Implied volatility surface and rough volatility

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Agenda

1. Introduction
2. Implied Volatility Surfaces and its patterns
3. RFSV Model - Introduction to Gatheral's paper
4. Compatibility of simple model with empirical variability scaling
5. Specification of the RFSV model - stationarity problem
6. RFSV vs. FSV
7. RFSV model autocovariance functions
8. Volatility and long memory
9. Forecasting using the RFSV model
10. Conclusions
11. Bibliography

Introduction

•What is volatility?
Volatility represents uncertainty in financial markets and is modeled as a semi-martingale, which assumes predictable price changes.

Volatility paths are rougher and performs better by fractional Brownian motion (fBM) with a small Hurst parameter.

•In the Black-Scholes formula implied volatility is supposed to be consistent across options with the same expiration. However in reality implied volatility varies with strike price and expiration

Implied Volatility Surfaces and its patterns

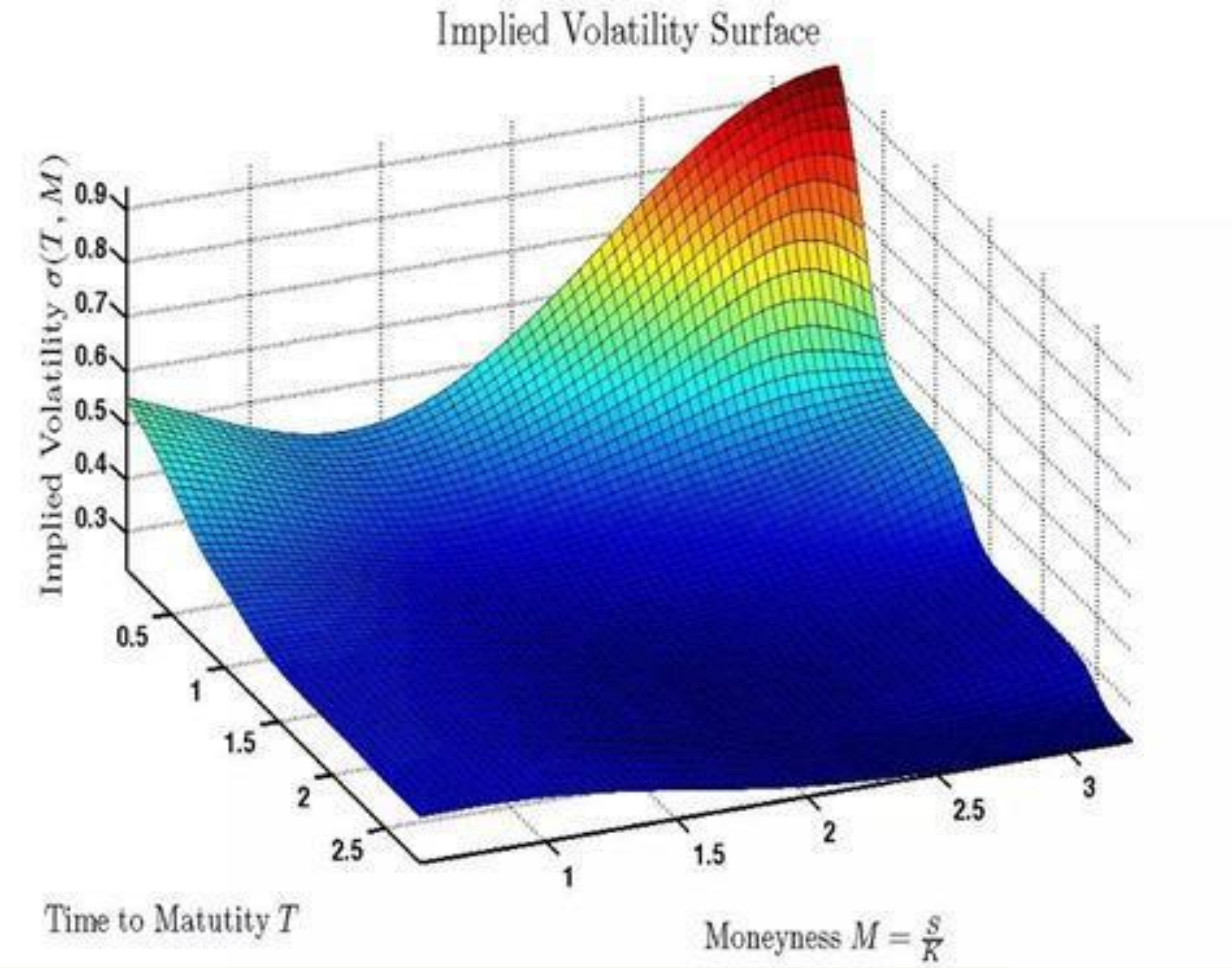


Figure 1.

Options with a different strike price (K), written on the same underlying asset and with the same time to expiration (T), should have the same implied volatility

We use the formula:

$$C(s,t) = sN(d_1(s,\tau)) - Ke^{-r\tau}N(s,\tau))$$

In reality the implied volatility surface varies significantly with K and T

Traders adjust the Black-Scholes model by using implied volatility for options with different strikes prices and maturities

Implied Volatility Surfaces and its patterns

Volatility Smile:

Implied volatility rises for options that are OTM or ITM compared to ATM; more pronounced for shorter maturities.

It has a U-shaped curve when plotting implied volatility against strike prices

Volatility Smirk (Reverse Skew)

Implied volatilities for OTM put options are higher compared to the implied volatilities for OTM call options. It reflects higher downside risk concerns, typical in equity markets; has a negative skew

Volatility Skew (Forward Skew):

Describes the slope of the implied volatility surface. Negatively skew surface indicates OTM'S higher implied volatility, positively surface for ITM's option.

