Wk 6: 11.1: ~~1-5 (Matlab)~~, ~~19a~~, ~~21-23~~, ~~41 (Matlab)~~, ~~44 (Matlab)~~
11.2: ~~4~~, ~~26-29 (Matlab)~~, ~~41~~, ~~42~~

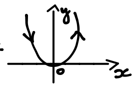
11.1:

Finding Cartesian from Parametric Equations

Exercises 1–18 give parametric equations and parameter intervals for the motion of a particle in the xy-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation. (The graphs will vary with the equation used.) Indicate the portion of the graph traced by the particle and the direction of motion.

1. $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$
2. $x = -\sqrt{t}, \quad y = t, \quad t \geq 0$
3. $x = 2t - 5, \quad y = 4t - 7, \quad -\infty < t < \infty$
4. $x = 3 - 3t, \quad y = 2t, \quad 0 \leq t \leq 1$
5. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \leq t \leq \pi$

(1) $y = \frac{\cancel{9}.x^2}{\cancel{9}} = x^2$:=

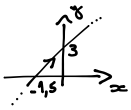


parabola, clockwise;

(2) $x = -\sqrt{y}$:=

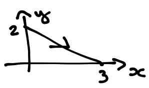
half parabola only with negatives values of x, ↖;

(3) $x = \cancel{2}\left(\frac{y+7}{\cancel{4}\,2}\right) - 5 = \frac{y+7-10}{2} \Rightarrow y = 2x+3$



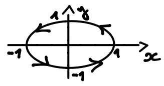
line, ↗;

(4) $x = 3 - \frac{3y}{2} \Rightarrow y = \frac{2x-6}{-3} = 2 - \frac{2}{3}x$ for $0 \leq x \leq 3$



line, ↘;

(5) $x^2 + y^2 = 1 \rightarrow \cos^2(2t) + \sin^2(2t) = 1$



circle, counter-clockwise.

$0 \leq t \leq \pi$, $x = \cos(2t)$, $y = \sin(2t)$; for $t = 0$, $(x=1, y=0)$; for $t = \frac{1}{2}$, $(x=0, y=1)$

Finding Parametric Equations

19. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the circle $\underbrace{x^2 + y^2 = a^2}_{\text{circulo}}$

a. once clockwise.

Parameter interval $I := 0 \leq t \leq 2\pi$

$x^2 + y^2 = (a.\cos(t))^2 + (-a.\sin(t))^2$
$= a^2.\cos^2(t) + (-a)^2.\sin^2(t)$
$= a^2.(\cancel{\cos^2(t) + \sin^2(t)}) = a^2$ (→1)

~~∴ $x = a.\cos(t)$ and $y = a.\sin(t)$~~
~~for $a=1, t=0, (x=1, y=0)$~~
~~for $a=1, t=1, (x=0, y=1)$~~ ~~counter-clockwise~~

∴ $x = a.\cos(t)$ and $y = -a.\sin(t)$
for $a=1, t=0$, $(1,0)$
for $a=1, t=1$, $(0,-1)$
clockwise ✓ ☒

In Exercises 21–26, find a parametrization for the curve.

21. the line segment with endpoints $(-1, -3)$ and $(4, 1)$
22. the line segment with endpoints $(-1, 3)$ and $(3, -2)$
23. the lower half of the parabola $x - 1 = y^2$

(21) $x = a + ct$
$y = b + dt$

$t=0$: $(-1,-3)$; $t=1$: $(4,1)$ ∴ $(4-(-1), 1-(-3)) = (5,4)$
↳ $x = -1 + 5t$ and $y = -3 + 4t$, for $0 \leq t \leq 1$.

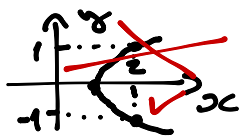
$4 = -1 + c1$ ∴ $c = 5$
$1 = -3 + d1$ ∴ $d = 4$
∴ $x = -1 + 5t$, $y = -3 + 4t$ for $0 \leq t \leq 1$ ☒

(22) $t=0$: $(-1,3)$; $t=1$: $(3,-2)$ ∴ $(3-(-1), (-2-3)) = (4,-5)$
↳ $x = -1 + 4t$ and $y = 3 - 5t$ for $0 \leq t \leq 1$ ☒

(23) $x - 1 = y^2$ ∴ $y = t$
$x = 1 + t^2$

for $\boxed{t \leq 0}$ → the lower half of the parabola:

~~$t=1$ ∴ $y=1$~~
~~$x=2$~~
$t=0$ ∴ $y=0$
$x=1$
$t=-1$ ∴ $y=-1$
$x=2$



☒

T GRAPHER EXPLORATIONS

If you have a parametric equation grapher, graph the equations over the given intervals in Exercises 41–48.

41. Ellipse $x = 4 \cos t, \quad y = 2 \sin t,$ over

a. $0 \le t \le 2\pi$

b. $0 \le t \le \pi$

c. $-\pi/2 \le t \le \pi/2.$

get values of x, y and draw on cartesian graphs.

```
Python: t = np.linspace(0, 2π, 1000)
        x = 4 * np.cos(t)
        y = 2 * np.sin(t)
        plt.plot(x, y)
        plt.show()
```

44. Cycloid $x = t - \sin t, \quad y = 1 - \cos t,$ over

a. $0 \le t \le 2\pi$

b. $0 \le t \le 4\pi$

c. $\pi \le t \le 3\pi.$

Same as 41!

11.2:

Tangents to Parametrized Curves

In Exercises 1–14, find an equation for the line tangent to the curve at the point defined by the given value of t. Also, find the value of d^2y/dx^2 at this point.

4. $x = \cos t, \quad y = \sqrt{3} \cos t, \quad t = 2\pi/3 = \frac{360}{3} = 120°$

sin 1 2 3
cos 3 2 $\frac{1}{2}$

$$\text{slope} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sqrt{3}\cdot\cancel{(-\sin(t))}}{\cancel{-\sin(t)}} = \sqrt{3}$$

$$x_0 = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} \; ?$$

$$y_0 = \sqrt{3}\cdot\cos\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2} \quad \therefore \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

The line tangent:

$$y + \frac{\sqrt{3}}{2} = \sqrt{3}\left(x + \frac{1}{2}\right) \longleftrightarrow 2y + \sqrt{3} = \cancel{2}\sqrt{3}\left(\frac{2x+1}{\cancel{2}}\right)$$

$$\cancel{2}y + \cancel{\sqrt{3}} = \cancel{2}x\sqrt{3} + \cancel{\sqrt{3}}$$

$$y = x\cdot\sqrt{3} \quad \blacksquare$$

$$\frac{d^2y}{dx^2} \Rightarrow 1°)\; y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \sqrt{3} \qquad 2°)\; \frac{dy'}{dt} = \frac{d}{dt}(y') = 0 \qquad 3°)\; \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = 0 \quad \blacksquare$$

Lengths of Curves

Find the lengths of the curves in Exercises 25–30.

26. $x = t^3, \quad y = 3t^2/2, \quad 0 \le t \le \sqrt{3}$

27. $x = t^2/2, \quad y = (2t + 1)^{3/2}/3, \quad 0 \le t \le 4$

28. $x = (2t + 3)^{3/2}/3, \quad y = t + t^2/2, \quad 0 \le t \le 3$

29. $x = 8 \cos t + 8t \sin t$
$y = 8 \sin t - 8t \cos t,$
$0 \le t \le \pi/2$

(26) $\frac{dx}{dt} = 3t^2$; $\frac{dy}{dt} = 3t$ $\therefore L = \int_0^{\sqrt{3}} \sqrt{[3t^2]^2 + [3t]^2}\, dt = \int_0^{\sqrt{3}} 3\cdot\sqrt{t^4+t^2}\, dt$

$$L = \int_0^{\sqrt{3}} 3t\cdot\sqrt{t^2+1}\, dt = \boxed{\begin{array}{ll} u = t^2+1 & dt = \frac{du}{2t} \\ du = 2t\, dt & \end{array}} = \boxed{\begin{array}{l} t = \sqrt{3} \to u = 4 \\ t = 0 \to u = 1 \end{array}} = \int_1^4 3\cancel{t}\cdot u^{1/2} \frac{du}{2\cancel{t}}$$

$$L = \int_1^4 \frac{3}{2} u^{1/2}\, du = \frac{\cancel{3}}{\cancel{2}}\left(\frac{\cancel{2}}{\cancel{3}}\cdot u^{3/2}\right)\Big|_1^4 = 2^3 - 1 = 7 \quad \blacksquare$$

(27) $\frac{dx}{dt} = t$; $\frac{dy}{dt} = \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}(2t+1)^{1/2}\cdot\cancel{2} = (2t+1)^{1/2}$

$$L = \int_0^4 \sqrt{t^2 + 2t + 1}\, dt = \int_0^4 \sqrt{(t+1)^2}\, dt$$

$$L = \int_0^4 t+1\, dt = \frac{t^2}{2} + t\Big|_0^4 = 12 \quad \blacksquare$$

chain rule:

$$\boxed{\left[(2t+1)^{3/2}\right]' = \frac{3}{2}(2t+1)^{1/2} \qquad u = 2t+1 \to u' = 2}$$

(28) $\frac{dx}{dt} = \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}(2t+3)^{1/2}\cdot\cancel{2} = (2t+3)$; $\frac{dy}{dt} = 1 + \frac{1}{\cancel{2}}\cdot\cancel{2}\cdot t = t+1$

$$L = \int_0^3 \sqrt{(2t+3) + (t+1)^2}\, dt = \boxed{\begin{array}{l} 2t+3+t^2+2t+1 \\ t^2+4t+4 \\ (t+2)^2 \end{array}} = \int_0^3 \sqrt{(t+2)^2}\, dt$$

$$L = \int_0^3 t+2\, dt = \frac{t^2}{2} + 2t\Big|_0^3 = \frac{9}{2} + 6 = \frac{21}{2} \quad \blacksquare$$

(29) $\frac{dx}{dt} = \cancel{-8\sin(t)} + \cancel{8\sin(t)} + 8t\cos(t)$; $\frac{dy}{dt} = \cancel{8\cos(t)} - \cancel{8\cos(t)} + 8t\sin(t) \longleftrightarrow L = \int_0^{\pi/2} \sqrt{8^2t^2\cos^2(t) + 8^2t^2\sin^2(t)}\, dt = \int_0^{\pi/2} 8t\, dt$

$$L = 4t^2\Big|_0^{\pi/2} = \frac{\cancel{4}\cdot\pi^2}{\cancel{4}} = \pi^2 \quad \blacksquare$$

Theory and Examples

41. **Length is independent of parametrization** To illustrate the fact that the numbers we get for length do not depend on the way we parametrize our curves (except for the mild restrictions preventing doubling back mentioned earlier), calculate the length of the semicircle $y = \sqrt{1-x^2}$ with these two different parametrizations:

a. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \le t \le \pi/2.$

b. $x = \sin \pi t, \quad y = \cos \pi t, \quad -1/2 \le t \le 1/2.$

(a) $\frac{dx}{dt} = -2\sin(2t)$; $\frac{dy}{dt} = 2\cos(2t)$

$$L = \int_0^{\pi/2} \sqrt{4\left(\cancel{\sin^2(2t)+\cos^2(2t)}\right)_{\to 1}}\, dt = 2t \Big|_0^{\frac{\pi}{2}} = \pi$$

(b) $\frac{dx}{dt} = -\pi\cos(\pi t)$; $\frac{dy}{dt} = \pi\sin(\pi t)$

$$L = \int_{-1/2}^{1/2} \sqrt{\pi^2 \cdot \left(\cancel{\cos^2(\pi t)+\sin^2(\pi t)}\right)_{\to 1}}\, dt = \int_{-\frac{1}{2}}^{1/2} \pi\, dt$$

$$L = \pi t \Big|_{-1/2}^{1/2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

42. a. Show that the Cartesian formula

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

for the length of the curve $x = g(y)$, $c \le y \le d$ (Section 6.3, Equation 4), is a special case of the parametric length formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Use this result to find the length of each curve.

b. $x = y^{3/2}, \quad 0 \le y \le 4/3$

c. $x = \frac{3}{2}y^{2/3}, \quad 0 \le y \le 1$

(a) $x = g(y)$; $y = y$ for $c \le y \le d$

$$L = \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\, dy = \int_c^d \sqrt{(g'(y))^2 + 1}\, dy$$

(b) $$L = \int_0^{4/3} \sqrt{\left(\frac{3}{2}\cdot y^{1/2}\right)^2 + 1}\, dy = \int_0^{4/3} \sqrt{\frac{9}{4}\cdot y + 1}\, dy$$

$u = \frac{9}{4}y + 1$

$\frac{du}{dy} = \frac{9}{4}$

$dy = \frac{4}{9}du$

$$L = \frac{4}{9}\cdot\frac{2}{3}\left(\frac{9}{4}y + 1\right)^{3/2}\Big|_0^{\frac{4}{3}}$$

$$L = \frac{8}{27}\left((4)^{3/2}\right) - \frac{8}{27} = \frac{56}{27}$$

$$\left[\frac{3}{2}y^{2/3}\right]' = \cancel{\frac{3}{2}}\cdot\cancel{\frac{2}{3}}\, y^{-1/3}$$

$$[t]' = 1$$

$$L = \int_0^1 \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt = \int_0^1 \sqrt{\left[y^{-1/3}\right]^2 + [1]^2}\, dy = \int_0^1 \sqrt{\frac{1+y^{2/3}}{y^{2/3}}}\, dy = \int_0^1 \sqrt{\frac{1}{y^{2/3}}\cdot(1+y^{2/3})}\, dy = \int_0^1 \frac{1}{y^{1/3}}\cdot\sqrt{1+y^{2/3}}\, dy \rightarrow$$

$$\rightarrow \boxed{\begin{array}{lll} u = 1+y^{2/3} & du = \frac{2y^{-1/3}\,dy}{3} & u(0) = 1+0^{2/3} = 1 \\ \frac{du}{dy} = \frac{2y^{-1/3}}{3} & dy = \frac{3\,du}{2\,y^{-1/3}} & u(1) = 1+1^{2/3} = 2 \end{array}}$$

$$\rightarrow = \int_1^2 \cancel{y^{-1/3}} \sqrt{u}\, \frac{du \cdot 3}{2\cdot \cancel{y^{-1/3}}} = \frac{3}{2}\int_1^2 \sqrt{u}\, du = \cancel{\frac{3}{2}}\cdot\left[\cancel{\frac{2}{3}}\, u^{3/2}\right]\Big|_1^2$$

$$L = u^{3/2}\Big|_1^2 = \sqrt{2^3} - \sqrt{1^3} = 2\sqrt{2} - 1$$

Wk 7: 11.3: ~~3, 5, 7, 11, 13, 16, 17, 19, 21, 23, 24, 29, 33, 36, 53–57~~
11.4: ~~1–5 (matlab), 22 (matlab), 25~~
11.5: ~~1 (matlab), 2 (matlab), 6 (matlab), 21–22, 29, 32~~

3. Plot the following points (given in polar coordinates). Then find all the polar coordinates of each point. $\to (r, \theta)$

a. $(2, \pi/2) \to (0, 2)$

b. $(2, 0) \to (2, 0)$

c. $(-2, \pi/2) \to (0, -2)$

d. $(-2, 0) \to (-2, 0)$

(a) $r = 2$, $\theta = \frac{\pi}{2} = 90°$; $x = r\cos\theta = 2\cdot\cos\frac{\pi}{2} = 0$; $y = r\sin\theta = 2\cdot\sin\frac{\pi}{2} = 2$

(b) $r = 2$, $\theta = 0$; $x = 2$, $y = 0$

(c) $r = -2$, $\theta = \frac{\pi}{2}$; $x = 0$, $y = -2$

(d) $r = -2$, $\theta = 0$; $x = -2$, $y = 0$

Polar to Cartesian Coordinates

5. Find the Cartesian coordinates of the points in Exercise 1.

1. Which polar coordinate pairs label the same point?

a. $(3, 0)$	b. $(-3, 0)$	c. $(2, 2\pi/3)$
d. $(2, 7\pi/3)$	e. $(-3, \pi)$	f. $(2, \pi/3)$
g. $(-3, 2\pi)$	h. $(-2, -\pi/3)$	

(a) $r = 3$, $\theta = 0$; $x = 3$, $y = 0$

(b) $r = -3$, $\theta = 0$; $x = -3$, $y = 0$

(c) $r = 2$, $\theta = \frac{2\pi}{3}$; $x = \cancel{2}\cdot -\frac{1}{\cancel{2}}$, $y = \cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}}$

(d) $r = 2$, $\theta = \frac{7\pi}{3} = \frac{7\cdot 180}{3} = 7\cdot(60)$; $x = 2\cdot\frac{1}{2} = 1$, $y = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$

(e) $r = -3$, $\theta = \pi$; $x = -3\cdot 1 = -3$, $y = -3\cdot 0 = 0$

(g) $r = -3$, $\theta = 2\pi$; $x = -3\cdot 1 = -3$, $y = -3\cdot 0 = 0$

(f) $r = 2$, $\theta = \frac{\pi}{3}$; $x = 2\cdot\frac{1}{2} = 1$, $y = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$

(h) $r = -2$, $\theta = -\frac{\pi}{3}$; $x = -2\cdot\frac{1}{2} = -1$ (4°Q), $y = -2\cdot -\frac{\sqrt{3}}{2} = \sqrt{3}$

Cartesian to Polar Coordinates

$P(r,\theta)$

7. Find the polar coordinates, $0 \le \theta < 2\pi$ and $r \ge 0$, of the following points given in Cartesian coordinates.

a. $(1, 1)$ **b.** $(-3, 0)$

c. $(\sqrt{3}, -1)$ **d.** $(-3, 4)$

(a) $r^2 = x^2 + y^2 \longrightarrow r = \sqrt{2}$ $\therefore$ $x = 1 = \sqrt{2}\cos\theta$

$\cos\theta = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$y = 1 = \sqrt{2}\sin\theta$ $\therefore$ $\theta = \frac{\pi}{4}$

$\sin\theta = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

(b) $r^2 = (-3)^2 + (0)^2 \longrightarrow r = 3$ $\therefore$ $x = -3 = 3 \cdot \cos\theta$

$\cos\theta = -1$ $\therefore$ $\theta = \pi$

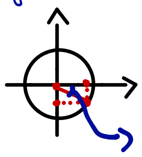
$y = 0 = 3\sin\theta$

$\sin\theta = 0$

(c) $r^2 = (\sqrt{3})^2 + (-1)^2 \longrightarrow r = \sqrt{4} = \pm 2$

$\therefore$ $x = \sqrt{3} = 2 \cdot \cos(\theta) \longrightarrow \cos(\theta) = \frac{\sqrt{3}}{2}$

$\therefore$ $y = -1 = 2 \cdot \sin(\theta) \longrightarrow \sin(\theta) = \frac{-1}{2}$



$\theta = 2\pi - \frac{\pi}{6} = \frac{12\pi - \pi}{6} = \frac{11\pi}{6}$

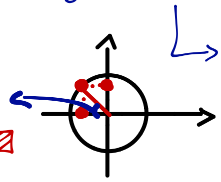
(d) $(-3,4) \to r^2 = (-3)^2 + (4)^2 = \sqrt{25} = \pm 5$

$\therefore$ $x = 5 \cdot \cos(\theta) = -3 \longrightarrow \cos(\theta) = \frac{-3}{5}$

$\therefore$ $y = 5 \cdot \sin(\theta) = 4 \longrightarrow \sin(\theta) = \frac{4}{5}$ $\therefore$ $tg(\theta) = \frac{4/\cancel{5}}{-3/\cancel{5}} = \frac{-4}{3}$

$\longrightarrow -\operatorname{arctg}\left(\frac{4}{3}\right)$

$\theta = \pi - \operatorname{arctg}\left(\frac{4}{3}\right)$



Graphing Sets of Polar Coordinate Points

Graph the sets of points whose polar coordinates satisfy the equations and inequalities in Exercises 11–26.

11. $r = 2$ **15.** $0 \le \theta \le \pi/6, \quad r \ge 0$

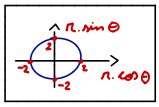
13. $r \ge 1$ **17.** $\theta = \pi/3, \quad -1 \le r \le 3$

19. $\theta = \pi/2, \quad r \ge 0$

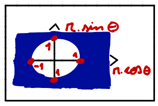
21. $0 \le \theta \le \pi, \quad r = 1$

23. $\pi/4 \le \theta \le 3\pi/4, \quad 0 \le r \le 1$

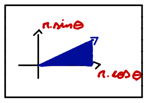
(11) $r = 2$ and $0 \le \theta \le 2\pi \longrightarrow$



(13) $r \ge 1$ and $0 \le \theta \le 2\pi \longrightarrow$



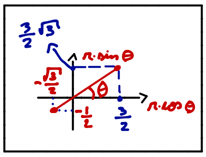
(15) $0 \le \theta \le \frac{\pi}{6}$ and $r \ge 0 \longrightarrow$



(17) $\theta = \frac{\pi}{3}$; $-1 \le r \le 3$

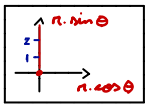
$\cos\frac{\pi}{3} = \frac{1}{2}$; $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$ $\longrightarrow$



(19) $\theta = \frac{\pi}{2}$, $r \ge 0$ $(0,0);(1,0);(2,0)$ $\longrightarrow$

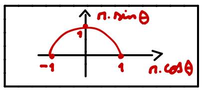
$\cos\frac{\pi}{2} = 0$; $\sin\frac{\pi}{2} = 1$



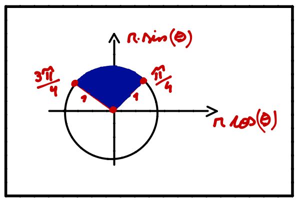
(21) $0 \le \theta \le \pi$; $r = 1$

$\cos 0 = 1$; $\cos\frac{\pi}{2} = 0$; $\cos\pi = -1$

$\sin 0 = 0$; $\sin\frac{\pi}{2} = 1$; $\sin\pi = 0$ $\longrightarrow$



(23) $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$, $0 \le r \le 1$



Polar to Cartesian Equations

Replace the polar equations in Exercises 27–52 with equivalent Cartesian equations. Then describe or identify the graph.

27. $r\cos\theta = 2$ **33.** $r\cos\theta + r\sin\theta = 1$

29. $r\sin\theta = 0$ **36.** $r^2 = 4r\sin\theta$

(27) $\underbrace{r.\cos(\theta)}_{x}=2$

$x = 2$

(29) $\underbrace{r.\sin(\theta)}_{y}=0$

$y = 0$

(33) $\underbrace{r.\cos(\theta)}_{x} + \underbrace{r.\sin(\theta)}_{y} = 1$

$x + y = 1$

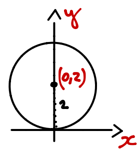
(36) $\underbrace{r^2}_{x^2+y^2} = 4\underbrace{r.\sin(\theta)}_{y}$

$x^2 + y^2 = 4 \cdot y$

$x^2 + (y^2 - 2 \cdot y \cdot 2 + 2^2) = 2^2$

$(x - x_c)^2 + (y - y_c)^2 = R^2 \longrightarrow$ circle's formula!

$(x-0)^2 + (y-2)^2 = 2^2 \longrightarrow$



Cartesian to Polar Equations

Replace the Cartesian equations in Exercises 53–66 with equivalent polar equations.

53. $x = 7$ **54.** $y = 1$ **55.** $x = y$

56. $x - y = 3$ **57.** $x^2 + y^2 = 4$

(53) $x = 7 = r.\cos\theta$

(54) $y = 1 = r\sin\theta$

(55) $\cancel{r}.\sin\theta = \cancel{r}.\cos\theta$

$\therefore$ $\theta = \frac{\pi}{4}$

(56) $r\cos\theta - r\sin\theta = 3$

(57) $r^2\cos^2\theta + r^2\sin^2\theta = 2^2$

$r^2 = 2^2 \longrightarrow r = \pm 2$

11.4: Symmetries and Polar Graphs

Identify the symmetries of the curves in Exercises 1–12. Then sketch the curves in the xy-plane.

1. $r = 1 + \cos\theta$
2. $r = 2 - 2\cos\theta$
3. $r = 1 - \sin\theta$
4. $r = 1 + \sin\theta$
5. $r = 2 + \sin\theta$
6. $r = 1 + 2\sin\theta$

① $r = 1 + \cos(\theta)$

Since $0 \leq \theta \leq 2\pi$, then:

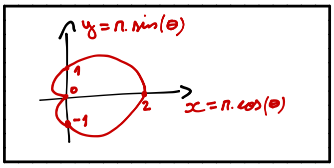
↳ $\cos(0) = 1 \longrightarrow r = 2$

↳ $\cos(\frac{\pi}{2}) = 0 \longrightarrow r = 1$

↳ $\cos(\pi) = -1 \longrightarrow r = 0$

↳ $\cos(\frac{3\pi}{2}) = 0 \longrightarrow r = 1$

↳ $\cos(2\pi) = 1 \longrightarrow r = 2$



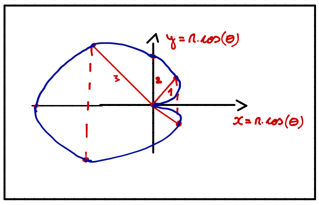
② $r = 2 - 2\cos(\theta)$

↳ $r = 2 - 2\cos(0) = 0$ | $r = 2 - 2\cos(\frac{\pi}{6}) = 1$

↳ $r = 2 - 2\cos(\frac{\pi}{2}) = 2$ | $r = 2 - 2\cos(\frac{5\pi}{6}) = 3$

↳ $r = 2 - 2\cos(\pi) = 4$ | $r = 2 - 2\cos(\frac{7\pi}{6}) = 3$

↳ $r = 2 - 2\cos(\frac{3\pi}{2}) = 0$ | $r = 2 - 2\cos(\frac{11\pi}{6}) = 1$

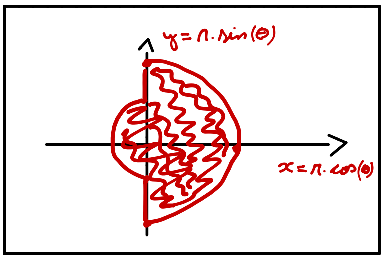


22. Cardioids

a. $r = 1 - \cos\theta$ **b.** $r = -1 + \sin\theta$

Graphing Polar Regions and Curves in the xy-Plane

25. Sketch the region defined by the inequalities $-1 \leq r \leq 2$ and $-\pi/2 \leq \theta \leq \pi/2$.

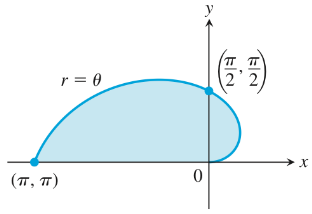


11.5:

Finding Polar Areas

Find the areas of the regions in Exercises 1–8.

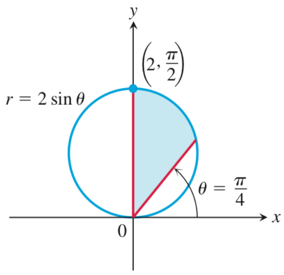
1. Bounded by the spiral $r = \theta$ for $0 \le \theta \le \pi$



① $A = \int_{\alpha}^{\beta} \frac{1}{2}\theta^2 \, d\theta = \frac{1}{2} \cdot \frac{\theta^3}{3}\Big|_0^{\pi} = \frac{\pi^3}{6}$ ▨

✳ $\boxed{\begin{array}{l} \sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \\ \cos^2(\theta) = \frac{1+\cos(2\theta)}{2} \end{array}}$

2. Bounded by the circle $r = 2\sin\theta$ for $\pi/4 \le \theta \le \pi/2$



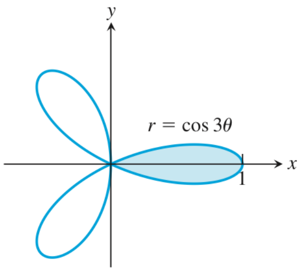
② $A = \int_{\frac{\pi}{4}}^{\pi/2} \frac{1}{\cancel{2}} \cdot \cancel{4}^{2} \sin^2\theta \, d\theta = \int_{\pi/4}^{\pi/2} 2 \cdot \sin^2\theta \, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cancel{2} \cdot \left(\frac{1-\cos 2\theta}{\cancel{2}}\right) d\theta$

$A = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 - \cos(2\theta) \, d\theta \rightarrow \boxed{\begin{array}{ll} u = 2 \cdot \theta & du = 2\,d\theta \\ \frac{du}{d\theta} = 2 & d\theta = \frac{du}{2} \end{array}} \rightarrow = \int_{\frac{\pi}{2}}^{\pi} 1 - \cos(u) \, \frac{du}{2}$

$A = \frac{1}{2}\int_{\pi/2}^{\pi} (2 - \cos(u)) \, du = \frac{1}{2}\left[2u - \sin(u)\right]\Big|_{\frac{\pi}{2}}^{\pi} = \frac{1}{2}\left[2\pi - \sin(2\pi) - \cancel{2}\frac{\pi}{\cancel{2}} + \sin\left(\frac{2\pi}{\cancel{2}}\right)\right]$

$A = \frac{1}{2}\left[\pi - \cancel{\sin(2\pi)}^{0} + \cancel{\sin(\pi)}^{0}\right] = \frac{\pi}{2} \ne \frac{\pi}{4} + \frac{1}{2}$?

6. Inside one leaf of the three-leaved rose $r = \cos 3\theta$



⑥ $A = \int_{\alpha}^{\beta} \frac{1}{2}\cos^2(3\theta) \cdot d\theta = \int_{\alpha}^{\beta} \frac{1}{2} \cdot \frac{1 + \cos(3 \cdot 2\theta)}{2} \, d\theta = \int_{\alpha}^{\beta} \frac{1}{4} \cdot (1 + \cos(6\theta)) \, d\theta$

$\boxed{\begin{array}{ll} u = 6\theta & du = 6\,d\theta \\ \frac{du}{d\theta} = 6 & d\theta = \frac{du}{6} \end{array}}$

$A = \frac{1}{4}\int_{\alpha}^{\beta} \underbrace{(1 + \cos(6\theta)) \cdot 6 \cdot d\theta}_{?} = \frac{3}{2} \cdot \left[\theta + \sin(6\theta)\right]\Big|_{\alpha = 0}^{\beta = 2\pi}$?

$A = \frac{3}{2} \cdot \left[2\pi + \cancel{\sin(6 \cdot 2\pi)}^{0} - 0 - \cancel{\sin(6 \cdot 0)}^{0}\right] = 3\pi \ne \frac{\pi}{12}$

Finding Lengths of Polar Curves

Find the lengths of the curves in Exercises 21–28.

21. The spiral $r = \theta^2$, $0 \le \theta \le \sqrt{5}$

22. The spiral $r = e^{\theta}/\sqrt{2}$, $0 \le \theta \le \pi$

$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2} \, d\theta$

$\left[\frac{e^{\theta}}{\sqrt{2}}\right]' = \frac{1}{\sqrt{2}} \cdot e^{\theta}$

㉑ $L = \int_0^{\sqrt{5}} \sqrt{\theta^4 + [2\theta]^2} \, d\theta = \int_0^{\sqrt{5}} \sqrt{\theta^4 + 2^2\theta^2} \, d\theta$

$L = \int_0^{\sqrt{5}} \theta \cdot \sqrt{\theta^2 + 4} \, d\theta \rightarrow \boxed{\begin{array}{ll} u = \theta^2 + 4 & du = 2\theta\,d\theta \\ \frac{du}{d\theta} = 2\theta & d\theta = \frac{du}{2\theta} \end{array}} \rightarrow \int_4^9 \cancel{\theta} \cdot \sqrt{u} \, \frac{du}{2\cancel{\theta}}$

$L = \frac{1}{2}\int_4^9 u^{1/2} \, du = \frac{1}{\cancel{2}} \cdot \left[\frac{u^{3/2}}{3/\cancel{2}}\right]\Big|_4^9 = \frac{9^{3/2}}{3} - \frac{4^{3/2}}{3} = \frac{1}{3} \cdot (27 - 8) = \frac{19}{3}$ ▨

㉒ $L = \int_0^{\pi} \sqrt{\left[\frac{e^{\theta}}{\sqrt{2}}\right]^2 + \left[\frac{e^{\theta}}{\sqrt{2}}\right]^2} \, d\theta = \int_0^{\pi} \sqrt{\frac{\cancel{2}e^{2\theta}}{\cancel{2}}} \, d\theta = \int_0^{\pi} e^{\theta} \, d\theta$

$L = e^{\pi} - 1$ ▨

29. **The length of the curve $r = f(\theta)$, $\alpha \le \theta \le \beta$** Assuming that the necessary derivatives are continuous, show how the substitutions

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta$$

(Equations 2 in the text) transform

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta$$

into

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta.$$

㉙ $\left[\frac{dx}{d\theta}\right]^2 = \left[f'(\theta)\cos\theta - f(\theta)\sin\theta\right]^2 = \cancel{f'(\theta)^2\cos^2\theta} - \cancel{2f'(\theta)\cos\theta f(\theta)\sin\theta}$
$+ f(\theta)^2 \cdot \sin^2(\theta)$

$\left[\frac{dy}{d\theta}\right]^2 = \left[f'(\theta)\sin\theta + f(\theta)\cos\theta\right]^2 = \cancel{f'(\theta)^2\sin^2\theta} + \cancel{2f'(\theta)\sin\theta f(\theta)\cos\theta}$
$+ f(\theta)^2\cos^2\theta$

⊕

$= f'(\theta)^2 \cdot (\cancel{\cos^2\theta + \sin^2\theta})^{1} + f(\theta)^2 \cdot (\cancel{\cos^2\theta + \sin^2\theta})^{1}$

$= f'(\theta)^2 + f(\theta)^2$

$\boxed{r = f(\theta)}$

$\therefore L = \int_{\alpha}^{\beta} \sqrt{f'(\theta)^2 + f(\theta)^2} \, d\theta = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} \, d\theta$ ▨

32. **$r = f(\theta)$ vs. $r = 2f(\theta)$** Can anything be said about the relative lengths of the curves $r = f(\theta)$, $\alpha \le \theta \le \beta$, and $r = 2f(\theta)$, $\alpha \le \theta \le \beta$? Give reasons for your answer.

