1 Electromagnetic Field

The electromagnetic field $(\vec{E}, \vec{B})$ can be expressed in terms of the electromagnetic four-potential A^μ given, in SI units, by

$$A^\mu = \left(\frac{\phi}{c}, \vec{A} \right) , \tag{1}$$

where ϕ is the electric scalar potential and $\vec{A}$ is the magnetic vector potential. The electric and magnetic fields are given by

$$\begin{aligned} \vec{E} &= -\nabla\phi - \partial_t \vec{A} , \\ \vec{B} &= \nabla \times \vec{A} . \end{aligned} \tag{2}$$

The Lagrangian of the electromagnetic field is

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu , \tag{3}$$

where the electromagnetic tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \tag{4}$$

and the four-current is

$$J^\mu = (c\rho, \vec{j}) , \tag{5}$$

where ρ is the charge density and $\vec{j}$ the conventional current density.

The Euler-Lagrangian field equations are

$$\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\mu)} \right) = 0 , \tag{6}$$

which, imposing the Lorentz gauge condition $\partial_\mu A^\mu = 0$, gives the Maxwell equations

$$\partial_\mu \partial^\mu A^\nu = \mu_0 J^\nu , \tag{7}$$

where

$$\partial_\mu \partial^\mu = \Box = \frac{1}{c^2} \partial_t^2 - \nabla^2 . \tag{8}$$

2 Charged particles

The effect of an electromagnetic field on a particle of mass m_a and charge q_a is given by the Lorentz force which can be derived from the Lagrangian

$$\mathcal{L}_a = \frac{1}{2} m_a \dot{\vec{x}}_a(t)^2 - q_a \phi(\vec{x}_a, t) + q_a \dot{\vec{x}}_a(t) \cdot \vec{A}(\vec{x}_a, t) . \tag{9}$$

From the Euler-Lagrangian equations

$$\frac{\partial \mathcal{L}}{\partial x_a^i} - \partial_t \frac{\partial \mathcal{L}}{\partial \dot{x}_a^i} = 0 , \tag{10}$$

one obtains the Lorentz force in terms of the electomagnetic potentials

$$m_a \ddot{x}^i_a = -q_a \left[\left(\frac{\partial}{\partial x^i_a} \phi(\vec{x}_a, t) + \partial_t A^i(\vec{x}_a, t) \right) - \dot{x}^j_a \left(\frac{\partial}{\partial x^i_a} A^j(\vec{x}_a, t) - \frac{\partial}{\partial x^j_a} A^i(\vec{x}_a, t) \right) \right] . \tag{11}$$

Using Eq. (2), the above expression can be rewritten more conventionally as

$$m_a \frac{d\vec{v}_a}{dt} = q_a \left(\vec{E} + \vec{v}_a \wedge \vec{B} \right) , \qquad \frac{d\vec{x}_a}{dt} = \vec{v}_a . \tag{12}$$

This expression can be made relativistic inserting the γ factor

$$m_a \frac{d\gamma \vec{v}_a}{dt} = q_a \left(\vec{E} + \vec{v}_a \wedge \vec{B} \right) , \tag{13}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} . \tag{14}$$

3 Plasma model

Consider a plasma composed by N identical particles (e.g. electrons) of mass m_e and charge q_e. The charge density and current density are given by

$$\begin{aligned} \rho_e(\vec{x}, t) &= \sum_{b=1}^{N} q_e \delta(\vec{x} - \vec{x}_b(t)) , \\ \vec{j}_e(\vec{x}, t) &= \sum_{b=1}^{N} q_e \dot{\vec{x}}_b(t) \delta(\vec{x} - \vec{x}_b(t)) . \end{aligned} \tag{15}$$

The Maxwell equations rewritten in terms of the electric and magnetic potentials are

$$\Box \phi = \frac{\rho_e}{\epsilon_0} , \qquad \Box \vec{A} = \mu_0 \vec{j}_e , \tag{16}$$

whose general (non retarded) solutions are

$$\begin{aligned} \phi(\vec{x}, t) &= \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho_e(\vec{x}', t)}{|\vec{x} - \vec{x}'|} , \\ \vec{A}(\vec{x}, t) &= \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{j}_e(\vec{x}', t)}{|\vec{x} - \vec{x}'|} . \end{aligned} \tag{17}$$

Substituting the plasma charge and current density one obtains

$$\begin{aligned} \phi(\vec{x}, t) &= \frac{q_e}{4\pi\epsilon_0} \sum_{b=1}^{N} \frac{1}{|\vec{x} - \vec{x}_b(t)|} , \\ \vec{A}(\vec{x}, t) &= \frac{\mu_0 q_e}{4\pi} \sum_{b=1}^{N} \frac{\dot{\vec{x}}_b(t)}{|\vec{x} - \vec{x}_b(t)|} . \end{aligned} \tag{18}$$