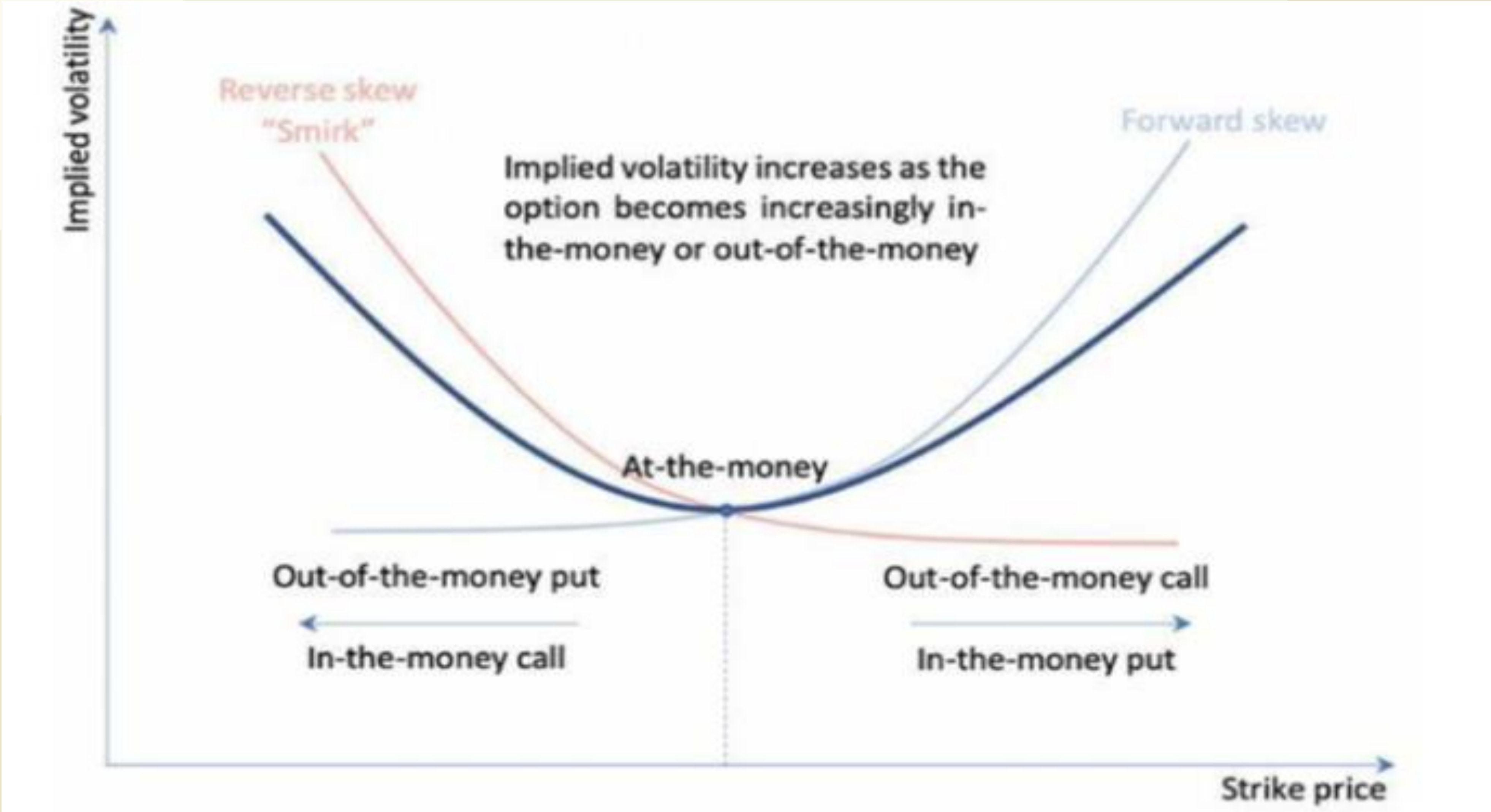


Figure 2.

RFSV Model - Introduction to Gatheral's paper

In Gatheral's paper they proxied the (true) spot volatility using two different kinds of assets: Future contracts (DAX and Bunds) and indices (S&P and NASDAQ).

- DAX and Bunds were used for Integrated Variance Estimator over one-hour windows and S&P and NASDAQ were used for Precomputed Variance over eight-hour trading windows

- They had access to discrete observations of volatility process, on a time grid with mesh Δ on $[0, T]$: $\mathrm{m}(\mathrm{q}, \Delta) = \frac{1}{N} \sum_{k=1}^{N} \left| \log(\sigma_{k\Delta}) - \log(\sigma_{(k-1)\Delta}) \right|^{q}$

- By assuming that for some s_q >0 and b_q >0 we have that Δ tends to zero, they get the equation:

$$N^{qSq} m(q, \Delta) \to b_q$$

(with Sq the smoothness parameter)

