Formal Language Selected Homework Chapter 5.1

7. Find context-free grammars for the following languages (with $n \geq 0$, $m \geq 0$).

(a) $L = \{a^n b^m : n \leq m + 3\}$.

(b) $L = \{a^n b^m : n \neq m - 1\}$.

(c) $L = \{a^n b^m : n \neq 2m\}$.

(d) $L = \{a^n b^m : 2n \leq m \leq 3n\}$.

(e) $L = \{w \in \{a, b\}^* : n_a(w) \neq n_b(w)\}$.

(f) $L = \{w \in \{a, b\}^* : n_a(v) \geq n_b(v)$, where v is any prefix of $w\}$.

Sol:

7. (a) First, solve the case $n = m + 3$. Then add more b's. This can be done by

$$\begin{aligned} S &\to aaaA, \\ A &\to aAb|B, \\ B &\to Bb|\lambda. \end{aligned}$$

But this is incomplete since it creates at least three a's. To take care of the cases $n = 0, 1, 2$, we add

$$S \to \lambda\, |aA|\, aaA.$$

(d) This has an unexpectedly simple solution

$$S \to aSbb\, |aSbbb|\, \lambda.$$

These productions nondeterministically produce either bb or bbb for each generated a.

(f) In p.128 Eg 5.4:

$$S \rightarrow aSb|SS|\lambda.$$ is for

$$L = \{w \in \{a,b\}^* : n_a(w) = n_b(w) \text{ and } n_a(v) \geq n_b(v), \text{ where } v \text{ is any prefix of } w\}.$$

So $S \rightarrow aSb|SS|aS|\lambda$ is for

$$L = \{w \in \{a,b\}^* : n_a(v) \geq n_b(v), \text{ where } v \text{ is any prefix of } w\}.$$

8. Find context-free grammars for the following languages (with $n \geq 0$, $m \geq 0$, $k \geq 0$).

(a) $L = \{a^n b^m c^k : n = m \text{ or } m \leq k\}$.

(b) $L = \{a^n b^m c^k : n = m \text{ or } m \neq k\}$.

(c) $L = \{a^n b^m c^k : k = n + m\}$.

(d) $L = \{a^n b^m c^k : n + 2m = k\}$.

(e) $L = \{a^n b^m c^k : k = |n - m|\}$.

(f) $L = \{w \in \{a, b, c\}^* : n_a(w) + n_b(w) \neq n_c(w)\}$.

(g) $L = \{a^n b^m c^k, k \neq n + m\}$.

(h) $L = \{a^n b^n c^k : k \geq 3\}$.

Sol:

8. (a) For the first case $n = m$ and k is arbitrary. This can be achieved by

$$\begin{aligned} S_1 &\rightarrow AC, \\ A &\rightarrow aAb|\lambda, \\ C &\rightarrow Cc|\lambda. \end{aligned}$$

In the second case, n is arbitrary and $m \leq k$. Here we use

$$\begin{aligned} S_2 &\rightarrow BD, \\ B &\rightarrow aB|\lambda, \\ D &\rightarrow bDc|E, \\ E &\rightarrow Ec|\lambda. \end{aligned}$$

Finally, we start productions with $S \rightarrow S_1|S_2$.

(e) Split the problem into two cases: $n = k + m$ and $m = k + n$. The first case is solved by

$$\begin{aligned} S &\rightarrow aSc\,|S_1|\,\lambda, \\ S_1 &\rightarrow aS_1b|\lambda. \end{aligned}$$

The second case is solved by

$$\begin{aligned} S' &\rightarrow S_2bS'c\,|\,\lambda \\ S_2 &\rightarrow aS_2b\,|\,\lambda \end{aligned}$$

Finally, we start the productions with $S_0 \rightarrow S\,|\,S'$.

(h)

$$S \to S_1 ccc S_2$$

$$S_1 \to aS_1b \mid \lambda$$

$$S_2 \to cS_2 \mid \lambda$$

13. Let $L = \{a^n b^n : n \geq 0\}$.

(a) Show that L^2 is context-free.

(b) Show that L^k is context-free for any given $k \geq 1$.

Sol: **13.** (a) If S derives L, then $S_1 \to SS$ derives L^2.

(b) If S derives L, then $S_1 \to \overbrace{SS \cdots S}^{k}$ derives L^k.

20. Consider the grammar with productions

$$S \to aaB,$$
$$A \to bBb|\lambda,$$
$$B \to Aa.$$

Show that the string *aabbabba* is not in the language generated by this grammar.

Sol:

20. The only possible derivations start with

$$S \Rightarrow aaB \Rightarrow aaAa \Rightarrow aabBba \Rightarrow aabAaba.$$

But this sentential form has the suffix *aba* so it cannot possibly lead to the sentence *aabbabba*.

23. Find a context-free grammar for the set of all regular expressions on the alphabet $\{a, b\}$.

Sol: **23.** $E \to E + E \,|E.E|\, E^* \,|(E)|\, \lambda|\varnothing|a|b.$