Basics: **Anti-derivatives:**

$$\boxed{\sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \qquad \cos^2(\theta) = \frac{1+\cos(2\theta)}{2}}$$

$$\boxed{e^{c\ln(z)} = z^c}$$

$$\int x^m\,dx = \frac{x^{m+1}}{m+1} + C \;;\; \int \frac{1}{x}\,dx = \ln(x) + C$$

$$\int e^x\,dx = e^x + C \;;\; \int a^x\,dx = \frac{a^x}{\ln a} + C \qquad \int \cos(x)\,dx = \sin(x) + C \;;\; \int \sin(x)\,dx = -\cos(x) + C$$

$$\int e^{cx}\,dx = \frac{1}{c}\cdot e^{cx} + C \;;\; \int a^{cx}\,dx = \frac{a^{cx}}{c\ln(a)} + C \qquad \int \sec^2(x)\,dx = tg(x) + C \;;\; \int tg(x)\,dx = -\ln|\cos(x)| + C \;;$$

Integral's techniques:

– **Substitution:** $\int x e^{x^2}\,dx \longrightarrow \boxed{u = x^2 \quad du = 2x\,dx \quad \frac{du}{dx} = 2x \quad dx = \frac{du}{2x}} \longrightarrow \int \cancel{x} e^u \frac{du}{2\cancel{x}} = \int e^u\,du = e^u + C$ ∎

– **By parts:** $\boxed{(fg)' = f'g + fg'}$ **product rule** $\rightarrow \int (fg)' = \int f'g + \int fg'$

$\boxed{\int fg' = fg - \int f'g}$ **integration by parts formula** $\longrightarrow \boxed{u = f(x) \quad v = g(x) \quad \frac{du}{dx} = f'(x) \quad \frac{dv}{dx} = g'(x) \quad du = f'(x)\,dx \quad dv = g'(x)\cdot dx} \longrightarrow \boxed{\int u\,dv = uv - \int v\,du}$

Chapter 11: **1. Parametrization of a curve:** $(x = f(t),\ y = g(t))$ for $t \in I$:= parametric equation of a curve;

parametrized; t: parameter; I: Domain / parameter interval

↳ if $a \le t \le b$, then $(f(a), g(a))$:= initial point

$(f(b), g(b))$:= Terminal point

∗ Line := $\begin{cases} y = a + mb \\ y - y_0 = m(x - x_0) \end{cases}$ where m := slope $= \frac{\Delta y}{\Delta x}$

∗ Parabola := $x = y^2$ ∗ Círculo := $(x - x_c)^2 + (y - y_c)^2 = r^2$ ∗ Hyperbola := $x^2 - y^2 = 4$

Cartesian equation:

↳ define $t = x - x_0$, then $y - y_0 = m(x - x_0)$, where $x = x_0 + t$, $y = y_0 + mt$ and $m = \frac{\Delta y}{\Delta x}$, for (x_0, y_0)

Cycloids:

↳ $x = at + a\cos\theta$, $y = a + a\sin\theta$

∗ **Parametric equation:** (x_0, y_0); (x_m, y_m) ∴ $((x_m - x_0), (y_m - y_0)) \longrightarrow$ line $\longrightarrow \begin{cases} x = x_0 + \Delta x\cdot t \\ y = y_0 + \Delta y\cdot t \end{cases}$

↳ parabola :=

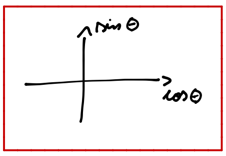
11.2: Calculus with parametric curves: slopes, lengths and areas associated with parametrized curves.

↳ Tangents and areas:

↳ Length := $\boxed{\text{for } x = f(t),\ y = g(t)}$ ∴ $\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$

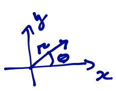
↳ $\boxed{\text{for } x = g(y);\ y = y}$ ∴ $L = \int_a^b \sqrt{\left[\frac{dx}{dy}\right]^2 + 1}\,dy$

$$\boxed{\text{The chain rule: } \frac{d\,f(u)}{dx} = \frac{df}{du}\cdot\frac{du}{dx}}$$



θ	2π	π/6	π/4	π/3	π/2
sin	0	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tg	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∄

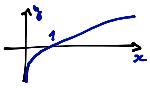
11.3:

↳ **Polar coordinates:** $(r, \theta) \longrightarrow$ 

↳ Equations related to (r, θ): $x = r\cos\theta$; $y = r\sin\theta$; $tg\,\theta = \frac{y}{x}$; $r^2 = x^2 + y^2$; $\sin^2\theta + \cos^2\theta = 1$

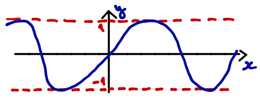
Remark:

⊛ $\ln(x)$: $D = (0, \infty)$; Range $= (-\infty, \infty)$



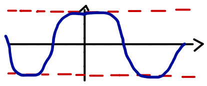
* $\ln(x \cdot y) = \ln(x) + \ln(y)$ * $[\ln(x)]' = \frac{1}{x}$
* $\ln(x/y) = \ln(x) - \ln(y)$ * $\int \ln(x)\,dx = x(\ln(x) - 1) + C$
* $\ln(x^y) = y \cdot \ln(x)$ * $e^{c \ln x} = x^c$ check!

⊛ $\sin(x)$: $D = (-\infty, \infty)$; Range $= [-1, 1]$



* $\sin(-x) = -\sin(x)$ (odd func.)
* $\sin^2(x) = \frac{1 - \cos(2x)}{2}$
* $\sin^2(x) + \cos^2(x) = 1$
* $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$
* $\sin(2x) = 2\sin(x)\cos(x)$
* law of sin: $\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} = 2r$

⊛ $\cos(x)$: $D = (-\infty, \infty)$; Range $= [-1, 1]$

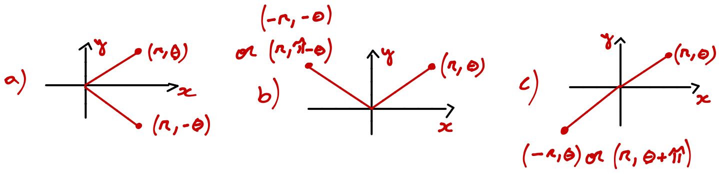


* $\cos(-x) = \cos(x)$ (even func.)
* $\cos^2(x) = \frac{1 + \cos(2x)}{2}$
* $\cos(2x) = \cos^2(x) - \sin^2(x)$
* $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$
* law of cos: $c^2 = a^2 + b^2 - 2ab\cos(C)$

Chapter 11.4: Graphing $P(r, \theta)$ equations:

↳ slope: $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

↳ Symmetry tests:



↳ $r^2 = x^2 + y^2$ where $x = r\cos\theta$ and $y = r\sin\theta$: $r^2 = r^2\cos^2\theta + r^2\sin^2\theta$
$\quad$ ↳ $= r^2 \cdot \underbrace{(\cos^2\theta + \sin^2\theta)}_{1}$

Chapter 11.5: Areas and lengths in $P(r, \theta)$:

↳ area of the fan-shaped region $r = f(\theta)$, $\alpha \le \theta \le \beta$:= $A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$

↳ area of the region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$:= $A = \int_\alpha^\beta \frac{1}{2} r_2^2 \, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2 \, d\theta = \int_\alpha^\beta \frac{1}{2}(r_2^2 - r_1^2)\, d\theta$

↳ length of $P(r, \theta)$:= $L = \int_\alpha^\beta \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2}\, d\theta$

$$\boxed{\text{Tangent line formula: } \vec{r}(t_0) + (t - t_0) \cdot \vec{r}\,'(t_0)}$$
$\rightarrow y = f(a) + f'(a) \cdot (x - a)$

Chapter 13: Vector Valued functions:

↳ Ch 13.1: Curves in space and their tangents:

↳ def. $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$:= Position vector function with $D := \mathbb{R}$ where $P(\overbrace{f(t), g(t), h(t)}^{\text{component functions}})$, $x = f(t)$, $y = g(t)$, $z = h(t)$

$\therefore \lim_{t \to t_0} \vec{r}(t) = L$ where for $\forall \varepsilon > 0\ \exists \delta > 0$: $|\vec{r}(t) - L| < \varepsilon \longrightarrow |t - t_0| < \delta$ → $\vec{r}(t)$ continuous if $\exists L$ and it is continuous over its interval domain.