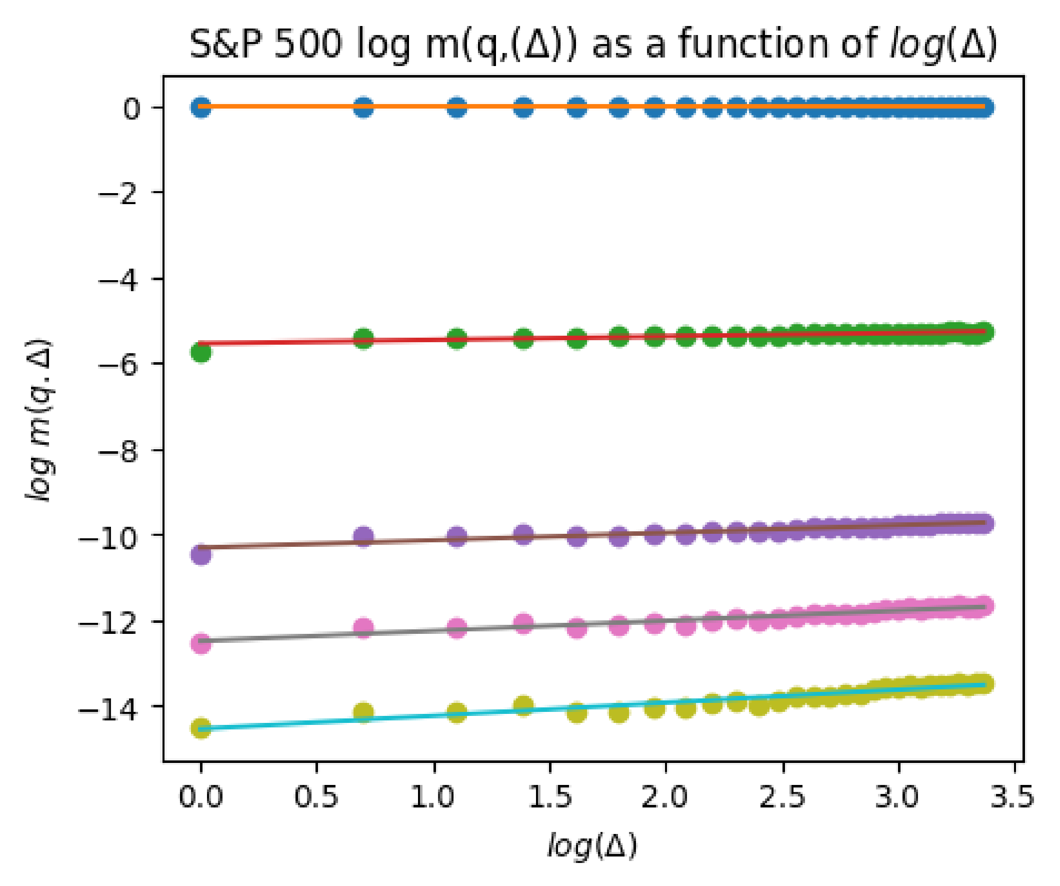


Figure 3.

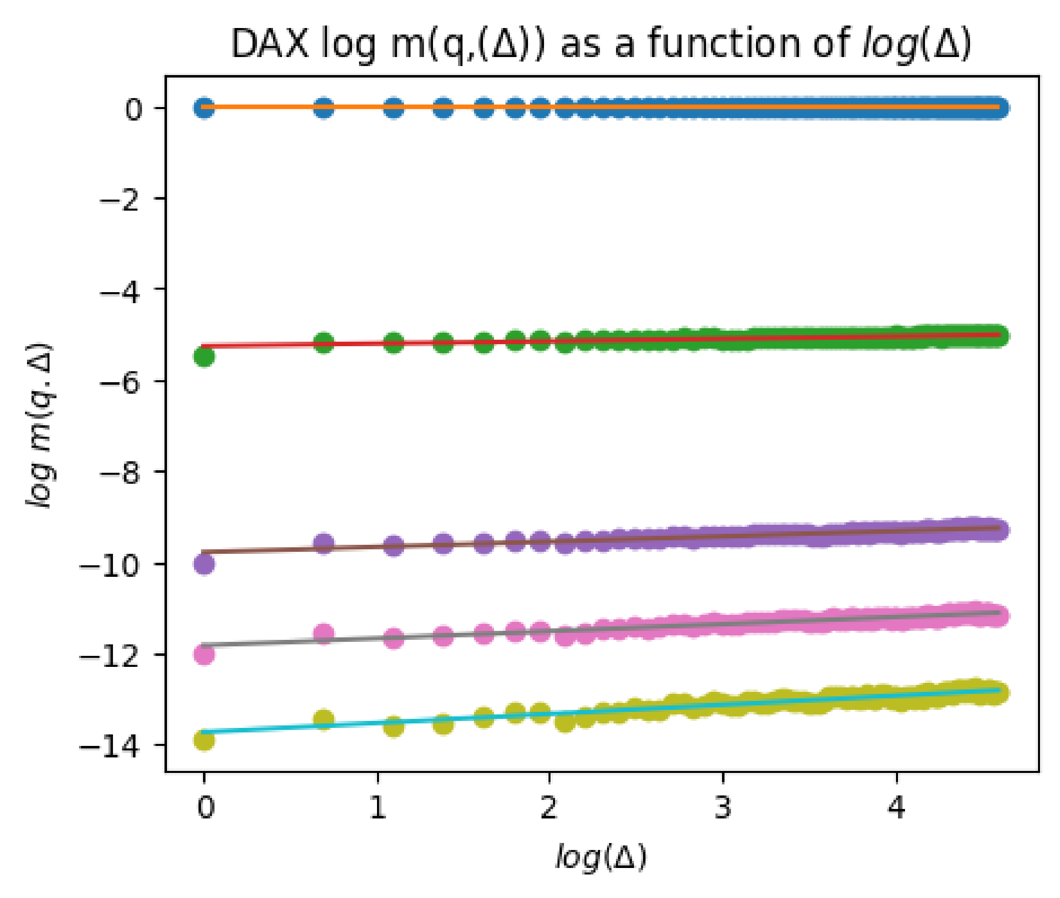


Figure 4.

With our data we obtain Figures 3 and 4, that show plots of log $m(q,\Delta)$ vs log for different values of q.

For both graphs the points lie on a straight line.

This implies that the log-volatility increments enjoy the following scaling property in expectation:

$$E[|log(\sigma_\Delta) - log(\sigma_0)|^q] = b_q \Delta^{\zeta_q},$$

where $\zeta_q = q s_q > 0$ is the slope of the line associated to q.

Compatibility of simple model with empirical variability scaling

After plotting ζq against q, we obtained that $\zeta q \sim Hq$.

We noticed that the slope of the two assets doesn't depend on q, but just on H. At the same time the differences between the estimations of H are not significant.

We observed the same scaling property for the S&P index as well as the DAX future.