$\int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2} \, d\theta = \int_{\alpha}^{\beta} \sqrt{2^2 f(\theta)^2 + 2^2 f'(\theta)^2} \, d\theta$

$\rightarrow = \int_{\alpha}^{\beta} \sqrt{4\left(f(\theta)^2 + f'(\theta)^2\right)} \, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2} \, d\theta \quad \therefore L_1 = \frac{1}{2}L_2$ ▨

Wu 8: C. 13.1: ~~1-3~~*, ~~9-11~~*, ~~19~~, 20, 25, ~~27~~, 28
C. 13.3: ~~1-5~~*, 9, ~~11~~, ~~12~~, 21

C. 14.1: 5, 6, 49-52*, 69 a, b*, 71* a, b.
C. 14.2: ~~13-14~~, 31, 35, 41*, 53, 60*.

Motion in the Plane

In Exercises 1–4, $\mathbf{r}(t)$ is the position of a particle in the xy-plane at time t. Find an equation in x and y whose graph is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of t.

1. $\mathbf{r}(t) = \underbrace{(t+1)}_{x}\mathbf{i} + \underbrace{(t^2-1)}_{y}\mathbf{j}, \quad t = 1$

$r'(t) = 1\vec{i} + 2t\vec{j} \therefore r'(1) = \vec{i} + 2\vec{j}$

$r''(t) = 2\vec{j} \therefore r''(1) = 2\vec{j}$

2. $\mathbf{r}(t) = \overbrace{\frac{t}{t+1}}^{x}\mathbf{i} + \overbrace{\frac{1}{t}}^{y}\mathbf{j}, \quad t = -\frac{1}{2}$

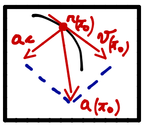
$r'(t) = \frac{\vec{i}}{(t+1)^2} - t^{-2}\vec{j} \Rightarrow r(-\tfrac{1}{2}) = 4\vec{i} - 4\vec{j}$

$r''(t) = -2\cdot(t+1)^{-3}\vec{i} + 2t^{-3}\vec{j}$
$= -2\left(\tfrac{1}{2}\right)^{-3}\vec{i} + 2\left(-\tfrac{1}{2}\right)^{-3}\vec{j}$
$= -16\vec{i} - 16\vec{j}$

3. $\mathbf{r}(t) = \underbrace{e^t}_{x}\mathbf{i} + \underbrace{\frac{2}{9}e^{2t}}_{y}\mathbf{j}, \quad t = \ln 3$

$r'(\ln 3) = e^{\ln 3}\vec{i} + \frac{4}{9}e^{2\ln 3}\vec{j} = 3\vec{i} + \frac{4\cdot\cancel{9}}{\cancel{9}}\vec{j}$

$r''(\ln 3) = e^{\ln 3}\vec{i} + \frac{8}{9}e^{2\ln 3}\vec{j} = 3\vec{i} + 8\vec{j}$



Motion in Space

In Exercises 9–14, $\mathbf{r}(t)$ is the position of a particle in space at time t. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t. Write the particle's velocity at that time as the product of its speed and direction.

9. $\mathbf{r}(t) = (t+1)\mathbf{i} + (t^2-1)\mathbf{j} + 2t\mathbf{k}, \quad t = 1$

↳ Velocity $:= v(t) = \vec{i} + 2t\vec{j} + 2\vec{k} \to v(1) = \vec{i} + 2\vec{j} + 2\vec{k}$

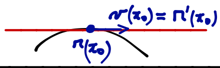
↳ Acceleration $:= a(t) = 2\vec{j} \to a(1) = 2\vec{j}$

↳ Speed $:= |v(1)| = \sqrt{1^2 + 2^2 + 2^2} = 3$

↳ direction $:= \frac{v(1)}{|v(1)|} = \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$

↳ Velocity = (speed).(direction) $= 3.\left(\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}\right)$

$\boxed{\text{Tangent line}: r(t_0) + (t - t_0)\cdot v(t_0):}$



10. $\mathbf{r}(t) = (1+t)\mathbf{i} + \frac{t^2}{\sqrt{2}}\mathbf{j} + \frac{t^3}{3}\mathbf{k}, \quad t = 1$

↳ Velocity: $v(t) = \vec{i} + \frac{2t}{\sqrt{2}}\vec{j} + t^2\vec{k}$

↳ Acceleration: $a(t) = \frac{2}{\sqrt{2}}\vec{j} + 2t\vec{k}$

↳ Speed: $|v(1)| = \sqrt{1^2 + 2 + 1^2} = 2$

↳ Direction: $\frac{v(1)}{|v(1)|} = \frac{\vec{i}}{2} + \frac{1}{\sqrt{2}}\vec{j} + \frac{1}{2}\vec{k}$

* Velocity = (speed).(direction)

$v(1) = 2.\left(\frac{\vec{i}}{2} + \frac{1\vec{j}}{\sqrt{2}} + \frac{1}{2}\vec{k}\right)$

11. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 4t\mathbf{k}, \quad t = \pi/2$

↳ $v(t) = -2\sin(t)\vec{i} + 3\cos(t)\vec{j} + 4\vec{k}$

↳ $a(t) = -2\cos(t)\vec{i} - 3\sin(t)\vec{j}$

↳ $|v(\tfrac{\pi}{2})| = \sqrt{(-2\sin(t))^2 + (3\cos(t))^2} = \sqrt{4\sin^2(t) + 9\cos^2(t) + 4^2} = 2\sqrt{5}$

↳ $\frac{v(\frac{\pi}{2})}{|v(\frac{\pi}{2})|} = \frac{-2\sin(\frac{\pi}{2})\vec{i} + 3\cos(\frac{\pi}{2})\vec{j} + 4\vec{k}}{2\sqrt{5}} = -\frac{\vec{i}}{\sqrt{5}} + \frac{2}{\sqrt{5}}\vec{k}$

↳ $v(\tfrac{\pi}{2}) = 2\sqrt{5}\left(-\frac{\vec{i}}{\sqrt{5}} + \frac{2}{\sqrt{5}}\vec{k}\right)$

Tangents to Curves

As mentioned in the text, the **tangent line** to a smooth curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ at $t = t_0$ is the line that passes through the point $(f(t_0), g(t_0), h(t_0))$ parallel to $\mathbf{v}(t_0)$, the curve's velocity vector at t_0. In Exercises 19–22, find parametric equations for the line that is tangent to the given curve at the given parameter value $t = t_0$.

19. $\mathbf{r}(t) = (\sin t)\mathbf{i} + (t^2 - \cos t)\mathbf{j} + e^t\mathbf{k}, \quad t_0 = 0$

19) $r(t) = (\sin(t))\vec{i} + (t^2 - \cos(t))\vec{j} + e^t\vec{k}$ $\quad t_0 = 0$ for $t_0 = 0$ $\to (0, -1, 1)$
$r'(t) = \cos(t)\vec{i} + (2t + \sin(t))\vec{j} + e^t\vec{k}$ $\to (1, 0, 1)$
$(0,-1,1) + (1,0,1)\cdot(t-0)$
$(t, -1, 1+t)$

$(x(t_0) + y(t_0) + z(t_0)) = \sin(t_0)\vec{i} + (t_0^2 - \cos(t_0))\vec{j} + e^{t_0}\vec{k} + (t-0)\left[\cos(t_0)\vec{i} + (2t_0 + \sin(t_0))\vec{j} + e^{t_0}\vec{k}\right]$

↳ (for $t=0$) $= \cancel{0\vec{i}} + (0^2 - 1)\vec{j} + 1\vec{k} + \cancel{t\vec{i}} + \cancel{t\,2\cdot 0\vec{j}} + \cancel{t\,0\vec{j}} + \cancel{t\,1\vec{k}}$

↳ $= t\vec{i} - 1\vec{j} + (1+t)\vec{k} \to (t, -1, 1+t)$

20. $\mathbf{r}(t) = t^2\mathbf{i} + (2t-1)\mathbf{j} + t^3\mathbf{k}, \quad t_0 = 2$

↳ Tangent line $:= r(t_0) + (t - t_0).v(t_0)$

$v(t_0) = 2t\,i + 2\,j + 3t^2\,k \to v(2) = (4; 2; 12)$

$(4; 3; 8) + (t-2).(4; 2; 12) = (4; 3; 8) + (-8 + 4t; -4 + 2t; -16 + 12t) = (-4 + 4t; -1 + 2t; -8 + 12t)$

$\neq (4+4t)i + (3+2t)j + (8+12t)k$?

25. **Motion along a parabola** A particle moves along the top of the parabola $y^2 = 2x$ from left to right at a constant speed of 5 units per second. Find the velocity of the particle as it moves through the point $(2, 2)$.

↳ $y^2 = 2x$; left to right at speed = 5 units/s

velocity at $(2,2)$ = ?

↳ $v(t)$ or (speed).(direction)

$r'(t) = |v(t)| \cdot \frac{v(t)}{|v(t)|}$

$f(x,y) = r(t) = t\,i + \sqrt{2t}\,j$ where $x = t$ and $y = \sqrt{2t}$

$r'(t) = v(t) = i + \frac{\sqrt{2}}{2}t^{-1/2}j$ at $(2,2) \to i + \sqrt{5}j \neq 2\sqrt{5}i + \sqrt{5}j$?

27. Let $\mathbf{r}$ be a differentiable vector function of t. Show that if $\mathbf{r}\cdot(d\mathbf{r}/dt) = 0$ for all t, then $|\mathbf{r}|$ is constant.

$|\vec{r}(t)| = c^2 \to \left(\sqrt{r(t)\cdot r(t)}\right)^2 = c^2 \to \vec{r}(t)\cdot\vec{r}(t) = c^2$

$\vec{r}'(t)\vec{r}(t) + \vec{r}(t)\vec{r}'(t) = 0$

$2r'(t)r(t) = 0$

$\therefore \vec{r}(t)\cdot\vec{r}'(t) = 0$

28. Derivatives of triple scalar products

a. Show that if **u**, **v**, and **w** are differentiable vector functions of t, then

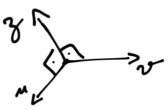
$$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}) = \frac{d\mathbf{u}}{dt}\cdot\mathbf{v}\times\mathbf{w} + \mathbf{u}\cdot\frac{d\mathbf{v}}{dt}\times\mathbf{w} + \mathbf{u}\cdot\mathbf{v}\times\frac{d\mathbf{w}}{dt}.$$

b. Show that

$$\frac{d}{dt}\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right) = \mathbf{r}\cdot\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right).$$

(*Hint:* Differentiate on the left and look for vectors whose products are zero.)

@ $\cdot$ = escalar $\rightarrow \underbrace{[0,1]}_{\vec{u}}\cdot\underbrace{[1,2]}_{\vec{v}} = \underbrace{[0,2]}_{\vec{z}}$ //



np.cross( ) = vetorial $\rightarrow \begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix} = \det = 2-1 = 1$ //

C. 13.3:

Finding Tangent Vectors and Lengths

In Exercises 1–8, find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve.

1. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + \sqrt{5}t\mathbf{k}, \quad 0 \le t \le \pi$
2. $\mathbf{r}(t) = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + 5t\mathbf{k}, \quad 0 \le t \le \pi$
3. $\mathbf{r}(t) = t\mathbf{i} + (2/3)t^{3/2}\mathbf{k}, \quad 0 \le t \le 8$
4. $\mathbf{r}(t) = (2+t)\mathbf{i} - (t+1)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 3$
5. $\mathbf{r}(t) = (\cos^3 t)\mathbf{j} + (\sin^3 t)\mathbf{k}, \quad 0 \le t \le \pi/2$

(1) $T = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{(-2\sin(t))\vec{i} + 2\cos(t)\vec{j} + \sqrt{5}\vec{k}}{\sqrt{+4\sin^2(t) + 4\cos^2(t) + 5} = 3} = -\frac{2}{3}\sin(t)\vec{i} + \frac{2}{3}\cos(t)\vec{j} + \frac{\sqrt{5}}{3}\vec{k}$ //

$\int_0^{\pi} |v|\, dt = \int_0^{\pi} 3\, dt \rightarrow [3t]_0^{\pi} = 3\pi$

(2) $\dfrac{6.2\cos(2t)\vec{i} - 6.2\sin(2t)\vec{j} + 5\vec{k}}{\sqrt{12^2\cos^2(2t) + 12^2\sin^2(2t) + 25}} = \frac{12}{13}\cos(2t)\vec{i} - \frac{12}{13}\sin(2t)\vec{j} + \frac{5}{13}\vec{k}$ //

$\sqrt{12^2\left(\frac{1+\cancel{\cos(4t)}}{2} + \frac{1-\cancel{\cos(4t)}}{2}\right) + 5^2} = 13 \therefore \int_0^{\pi} 13\, dt = 13\pi$

(3) $\dfrac{\vec{i} + \cancel{\frac{2}{3}}\cdot\cancel{\frac{3}{2}}t^{1/2}\vec{k}}{\sqrt{1^2 + t} = \sqrt{1+t}} = \frac{\vec{i}}{\sqrt{1+t}} + \frac{\sqrt{t}}{\sqrt{1+t}}\vec{k}$ // $\quad \int_0^8 (1+t)^{1/2}\, dt \quad \boxed{\begin{matrix} u = 1+t \\ du = dt \end{matrix}}$

$\int_1^9 u^{1/2}\, du = \left[\frac{2u^{3/2}}{3}\right]_1^9 = \frac{2\cdot 9^{3/2}}{3} - \frac{2\cdot 1^{3/2}}{3} = \frac{2}{3}\left(\cancel{9^{3/2}}^{27} - 1\right) = \frac{2}{\cancel{3}}\cdot\cancel{26}^{8} = 16$

Onde errei aqui?

(4) $\dfrac{\vec{i} - \vec{j} + \vec{k}}{\sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}} = \dfrac{\vec{i} - \vec{j} + \vec{k}}{\sqrt{3}} \rightarrow \int_0^3 \sqrt{3}\, dt = [\sqrt{3}t]_0^3 = 3\sqrt{3}$

(5) (?)

9. Find the point on the curve

$$\mathbf{r}(t) = (5\sin t)\mathbf{i} + (5\cos t)\mathbf{j} + 12t\mathbf{k}$$

at a distance 26π units along the curve from the point $(0, 5, 0)$ in the direction of increasing arc length.

Speed := $|\vec{v}| = \sqrt{5^2\cdot\left(\frac{1+\cancel{\cos(2t)}}{2} + \frac{1-\cancel{\cos(2t)}}{2}\right) + 12^2} = 13$ //

Direction := $\dfrac{v(t)}{|v(t)|} = \dfrac{5\cos(t)\vec{i} - 5\sin(t)\vec{j} + 12\vec{k}}{\sqrt{5^2\cos^2(t) + 5^2\sin^2(t) + 12^2}} = \frac{5}{13}\cos(t)\vec{i} - \frac{5}{13}\sin(t)\vec{j} + 12\vec{k}$ //

arc length := $\int_0^{t_0} 13\, dt = 13t_0$

$\rightarrow$ arc length $= 26\pi = 13t_0 \therefore t_0 = 2\pi$

$\rightarrow \vec{r}(0) = (0,5,0)$

$\rightarrow \vec{r}(t_0) = ? = (5\sin(2\pi); 5\cos(2\pi); 12\cdot 2\pi) = (0,5,24\pi)$

Arc Length Parameter

In Exercises 11–14, find the arc length parameter along the curve from the point where $t = 0$ by evaluating the integral

$$s = \int_0^t |\mathbf{v}(\tau)|\, d\tau$$

from Equation (3). Then find the length of the indicated portion of the curve.

11. $\mathbf{r}(t) = (4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3t\mathbf{k}, \quad 0 \le t \le \pi/2$
12. $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j}, \quad \pi/2 \le t \le \pi$

(11) speed $|\vec{v}| = \sqrt{4^2\sin^2(t) + 4^2\cos^2(t) + 9} = \sqrt{4^2\cdot\cancel{\left(\frac{1-\cos(2t)}{2} + \frac{1+\cos(2t)}{2}\right)}^{1} + 9} = 5$ //

arc length := $\int_0^{\pi/2} |\vec{v}|\, dt = \frac{5\pi}{2}$

(12) speed := $|\vec{v}| = \sqrt{(\cancel{-\sin(t)} + \cancel{1.\sin(t)} + t\cos(t))^2 + (\cancel{\cos(t)} - \cancel{1.\cos(t)} - t\sin(t))^2}$

$\rightarrow = \sqrt{t^2\cancel{(\cos^2(t) + \sin^2(t))}^{1}} = t$ //

arc length := $\int_{\pi/2}^{\pi} t\, dt = \left[\frac{t^2}{2}\right]_{\pi/2}^{\pi} = \frac{\pi^2}{2} - \frac{\pi^2}{8} = \frac{3\pi^2}{8}$

21. Distance along a line Show that if **u** is a unit vector, then the arc length parameter along the line $\mathbf{r}(t) = P_0 + t\mathbf{u}$ from the point $P_0(x_0, y_0, z_0)$ where $t = 0$, is t itself.

C.14.1: ~~5, 6, 49-52*~~, 69*, 70*

C.14.2: ~~13-14, 31, 35~~, 41*, 53, 60*.

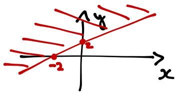
In Exercises 5–12, find and sketch the domain for each function.

5. $f(x, y) = \sqrt{y - x - 2}$

↳ Since the square root can not be negative, then $y \geq x+2$;

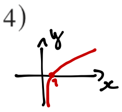
↳ Since $y \geq x+2$, as $y = mx + c$, then slope $= m = 1$ and y-intercept $= 2$;

↳ x-intercept is -2 because $0 = x+2 \rightarrow x = -2$;

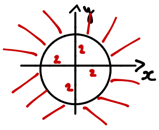
∴ 

* $D_{f(x,y)}$ is all the points above the line $y = x+2$, because $y \geq x+2$ ▨

6. $f(x, y) = \ln(x^2 + y^2 - 4)$

↳ Since the $\left(\ln(\theta) := \right.$  $\left.\right)$ has $D(0,\infty)$ and $R(-\infty,\infty)$, then x^2+y^2-4 can not be less or equal to 0;

↳ x^2+y^2-4 is a circle $(x-x_c)^2+(y-y_c)^2 = 2^2$ with center coordinates $(0,0)$ and radius 2;

∴ 

* $D_{f(x,y)}$ is all points outside the circle because $x^2+y^2-4 > 0$ ▨

PYTHON or Matlab:

Finding Level Curves

In Exercises 49–52, find an equation for and sketch the graph of the level curve of the function $f(x, y)$ that passes through the given point.

49. $f(x, y) = 16 - x^2 - y^2, \quad (2\sqrt{2}, \sqrt{2})$

↳ $f(x,y) = R$

(49) $f(x,y) = R = 16 - x^2 - y^2$

$x^2 + y^2 = 16 - R$ where $\begin{cases} r^2 = 16 - R \\ r = \sqrt{16-R} \rightarrow R > 16 \ \nexists \\ R = 16 \rightarrow r = 0 \\ R < 16 \rightarrow \text{circle} \end{cases}$

∴ $f(2\sqrt{2}, \sqrt{2}) = R = 16 - (2\sqrt{2})^2 - (\sqrt{2})^2$

$R = 6$ // → circle with radius $= 10$:

$x^2 + y^2 = 10$ ▨

50. $f(x, y) = \sqrt{x^2 - 1}, \quad (1, 0)$

51. $f(x, y) = \sqrt{x + y^2 - 3}, \quad (3, -1)$

52. $f(x, y) = \dfrac{2y - x}{x + y + 1}, \quad (-1, 1)$

(50) $f(x,y) = \sqrt{1^2 - 1} = \sqrt{0}$

∴ $\sqrt{x^2-1} = 0 \rightarrow x^2 - 1 = 0 \rightarrow x = -1$ and $x = +1$ ▨

(51) $f(x,y) = \sqrt{3+1-3} = 1$ ∴ $\sqrt{x+y^2-3} = 1 \rightarrow x + y^2 = 4$ ▨

(52) $f(x,y) = \dfrac{2+1}{-1+1+1} = 3$ ∴ $2y - x = 3x + 3y + 3 \rightarrow 4x - y = -3$ ▨

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for each of the functions in Exercises 69–72.

a. Plot the surface over the given rectangle.

b. Plot several level curves in the rectangle.

c. Plot the level curve of f through the given point.

69. $f(x, y) = x \sin \dfrac{y}{2} + y \sin 2x, \quad 0 \leq x \leq 5\pi, \quad 0 \leq y \leq 5\pi,$
$P(3\pi, 3\pi)$

71. $f(x, y) = \sin(x + 2\cos y), \quad -2\pi \leq x \leq 2\pi,$
$-2\pi \leq y \leq 2\pi, \quad P(\pi, \pi)$

Ch. 14.2:

Limits of Quotients

Find the limits in Exercises 13–24 by rewriting the fractions first.

13. $\lim\limits_{\substack{(x,y)\to(1,1) \\ x\neq y}} \dfrac{x^2 - 2xy + y^2}{x - y}$

14. $\lim\limits_{\substack{(x,y)\to(1,1) \\ x\neq y}} \dfrac{x^2 - y^2}{x - y}$

(13) $x^2 - 2xy + y^2 = (x-y)^2$

$$\lim_{\substack{x,y\to 1,1 \\ x\neq y}} \frac{(x-y)^{\cancel{2}}}{\cancel{(x-y)}} = (1-1) = 0$$

(14) $x^2 - y^2 = (x+y)(x-y)$

$$\lim_{\substack{x,y\to 1,1 \\ x\neq y}} \frac{(x+y)\cancel{(x-y)}}{\cancel{(x-y)}} = (1+1) = 2$$

Continuity for Two Variables

At what points (x, y) in the plane are the functions in Exercises 31–34 continuous? $\longrightarrow f'(+x_0) = f'(-x_0) = f'(x_0)$

31. **a.** $f(x, y) = \sin(x + y)$ **b.** $f(x, y) = \ln(x^2 + y^2)$

(a)

↳ Since $\sin(\theta)$ and $\cos(\theta)$ are continuous for all real values, then $\sin(x+y)$ is also continuous for all real values.

(b) Since $D_{\ln} = (0, \infty)$ and Range $(-\infty, \infty)$ and x^2 and y^2 can only be positive numbers, then (x,y) can be any positive real value, ecluded 0.

Continuity for Three Variables

At what points (x, y, z) in space are the functions in Exercises 35–40 continuous?

35. **a.** $f(x, y, z) = x^2 + y^2 - 2z^2$

b. $f(x, y, z) = \sqrt{x^2 + y^2 - 1}$

(a) $x^2 + y^2 = 2 \cdot z^2 \longrightarrow (x - x_c)^2 + (y - y_c)^2 = 2 \cdot z^2$

where $(x_c, y_c) = (0,0)$ and $r^2 = 2z^2 \to r = \sqrt{2} \cdot z$

↳ This is a circle with center points on $(0,0)$ and radius $\sqrt{2} \cdot z$;

↳ Then, any real number can be (x, y, z)

(b) * Square root can not be less than zero;

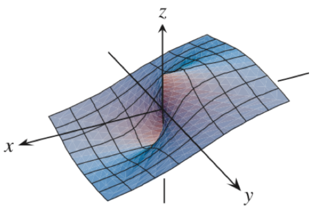
* $x^2 + y^2 - 1$ is a circle with center $(0,0)$ and radius 1;

* $f(x,y,z)$ is continuous for all points outside of the circle, in other words for $\forall (x,y,z) \subset \mathbb{R}$ given $x^2 + y^2 - 1 \geq 0$

No Limit Exists at the Origin

By considering different paths of approach, show that the functions in Exercises 41–48 have no limit as $(x, y) \to (0, 0)$.

41. $f(x, y) = -\dfrac{x}{\sqrt{x^2 + y^2}}$



53. Show that the function in Example 6 has limit 0 along every straight line approaching $(0, 0)$.

60. Continuous extension Define $f(0, 0)$ in a way that extends

$$f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2}$$

to be continuous at the origin.

~~14.3: 1-10, 23-26, 31, 32, 41, 43, 45, 51-53, 62*, 73~~
~~14.4: 1-3, 7-10, 13-15, 25, 26, 29, 30~~

Calculating First-Order Partial Derivatives

In Exercises 1–22, find $\partial f/\partial x$ and $\partial f/\partial y$.

1. $f(x, y) = 2x^2 - 3y - 4$
2. $f(x, y) = x^2 - xy + y^2$
3. $f(x, y) = (x^2 - 1)(y + 2)$
4. $f(x, y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$
5. $f(x, y) = (xy - 1)^2$
6. $f(x, y) = (2x - 3y)^3$
7. $f(x, y) = \sqrt{x^2 + y^2}$
8. $f(x, y) = (x^3 + (y/2))^{2/3}$
9. $f(x, y) = 1/(x + y)$
10. $f(x, y) = x/(x^2 + y^2)$

① $\dfrac{\partial f(x,y)}{\partial x} = 4x$ and $\dfrac{\partial f}{\partial y} = -3$

② $\dfrac{\partial f}{\partial x} = 2x - 1y$ and $\dfrac{\partial f}{\partial y} = -x + 2y$

③ $\dfrac{\partial f}{\partial x} = 2x(y+2)$ and $\dfrac{\partial f}{\partial y} = (x^2-1)$

④ $\dfrac{\partial f}{\partial x} = 5y - 14x + 3$ and $\dfrac{\partial f}{\partial y} = 5x - 2y - 6$

⑤ $\dfrac{\partial f}{\partial x} = \left[x^2y^2 - 2xy + 1^2\right]'_x = 2xy^2 - 2y = 2y(xy-1)$

$\dfrac{\partial f}{\partial y} = 2x^2y - 2x = 2x(xy-1)$

⑥ $\dfrac{\partial f}{\partial x} = \dfrac{\partial (\mu)^3}{\partial \mu} \cdot \dfrac{\partial (2x-3y)}{\partial x} = 3.\mu^2 . 2 = 6(2x-3y)^2$

$\dfrac{\partial f}{\partial y} = 3.\mu^2 . (-3) = -9(2x-3y)^2$

⑦ $\dfrac{\partial f}{\partial x} = \sqrt{x^2+y^2} \xrightarrow{\text{chain rule}} \dfrac{d(\sqrt{\mu})}{d\mu} \cdot \dfrac{d(x^2+y^2)}{dx} = \dfrac{1}{\cancel{2}\sqrt{\mu}} \cdot \cancel{2}x = \dfrac{x}{\sqrt{x^2+y^2}}$ and $\dfrac{\partial f}{\partial y} = \dfrac{1}{2\sqrt{\mu}} \cdot 2y = \dfrac{y}{\sqrt{x^2+y^2}}$

⑧ $\dfrac{\partial (u)^{2/3}}{\partial \mu} \cdot \dfrac{\partial (x^3 + y/2)}{\partial x} = \dfrac{2}{\cancel{3}} u^{-1/3} . \cancel{3} . x^2 = \dfrac{2x^2}{\sqrt[3]{x^3+y/2}}$ and $\dfrac{\partial f}{\partial y} = \dfrac{2}{3\mu^{1/3}} \cdot \dfrac{1}{2} = \dfrac{1}{3\left(\sqrt[3]{x^3+y/2}\right)}$

exponent rule $\frac{1}{\mu} = \mu^{-1}$

⑨ $\dfrac{\partial \frac{1}{x+y}}{\partial x} = \dfrac{\partial (\mu)^{-1}}{\partial \mu} \cdot \dfrac{\partial\, x+y}{\partial x} = -\mu^{-2} . 1 = \dfrac{-1}{(x+y)^2}$ and $\dfrac{\partial f}{\partial y} = -\mu^{-2} . 1 = \dfrac{-1}{(x+y)^2}$

⑩ $\dfrac{\partial \left(x/(x^2+y^2)\right)}{\partial x} = \dfrac{1.(x^2+y^2) - x . 2x}{(x^2+y^2)^2} = \dfrac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$\dfrac{\partial f}{\partial y} = \left[x . \dfrac{1}{x^2+y^2}\right]'_y = x . \dfrac{2y}{-(x^2+y^2)} = -\dfrac{2xy}{x^2+y^2}$

In Exercises 23–34, find f_x, f_y, and f_z.

23. $f(x, y, z) = 1 + xy^2 - 2z^2$
24. $f(x, y, z) = xy + yz + xz$
25. $f(x, y, z) = x - \sqrt{y^2 + z^2}$
26. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$

31. $f(x, y, z) = e^{-(x^2+y^2+z^2)}$
32. $f(x, y, z) = e^{-xyz}$

㉓ $f_x = y^2$; $f_y = 2xy$; $f_z = -4z$

㉔ $f_x = y + z$; $f_y = x + z$; $f_z = y + x$

㉕ $f_x = 1$; $f_y = -\dfrac{1}{\cancel{2}\sqrt{\mu}} . \cancel{2}y = \dfrac{-y}{\sqrt{y^2+z^2}}$; $f_z = \dfrac{-z}{\sqrt{y^2+z^2}}$

㉖ $f_x = -\dfrac{1}{\cancel{2}\mu^{-3/2}} . \cancel{2}x = \dfrac{-x}{(x^2+y^2+z^2)^{-3/2}}$

㉛ $f(x,y,z) = e^{-x^2} . e^{-y^2} . e^{-z^2} = e^{-\mu^2}$

$f_x = -1.e^{-2\mu} . 2x = -2x e^{-2(x^2+y^2+z^2)}$ and $f_y = -2y e^{-2(x^2+y^2+z^2)}$ and $f_z = -2z e^{-2(x^2+y^2+z^2)}$

㉜ $f_x = -yz e^{-xyz}$; $f_y = -xz e^{-xyz}$; $f_z = -xy e^{-xyz}$

Calculating Second-Order Partial Derivatives

Find all the second-order partial derivatives of the functions in Exercises 41–50.

41. $f(x, y) = x + y + xy$ **42.**

43. $g(x, y) = x^2y + \cos y + y \sin x$

45. $r(x, y) = \ln(x + y)$

$\to r_x = \frac{1}{x+y} = (x+y)^{-1} \to r_{xx} = r_{xy} = \frac{-1}{(x+y)^2}$

$r_y = \frac{1}{x+y} = (x+y)^{-1} \to r_{yx} = r_{yy} = \frac{-1}{(x+y)^2}$

(41) $f_x = 1+y \to f_{xx} = 0$; $f_{xy} = 1$

$f_y = 1+x \to f_{yy} = 0$; $f_{yx} = 1$

(43) $g_x = 2xy + y\cos(x) \to g_{xx} = 2y - y\sin(x)$

$g_{xy} = 2x + \cos(x)$

$g_y = x^2 - \sin(y) + \sin(x) \to g_{yy} = -\cos(y)$

$g_{yx} = 2x + \cos(x)$

Mixed Partial Derivatives

In Exercises 51–54, verify that $w_{xy} = w_{yx}$.

51. $w = \ln(2x + 3y)$ **52.** $w = e^x + x \ln y + y \ln x$

53. $w = xy^2 + x^2y^3 + x^3y^4$ **54.** $w = x \sin y + y \sin x + xy$

(51) $w_x = \frac{2}{2x+3y} \to w_{xy} = -\frac{6}{(2x+3y)^2}$

$w_y = \frac{3}{2x+3y} \to w_{yx} = \frac{-6}{(2x+3y)^2}$

$\therefore w_{xy} = w_{yx}$

(52) $w_x = e^x + \ln(y) + \frac{y}{x} \to w_{xy} = \frac{1}{y} + \frac{1}{x}$

$w_y = \frac{x}{y} + \ln(x) \to w_{yx} = \frac{1}{y} + \frac{1}{x}$

(53) $w_x = y^2 + 2xy^3 + 3x^2y^4 \to w_{xy} = 2y + 6xy^2 + 12x^2y^3$

$w_y = 2xy + 3x^2y^2 + 4x^3y^3 \to w_{yx} = 2y + 6xy^2 + 12x^2y^3$

(54) $w_x = \sin(y) + y\cos(x) + y \to w_{xy} = \cos(y) + \cos(x) + 1$ $w_y = x\cos(y) + \sin(x) + x \to w_{yx} = \cos(y) + \cos(x) + 1$

62. Let $f(x, y) = x^2 + y^3$. Find the slope of the line tangent to this surface at the point $(-1, 1)$ and lying in the **a.** plane $x = -1$ **b.** plane $y = 1$.

Recall: $y = mx + b$ where m slope and b y-intercept;

$m = \frac{y-y_1}{x-x_1}$ and $f'(x)$ = slope of tangent line at x.

tangent line $\to y - y_1 = m.(x - x_1)$

or $r(t) = (1,0,-2)$

(62) $f_x(x,y) = 2x$ and $f_y(x,y) = 3y^2$

$f_x(-1,1) = 2 \cdot -1 = -2$ and $f_y(-1,1) = 3 \cdot 1^2 = 3$

(a)

Show that each function in Exercises 73–80 satisfies a Laplace equation. $\to f_{xx} + f_{yy} + f_{zz} = 0$

73. $f(x, y, z) = x^2 + y^2 - 2z^2 \to$

$$\begin{cases} f_x = 2x \to f_{xx} = 2 \\ f_y = 2y \to f_{yy} = 2 \\ f_z = -4z \to f_{zz} = -4 \end{cases}$$

$\therefore f_{xx} + f_{yy} + f_{zz} = 2 + 2 - 4 = 0$

Chain Rule: One Independent Variable

In Exercises 1–6, **(a)** express dw/dt as a function of t, both by using the Chain Rule and by expressing w in terms of t and differentiating directly with respect to t. Then **(b)** evaluate dw/dt at the given value of t.