↳ def.: $\vec{r}(t)$ is differentiable at t $\longleftrightarrow$ $f(t), g(t), h(t)$ are differentiable at t.

$\therefore \frac{d\vec{r}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \frac{df(t)}{dt}\vec{i} + \frac{dg(t)}{dt}\vec{j} + \frac{dh(t)}{dt}\vec{k}$

> *** Remark:** $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$
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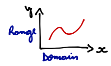
> 
>
>
> or $x = x_0 + h$
>
> $f'(a) = \lim_{h \to a} \frac{f(a+h) - f(a)}{h}$

↳ def.: $v(t) = r'(t)$:= velocity vector (derivative of the position $\vec{r}(t)$)

$a(t) = r''(t)$:= acceleration vector

Direction vector := $\frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\vec{v}(t)}{\sqrt{f(t)^2 + g(t)^2 + h(t)^2}}$:= speed

* Dot product differentiation rule: $\vec{u}(t) = u_1(t)\vec{i} + u_2(t)\vec{j} + u_3(t)\vec{k}$ and $\vec{v}(t) = v_1(t)\vec{i} + v_2(t)\vec{j} + v_3(t)\vec{k}$

$$[\vec{u} \cdot \vec{v}]' = \underbrace{\vec{u}_1'\vec{v}_1 + \vec{u}_2'\vec{v}_2 + \vec{u}_3'\vec{v}_3}_{\vec{u}' \cdot \vec{v}} + \underbrace{\vec{u}_1\vec{v}_1' + \vec{u}_2\vec{v}_2' + \vec{u}_3\vec{v}_3'}_{\vec{u} \cdot \vec{v}'} \quad \boxtimes$$

* Cross product rule: $[\vec{u} \times \vec{v}]' = \lim_{h \to 0}\left[\frac{\vec{u}(t+h) \times \vec{v}(t+h) - \vec{u}(t) \times \vec{v}(t)}{h}\right] = \lim_{h \to 0}\left[\frac{\vec{u}(t+h) \times \vec{v}(t+h) - \vec{u}(t) \times \vec{v}(t+h) + \vec{u}(t) \times \vec{v}(t+h) - \vec{u}(t) \times \vec{v}(t)}{h}\right]$

? $= \lim_{h \to 0}\left[\frac{\vec{u}(t+h) - \vec{u}(t)}{h} \times \vec{v}(t+h) + \vec{u}(t) \times \frac{\vec{v}(t+h) - \vec{v}(t)}{h}\right] = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t) \quad \boxtimes$

* Vector function of constant length: $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$

↳ Ch. 13.3: Arc Length in Space:

↳ Length of smooth curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ in $a \le t \le b$: $L = \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2} = \int_a^b |\vec{v}|\, dt$:= arc length

↳ Unit tangent vector := $T = \frac{\vec{v}}{|\vec{v}|}$

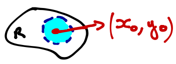
↳ Speed := $|\vec{v}|$

Chapter 14: Partial Derivative:

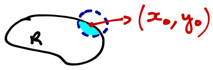
- Level curve: $f(x,y) = c$
- Surface $z = f(x,y)$:= set of all points $(x, y, f(x,y))$ in space
  ↳ graph of f
- $f(x,y,z) = c$:= level surface of f.

↳ Ch. 14.1: Functions of several variables:

↳ def: – Interior point:



– Boundary point:



↳ Ch. 14.2: Limits and Continuity in higher Dimension:

↳ def: $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \forall \varepsilon > 0\ \exists \delta > 0: |f(x,y) - L| < \varepsilon \to 0 < \underbrace{\sqrt{(x-x_0)^2 + (y-y_0)^2}}_{|x-x_0| + |y-y_0|} < \delta$

✱ Remark:
$|x - x_0| = \sqrt{(x-x_0)^2}$
$|x| = \sqrt{x^2}$

↳ Theorem 1: $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ | $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$ ∴

1.) $L+M$
2.) $L-M$
3.) cL or cM
4.) $L \cdot M$
5.) $\frac{L}{M}$ for $M \neq 0$
6.) L^m for $m>0$
7.) $\sqrt[n]{L} = L^{1/m}$ for $m>0$. If m odd, then $L>0$.

↳ Continuity: $f(x,y)$ is continuous at (x_0,y_0) if ① $f : \to (x_0,y_0)$; ② L exists; ③ $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

✱ a function is continuous if it is continuous at every point of its domains.

✱ Two-path test for nonexistence of a limit := If $f(x,y)$ has different limits along two different paths in the domain of f as (x,y) approaches (x_0,y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exists.

✱ Continuity of composite := If $f(x,y)$ is continuous at (x_0,y_0) and g is a single variable function continuous at $f(x_0,y_0)$, then the composite function $h = g \circ f$ defined by $h(x,y) = g(f(x,y))$ is continuous at (x_0,y_0).

↳ Ch. 14.3: Partielle deriverte:

Remark: ① If $f(x)$ is differentiable at point x_0, then $f(x)$ must be continuous at x_0.
↳ The converse does not hold! It doesn't hold for partial derivatives! If $f(x,y)$ is differentiable at (x_0,y_0), then $f(x,y)$ is continuous at (x_0,y_0).

② The chain rule: $[f(g(x))]' = \frac{dz}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$:= If $z = f(y)$ and $y = g(x)$, then $z = f(g(x)) = (f \circ g)(x)$.

③ $f'(x) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$

def: The partial derivative of $f(x,y)$ with respect of x at the point (x_0,y_0) is:

$$\left.\frac{\partial f}{\partial x}\right|_{x_0,y_0} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$; provided that limit exists.

def: The partial derivative of $f(x,y)$ with respect of y at point (x_0,y_0) is:

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{d f(x_0,y)}{d y}\right|_{y=y_0} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$; provided that limit exists.

Second-order partial derivatives:

↳ $f_{xx} = \frac{\partial^2 f}{\partial x \partial x} = \frac{\partial^2 f}{\partial x^2}$

↳ $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$

↳ $f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$

↳ $f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial y \partial y}$

Slope of a tang. of a surface: $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$

Ch. 14.3: Continuation:

↳ Laplace equation: $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$

↳ The wave equation: $\dfrac{\partial^2 w}{\partial t^2} = c^2 \dfrac{\partial^2 w}{\partial x^2}$ where $w :=$ height; $x :=$ distance variable; $t :=$ time variable; $c :=$ velocity of the wave.