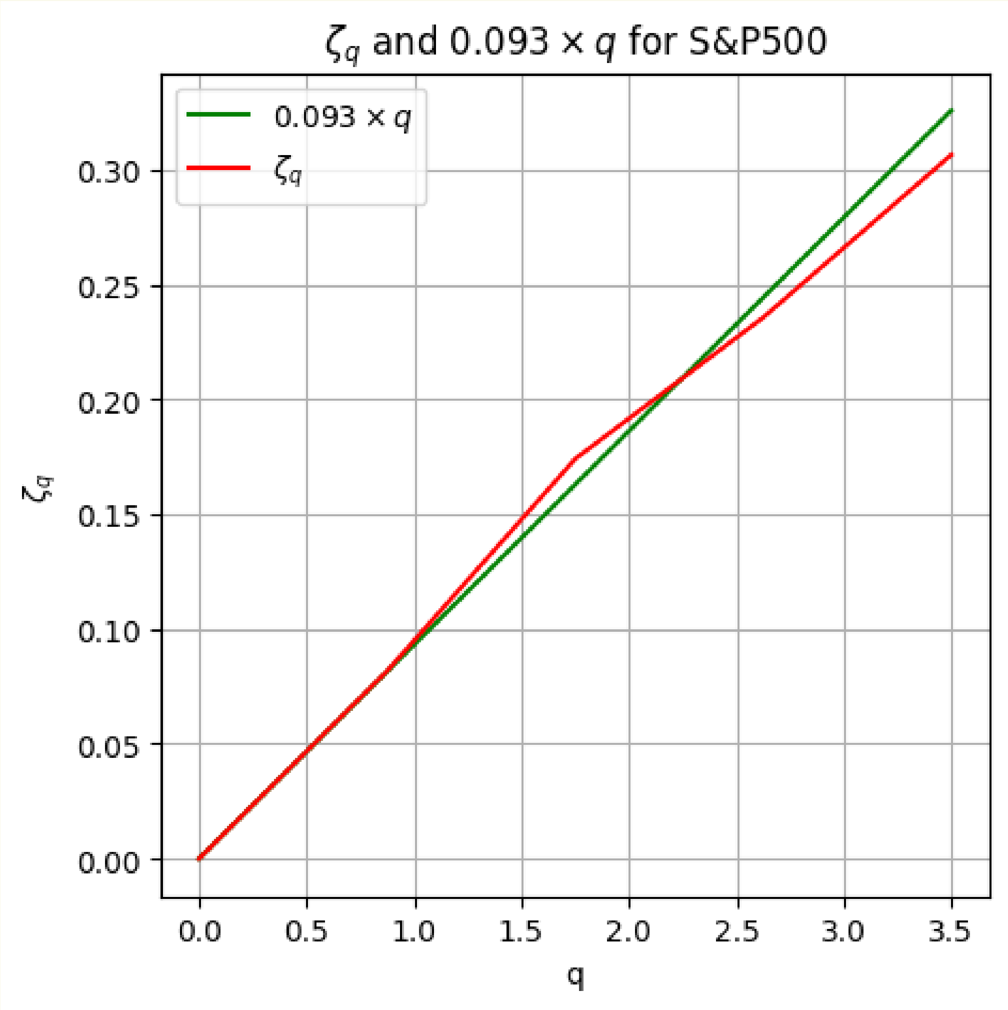


Figure 5.

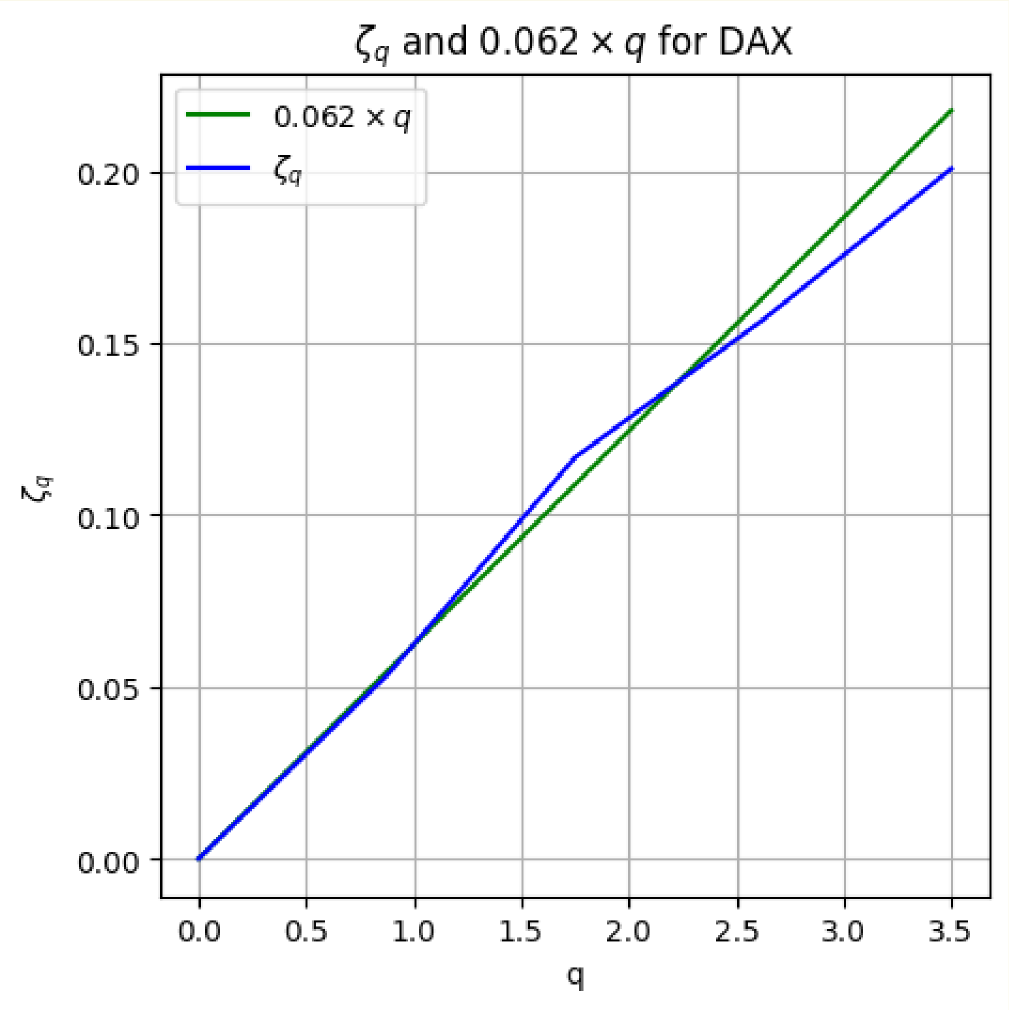


Figure 6.