1. $w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t; \quad t = \pi$
2. $w = x^2 + y^2, \quad x = \cos t + \sin t, \quad y = \cos t - \sin t; \quad t = 0$
3. $w = \frac{x}{z} + \frac{y}{z}, \quad x = \cos^2 t, \quad y = \sin^2 t, \quad z = 1/t; \quad t = 3$

$\hookrightarrow w(t) = \cos^2(t)\cdot t + \sin^2(t)\cdot t = t$ ⓐ $\frac{dw(t)}{dt} = 1$ and ⓑ $\frac{dw(3)}{dt} = 1$

① $w(t) = \cos^2(t) + \sin^2(t) = 1$

ⓐ $\frac{dw(t)}{dt} = 0$ ⓑ $\frac{dw(\pi)}{dt} = 0$

② $w(t) = (\cos(t) + \sin(t))^2 + (\cos(t) - \sin(t))^2$

$w(0) = (1+0)^2 + (1-0)^2 = 2$

ⓐ $\frac{dw(t)}{dt} = 0$ and $\frac{dw(0)}{dt} = 0$

Chain Rule: Two and Three Independent Variables

In Exercises 7 and 8, **(a)** express $\partial z/\partial u$ and $\partial z/\partial v$ as functions of u and v both by using the Chain Rule and by expressing z directly in terms of u and v before differentiating. Then **(b)** evaluate $\partial z/\partial u$ and $\partial z/\partial v$ at the given point (u, v).

7. $z = 4e^x \ln y, \quad x = \ln(u \cos v), \quad y = u \sin v;$
$(u, v) = (2, \pi/4)$

⑦ ⓐ $\frac{\partial z}{\partial u} = \left[4 e^{\ln(u\cdot\cos(v))} \ln(u\cdot\sin(v))\right]' =$

$= \left[4\cdot u\cdot\cos(v)\cdot\ln(u\cdot\sin(v))\right]'_u =$

$= 4\cos(v)\left[u\cdot\ln(u\cdot\sin(v))\right]'_u =$

$= 4\cos(v)\left[\ln(u\sin(v)) + \frac{\cancel{u}\cdot\cancel{\sin(v)}\,1}{\cancel{u}\cdot\sin(v)}\right] =$

$= 4\cos(v)\cdot\left[\ln(u\cdot\sin(v)) + 1\right]$

$\frac{\partial z}{\partial v} = 4u\left[-\sin(v)\cdot\ln(u\sin(v)) + \frac{\cos(v)\cdot\cancel{u}\cdot\cos(v)}{\cancel{u}\sin(v)}\right]$

$= 4\cdot u\cdot\left[-\sin(v)\ln(u\sin(v)) + \frac{\cos^2(v)}{\sin(v)}\right]$

$z(u,v) = 4\cdot u\cdot\cos(v)\cdot\ln(u\sin(v))$

ⓑ $z_u = 4\cdot\cos\left(\frac{\pi}{4}\right)\cdot\left[\ln\left(2\cdot\sin\left(\frac{\pi}{4}\right)\right) + 1\right] = 2\sqrt{2}\left[\ln\left(\cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}}\right) + 1\right] = 2\sqrt{2}\cdot\ln(\sqrt{2}) + 2\sqrt{2} = \sqrt{2}(\ln(2) + 2)$

$z_v = 8\left[-\frac{\sqrt{2}}{2}\cdot\ln\left(\cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}}\right) + \frac{\left(\frac{\sqrt{2}}{2}\right)^{\cancel{2}}}{\frac{\cancel{\sqrt{2}}}{\cancel{2}}}\right] = \cancel{8}^{4}\left[-\frac{\sqrt{2}}{\cancel{2}}\cdot\ln(\sqrt{2}) + \frac{\sqrt{2}}{\cancel{2}}\right] = -4\sqrt{2}\ln(\sqrt{2}) + 4\sqrt{2} = -2\sqrt{2}\ln(2) + 4\sqrt{2}$

8. $z = \tan^{-1}(x/y), \quad x = u\cos v, \quad y = u \sin v;$
$(u, v) = (1.3, \pi/6)$

$z = \tan^{-1}\left(\frac{\cancel{u}\cdot\cos(v)}{\cancel{u}\cdot\sin(v)}\right) = \arctan(\cot v)$

Remark: $\left[\tan^{-1}(x)\right]' = \left[\arctan(x)\right]' = \frac{1}{x^2+1}$ $\quad \csc^2(x) - \cot^2(x) = 1$

$\left[\frac{\cos(x)}{\sin(x)}\right]' = \left[\cot(x)\right]' = -\csc^2(v)$

ⓐ $\frac{\partial z}{\partial u} = 0$; $\frac{\partial z}{\partial v} = \left[\arctan(\cot(v))\right]'_v = \frac{\partial \arctan(u)}{\partial u}\cdot\frac{\partial \cot(v)}{\partial v} = \frac{1}{\cot^2(v)+1}\cdot(-\csc^2(v))$

$\hookrightarrow = -\frac{\csc^2(v)}{\cot^2(v)+1} = \frac{-\csc^2(v)}{\csc^2(v)} = -1$

ⓑ $z_u = 0$; $z_v = -1$

In Exercises 9 and 10, **(a)** express $\partial w/\partial u$ and $\partial w/\partial v$ as functions of u and v both by using the Chain Rule and by expressing w directly in terms of u and v before differentiating. Then **(b)** evaluate $\partial w/\partial u$ and $\partial w/\partial v$ at the given point (u, v).

9. $w = xy + yz + xz, \quad x = u + v, \quad y = u - v, \quad z = uv;$
$(u, v) = (1/2, 1)$

(a)
$$\frac{\partial w(x,y,z)}{\partial u} = \frac{\partial w(x,y,z)}{\partial x}\cdot\frac{\partial x(u,v)}{\partial u} = (y+z)\cdot(1) = u - v + uv$$
$$+\ \frac{\partial w(x,y,z)}{\partial y}\cdot\frac{\partial y(u,v)}{\partial u} = (x+z)\cdot(1) = u + v + uv$$
$$+\ \frac{\partial w(x,y,z)}{\partial z}\cdot\frac{\partial z(u,v)}{\partial u} = (y+x)\cdot(v) = (u - \cancel{v} + u + \cancel{v})\cdot v = 2uv$$

$$\therefore\ \frac{\partial w(x,y,z)}{\partial u} = u - \cancel{v} + uv + u + \cancel{v} + uv + 2uv = 2u + 4uv = 2u(1+2v)$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} = (y+z)\cdot(1) = u - v + uv$$
$$+\ \frac{\partial w(x,y,z)}{\partial y}\cdot\frac{\partial y(u,v)}{\partial v} = (x+z)\cdot(-1) = -u - v - uv$$
$$+\ \frac{\partial w(x,y,z)}{\partial z}\cdot\frac{\partial z(u,v)}{\partial v} = (y+x)\cdot(u) = (u - \cancel{v} + u + \cancel{v})\cdot u = 2u^2$$

$$\therefore\ \frac{\partial w}{\partial v} = \cancel{u} - v + \cancel{uv} - \cancel{u} - v - \cancel{uv} + 2u^2 = 2u^2 - 2v = 2(u^2 - v)$$

$$w(u,v) = \big[(u+v)(u-v)\big] + \big[(u-v)\cdot uv\big] + \big[(u+v)\,uv\big] = u^2 - v^2 + u^2v - \cancel{uv^2} + u^2v + \cancel{uv^2} = u^2 + 2u^2v - v^2$$

$$\frac{\partial w(u,v)}{\partial u} = 2u + 4uv \quad \text{and} \quad \frac{\partial w(u,v)}{\partial v} = 2u^2 - 2v$$

(b) $(u, v) = (\frac{1}{2}, 1)$

$$\frac{\partial w(u,v)}{\partial u} = 1 + 2 = 3 \quad \text{and} \quad \frac{\partial w(u,v)}{\partial v} = \frac{1}{2} - 2 = -\frac{3}{2}$$

10. $w = \ln(x^2 + y^2 + z^2), \quad x = ue^v \sin u, \quad y = ue^v \cos u,$
$z = ue^v; \quad (u, v) = (-2, 0)$

(a) $$w(u,v) = \ln\left(u^2 e^{2v}\sin^2(u) + u^2 e^{2v}\cos^2(u) + u^2 e^{2v}\right) = \ln\left(u^2\left(e^{2v} + e^{2v} + e^{2v}\right)\right) = \ln\left(3\cdot u^2\cdot e^{2v}\right)$$

$$\therefore\ \frac{\partial w(u,v)}{\partial u} = \frac{1}{\cancel{3}u^{\cancel{2}}\cancel{e^{2v}}}\cdot \cancel{6}^{\,2}\cdot\cancel{u}\cdot\cancel{e^{2v}} = \frac{2}{u} \quad \text{and} \quad \frac{\partial w(u,v)}{\partial v} = \frac{1}{\cancel{3\cdot u^2 e^{2v}}}\cdot \cancel{3u^2}\ 2\cancel{e^{2v}} = 2$$

(b) $(u, v) = (-2, 0)$

$$\therefore\ \frac{\partial w(u,v)}{\partial u} = \frac{2}{-2} = -1 \quad \text{and} \quad \frac{\partial w(u,v)}{\partial v} = 2$$

Using a Branch Diagram

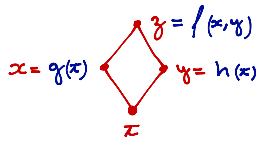
In Exercises 13–24, draw a branch diagram and write a Chain Rule formula for each derivative.

13. $\dfrac{dz}{dt}$ for $z = f(x, y), \quad x = g(t), \quad y = h(t)$

14. $\dfrac{dz}{dt}$ for $z = f(u, v, w), \quad u = g(t), \quad v = h(t), \quad w = k(t)$

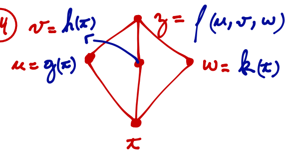
15. $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$ for $w = h(x, y, z), \quad x = f(u, v), \quad y = g(u, v), \quad z = k(u, v)$

(13)



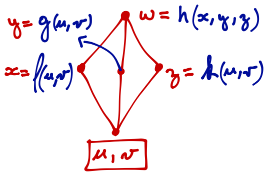
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

(14)



$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial t} + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial t}$$

(15)



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

and

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

Implicit Differentiation

Assuming that the equations in Exercises 25–28 define y as a differentiable function of x, use Theorem 8 to find the value of dy/dx at the given point.

25. $x^3 - 2y^2 + xy = 0, \quad (1, 1)$

26. $xy + y^2 - 3x - 3 = 0, \quad (-1, 1)$

(25) $$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2+y}{-4y+x} = \frac{-4}{-3} = \frac{4}{3}$$

(26) $$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y-3}{x+2y} = -\frac{1-3}{-1+2} = -\frac{-2}{1} = 2$$

Find the values of $\partial z/\partial x$ and $\partial z/\partial y$ at the points in Exercises 29–32.

29. $z^3 - xy + yz + y^3 - 2 = 0, \quad (1, 1, 1)$

30. $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} - 1 = 0, \quad (2, 3, 6)$

(29) $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-y}{3z^2+y} = -\frac{-1}{3+1} = \frac{1}{4}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-x+z+3y^2}{3z^2+y} = -\frac{-1+1+3}{4} = -\frac{3}{4}$$

(30) $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\frac{1}{x^2}}{\frac{1}{z^2}} = -\frac{z^2}{x^2} = -\frac{36}{4} = -9$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_x} = -\frac{\frac{1}{y^2}}{\frac{1}{z^2}} = -\frac{z^2}{y^2} = -\frac{36}{9} = -4$$

~~14.5: 1-3, 7, 9-13, 15-17, 19-22, 39*~~

~~14.7: 1-3**, 13-16, 18-19, 31-32, 34-35**, 41, 47, 54~~

Calculating Gradients

In Exercises 1–6, find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point.

1. $f(x, y) = y - x, \quad (2, 1)$
2. $f(x, y) = \ln(x^2 + y^2), \quad (1, 1)$
3. $g(x, y) = xy^2, \quad (2, -1)$
4. $g(x, y) = \dfrac{x^2}{2} - \dfrac{y^2}{2}, \quad (\sqrt{2}, 1)$

(1) $\nabla f(x,y) = \dfrac{\partial f}{\partial x}\vec{i} + \dfrac{\partial f}{\partial y}\vec{j}$

$\dfrac{\partial f}{\partial x} = -1 \,;\, \dfrac{\partial f}{\partial y} = 1 \;\therefore\; \nabla f = -\vec{i} + \vec{j}$

⊛ $f(2,1) = 1 - 2 = -1$

$-1 = y - x \rightarrow$ level curve

$(2,1)\quad 1-2=-1$ ✓

$(1,0)\quad 0-1=-1$ ✓

$(0,-1)\quad -1-0=-1$ ✓ ▨

(2) $\dfrac{\partial f}{\partial x} = [\ln(u)]'_u \cdot [x^2+y^2]'_x = \dfrac{1}{u}\cdot 2x = \dfrac{2x}{x^2+y^2}$ and $\dfrac{\partial f}{\partial y} = \dfrac{2y}{x^2+y^2}$

$\nabla f(x,y) = \dfrac{2x}{x^2+y^2}\vec{i} + \dfrac{2y}{x^2+y^2}\vec{j} \rightarrow \nabla f(1,1) = \vec{i} + \vec{j}$ and $f(1,1) = \ln(1^2+1^2) = \ln(2) \rightarrow \ln(2) = \ln(x^2+y^2) \rightarrow 2 = x^2+y^2 \rightarrow$ level curve ↳ circle

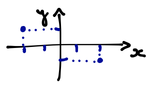
(3) $\dfrac{\partial g}{\partial x} = y^2$ and $\dfrac{\partial g}{\partial y} = 2xy \rightarrow \nabla g(x,y) = y^2\vec{i} + 2xy\vec{j} \rightarrow \nabla g(2,-1) = 1.\vec{i} - 4\vec{j}$

$g(2,-1) = -2 \rightarrow xy^2 = -2 \rightarrow$ level curve

⊛ como acho desenho?

$(2,-1)\quad -2=-2$

$(-2,1)\quad -2=-2$



(4) $\dfrac{\partial g}{\partial x} = x$ and $\dfrac{\partial g}{\partial y} = -y \;\therefore\; \nabla g(x,y) = x\vec{i} - y\vec{j} \rightarrow \nabla g(\sqrt{2},1) = \sqrt{2}\,\vec{i} - \vec{j}$

$g(x,y) = \dfrac{x^2}{2} - \dfrac{y^2}{2} \rightarrow g(\sqrt{2},1) = \dfrac{1}{2} \;\therefore\; x^2 - y^2 = 1 \rightarrow$ level curve

$y^2 = x^2 - 1$

⊛ como acho desenho?

In Exercises 7–10, find ∇f at the given point.

7. $f(x, y, z) = x^2 + y^2 - 2z^2 + z\ln x, \quad (1, 1, 1)$
9. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} + \ln(xyz), \quad (-1, 2, -2)$
10. $f(x, y, z) = e^{x+y}\cos z + (y + 1)\sin^{-1} x, \quad (0, 0, \pi/6)$

(7) $\dfrac{\partial f}{\partial x} = 2x + \dfrac{z}{x} \,;\, \dfrac{\partial f}{\partial y} = 2y \,;\, \dfrac{\partial f}{\partial z} = -4z + \ln(x)$

$\nabla f = 3\vec{i} + 2\vec{j} - 4\vec{k}$ ▨

(9) $\dfrac{\partial f}{\partial x} = \left[(u)^{-1/2}\right]'_u \cdot \left[x^2+y^2+z^2\right]'_x + \left[\ln(v)\right]'_v \cdot \left[xyz\right]'_x = -\dfrac{(u)^{-3/2}}{2}\, 2x + \dfrac{1}{v}\cdot yz = \dfrac{-x}{(x^2+y^2+z^2)^{3/2}} + \dfrac{1}{x}$ where $\begin{cases} u = x^2+y^2+z^2 \\ v = xyz \end{cases}$

$\dfrac{\partial f}{\partial x}(-1,2,-2) = \dfrac{-(-1)}{(9)^{3/2}} - 1 = \dfrac{1}{\sqrt{9^3}} - \dfrac{\sqrt{9^3}}{\sqrt{9^3}} = \dfrac{1-\sqrt{9^3}}{\sqrt{9^3}} = \dfrac{1-27}{27} = -\dfrac{26}{27}\,\vec{i}$

$\dfrac{\partial f}{\partial y} = \dfrac{-y}{(x^2+y^2+z^2)^{3/2}} + \dfrac{1}{y} \rightarrow \dfrac{\partial f}{\partial y}(-1,2,-2) = \dfrac{-2}{27} + \dfrac{1}{2} = \dfrac{-4+27}{27.2} = \dfrac{23}{54}$

$\dfrac{\partial f}{\partial z} = \dfrac{-z}{(x+y+z)^{3/2}} + \dfrac{1}{z} = \dfrac{4}{2\,27} - \dfrac{27}{27.2} = \dfrac{-23}{54}$

$\therefore\; \nabla f(-1,2,-2) = -\dfrac{26}{27}\vec{i} + \dfrac{23}{54}\vec{j} - \dfrac{23}{54}\vec{k}$ ▨

(10) $e^{x+y}\cdot\cos(z) + (y+1)\sin^{-1}(x)$ for $(0, 0, \frac{\pi}{6})$

$e^x \cdot e^y \cdot \cos(z) + (y+1)\cdot \text{arc}\sin(x)$

$\dfrac{\partial f}{\partial x} = e^y \cdot e^x \cdot \cos(z) + (y+1)\left(\dfrac{1}{\sqrt{1-x^2}}\right) \,;\, \dfrac{\partial f}{\partial y} = e^x e^y \cos(z) + \sin^{-1}(x) \,;\, \dfrac{\partial f}{\partial z} = -e^x e^y \sin(z)$

$\nabla f = \left(\dfrac{\sqrt{3}}{2} + 1\right)\vec{i} + \left(\dfrac{\sqrt{3}}{2} + 0\right)\vec{j} - \dfrac{1}{2}\vec{k}$ ▨

Remark:

$\sin^{-1}(x) = \text{arc}\sin(x)$

$[\text{arc}\sin x]' = \dfrac{1}{\sqrt{1-x^2}}$

Finding Directional Derivatives

In Exercises 11–18, find the derivative of the function at P_0 in the direction of $\mathbf{u}$.

11. $f(x, y) = 2xy - 3y^2$, $P_0(5, 5)$, $\mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$
12. $f(x, y) = 2x^2 + y^2$, $P_0(-1, 1)$, $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$
13. $g(x, y) = \dfrac{x - y}{xy + 2}$, $P_0(1, -1)$, $\mathbf{u} = 12\mathbf{i} + 5\mathbf{j}$

(11) $\dfrac{\partial f}{\partial x} = 2y$; $\dfrac{\partial f(5,5)}{\partial x} = 10$

$\dfrac{\partial f}{\partial y} = 2x - 6y$; $\dfrac{\partial f(5,5)}{\partial y} = 10 - 30 = -20$

$\nabla f(5,5) = 10\vec{i} - 20\vec{j}$

⊛ from direction vector to unit direction vector:

$\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} := \dfrac{4i + 3j}{\sqrt{4^2 + 3^2}} = \dfrac{4}{5}\vec{i} + \dfrac{3}{5}\vec{j}$

⊛ $(D_u f)_{P_0} = \nabla f(5;5) \cdot \vec{u}$

$(D_u f)_{P_0} = (10\vec{i} - 20\vec{j}) \cdot \left(\frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}\right) = 8 - 12 = 4$ ▨

(12) $\dfrac{\partial f}{\partial x} = 4x \therefore f_x(-1,1) = -4$; $\dfrac{\partial f}{\partial y} = 2y \therefore f_y(-1,1) = 2$

$\hookrightarrow \nabla f(-1,1) = -4\vec{i} + 2\vec{j}$

$\rightarrow \dfrac{u}{|u|} = \dfrac{3\vec{i} - 4\vec{j}}{\sqrt{3^2 + (-4)^2}} = \dfrac{3}{5}\vec{i} - \dfrac{4}{5}\vec{j}$

$(D_u f)_{P_0} = (-4) \cdot \frac{3}{5} + 2 \cdot \left(-\frac{4}{5}\right)$

$= -\dfrac{12}{5} - \dfrac{8}{5} = \dfrac{-20}{5} = -4$ ▨

(13) $\dfrac{\partial f}{\partial x} = \dfrac{1 \cdot (xy+2) - (x-y) \cdot (y)}{(xy+2)^2} = \dfrac{\cancel{xy} + 2 - \cancel{xy} + y^2}{xy^2 + 4xy + 4} = \dfrac{y^2 + 2}{(xy+2)^2} \rightarrow \dfrac{\partial f}{\partial x}(1,-1) = \dfrac{(-1)^2 + 2}{(-1+2)^2} = 3$

$\dfrac{\partial f}{\partial y} = \dfrac{-1 \cdot (xy+2) - (x-y) \cdot x}{(xy+2)^2} = \dfrac{\cancel{-xy} - 2 - x^2 + \cancel{yx}}{(xy+2)^2} = \dfrac{-x^2 - 2}{(xy+2)^2} \longrightarrow \dfrac{\partial f(1,-1)}{\partial y} = \dfrac{-1-2}{1} = -3$

$\hookrightarrow \nabla f(1,-1) = 3\vec{i} - 3\vec{j}$

$\hookrightarrow \dfrac{u}{|u|} = \dfrac{12\vec{i} + 5\vec{j}}{\sqrt{12^2 + 5^2}} = \dfrac{12\vec{i} + 5\vec{j}}{\sqrt{144 + 25}} = \dfrac{12}{13}\vec{i} + \dfrac{5}{13}\vec{j}$

$(D_u f)_{P_0} = 3 \cdot \frac{12}{13} - 3 \cdot \frac{5}{13} = \frac{36 - 15}{13} = \frac{21}{13}$ ▨

15. $f(x, y, z) = xy + yz + zx$, $P_0(1, -1, 2)$, $\mathbf{u} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$
16. $f(x, y, z) = x^2 + 2y^2 - 3z^2$, $P_0(1, 1, 1)$, $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
17. $g(x, y, z) = 3e^x \cos yz$, $P_0(0, 0, 0)$, $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

(15) $\dfrac{\partial f}{\partial x} = y + z \rightarrow \dfrac{\partial f(1,-1,2)}{\partial x} = 1$

$\dfrac{\partial f}{\partial y} = x + z \rightarrow \dfrac{\partial f(1,-1,2)}{\partial y} = 3$

$\dfrac{\partial f}{\partial z} = y + x \rightarrow \dfrac{\partial f(1,-1,2)}{\partial z} = 0$

$\therefore \nabla f(1,-1,2) = \vec{i} + 3\vec{j} + 0\vec{k}$

$\dfrac{u}{|u|} = \dfrac{3\vec{i} + 6\vec{j} - 2\vec{k}}{\sqrt{3^2 + 6^2 + (-2)^2}} = \dfrac{3}{7}\vec{i} + \dfrac{6}{7}\vec{j} - \dfrac{2}{7}\vec{k}$

$\therefore (D_u f)_{P_0} = 1 \cdot \frac{3}{7} + 3 \cdot \frac{6}{7} + 0 \cdot -\frac{2}{7} = \frac{21}{7} = 3$ ▨

(16) $\dfrac{\partial f(1,1,1)}{\partial x} = 2x = 2$; $\dfrac{\partial f(1,1,1)}{\partial y} = 4y = 4$; $\dfrac{\partial f(1,1,1)}{\partial z} = -6z = -6$

$\therefore \nabla f(1,1,1) = 2\vec{i} + 4\vec{j} - 6\vec{k}$

$\hookrightarrow \dfrac{u}{|u|} = \dfrac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{1^2 + 1^2 + 1^2}} = \dfrac{1}{\sqrt{3}}\vec{i} + \dfrac{1}{\sqrt{3}}\vec{j} + \dfrac{1}{\sqrt{3}}\vec{k}$

$(D_u f)_{P_0} = 2 \cdot \frac{1}{\sqrt{3}} + 4\frac{1}{\sqrt{3}} + (-6) \cdot \frac{1}{\sqrt{3}} = 0$ ▨

(17) $\dfrac{\partial f(0,0,0)}{\partial x} = 3e^x \cos(yz) = 3$; $\dfrac{\partial f(0,0,0)}{\partial y} = -3e^x \sin(yz) \cdot z = 0$; $\dfrac{\partial f(0,0,0)}{\partial z} = 0$

$\therefore \nabla f(0,0,0) = 3i$

$\dfrac{u}{|u|} = \dfrac{2\vec{i} + \vec{j} - 2\vec{k}}{\sqrt{2^2 + 1^2 + (-2)^2}} = \dfrac{2}{3}\vec{i} + \dfrac{1}{3}\vec{j} - \dfrac{2}{3}\vec{k}$

$(D_u f)_{P_0} = \cancel{3} \cdot \frac{2}{\cancel{3}} = 2$ ▨

In Exercises 19–24, find the directions in which the functions increase and decrease most rapidly at P_0. Then find the derivatives of the functions in these directions.

19. $f(x, y) = x^2 + xy + y^2, \quad P_0(-1, 1)$

20. $f(x, y) = x^2y + e^{xy}\sin y, \quad P_0(1, 0)$

21. $f(x, y, z) = (x/y) - yz, \quad P_0(4, 1, 1)$

22. $g(x, y, z) = xe^y + z^2, \quad P_0(1, \ln 2, 1/2)$

(19)

$$\left.\begin{aligned}\frac{\partial f}{\partial x}(-1,1) &= 2x+y = -1\\ \frac{\partial f}{\partial y}(-1,1) &= x+2y = 1\end{aligned}\right] \quad \nabla f(-1,1) = -\vec{i}+\vec{j}$$

direction: $\dfrac{-\vec{i}+\vec{j}}{\sqrt{(-1)^2+1^2}} = -\dfrac{1}{\sqrt{2}}\vec{i}+\dfrac{1}{\sqrt{2}}\vec{j}$

increase most rapidly $:= -\dfrac{1}{\sqrt{2}}\vec{i}+\dfrac{1}{\sqrt{2}}\vec{j} = \vec{\mu}$

$\hookrightarrow (D_\mu f)_{P_0} = (-1)\cdot\left(-\dfrac{1}{\sqrt{2}}\right) + 1\cdot\dfrac{1}{\sqrt{2}} = \dfrac{2}{\sqrt{2}}\cdot\dfrac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$

decrease most rapidly $:= \dfrac{1}{\sqrt{2}}\vec{i} - \dfrac{1}{\sqrt{2}}\vec{j} = -\vec{\mu}$

$\hookrightarrow (D_{-\mu} f)_{P_0} = (-1)\cdot\dfrac{1}{\sqrt{2}} + 1\cdot -\dfrac{1}{\sqrt{2}} = -\dfrac{2}{\sqrt{2}}\cdot\dfrac{\sqrt{2}}{\sqrt{2}} = -\sqrt{2}$ ▨

(20) $\dfrac{\partial f}{\partial x}(1,0) = 2xy + ye^{xy}\sin(y) = 2.1.0 + 0.e^0.\sin(0) = 0$

$\dfrac{\partial f}{\partial y}(1,0) = x^2 + \left(xe^{xy}\sin y + e^{xy}\cos(y)\right) = 1 + \left(1.e^0.\sin(0) + e^0.\cos(0)\right)$

$\hookrightarrow = 1+1 = 2$

$\nabla f(1,0) = 2\vec{j} \rightarrow \dfrac{2\vec{j}}{\sqrt{2^2}} = \vec{j} = \mu$ and $-\mu = -\vec{j}$

$(D_\mu f)_{P_0} = 2.1 = 2$ and $(D_{-\mu} f)_{P_0} = 2.(-1) = -2$ ▨

(21) $\dfrac{\partial f}{\partial x}(4,1,1) = \dfrac{1}{y} = 1$; $\dfrac{\partial f}{\partial y}(4,1,1) = -x-z = -5$ and $\dfrac{\partial f}{\partial z}(4,1,1) = -y = -1$

$\nabla f(4,1,1) = \vec{i} - 5\vec{j} - \vec{k}$

$\mu = \dfrac{\vec{i}-5\vec{j}-\vec{k}}{\sqrt{1^2+5^2+(-1)^2}} = \dfrac{1}{\sqrt{27}}\vec{i} - \dfrac{5}{\sqrt{27}}\vec{j} - \dfrac{\vec{k}}{\sqrt{27}} \longrightarrow (D_\mu f)_{P_0} = 1.\dfrac{1}{\sqrt{27}} + (-5)\left(\dfrac{-5}{\sqrt{27}}\right) + (-1)\left(\dfrac{-1}{\sqrt{27}}\right) = \dfrac{27}{3\sqrt{3}}$

$-\mu = -\dfrac{\vec{i}}{\sqrt{27}} + \dfrac{5\vec{j}}{\sqrt{27}} + \dfrac{\vec{k}}{\sqrt{27}} \longrightarrow (D_{-\mu} f)_{P_0} = 1\left(\dfrac{-1}{\sqrt{27}}\right) + (-5)\left(\dfrac{5}{\sqrt{27}}\right) + (-1)\cdot\dfrac{1}{\sqrt{27}} = -\dfrac{27}{3\sqrt{3}}\cdot\dfrac{3\sqrt{3}}{3\sqrt{3}} = \dfrac{27}{27}(-3\sqrt{3}) = -\sqrt{3}$ ▨

(22)

$$\left.\begin{aligned}\frac{\partial f}{\partial x}(1, \ln(2), 1/2) &= e^y = e^{\ln 2} = 2\\ \frac{\partial f}{\partial y} &= xe^y = 2\\ \frac{\partial f}{\partial z} &= 2z = 1\end{aligned}\right]$$

$\nabla f(1, \ln(2), 1/2) = 2\vec{i} + 2\vec{j} + \vec{k}$

$\mu = \dfrac{2\vec{i}+2\vec{j}+\vec{k}}{\sqrt{2^2+2^2+1^2}} = \dfrac{2}{3}\vec{i} + \dfrac{2}{3}\vec{j} + \dfrac{1}{3}\vec{k}$ and $-\mu = -\dfrac{2}{3}\vec{i} - \dfrac{2}{3}\vec{j} - \dfrac{1}{3}\vec{k}$

$(D_\mu f)_{P_0} = 2.\dfrac{2}{3} + 2.\dfrac{2}{3} + 1.\dfrac{1}{3} = 3$ and $(D_\mu f)_{P_0} = 2.\left(-\dfrac{2}{3}\right) + 2.\left(-\dfrac{2}{3}\right) + 1.-\dfrac{1}{3} = -3$ ▨

39. **Lines in the *xy*-plane** Show that $A(x - x_0) + B(y - y_0) = 0$ is an equation for the line in the xy-plane through the point (x_0, y_0) normal to the vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j}$.

$A(x-x_0) + B(y-y_0) = 0$:= tangent line to level curve

Since $\dfrac{\partial f}{\partial x}(x,y) = A$ and $\dfrac{\partial f}{\partial y}(x,y) = B$, Then $\nabla f(x,y) = A\vec{i} + B\vec{j}$ ▨

Ch. 14.7)

Finding Local Extrema

Find all the local maxima, local minima, and saddle points of the functions in Exercises 1–30.

1. $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$
2. $f(x, y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$
3. $f(x, y) = x^2 + xy + 3x + 2y + 5$

③ $f_x = 2x + y + 3 = 0 \to y = 1$; $f_{xx} = 2$; $f_{xy} = 1$

$f_y = x + 2 = 0 \to x = -2$; $f_{yy} = 0$

Since $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 = 2.0 - 1^2 = -1 < 0$,

then $f(-2,1) = (-2)^2 + (-2)1 + 3.(-2) + 2.1 + 5 = 3$ is saddle point.

① $f_x(x,y) = 2x + y + 3 = 0$

$f_y(x,y) = x + 2y - 3 = 0$

∴ $2(3-2y) + y + 3 = 0$

$6 - 4y + y + 3 = 0$

$3y = 9 \to y = 3$

and

$x + 2.(3) - 3 = 0$

$x = -3$

⊛ Critical points $(-3;3)$

$f_{xx} = 2$ and $f_{yy} = 2$

and $f_{xy} = 1$

∴ Since $f_{xx}f_{yy} - 1^2 = 2.2 - 1 = 3 > 0$ and $f_{xx} > 0$,

then $f(-3;3) = -5$ is the local minimum.

② $f_x(x,y) = 2y - 10x + 4 = 0 \to y = \frac{10x-4}{2} = 5x - 2$

$f_y(x,y) = 2x - 4y + 4 = 0 \to 2x - 4(5x-2) + 4 = 0$

$2x - 20x + 8 + 4 = 0$

$x - 10x + 6 = 0$

$3\cancel{9}x = \cancel{6}2$

$x = \frac{2}{3}$ and $y = 5.\frac{2}{3} - 2$

$y = \frac{10-6}{3}$

$y = 4/3$

$f_{xx} = -10$; $f_{yy} = -4$

$f_{xy} = 2$

∴ Since $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 = 40 - 4 = 38 > 0$, then $f(\frac{2}{3};\frac{4}{3}) = 2.\frac{2}{3}.\frac{4}{3} - 5.\frac{2^2}{3^2} - 2\frac{4^2}{3^2} + 4.\frac{2}{3} + 4.\frac{4}{3} - 4$

$f(\frac{2}{3};\frac{4}{3}) = \frac{16}{9} - \frac{20}{9} - \frac{32}{9} + \frac{8}{3} + \frac{16}{3} - 4 = \frac{-12}{3} + \frac{8}{3} + \frac{16}{3} - 4$

$= \frac{12}{3} - 4 = 0$ is local maximum.