↳ The heat equation: $\dfrac{\partial f}{\partial t} = \dfrac{\partial^2 f}{\partial x^2}$

Ch. 14.4: The Chain Rule:

Remark: $\left[\tan^{-1}(x)\right]' = \left[\arctan(x)\right]' = \dfrac{1}{x^2+1}$ $\quad \csc^2(x) - \cot^2(x) = 1$

$\left[\dfrac{\cos(x)}{\sin(x)}\right]' = \left[\cot(x)\right]' = -\csc^2(x)$

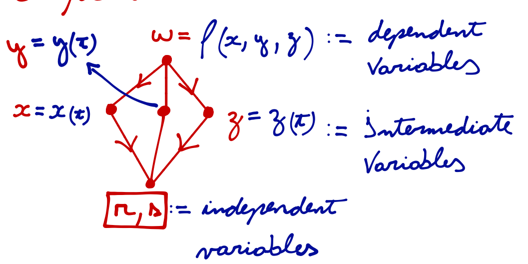
The chain rule for multiple variables:

↳ For $w = f(x,y)$ and $x = x(t)$ and $y = y(t)$, then $w = f(x(t), y(t))$:

↳ $$\frac{\partial w(x,y,z)}{\partial t} = \frac{\partial w(x,y)}{\partial x} \cdot \frac{\partial x(t)}{\partial t} + \frac{\partial w(x,y)}{\partial y} \cdot \frac{\partial y(t)}{\partial t}$$

or $\left[w(x,y,z)\right]'_{u} \xrightarrow{=} \left[w(t)\right]'_{t}$

Surface:



Implicit differentiation:

↳ if $\begin{cases} 1.\ F(x,y) \text{ is differentiable} \\ 2.\ F(x,y) = 0 \longrightarrow y = h(x) \end{cases}$ then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$ for $F_y \neq 0$ or $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$ for $F_z \neq 0$.

Ch. 14.5: Directional Derivatives and Gradient Vectors:

↳ def. gradient vector of $f(x,y)$ at $P_0(x_0,y_0)$ is the vector: $\nabla f = \dfrac{\partial f}{\partial x}\vec{i} + \dfrac{\partial f}{\partial y}\vec{j}$

↳ Remark: Unit vector $\dfrac{\vec{u}}{|\vec{u}|}$

↳ Theorem: The directional derivative is a Dot product: If $f(x,y)$ is differentiable in open region containing $P_0(x_0,y_0)$, then:

↳ $\left(\dfrac{df}{ds}\right)_{\vec{u}, P_0} = (\nabla f)_{P_0} \cdot \vec{u}$

↳ Properties: ① $D_u f = |\nabla f| \cdot \vec{u} = |\nabla f| \cdot \cos\theta = |\nabla f| \cdot \cos(0) = |\nabla f|$ → increase most rapidly

② $D_u f = |\nabla f| \cdot \cos(\pi) = -|\nabla f|$ → decrease most rapidly

③ $D_u f = |\nabla f| \cdot \cos\left(\frac{\pi}{2}\right) = |\nabla f| \cdot 0 = 0$ → no change

Ch. 14.5:

↳ Tangent line to a level curve: $f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)=0$

↳ Derivative along a path: $\frac{d}{dx}f(\vec{r}(x))=\nabla f(\vec{r}(x))\cdot\vec{r}\,'(x)$

Remark:

$\sin^{-1}(x)=\arcsin(x)$

$[\arcsin x]'=\frac{1}{\sqrt{1-x^2}}$

Chapter 14.7: Extreme Values and Saddle Points:

↳ def.: $f(x,y):=$ region R containing (a,b), then:

① $f(a,b)$ is local maximum of f if $f(a,b)\geq f(x,y)$ for $\forall(x,y)$ in an open disk centered at (a,b);

② $f(a,b)$ is local minimum of f if $f(a,b)\leq f(x,y)$ for $\forall(x,y)$ in an open disk centered at (a,b);

↳ First derivative test for local extreme values:

↳ Since $\underbrace{f_x(a,b)=0 \text{ and } f_y(a,b)=0}_{\text{or one or both of them does not exists}}$, then $f(x,y)$ has $\underbrace{\text{a local minimum or maximum at interior point } (a,b)}_{\text{critical point}}$;

↳ def.:= Since $\exists$ some (x,y) where $f(x,y)>f(a,b)$ and some (x,y) where $f(x,y)<f(a,b)$,
then $(a,b,f(a,b))$ on the surface $z=f(x,y)$ is called saddle point.

↳ Theorem: 2^{nd} derivative test for local extreme values:

↳ Suppose $f_x(a,b)=f_y(a,b)=0$ and both are continuous throughout a disk centered at (a,b),
then (i) f has local maximum at (a,b) if $f_{xx}<0$ and $f_{xx}f_{yy}-f_{xy}^2>0$;

(ii) f has local minimum at (a,b) if $f_{xx}>0$ and $f_{xx}f_{yy}-f_{xy}^2>0$;

(iii) f has saddle point at (a,b) if $f_{xx}f_{yy}-f_{xy}^2<0$.

(iv) inconclusive if $f_{xx}f_{yy}-f_{xy}^2=0$.

⊛ where $f_{xx}f_{yy}-f_{xy}^2=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy}\end{vmatrix}$ → the discriminant or Hessian of f.

Chapter 14.8: Lagrange Multipliers:

↳ To find the extreme values of a function whose domain is constrained to lie within some particular subset of the plane;

↳ $\nabla f=\lambda\nabla g$ where the local extreme value of $f(x,y,z)$ whose variables are subject to a constraint $g(x,y,z)=0$;

↳ $\underbrace{\nabla f \text{ is orthogonal to}}_{*\ \nabla f\cdot\vec{r}\,'=0}$ C: $\underbrace{\vec{r}(x)=x(x)\vec{i}+y(x)\vec{j}+z(x)\vec{k}}_{\text{smooth curve}}$ at point P_0;

↳ Lagrange multipliers with two constraints := $\nabla f=\lambda\nabla g_1+\mu\nabla g_2$ where $g_1=0$ and $g_2=0$

Chapter 15: Multiple Integrals:

15.1: Double and integrated integrals over rectangles:

↳ Fubini's theorem: If $f(x,y)$ is continuous throughout the rectangular region R: $a\leq x\leq b$, $c\leq y\leq d$,
then: $\iint_R f(x,y)\,dA=\int_c^d\int_a^b f(x,y)\,dx\,dy=\int_a^b\int_c^d f(x,y)\,dy\,dx$

Ch. 15,2: Double Integrals over General regions:

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx. \longrightarrow \text{vertical selection}$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy. \longrightarrow \text{horizontal selection}$$

Chap. 15,3: Area by double integration:

DEFINITION The **area** of a closed, bounded plane region R is

$$A = \iint_R dA.$$

$$\textbf{Average value of } f \textbf{ over } R = \frac{1}{\text{area of } R} \iint_R f\, dA. \qquad (3)$$

Chap. 15,4: Double integrals in Polar form:

↳ Polar form := $x = r\cos\theta$ and $y = r\sin\theta$ and $r = \sqrt{x^2+y^2}$

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r\, dr\, d\theta. \longrightarrow \int_{\theta_0}^{\theta} \int_{r_0}^{r} dr\, d\theta$$

Chap. 15,5: Triple integral in rectangular Coordinates:

DEFINITION The **volume** of a closed, bounded region D in space is

$$V = \iiint_D dV.$$

→ where $\Delta V_k = \Delta x_k\, \Delta y_k\, \Delta z_k$

$$\longrightarrow S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\, \Delta V_k. \longrightarrow \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x, y)}^{z=f_2(x, y)} F(x, y, z)\, dz\, dy\, dx.$$

Chap. 15,6: Moments and Centers of mass:

TABLE 15.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint_D \delta\, dV$ $\qquad \delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x\,\delta\, dV, \qquad M_{xz} = \iiint_D y\,\delta\, dV, \qquad M_{xy} = \iiint_D z\,\delta\, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint_R \delta\, dA$ $\qquad \delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint_R x\,\delta\, dA, \qquad M_x = \iint_R y\,\delta\, dA$