In general the electromagnetic field A^μ is the sum of an external field A^μ_{ext} and the internal field A^μ_{int} generated by the plasma particles and calculated above. The Lorentz force, which determines the charge and current density of the plasma particles, can also be split in the sum of the two contributions

$$m_a\ddot{\vec{x}}_a = \vec{F}_{ext}+\vec{F}_{int} = q_a\left[\vec{E}_{ext}(\vec{x}_a,t)+\dot{\vec{x}}_a\times\vec{B}_{ext}(\vec{x}_a,t)\right]+q_a\left[\vec{E}_{int}(\vec{x}_a,t)+\dot{\vec{x}}_a\times\vec{B}_{int}(\vec{x}_a,t)\right] . \tag{19}$$

The internal contribution $\vec{F}_{int}$ can be computed inserting the expressions of the internal electric and magnetic potential calculated above. In particular, we need to compute the quantities

$$F^i_{1a} = \frac{\partial}{\partial x^i}\phi(\vec{x},t)\Big|_{\vec{x}=\vec{x}_a} , \qquad F^i_{2a} = \frac{\partial}{\partial t}A^i(\vec{x},t)\Big|_{\vec{x}=\vec{x}_a} , \qquad F^{ij}_{3a} = \frac{\partial}{\partial x^i}A^j(\vec{x},t)\Big|_{\vec{x}=\vec{x}_a} , \tag{20}$$

which give

$$\vec{F}_{int} = -q_a\left[F^i_{1a}+F^i_{2a}-\dot{x}^j_a\left(F^{ij}_{3a}-F^{ji}_{3a}\right)\right] . \tag{21}$$

Substituting the expressions of the charge and current densities, one obtains

$$\begin{aligned}
F^i_{1a} &= \frac{\partial}{\partial x^i}\phi(\vec{x},t)\Big|_{\vec{x}=\vec{x}_a} = -\frac{q_e}{4\pi\epsilon_0}\sum_{b\neq a}\frac{x^i-x^i_b(t)}{|\vec{x}-\vec{x}_b(t)|^3}\Bigg|_{\vec{x}=\vec{x}_a} , \\
F^i_{2a} &= \frac{\partial}{\partial t}A^i(\vec{x},t)\Big|_{\vec{x}=\vec{x}_a} = \frac{\mu_0 q_e}{4\pi}\sum_{b\neq a}\frac{\ddot{x}^i_b|\vec{x}-\vec{x}_b(t)|^2+\dot{x}^i_b(\vec{x}-\vec{x}_b(t))\cdot\dot{\vec{x}}_b}{|\vec{x}-\vec{x}_b(t)|^3}\Bigg|_{\vec{x}=\vec{x}_a} , \\
F^{ij}_{3a} &= \frac{\partial}{\partial x^i}A^j(\vec{x},t)\Big|_{\vec{x}=\vec{x}_a} = -\frac{\mu_0 q_e}{4\pi}\sum_{b\neq a}\frac{(x^i-x^i_b(t))\dot{x}^j_b}{|\vec{x}-\vec{x}_b(t)|^3}\Bigg|_{\vec{x}=\vec{x}_a} .
\end{aligned} \tag{22}$$

3.1 Boris Method

In order to solve the Lorentz force equation (19) numerically we use an efficient method introduced by Boris.

Let us start by rewriting the Lorentz force equation as a system of first order differential equations

$$\begin{aligned}
\dot{\vec{v}}_a &= \frac{q_a}{m_a}\left(\vec{E}+\vec{v}_a\wedge\vec{B}\right) , \\
\dot{\vec{x}}_a &= \vec{v}_a .
\end{aligned} \tag{23}$$

By rewriting the differential operators as central finite difference operators we obtain

$$\dot{\vec{v}}_a = \frac{\vec{v}_a(t+\frac{dt}{2})-\vec{v}_a(t-\frac{dt}{2})}{dt} = \frac{q_a}{m_a}\left(\vec{E}+\frac{\vec{v}_a(t+\frac{dt}{2})+\vec{v}_a(t-\frac{dt}{2})}{2}\wedge\vec{B}\right) , \tag{24}$$

where in the right-hand side we used the average value in place of $\vec{v}_a(t)$. Introducing the Boris transformations

$$\begin{aligned}
\vec{v}_a(t+\frac{dt}{2}) &= \vec{v}^{\,+}_a+\frac{q_a}{m_a}\frac{dt}{2}\vec{E} , \\
\vec{v}_a(t-\frac{dt}{2}) &= \vec{v}^{\,-}_a-\frac{q_a}{m_a}\frac{dt}{2}\vec{E} ,
\end{aligned} \tag{25}$$

we can eliminate the electric field from the Lorentz force which becomes

$$\frac{\vec{v}_a^+ - \vec{v}_a^-}{dt} = \frac{q_a}{m_a} \frac{\vec{v}_a^+ + \vec{v}_a^-}{2} \wedge \vec{B} \,. \tag{26}$$