13. $f(x, y) = x^3 - y^3 - 2xy + 6$
14. $f(x, y) = x^3 + 3xy + y^3$

⑬ $f_x = 3x^2 - 2y = 0 \longrightarrow 3(-\frac{3}{2}y^2)^2 - 2y = 0$

$f_y = -3y^2 - 2x = 0 \to x = -\frac{3}{2}y^2$

$3.\frac{9.y^4}{4} - 2y = 0 \longrightarrow \frac{27y^4}{4} - 2y = 0$

$y.(\frac{27}{4}y^3 - 2) = 0 \longrightarrow y = 0$ and $27y^3 = 8 \to y = \frac{2}{3}$

for $y = 0 \to x = -\frac{3}{2}.0^2 = 0$

for $y = \frac{2}{3} \longrightarrow x = -\frac{3}{2}.(\frac{2}{3})^2 = -\frac{\cancel{3}^1}{\cancel{2}_1}.\frac{\cancel{4}^2}{\cancel{9}_3} = \frac{-2}{3}$

Critical points are $(0,0)$ and $(-\frac{2}{3};\frac{2}{3})$

$f_{xx} = 6x \to f_{xx}(0;0) = 0$ and $f_{xx}(-\frac{2}{3};\frac{2}{3}) = 6.(-\frac{2}{3}) = -4$ and $f_{xy}(0,0) = f_{xy}(-\frac{2}{3};\frac{2}{3}) = -2$

$f_{yy} = -6y \longrightarrow f_{yy}(0;0) = 0$ and $f_{yy}(-\frac{2}{3};\frac{2}{3}) = -6.\frac{2}{3} = -4$

⊛ Since $f_{xx}(0;0)f_{yy}(0;0) - f_{xy}(0;0)^2 = -4 < 0$, then $f(0,0) = 6$ is saddle point;

⊛ Since $f_{xx}(-\frac{2}{3};\frac{2}{3}).f_{yy}(-\frac{2}{3};\frac{2}{3}) - f_{xy}(-\frac{2}{3};\frac{2}{3})^2 = (-4).(-4) - 2 = 14 > 0$ and $f_{xx}(-\frac{2}{3};\frac{2}{3}) = -4 < 0$, then $f(-\frac{2}{3};\frac{2}{3}) = (-\frac{2}{3})^3 - (\frac{2}{3})^3 - 2.(-\frac{2}{3}).\frac{2}{3} + 6$

$f(-\frac{2}{3};\frac{2}{3}) = -\frac{8}{27} - \frac{8}{27} + \frac{8}{9} + 6 = -\frac{16}{27} + \frac{24}{27} + \frac{27.6}{27} = \frac{8+27.6}{27} = \frac{170}{27}$ is local maximum.

⑭ $f(x,y) = x^3 + 3xy + y^3$

$f_x = 3x^2 + 3y = 0 \longrightarrow 3(-y^2)^2 + 3y = 0 \to 3y(y^3 + 1) = 0 \longrightarrow y_1 = 0$ and $y_2 = -1$

$f_y = 3y^2 + 3x = 0 \to x = -y^2 \to x = 0$ and $x = -1$

⊛ $f_{xx}(0;0) = 6.0 = 0$ and $f_{yy}(0;0) = 6.0 = 0$ and $f_{xy}(0;0) = 3$

Since $f_{xx}(0,0).f_{yy}(0,0) - f_{xy}(0,0)^2 = 0.0 - 3^2 = -9 < 0$, then $f(0,0) = 0$ is saddle point.

⊛ $f_{xx}(-1,-1) = 6.(-1) = -6$ and $f_{yy}(-1;-1) = 6.(-1) = -6$ and $f_{xy}(1;-1) = 3$

Since $f_{xx}(1;-1).f_{yy}(1;-1) - f_{xy}(1;-1)^2 = (-6).(-6) - 3^2 = +36 - 9 = +27 > 0$ and $f_{xx}(-1;-1) = -6 < 0$, then

$f(-1;-1) = (-1)^3 + 3.1.1 + (-1)^3 = 1$ is local maximum.

15. $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$

16. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$

(15) $f_x(x,y) = 12x - 2x^2 + 6y = 0 \longrightarrow 12\cdot(-y) - 2\cdot(-y)^2 + 6y = 0 \rightarrow -\cancel{12}^{6}y - \cancel{2}y^2 + \cancel{6}^{3}y = 0 \rightarrow -y^2 - 3y = 0$

$-y\cdot(y+3) = 0$

$f_y(x,y) = 6y + 6x = 0 \longrightarrow x = -y \rightarrow x_1 = 0$ and $x_2 = -(-3) = 3$

$\therefore y_1 = 0$ and $y_2 = -3$

$f_{xx}(0;0) = 12 - 4x = 12$ and $f_{yy}(0;0) = 6$ and $f_{xy}(0;0) = 6$

Since $12\cdot 6 - 6^2 = 72 - 36 = 36 > 0$ and $f_{xx}(0;0) = 6 > 0$, then $f(0,0) = 0$ is local minimum.

$f_{xx}(3;-3) = 12 - 4\cdot 3 = 0$ and $f_{yy} = 6$ and $f_{xy} = 6$

Since $0\cdot 6 - 6^2 = -36 < 0$, then $f(3;-3) = 6\cdot(+3)^2 - 2(+3)^3 + 3\cdot(-3)^2 + 6\cdot 3\cdot(-3) = \cancel{6}^{5}\cdot 27 - \cancel{2\cdot 27} + \cancel{3\cdot 9} - 54 = 81$ is saddle point.

(16) $f(x;y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$

$f_x = 3x^2 + 6x = 0 \longrightarrow 3x(x+2) = 0 \longrightarrow x_1 = 0$ and $x_2 = -2$

$f_y = 3y^2 - 6y = 0 \longrightarrow 3y(y-2) = 0 \longrightarrow y_1 = 0$ and $y_2 = 2$

Critical points: $(0;0)$; $(-2;2)$; $(0;2)$; $(-2;0)$

$f_{xx}(0;0) = 6x + 6 = 6$; $f_{yy}(0;0) = 6y - 6 = -6$; $f_{xy}(0;0) = 0$

Since $6\cdot(-6) - 0^2 = -36 < 0$, then $f(0;0) = -8$ is saddle point.

$f_{xx}(-2;2) = 6\cdot(-2) + 6 = -6$; $f_{yy}(-2;2) = 6\cdot 2 - 6 = 6$ and $f_{xy}(-2;2) = 0$

Since $(-6)\cdot 6 - 0^2 = -36 < 0$, then $f(-2;2) = \cancel{(-2)^3} + \cancel{2^3} + 3\cdot(-2)^2 - 3\cdot 2^2 - 8 = -12 - 12 - 8 = -32$ is saddle point.

$f_{xx}(0;2) = 6\cdot 0 + 6 = 6$; $f_{yy}(0;2) = 6\cdot 2 - 6 = 6$; $f_{xy}(0;2) = 0$

Since $6\cdot 6 - 0^2 = 36 > 0$ and $f_{xx} = 6 > 0$, then $f(0;2) = \cancel{2^3} - 3\cdot 2^2 - \cancel{8} = -12$ is local minimum.

$f_{xx}(-2;0) = 6\cdot(-2) + 6 = -6$ and $f_{yy}(-2;0) = 6\cdot 0 - 6 = -6$; $f_{xy}(-2;0) = 0$

Since $(-6)\cdot(-6) - 0^2 = 36 > 0$ and $f_{xx} = -6 < 0$, then $f(-2;0) = (-2)^3 + 3(-2)^2 - 8 = -4$ is local maximum.

18. $f(x, y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$

(18) $f_x = 6x^2 - 18x = 0 \longrightarrow 6x(x-3) = 0 \rightarrow x_1 = 0$; $x_2 = 3$

$f_y = 6y^2 + 6y - 12 = 0 \longrightarrow y^2 + y - 2 = 0 \longrightarrow S = \frac{-b}{a} = -1$, $P = \frac{c}{a} = -2$; $y_1 = +1$, $y_2 = -2$

Critical points: $(0;1)$; $(0;-2)$; $(3;1)$; $(3;-2)$

$f_{xx}(0;1) = 12x - 18 = -18$; $f_{yy}(0;1) = 12y + 6 = 18$ and $f_{xy}(0;1) = 0$

Since $(-18)\cdot(18) - 0^2 = -324 < 0$, then $f(0,1) = 2\cdot 1^3 + 3\cdot 1^2 - 12\cdot 1 = 7$ is saddle point.

$f_{xx}(0;-2) = 12x - 18 = -18$; $f_{yy}(0;1) = 12y + 6 = -18$ and $f_{xy}(0;1) = 0$

Since $(-18)\cdot(-18) - 0^2 = +324 > 0$ and $f_{xx}(0;-2) = -18 < 0$, then $f(0;-2) = 2\cdot(-2)^3 + 3\cdot(-2)^2 - 12(-2) = -16 + 12 + 24 = 20$ is local maximum.

$f_{xx}(3;1) = 12x - 18 = +18$; $f_{yy}(0;1) = 12y + 6 = 18$ and $f_{xy}(0;1) = 0$

$\cancel{72}$
59
23

Since $(+18)\cdot(18) - 0^2 = +324 > 0$ and $f_{xx}(3;1) = 18 > 0$, then $f(3;1) = 2\cdot(3)^3 + 2\cdot 1^3 - 9\cdot 3^2 + 3\cdot 1^2 - 12\cdot 1 = \cancel{54} + \cancel{2} - \cancel{81} + \cancel{3} - \cancel{12} = 11$ is local minimum.

$f_{xx}(3;-2) = 12x - 18 = +18$; $f_{yy}(0;1) = 12y + 6 = -18$ and $f_{xy}(0;1) = 0$

Since $(+18)\cdot(-18) - 0^2 = -324 < 0$, then $f(3;-2) = 2\cdot(3)^3 + 2\cdot(-2)^3 - 9\cdot 3^2 + 3\cdot(-2)^2 - 12\cdot(-2) = 54 - \cancel{16} - 82 + \cancel{12} + \cancel{24} = -8$ is saddle point.

19. $f(x, y) = 4xy - x^4 - y^4$

$f_x = 4y - 4x^3 = 0 \to -x^3 + y = 0 \to -(y^3)^3 + y = 0 \to -y^9 + y = 0 \to y(1 - y^8) = 0 \to y_1 = 0$ and $y_2 = 1$ and $y_3 = -1$

$f_y = 4x - 4y^3 = 0 \to x - y^3 = 0 \to x = y^3 \to x_1 = 0^3 = 0$ and $x_2 = 1^3 = 1$ and $x_3 = -1$

Critical points: $(0;0)$; $(1;1)$; $(-1;-1)$

* $f_{xx}(0;0) = -12x^2 = 0$ and $f_{yy}(0;0) = -12y^2 = 0$ and $f_{xy}(0;0) = 4$

→ Since $0.0 - 4^2 = -16 < 0$, then $f(0;0) = 0$ is saddle point.

* $f_{xx}(1;1) = -12.1^2 = -12$ and $f_{yy}(1;1) = -12.1^2 = -12$ and $f_{xy}(1;1) = 4$

→ Since $(-12).(-12) - 4^2 = 128 > 0$ and $f_{xx}(1;1) = -12 < 0$, then $f(1;1) = 4.1.1 - 1^4 - 1^4 = 2$ is local maximum.

* $f_{xx}(-1;-1) = -12.(-1)^2 = -12$ and $f_{yy}(-1;-1) = -12(-1)^2 = -12$ and $f_{xy}(1;1) = 4$

→ Since $(-12).(-12) - 4^2 = 128 > 0$ and $f_{xx}(-1;-1) = -12 < 0$, then $f(1;1) = 4.(-1).(-1) - (-1)^4 - (-1)^4 = 2$ is local maximum.

Finding Absolute Extrema

In Exercises 31–38, find the absolute maxima and minima of the functions on the given domains.

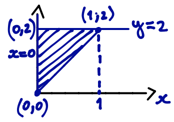
31. $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant

32. $D(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines $x = 0$, $y = 4$, $y = x$

34. $T(x, y) = x^2 + xy + y^2 - 6x$ on the rectangular plate $0 \leq x \leq 5$, $-3 \leq y \leq 3$

35. $T(x, y) = x^2 + xy + y^2 - 6x + 2$ on the rectangular plate $0 \leq x \leq 5$, $-3 \leq y \leq 0$

31



$f_x = 4x - 4 = 0 \to f_x(0,2) = -4$

$f_x(0;0) = -4$

$f_x(1;2) = 4$

$f_y = 2y - 4 = 0 \to f_y(0;2) = 0$

$f_y(0;0) = -4$

$f_y(1;2) = 0$

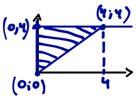
∴ Critical points := $x = 1$ and $y = 2$

∴ $f_{xx} = 4$; $f_{yy} = 2$; $f_{xy} = 0$

∴ For $(1;2; f(1;2) = -5)$: Since $4.2 - 0^2 = 8 > 0$ and $f_{xx} = 4 > 0$,

Then $f(1,2) = \cancel{2} - \cancel{4} + \cancel{4} - \cancel{8} + \cancel{1} = -5$ is local minimum.

32



$f_x = 2x - y = 0 \longrightarrow 2(2y) - y = 0 \to y = 0$

$f_y = -x + 2y = 0 \to x = 2y \to x = 0$

Critical point $(0;0)$: $f_{xx} = 2$; $f_{yy} = 2$; $f_{xy} = -1$

Since $2.2 - (-1)^2 = 3 > 0$ and $f_{xx} = 2 > 0$, then $f(0;0) = 1$ is the local minimum.

34 $f_x = 2x + y - 6 = 0 \longrightarrow 2(-2y) + y - 6 = 0 \to y = -2$

$f_y = x + 2y = 0 \to x = -2y \to x = 4$

→ Critical point: $(4;-2)$: $f_{xx} = 2$; $f_{yy} = 2$; $f_{xy} = 1$

→ Since $2.2 - 1^2 = 3 > 0$ and $f_{xx} = 2 > 0$, then $f(4;-2) = 4^2 + 4.(-2) + (-2)^2 - 6.4 = 16 - 8 + 4 - 20 = -10$ is local minimum.

35 $f_x = 2x + y - 6 = 0 \longrightarrow 2(-2y) + y = 6 \to y = -2$

$f_y = x + 2y = 0 \to x = -2y \to x = 4$

critical point $(4;-2)$: $f_{xx} = 2$; $f_{yy} = 2$; $f_{xy} = 1$

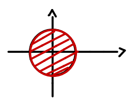
Since $2.2 - 1^2 = 3 > 0$ and $f_{xx} = 2 > 0$, then $f(4;-2) = -10$ is local minimum.

41. Temperatures A flat circular plate has the shape of the region $x^2 + y^2 \leq 1$. The plate, including the boundary where $x^2 + y^2 = 1$, is heated so that the temperature at the point (x, y) is

$$T(x, y) = x^2 + 2y^2 - x.$$

* Find the temperature of the hottest and coldest points on the plane.

41



$f_x = 2x - 1 = 0 \to x = \frac{1}{2}$

$f_y = 4y = 0 \to y = 0$

Critical point: $(\frac{1}{2};0)$

→ $f_{xx} = 2$; $f_{yy} = 4$; $f_{xy} = 0$

→ Since $2.4 - 0^2 = 8 > 0$ and $f_{xx} = 2 > 0$, then $f(\frac{1}{2};0) = \frac{1}{4} + 2.0^2 - \frac{1}{2} = -\frac{3}{4}$ is the local minimum.

47. If $f_x(a, b) = f_y(a, b) = 0$, must f have a local maximum or minimum value at (a, b)? Give reasons for your answer.

→ for example: $f(x,y) = xy$; $f_x(a,b) = y = b$; $f_y(a,b) = x = a$; $f_{xx}(a,b) = 0$; $f_{yy}(a,b) = 0$ and $f_{xy}(a,b) = 1$.

→ Since $f_{xx}(a,b) \cdot f_{yy}(a,b) - f_{xy}(a,b)^2 = 0.0 - 1^2 = -1 < 0$, then $f(a,b)$ is a saddle point. ▨

54. Find three positive numbers whose sum is 3 and whose product is a maximum.

→ $f(x,y,z) = xyz$

$x + y + z = 3 \longrightarrow z = 3 - x - y$

where $x, y, z > 0$

$\therefore f(x,y) = xy(3-x-y) = 3xy - x^2y - xy^2$

$f_x = 3y - 2xy - y^2 = 0 \rightarrow y(3-2x-y) = 0 \rightarrow y = 0$ and $y = 3 - 2x$

$f_y = 3x - x^2 - 2xy = 0 \rightarrow x(3-x-2y) = 0 \rightarrow x = 0$ and $x = 3 - 2y$

$y = 3 - 2(3-2y) \rightarrow y = 3 - 6 + 4y \rightarrow 3y = 3 \rightarrow y = 1$ and $x = 1$

$f_{xx} = -2y$; $f_{yy} = -2x$; $f_{xy} = 3 - 2x - 2y$

For $x = 1$; $y = 1$; $z = 1$: $-2 \cdot (-2) - (3-2-2)^2 = 4 - 1 = 3 > 0$ and $f_{xx} = -2 < 0 \longrightarrow$ local maximum ▨

Ch. 14.8: ~~1-6, 17-20, 37, 38, 41~~

Ch. 15.1: ~~1, 3, 9, 15, 23-25~~

Two Independent Variables with One Constraint

1. **Extrema on an ellipse** Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x, y) = xy$ has its extreme values.

$$① \quad \nabla f = (y, x) = \lambda \nabla g = (\lambda 2x, \lambda 4y)$$

$$\therefore \begin{cases} y = \lambda 2x \\ x = \lambda 4y \\ \underbrace{x^2 + 2y^2 = 1} \end{cases} \qquad (\lambda 4y)^2 + 2(\lambda 2x)^2 = 1$$

$$\lambda^2 16y^2 + \lambda^2 8x^2 = 1$$

$$8\lambda^2 \underbrace{(x^2 + 2y^2)}_{1} = 1$$

$$\therefore 8\lambda^2 = 1 \rightarrow \lambda = \pm \frac{1}{2\sqrt{2}} \cdot \frac{2\sqrt{2}}{2\sqrt{2}} = \pm\frac{\cancel{2}\sqrt{2}}{4 \cdot \cancel{2}} = \pm\frac{\sqrt{2}}{4}$$

$$\therefore \begin{cases} y = \pm\frac{\sqrt{2}}{\cancel{4}\,2} \cdot \cancel{2}x \rightarrow y^2 = \left(\pm\frac{\sqrt{2}}{2}x\right)^2 = +\frac{1}{2}x^2 \\ x = \pm\frac{\sqrt{2}}{\cancel{4}} \cdot \cancel{4}y \rightarrow x^2 = \left(\pm\sqrt{2}\,y\right)^2 = +2y^2 \\ x^2 + 2y^2 = 1 \end{cases}$$

$$\therefore 2y^2 + 2y^2 = 1 \rightarrow 4y^2 = 1 \rightarrow y = \pm\frac{1}{2}$$

$$\text{and } x = \pm\sqrt{2} \cdot \left(\pm\frac{1}{2}\right) = \pm\frac{\sqrt{2}}{2}$$

∴ The extreme values are $\left(\pm\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ and $\left(\pm\frac{\sqrt{2}}{2}, -\frac{1}{2}\right)$

$$\therefore f\left(\pm\frac{\sqrt{2}}{2}; \pm\frac{1}{2}\right) = \pm\frac{\sqrt{2}}{4} \neq \pm\frac{\sqrt{2}}{2} \quad ?$$

2. **Extrema on a circle** Find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 10 = 0$.

$$\nabla f = (y, x) = \lambda \nabla g = (\lambda 2x, \lambda 2y)$$

$$\begin{cases} y = \lambda 2x \\ x = \lambda 2y \\ x^2 + y^2 = 10 \end{cases} \qquad y = \lambda 2 \cdot (\lambda y) \rightarrow y = \lambda^2 \cdot 2y \therefore y = 0 \text{ or } \lambda = \pm\frac{1}{2}$$

for $y = 0 \rightarrow x = 0 \rightarrow$ not possible because $0^2 + 0^2 - 10 \neq 0$

$$\text{for } \lambda = \pm\frac{1}{2} \rightarrow (x)^2 = \left(2y \cdot \left(\pm\frac{1}{2}\right)\right)^2 \rightarrow x^2 = \cancel{4}y^2 \cdot \frac{1}{\cancel{4}} = y^2$$

$$\therefore y^2 + y^2 = 10 \rightarrow y = \pm\sqrt{5} \text{ and } x = \cancel{2}(\pm\sqrt{5}) \cdot \left(\pm\frac{1}{\cancel{2}}\right) = \pm\sqrt{5}$$

∴ Extreme values: $(\pm\sqrt{5}, \sqrt{5})$; $(\pm\sqrt{5}, -\sqrt{5})$

$$\therefore f(\pm\sqrt{5}, \pm\sqrt{5}) = \pm\sqrt{5} \cdot (\pm\sqrt{5}) = \pm 5$$

3. **Maximum on a line** Find the maximum value of $f(x, y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$.

$$\nabla f = (2x, 2y);\ \nabla g = (1, 3) \rightarrow (2x, 2y) = (\lambda, 3\lambda) \rightarrow x = \frac{\lambda}{2} \text{ and } y = \frac{3}{2}\lambda \rightarrow \lambda = 2x = \frac{2}{3}y$$

$$\frac{\lambda}{2} + 3 \cdot \frac{3}{2}\lambda = 10 \rightarrow \lambda = 2 \therefore x = 1 \text{ and } y = 3 \therefore f(1;3) = 49 - 1 - 9 = 39$$

4. **Extrema on a line** Find the local extreme values of $f(x, y) = x^2y$ on the line $x + y = 3$.

$$\nabla f = (2xy, x^2) = \lambda \nabla g = (\lambda, \lambda) \rightarrow \begin{cases} 2xy = \lambda \\ x^2 = \lambda \\ x + y = 3 \end{cases} \rightarrow 2xy = x^2 \rightarrow \text{for } x \neq 0:\ y = \frac{x^{\cancel{2}}}{\cancel{x}\,2} = \frac{x}{2}$$

$$\therefore \frac{2x}{2} + \frac{x}{2} = 3 \rightarrow x = 2$$

$$\text{for } x = 0:\ x + y = 0 + y = y = 3$$

∴ Extreme points: $(2, 1)$; $(0, 3)$

$$\hookrightarrow f(2;1) = 4 \text{ and } f(0;3) = 0$$

5. **Constrained minimum** Find the points on the curve $xy^2 = 54$ nearest the origin.

$$f(x;y) = \sqrt{x^2 + y^2} \xrightarrow{\sqrt{\ }\text{ is crescent function, then equals}} f(x;y) = x^2 + y^2$$

$$\nabla f = (2x, 2y) = \lambda \nabla g = (\lambda y^2, \lambda x 2y) \rightarrow \begin{matrix} 2x = \lambda y^2 \\ y = \lambda x y \end{matrix}$$

for $y = 0 \rightarrow x = 0 \rightarrow 0.0^2 \neq 54 \rightarrow$ not possible, then $y \neq 0$ and

$$\cancel{y} = \lambda x \cancel{y} \rightarrow \lambda x = 1 \rightarrow x = \frac{1}{\lambda}$$

$$\text{for } y \neq 0,\ x = \frac{1}{\lambda}:\ 2 \cdot \frac{1}{\lambda} = \lambda \cdot y^2 \rightarrow \frac{2}{\lambda^2} = y^2$$

$$\therefore \frac{1}{\lambda} \cdot \frac{2}{\lambda^2} = 54 \rightarrow \lambda^3 = \frac{2}{54} \rightarrow \lambda = \frac{1}{3}$$

$$\therefore x = \frac{1}{6}y^2 \text{ and } \cancel{y} = \frac{1}{3} \cdot \frac{1}{6}y^2 \cdot \cancel{y} \rightarrow 18 = y^2 \rightarrow y = \pm 3\sqrt{2}$$

$$x = \frac{1}{6}(\pm 3\sqrt{2})^2 = \frac{\cancel{9}^{3} \cdot \cancel{2}^{1}}{\cancel{3}\,\cancel{6}} = 3$$

∴ Extreme points: $(3, \pm 3\sqrt{2})$

6. Constrained minimum Find the points on the curve $x^2y = 2$ nearest the origin.

$\nabla f = (2x, 2y) = \lambda \nabla g = (\lambda 2xy, \lambda x^2) \to \begin{array}{l} 2x = 2xy\lambda \to 2x = \lambda x^2 \cdot x\lambda \\ 2y = \lambda x^2 \end{array}$

for $x = 0 \to y = 0 \to 0^2 \cdot 0 \neq 2$ not possible

then $x \neq 0$

$\therefore \begin{array}{l} 2 = \lambda^2 x^2 \\ 2y = \lambda x^2 \\ x^2 y = 2 \end{array}$ $\quad \dfrac{\cancel{2}}{\cancel{2}y} = \dfrac{\lambda^{\cancel{2}} x^{\cancel{2}}}{\cancel{\lambda} \cancel{x^2}} \to \dfrac{1}{y} = \lambda \to y = \dfrac{1}{\lambda}$ and $x = \dfrac{\pm\sqrt{2}}{\lambda}$

$\left(\dfrac{\pm\sqrt{2}}{\lambda}\right)^2 \cdot \dfrac{1}{\lambda} = 2 \to \dfrac{2}{\lambda^2} \cdot \dfrac{1}{\lambda} = 2 \to \lambda^3 = 1 \to \lambda = 1$ $\therefore x = \pm\sqrt{2}$ and $y = 1$

$\therefore$ Extreme points: $(\pm\sqrt{2}, 1)$ ▨

Three Independent Variables with One Constraint

17. Minimum distance to a point Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1, 1, 1)$.

$f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$

$\nabla f = (2x-2, 2y-2, 2z-2) = \lambda \nabla g = (\lambda, 2\lambda, 3\lambda)$

$\therefore \begin{cases} 2x-2 = \lambda \\ \cancel{2}y - \cancel{2}^1 = \cancel{2}\lambda \\ 2z-2 = 3\lambda \\ x+2y+3z = 13 \end{cases}$ $\quad \begin{cases} y-1 = 2x-2 \\ 2z-2 = 6x-6 \\ x+2y+3z = 13 \end{cases}$ $\quad \begin{cases} y = 2x-1 \\ z = \dfrac{6x-4}{2} = 3x-2 \\ x+2y+3z = 13 \end{cases} \to x + 4x - 2 + 9x - 6 = 13 \to 14x = 21 \to x = \dfrac{3}{2}$

for $x = \dfrac{3}{2}$, then $y = \cancel{2} \cdot \dfrac{3}{\cancel{2}} - 1 = 2$, and $z = \dfrac{9}{2} - \dfrac{4}{2} = \dfrac{5}{2} \to f\left(\dfrac{3}{2}, 2, \dfrac{5}{2}\right) = \left(\dfrac{3}{2} - 1\right)^2 + (2-1)^2 + \left(\dfrac{5}{2} - 1\right)^2 = \left(\dfrac{1}{2}\right)^2 + 1 + \left(\dfrac{3}{2}\right)^2 =$

$\hookrightarrow = \dfrac{1}{4} + \dfrac{4}{4} + \dfrac{9}{4} = \dfrac{14}{4} = \dfrac{7}{2}$ ▨

18. Maximum distance to a point Find the point on the sphere $x^2 + y^2 + z^2 = 4$ farthest from the point $(1, -1, 1)$.

$f(x,y,z) = (x-1)^2 + (y+1)^2 + (z-1)^2 \to \nabla f = (2x-2, 2y+2, 2z-2) = \lambda \nabla g = (2x\lambda, 2y\lambda, 2z\lambda)$

$\therefore$
$2x - 2 = 2x\lambda \to \cancel{2}(x-1) = \cancel{2}x\lambda \to x - 1 = x\lambda \to \dfrac{x-1}{x} = \lambda \to 1 - \dfrac{1}{x} = \lambda \to (\lambda - 1) \cdot x = -1 \to x = \dfrac{-1}{\lambda - 1}$ for $x \neq 0$ and $\lambda \neq 0$

$2y + 2 = 2y\lambda \to \cancel{2}(y+1) = \cancel{2}y\lambda \to 1 + \dfrac{1}{y} = \lambda \to y = \dfrac{1}{\lambda - 1}$ for $y \neq 0$ and $\lambda \neq 0$

$2z - 2 = 2z\lambda \to z = \dfrac{-1}{\lambda - 1}$ for $z \neq 0$ and $\lambda \neq 0$

$x^2 + y^2 + z^2 = 4$

$\hookrightarrow \left(\dfrac{-1}{\lambda-1}\right)^2 + \left(\dfrac{1}{\lambda-1}\right)^2 + \left(\dfrac{1}{\lambda-1}\right)^2 = 4 \to 3 \cdot \left(\dfrac{1}{\lambda-1}\right)^2 = 4 \to \dfrac{1}{\lambda-1} = \dfrac{\sqrt{4}}{\sqrt{3}} = \pm\dfrac{2}{\sqrt{3}}$

$\therefore x = -\left(\pm\dfrac{2}{\sqrt{3}}\right) = \pm\dfrac{2}{\sqrt{3}}$; $y = \pm\dfrac{2}{\sqrt{3}}$; $z = \pm\dfrac{2}{\sqrt{3}}$, then the largest/farthest value of $f(x,y,z)$ is with $\left(-\dfrac{2}{\sqrt{3}}, \dfrac{2}{\sqrt{3}}, \dfrac{2}{\sqrt{3}}\right)$. ▨

19. Minimum distance to the origin Find the minimum distance from the surface $x^2 - y^2 - z^2 = 1$ to the origin.

$\nabla f = (2x, 2y, 2z) = \lambda \nabla g = (\lambda 2x, -\lambda 2y, -\lambda 2z)$

$$\therefore \begin{cases} 2x = \lambda 2x \longrightarrow x = \lambda x \longrightarrow x - \lambda x = 0 \longrightarrow x(1-\lambda) = 0 \longrightarrow x = 0 \text{ or } \lambda = 1 \\ 2y = -\lambda 2y \\ 2z = -\lambda 2z \\ x^2 - y^2 - z^2 = 1 \end{cases}$$

for $x = 0$, then $\begin{cases} y = -\lambda y \\ z = -\lambda z \\ -y^2 - z^2 = 1 \end{cases} \longrightarrow -(-\lambda y)^2 - (-\lambda z)^2 = \overbrace{-\lambda^2}^{\ominus}\overbrace{y^2 - \lambda^2}^{\ominus}\overbrace{z^2}^{\ominus} = 1$ not possible!

for $\lambda = 1$, then $\begin{cases} y = -1 \cdot y \longrightarrow y + y = 0 \longrightarrow y = 0 \\ z + z = 0 \longrightarrow z = 0 \\ x^2 - y^2 - z^2 = 1 \end{cases}$ $\begin{array}{l} x^2 - 0^2 - 0^2 = 1 \\ \therefore x = \pm 1 \end{array}$ $\therefore f(1,0,0) = 1^2 + 0^2 + 0^2 = 1$ //

$\therefore$ The minimum distance to $g(x,y,z)$ is $f(1,0,0) = 1$ ▨

20. Minimum distance to the origin Find the point on the surface $z = xy + 1$ nearest the origin.

$\nabla f = (2x, 2y, 2z) = \lambda \nabla g = (-\lambda y, -\lambda x, \lambda)$

$$\therefore \begin{cases} 2x = -\lambda y \\ 2y = -\lambda x \\ 2z = \lambda \\ z - xy = 1 \end{cases} \qquad \cancel{2}x = -\lambda\left(\frac{-\lambda x}{\cancel{2}}\right) \longrightarrow x = \lambda^2 x \longrightarrow x - \lambda^2 x = 0 \longrightarrow x(1-\lambda^2) = 0 \longrightarrow x = 0 \text{ or } \lambda = \pm 1$$

for $x = 0$, then $y = 0$ and $z = 1 \longrightarrow 1 - 0 \cdot 0 = 1$ ✓ $\longrightarrow f(0,0,1) = 0^2 + 0^2 + 1^2 = 1$ //

for $\lambda = 1$, then $\begin{cases} y = \frac{-x}{2} \\ z = \frac{1}{2} \\ \frac{1}{2} - x \cdot \left(\frac{-x}{2}\right) = 1 \end{cases}$ $\frac{1}{2} + \frac{x^2}{2} = 1 \longrightarrow 1 + x^2 = 2 \longrightarrow x = \pm 1$ //

for $z = \frac{1}{2}$ and $x = 1$, then $y = -\frac{1}{2}$ and $f\left(1, -\frac{1}{2}, \frac{1}{2}\right) = 1^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1 + \frac{1}{4} + \frac{1}{4} = \frac{6}{4} = \frac{3}{2} = 1{,}5$ //

for $z = \frac{1}{2}$ and $x = -1$, then $y = \frac{1}{2}$ and $f\left(-1, \frac{1}{2}, \frac{1}{2}\right) = (-1)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1{,}5$ //

for $\lambda = -1$, then $\begin{cases} y = \frac{x}{2} \\ z = -\frac{1}{2} \\ \frac{1}{2} - x \cdot \left(\frac{x}{2}\right) = 1 \end{cases}$ $\frac{1}{2} - \frac{x^2}{2} = 1 \longrightarrow 1 - x^2 = 2 \longrightarrow x^2 = -1$ not possible because $x = \sqrt{-1}$ is not $\mathbb{R}$.