Center of mass: $\bar{x} = \dfrac{M_y}{M}, \qquad \bar{y} = \dfrac{M_x}{M}$

TABLE 15.2 Moments of inertia (second moments) formulas

THREE-DIMENSIONAL SOLID

About the x-axis:	$I_x = \iiint (y^2 + z^2)\,\delta\, dV$	$\delta = \delta(x, y, z)$
About the y-axis:	$I_y = \iiint (x^2 + z^2)\,\delta\, dV$	
About the z-axis:	$I_z = \iiint (x^2 + y^2)\,\delta\, dV$	
About a line L:	$I_L = \iiint r^2(x, y, z)\,\delta\, dV$	$r(x, y, z) =$ distance from the point (x, y, z) to line L

TWO-DIMENSIONAL PLATE

About the x-axis:	$I_x = \iint y^2\,\delta\, dA$	$\delta = \delta(x, y)$
About the y-axis:	$I_y = \iint x^2\,\delta\, dA$	
About a line L:	$I_L = \iint r^2(x, y)\,\delta\, dA$	$r(x, y) =$ distance from (x, y) to L
About the origin (polar moment):	$I_0 = \iint (x^2 + y^2)\,\delta\, dA = I_x + I_y$	

Chapter 15.7: Triple integrals in Cylindrical and Spherical Coordinates:

DEFINITION **Cylindrical coordinates** represent a point P in space by ordered triples (r, θ, z) in which $r \geq 0$,

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$

Volume Differential in Cylindrical Coordinates

$$dV = dz\, r\, dr\, d\theta$$

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k, z_k)\, \Delta z_k\, r_k\, \Delta r_k\, \Delta\theta_k. \longrightarrow \lim_{n\to\infty} S_n = \iiint_D f\, dV = \iiint_D f\, dz\, r\, dr\, d\theta.$$

$$\iiint_D f(r, \theta, z)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

DEFINITION **Spherical coordinates** represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin $(\rho \geq 0)$.
2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis $(0 \leq \phi \leq \pi)$.
3. θ is the angle from cylindrical coordinates.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$r = \rho\sin\phi, \qquad x = r\cos\theta = \rho\sin\phi\cos\theta,$$
$$z = \rho\cos\phi, \qquad y = r\sin\theta = \rho\sin\phi\sin\theta,$$
$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

Volume Differential in Spherical Coordinates

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$S_n = \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k)\, \rho_k^2 \sin\phi_k\, \Delta\rho_k\, \Delta\phi_k\, \Delta\theta_k. \longrightarrow \lim_{n\to\infty} S_n = \iiint_D f(\rho, \phi, \theta)\, dV = \iiint_D f(\rho, \phi, \theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\, dy\, dz \\ &= dz\, r\, dr\, d\theta \\ &= \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \end{aligned}$$

Chapter 15.8: Substitutions in multiple Integrals:

$x = g(u, v), \qquad y = h(u, v), \longrightarrow \displaystyle\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(u))\,g'(u)\,du. \qquad x = g(u),\ dx = g'(u)\,du$

DEFINITION The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}. = \frac{\partial(x, y)}{\partial(u, v)} \qquad (1)$$

THEOREM 3—Substitution for Double Integrals Suppose that $f(x, y)$ is continuous over the region R. Let G be the preimage of R under the transformation $x = g(u, v)$, $y = h(u, v)$, assumed to be one-to-one on the interior of G. If the functions g and h have continuous first partial derivatives within the interior of G, then

$$\iint_R f(x, y)\,dx\,dy = \iint_G f(g(u, v), h(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv. \qquad (2)$$

Differential Area Change Substituting $x = g(u, v), y = h(u, v)$

$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv$$

Substitutions in Triple Integrals

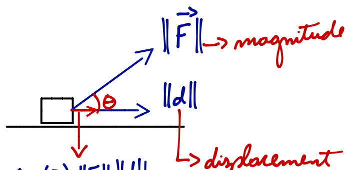
$$x = g(u, v, w), \qquad y = h(u, v, w), \qquad z = k(u, v, w),$$

$$\hookrightarrow \iiint_D F(x, y, z)\,dx\,dy\,dz = \iiint_G H(u, v, w)\,|J(u, v, w)|\,du\,dv\,dw.$$

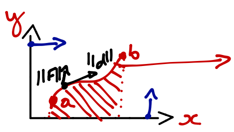
$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi$$

$$\hookrightarrow \iiint_D F(x, y, z)\,dx\,dy\,dz = \iiint_G H(\rho, \phi, \theta)\,|\rho^2\sin\phi|\,d\rho\,d\phi\,d\theta.$$

Chapter 16.1: Line Integrals:



$W = \underbrace{\cos(\theta)\cdot\|F\|\cdot\|d\|}_{\text{dot product}}\ \vec{d}\cdot\vec{F}$



$\longrightarrow \displaystyle\int_a^b F(r(t))\cdot\frac{dr(t)}{dt}\,dt$

$\vec{f}(x, y) = y\,\vec{i} + x\,\vec{j}$

Scalar field $:= \displaystyle\int_a^b f(x(t), y(t))\,\|v(t)\|\,dt$

$r(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$

$r(t) = (1-t)\,P + t.Q$ for $\overrightarrow{PQ}$

TABLE 16.1 Mass and moment formulas for coil springs, wires, and thin rods lying along a smooth curve C in space

Mass: $M = \int_C \delta\, ds$ $\quad$ $\delta = \delta(x, y, z)$ is the density at (x, y, z)

First moments about the coordinate planes:

$$M_{yz} = \int_C x\,\delta\, ds, \qquad M_{xz} = \int_C y\,\delta\, ds, \qquad M_{xy} = \int_C z\,\delta\, ds$$

Coordinates of the center of mass:

$$\bar{x} = M_{yz}/M, \qquad \bar{y} = M_{xz}/M, \qquad \bar{z} = M_{xy}/M$$

Moments of inertia about axes and other lines:

$$I_x = \int_C (y^2 + z^2)\,\delta\, ds, \qquad I_y = \int_C (x^2 + z^2)\,\delta\, ds, \qquad I_z = \int_C (x^2 + y^2)\,\delta\, ds,$$

$I_L = \int_C r^2 \delta\, ds$ $\quad$ $r(x, y, z)$ = distance from the point (x, y, z) to line L

Chap 16.2: Vector fields and line integrals – Work, Circulation, Flux:

Remark! (movie 23)

$r(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \longrightarrow$ for $0 \le t \le 1 \longrightarrow r(t) = (1-t)P + t \cdot Q$ where $P \bullet\!\longrightarrow\!\bullet\, Q$

Positional vector $:= \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$

What is smooth curve?

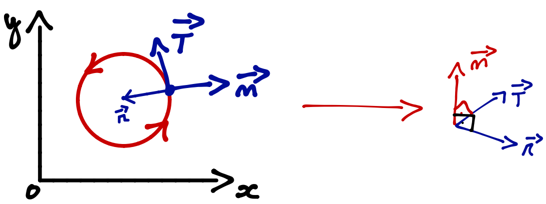
Gradient field $:= \vec{F} = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = \vec{f}(x(t), y(t))$ → it has direction for the max. change in a point!