Our diagrams show that the increments of the log-volatility of various assets enjoy a scaling property with constant smoothness parameter and their distribution is close to Gaussian. This suggests the simple model:

$$\log \sigma_{t+\Delta} - \log \sigma_t = \nu \left(W_{t+\Delta}^H - W_t^H\right)$$

where W^H is a fractional Brownian motion with Hurst parameter equal to the measured smoothness of the volatility and ν is a positive constant.

Figure 7.

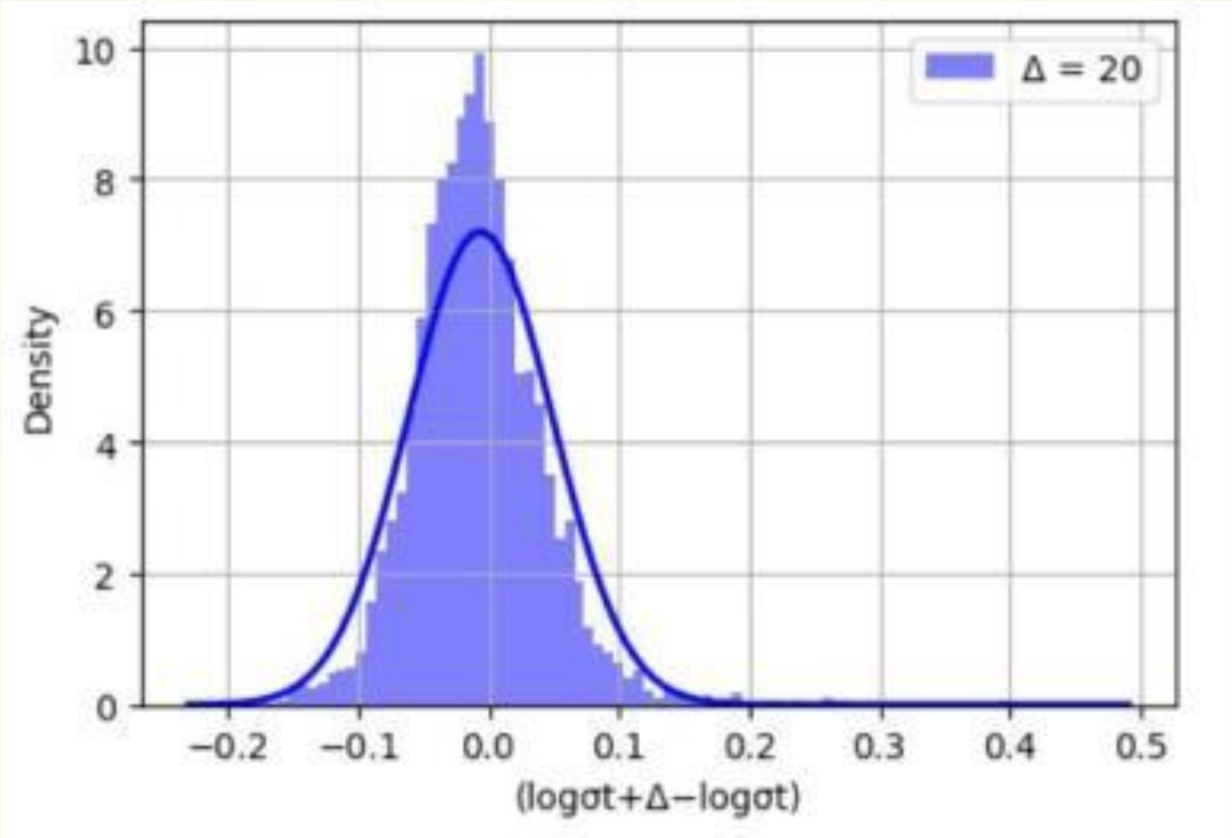


Figure 8.