$\therefore$ The minimum distance to $g(x,y,z)$ is $f(0,0,1) = 1$ ▨

Extreme Values Subject to Two Constraints

37. Maximize the function $f(x, y, z) = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.

$$\nabla f = (2x, 2, -2z) = \nabla g\,\lambda + \nabla k\,\mu = (2\lambda, -\lambda, 0) + (0, \mu, \mu)$$

$$\therefore \begin{cases} \cancel{2}x = \cancel{2}\lambda \rightarrow x = \lambda \\ 2 = -\lambda + \mu \\ -2z = \mu \\ 2x - y = 0 \\ y + z = 0 \end{cases} \qquad \begin{cases} 2 = -x + \mu \\ -2z = \mu \longrightarrow \mu = -2z \\ 2x - y = 0 \\ y + z = 0 \end{cases} \qquad \begin{cases} +x + 2z = -2 \\ 2x - y = 0 \\ y + z = 0 \rightarrow y = -z \end{cases}$$

$$(-2)\cdot\begin{cases} x - 2y = -2 \\ 2x - y = 0 \end{cases} \qquad \begin{array}{r} -\cancel{2x} + 4y = +4 \\ \cancel{2x} - y = 0 \\ \hline 3y = 4 \\ y = \frac{4}{3} \mathbin{//} \end{array}$$

$$\therefore y = \frac{4}{3} \mathbin{//},\ x = -2 + 2\cdot\frac{4}{3} = \frac{-6}{3} + \frac{8}{3} = \frac{2}{3} \mathbin{//} \text{ and } z = -\frac{4}{3} \mathbin{//} \quad \therefore f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \frac{4}{3} \ \blacksquare$$

38. Minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.

$$\nabla f = (2x, 2y, 2z) = \lambda \nabla g + \mu \nabla h = (\lambda, 2\lambda, 3\lambda) + (\mu, 3\mu, 9\mu)$$

$$\therefore \begin{cases} 2x = \lambda + \mu \longrightarrow x = \frac{\lambda+\mu}{2} \\ 2y = 2\lambda + 3\mu \longrightarrow y = \frac{2\lambda + 3\mu}{2} \\ 2z = 3\lambda + 9\mu \longrightarrow z = \frac{3\lambda + 9\mu}{2} \\ x + 2y + 3z - 6 = 0 \\ x + 3y + 9z - 9 = 0 \end{cases}$$

$$\begin{cases} \frac{\lambda+\mu}{2} + 2\cdot\left(\frac{2\lambda+3\mu}{2}\right) + 3\cdot\frac{3\lambda+9\mu}{2} = 6 \longrightarrow \lambda + \mu + 4\lambda + 6\mu + 9\lambda + 27\mu = 12 \rightarrow 14\lambda + 34\mu = 12 \\ \frac{\lambda+\mu}{2} + 3\cdot\left(\frac{2\lambda+3\mu}{2}\right) + 9\left(\frac{3\lambda+9\mu}{2}\right) = 9 \rightarrow \lambda + \mu + 6\lambda + 9\mu + 27\lambda + 81\mu = 18 \rightarrow 34\lambda + 91\mu = 18 \end{cases}$$

$$\therefore \begin{cases} 7\lambda + 17\mu = 6 \longrightarrow 7\left(\frac{18 - 91\mu}{34}\right) + 17\mu = 6 \longrightarrow 126 - 637\mu + 578\mu = 204 \rightarrow -59\mu = 78 \rightarrow \mu = -\frac{78}{59} \mathbin{//} \\ 34\lambda + 91\mu = 18 \longrightarrow \lambda = \frac{18 - 91\mu}{34} \end{cases}$$

$$\therefore \lambda = \frac{18 - \left(-\frac{7098}{59}\right)}{34} = \frac{1062 + 7098}{59}\cdot\frac{1}{34} = \frac{\overset{240}{\cancel{8160}}}{59\cdot\cancel{34}} = \frac{240}{59} \mathbin{//}\ ; \quad x = \left(\frac{240}{59} + \left(\frac{-78}{59}\right)\right)\cdot\frac{1}{2} = \frac{\cancel{162}}{59}\cdot\frac{1}{\cancel{2}} = \frac{81}{59} \mathbin{//}$$

$$\therefore y = \left(2\cdot\frac{240}{59} + 3\cdot\left(\frac{-78}{59}\right)\right)\cdot\frac{1}{2} = \left(\frac{480}{59} - \frac{234}{59}\right)\cdot\frac{1}{2} = \frac{246}{59}\cdot\frac{1}{2} = \frac{123}{59} \mathbin{//}\ ; \quad z = \left(3\cdot\frac{240}{59} + 9\cdot\left(\frac{-78}{59}\right)\right)\cdot\frac{1}{2} = \left(\frac{720}{59} - \frac{702}{59}\right)\cdot\frac{1}{2} = \frac{9}{59} \mathbin{//}$$

$$\therefore f\left(\frac{81}{59}; \frac{123}{59}; \frac{9}{59}\right) = \left(\frac{81}{59}\right)^2 + \left(\frac{123}{59}\right)^2 + \left(\frac{9}{59}\right)^2 = \frac{369}{59} \ \blacksquare$$

41. **Extrema on a curve of intersection** Find the extreme values of $f(x, y, z) = x^2yz + 1$ on the intersection of the plane $z = 1$ with the sphere $x^2 + y^2 + z^2 = 10$.

$$\nabla f = (2xyz,\ x^2z,\ x^2y) = \lambda \nabla g + \mu \nabla h = (0, 0, \lambda) + (2x\mu, 2y\mu, 2z\mu)$$

$$\therefore \begin{cases} \cancel{2}xyz = \cancel{2}x\mu \\ x^2z = 2y\mu \\ x^2y = \lambda + 2z\mu \\ z = 1 \\ x^2+y^2+z^2 = 10 \end{cases} \qquad \begin{cases} \cancel{xy = x\mu} \longrightarrow \text{for } x \neq 0 \longrightarrow y = \mu \text{ or } x = 0 \longrightarrow y = 0 \\ x^2 = 2y\mu \\ x^2y = \lambda + 2\mu \\ x^2 + y^2 = 9 \end{cases}$$

$\therefore$ for $x = 0$, then $y = \pm 3$ and $z = 1 \longrightarrow f(0,3,1) = 1$ // or $f(0,-3,1) = 1$ //

$\therefore$ for $x \neq 0$, then $y = \mu \longrightarrow \begin{cases} x^2 = 2y^2 \\ x^2y = \lambda + 2y \\ x^2 + y^2 = 9 \end{cases} \quad \begin{cases} x^2y = \lambda + 2y \\ 2y^2 + y^2 = 9 \end{cases} \longrightarrow 3y^2 = 9 \longrightarrow y = \pm\sqrt{3}$

$\therefore$ for $x \neq 0$ and $y = \pm\sqrt{3}$ and $z = 1$, then $x^2 = 2.(\pm\sqrt{3})^2 = 2.3 = 6 \longrightarrow x = \pm\sqrt{6}$

$\therefore \underbrace{f(\sqrt{6}, \sqrt{3}, 1)}_{max} = 6\cdot\sqrt{3}\cdot 1 + 1 = 6\sqrt{3} + 1$ // ; $\underbrace{f(\sqrt{6}, -\sqrt{3}, 1)}_{min} = 6\cdot(-\sqrt{3})\cdot 1 + 1 = -6\sqrt{3} + 1$ // ; $\underbrace{f(-\sqrt{6}, \sqrt{3}, 1)}_{max} = 6\cdot\sqrt{3}\cdot 1 + 1 = 6\sqrt{3} + 1 = 11{,}39$ //

$\underbrace{f(-\sqrt{6}, -\sqrt{3}, 1)}_{min} = 6\cdot(-\sqrt{3})\cdot 1 + 1 = -6\sqrt{3} + 1$ //

Ch. 15.1: ~~1~~, ~~3~~, ~~9~~, ~~15~~, ~~23-25~~.

Evaluating Iterated Integrals

In Exercises 1–14, evaluate the iterated integral.

1. $$\int_1^2 \int_0^4 2xy\, dy\, dx = \int_1^2 2x \int_0^4 y\, dy\, dx \to \int_1^2 2x \left[\frac{y^2}{2}\right]_0^4 dx \to \int_1^2 2x \cdot 8\, dx = \cancel{16}^{8}\left[\frac{x^2}{\cancel{2}}\right]_1^2 = 24$$

3. $$\int_{-1}^0 \int_{-1}^1 (x + y + 1)\, dx\, dy = \int_{-1}^0 \left[\frac{x^2}{2} + yx + x\right]_{-1}^1 dy = \int_{-1}^0 \frac{1}{2} + y + 1 - \left(\frac{1}{2} - y - 1\right) dy = \int_{-1}^0 2 + 2y\, dy = \left[2y + \frac{\cancel{2}y^2}{\cancel{2}}\right]_{-1}^0 = -(-2+1) = 1$$

9. $$\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y}\, dy\, dx = \int_0^{\ln 2}\int_1^{\ln 5} e^{2x} \cdot e^{y}\, dy\, dx = \int_0^{\ln 2} e^{2x} \cdot \left[e^{y}\right]_1^{\ln 5} dx = \int_0^{\ln 2} e^{2x} \cdot (5 - e^1)\, dx$$

$$\to = (5 - e^1)\int_0^{\ln 2} e^{2x} dx = (5 - e^1) \cdot \frac{1}{2}\left[e^{2x}\right]_0^{\ln 2} = \frac{1}{2} \cdot (5 - e^1) \cdot [1 - 4] = \frac{3}{2} \cdot (5 - e^1)$$

Evaluating Double Integrals over Rectangles

In Exercises 15–22, evaluate the double integral over the given region R.

15. $\iint_R (6y^2 - 2x)\, dA, \quad R: \; 0 \le x \le 1, \; 0 \le y \le 2$

$$\int_0^1 \int_0^2 6y^2 - 2x\, dy\, dx = \int_0^1 \left[6 \cdot \frac{y^3}{3} - 2xy\right]_0^2 dx = \int_0^1 \frac{6 \cdot 2^3}{3} - 2x\,2\, dx = \int_0^1 16 - 4x\, dx = \left[16x - \frac{4x^2}{2}\right]_0^1 = 16 - 2 = 14$$

In Exercises 23 and 24, integrate f over the given region.

23. **Square** $f(x, y) = 1/(xy)$ over the square $1 \le x \le 2$, $1 \le y \le 2$

$$\int_1^2 \int_1^2 \frac{1}{xy}\, dy\, dx = \int_1^2 \frac{1}{x}\left[\ln(y)\right]_1^2 dx = \int_1^2 \frac{1}{x} \cdot \ln(2)\, dx = \ln(2) \cdot \ln(2) = \ln^2(2) = 2\ln(2)$$

check!

24. **Rectangle** $f(x, y) = y \cos xy$ over the rectangle $0 \le x \le \pi$, $0 \le y \le 1$

$u = xy \quad \frac{du}{dx} = \frac{y}{x} \to du = \frac{y}{x} \cdot dx \to dx = \frac{du \cdot x}{y}$

$$\int_0^1 \int_0^{\pi} y \cos(xy)\, dx\, dy = \int_0^1 \int_0^{\pi} \cancel{y} \cos(u) \cdot \frac{x}{\cancel{y}}\, du\, dy = \int_0^1 \int_0^{\pi y} \cos(u) \cdot x\, du\, dy = \int_0^1 \left[\sin(u)\right]_0^{\pi y} dy = \int_0^1 \sin(\pi y)\, dy$$

$u = \pi y \to du = \pi\, dy \to dy = \frac{du}{\pi}$

$$= \int_0^{\pi} \sin(u) \frac{du}{\pi} = \frac{1}{\pi} \cdot \left[-\cos(\pi) + \cos(0)\right] = \frac{1}{\pi} \cdot (1 + 1) = \frac{2}{\pi}$$

25. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the square $R: -1 \le x \le 1$, $-1 \le y \le 1$.

$$\int_{-1}^1 \int_{-1}^1 x^2 + y^2\, dx\, dy = \int_{-1}^1 \left[\frac{x^3}{3} + y^2 x\right]_{-1}^1 dy = \int_{-1}^1 \left[\frac{1}{3} + y^2 + \frac{1}{3} + y^2\right] dy = \int_{-1}^1 \frac{2}{3} + 2y^2\, dy = \left[\frac{2}{3}y + \frac{2y^3}{3}\right]_{-1}^1 =$$

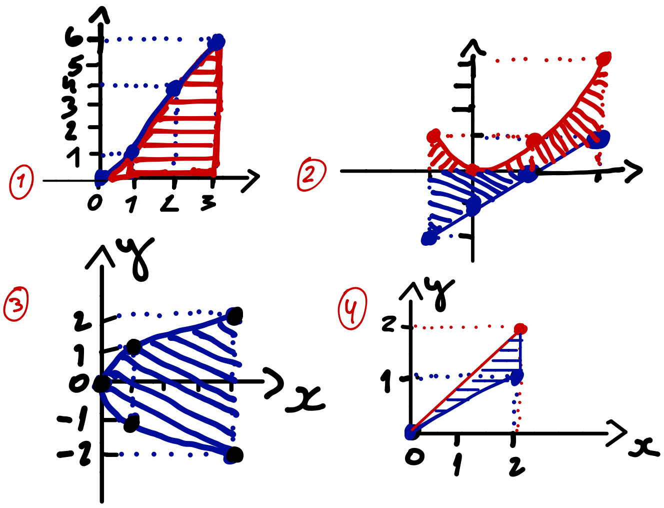
$$= \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{8}{3}$$

Ch. 15.2: ~~1-4~~, ~~9-12~~, ~~23-26~~, 57-58

Ch. 15.3: ~~3, 5, 7~~

Ch. 15.4: ~~1, 2, 7-10~~, 27, 30, 31

Ch. 15.2:



Sketching Regions of Integration

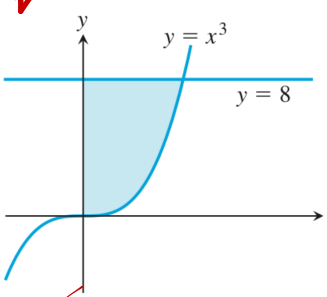
In Exercises 1–8, sketch the described regions of integration.

1. $0 \le x \le 3, \quad 0 \le y \le 2x$
2. $-1 \le x \le 2, \quad x - 1 \le y \le x^2$
3. $-2 \le y \le 2, \quad y^2 \le x \le 4$
4. $0 \le y \le 1, \quad y \le x \le 2y$

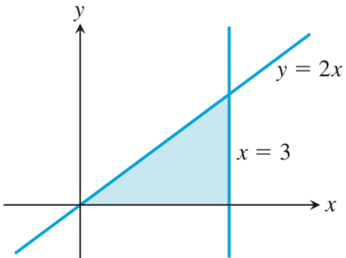
Finding Limits of Integration

In Exercises 9–18, write an iterated integral for $\iint_R dA$ over the described region R using (a) vertical cross-sections, (b) horizontal cross-sections.

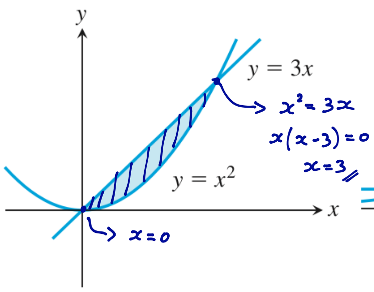
9.



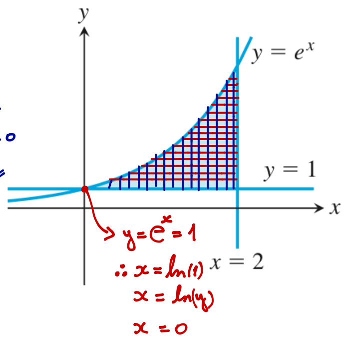
10.



11.



12.



(9a) vertical $\iint_R f(x,y)\,dy\,dx$ for $y = x^3$ and $y = 8$:

→ $0 \le x \le \sqrt[3]{y}$; $x^3 \le y \le 8$

$\therefore \int_0^{\sqrt[3]{8}=2}\int_{x^3}^{8} dx\,dy$ ▨

(9b) Horizontal $\iint_R f(x,y)\,dx\,dy$ for $y = x^3$ and $y = 8$:

→ $0 \le y \le 8$; $0 \le x \le \sqrt[3]{y}$

$\therefore \int_0^{8}\int_0^{\sqrt[3]{y}} dx\,dy$ ▨

(10a) $0 \le x \le 3$ and $0 \le y \le 2x$

$\therefore \int_0^3\int_0^{2x} dy\,dx$ ▨

(10b) $0 \le y \le 6$ and $\frac{y}{2} \le x \le 3$

$\therefore \int_0^6\int_{y/2}^{3} dx\,dy$ ▨

(12a) Vertical (y first integrated) $\iint_R dy\,dx$;

$0 \le x \le 2$ and $1 \le y \le e^x$, $\int_0^2\int_1^{e^x} dy\,dx$ ▨

(12b) horizontal (x integrated first); $1 \le y \le e^2$ and $\ln(y) \le x \le 2 \to \int_1^{e^2}\int_{\ln(y)}^{2} dx\,dy$ ▨

(11a) Vertical (y integrated first); $0 \le x \le 3$ and $x^2 \le y \le 3x$

$\therefore \int_0^3\int_{x^2}^{3x} dy\,dx$ ▨

(11b) horizontal (x integrated first); $0 \le y \le 9$ and $\frac{y}{3} \le x \le \sqrt{y}$

$\therefore \int_0^9\int_{y/3}^{\sqrt{y}} dx\,dy$ ▨

Finding Regions of Integration and Double Integrals

In Exercises 19–24, sketch the region of integration and evaluate the integral.

23. $\int_0^1\int_0^{y^2} 3y^3 e^{xy}\,dx\,dy$

24. $\int_1^4\int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\,dy\,dx$

$u = xy \to du = y\,dx \to dx = \frac{du}{y}$

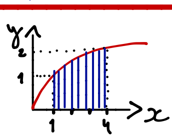
(23) $\int_0^1\int_0^{y^2} 3y^3 e^{xy}\,dx\,dy = \int_0^1 3y^{\cancel{3}2}\int_0^{y^2} e^u \frac{du}{\cancel{y}}\,dy =$

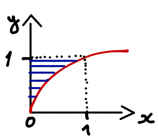
$= \int_0^1 3y^2\left[e^{y^3} - \cancel{e^0}^{1}\right]dy = \int_0^1 3y^2 e^{y^3} - 3y^2\,dy$

$u = y^3 \to du = 3y^2\,dy \to dy = \frac{du}{3y^2}$

$= \int_0^1 \cancel{3y^2} e^u \frac{du}{\cancel{3y^2}} - \int_0^1 3y^2\,dy = (e^1 - e^0) - (1^3 - 0^3)$

$= (e-1) - 1 = e - 2$ ▨





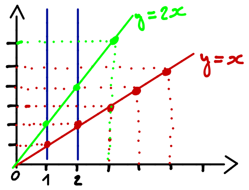
$u = y x^{-1/2} \to du = x^{-1/2}\,dy \Rightarrow dy = x^{1/2}\,du$

(24) $\int_1^4\int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\,dy\,dx = \int_1^4\int_0^1 \frac{3}{2} e^u \cdot x^{1/2}\,du\,dx$

$= \int_1^4 \frac{3}{2}x^{1/2}\left[e^1 - \cancel{e^0}^{1}\right]dx = \int_1^4 \frac{3}{2}x^{1/2}(e-1)\,dx = \frac{3}{2}(e-1)\int_1^4 x^{1/2}\,dx$

$= \frac{3}{2}(e-1)\cdot\left[\frac{2x^{3/2}}{3}\right]_1^4 = \frac{3}{2}(e-1)\cdot\left[\frac{16}{3} - \frac{2}{3}\right] = \cancel{\frac{3}{2}}\cdot\cancel{\frac{14}{3}}^{7}(e-1) = 7e - 7$ ▨

In Exercises 25–28, integrate f over the given region.

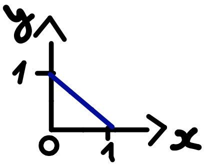


~~25~~. **Quadrilateral** $f(x, y) = x/y$ over the region in the first quadrant bounded by the lines $y = x$, $y = 2x$, $x = 1$, and $x = 2$

$1 \le x \le 2$ and $x \le y \le 2x$

$$\int_1^2\int_x^{2x} \frac{x}{y}\,dy\,dx = \int_1^2 x\cdot\Big[\underbrace{\ln(2x)}_{\ln(2)+\cancel{\ln(x)}} - \cancel{\ln(x)}\Big]\,dx = \int_1^2 x\cdot\ln(2)\,dx = \ln(2)\cdot\left[\frac{x^2}{2}\right]_1^2 = \ln(2)\left[\frac{2^2}{2}-\frac{1^2}{2}\right] = \ln(2)\left[\frac{4}{2}-\frac{1}{2}\right] = \frac{3}{2}\ln 2 \quad \blacksquare$$

~~26~~. **Triangle** $f(x, y) = x^2 + y^2$ over the triangular region with vertices (0, 0), (1, 0), and (0, 1)



$0 \le x \le 1$ and $0 \le y \le 1$

$$\int_0^1\int_0^1 x^2+y^2\,dx\,dy = \int_0^1\left[\frac{x^3}{3}+y^2x\right]_0^1 dy = \int_0^1 \frac{1}{3}+y^2\,dy = \frac{1}{3}\left[\frac{y^3}{3}\right]_0^1 = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{6} \quad \blacksquare$$

Volume Beneath a Surface $z = f(x, y)$

57. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

$z = x^2+y^2$
$y = x$ below +
$x = 0$ below
$x+y = 2$ below

$x + x = 2 \to x = 1 \to 0 \le x \le 1$ and $x \le y \le 2-x$

$$V = \int_0^1\int_x^{2-x} x^2+y^2\,dy\,dx = \int_0^1\left[x^2y+\frac{y^3}{3}\right]_x^{2-x} dx = \int_0^1\left[\left(x^2(2-x)+\frac{(2-x)^3}{3}\right)-\left(x^2\cdot x+\frac{x^3}{3}\right)\right]dx$$

$$\boxed{\begin{aligned}&(2-x)^3 = \\ &2^3-3\cdot2^2\cdot x+3\cdot2^1\cdot x^2-x^3\\ &8-12x+48x^2-x^3\end{aligned}}$$

$$V = \int_0^1\left(2x^2-x^3+\frac{8}{3}-\frac{12x}{3}+\frac{48x^2}{3}-\frac{x^3}{3}\right)-\frac{4x^3}{3}\,dx = \int_0^1\left(\cancel{\frac{6x^2}{3}}-\cancel{\frac{3x^3}{3}}+\cancel{\frac{8}{3}}-\cancel{\frac{12x}{3}}+\cancel{\frac{48x^2}{3}}-\cancel{\frac{x^3}{3}}\right)-\cancel{\frac{4x^3}{3}}\,dx =$$

$$V = \int_0^1 -\frac{8x^3}{3}+\frac{54x^2}{3}-\frac{12x}{3}+\frac{8}{3}\,dx = \left[-\frac{\overset{2}{\cancel{8}}x^4}{3\cdot\cancel{4}}+\frac{\overset{6}{\cancel{54}}x^3}{\cancel{3}\cdot\cancel{3}}-\frac{\overset{2}{\cancel{12}}x^2}{\cancel{3}\cdot\cancel{2}}+\frac{8}{3}x\right]_0^1$$

$$V = -\frac{2}{3}+6-2+\frac{8}{3} = \frac{6}{3}+4 = \frac{18}{3} = 6 \neq \frac{4}{3}$$

~~58~~. Find the volume of the solid that is bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy-plane.

$z = x^2$ above
$y = 2-x^2$ below
$y = x$ below +

$x = 2-x^2 \to x^2+x-2 = 0 \to \left.\begin{matrix}S=-1\\P=-2\end{matrix}\right\}\ -2 \le x \le 1$ //

$$V = \int_{-2}^1\int_x^{2-x^2} x^2\,dy\,dx = \int_{-2}^1\left[x^2y\right]_x^{2-x^2}dx = \int_{-2}^1 x^2\cdot(2-x^2-x)\,dx = \int_{-2}^1 2x^2-x^4-x^3\,dx = \left[\frac{2}{3}x^3-\frac{x^5}{5}-\frac{x^4}{4}\right]_{-2}^1$$

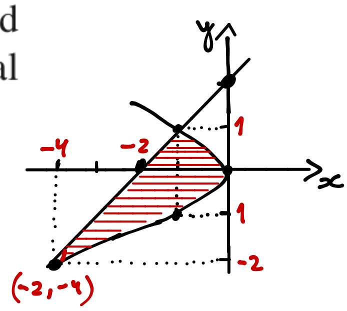
$$V = \left(\frac{2}{3}\cdot1-\frac{1}{5}-\frac{1}{4}\right)-\left(-\frac{16}{3}+\frac{2^5}{5}-\frac{16}{4}\right) = \left(\frac{7}{15}-\frac{1}{4}\right)-\left(-\frac{16}{3}+\frac{32}{5}-\frac{16}{4}\right) = \left(\frac{28-15}{15\cdot4}\right)-\left(\frac{-80+96}{15}-\frac{16}{4}\right)$$

$$V = \frac{13}{60}-\frac{(64-240)}{60} = \frac{13}{60}+\frac{176}{60} = \frac{189}{60} = \frac{63}{20} \quad \blacksquare$$

Chap. 15.3:

Area by Double Integrals

In Exercises 1–12, sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral.



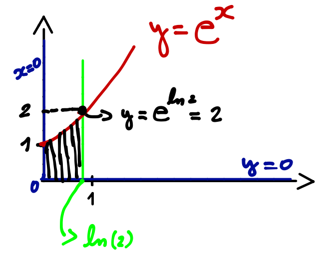
3. The parabola $\overset{\text{2nd}}{x = -y^2}$ and the line $\overset{\text{1st}}{y = x + 2}$

$y = -y^2 + 2$
$y^2 + y - 2 = 0$
$S = -1$, $P = -2$ $\Big\}$ $-2 \le y \le 1$

$-2 \le y \le 1$ and $y-2 \le x \le -y^2 \longrightarrow \int_{-2}^{1}\int_{y-2}^{-y^2} dx\,dy =$

$= \int_{-2}^{1}\left[x\right]_{y-2}^{-y^2} dy = \int_{-2}^{1} -y^2 - y + 2\, dy = \left[-\frac{y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^{1} =$

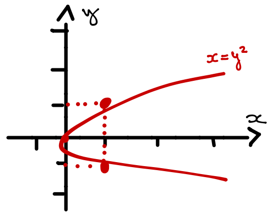
$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{-(-2)^3}{3} - \frac{(-2)^2}{2} - 4\right) = \left(\frac{-2-3+12}{6}\right) - \left(\frac{8}{3} - \frac{4}{2} - 4\right) = \frac{7}{6} - \left(\overbrace{\frac{16-12-24}{6}}^{-20}\right) = \frac{27}{6} = \frac{9}{2}$ ∎



5. The curve $\overset{\text{2nd}}{y = e^x}$ and the lines $\overset{\text{1st}}{y = 0}$, $x = 0$, and $x = \ln 2$

$0 \le x \le \ln(2)$ and $0 \le y \le e^x$

$\int_0^{\ln(2)}\int_0^{e^x} dy\,dx = \int_0^{\ln 2} e^x\,dx = 2 - 1 = 1$ ∎



7. The parabolas $\overset{\text{1st}}{x = y^2}$ and $\overset{\text{2nd}}{x = 2y - y^2}$ $\longrightarrow \int_0^1\int_{y^2}^{2y-y^2} dx\,dy$

$y^2 = 2y - y^2 \longrightarrow 2y^2 - 2y = 0 \longrightarrow 2y(y-1) = 0 \therefore 0 \le y \le 1$ and $y^2 \le x \le 2y - y^2$

$\int_0^1\int_{y^2}^{2y-y^2} dx\,dy = \int_0^1 2y - y^2 - y^2\,dy = \int_0^1 2y - 2y^2\,dy = \left[y^2 - \frac{2}{3}y^3\right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$ ∎

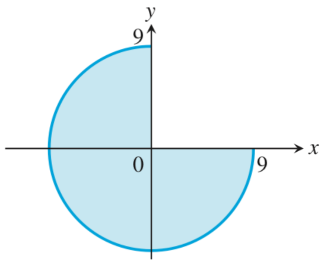
Chap. 15,4:

Regions in Polar Coordinates

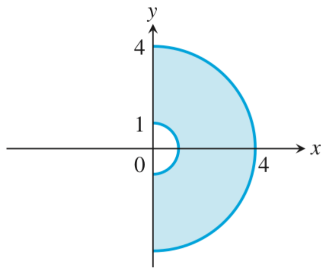
$\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \end{cases}$ and $r = \sqrt{x^2+y^2}$

In Exercises 1–8, describe the given region in polar coordinates.

1.



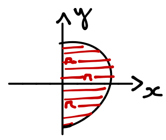
2.



① $r = 9$; $x_0 = 0$; $y_0 = 0 \therefore x^2 + y^2 = 9^2$

$\frac{\pi}{2} \le \theta \le 2\pi$ and $0 \le r \le 9$ ∎

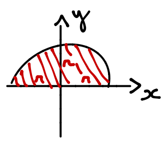
② $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $1 \le r \le 4$ ∎



7. The region enclosed by the circle $x^2 + y^2 = 2x$ $= r^2 \therefore r = \sqrt{2x}$, then $x \ge 0$

$\begin{cases} 2x = r^2 \\ x = r\cos\theta \longrightarrow r = \frac{x}{\cos\theta} \longrightarrow \cancel{r} = \frac{r^{\cancel{2}}}{2\cdot\cos\theta} \longrightarrow r = 2\cos\theta \end{cases}$ //

$\therefore -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 2\cos\theta$ ∎



8. The region enclosed by the semicircle $x^2 + y^2 = 2y$, $y \ge 0$

$\longrightarrow 2y = r^2 \therefore r = \sqrt{2y}$, then $y \ge 0$

$\begin{cases} 2y = r^2 \longrightarrow y = \frac{r^2}{2} \\ y = r\cdot\sin\theta \longrightarrow \frac{r^{\cancel{2}}}{2} = \cancel{r}\cdot\sin\theta \longrightarrow r = 2\sin\theta \end{cases}$ //

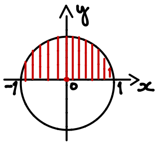
$\therefore 0 \le \theta \le \pi$ and $0 \le r \le 2\sin\theta$ ∎

Evaluating Polar Integrals

In Exercises 9–22, change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

9. $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\,dx$ **10.** $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy$

(9) $-1 \le x \le 1$ and $0 \le y \le \sqrt{1-x^2} \to y^2 = 1-x^2 \to \underbrace{x^2+y^2=1}_{\text{circle}}$

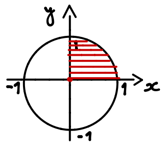


$\therefore\ 0 \le \theta \le \pi$ and $0 \le r \le \sqrt{1-x^2}$

$\therefore\ \displaystyle\int_0^{\pi}\int_0^{\sqrt{1-x^2}} r\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^1 d\theta = \int_0^{\pi}\frac{1}{2}\,d\theta$

$= \left[\frac{\theta}{2}\right]_0^{\pi} = \frac{\pi}{2}$ $\blacksquare$

(10) $0 \le y \le 1$ and $0 \le x \le \sqrt{1-y^2} \to x^2+y^2=1$ and $x^2+y^2=r^2$



$\therefore\ 0 \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 1$

$\therefore\ \displaystyle\int_0^{\pi/2}\int_0^{1} r^2\, r\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^1 d\theta = \int_0^{\pi/2}\frac{1}{4}\,d\theta$

$= \left[\frac{\theta}{4}\right]_0^{\pi/2} = \frac{\pi}{8}$ $\blacksquare$

Area in Polar Coordinates

27. Find the area of the region cut from the first quadrant by the curve $r = 2(2-\sin 2\theta)^{1/2}$.

$\to 0 \le \theta \le \pi/2$

$0 \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 2(2-\sin 2\theta)^{1/2}$

$u = 2\theta \to du = 2\,d\theta \to d\theta = \frac{du}{2}$

$\therefore\ \displaystyle\int_0^{\pi/2}\int_0^{2(2-\sin 2\theta)^{1/2}} r\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{2(2-\sin 2\theta)^{1/2}} d\theta = \int_0^{\pi/2}\frac{\cancel{4}^{\,2}(2-\sin 2\theta)}{\cancel{2}}\,d\theta = \cancel{2}\int_0^{\pi}(2-\sin(u))\frac{du}{\cancel{2}}\,d\theta = \Big[2\theta+\cos(u)\Big]_0^{\pi}$

$= (2\pi-1)-1 = 2(\pi-1)$ $\blacksquare$

30. Snail shell Find the area of the region enclosed by the positive x-axis and spiral $r = 4\theta/3,\ 0 \le \theta \le 2\pi$. The region looks like a snail shell.

$0 \le \theta \le 2\pi$ and $0 \le r \le 4\theta/3$

$\displaystyle\int_0^{2\pi}\int_0^{4\theta/3} r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^{4\theta/3} d\theta = \int_0^{2\pi}\frac{8\theta^2}{9}\,d\theta = \frac{8}{9}\left[\frac{\theta^3}{3}\right]_0^{2\pi} = \frac{8}{9}\cdot\frac{8\pi^3}{3} = \frac{64\pi^3}{27}$ $\blacksquare$

31. Cardioid in the first quadrant Find the area of the region cut from the first quadrant by the cardioid $r = 1 + \sin\theta$.