↳ it has continuous and differentiable components and C is a smooth curve, then every point at C has a unit tangent vector $:= \vec{T} = \frac{d\vec{r}}{ds} = \frac{\vec{v}}{|\vec{v}|}$ → unit vector in the velocity direction

Curve Integral of $\vec{F}$ over $C := \int_C \vec{F}\cdot\vec{T}\, ds = \int_C \underbrace{\vec{F}}_{\text{scalar field}} \frac{d\vec{r}}{ds}\, ds = \int_C \vec{F}(\vec{r}(t))\, \frac{d\vec{r}}{dt}\, dt \longrightarrow F(\vec{r}(t)) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

Work along a curve $:= \int_a^b \vec{f}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$

Flow along a curve $:= \int_C \vec{F}\cdot\vec{T}\, ds$ or $\oint_C :=$ Circulation $:= \oint_C \vec{F}\cdot\vec{T}\, ds$



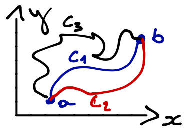
normal unit vector $\quad$ cross multiplication

$$\vec{m} = \vec{T} \times \vec{k} = \frac{dy}{ds}\vec{i} + \frac{dx}{ds}\vec{j}$$

$\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$; $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$

Flux := crosses curve $:= \oint_C \vec{F}\cdot\vec{m}\, ds = \oint_C \left[M(r(t)) \cdot \frac{dy(t)}{dt} - N(r(t)) \cdot \frac{dx(t)}{dt} \right] dt$

Chap. 16.3: Path Independence, Conservative Field, Potential Functions:



$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\, d\vec{r} = \int_{C_3} \vec{F}\, d\vec{r}$ if $\vec{F}$ is a conservative field!

$\longrightarrow \oint \vec{F}\, d\vec{r} = 0$ ∴ If $\vec{F}$ is a conservative field, then $\vec{F}$ is a gradient field $\nabla f = \vec{F}$. //

Fundamental theorem for curve integral:

↳ C is a smooth curve between A and B with parametrised $\vec{r}(t)$.

↳ F is differentiable with continuous gradient vector $\nabla f = \vec{F}$: $\int_C \vec{F}\, d\vec{r} = f(B\, r(t)) - f(A\, r(t))$

How do we check if F is conservative?

↳ Component test := $\nabla \times \vec{F} = \vec{0} \longrightarrow \nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$; $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$

$$\longrightarrow \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \vec{i}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) - \vec{j}\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right) + \vec{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \vec{0} \quad \therefore \left(\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} ; \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z} ; \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}\right) //$$

→ Potential function: ⊛ Example: $F = e^{y+2z}\vec{i} + e^{y+2z}\cdot x\,\vec{j} + e^{y+2z}\,2x\,\vec{k}$ → conservative / path independent and open simple connected domain!

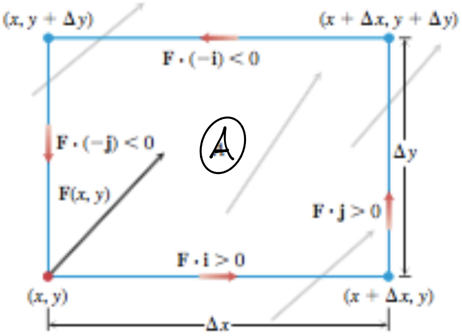
$$\therefore \frac{\partial f}{\partial x} = M = e^{y+2z} \longrightarrow f(x,y,z) = x e^{y+2z} + C(y,z)$$

$$\hookrightarrow \frac{\partial f}{\partial y} = N \longrightarrow \cancel{x\, e^{y+2z}} + \frac{\partial C}{\partial y} = \cancel{e^{y+2z}\cdot x} \longrightarrow f(x,y,z) = x e^{y+2z} + 0 + C(z)$$

$$\hookrightarrow \frac{\partial f}{\partial z} = P \longrightarrow \cancel{2\cdot x e^{y+2z}} + \frac{\partial C}{\partial z} = \cancel{2x\, e^{y+2z}} \longrightarrow f(x,y,z) = x e^{y+2z} + 0 + 0 + C \; ▨$$

Chapter 16.4 : Green's Theorem in the plane

↳ $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ ⟶ Velocity field ; (x,y) := point in a region R forming a rectangle A.



$$\underbrace{\vec{F}(x,y)}_{\text{velocity}} \cdot \vec{i}\ \overbrace{\Delta x}^{\text{length of the segment}} = M(x,y)\,\Delta x := \text{flow rate}$$

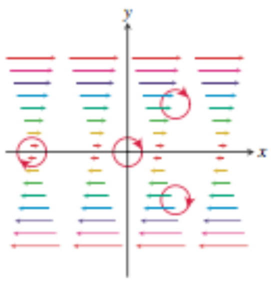
$\vec{i}$ ↳ tangent direction

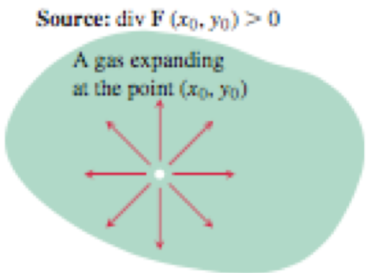
⊛ Circulation density $\left(\text{curl } \vec{F}(x,y) \cdot \vec{k}\right) := \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$:= the k-component of the curl

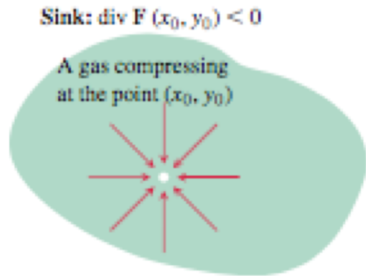
↳ how the fluid is circulating around axes located at ≠ points and perpendicular the plane.

⊛ Divergence (flux density) $\text{div } \vec{F}(x,y) := \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}$

↳ The rate at which the fluid leaves A.







⊛ $\oint_C$ positively oriented := The curve C is traversed counterclockwise. Conversely, negative oriented.

→ green's theorem for circulation : $\displaystyle\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R \left(\text{curl } \vec{F}\right)\cdot\vec{k}\,dA$

$\vec{T}$ ↳ displacement $\vec{T} = \overrightarrow{P_1P_2} = (x_2-x_1)\vec{i} + (y_2-y_1)\vec{j} + (z_2-z_1)\vec{k}$

→ green's theorem for flux : $\displaystyle\oint_C \vec{F}\cdot\vec{m}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \iint_R \nabla\cdot\vec{F}\,dA$

$\vec{m}$ ↳ Normal

Ch. 16.5: Surfaces and Area:

↳ curves in the plane: ① $y = f(x)$; ② $F(x,y) = 0$; ③ $\vec{r}(x) = f(x)\,\vec{i} + g(x)\,\vec{j}$ $\quad a \le x \le b$

↳ surface in space: ① $z = f(x,y)$; ② $F(x,y,z) = 0$; ③ $\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}$

↳ Cone: $z = \sqrt{x^2+y^2} = r$ → parametrized using cylindrical coordinates: $\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + r\,\vec{k}$

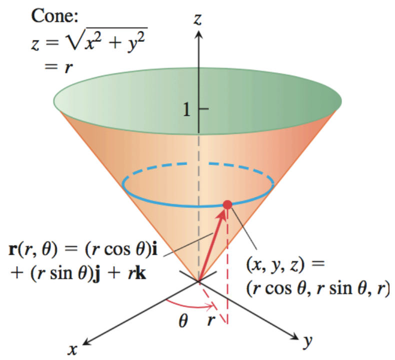
↳ for $0 \le r \le 1$ and $0 \le \theta \le 2\pi$