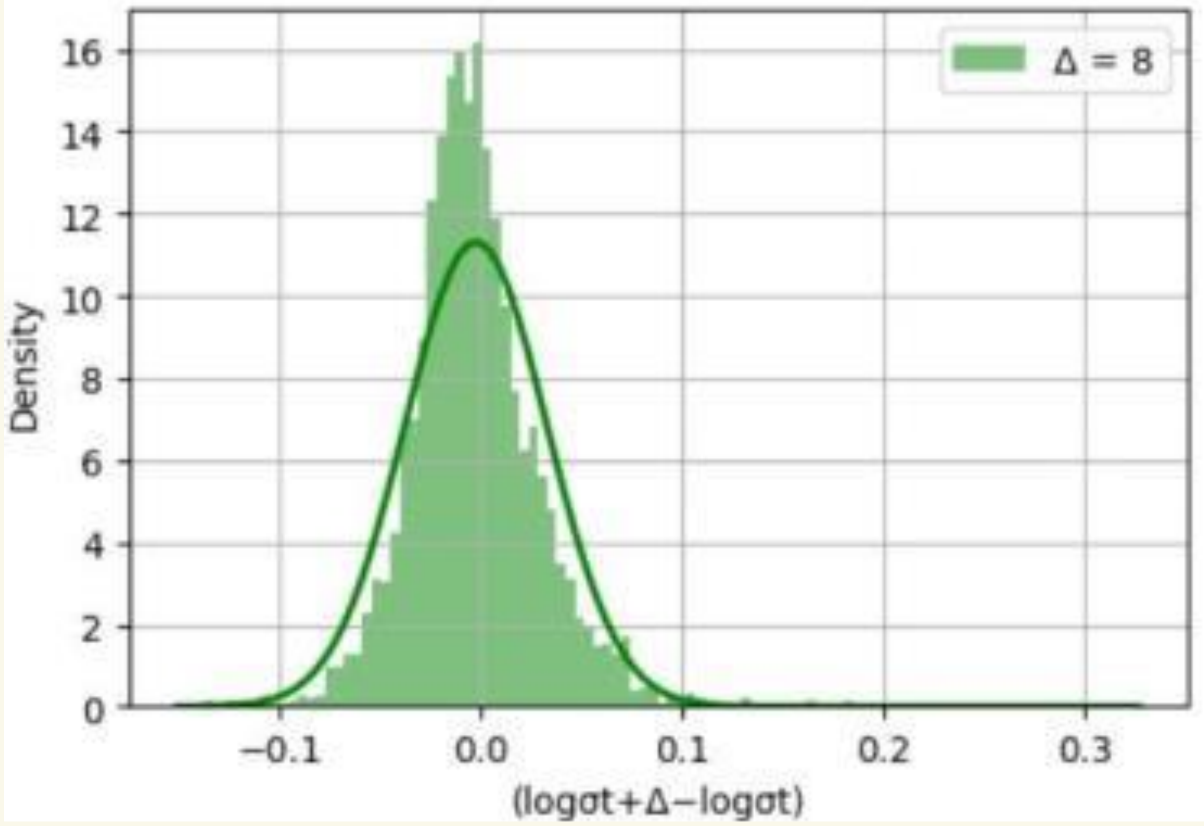
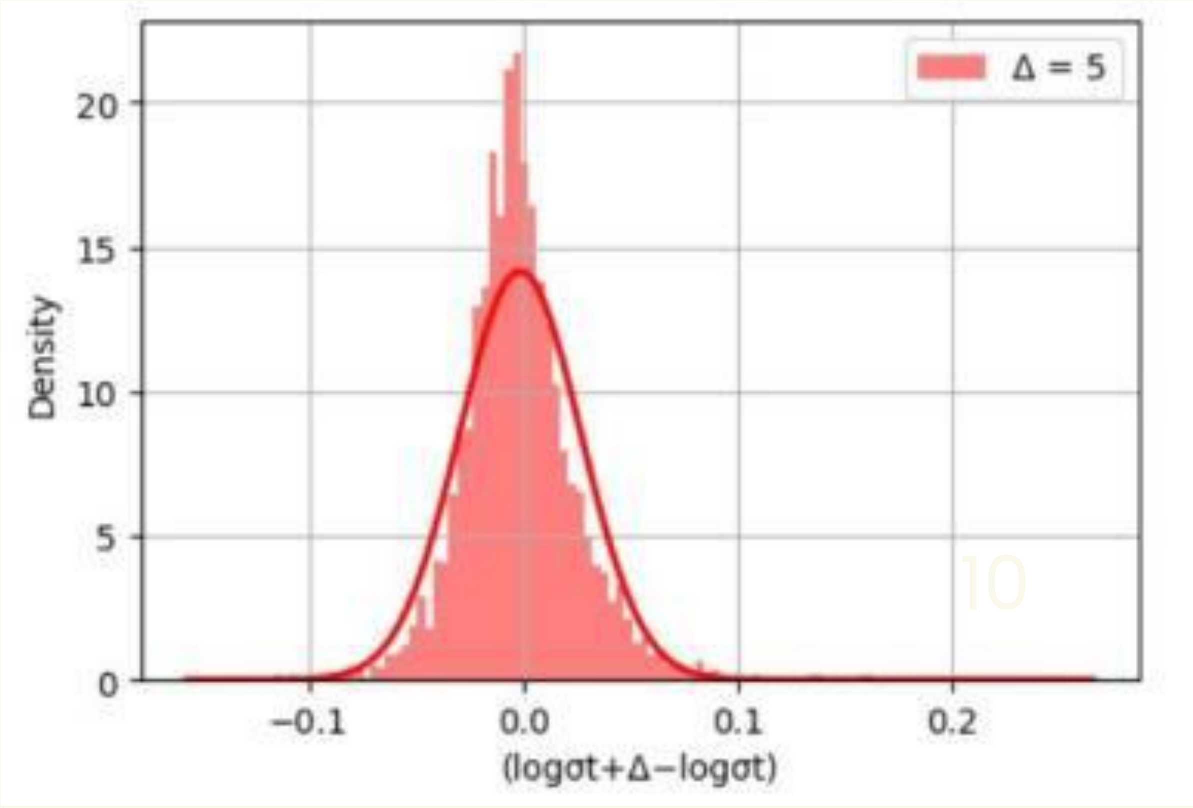


Figure 9.



Specification of the RFSV model - stationarity problem

The outcome of the model presented before(S&P) was not stationary.

Therefore, it was needed to impose stationary by modeling the log-volatility as a fractional Ornstein-Uhlenbeck process (Xt) with a very long reversion timescale, defined as:

$$X_t = v \int_{-\infty}^{t} e^{-\alpha(t-s)} dW_t^H + m$$

The final specification of Rough Fractional Stochastic Volatility (RFSV) model for volatility over the time frame of interest was obtained:

$$\sigma_t = exp(X_t), \qquad t \epsilon [0, T]$$

where (Xt) satisfies the explicit stationary solution of the stochastic differential equation, for some $v > 0$, $\alpha > 0$.

RFSV vs FSV

The key difference between FSV model (Comte and Renault) and RFSV model is in the assumption of H and α (H<1/2 and $\alpha \ll 1/T$ for RFSV).

1. Assumption that H<1/2	2. Assumption that $\alpha \ll 1/T$
It is consistent with the mean reversion and allowes to generate a term structure of volatility skew in agreement with the observed one.	This suggests that the dynamics of the process is similar to that of fBM. Then we have simultaneous stationarity by modeling logarithmic variation as an Ornstein-Uhlenbeck process and the ability to capture the roughness of fBM variation.