$\to 0 \le \theta \le \pi/2$

$0 \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 1+\sin\theta$

$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

$u = 2\theta \to du = 2\,d\theta \to d\theta = du/2$

$\displaystyle\int_0^{\pi/2}\int_0^{1+\sin\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{1+\sin\theta} d\theta = \int_0^{\pi/2}\frac{1+\sin^2\theta}{2}\,d\theta = \int_0^{\frac{\pi}{2}}\frac{1}{2}+\frac{1-\cos(2\theta)}{2}\,d\theta = \int_0^{\pi}\left(\frac{1}{2}+\frac{1-\cos(u)}{2}\right)\cdot\frac{du}{2} =$

$\displaystyle= \int_0^{\pi}\frac{1}{4}+\left(\frac{1-\cos(u)}{4}\right)\cdot du = \left[\frac{u}{4}+\left(\frac{u-\sin(u)}{4}\right)\right]_0^{\pi} = \left(\frac{\pi}{4}+\frac{\pi}{4}-\frac{\sin\pi}{4}\right)-\left(\frac{0}{4}+\frac{0}{4}-\frac{\sin(0)}{4}\right) = \frac{2\pi}{4} \neq \frac{3\pi}{8}+1$

Chap. 15,5 : ~~7, 8, 13, 14, 23~~, ~~24-26~~
Chap. 15,6 : ~~1, 29~~
Chap. 15,7 : ~~15-17, 21, 22, 23, 33, 37~~, ~~39~~

Evaluating Triple Iterated Integrals

Evaluate the integrals in Exercises 7–20.

7. $\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx = \int_0^1\int_0^1 \left[x^2z+y^2z+\frac{z^3}{3}\right]_0^1 dy\,dx$

$= \int_0^1\int_0^1 x^2+y^2+\frac{1}{3}\,dy\,dx = \int_0^1\left[x^2y+\frac{y^3}{3}+\frac{y}{3}\right]_0^1 = \int_0^1 x^2+\frac{1}{3}+\frac{1}{3}\,dx = \int_0^1 x^2+\frac{2}{3}\,dx = \left[\frac{x^3}{3}+\frac{2}{3}x\right]_0^1 = \frac{1}{3}+\frac{2}{3} = 1$

8. $\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy = \int_0^{\sqrt{2}}\int_0^{3y}[z]_{x^2+3y^2}^{8-x^2-y^2}dx\,dy = \int_0^{\sqrt{2}}\int_0^{3y}(8-\cancel{x^2}-y^2)-\cancel{x^2}-3y^2\,dx\,dy = \int_0^{\sqrt{2}}\int_0^{3y} -2x^2-4y^2+8\,dx\,dy$

$= \int_0^{\sqrt{2}}\left[\frac{-2x^3}{3}-4y^2x+8x\right]_0^{3y} dy = \int_0^{\sqrt{2}} -\frac{2\cdot\cancel{3^3}^{3^2}\cdot y^3}{\cancel{3}} - 4y^2\cdot 3y + 8\cdot 3\cdot y\,dy = \int_0^{\sqrt{2}} -18y^3-12y^3+24y\,dy = \int_0^{\sqrt{2}}-30y^3+24y\,dy$

$= \left[\frac{-30}{4}y^4+\frac{\cancel{24}^{12}}{\cancel{2}}y^2\right]_0^{\sqrt{2}} = \frac{-30}{\cancel{4}}\cdot(\cancel{\sqrt{2}})^{\cancel{4}} + 12\cdot(\cancel{\sqrt{2}})^{\cancel{2}} = -30+24 = -6$

13. $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dz\,dy\,dx = \int_0^3\int_0^{\sqrt{9-x^2}}[z]_0^{\sqrt{9-x^2}}dy\,dx = \int_0^3\int_0^{\sqrt{9-x^2}}\sqrt{9-x^2}\,dy\,dx = \int_0^3\left[(9-x^2)^{1/2}\cdot y\right]_0^{\sqrt{9-x^2}}dx =$

$= \int_0^3(9-x^2)^{1/2}\cdot(9-x^2)^{1/2}\,dx = \int_0^3 9-x^2\,dx = \left[9x-\frac{x^3}{3}\right]_0^3 = 27-\frac{\cancel{27}^{9}}{\cancel{3}} = 18$

14. $\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^{2x+y} dz\,dx\,dy = \int_0^2\int_{-(4-y^2)^{1/2}}^{(4-y^2)^{1/2}}[z]_0^{2x+y}dx\,dy = \int_0^2\int_{-(4-y^2)^{1/2}}^{(4-y^2)^{1/2}} 2x+y\,dx\,dy =$

$= \int_0^2[x^2+yx]_{-(4-y^2)^{1/2}}^{(4-y^2)^{1/2}}dy = \int_0^2\left[(4-y^2)+y\cdot(4-y^2)^{1/2}\right]-\left[(4-y^2)-y(4-y^2)^{1/2}\right]dy = \int_0^2 \cancel{4}-\cancel{y^2}+y(4-y^2)^{1/2}-\cancel{4}+\cancel{y^2}+y(4-y^2)^{1/2}\,dy$

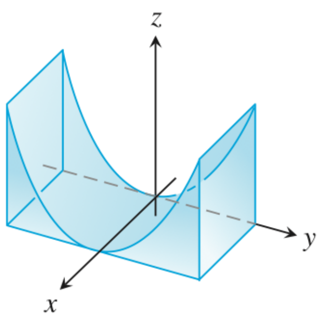
$\to u=4-y^2 \to du=-2y\,dy \to dy = du/-2y$

$= 2\int_0^2 y\cdot(4-y^2)^{1/2}\,dy = -\cancel{2}\int_4^0 \cancel{y}\cdot(u)^{1/2}\frac{du}{+\cancel{2}\cancel{y}} = -\frac{2}{3}\left[u^{3/2}\right]_4^0 = -\frac{2}{3}\cdot(0-8) = \frac{16}{3}$

Finding Volumes Using Triple Integrals

Find the volumes of the regions in Exercises 23–36.

23. The region between the cylinder $z = y^2$ and the xy-plane that is bounded by the planes $x = 0$, $x = 1$, $y = -1$, $y = 1$

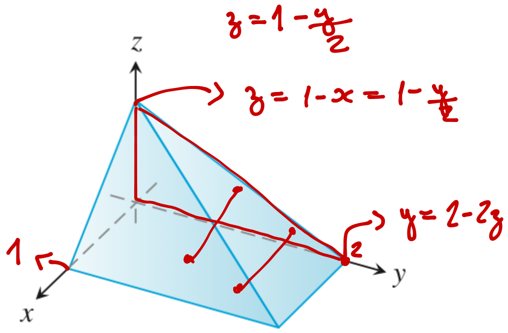


$\int_0^1\int_{-1}^1\int_0^{y^2}dz\,dy\,dx = \int_0^1\int_{-1}^1[z]_0^{y^2}dy\,dx$

$= \int_0^1\int_{-1}^1 y^2\,dy\,dx = \int_0^1\left[\frac{y^3}{3}\right]_{-1}^1dx =$

$= \int_0^1\frac{1}{3}+\frac{1}{3}\,dx = \frac{2}{3}[x]_0^1 = \frac{2}{3}$

24. The region in the first octant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$



$0 \le x \le 1 \longrightarrow x = 1 + \cancel{z}^{\,0}$

$0 \le y \le 2 - 2z$

$0 \le z \le 1 - x$

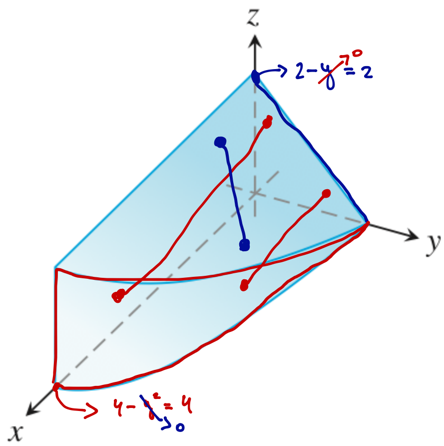
$$\int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\,dz\,dx = \int_0^1 \int_0^{1-x} 2 - 2z \; dy\,dx =$$

$$= \int_0^1 \left[2z - z^2\right]_0^{1-x} dx = \int_0^1 2.(1-x) - (1-x)^2 \, dx =$$

$$= \int_0^1 2 - 2x - (1 - 2x + x^2)\, dx = \int_0^1 2 - \cancel{2x} - 1 + \cancel{2x} - x^2 \, dx =$$

$$= \int_0^1 1 - x^2 \, dx = \left[x - \frac{x^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

25. The region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$



$0 \le x \le 4$; $0 \le y \le \sqrt{4-x}$ and $0 \le z \le 2 - y$

$$\int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dz\,dy\,dx = \int_0^4 \int_0^{\sqrt{4-x}} 2 - y \; dy\,dx = \int_0^4 \left[2y + \frac{y^2}{2}\right]_0^{\sqrt{4-x}} dx$$

$$= \int_0^4 2.\sqrt{4-x} - \frac{(\sqrt{4-x})^2}{2} \, dx = \int_0^4 2\sqrt{4-x} - 2 + \frac{x}{2} \, dx$$

$$= \int_0^4 2\sqrt{4-x}\, dx + \int_0^4 -2 \, dx + \int_0^4 \frac{x}{2} \, dx =$$

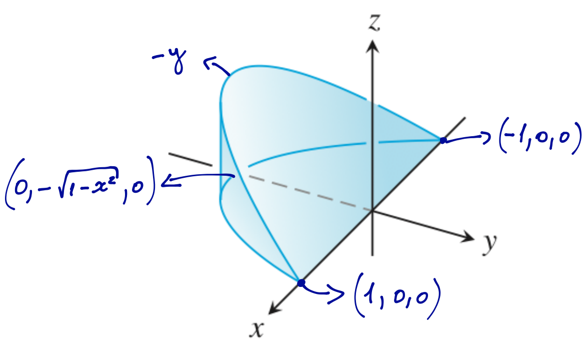
$u = 4 - x \longrightarrow du = -dx \longrightarrow dx = -du$

$$= \int_4^0 -2.\, u^{1/2} du - \left[2x\right]_0^4 + \left[\frac{x^2}{4}\right]_0^4 = \left[-2.\, \frac{u^{3/2}}{3}.2\right]_4^0 - 8 + 4$$

$$= \frac{4.8}{3} - \frac{4.3}{3} = \frac{20}{3}$$

26. The wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y$ and $z = 0$

$-1 \le x \le 1$, $-\sqrt{1-x^2} \le y \le 0$ and $0 \le z \le -y$



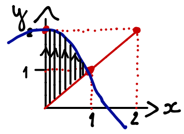
$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^0 \int_0^{-y} dz\,dy\,dx = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^0 -y \; dy\,dx =$$

$$= \int_{-1}^1 \left[-\frac{y^2}{2}\right]_{-\sqrt{1-x^2}}^0 dx = \int_{-1}^1 \left[0 + \frac{(-\sqrt{1-x^2})^2}{2}\right] dx =$$

$$= \frac{1}{2} \int_{-1}^1 -x^2 + 1 \, dx = \frac{1}{2}\left[-\frac{x^3}{3} + x\right]_{-1}^1 = \frac{1}{2}\cdot\left[\left(\frac{-1}{3} + 1\right) - \left(\frac{-(-1)^3}{3} - 1\right)\right]$$

$$= \frac{1}{2}\cdot\left[\left(+\frac{2}{3}\right) - \left(\frac{+1}{3} - \frac{3}{3}\right)\right] = \frac{1}{2}\left(+\frac{2}{3} + \frac{2}{3}\right) = +\frac{2}{3}$$

Chapter 15.6 :



$\therefore\ 0 \le x \le 1$ and $2-x^2 \le y \le x$

Plates of Constant Density

1. **Finding a center of mass** Find the center of mass of a thin plate of density $\delta = 3$ bounded by the lines $x = 0$, $y = x$, and the parabola $y = 2 - x^2$ in the first quadrant.

$$M = \int_0^1 \int_x^{2-x^2} 3\, dy\, dx = \int_0^1 3 \cdot \left[y\right]_x^{2-x^2} dx = 3\int_0^1 (2-x^2) - x\, dx$$

$$= 3\left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 = 3\left[2 - \frac{1}{3} - \frac{1}{2}\right] = \cancel{3}\left[\frac{12}{\cancel{6}_2} - \frac{2}{\cancel{6}_2} - \frac{3}{\cancel{6}_2}\right] = \frac{7}{2}$$ //

$$M_x = \int_0^1 \int_x^{2-x^2} 3y\, dy\, dx = 3\int_0^1 \left[\frac{y^2}{2}\right]_x^{2-x^2} dx = 3\int_0^1 \frac{(2-x^2)^2}{2} - \frac{x^2}{2}\, dx = 3\int_0^1 \frac{4-4x^2+x^4}{2} - \frac{x^2}{2}\, dx = \frac{3}{2}\int_0^1 4 - 5x^2 + x^4\, dx$$

$$= \frac{3}{2}\left[4x - \frac{5x^3}{3} + \frac{x^5}{5}\right]_0^1 = \frac{3}{2}\left[4 - \frac{5}{3} + \frac{1}{5}\right] = \frac{\cancel{3}}{2}\left[\frac{60}{\cancel{15}_5} - \frac{25}{\cancel{15}_5} + \frac{3}{\cancel{15}_5}\right] = \frac{1}{\cancel{2}} \cdot \frac{\cancel{38}}{5} = \frac{19}{5}$$ //

$$M_y = \int_0^1 \int_x^{2-x^2} 3x\, dy\, dx = 3\int_0^1 x\left[y\right]_x^{2-x^2} dx = 3\int_0^1 x\left[2-x^2-x\right] dx = 3\int_0^1 2x - x^3 - x^2\, dx = 3\left[\frac{2x^2}{2} - \frac{x^4}{4} - \frac{x^3}{3}\right]_0^1$$

$$= \cancel{3}\left[\frac{12-3-4}{\cancel{12}_4}\right] = \frac{5}{4}$$ //

$\therefore\ \bar{x} = \frac{M_y}{M} = \frac{5}{4}\cdot\frac{2}{7} = \frac{10}{28} = \frac{5}{14}$ // and $\bar{y} = \frac{M_x}{M} = \frac{19}{5}\cdot\frac{2}{7} = \frac{38}{35}$ $\blacksquare$

Solids with Varying Density

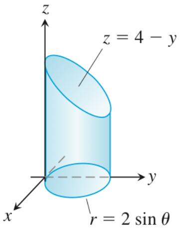
In Exercises 29 and 30, find

a. the mass of the solid. **b.** the center of mass.

29. A solid region in the first octant is bounded by the coordinate planes and the plane $x + y + z = 2$. The density of the solid is $\delta(x, y, z) = 2x$.

$$\int_0 \int_0 \int_0^{2-x-y} 2x\, dz\, dy\, dx$$

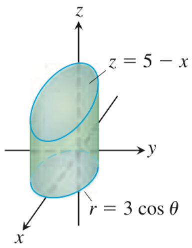
15. D is the right circular cylinder whose base is the circle $r = 2\sin\theta$ in the xy-plane and whose top lies in the plane $z = 4 - y$.



$0 \le \theta \le \pi$; $0 \le r \le 2\sin\theta$; $0 \le z \le 4 - r\sin\theta$

$$\therefore \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{4-r\sin\theta} f(r,\theta,z)\,dz\,r\,dr\,d\theta$$

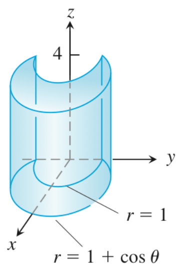
16. D is the right circular cylinder whose base is the circle $r = 3\cos\theta$ and whose top lies in the plane $z = 5 - x$.



$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$; $0 \le r \le 3\cos\theta$; $0 \le z \le 5 - r\cos\theta$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{3\cos\theta}\int_0^{5-r\cos\theta} f(z,r,\theta)\,dz\,r\,dr\,d\theta$$

17. D is the solid right cylinder whose base is the region in the xy-plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$.



$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$; $1 \le r \le 1+\cos\theta$; $0 \le z \le 4$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_1^{1+\cos\theta}\int_0^{4} f(r,z,\theta)\,dz\,r\,dr\,d\theta$$

Evaluating Integrals in Spherical Coordinates

Evaluate the spherical coordinate integrals in Exercises 21–26.

21. $$\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi}\int_0^{\pi}\sin(\phi)\left[\frac{\rho^3}{3}\right]_0^{2\sin\phi}d\phi\,d\theta = \int_0^{\pi}\int_0^{\pi}\sin(\phi)\cdot\left(\frac{8\cdot\sin^3\phi}{3}\right)d\phi\,d\theta$$

$\to \sin^2(\phi) = \frac{1-\cos(2\phi)}{2}$

$$= \frac{8}{3}\int_0^{\pi}\int_0^{\pi}\left(\sin^2(\phi)\right)^2 d\phi\,d\theta = \frac{8}{3}\int_0^{\pi}\int_0^{\pi}\left(\frac{1-\cos(2\phi)}{2}\right)^2 d\phi\,d\theta = \frac{\cancel{8}^{2}}{3\cancel{4}}\int_0^{\pi}\int_0^{\pi}(1-\cos(2\phi))^2\,d\phi\,d\theta = \frac{2}{3}\int_0^{\pi}\int_0^{\pi}1-2\cos(2\phi)+\cos^2(2\phi)\,d\phi\,d\theta$$

$\to \cos^2(\phi) = \frac{1+\cos(2\phi)}{2}$

$$= \frac{2}{3}\int_0^{\pi}\int_0^{\pi}1-2\cos(2\phi)+\left(\frac{1+\cos(4\phi)}{2}\right)d\phi\,d\theta = \frac{2}{3}\left[\int_0^{\pi}\left[\phi-\sin(2\phi)+\frac{\phi}{2}\right]_0^{\pi}d\theta + \int_0^{\pi}\int_0^{\pi}\frac{\cos(4\phi)}{2}\,d\phi\,d\theta\right] =$$

$\to u = 4\phi$; $\frac{du}{d\phi} = 4$ $\to du = 4\,d\phi$; $d\phi = \frac{du}{4}$

$$= \frac{2}{3}\left[\int_0^{\pi}\pi-\cancel{\sin(2\pi)}^{\to 0}+\frac{\pi}{2}\,d\theta+\int_0^{\pi}\int_0^{4\pi}\frac{1}{2}\cos(u)\frac{du}{4}\,d\theta\right] = \frac{2}{3}\left[\int_0^{\pi}\frac{3\pi}{2}\,d\theta + \cancel{\int_0^{\pi}\frac{1}{8}\left[\sin(4\phi)\right]_0^{\pi}}^{\to 0}\right] = \frac{\cancel{2}}{\cancel{3}}\left[\frac{\cancel{3}\pi\theta}{\cancel{2}}\right]_0^{\pi} = \pi^2$$

22. $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2}(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\sin(\phi)\int_0^{2}(\rho\cdot\cos(\phi))\cdot\rho^2\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\rho^3\cdot\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta$$

$\to u = \sin(\phi) \to du = \cos(\phi)\,d\phi \to d\phi = du/\cos(\phi)$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\cos(\phi)\sin(\phi)\left[\frac{\rho^4}{4}\right]_0^{2}d\phi\,d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\cos(\phi)\sin(\phi)\cdot 4\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\frac{\sqrt{2}}{2}}+4u\,du\,d\theta = \int_0^{2\pi}+4\left[\frac{u^2}{2}\right]_0^{\frac{\sqrt{2}}{2}}d\theta = \int_0^{2\pi}+\cancel{4}\cdot\frac{2}{\cancel{8}_{2}}\,d\theta$$

$$= \left[+\frac{2\theta}{2}\right]_0^{2\pi} = 2\pi$$

23. $$\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\sin(\phi)\left[\frac{\rho^3}{3}\right]_0^{1-\cos(\phi)/2} d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\sin(\phi)\cdot\frac{(1-\cos(\phi))^3}{24}\,d\phi\,d\theta$$

$\to u=\cos(\phi)\to du=-\sin(\phi)\,d\phi \to d\phi = du/-\sin(\phi)$

$$= \frac{1}{24}\int_0^{2\pi}\int_0^\pi \sin(\phi)\cdot(1-\cos(\phi))^3\,d\phi\,d\theta = \frac{1}{24}\int_0^{2\pi}\int_1^{-1}\cancel{\sin(\phi)}\cdot(1-u)^3\frac{du}{-\cancel{\sin(\phi)}}\,d\theta = \frac{1}{24}\int_0^{2\pi}\int_{-1}^1 1-3u+3u^2-u^3\,du\,d\theta =$$

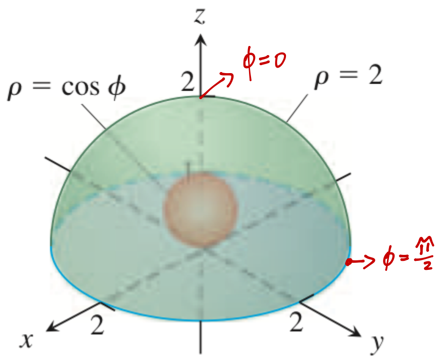
$$=\frac{1}{24}\int_0^{2\pi}\left[u-\frac32u^2+u^3-\frac{u^4}{4}\right]_{-1}^1 d\theta = \frac1{24}\int_0^{2\pi}\left(1-\frac32+1-\frac14\right)-\left(-1-\frac32-1-\frac14\right)d\theta = \frac1{24}\int_0^{2\pi}\left(\overbrace{\frac{4-6+4-1}{4}}^{1}\right)-\left(\overbrace{\frac{-4-6-4-1}{4}}^{-15}\right)d\theta =$$

$$=\frac1{24}\int_0^2 \frac{\cancelto{4}{16}}{\cancel{4}}\,d\theta = \frac{1\cdot4}{24}\left[\theta\right]_0^{2\pi} = \frac{\cancel{4}}{\cancel{24}_{\cancel{12}\,3}}\cdot\cancel{2}\pi = \frac{\pi}{3}$$ ▨

Finding Iterated Integrals in Spherical Coordinates

In Exercises 33–38, **(a)** find the spherical coordinate limits for the integral that calculates the volume of the given solid and then **(b)** evaluate the integral.

33. The solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2, z \geq 0$



$0\le\theta\le2\pi;\ 0\le\phi\le\frac\pi2;\ \cos\phi\le\rho\le2$

$$V=\int_0^{2\pi}\int_0^{\pi/2}\int_{\cos\phi}^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\frac\pi2}\sin(\phi)\left[\frac{\rho^3}{3}\right]_{\cos\phi}^2 d\phi\,d\theta=$$

$\to u=\cos(\phi)\to du=-\sin(\phi)\,d\phi$, $d\phi = du/-\sin(\phi)$

$$=\int_0^{2\pi}\int_0^{\frac\pi2}\sin(\phi)\left(\frac83-\frac{\cos^3\phi}{3}\right)d\phi\,d\theta = -\int_0^{2\pi}\int_1^0 \cancel{\sin(\phi)}\cdot\left(\frac83-\frac{u^3}{3}\right)\frac{du}{-\cancel{\sin(\phi)}}\,d\theta$$

$$=\frac{-1}{3}\int_0^{2\pi}\int_1^0 8-u^3\,du\,d\theta = \frac{-1}{3}\int_0^{2\pi}\left[8\cdot u-\frac{u^4}{4}\right]_1^0 d\theta = \frac{+1}{3}\int_0^{2\pi}+\left(8-\frac14\right)d\theta$$

$$=\frac13\int_0^{2\pi}\frac{32-1}{4}\,d\theta = \frac{31}{12}\left[\theta\right]_0^{2\pi} = \frac{31}{\cancel{12}_6}\cdot\cancel{2}\pi = \frac{31}{6}\pi$$ ▨

37. The solid bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone $z = \sqrt{x^2 + y^2}$

$\phi = z=\sqrt{x^2+y^2}$

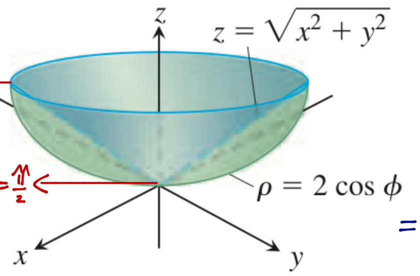
$\rho\cos(\phi)=\sqrt{r^2\cos^2(\theta)+r^2\sin(\theta)}$

$\cancel{\rho}\cos(\phi)=r=\cancel{\rho}\sin(\phi)$

$\frac{\cos(\phi)}{\sin(\phi)}=1=tg(\phi)$

$\therefore\ \phi=\frac\pi4$ //

$\phi=\frac\pi2$



$0\le\theta\le2\pi;\ \frac\pi4\le\phi\le\frac\pi2;\ 0\le\rho\le2\cos\phi$

$$V=\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\cos(\phi)}\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\sin(\phi)\cdot\left[\frac{\rho^3}{3}\right]_0^{2\cos(\phi)}d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\sin(\phi)\cdot\frac83\cdot\cos^3(\phi)\,d\phi\,d\theta =$$

$\to u=\cos(\phi)$, $du=-\sin(\phi)\,d\phi$, $d\phi=du/-\sin(\phi)$

$$=-\frac83\int_0^{2\pi}\int_{\sqrt2/2}^0\cancel{\sin(\phi)}\cdot u^3\cdot\frac{du}{-\cancel{\sin(\phi)}}\,d\theta = -\frac83\int_0^{2\pi}\int_{\sqrt2/2}^0 u^3\,du\,d\theta$$

$$=-\frac83\int_0^{2\pi}\left[\frac{u^4}{4}\right]_{\sqrt2/2}^0 d\theta = +\frac83\int_0^{2\pi}+\left(\frac{4}{16\cdot4}\right)d\theta = \frac{\cancel{8}\cdot\cancel{4}}{3\cdot\cancel{4}\cdot\cancel{16}_2}\left[\theta\right]_0^{2\pi} = \frac{1}{3\cdot\cancel{2}}\cancel{2}\pi = \frac\pi3$$ ▨

Finding Triple Integrals

39. Set up triple integrals for the volume of the sphere $\rho = 2$ in **(a)** spherical, **(b)** cylindrical, and **(c)** rectangular coordinates.

$0\le\theta\le2\pi;\ 0\le\phi\le\frac\pi2;\ 0\le\rho\le2 \to$ (a) $\int_0^{2\pi}\int_0^{\frac\pi2}\int_0^2\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$ //

$0\le\theta\le2\pi;\ 0\le r\le \underbrace{2\cdot\sin\left(\frac\pi2\right)}_{\rho\sin(\phi)}=2;\ 0\le z\le\underbrace{2\cdot\cos\frac\pi2}_{\rho\cdot\cos(\phi)}=0 \to$ (b) $\int_0^{2\pi}\int_0^2\int_0^{-} f(r,z,\theta)\,dz\,r\,dr\,d\theta$ //

$\to$?

$0\le x\le -;\ 0\le y\le -;\ 0\le z\le - \to$ (c) $\int_0^-\int_0^-\int_0^- dz\,dy\,dx$ ▨

Chap. 15.8: ~~1~~, ~~3~~, ~~6~~, ~~7~~

Chap. 16.1: ~~10-13~~, ~~15-16~~, ~~33~~, ~~35~~

Chap. 15.8:

Jacobians and Transformed Regions in the Plane

1. a. Solve the system

$$u = x - y, \qquad v = 2x + y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = x - y$, $v = 2x + y$ of the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(1, -2)$ in the xy-plane. Sketch the transformed region in the uv-plane.

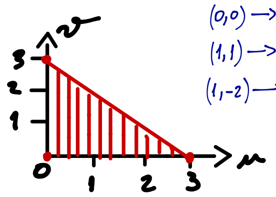
① $\begin{cases} u = x - y \\ v = 2x + y \end{cases}$ ⊕ $\quad f(g(u,v), h(u,v))$

$u + v = 3x$

$\therefore x = \dfrac{(u+v)}{3}$ // $\longrightarrow y = \dfrac{u+v}{3} - \dfrac{u3}{3} = \dfrac{v - 2u}{3}$ //

ⓐ $J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}\cdot\frac{1}{3} - \left(-\frac{2}{3}\cdot\frac{1}{3}\right) = \frac{1}{9} + \frac{2}{9} = \frac{3}{9} = \frac{1}{3}$ //

ⓑ



$(0,0) \longrightarrow u = 0-0 = 0;\ v = 2.0+0 = 0$

$(1,1) \longrightarrow u = (1-1) = 0;\ v = 2.1+1 = 3$

$(1,-2) \longrightarrow u = 1-(-2) = 3;\ v = 2.1 + -2 = 0$

3. a. Solve the system

$$u = 3x + 2y, \qquad v = x + 4y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = 3x + 2y$, $v = x + 4y$ of the triangular region in the xy-plane bounded by the x-axis, the y-axis, and the line $x + y = 1$. Sketch the transformed region in the uv-plane.

3a $\quad u = 3x + 2y$

$(-3)\times \quad v = x + 4y \quad \times(-3)$ ⊕

$u - 3v = \cancel{3x} - \cancel{3x} + 2y - 12y$

$u - 3v = -10y$

$y = \dfrac{-u + 3v}{10}$ //

$u = 3x + 2\cdot\left(\dfrac{-u+3v}{10}\right)$

$u = 3x - \dfrac{2u}{10} + \dfrac{6v}{10}$

$-3x = \dfrac{-10u}{10} - \dfrac{2u + 6v}{10}$

$-3x = \dfrac{-12u}{10} + \dfrac{6v}{10}$

$x = \dfrac{\cancel{12}u}{\cancel{30}} - \dfrac{\cancel{6}v}{\cancel{30}} = \dfrac{2}{5}u - \dfrac{1}{5}v$ //

ⓐ $J(u,v) = \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{vmatrix}$

$= \left(\frac{2}{5}\cdot\frac{3}{10}\right) - \left(-\frac{1}{10}\left(-\frac{1}{5}\right)\right)$

$= \frac{6}{50} - \frac{1}{50} = \frac{5}{50} = \frac{1}{10}$ //

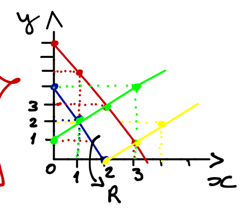
3b

6. Use the transformation in Exercise 1 to evaluate the integral

$$\iint_R (2x^2 - xy - y^2)\, dx\, dy \longrightarrow \iint_G f(g(u,v), h(u,v))\cdot J(u,v)\, du\, dv$$

for the region R in the first quadrant ($0 \le \theta \le \frac{\pi}{2}$) bounded by the lines $\boxed{y = -2x + 4}$, $\boxed{y = -2x + 7}$, $\boxed{y = x - 2}$, and $\boxed{y = x + 1}$.

$y=-2x+4$	$y=-2x+7$	$y=x-2$	$y=x+1$
$x=0;\ y=4$	$x=0;\ y=7$	$x=0;\ y=-2$	$x=0;\ y=1$
$x=1;\ y=2$	$x=1;\ y=5$	$x=1;\ y=-1$	$x=1;\ y=2$
$x=2;\ y=0$	$x=2;\ y=3$	$x=2;\ y=0$	$x=2;\ y=3$
	$x=3;\ y=1$	$x=3;\ y=1$	$x=3;\ y=4$
	$x=4;\ y=-1$	$x=4;\ y=2$	



① $2x + y = 4 \longrightarrow 2\left(\dfrac{u+v}{3}\right) + \left(\dfrac{v - 2u}{3}\right) = 4$

$\longrightarrow \dfrac{2u + 2v}{3} + \dfrac{v - 2u}{3} = \dfrac{3}{3}v = v = 4$ //

② $2x + y = 7 \longrightarrow 2\left(\dfrac{u+v}{3}\right) + \left(\dfrac{v-2u}{3}\right) = 7$

$\longrightarrow v = 7$ //

③ $x - y = 2 \longrightarrow \dfrac{u+v}{3} - \left(\dfrac{v-2u}{3}\right) = 2$

$\longrightarrow u + \cancel{v} - \cancel{v} + 2u = 6 \longrightarrow \cancel{3}u = \cancel{6}2 \longrightarrow u = 2$ //

④ $x - y = -1 \longrightarrow \dfrac{u+v}{3} - \left(\dfrac{v-2u}{3}\right) = -1$

$\longrightarrow \dfrac{u + \cancel{v} - \cancel{v} + 2u}{3} = -1 \longrightarrow \cancel{3}u = -\cancel{3}1$

$\longrightarrow u = -1$ //

⑥ $\iint_R 2x^2 - xy - y^2\, dx\, dy = \iint_R \overbrace{(x-y)}^{u}\cdot\overbrace{(2x+y)}^{v}\, dx\, dy$

$= \iint_G u\cdot v\cdot\frac{1}{3}\, du\, dv = \frac{1}{3}\int_4^7\int_{-1}^2 u\cdot v\, du\, dv$

$= \frac{1}{3}\int_4^7 v\left[\frac{u^2}{2}\right]_{-1}^2 dv = \frac{1}{3}\int_4^7 v\left(\frac{2^2}{2} - \frac{(-1)^2}{2}\right) dv$

$= \frac{1}{3}\int_4^7\left(\frac{4}{2} - \frac{1}{2}\right) v\, dv = \frac{\cancel{3}1}{2\cancel{6}}\left[\frac{v^2}{2}\right]_4^7$

$= \frac{1}{2}\left(\frac{7^2}{2} - \frac{4^2}{2}\right) = \frac{1}{2}\cdot\left(\frac{49-16}{2}\right) = \frac{33}{4}$ ▨

7. Use the transformation in Exercise 3 to evaluate the integral

$$\iint_R (3x^2 + 14xy + 8y^2)\, dx\, dy$$

for the region R in the first quadrant bounded by the lines ① $y = -(3/2)x + 1$, ② $y = -(3/2)x + 3$, ③ $y = -(1/4)x$, and ④ $y = -(1/4)x + 1$.