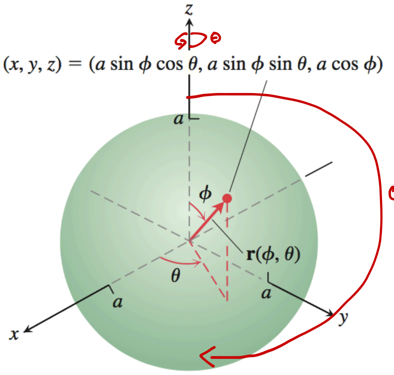
↳ sphere := $x^2+y^2+z^2 = a^2$ ($a = \rho$) → parametrized using spherical coordinates: $\vec{r}(\phi,\theta) = a\sin\phi\cos\theta\,\vec{i} + a\sin\phi\sin\theta\,\vec{j} + a\cos\phi\,\vec{k}$

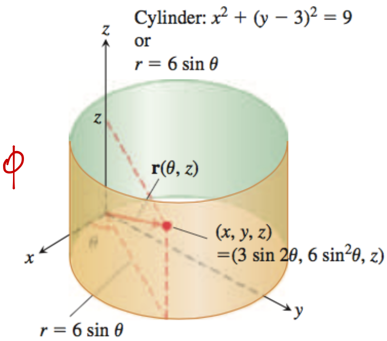
↳ for $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$

↳ Cylinder := $x^2+y^2 = r^2$ → parametrized using cylindrical coordinates: $\vec{r}(\theta,z) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + z\,\vec{k}$

↳ for $0 \le \theta \le 2\pi$ and $a \le z \le b$







⊛ Smooth := $\vec{r}(u,v)$ is smooth if $\vec{r}_u$ and $\vec{r}_v$ are continuous and $\vec{r}_u \times \vec{r}_v \neq 0$;

↳ $\vec{r}_u = \dfrac{\partial \vec{r}}{\partial u} = \dfrac{\partial f}{\partial u}\vec{i} + \dfrac{\partial g}{\partial u}\vec{j} + \dfrac{\partial h}{\partial u}\vec{k}$ and $\vec{r}_v = \dfrac{\partial \vec{r}}{\partial v} = \dfrac{\partial f}{\partial v}\vec{i} + \dfrac{\partial g}{\partial v}\vec{j} + \dfrac{\partial h}{\partial v}\vec{k}$

⊛ Area := $\iint_R |\vec{r}_u \times \vec{r}_v|\,dA = \int_c^d \int_a^b |\vec{r}_u \times r_v|\,du\,dv = \iint_S d\sigma$

⊛ Vector cross-multiplication :=
$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial f}{\partial u} & \frac{\partial g}{\partial u} & \frac{\partial h}{\partial u} \\ \frac{\partial f}{\partial v} & \frac{\partial g}{\partial v} & \frac{\partial h}{\partial v} \end{vmatrix} = +\left(\frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u}\right)\vec{i} - \left(\frac{\partial f}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial h}{\partial u}\right)\vec{j} + \left(\frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial g}{\partial u}\right)\vec{k}$$
$= a\,\vec{i}$, $= -b\,\vec{j}$, $= c\,\vec{k}$

⊛ Magnitude $|\vec{r}_u \times \vec{r}_v| = \sqrt{a^2 + (-b)^2 + c^2}$

→ Implicit Surfaces: → $F(x,y,z) = c$:= level surface and $\nabla F \cdot \vec{p} = \nabla F \cdot \vec{k} = F_z \neq 0$ on S.

↳ $\vec{r}(u,v) = u\,\vec{i} + v\,\vec{j} + h(u,v)\,\vec{k}$ where $x = u$ and $y = v$ → $\vec{r}_u = \vec{i} + \dfrac{\partial h}{\partial u}\vec{k}$ and $\vec{r}_v = \vec{j} + \dfrac{\partial h}{\partial v}\vec{k}$

↳ Chain rule for implicit differentiation := $\dfrac{\partial h}{\partial u} = -\dfrac{F_x}{F_z}$ and $\dfrac{\partial h}{\partial v} = -\dfrac{F_v}{F_z}$

↳ $\vec{r}_u \times \vec{r}_v = \cdots = \dfrac{\nabla F}{\nabla F \cdot \vec{p}}$ for $\nabla F = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ and $\vec{p} = \vec{k}$ and $F_z \neq 0$ → $|\vec{r}_u \times \vec{r}_v| = \left|\dfrac{\nabla F}{\nabla F \cdot \vec{p}}\right|$

∴ $\iint_R \left|\dfrac{\nabla F}{\nabla F \cdot \vec{p}}\right| dA$:= Surface Area //

⊛ $\nabla F = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ where $F_z = \nabla F \cdot \vec{k}$

Ch. 16,6: Surface Integrals:

$$\boxed{\begin{aligned} d\sigma &= |\vec{r}_u \times \vec{r}_v| \cdot du\,dv \\ dA &= du\,dv \end{aligned}}$$

$A :=$ 2-dim.
$S :=$ 3-dim.
$R :=$ region

$$A = \iint_S d\sigma = \iint_R |\vec{r}_u \times \vec{r}_v|\, dA$$

* If $S := \vec{r}(u,v)$: $\iint_S f(x,y,z)\, d\sigma = \iint_R f(x,y,z) \cdot |\vec{r}_u \times \vec{r}_v|\, dA = \iint_R f(x(u,v), y(u,v), z(u,v)) \cdot |\vec{r}_u \times \vec{r}_v|\, dA$

* If $S :=$ implicit as $G(x,y,z) = C$: $\iint_S f(x,y,z)\, d\sigma = \iint_R f(x,y,z) \cdot |\vec{r}_u \times \vec{r}_v|\, dA = \iint_R f(x,y,z) \cdot \dfrac{|\nabla G|}{|\nabla G \cdot \vec{p}|}\, dA$

↳ where $\nabla G = z$ and $\vec{p} = \vec{k}$ and $\vec{r}(u,v) = u\,\vec{i} + v\,\vec{j} + z\,\vec{k}$

Flux-Integral: $\vec{F} = \vec{F}(x,y)$

↳ If $C \longrightarrow r(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$

↳ If $S \longrightarrow r(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$;

↳ If S is closed: $\int_C \vec{F} \cdot \vec{n}\, ds = \iint_S \vec{F} \cdot \vec{n}\, d\sigma$; $\vec{n} = (\vec{r}_u \times \vec{r}_v) \cdot \dfrac{1}{|\vec{r}_u \times \vec{r}_v|}$; $d\sigma = |\vec{r}_u \times \vec{r}_v|\, du\,dv$

$$\left.\begin{aligned} x &= x \\ y &= x^2 \\ z &= z \end{aligned}\right\} \vec{r}(u,v) = \vec{r}(x,z) = x\,\vec{i} + x^2\,\vec{j} + z\,\vec{k}$$ $\longrightarrow$ Example of parametrization

↳ If S is implicity: $\vec{n} = \dfrac{\nabla G}{|\nabla G|}$; $\int_S \vec{F} \cdot \vec{n}\, d\sigma = \iint_R \vec{F} \cdot \dfrac{\nabla G}{\cancel{|\nabla G|}} \cdot \dfrac{\cancel{|\nabla G|}}{|\nabla G \cdot \vec{p}|}\, dA = \iint_R \vec{F} \cdot \dfrac{\nabla G}{|\nabla G \cdot \vec{p}|} \cdot dA$

PARAMETRIZATION USING THE FOLLOWING COORDINATES :

* if $C \longrightarrow R(x) = x(x)\vec{i} + y(x)\vec{j} + z(x)\vec{k}$;

* if $S \longrightarrow R(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$

Coordinate Conversion Formulas

CYLINDRICAL TO RECTANGULAR	SPHERICAL TO RECTANGULAR	SPHERICAL TO CYLINDRICAL
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