RFSV model autocovariance functions

The goal of Gatheral's paper was to demonstrate that data from RFSV model behaved like empirical data

Let $q > 0$, $\Delta > 0$, $t > 0$, as $\alpha \to 0$, we have the same relation:

$$Cov[X_t^\alpha, X_{t+\Delta}^\alpha] = Var[X_t^\alpha] - \frac{1}{2} v^2 \Delta^{2H} + o(1)$$

In the RFSV model, for fixed t, the covariance between X_t and $X_{t+\Delta}$ is linear with respect to Δ^{2H}

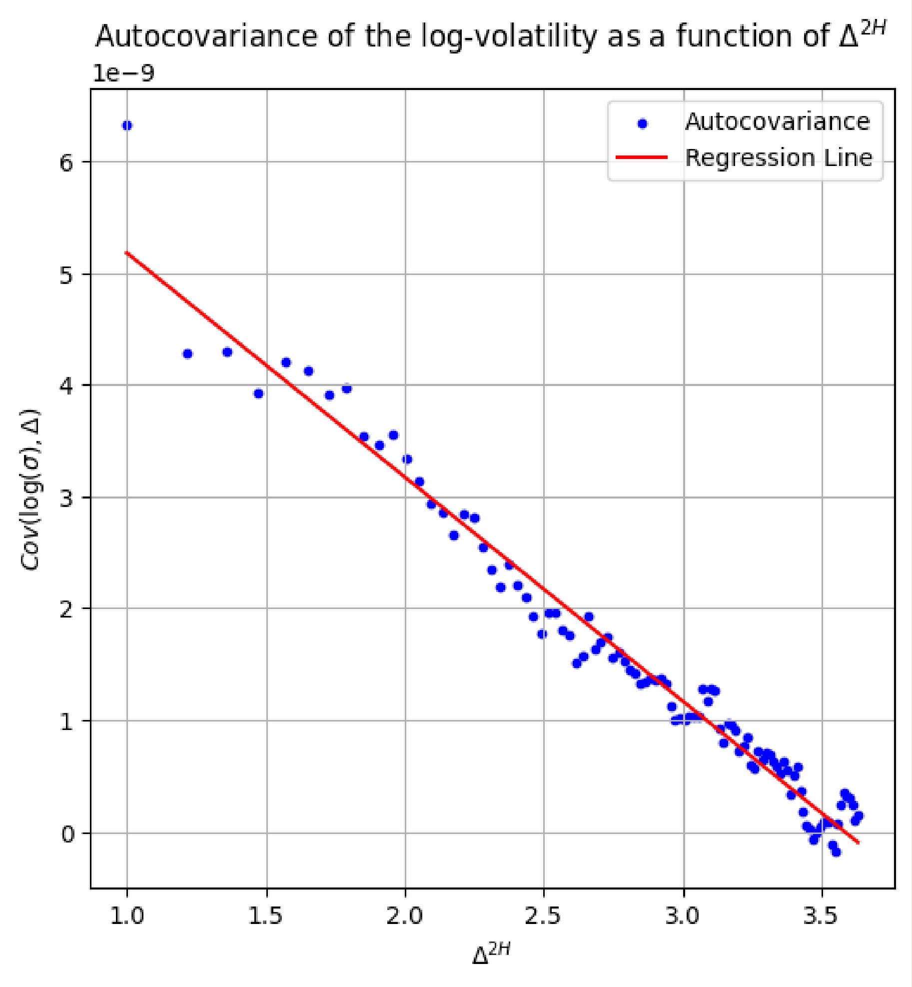


Figure 10.

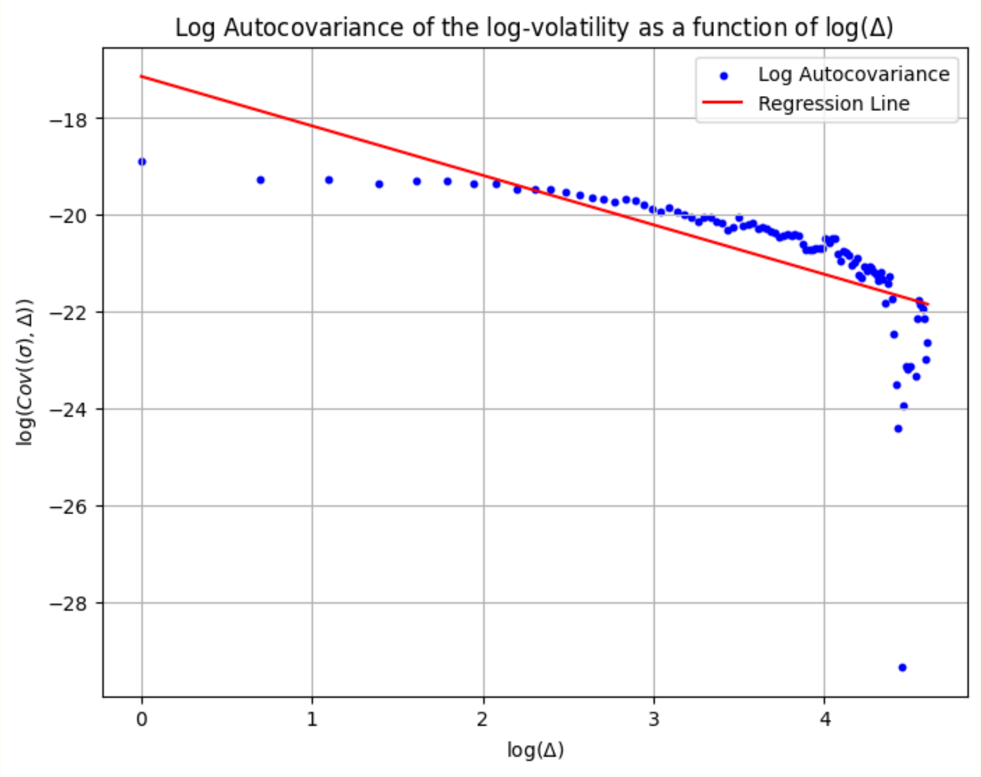


Figure 11.