$u = 3x + 2y$; $v = x + 4y$; $x = \frac{2}{5}u - \frac{1}{5}v$; $y = \frac{-u+3v}{10}$

$J(u,v) = 1/10$

① $\frac{3}{2}x + y = 1 \rightarrow \frac{3}{2}\left(\frac{2}{5}u - \frac{1}{5}v\right) + \left(\frac{-u+3v}{10}\right) = 1 \rightarrow \frac{6}{10}u - \cancel{\frac{3}{10}v} - \frac{u}{10} + \cancel{\frac{3}{10}v} = 1 \rightarrow \frac{5}{10}u = 1 \rightarrow u = 2$

② $\frac{3}{2}x + y = 3 \rightarrow \frac{5}{10}u = 3 \rightarrow u = \frac{30}{5} = 6$

③ $\frac{1}{4}x + y = 0 \rightarrow \frac{1}{4}\left(\frac{2}{5}u - \frac{1}{5}v\right) + \left(\frac{-u+3v}{10}\right) = 0 \rightarrow \cancel{\frac{2}{20}u} - \frac{1}{20}v - \cancel{\frac{2u}{20}} + \frac{6v}{20} = 0 \rightarrow \frac{5}{20}v = 0 \rightarrow v = 0$

④ $\frac{1}{4}x + y = 1 \rightarrow \frac{5}{20}v = 1 \rightarrow v = \frac{20}{5} = 4$

$$\iint_R (3x^2 + 14xy + 8y^2)\, dx\, dy = \int_0^4 \int_2^6 \frac{1}{10} \cdot u \cdot v \, du\, dv = \frac{1}{10}\int_0^4 v\left[\frac{u^2}{2}\right]_2^6 dv = \frac{1}{10}\int_0^4 v\left[\frac{6^2}{2} - \frac{4}{2}\right] dv = \frac{1}{10}\int_0^4 \left[\frac{36}{2} - \frac{4}{2}\right] v\, dv$$

$$= \frac{32}{20}\left[\frac{v^2}{2}\right]_0^4 = \frac{\cancel{32}^{\cancel{16}\,8}}{\cancel{20}_{\cancel{10}\,5}}\left[\frac{\cancel{16}^{8}}{\cancel{2}} - 0\right] = \frac{64}{5}$$

Chap. 16.1:

10. Evaluate $\int_C (x - y + z - 2)\, ds$ where C is the straight-line segment $x = t$, $y = (1 - t)$, $z = 1$, from $(0, 1, 1)$ to $(1, 0, 1)$.
11. Evaluate $\int_C (xy + y + z)\, ds$ along the curve $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2 - 2t)\mathbf{k}$, $0 \le t \le 1$.
12. Evaluate $\int_C \sqrt{x^2 + y^2}\, ds$ along the curve $\mathbf{r}(t) = (4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3t\mathbf{k}$, $-2\pi \le t \le 2\pi$.
13. Find the line integral of $f(x, y, z) = x + y + z$ over the straight-line segment from $(1, 2, 3)$ to $(0, -1, 1)$.

⑩ $\vec{r}(t) = t\vec{i} + (1-t)\vec{j} + \vec{k}$

$|v(t)| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$

$\int_a^b t - (1-t) + 1 - 2) \cdot \sqrt{2}\, dt =$

$= \int_a^b (2t - 2) \cdot \sqrt{2}\, ds = \sqrt{2} \cdot \left[t^2 - 2t\right]_0^1 = -\sqrt{2}$

⑪ $|v(t)| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$ ∴ $\int_0^1 (2t) \cdot t + t + (2 - 2t) \cdot |v(t)|\, dt = \int_0^1 (2t^2 + 2 - t) \cdot 3\, dt = 3\left[\frac{2}{3}t^3 + 2t - \frac{t^2}{2}\right]_0^1 = 3 \cdot \left[\frac{2}{3} + 2 - \frac{1}{2}\right]$

$= \cancel{3} \cdot \left[\frac{4}{\cancel{6}_2} + \frac{12}{\cancel{6}_2} - \frac{3}{\cancel{6}_2}\right] = \frac{13}{2}$

⑫ $|v(t)| = \sqrt{(-4\sin(t))^2 + (4\cos(t))^2 + (3)^2} = \sqrt{16\sin^2(t) + 16 \cdot \cos^2(t) + 9} = \sqrt{16 \cdot (\sin^2(t) + \cos^2(t)) + 9} = \sqrt{16 + 9} = 5$

∴ $\int_{-2\pi}^{2\pi} \sqrt{(4\cos(t))^2 + (4\sin(t))^2} \cdot 5\, dt = 5\int_{-2\pi}^{2\pi} \sqrt{16}\, dt = 5 \cdot \sqrt{16}\, [t]_{-2\pi}^{2\pi} = 20\,[2\pi + 2\pi] = 20 \cdot 4\pi = 80\pi$

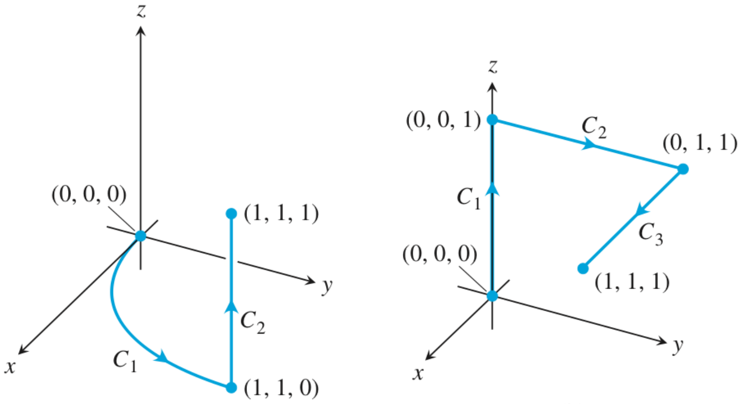
⑬ $f(x, y, z) = x + y + z$ and $(1, 2, 3)$ to $(0, -1, 1)$

$\vec{r}(t) = \overset{(1-t)}{\cancel{(t-1)}}\,(1, 2, 3) + t\,(0, -1, 1) =$

15. Integrate $f(x, y, z) = x + \sqrt{y} - z^2$ over the path from $(0, 0, 0)$ to $(1, 1, 1)$ (see accompanying figure) given by

C_1: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1$

C_2: $\mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1$



(a) (b)

The paths of integration for Exercises 15 and 16.

15 $C_1: \int_0^1 \left(x+\sqrt{x^2}-0^2\right)\cdot\left[\sqrt{1^2+4x^2}\right]\cdot dx$

$\hookrightarrow 2|x|$, but $x>0$, then $2x$

$\therefore \int_0^1 2x\cdot(1+4x^2)^{1/2}\,dx \rightarrow u = 1+4x^2 \rightarrow du = 8x\,dx$

$dx = du/8x$

$\therefore \int_1^5 \cancel{2x}\cdot(u)^{1/2}\cdot\frac{du}{4\cancel{8x}} = \frac{1}{4}\int_1^5 (u)^{1/2}\,du = \frac{1}{4}\left[\frac{2}{3}\cdot u^{3/2}\right]_1^5 =$

$= \frac{\cancel{2}}{6\cancel{12}}\left[u^{3/2}\right]_1^5 = \frac{1}{6}\cdot\left(\sqrt{5\cdot5^2}-1\right) = \frac{1}{6}\left[5\cdot\sqrt{5}-1\right]$ //

$C_2: \int_0^1 1+|1|-x^2\cdot[1]\,dx = \int_0^1 2-x^2\,dx$

$= \left[2x-\frac{x^3}{3}\right]_0^1 = 2-\frac{1}{3} = \frac{6-1}{3} = \frac{5}{3}$ //

$\therefore \int_C f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds$

$= \frac{1}{6}\left[5\sqrt{5}-1\right]+\frac{5}{3} = \frac{5\sqrt{5}}{6}-\frac{1}{6}+\frac{5}{3}$

$= \frac{5}{6}\sqrt{5}+\left(\frac{-1+10}{6}\right) = \frac{5}{6}\sqrt{5}+\frac{\cancel{9}\,3}{\cancel{6}\,2}$

$= \frac{5}{6}\sqrt{5}+\frac{3}{2}$ ▨

16. Integrate $f(x, y, z) = x + \sqrt{y} - z^2$ over the path from $(0, 0, 0)$ to $(1, 1, 1)$ (see accompanying figure) given by

C_1: $\mathbf{r}(t) = t\mathbf{k}, \quad 0 \le t \le 1$

C_2: $\mathbf{r}(t) = t\mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1$

C_3: $\mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1$

$C_1: \int_0^1 -x^2\cdot 1\,dx = \left[\frac{-x^3}{3}\right]_0^1 = \frac{-1}{3}$ //

$C_2: \int_0^1 \left(\sqrt{x}-1\right)\cdot 1\,dx = \left[\frac{2}{3}x^{3/2}-x\right]_0^1 = \left[\frac{2}{3}-1\right] = \frac{2-3}{3} = -\frac{1}{3}$ //

$C_3: \int_0^1 (x+|1|-1)\cdot[1]\,dx = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$ //

$\therefore \int_C f(x,y,z)\,ds = -\frac{1}{3}-\frac{1}{3}+\frac{1}{2} = \frac{-2-2+3}{6} = \frac{-1}{6}$ ▨

Masses and Moments

33. **Mass of a wire** Find the mass of a wire that lies along the curve $\mathbf{r}(t) = (t^2 - 1)\mathbf{j} + 2t\mathbf{k}$, $0 \le t \le 1$, if the density is $\delta = (3/2)t$.

$|v(x)| = \sqrt{(2x)^2+(2)^2} = 2\sqrt{x^2+1}$

$\rightarrow u = x^2+1 \rightarrow du = 2x\,dx \rightarrow dx = du/2x$

$\int_0^1 \frac{3}{\cancel{2}}\cdot x\cdot\cancel{2}\cdot\sqrt{x^2+1}\,dx = 3\int_0^1 x\cdot(x^2+1)^{1/2}\,dx = 3\int_1^2 \cancel{x}\cdot(u)^{1/2}\frac{du}{2\cancel{x}} = \frac{3}{2}\int_1^2 (u)^{1/2}\,du = \frac{\cancel{3}}{\cancel{2}}\left[\frac{\cancel{2}}{\cancel{3}}u^{3/2}\right]_1^2 = \sqrt{2^3}-1 = 2\sqrt{2}-1$ ▨

35. **Mass of wire with variable density** Find the mass of a thin wire lying along the curve $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + \sqrt{2}t\mathbf{j} + (4 - t^2)\mathbf{k}$, $0 \le t \le 1$, if the density is (a) $\delta = 3t$ and (b) $\delta = 1$.

a $\int_0^1 3x\left[\sqrt{2+2+(-2x)^2}\right]dx = 3\int_0^1 x\cdot\sqrt{4+4x^2}\,dx \rightarrow u = 4+4x^2 \rightarrow du = 8x\,dx \rightarrow dx = du/8x$

$= 3\int_4^8 \cancel{x}\cdot(u)^{1/2}\frac{du}{8\cancel{x}} = \frac{\cancel{3}}{4\cancel{8}}\left[\frac{\cancel{2}}{\cancel{3}}u^{3/2}\right]_4^8 = \frac{1}{4}\left[\sqrt{8^3}-\sqrt{4^3}\right] = \frac{1}{\cancel{4}}\left[\overset{2}{\cancel{8}}\sqrt{8}-\cancel{4}\sqrt{\cancel{4}^{\,2}}\right] = 4\sqrt{2}-2$ //

b $|v(x)| = \sqrt{(\sqrt{2})^2+(\sqrt{2})^2+(-2x)^2} = \sqrt{4-4x^2} \longrightarrow 2\sqrt{1-x^2}$

$M = \int_0^1 (1)\cdot\left(2\cdot\sqrt{1-x^2}\right)dx \longrightarrow \cdots$ ▨

Ch. 16,2: ~~3, 4, 7a, b, 19, 20, 29a, b, 38b~~

Ch. 16,3: ~~1-4, 7-9, 19, 25, 29a, b~~

Vector Fields

Find the gradient fields of the functions in Exercises 1–4

3. $g(x, y, z) = e^z - \ln(x^2 + y^2)$

4. $g(x, y, z) = xy + yz + xz$

(3) $\nabla g = \frac{-2x}{x^2+y^2}\vec{i} - \frac{2y}{x^2+y^2}\vec{j} + e^z\vec{k}$

(4) $\nabla g = (y+z)\vec{i} + (z+x)\vec{j} + (y+x)\vec{k}$

Line Integrals of Vector Fields

In Exercises 7–12, find the line integrals of **F** from (0, 0, 0) to (1, 1, 1) over each of the following paths in the accompanying figure.

a. The straight-line path C_1: $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$

b. The curved path C_2: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$, $0 \le t \le 1$

7. $\mathbf{F} = 3y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}$

(7) $\int_C F(r(t))\cdot\frac{dr}{dt}\,dt$

(a) $\therefore F(r(t)) = 3t\vec{i} + 2t\vec{j} + 4t\vec{k}$ and $\frac{dr(t)}{dt} = \vec{i} + \vec{j} + \vec{k}$

$\hookrightarrow \int_0^1 (3t, 2t, 4t)\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} dt = \int_0^1 9t\,dt = 9\left[\frac{1}{2}\right] = \frac{9}{2}$

(b) $\therefore \frac{dr(t)}{dt} = (1, 2t, 4t^3) \therefore \int_0^1 (3t^2, 2t, 4t^4)\cdot\begin{bmatrix}1\\2t\\4t^3\end{bmatrix} dt$

$\hookrightarrow = \int_0^1 3t^2 + 4t^2 + 16t^7\,dt = \int_0^1 7t^2 + 16t^7\,dt$

$= \left[\frac{7}{3}t^3 + \frac{16}{8}t^8\right]_0^1 = \frac{7}{3} + \frac{2.3}{3} = \frac{13}{3}$

Work

In Exercises 19–22, find the work done by **F** over the curve in the direction of increasing t.

19. $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$

20. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x + y)\mathbf{k}$

$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/6)\mathbf{k}$, $0 \le t \le 2\pi$

(19) $F(r(t)) = (t^3, t^2, -t^3)$ and $\frac{dr(t)}{dt} = (1, 2t, 1)$

$\therefore \int_0^1 \cancel{t^3} + 2t^3 + \cancel{(-t^3)}\,dt = \left[\frac{2}{4}t^4\right]_0^1 = \frac{1}{2}$

(20) $F(r(t)) = (2\cdot\sin(t), 3\cos(t), \cos(t) + \sin(t))$

$\frac{dr(t)}{dt} = \left(-\sin(t), \cos(t), \frac{1}{6}\right) \therefore \int_0^{2\pi} -2\sin^2(t) + 3\cos^2(t) + \frac{1}{6}\cdot(\cos(t) + \sin(t))\,dt = \int_0^{2\pi} -\cancel{2}\cdot\left(\frac{1-\cos(2t)}{\cancel{2}}\right) + 3\cdot\left(\frac{1+\cos(2t)}{2}\right) + \frac{1}{6}(\cos(t) + \sin(t))\,dt$

$= \int_0^{2\pi} -1 + \cos(2t) + \frac{3}{2} + \frac{3\cos(2t)}{2}\,dt + \frac{1}{6}\int_0^{2\pi} \cos(t) + \sin(t)\,dt = \int_0^{2\pi} \frac{1}{2} + \frac{5\cos(2t)}{2}\,dt + \cancel{\frac{1}{6}\left[\sin(t) - \cos(t)\right]_0^{2\pi}} \to 0 =$

$u = 2t \to du = 2dt \to dt = du/2$

$= \int_0^{2\pi} \frac{1}{2}\left[1 + 5\cos(2t)\right] dt = \int_0^{4\pi} \frac{1}{2}\cdot\left[1 + 5\cos(u)\right]\cdot\frac{du}{2} = \frac{1}{4}\int_0^{4\pi} 1 + 5\cos(u)\,du = \frac{1}{4}\left[u + 5\cdot\sin(u)\right]_0^{4\pi} = \frac{1}{4}\cdot 4\pi = \pi$

29. **Circulation and flux** Find the circulation and flux of the fields

$$\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$$

around and across each of the following curves.

a. The circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$

b. The ellipse $\mathbf{r}(t) = (\cos t)\mathbf{i} + (4\sin t)\mathbf{j}$, $0 \le t \le 2\pi$

(29) (a) $\frac{dr(t)}{t} = (-\sin(t), \cos(t))$; $F_1 = (\cos(t), \sin(t))$;

$F_2 = (-\sin(t), \cos(t)) \therefore F_1\cdot\frac{dr}{dt} = \left[(-\sin(t)\cdot\cos(t)) + (\cos(t)\sin(t))\right] = 0$ //

$F_2\cdot\frac{dr}{dt} = \left[\sin^2(t), \cos^2(t)\right] = 1$ //; $C_1: \int_0^{2\pi} 0\,dt = 0$ //; $C_2: \int_0^{2\pi} dt = 2\pi$ //

Flux: $\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C \left[M(r(t))\cdot\frac{dy(t)}{dt} - N(r(t))\cdot\frac{dx(t)}{dt}\right]\cdot dt$

$\hookrightarrow F(x,y,z) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$

$F(\vec{r}(t)) = M(r(t))\,\vec{i} + N(r(t))\,\vec{j}$

$F_1 := \begin{cases} M(r(t)) = \cos(t);\ N(r(t)) = \sin(t) \\ x(t) = \cos(t);\ y(t) = \sin(t) \end{cases}$

$F_2 : \begin{cases} M(r(t)) = -\sin(t);\ N(r(t)) = \cos(t) \\ x(t) = \cos(t);\ y(t) = \sin(t) \end{cases}$

$\therefore F_1 : \oint_C \vec{F}\cdot\vec{n}\,ds = \oint_0^{2\pi} \cos(t)\cdot\cos(t) - \sin(t)\cdot(-\sin(t))\,dt$

$= \oint_0^{2\pi} \cos^2(t) + \sin^2(t)\,dt = \oint_0^{2\pi} dt = \left[t\right]_0^{2\pi} = 2\pi$

$\therefore F_2 : \oint_0^{2\pi} \vec{F}\cdot\vec{n}\,ds = \oint_0^{2\pi} -\sin(t)\cdot\cos(t) - \cos(t)\cdot(-\sin(t))\,dt$

$= \oint_0^{2\pi} 0\,dt = 0$

29. **Circulation and flux** Find the circulation and flux of the fields

$$\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$$

around and across each of the following curves.

b. The ellipse $\mathbf{r}(t) = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j}, \quad 0 \le t \le 2\pi$

(b) $\frac{dr(t)}{dt} = \left(-\sin(t), 4\cos(t)\right)$; $F_1: \left(\cos(t), 4\sin(t)\right)$; $F_2: \left(-4\sin(t), \cos(t)\right)$;

$F_1 \cdot \frac{dr}{dt} = -\sin(t)\cos(t) + 16\sin(t)\cos(t) = 15\sin(t)\cos(t)$ //

$F_2 \cdot \frac{dr}{dt} = 4\sin^2(t) + 4\cos^2(t) = 4\left(\sin^2(t)\cdot\cos^2(t)\right) = 4$ //;

$C_1: \int_0^{2\pi} 15\sin(t)\cdot\cos(t)\,dt \overset{u=\cos(t)}{=} \underset{dt = du/-\sin(t)}{} = \int_1^1 -15\cancel{\sin(t)}\cdot u \cdot \frac{du}{\cancel{-\sin(t)}} = -15\left[\frac{u^2}{2}\right]_1^1 = 0$ //

$C_2: \int_0^{2\pi} 4\,dt = 4\left[t\right]_0^{2\pi} = 8\pi$ //

Flux:

$F(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$; $F(r(t)) = M(r(t))\vec{i} + N(r(t))\vec{j}$

$\vec{r}(t) = \cos(t)\vec{i} + 4\sin(t)\vec{j}$; $x(t) = \cos(t)$; $y(t) = 4\sin(t)$

$F_1 = \left(\cos(t), 4\sin(t)\right)$; $F_2 = \left(-4\sin(t), \cos(t)\right)$; $\frac{dx}{dt} = -\sin(t)$; $\frac{dy}{dt} = 4\cos(t)$

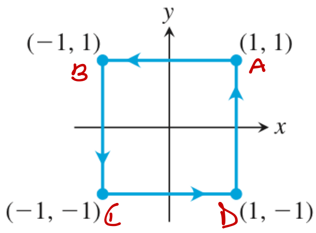
$\therefore F_1: \oint_0^{2\pi} \cos(t)\cdot 4\cos(t) - 4\sin(t)\cdot(-\sin(t))\,dt = \oint_0^{2\pi} 4\left[\cos^2(t) + \sin^2(t)\right]dt = 4\int_0^{2\pi} dt = 8\pi$ //

$\therefore F_2: \oint_0^{2\pi} \vec{F}\cdot\vec{n}\,ds = \int_0^{2\pi} +4\sin^2(t) - 4\cos^2(t)\,dt = 4\int_0^{2\pi} \sin^2(t) - \cos^2(t)\,dt \overset{\sin^2(t) = 1-\cos^2(t)}{=} 4\int_0^{2\pi} 1 - \cos^2(t) - \cos^2(t)\,dt =$

$= 4\int_0^{2\pi} 1 - 2\cos^2(t)\,dt = 4\int_0^{2\pi} \cancel{1} - \cancel{2}\left[\frac{\cancel{1} + \cos(2t)}{\cancel{2}}\right]dt = 4\int_0^{2\pi} \cos(2t)\,dt \overset{u = 2t \to du = 2\,dt \to dt = du/2}{=} \cancel{4}^{2}\int_0^{4\pi} \cos(u)\cdot\frac{du}{\cancel{2}} = 2\cdot\left[\sin(u)\right]_0^{4\pi} = 0$ ▨

38. Find the circulation of the field $\mathbf{F} = y\mathbf{i} + (x + 2y)\mathbf{j}$ around each of the following closed paths.

a.



$A \to B$ $r(t) := (1-t)\cdot(1,1) + (-1,1)\cdot t = \left((1-2t), (1)\right) \to \frac{dr(t)}{dt} = (-2, 0) \to F(r(t)) = (1, (1-2t))$

$\to F\cdot\frac{dr}{dt} = -2 + 0 = -2 \quad \therefore \int_0^1 -2\,dt = \left[-2t\right]_0^1 = -2$ //

$B \to C$ $r(t) = (1-t)\cdot(-1,1) + (-1,-1)t = \left(-1, (1-2t)\right) \to \frac{dr}{dt} = (0,-2) \to F(r(t)) = \left((1-2t), (-1+2-4t)\right)$

$\to F\cdot\frac{dr(t)}{dt} = (-2+8t) \quad \therefore \int_0^1 -2+8t\,dt = \left[-2t+4t^2\right]_0^1 = 2$ //

$C \to D$ $r(t) = (1-t)\cdot(-1,-1) + (1,-1)t = \left((-1+2t), (-1+0)\right) \to \frac{dr}{dt} = (2,0) \to F(r(t)) = (-1, (-3+2t))$

$\to F\cdot\frac{dr}{dt} = -2 \quad \therefore \int_0^1 -2\,dt = \left[-2t\right]_0^1 = -2$ //

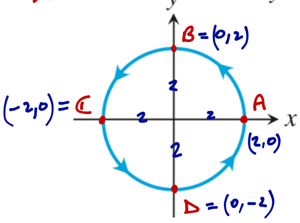
$D \to A$ $r(t) = (1-t)\cdot(1,-1) + (1,1)t = \left(1, (-1+2t)\right) \to \frac{dr}{dt} = (0,2) \to F(r(t)) = \left((-1+2t), (-1+4t)\right)$

$\to F\cdot\frac{dr}{dt} = 0 + 2\cdot(-1+4t) = -2+8t \quad \therefore \int_0^1 -2+8t\,dt = \left[-2t+4t^2\right]_0^1 = 2$ //

$\therefore$ Circulation $= -2 + 2 - 2 + 2 = 0$ ▨

$\boxed{\begin{aligned} r^2 &= 4 \to r = 2 \\ x &= r\cdot\cos(t) \\ y &= r\cdot\sin(t) \end{aligned}}$

b. $x^2 + y^2 = 4$



$r(t) = 2\cos(\theta)\vec{i} + 2\sin(\theta)\vec{j} \to \vec{F} = \left(2\sin(\theta), (2\cos(\theta) + 4\sin(\theta))\right)$; $\frac{d\vec{r}}{d\theta} = \left(-2\sin(\theta), 2\cos(\theta)\right)$

$\therefore \int_0^{2\pi} -4\sin^2(\theta) + \left(4\cos^2(\theta) + 8\sin(\theta)\cos(\theta)\right)d\theta = \int_0^{2\pi} 4\cdot\left(\cancel{-\sin^2(\theta) + \cos^2(\theta)}^{-1}\right) + 2\sin(\theta)\cos(\theta)\,d\theta$

$= \int_0^{2\pi} 4\cdot\left[-1 + 2\sin(\theta)\cos(\theta)\right]d\theta \overset{u = \cos(\theta) \to du = -\sin(\theta)d\theta \to d\theta = du/-\sin(\theta)}{=} 4\int_1^1 -d\theta + 4\int_1^1 \cancel{\sin(\theta)}\cdot u\,\frac{du}{\cancel{-\sin(\theta)}} = 0 + 0 = 0$ ▨

Ch. 16,3:

Component Test $\nabla \times F := \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = +\vec{i}\left(\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z}\right)-\vec{j}\left(\frac{\partial P}{\partial x}-\frac{\partial M}{\partial z}\right) + \vec{k}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = 0$

Testing for Conservative Fields

Which fields in Exercises 1–6 are conservative, and which are not?

1. $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ → Conservative
2. $\mathbf{F} = (y\sin z)\mathbf{i} + (x\sin z)\mathbf{j} + (xy\cos z)\mathbf{k}$ → Conservative
3. $\mathbf{F} = y\mathbf{i} + (x+z)\mathbf{j} - y\mathbf{k}$ → Not conservative
4. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ → Not conservative

① $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} = z - z = 0$ ✓ ; $\frac{\partial P}{\partial x}-\frac{\partial M}{\partial z} = y - y = 0$ ✓

$\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z} = x - x = 0$ ✓ → Conservative

③ $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} = 1-1=0$ ✓ ; $\frac{\partial P}{\partial x}-\frac{\partial M}{\partial z} = 0-0=0$ ✓ ;

$\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z} = -1-0=-1$ ✗

② $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} = \sin(z)-\sin(z)=0$ ✓ ; $\frac{\partial P}{\partial x}-\frac{\partial M}{\partial z} = y\cos(z)-y\cos(z) = 0$ ✓

$\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z} = x\cos(z)-x\cos(z)=0$ ✓

④ $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} = -1-1=-2$ ✗

Finding Potential Functions

In Exercises 7–12, find a potential function f for the field $\mathbf{F}$.

7. $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$ → Conservative
8. $\mathbf{F} = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$ → Conservative
9. $\mathbf{F} = e^{y+2z}(\mathbf{i} + x\mathbf{j} + 2x\mathbf{k})$

⑦ $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} = 0$ ✓, $\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z} = 0$ ✓, $\frac{\partial P}{\partial x}-\frac{\partial M}{\partial z} = 0$ ✓

↳ $\frac{\partial f}{\partial x} = M = 2x \to f(x,y,z) = x^2 + C(y,z)$

↳ $\frac{\partial f}{\partial y} = N \Rightarrow 0 + \frac{\partial C}{\partial y} = 3y \to f(x,y,z) = x^2 + \frac{3}{2}y^2 + C(z)$

↳ $\frac{\partial f}{\partial z} = P \Rightarrow 0+0+\frac{\partial C}{\partial z} = 4z \to f(x,y,z) = x^2+\frac{3}{2}y^2+2z^2+C$ ▨

⑧ $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} = 0$ ✓, $\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z} = 0$ ✓, $\frac{\partial P}{\partial x}-\frac{\partial M}{\partial z} = 0$ ✓

∴ $f(x,y,z) = C(x,y,z)$;

→ $\frac{\partial f(x,y,z)}{\partial x} = M \to \frac{\partial f}{\partial x} = (y+z) \to f(x,y,z) = (y+z)x + C(y,z)$

→ $\frac{\partial f}{\partial y} = N \to \cancel{x} + \frac{\partial C}{\partial y} = \cancel{x} + z \to \frac{\partial C}{\partial y} = z \to f(x,y,z) = (y+z)x + zx + C(z)$

→ $\frac{\partial f}{\partial z} = P \to x + \cancel{x} + \frac{\partial C}{\partial z} = \cancel{x} + y \to \frac{\partial C}{\partial z} = y - x \to f(x,y,z) = (y+z)x + zx + (y-x)z + C$

↳ $= yx + zx + \cancel{zx} + yz - \cancel{xz} + C$

↳ $= xy + xz + yz + C = x(y+z) + yz + C$ ▨

⑨ $F = e^{y+2z}\vec{i} + e^{y+2z}x\,\vec{j} + e^{y+2z}2x\,\vec{k}$

$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} = \left[e^y e^{2z}\right]'_y - \left[e^{y+2z}x\right]'_x = 1e^{y+2z} - 1e^{y+2z} = 0$ ✓

$\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z} = \left[2x e^y e^{2z}\right]'_y - \left[e^y e^{2z} x\right]'_z = 2xe^{y+2z} - 2xe^{y+2z} = 0$ ✓

$\frac{\partial P}{\partial x}-\frac{\partial M}{\partial z} = \left[2x e^{y+2z}\right]'_x - \left[e^y e^{2z}\right]'_z = 2e^{y+2z} - 2e^{y+2z} = 0$ ✓

} conservative and open simple connected domain

∴ $\frac{\partial f}{\partial x} = M = e^{y+2z} \to f(x,y,z) = xe^{y+2z} + C(y,z)$

→ $\frac{\partial f}{\partial y} = N \to \cancel{xe^{y+2z}} + \frac{\partial C}{\partial y} = \cancel{e^{y+2z}x} \to f(x,y,z) = xe^{y+2z} + 0 + C(z)$

→ $\frac{\partial f}{\partial z} = P \to \cancel{2xe^{y+2z}} + \frac{\partial C}{\partial z} = \cancel{2xe^{y+2z}} \to f(x,y,z) = xe^{y+2z} + 0 + 0 + C$ ▨

Finding Potential Functions to Evaluate Line Integrals

Although they are not defined on all of space R^3, the fields associated with Exercises 18–22 are conservative. Find a potential function for each field and evaluate the integrals as in Example 6.

19. $\displaystyle\int_{(1,1,1)}^{(1,2,3)} \underbrace{3x^2\,dx}_{M} + \underbrace{\frac{z^2}{y}\,dy}_{N} + \underbrace{2z\ln y\,dz}_{P} = \Big[x^3+z^2\ln(y)\Big]_{1,1,1}^{1,2,3} = \left(1^3+9\ln(2)\right)-\left(1^3+\cancel{\ln(1)}^{\,0}\right) = 9\ln(2)$ ▨

$\rightarrow \dfrac{\partial f}{\partial x} = M = 3x^2 \rightarrow f(x,y,z) = x^3 + C(y,z)$

$\rightarrow \dfrac{\partial f}{\partial y} = 0 + \dfrac{\partial C}{\partial y} = N = \dfrac{z^2}{y} \rightarrow \dfrac{\partial C}{\partial y} = \dfrac{z^2}{y} \;\therefore\; f(x,y,z) = x^3 + z^2\cdot\ln(y)$

$\rightarrow \dfrac{\partial f}{\partial z} = P \rightarrow \cancel{2z\ln(y)} + \dfrac{\partial C}{\partial z} = \cancel{2z\ln(y)} = 0 \;\therefore\; f(x,y,z) = x^3 + z^2\ln(y) + C$ //

Independence of path Show that the values of the integrals in Exercises 25 and 26 do not depend on the path taken from A to B.

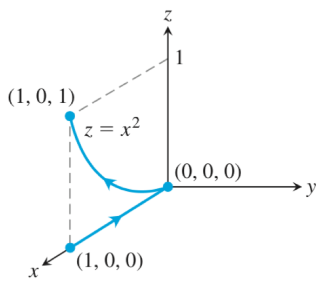
25. $\displaystyle\int_A^B \underbrace{z^2}_{M}\,dx + \underbrace{2y}_{N}\,dy + \underbrace{2xz}_{P}\,dz$

$\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 0-0 = 0$ ✓ ; $\dfrac{\partial P}{\partial y} - \dfrac{\partial N}{\partial z} = 0-0 = 0$ ✓ ; $\dfrac{\partial P}{\partial x} - \dfrac{\partial M}{\partial z} = 2z - 2z = 0$ ✓ $\therefore F = \nabla f \rightarrow$ conservative or path independent ▨

29. **Work along different paths** Find the work done by $\mathbf{F} = (x^2+y)\mathbf{i} + (y^2+x)\mathbf{j} + ze^z\mathbf{k}$ over the following paths from $(1, 0, 0)$ to $(1, 0, 1)$.

a. The line segment $x = 1, y = 0, 0 \le z \le 1$

b. The helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/2\pi)\mathbf{k}, 0 \le t \le 2\pi$



Ch. 16,4: ~~5–8, 19–23; 29a*, 33*~~

Circulation and Flux

In Exercises 5–14, use Green's Theorem to find the counterclockwise circulation and outward flux for the field **F** and curve C.

