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) + 0 = P(A) + P(B).$$

1.2 EXERCISE 2

If A, B are not exclusive events i.e. $A \cap B \neq \emptyset$, then

$$P(A \cup B) \leq P(A) + P(B) - P(A \cap B).$$

1.3 EXERCISE 3

If A, B are two events, with $P(B) > 0$, then conditional probability is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

1.4 EXERCISE 4

Let $A_1, ..., A_n$ be mutually exclusive events so that $\bigcup_{i=1}^{n} A_i = \Omega$ and $B \subset \Omega$ an arbitrary other event. Then:

$$P(B) = \sum_{i=1}^{n} P(B, A_i).$$

2 Random variables and statistics

2.1 EXERCISE 5

A real-valued random variable X is simply a mapping $X : \Omega \to \mathbb{R}$. Write the definition of the expectation of X drawn from $\mathbb{P}$, where $\mathbb{P}$ is a probability measure on (Ω, Σ) and Σ is the σ-algebra generated by Ω.

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(w)\mathbb{P}(w).$$

2.2 EXERCISE 6

The sample mean μ_n of n i.i.d. random variables $X_1, X_2, ..., X_n$ is defined as

$$\mu_n \triangleq \frac{1}{n} \sum_{n=1}^{N} X_n.$$

2.3 EXERCISE 7

Write the expectation of the sample mean μ_n in relation to $X_1, X_2, ...$

$$\mathbb{E}\mu_n = \sum_{n \in \mathbb{N}} X_n \mu(X_n).$$

2.4 EXERCISE 8

A null hypothesis test at significance level p is constructed by using a test statistic $\pi : \chi \to [0, 1)$ mapping from the space of possible data to the interval $[0, 1)$, so that the test rejects the null hypothesis whenever $\pi(x) < p$. Which of the following statements are true?

1) The probability that the test will falsely reject the null hypothesis is p.

False, because the type I error $\mathbb{P}$(rejects H_0 when H_0 is true) is not the same parameter/measure as the significance level p.

2) The probability that the test will falsely accept the null hypothesis is p.

False, because the type II error $\mathbb{P}$(not reject H_0 when H_0 is false) is not the same parameter/measure as the significance level p.

3) The probability that the test will falsely reject the alternative hypothesis is p.

False, because the alternative hypothesis is not subject to rejection or acceptance in hypothesis testing.

4) The probability that the test will falsely accept the alternative hypothesis is p.

False, because the alternative hypothesis is not subject to rejection or acceptance in hypothesis testing.

5) Given the data x, the probability that the null hypothesis is true is $\pi(x)$.

True. Since the null hypothesis is $H_0 : \pi(x) \geq p$, the alternative hypothesis is $H_a : \pi(x) < p$, and $\pi : \chi \to [0, 1)$, we can say that $\pi(x)$ is a probability measure of H_0 or H_a being true or false.

6) Given the data x, the probability that the null hypothesis is false is $\pi(x)$.

True. Since the null hypothesis is $H_0 : \pi(x) \geq p$, the alternative hypothesis is $H_a : \pi(x) < p$, and $\pi : \chi \to [0, 1)$, we can say that $\pi(x)$ is a probability measure of H_0 or H_a being true or false.

7) Given the data x, the probability that the alternative hypothesis is true is $\pi(x)$.

True. Since the null hypothesis is $H_0 : \pi(x) \geq p$, the alternative hypothesis is $H_a : \pi(x) < p$, and $\pi : \chi \to [0, 1)$, we can say that $\pi(x)$ is a probability measure of H_0 or H_a being true or false.

8) Given the data x, the probability that the alternative hypothesis is false is $\pi(x)$.

True. Since the null hypothesis is $H_0 : \pi(x) \geq p$, the alternative hypothesis is $H_a : \pi(x) < p$, and $\pi : \chi \to [0, 1)$, we can say that $\pi(x)$ is a probability measure of H_0 or H_a being true or false.

3 Linear Algebra

3.1 EXERCISE 9

If $\mathbf{x} = x_1, ..., x_n$, $\mathbf{y} = y_1, ..., y_n$ are two column vectors in $\mathbb{R}^n$, what is their inner product:

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.$$

3.2 EXERCISE 10

The matrix

$$\mathbf{A}^+ \triangleq (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$$

is the left-pseudo-inverse of $\mathbf{A}$. Complete the following:

$$\mathbf{A}^+ \mathbf{A} = (\mathbf{A}^T \mathbf{A})^{-1} (\mathbf{A}^T \mathbf{A}) = \mathbf{I}.$$

4 Calculus

4.1 EXERCISE 11

If $f : \mathbb{R} \to \mathbb{R}$ is a real-valued twice-differentiable function, what are *sufficient* conditions for x_0 to be a local maximum of the function i.e. there exists $\epsilon > 0$ so that $f(x_0) \geq f(x)$ for all $x : |a - a_0| < \epsilon$?

The *sufficient* conditions are that the first derivative is zero and the second is negative.

4.2 EXERCISE 12

Solve the following integral, for $T > 0$

$$\int_1^T \frac{1}{x} dx = ln|T| - ln|1| = ln|T|.$$