$\int_C f(R(x))\cdot|v|\cdot dA$

Curve Integral of $\vec{F}$ over C $:= \int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\,\frac{d\vec{R}}{ds}\,ds = \int_C \underbrace{\vec{F}(\vec{R}(x))}_{\to\text{ scalar field}}\frac{d\vec{R}}{dx}\,dx \longrightarrow F(\vec{R}(x)) = x(x)\vec{i} + y(x)\vec{j} + z(x)\vec{k}$

Work along a curve $:= \int_a^b \vec{f}(\vec{R}(x))\cdot\frac{d\vec{R}}{dx}\,dx$

Flow along a curve $:= \int_C \vec{F}\cdot\vec{T}\,ds$ or $\oint_C :=$ Circulation $:= \oint_C \vec{F}\cdot\vec{T}\,ds$

SURFACE INTEGRALS :

* if $S := \vec{R}(u,v)$: $\iint_S f(x,y,z)\,d\sigma = \iint_R f(R(u,v))\cdot|\vec{R}_u \times \vec{R}_v|\,dA = \iint_R f(x(u,v), y(u,v), z(u,v))\cdot|\vec{R}_u\times\vec{R}_v|\,dA$

* if $S :=$ implicit as $G(x,y,z) = C$: $\iint_S f(x,y,z)\,d\sigma = \iint_R f(R(u,v))\cdot|\vec{R}_u\times\vec{R}_v|\,dA = \iint_R f(R(u,v))\cdot\frac{|\nabla G|}{|\nabla G\cdot\vec{p}|}\,dA$

  ↳ where $\nabla G = z$ and $\vec{p} = \vec{k}$ and $\vec{R}(u,v) = u\vec{i} + v\vec{j} + z\vec{k}$

Flux : $\int_C \vec{F}\cdot\vec{n}\,dV =$

$d\sigma = |R_u \times R_v|\,dA$ where $dA = dv\,du$

Flux in surface : $\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_R \vec{F}(\vec{R}(u,v))\cdot\frac{R_u\times R_v}{\cancel{|R_u\times R_v|}}\,\cancel{|R_u\times R_v|}\,dv\,du$

↳ Implicity : $\iint_R \vec{F}\cdot\frac{\nabla G}{\cancel{|\nabla G|}}\cdot\frac{\cancel{|\nabla G|}}{|\nabla G\cdot\vec{p}|}\,dA = \iint_R \vec{F}(R(u,v))\cdot\frac{\nabla G}{|\nabla G\cdot\vec{p}|}\cdot dv\,du$

Ch. 16.7: STOKES TEOREM:

$\oint_C \vec{F}\, d\vec{r}$ = curlens z komponent where $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \vec{i}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) - \vec{j}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \vec{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

if curl $\vec{F} = 0 \longrightarrow \vec{F}$ is conservative field

STOKES: $\underbrace{\boxed{\oint_C \vec{F}\, d\vec{r}}}_{\text{but calculate only this}} = \underbrace{\boxed{\iint_S \nabla \times \vec{F} \cdot \vec{n}\, d\sigma}}_{\text{we have this}}$ where $\begin{cases} C: \text{smooth and single relationship} \\ S: \text{Smooth surface} \\ C \text{ limits } S \end{cases}$

- 1st Identify C, S and F as functions of x, y, z;
- 2nd Parametrize C and F and calculate $\int_C \vec{F}_{(r(t))} \cdot \frac{d\vec{r}}{dt}\, dt$;
- 3rd Parametrize S and F and calculate $\iint_S \nabla \times \vec{F}\, \vec{n}\, d\sigma$:

where $\nabla \times \vec{F} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$ and $\vec{n} = \pm \frac{\nabla F}{|\nabla F|}$ and $d\sigma = \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|}$ where $\vec{p} = \vec{k}$

Ch. 16.8: Divergensteoremet (Gauss' sats):

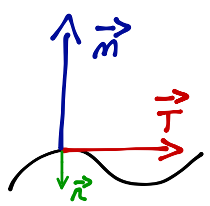
→ Divergens i planet: $\frac{\text{Fluks}}{\text{areal}}$; $\vec{F} = M\vec{i} + N\vec{j} \longrightarrow \text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$

→ Divergens i rommet (space): $\frac{\text{Fluks}}{\text{Volum}}$; $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k} \longrightarrow \text{div } \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$

$$\therefore \oiint_S \vec{F}\, \vec{n}\, d\sigma = \iiint_D \overbrace{\nabla \cdot \vec{F}}^{\text{div } \vec{F}}\, dV$$

div: flux density

curl: circulation / rotation: Tangent density



and $\text{div}(\text{curl}) \overset{\text{always}}{=} 0$

MATH112
SUMMARY

Parametrizations:

↳ Cone: $z = \sqrt{x^2+y^2} = r$ → parametrized using cylindrical coordinates: $\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + \overset{r}{\overset{\shortparallel}{z}}\,\vec{k}$

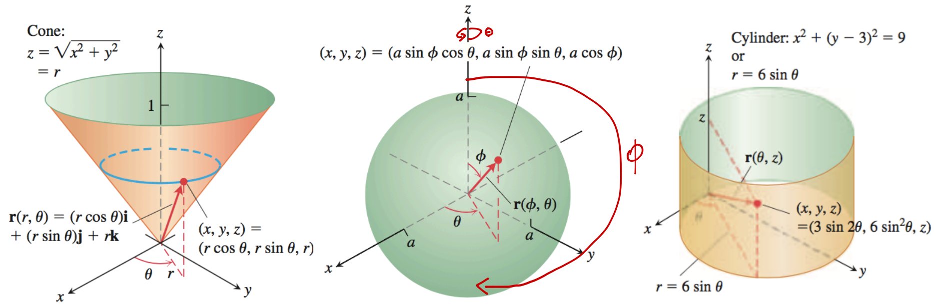
↳ for $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$

↳ sphere := $x^2+y^2+z^2 = \overset{a=\rho}{a^2}$ → parametrized using spherical coordinates: $\vec{r}(\phi,\theta) = a\sin\phi\cos\theta\,\vec{i} + a\sin\phi\sin\theta\,\vec{j} + a\cos\phi\,\vec{k}$

↳ for $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$

↳ Cylinder := $x^2+y^2 = r^2$ → parametrized using cylindrical coordinates: $\vec{r}(\theta,z) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + z\,\vec{k}$

↳ for $0 \leq \theta \leq 2\pi$ and $a \leq z \leq b$



Integrals in scalar field: CH 15

1) Line Integrals: → $\int_a^b f(x)\,dx$

2) Curve integrals:

↳ Area: $\iint_R dA$ where $dA = dy\,dx$

↳ Multiple integrals: $\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_a^b\int_c^d f(x,y)\,dx\,dy$

Only if R continuous in a retangular region

↳ Vertical selection: $\iint_R f(x,y)\,dy\,dx$ → Horizontal selection: $\iint_R f(x,y)\,dx\,dy$

3) Surface Integrals:

↳ Volum: $\iiint_R dV$ where $dV = dz\,dy\,dx$

↳ Cylindrical coordinates: $\iiint_D f(z,r,\theta)\,dV$ where $dV = dz\,r\,dr\,d\theta$

↳ Spherical coordinates: $\iiint_D f(\rho,\phi,\theta)\,dV$ where $dV = \rho\,d\rho\sin(\phi)\,d\phi\,d\theta$