5. $\mathbf{F} = (x - y)\mathbf{i} + (y - x)\mathbf{j}$

C: The square bounded by $x = 0, x = 1, y = 0, y = 1$

$$\text{curl } F = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -1+1 = 0 \quad \therefore \int_0^1\int_0^1 \text{curl } F \, dx\, dy = 0$$

$$\text{div } F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 1+1 = 2 \quad \therefore \int_0^1\int_0^1 \text{div } F \, dx\, dy = 2$$

6. $\mathbf{F} = (x^2 + 4y)\mathbf{i} + (x + y^2)\mathbf{j}$

C: The square bounded by $x = 0, x = 1, y = 0, y = 1$

$$\int_0^1\int_0^1\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \int_0^1\int_0^1 1-4 \, dx\,dy = -3 = \text{curl } \vec{F}$$

$$\int_0^1\int_0^1\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy = \int_0^1\int_0^1 2x+2y \, dx\,dy = 2\int_0^1\left[\frac{x^2}{2}+yx\right]_0^1 dy = 2\int_0^1 \frac{1}{2}+y\,dy = \cancel{2}\left[\frac{y}{\cancel{2}}+\frac{y^2}{\cancel{2}}\right]_0^1 = 2$$

7. $\mathbf{F} = (y^2 - x^2)\mathbf{i} + (x^2 + y^2)\mathbf{j}$

C: The triangle bounded by $y = 0, x = 3$, and $y = x$

$$\int_0^3\int_0^x (2x-2y)\,dy\,dx = \int_0^3\left[2xy - \frac{\cancel{2}y^2}{\cancel{2}}\right]_0^x dx = \int_0^3 2x^2 - x^2\,dx = \left[\frac{x^3}{3}\right]_0^3 = 9 = \text{curl } \vec{F}$$

$$\int_0^3\int_0^x (-2x+2y)\,dy\,dx = \int_0^3\left[-2xy+y^2\right]_0^x dx = \int_0^3(-2x^2+x^2)\,dx = \left[-\frac{x^3}{3}\right] = -9 = \text{div } \vec{F}$$

8. $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j}$

C: The triangle bounded by $y = 0, x = 1$, and $y = x$

$$\int_0^1\int_0^x (-2x-1)\,dy\,dx = \int_0^1\left[-2xy-y\right]_0^x dx = \int_0^1 -2x^2-x\,dx = \left[-\frac{2}{3}x^3 - \frac{x^2}{2}\right]_0^1 = -\frac{2}{3}-\frac{1}{2} = -\frac{4}{6}-\frac{3}{6} = -\frac{7}{6} = \text{curl } \vec{F}$$

$$\int_0^1\int_0^x 1+(-2y)\,dy\,dx = \int_0^1\left[y-y^2\right]_0^x dx = \int_0^1 x-x^2\,dx = \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 = \frac{1}{2}-\frac{1}{3} = \frac{3}{6}-\frac{2}{6} = \frac{1}{6} = \text{div } \vec{F}$$

Work

In Exercises 19 and 20, find the work done by **F** in moving a particle once counterclockwise around the given curve.

19. $\mathbf{F} = 2xy^3\mathbf{i} + 4x^2y^2\mathbf{j}$

C: The boundary of the "triangular" region in the first quadrant → $0 \le \theta \le \frac{\pi}{2}$
enclosed by the x-axis, the line $x = 1$, and the curve $y = x^3$

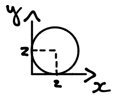
$$M(x,y) = 2xy^3 \longrightarrow \frac{\partial M}{\partial x} = 2y^3 \,;\; \frac{\partial M}{\partial y} = 6xy^2$$

$$N(x,y) = 4x^2y^2 \longrightarrow \frac{\partial N}{\partial x} = 8xy^2 \,;\; \frac{\partial N}{\partial y} = 8x^2y$$

$$\int_0^1\int_0^{x^3} \text{curl } \vec{F}\,dy\,dx = \int_0^1\int_0^{x^3} 8xy^2 - 6xy^2\,dy\,dx = \int_0^1 2\left[\frac{xy^3}{3}\right]_0^{x^3}dx = \int_0^1 2\cdot\frac{x^{10}}{3}\,dx = 2\left[\frac{x^{11}}{3\cdot 11}\right]_0^1 = 2\cdot\frac{1}{33} = \frac{2}{33}$$

20. $\mathbf{F} = (4x - 2y)\mathbf{i} + (2x - 4y)\mathbf{j}$

C: The circle $(x - 2)^2 + (y - 2)^2 = 4$ ⊛ $0 \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 2$



$M(x,y) = 4x - 2y \to \frac{\partial M}{\partial x} = 4$; $\frac{\partial M}{\partial y} = -2$; $N(x,y) = 2x - 4y \to \frac{\partial N}{\partial x} = 2$; $\frac{\partial N}{\partial y} = -4$

$\vec{r}(r,\theta) = 2\cos(\theta)\vec{i} + 2\sin(\theta)\vec{j} \to \vec{F}(r(r,\theta)) = (8\cos(\theta) - 4\sin(\theta))\vec{i} + (4\cos(\theta) - 8\sin(\theta))\vec{j}$

$\oint_C (4x-2y)\,dx + 2x - 4y\,dy = \iint_R \text{curl}\,\vec{F}\,dx\,dy = \iint_R 2+2\,dx\,dy = 4\cdot(\text{area of circle}) = 4\cdot(4\pi) = 16\pi$ ■

Using Green's Theorem

Apply Green's Theorem to evaluate the integrals in Exercises 21–24.

21. $\oint_C (y^2\,dx + x^2\,dy)$

C: The triangle bounded by $x = 0$, $x + y = 1$, $y = 0$

$\int_0^1 \int_0^{1-x} (2x - 2y)\,dy\,dx = \int_0^1 \left[2xy - \frac{2y^2}{2}\right]_0^{1-x} dx = \int_0^1 (2x - 2x^2) - (1-x)^2\,dx$

$= \int_0^1 2x - 2x^2 - (1 - 2x + x^2)\,dx = \int_0^1 -3x^2 + 4x - 1\,dx$

$= \left[-x^3 + 2x^2 - x\right]_0^1 = 0$ ■

22. $\oint_C (3y\,dx + 2x\,dy)$ $\quad \frac{\partial N}{\partial x} = 2$ $\quad \frac{\partial M}{\partial x} = 3$

C: The boundary of $0 \le x \le \pi$, $0 \le y \le \sin x$

$\int_0^\pi \int_0^{\sin(x)} 2 - 3\,dy\,dx = \int_0^\pi \left[-1y\right]_0^{\sin(x)} dx = -1\int_0^\pi \sin(x)\,dx = -1\cdot\left[\cos(x)\right]_0^\pi$

$= -1\left[-\cos(\pi) + \cos(0)\right] = -2$ ■

23. $\oint_C \underbrace{(6y + x)}_{\frac{\partial M}{\partial y} = 6}\,dx + \underbrace{(y + 2x)}_{\frac{\partial N}{\partial x} = 2}\,dy$

C: The circle $(x - 2)^2 + (y - 3)^2 = 4 \to r = 2$

$\oint_C (6y + x)\,dx + (y + 2x)\,dy = \iint_R 2 - 6\,dy\,dx = -4\iint_R dy\,dx = -4\cdot(\pi\cdot 2^2)$

$= -16\pi$ ■

29. Let C be the boundary of a region on which Green's Theorem holds. Use Green's Theorem to calculate

a. $\oint_C f(x)\,dx + g(y)\,dy$ $\quad \iint_R \frac{\partial g(y)}{\partial x} - \frac{\partial f(x)}{\partial y}\,dy\,dx = 0$ ■ $\quad$ 0 $\quad$ 0

33. Area as a line integral Show that if R is a region in the plane bounded by a piecewise smooth, simple closed curve C, then

$$\text{Area of } R = \oint_C x\,dy = -\oint_C y\,dx.$$

$\oint_C -y\,dx + x\,dy = \oint_C M(x,y)\,dx + N(x,y)\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dy\,dx = \iint_R 1+1\,dy\,dx = 2 \neq 1$

Ch: 16,5 : ~~1, 2, 17, 19, 21~~

Finding Parametrizations

In Exercises 1–16, find a parametrization of the surface. (There are many correct ways to do these, so your answers may not be the same as those in the back of the book.)

1. The paraboloid $z = x^2 + y^2, z \leq 4$ $\longrightarrow$ $\vec{r}(r,\theta) = r\cos(\theta)\vec{i} + r\sin(\theta)\vec{j} + r^2\vec{k}$ $\begin{cases} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{cases}$

2. The paraboloid $z = 9 - x^2 - y^2, z \geq 0$ $\longrightarrow$ $\vec{r}(r,\theta) = r\cos(\theta)\vec{i} + r\sin(\theta)\vec{j} + 9 - r^2\vec{k}$ $\begin{cases} 0 \le r \le 3 \\ 0 \le \theta \le 2\pi \end{cases}$

Surface Area of Parametrized Surfaces

In Exercises 17–26, use a parametrization to express the area of the surface as a double integral. Then evaluate the integral. (There are many correct ways to set up the integrals, so your integrals may not be the same as those in the back of the book. They should have the same values, however.)

17. **Tilted plane inside cylinder** The portion of the plane $y + 2z = 2$ inside the cylinder $x^2 + y^2 = 1$

r, θ

$z = \frac{2-y}{2} = \frac{2 - r\sin(\theta)}{2}$ and

$\vec{r}(r,\theta) = \left(r\cos(\theta); r\sin(\theta); \frac{2-r\sin(\theta)}{2}\right)$

$\vec{r}_r = \left(\cos(\theta); \sin(\theta); -\frac{\sin(\theta)}{2}\right)$

$\vec{r}_\theta = \left(-r\sin(\theta); r\cos(\theta); -\frac{r\cos(\theta)}{2}\right)$

$$r_r \times r_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(\theta) & \sin(\theta) & -\frac{\sin(\theta)}{2} \\ -r\sin(\theta) & r\cos(\theta) & -\frac{r\cos(\theta)}{2} \end{vmatrix}$$

$$= \vec{i}\left(\frac{-r\cos(\theta)\sin(\theta)}{2} - \left(\frac{-r\cos(\theta)\sin(\theta)}{2}\right)\right) - \vec{j}\left(\frac{-r\cos^2(\theta)}{2} - \left(\frac{-r\sin^2(\theta)}{2}\right)\right) + \vec{k}\left(r\cos^2(\theta) - (-r\sin^2(\theta))\right)$$

$$= \vec{i}\,0 - \vec{j}\left(\frac{r}{2}\left(-\cos^2(\theta) + \sin^2(\theta)\right)\right) + \vec{k}\,r\left(\cos^2(\theta) + \sin^2(\theta)\right) = \vec{i}\,0 + \vec{j}\,\frac{r}{2} + \vec{k}\,r$$

$$\therefore |r_r \times r_\theta| = \sqrt{0^2 + \frac{r^2}{4} + r^2} = \frac{r}{2}\sqrt{5} \longrightarrow \iint_S d\sigma = \iint_R |r_r \times r_\theta|\, dr\, d\theta =$$

$$= \int_0^{2\pi}\int_0^1 \frac{r}{2}\sqrt{5}\, dr\, d\theta = \int_0^{2\pi}\left[\frac{r^2}{4}\sqrt{5}\right]_0^1 d\theta = \left[\frac{\sqrt{5}}{4}\theta\right] = \frac{\sqrt{5}\pi}{2}$$ ▨

19. **Cone frustum** The portion of the cone $z = 2\sqrt{x^2 + y^2} = 2r$ between the planes $z = 2$ and $z = 6$

r, θ

$1 \le r \le 3$ and $0 \le \theta \le 2\pi$; $\vec{r}(r,\theta) = (r\cos(\theta); r\sin(\theta); 2r)$

$\vec{r}_r = (\cos(\theta); \sin(\theta); 2)$; $\vec{r}_\theta = (-r\sin(\theta); r\cos(\theta); 0)$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(\theta) & \sin(\theta) & 2 \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} = \vec{i}\left[0 - 2r\cos(\theta)\right] - \vec{j}\left[0 + 2r\sin(\theta)\right] + \vec{k}\left[r\cos^2(\theta) + r\sin^2(\theta)\right]$$

$$= \vec{i}\,2r\cos(\theta) - \vec{j}\,2r\sin(\theta) + \vec{k}\,r$$

$$\therefore |\vec{r}_r \times \vec{r}_\theta| = \sqrt{4r^2\cos^2(\theta) + 4r^2\sin^2(\theta) + r^2} = \sqrt{4r^2 \cdot \cancelto{1}{(\cos^2(\theta) + \sin^2(\theta))} + r^2} = r\sqrt{5}$$ //

$$\iint_S d\sigma = \int_0^{2\pi}\int_1^3 r\sqrt{5}\, dr\, d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\sqrt{5}\right]_1^3 d\theta = \int_0^{2\pi}\sqrt{5}\left(\cancelto{4}{\frac{9}{2} - \frac{1}{2}}\right)d\theta = \left[\sqrt{5}\cdot 4\right]_0^{2\pi} = 8\pi\sqrt{5}$$ ▨

21. Circular cylinder band The portion of the cylinder (θ, z) $x^2 + y^2 = 1$ between the planes $z = 1$ and $z = 4$

$z = 1 \longrightarrow 0 \le \theta \le 2\pi$ and $1 \le z \le 4 \longrightarrow \vec{r}(\theta; z) = r\cos(\theta)\,\vec{i} + r\sin(\theta)\,\vec{j} + z\,\vec{k}$

$\vec{r}_\theta = (-r\sin(\theta); r\cos(\theta); 0)$ and $\vec{r}_r = (0; 0; 1)$

$$\left|\vec{r}_\theta \times \vec{r}_z\right| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i}\, r\cos(\theta) - \vec{j}\,(-r\sin(\theta)) + \vec{k}\,0 = \sqrt{r^2\cos^2(\theta) + r^2\sin^2(\theta)}$$

$$= r = 1 \quad r^2 = x^2 + y^2 = 1$$

$$\therefore \iint_S d\sigma = \iint_R |r_\theta \times r_z|\, d_\theta\, d_z = \int_1^4 \int_0^{2\pi} d_\theta\, d_z = \int_1^4 2\pi\, d_z = 6\pi \quad \boxtimes$$

$\boxed{\text{Ch.: 16,6: } \cancel{1}, \cancel{3}, \cancel{5}, \cancel{6}, \cancel{21}, \cancel{23}}$

Surface Integrals of Scalar Functions

In Exercises 1–8, integrate the given function over the given surface.

1. Parabolic cylinder $G(x, y, z) = x$, over the parabolic cylinder (θ, z) $y = x^2, 0 \le x \le 2, 0 \le z \le 3$

$\iint_C d\sigma \rightarrow$ line

$\iint_S f(x,y,z)\, d\sigma \rightarrow$ surface

$= \iint_R f(x(u,v), y(u,v), z(u,v))\, |r_u \times r_v|\, du\, dv$

or $\iint_R f(x(u,v), y(u,v), z(u,v))\, \dfrac{|\nabla g|}{|\nabla G \cdot \vec{p}|}\, du\, dv$

$\vec{r}(x,z) = (x; x^2; z) \longrightarrow \vec{r}_x = (1, 2x, 0)$; $\vec{r}_z = (0, 0, 1)$

$$\left|\vec{r}_x \times \vec{r}_z\right| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1, 2x, 0) = \sqrt{1 + 4x^2}$$

$$\iint_S G \cdot d\sigma = \int_0^3 \int_0^2 x \cdot \sqrt{1 + 4x^2}\, dx\, dz = \quad u = 1 + 4x^2 \rightarrow du = 8x \rightarrow dx = du/8x$$

$$= \int_0^3 \int_1^{17} \cancel{x} \cdot u^{1/2} \frac{du}{8\cancel{x}} = \int_0^3 \frac{\overset{1}{\cancel{2}}}{\underset{12}{\cancel{24}}} \left[u^{3/2}\right]_1^{17} dz = \frac{1}{12} \int_0^3 17\sqrt{17} - 1\, dz = \frac{1}{\underset{4}{\cancel{12}}} \left[\cancel{3} \cdot 17\sqrt{17} - 1 \cdot \cancel{3}\right]$$

$$= \frac{17\sqrt{17} - 1}{4} \quad \boxtimes$$

3. **Sphere** $G(x, y, z) = x^2$, over the unit sphere $x^2 + y^2 + z^2 = 1$ $\rightarrow \rho = 1;\ 0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

$\hookrightarrow \vec{r}(\phi,\theta) = (\rho\sin(\phi)\cos(\theta);\ \rho\sin(\phi)\sin(\theta);\ \rho\cos(\phi))$

$\vec{r}_\phi = (\rho\cos(\phi)\cos(\theta);\ \rho\cos(\phi)\sin(\theta);\ -\rho\sin(\phi))$ and $\vec{r}_\theta = (-\rho\sin(\phi)\sin(\theta);\ \rho\sin(\phi)\cos(\theta);\ 0)$

$$r_\phi \times r_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \rho\cos(\phi)\cos(\theta) & \rho\cos(\phi)\sin(\theta) & -\rho\sin(\phi) \\ -\rho\sin(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) & 0 \end{vmatrix} = \vec{i}\cdot(+\rho^2\sin^2(\phi)\cos(\theta)) - \vec{j}\cdot(+\rho^2\sin^2(\phi)\cdot\sin(\theta)) + \vec{k}\cdot(\rho^2\sin(\phi)\cdot\cos(\phi)\cdot\cos^2(\theta) + \rho^2\sin(\phi)\cos(\phi)\sin^2(\theta))$$

$$= \left(\rho^2\sin^2(\phi)\cos(\theta);\ -\rho^2\sin^2(\phi)\sin(\theta);\ \rho^2\sin(\phi)\cos(\phi)\cdot\underbrace{\cancel{(\cos^2(\theta)+\sin^2(\theta))}}_{1}\right)$$

$$\therefore |\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{\cancel{\rho^4}^{1}\sin^4(\phi)\cos^2(\theta) + \cancel{\rho^4}^{1}\sin^4(\phi)\sin^2(\theta) + \cancel{\rho^4}^{1}\sin^2(\phi)\cos^2(\phi)} \quad \boxed{\rho = 1}$$

$$= \sqrt{\sin^4(\phi)\cdot\underbrace{\cancel{(\cos^2(\theta)+\sin^2(\theta))}}_{1} + \sin^2(\phi)\cos^2(\phi)} = \sqrt{\sin^2(\phi)(\sin^2(\phi)+\cos^2(\phi))} = \sin(\phi) //$$

$$\therefore \iint_S G\,d\sigma = \iint_S G(r(\phi,\theta))\cdot|r_\phi \times r_\theta|\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi} \sin^2(\phi)\cdot\cos^2(\theta)\cdot\sin(\phi)\,d\phi\,d\theta = \frac{4\pi}{3} \blacksquare$$

or by $\vec{k}$ as implicit:

$$F(x,y,z) = x^2+y^2+z^2-1 \rightarrow \nabla F = (2x, 2y, \underbrace{2z}_{=|\nabla F\cdot\vec{p}| \text{ for } \vec{p}=\vec{k}}) \rightarrow |\nabla F| = \sqrt{4x^2+4y^2+4z^2} = 2\underbrace{\sqrt{x^2+y^2+z^2}}_{=1}$$

$$\therefore \frac{|\nabla F|}{|\nabla F\cdot\vec{p}|} = \frac{2}{2z} = \frac{1}{z} //$$

$0 \le \theta \le 2\pi;\ 0 \le r \le 1;\ \vec{r}(r,\theta) = (r\cos(\theta);\ r\sin(\theta);\ z)$ where $z = \sqrt{1-x^2-y^2} = \sqrt{1-r^2}$

$$\iint_S G\,d\sigma = \iint_S G\cdot\frac{|\nabla F|}{|\nabla F\cdot\vec{p}|}\cdot dA = \iint_S x^2\cdot\frac{1}{z}\cdot dA = \int_0^{2\pi}\int_0^1 \frac{r^2\cdot\cos^2(\theta)}{\sqrt{1-r^2}}\cdot r\,dr\,d\theta = \frac{2\pi}{3} //$$

$$\therefore 2\cdot\frac{2\pi}{3} = \frac{4\pi}{3} \blacksquare$$

$\hookrightarrow$ upper half and lower half of the ellipse.

5. **Portion of plane** $F(x, y, z) = z$, over the portion of the plane $x + y + z = 4$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$, in the xy-plane

$\rho = 2$; $0 \le x \le 1$ and $0 \le y \le 1$

$r(x,y) = x\vec{i} + y\vec{j} + (4-x-y)\vec{k}$; $r_x = (1,0,-1)$; $r_y = (0,1,-1)$

$$r_x \times r_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i}\,1 + \vec{j} + \vec{k} \quad \therefore \; |r_x \times r_y| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$\iint_S F(x,y,z)\,d\sigma = \int_0^1\int_0^1 F(x,y,z)\cdot|r_x \times r_y|\,dy\,dx = \int_0^1\int_0^1 (4-x-y)\cdot\sqrt{3}\,dy\,dx = 3\sqrt{3}$$

6. **Cone** $F(x, y, z) = z - x$, over the cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$

$x^2+y^2 = r^2$

$r(r,\theta) = (r\cos(\theta)\,;\, r\sin(\theta)\,;\, r)$

$r_r = (\cos(\theta); \sin(\theta); 1)$; $r_\theta = (-r\sin\theta; r\cos(\theta); 0)$

$$r_r \times r_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(\theta) & \sin(\theta) & 1 \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} = \vec{i}\,r\cos(\theta) + \vec{j}\,r\sin(\theta) + \vec{k}\left(r\cos^2(\theta) + r\sin^2(\theta)\right)$$

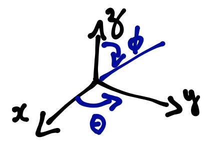
$$|r_r \times r_\theta| = \sqrt{r^2\cos^2(\theta) + r^2\sin^2(\theta) + r^2} = r\sqrt{2}$$

$$\int_0^{2\pi}\int_0^1 F(\vec{r}(r,\theta))\cdot r\sqrt{2}\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r - r\cos(\theta))\cdot r\sqrt{2}\,dr\,d\theta = \sqrt{2}\cdot\frac{2\pi}{3}$$

Finding Flux or Surface Integrals of Vector Fields

In Exercises 19–28, use a parametrization to find the flux $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ across the surface in the specified direction.

$\rightarrow r(\phi,\theta)$



21. Sphere $\mathbf{F} = z\mathbf{k}$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin

$\hookrightarrow 0 \le \phi \le \pi/2$ and $0 \le \theta \le \pi/2$ $\hookrightarrow \oplus$

$$r(\phi,\theta) = \left(\rho\sin(\phi)\cos(\theta);\ \rho\sin(\phi)\sin(\theta);\ \rho\cos(\phi)\right) \text{ where } \rho = a \text{ and } a \ge 0$$

$$r_\phi \times r_\theta = \left(a^2\sin^2(\phi)\cos(\theta);\ a^2\sin^2(\phi)\sin(\theta) + a^2\sin(\phi)\cos(\phi)\right) \text{ and } \vec{F} = (a\cos(\phi))\vec{k}$$

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \int_0^{\pi/2}\int_0^{\pi/2} \vec{F}\cdot\frac{r_\phi\times r_\theta}{\cancel{|r_\phi\times r_\theta|}}\,\cancel{|r_\phi\times r_\theta|}\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} a^3\sin(\phi)\cos^2(\phi)\,d\theta\,d\phi = \frac{\pi a^3}{6} \quad \blacksquare$$

23. Plane $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ upward across the portion of the plane $x + y + z = 2a$ that lies above the square $0 \le x \le a$, $0 \le y \le a$, in the xy-plane

$$r(x,y) = (x;\ y;\ 2a - x - y) \longrightarrow r_x = (1,0,-1) \text{ and } r_y = (0,1,-1)$$

$$r_x \times r_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (1,-1,1) \quad \text{and } F(\vec{r}(x,y)) = \left(2xy,\ 2y(2a-x-y),\ 2x(2a-x-y)\right)$$

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \int_0^a\int_0^a \vec{F}\cdot\frac{r_x\times r_y}{\cancel{|r_x\times r_y|}}\,\cancel{|r_x\times r_y|}\,dy\,dx$$

$$= \int_0^a\int_0^a 2xy - 2y(2a-x-y) + 2x(2a-x-y)\,dy\,dx = \frac{13a^4}{6} \quad \blacksquare$$

Ch. 16.7 : ~~1~~, ~~2~~, ~~5~~, ~~7~~, ~~9~~, ~~13~~

Using Stokes' Theorem to Find Line Integrals

In Exercises 1–6, use the surface integral in Stokes' Theorem to calculate the circulation of the field **F** around the curve C in the indicated direction.

1. $\mathbf{F} = x^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$

   C: The ellipse $4x^2 + y^2 = 4$ in the xy-plane, counterclockwise when viewed from above

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \rightarrow \frac{4x^2}{4}+\frac{y^2}{4}=\frac{4}{4} \rightarrow \frac{x^2}{1^2}+\frac{y^2}{2^2}=1$$

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2x & z^2 \end{vmatrix} = (0;0;2)$$

$$\oint_C \vec{F}\,d\vec{r} = \iint_S \nabla\times F\cdot\vec{n}\,d\sigma = \iint_S 2\,d\sigma = 2\iint_S d\sigma = 2\cdot\text{area ellipse} = 2\cdot1\cdot2\pi = 4\pi$$

or

$a=1$; $b=2$; $r(\theta)=(\cancel{a}^{1}\cos(\theta)\,;\,\cancel{b}^{2}\sin(\theta)\,;\,z=0)$ for $0\le\theta\le2\pi$

$\vec{F}(\theta)=(\cos^2(\theta)\,;\,2\cos(\theta)\,;\,0)$ and $\vec{v}(\theta)=(-\sin(\theta)\,;\,2\cos(\theta)\,;\,0)$

$$\oint_C \vec{F}\,d\vec{r} = \int_0^{2\pi}\vec{F}(\theta)\cdot\vec{v}(\theta)\,d\theta = \int_0^{2\pi}-\sin(\theta)\cdot\cos^2(\theta)+4\cos^2(\theta)\,d\theta$$

$$=\int_0^{2\pi}\cos^2(\theta)\cdot(4-\sin(\theta))\,d\theta = 4\pi$$

2. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k}$

   C: The circle $x^2 + y^2 = 9$ in the xy-plane, counterclockwise when viewed from above

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & -z^2 \end{vmatrix} = (0;0;1)$$

$$\oint_C \vec{F}\,d\vec{r} = \iint_S \nabla\times F\cdot\vec{n}\,d\sigma = \iint_S d\sigma = \text{area of the circle} = \pi r^2 = 9\pi$$

5. $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + y^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

C: The square bounded by the lines $x = \pm 1$ and $y = \pm 1$ in the xy-plane, counterclockwise when viewed from above

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+y^2 & x^2+y^2 \end{vmatrix} = (2y\,;\, 2x-2z\,;\, 2x-2y)$$

$$\oint_C \vec{F}\, d\vec{r} = \iint_S \nabla \times F \cdot \vec{n}\, d\sigma = \iint_S 2(x-y)\, d\sigma = 2\int_{-1}^{1}\int_{-1}^{1} x-y \; dy\, dx = 2\int_{-1}^{1} \left[xy - \frac{y^2}{2}\right]_{-1}^{1} dx$$

$$= 2\int_{-1}^{1} \left(x-\frac{1}{2}\right) - \left(-x-\frac{1}{2}\right) dx = 2\int_{-1}^{1} 2x\, dx = 2\left[\frac{\cancel{2}x^2}{\cancel{2}}\right]_{-1}^{1} = 2\,[1-1] = 0$$

Integral of the Curl Vector Field

7. Let $\mathbf{n}$ be the outer unit normal of the elliptical shell

$$S: \quad 4x^2 + 9y^2 + 36z^2 = 36, \qquad z \geq 0,$$

and let

$$\mathbf{F} = y\mathbf{i} + x^2\mathbf{j} + (x^2 + y^4)^{3/2} \sin e^{\sqrt{xyz}}\,\mathbf{k}.$$

Find the value of

$$\oint_C \vec{F}\, d\vec{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

(*Hint:* One parametrization of the ellipse at the base of the shell is $x = 3\cos t$, $y = 2\sin t$, $0 \leq t \leq 2\pi$.) $\rightarrow r(t) = (3\cos(t)\,;\, 2\sin(t)\,;\, 0)$

$\vec{T}(t) = (-3\sin(t)\,;\, 2\cos(t)\,;\, 0)$ and $\vec{F}(t) = (2\sin(t)\,;\, 9\cos^2(t)\,;\, 0)$

$$\therefore \oint_C \vec{F}\, d\vec{r} = \int_R \vec{F}\cdot\vec{T}\, dt = \int_0^{2\pi} 2\sin(t)\cdot(-3\sin(t)) + 9\cos^2(t)\cdot 2\cos(t)\, dt$$

$$= \int_0^{2\pi} 18\cos^3(t) - 6\sin^2(t)\, dt = -6\pi$$

$r(\theta, z)$

9. Let S be the cylinder $x^2 + y^2 = a^2$, $0 \le z \le h$, together with its top, $x^2 + y^2 \le a^2$, $z = h$. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + x^2\mathbf{k}$. Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through S.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & x^2 \end{vmatrix} = (0\,;\,2x\,;\,2)$$

$$\oint_C \vec{F}\,d\vec{r} = \iint_S \nabla \times F . \vec{n} . d\sigma = 2\iint_S d\sigma = 2\pi a^2$$

Stokes' Theorem for Parametrized Surfaces

In Exercises 13–18, use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field $\mathbf{F}$ across the surface S in the direction of the outward unit normal $\mathbf{n}$.

13. $\mathbf{F} = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$

S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (4 - r^2)\mathbf{k}$,
$0 \le r \le 2$, $0 \le \theta \le 2\pi$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} = (5\,;\,-2\,;\,3)$$

$$\vec{r}_r = (\cos(\theta)\,;\,\sin(\theta)\,;\,-2r)$$

$$\vec{r}_\theta = (-r\sin(\theta)\,;\,r\cos(\theta)\,;\,0)$$

$$r_r \times r_\theta = \begin{vmatrix} i & j & k \\ \cos(\theta) & \sin(\theta) & -2r \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} = (2r^2\cos(\theta)\,;\,-2r^2\sin(\theta)\,;\,r\overbrace{\cancel{(\cos^2(\theta)+\sin^2(\theta))}}^{1})$$

$$\oint_C \vec{F}\,d\vec{r} = \iint_S \nabla \times F\,\vec{n}\,d\sigma = \iint_S (\nabla \times F) . \frac{r_r \times r_\theta}{\cancel{|r_r \times r_\theta|}} . \cancel{|r_r \times r_\theta|}\,dA$$

$$= \int_0^{2\pi}\int_0^2 (5\,;\,-2\,;\,3) . (2r^2\cos(\theta)\,;\,-2r^2\sin(\theta)\,;\,r)\,dr\,d\theta = \int_0^{2\pi}\int_0^2 10r^2\cos(\theta) + 4r^2\sin(\theta) + 3r\,dr\,d\theta$$

$$= 12\pi$$

Ch. 16,8: ~~5,7,9~~

Calculating Flux Using the Divergence Theorem

In Exercises 5–16, use the Divergence Theorem to find the outward flux of **F** across the boundary of the region D.

~~5~~. **Cube** $\mathbf{F} = (y - x)\mathbf{i} + (z - y)\mathbf{j} + (y - x)\mathbf{k}$

D: The cube bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$

$$\nabla\cdot\vec{F} = \text{div}\,\vec{F} = -1-1+0 = -2$$

$$\therefore \oiint_S \vec{F}\,\vec{n}\,d\sigma = \iiint_D \overbrace{\nabla\cdot\vec{F}}^{\text{div}\,\vec{F}}\,dV = -2\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} dz\,dy\,dx = -2\int_{-1}^{1}\int_{-1}^{1}\left[z\right]_{-1}^{1} dy\,dx$$

$$= -2\int_{-1}^{1}\int_{-1}^{1} 2\,dy\,dx = -2\cdot 2\cdot 2\cdot 2 = -16 \;\blacksquare$$

~~7~~. **Cylinder and paraboloid** $\mathbf{F} = y\mathbf{i} + xy\mathbf{j} - z\mathbf{k}$

D: The region inside the solid cylinder $x^2 + y^2 \le \overset{r=2}{4}$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2 = r^2$

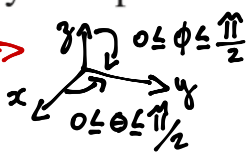
$$\vec{r}(\theta,r) = (r\cos(\theta);\, r\sin(\theta);\, r^2)$$

$$\oiint_S \vec{F}\,\vec{n}\,d\sigma = \iiint_D \nabla\cdot\vec{F}\,d\sigma = \iiint_D (x-1)\,dz\,dy\,dx = \int_0^{2\pi}\int_0^{2}\int_0^{r^2} (r\cos(\theta)-1)\,dz\,r\,dr\,dz$$

$$= -8\pi \;\blacksquare$$

~~9~~. **Portion of sphere** $\mathbf{F} = x^2\mathbf{i} - 2xy\mathbf{j} + 3xz\mathbf{k}$

D: The region cut from the first octant by the sphere $x^2 + y^2 + z^2 = 4$



$$\vec{r}(\phi,\theta) = (\rho\sin(\phi)\cos(\theta);\, \rho\sin(\phi)\sin(\theta);\, \rho\cos(\phi))$$

$$\oiint_S \vec{F}\,\vec{n}\,d\sigma = \iiint_D \nabla\cdot\vec{F}\,d\sigma = \iiint_D 2x-2x+3x\,dz\,dy\,dx = \iiint_D 3x\,dz\,dy\,dx$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} 3\rho\sin(\phi)\cos(\theta)\cdot\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta = 3\pi \;\blacksquare$$