Formal Language Selected Homework Chapter 4.3

2. ~~Prove the following generalization of the pumping lemma, which includes Theorem 4.8 as well as Exercise 1 as special cases.~~

~~If L is regular, then there exists an m, such that the following holds for every sufficiently long $w \in L$ and every one of its decompositions $w = u_1 v u_2$, with $u_1, u_2 \in \Sigma^*$, $|v| \geq m$. The middle string v can be written as $v = xyz$, with $|xy| \leq m$, $|y| \geq 1$, such that $u_1 x y^i z u_2 \in L$ for all $i = 0, 1, 2, \ldots$.~~

4. Prove that the following languages are not regular.

(a) $L = \{a^n b^l a^k : k \geq n + l\}$.

(b) $L = \{a^n b^l a^k : k \neq n + l\}$.

(c) $L = \{a^n b^l a^k : n = l \text{ or } l \neq k\}$.

(d) $L = \{a^n b^l : n \leq l\}$.

(e) $L = \{w : n_a(w) \neq n_b(w)\}$.

Sol. **4.** (a) Given m, pick $w = a^m b^m a^{2m}$. The string y must then be a^k and the pumped strings will be

$$w_i = a^{m+(i-1)k} b^m a^{2m}.$$

If we take $i \geq 2$, then $m + (i-1)k > m$, and then w_i is not in L.

(e) It does not seem easy to apply the pumping lemma directly, so we proceed indirectly. Suppose that L were regular. Then by the closure of regular languages under complementation, $\overline{L}$ would also be regular. But $\overline{L} = \{w : n_a(w) = n_b(w)\}$, which, as is easily shown, is not regular. By contradiction, L is not regular.

(d) Given m, pick $w = a^m b^m$. The string y must then be a^k and $w_i = a^{m+(i-1)k} b^m$. But $w_2 = a^{m+k} b^m$ is not in L.

5. Determine whether or not the following languages on $\Sigma = \{a\}$ are regular.

(a) $L = \{a^n : n \geq 2,$ is a prime number$\}$.

(b) $L = \{a^n : n$ is not a prime number$\}$.

(c) $L = \{a^n : n = k^3$ for some $k \geq 0\}$.

(d) $L = \{a^n : n = 2^k$ for some $k \geq 0\}$.

14. Prove or disprove the following statement: If L_1 and L_2 are nonregular languages, then $L_1 \cup L_2$ is also nonregular.

Sol. **5.** (a) Take p to be the smallest prime number greater or equal to m and choose $w = a^p$. Now y is a string of a's of length k, so that

$$w_i = a^{p+(i-1)k}.$$

If we take $i - 1 = p$, then $p + (i-1)k = p(k+1)$ is composite and w_{p+1} is not in the language. (b)(d) in next page.

Sol. **14.** The proposition is false. As a counterexample, take $L_1 = \{a^n b^m : n \le m\}$ and $L_2 = \{a^n b^m : n > m\}$, both of which are non-regular. But $L_1 \cup L_2 = L(a^* b^*)$, which is regular.

Sol. 5.(b) $L = \{a^n : n \text{ is not a prime number}\}$

L is not regular.

$\because$ If L is reglar, $\bar{L} = \lambda \cup \{a^n : n \text{ is prime}\}$

will be regular. But by (a), we can prove $\bar{L}$ is

not.

(d). Given m, pick s as smallest integer s.t. $2^s \geq m$

consider $w_i = a^{2^s + (i-1)k}$, where $y = a^k$, $k > 1$, $k \leq m$

But $w_2 = a^{2^s + k}$ cannot be in L.

15. Consider the languages below. For each, make a conjecture whether or not it is regular. Then prove your conjecture.

(a) $L = \{a^n b^l a^k : n + l + k > 5\}$.

(b) $L = \{a^n b^l a^k : n > 5, l > 3, k \leq l\}$.

(c) $L = \{a^n b^l : n/l \text{ is an integer}\}$.

(d) $L = \{a^n b^l : n + l \text{ is a prime number}\}$.

(e) $L = \{a^n b^l : n \leq l \leq 2n\}$.

Sol.

15. (a) The language is regular. This is most easily seen by splitting the problem into cases such as $l = 0, k = 0, n > 5$, for which one can easily construct regular expressions.

(b) This language is not regular. If we choose $w = aaaaaab^m a^m$, our opponent has several choices. If y consists of only a's, we use $i = 0$ to violate the condition $n > 5$. If the opponent chooses y as consisting of b's, we can then violate the condition $k \le l$.

(e) $L = \{a^n b^l : n \le l \le 2n\}$ is not regular.

Given m, choose $w = a^m b^m$ and $w_0 = a^{m-k} b^m \notin L$.

17. Let L_1 and L_2 be regular languages. Is the language $L = \{w : w \in L_1, w^R \in L_2\}$ necessarily regular?

Sol. **17.** L is regular. We see this from $L = L_1 \cap L_2^R$ and the known closures for regular languages.

(Skip 19. 21. 25.)