⊛ Substitution: $\iiint_R f(x,y,z)\,dV = \iiint_G f\big(g(u,v,w); h(u,v,w); k(u,v,w)\big)\,|J(u,v,w)|\,du\,dv\,dw$

where $x = g(u,v,w)$; $y = h(u,v,w)$; $z = k(u,v,w)$; $dV = |J(u,v,w)|\,du\,dv\,dw$

$$J(u,v,w) = \det\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

⊛ Same for $\iint_R f(x,y)\,dA$

where $dA = |J(u,v)|\,du\,dv$

Integrals in vector field : CH.16.1-3

↳ Parametrization: $\vec{r}(t) = (x(t); y(t); z(t))$ where $\vec{r}(t) = (1-t).\vec{P} + t.\vec{Q}$ for $\vec{P} \to \vec{Q}$ and $0 \le t \le 1$

Curve integrals of $\vec{F}$ over C:

$\vec{T}$:= displacement

↳ Circulation / flow / work: $\oint_C \vec{F}\,\vec{T}\,ds = \oint_C \vec{F}\,d\vec{r} = \oint_R \vec{F}_{(t)}\,v_{(t)}\,dt$

where $\vec{F} = M_{(x,y)}\,\vec{i} + N_{(x,y)}\,\vec{j} + P_{(x,y)}\,\vec{k}$ and $\vec{T} = \vec{v}.\dfrac{d\vec{r}}{ds}$

↳ Flux: $\oint_C \vec{F}.\vec{n}\,ds = \oint_C \left[M(r(t)).\dfrac{\partial y(t)}{\partial t} - N(r(t)).\dfrac{\partial x(t)}{\partial t}\right]dt$ where $\vec{n} = \vec{T} \times \vec{r}$

↳ Conservative / Path independent: $\oint_C \vec{F}\,d\vec{r} = 0$

Component test: $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \vec{i}\left(\dfrac{\partial P}{\partial y} - \dfrac{\partial N}{\partial z}\right) - \vec{j}\left(\dfrac{\partial P}{\partial x} - \dfrac{\partial M}{\partial z}\right) + \vec{k}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right) = \vec{0}$

Potential Function: $F = e^{y+2z}\,\vec{i} + e^{y+2z}.x\,\vec{j} + e^{y+2z}\,2x\,\vec{k}$ → conservative / path independent and open simple connected domain!

$\therefore \dfrac{\partial f}{\partial x} = M = e^{y+2z} \longrightarrow f(x,y,z) = x e^{y+2z} + C(y,z)$

↳ $\dfrac{\partial f}{\partial y} = N \longrightarrow \cancel{x.e^{y+2z}} + \dfrac{\partial C}{\partial y} = \cancel{e^{y+2z}.x} \longrightarrow f(x,y,z) = x e^{y+2z} + 0 + C(z)$

↳ $\dfrac{\partial f}{\partial z} = P \longrightarrow \cancel{2.x e^{y+2z}} + \dfrac{\partial C}{\partial z} = \cancel{2x e^{y+2z}} \longrightarrow f(x,y,z) = x e^{y+2z} + 0 + 0 + C$

Green's Theorem: CH.16.4

for circulation: $\oint_C \vec{F}\,\vec{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right) dA = \iint_R (\text{curl}\,\vec{F}).\vec{k}\,dA$

for flux: $\oint_C \vec{F}.\vec{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}\right) dA = \iint_R \nabla\cdot\vec{F}\,dA$

Surface areas: CH.16.5

$\iint_S d\sigma = \iint_R |\vec{r}_u \times \vec{r}_v|\,dA = \iint_R \left|\dfrac{\nabla G}{\nabla G.\vec{p}}\right| dA$

where: $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial f}{\partial u} & \frac{\partial g}{\partial u} & \frac{\partial h}{\partial u} \\ \frac{\partial f}{\partial v} & \frac{\partial g}{\partial v} & \frac{\partial h}{\partial v} \end{vmatrix} = +\left(\dfrac{\partial g}{\partial u}\dfrac{\partial h}{\partial v} - \dfrac{\partial g}{\partial v}\dfrac{\partial h}{\partial u}\right)\vec{i} - \left(\dfrac{\partial f}{\partial u}\dfrac{\partial h}{\partial v} - \dfrac{\partial f}{\partial v}\dfrac{\partial h}{\partial u}\right)\vec{j}$

$+\left(\dfrac{\partial f}{\partial u}\dfrac{\partial g}{\partial v} - \dfrac{\partial f}{\partial v}\dfrac{\partial g}{\partial u}\right)\vec{k}$

Surface Integrals: $\iint_S f(x,y,z)\,d\sigma = \iint_R f(x(u,v), y(u,v), z(u,v)) \cdot |\vec{r}_u \times \vec{r}_v|\,dA$

CH. 16.6

$$= \iint_R f(x(u,v), y(u,v), z(u,v)) \cdot \frac{|\nabla G|}{|\nabla G \cdot \vec{p}|}\,dA$$

where $\nabla G = z$ and $\vec{p} = \vec{k}$ and $\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$

Flux-Integrals: $\int_C \vec{F} \cdot \vec{n}\,ds = \iint_S \vec{F} \cdot \vec{n}\,d\sigma$; $\vec{n} = (\vec{r}_u \times \vec{r}_v) \cdot \frac{1}{|\vec{r}_u \times \vec{r}_v|}$; $d\sigma = |\vec{r}_u \times \vec{r}_v|\,du\,dv$

↳ If S is implicity: $\vec{n} = \frac{\nabla G}{|\nabla G|}$; $\int_S \vec{F} \cdot \vec{n}\,d\sigma = \iint_R \vec{F} \cdot \frac{\nabla G}{\cancel{|\nabla G|}} \cdot \frac{\cancel{|\nabla G|}}{|\nabla G \cdot \vec{p}|}\,dA = \iint_R \vec{F} \cdot \frac{\nabla G}{|\nabla G \cdot \vec{p}|}\,dA$

STOKES: $\underbrace{\oint_C \vec{F}\,dr}_{1} = \iint_S \underbrace{\nabla \times \vec{F} \cdot \vec{n}}_{2}\,d\sigma$ where
- C: smooth and single relationship
- S: Smooth surface
- C limits S

CH. 16.7

$\boxed{\vec{T}(x) = \frac{\vec{v}_x\,dx}{ds}}$

1. $\oint_C \vec{F}\,d\vec{r} =$ For circulation: $\int_C \vec{F}(r(x)) \cdot \vec{T}(r(x))\,ds$; For flux: $\int_C F(r(x)) \cdot \vec{n}\,ds$

   ↳ $= \oint_C \left[M(r(x)) \cdot \frac{\partial y(x)}{\partial x} - N(r(x)) \cdot \frac{\partial x(x)}{\partial x}\right] dx$

2. For circulation or flux: $\iint_S \nabla \times \vec{F} \cdot \vec{n}\,d\sigma$

⊛ If $S := (x,y,z)$, then $\nabla \times \vec{F} \cdot \vec{n}$ is equal to the 3rd component ($\vec{k}$) of $\nabla \times F$;

⊛ If $S :=$ parametrized, then $\nabla \times \vec{F} \cdot \vec{n}$ is equal to $(\nabla \times \vec{F}) \cdot \underbrace{\frac{r_u \times r_v}{|r_u \times r_v|}}_{\vec{n}} \cdot \underbrace{|r_u \times r_v|\,dA}_{d\sigma}$.

where $\nabla \times \vec{F} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$ and $\vec{n} = \pm \frac{r_u \times r_v}{|r_u \times r_v|}$ and $d\sigma = |r_u \times r_v|\,dA$

Ch. 16.8: Divergensteoremet (Gauss' sats):

$$\oiint_S \vec{F} \cdot \vec{n}\,d\sigma = \iiint_D \overbrace{\nabla \cdot \vec{F}}^{\text{div } \vec{F}}\,dV \text{ for } \nabla = \left(\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right) \text{ and } \vec{F} = (M; N; P)$$

dot product