We showed in Figure 11 that a log–log plot of the autocovariance function does not yield a straight line.

The autocovariance function of the volatility does not decay as a power law.

Volatility and long memory

Our main finding is that although the RFSV model does not have any long memory property, classical statistical procedures (aiming at detecting volatility persistence) tend to conclude the presence of long memory in data generated from it.

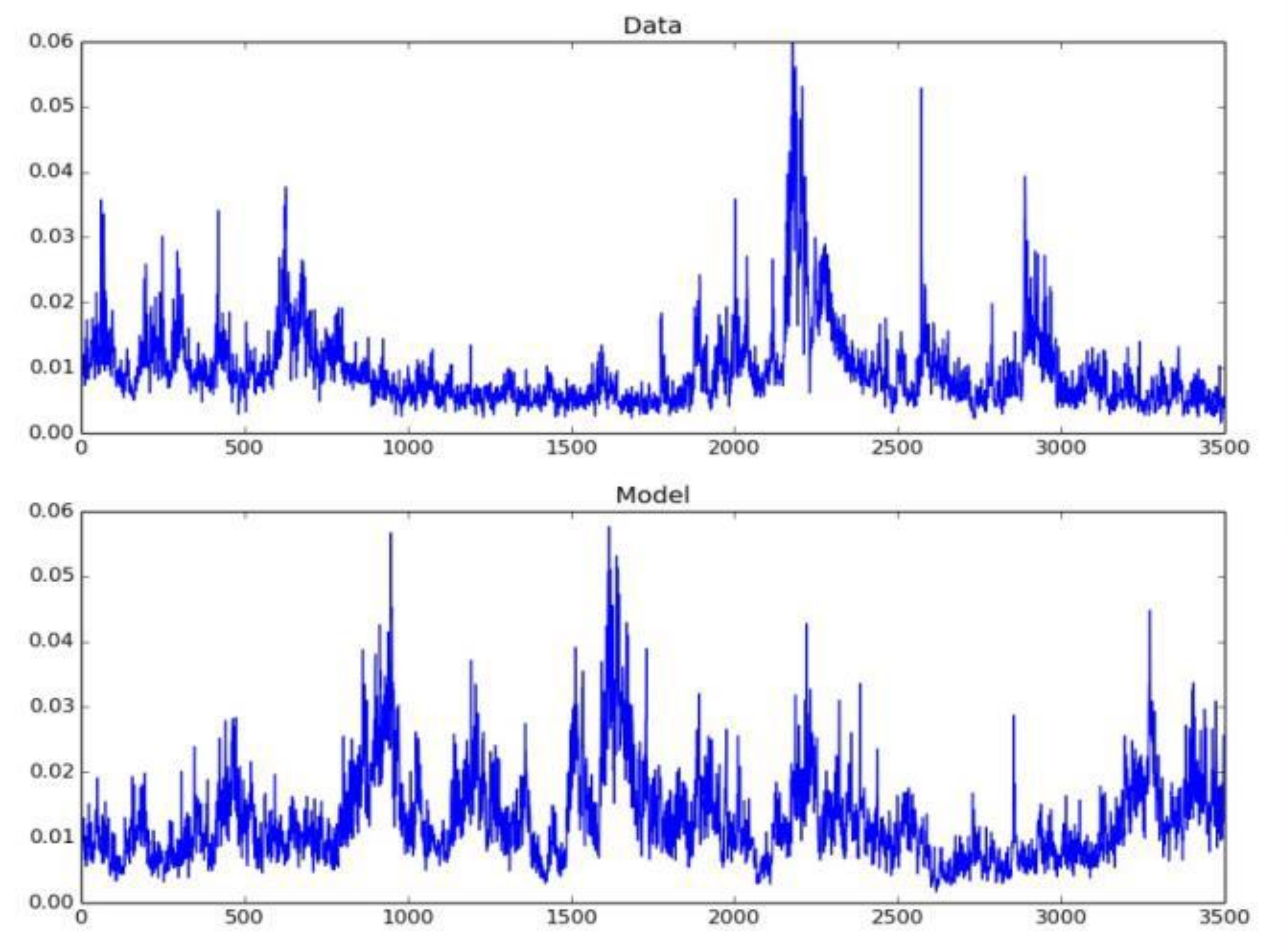


Figure 12.

In this diagram, they plotted a sample path of the model-generated volatility together with a graph of S&P volatility over 3500 days.

They observed that sample path of volatility, in a restricted time window, behaves the same as in a longer timescale (fractal-type behaviour).

Forecasting using the RFSV model

Log-Volatility Forecasting Formula (used for predicting 1, 5, and 20 days ahead (Δ=1,5,20):

$$E[log\sigma^2_{t+\Delta}|F_t] = \frac{cos(H\pi)}{\pi}\Delta^{H+1/2}\int_{-\infty}^{t}\frac{log\sigma^2_s}{(t-s+\Delta)(t-s)^{H+1/2}}ds$$

In Gatheral’s paper they compared RFSV model with Autoregressive (AR) and Heterogeneous Autoregressive (HAR) models.

The results shows that:

- RFSV consistently outperforms AR and HAR, especially for longer forecasts.

- RFSV only needs parameter H, while AR and HAR require recalculating coefficients for different forecast horizons.

Conclusions

- The distributions of increments of log-volatility are Gaussian.

- The monofractal scaling relationship is

$$E[|log(\sigma_\Delta) - log(\sigma_0)|^q] = K_q v^q \Delta^{qH}$$

  where H can be seen as a measure of smoothness characteristic of the underlying volatility process

- Log-volatility can be modeled using fractional Brownian motion.

- The differences between RFSV and FSV models are:
  - In FSV, $H > ½$ for long memory, in RFSV $H < ½$.
  - FSV requires $\alpha >> 1/T$, while in RFSV $\alpha << 1/T$ and α can be considered 0 for practical timescales.
  - Volatility skew decays naturally in RFSV, matching observed volatility skew.

- Empirical daily values of realized variance are more likely to come from an approximate variation process than from a smooth one.

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