Introducing the two magnetic vectors

$$\vec{t} = \frac{q_a}{m_a} \frac{dt}{2} \vec{B} \,, \qquad \vec{s} = \frac{2\vec{t}}{1+t^2} \,, \tag{27}$$

we obtain finally

$$\vec{v}_a' = \vec{v}_a^- + \vec{v}_a^- \wedge \vec{t} \,, \qquad \vec{v}_a^+ = \vec{v}_a^- + \vec{v}_a' \wedge \vec{s} \,. \tag{28}$$

We can use these relations to write explicitly the Boris algorithm

Boris Algorithm (input: $\vec{v}_a(t - \frac{dt}{2})$, output: $\vec{v}_a(t + \frac{dt}{2})$)

- $\vec{v}_a^- = \vec{v}_a(t - \frac{dt}{2}) + \frac{q_a}{m_a} \frac{dt}{2} \vec{E}$,
- $\vec{v}_a' = \vec{v}_a^- + \vec{v}_a^- \wedge \vec{t}$,
- $\vec{v}_a^+ = \vec{v}_a^- + \vec{v}_a' \wedge \vec{s}$,
- $\vec{v}_a(t + \frac{dt}{2}) = \vec{v}_a^+ + \frac{q_a}{m_a} \frac{dt}{2} \vec{E}$.

The last step is to obtain the position $\vec{x}_a$ from the velocity, this is simply given by

$$\vec{x}_a(t+dt) = \vec{x}_a(t) + \vec{v}_a(t + \frac{dt}{2})dt \,. \tag{29}$$

3.2 Vlasov equation

An approximation of the plasma is obtained considering the distribution function $f(\vec{x}, \dot{\vec{x}}, t)$ of the plasma particles in phase space. We have that $f(\vec{x}, \vec{v}, t)d^3x d^3v$ represents the number of particles with position in the volume d^3x around $\vec{x}$ and velocities in the volume d^3v around $\vec{v}$.

The distribution function satisfies, in general, the equation

$$\frac{\partial f}{\partial_t} + \dot{\vec{x}} \cdot \nabla_x f + \ddot{\vec{x}} \cdot \nabla_v f = 0 \,. \tag{30}$$

Substituting into this equation the Lorentz force equation in place of $\ddot{\vec{x}}$, one obtain the Vlasov equation

$$\frac{\partial f}{\partial_t} + \dot{\vec{x}} \cdot \nabla_x f - \frac{q_e}{m_e} \left[\nabla_x \phi + \partial_t \vec{A} - \vec{v} \times (\nabla_x \times \vec{A}) \right] \cdot \nabla_v f = 0 \,. \tag{31}$$

The charge and current densities are related to the distribution function by

$$\rho_e(\vec{x}, t) = q_e \int d^3v f(\vec{x}, \vec{v}, t) \,, \qquad j_e(\vec{x}, t) = q_e \int d^3v \vec{v} f(\vec{x}, \vec{v}, t) \,. \tag{32}$$

4 Magnetic Field from current loop

Considering the spatial part of Eq. 7, a static magnetic potential satisfies the equation:

$$\partial^2 \vec{A} = -\mu_0 \vec{J}\,, \tag{33}$$

where ∂^2 is the vector Laplacian. From Eq. 2, the magnetic field is related to the magnetic potential by

$$\vec{B} = \vec{\nabla} \wedge \vec{A}\,. \tag{34}$$

The current density circulating (counterclockwise) in a loop of radius R in the plane (x, y) is

$$\vec{J} = JR(-\sin\phi\hat{x} + \cos\phi\hat{y})\delta(x^2 + y^2 - R^2)\delta(z) = J\delta(r' - R)\delta(\theta' - \frac{\pi}{2})\hat{\phi}'\,. \tag{35}$$

Note that $\hat{\phi}'$ is the base vector of the current element and is different from the base vector $\hat{\phi}$ of the observation point $\vec{r}$.

From Green's function theory, we know that the solution of a differential equation of the form

$$Lu(x) = f(x)\,, \tag{36}$$

can be obtained from the Green's function of the differential operator L:

$$LG(x, x') = \delta(x - x')\,, \tag{37}$$

by convolution with the function $f(x)$:

$$u(x) = \int dx' G(x, x') f(x')\,. \tag{38}$$

The Green's function of the Laplacian operator in 3 dimension is

$$G(\vec{r}, \vec{r'}) = -\frac{1}{4\pi|\vec{r} - \vec{r'}|}\,, \tag{39}$$

and consequently:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{J}(\vec{r'})}{|\vec{r} - \vec{r'}|}\,. \tag{40}$$

Given the rotational symmetry for rotation around the z axis, we can look for solution in the plane (x, z) $(\phi = 0)$. The current's base vector $\hat{\phi}'$ in the $\vec{r}$ reference frame is then given by

$$\hat{\phi}' = -\sin\phi'\hat{r} + \cos\phi'\hat{\phi}\,. \tag{41}$$

Substituting into Eq. 40, we get

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d\phi' d\theta' \sin\theta' dr' r'^2 \frac{J\delta(r' - R)\delta(\theta' - \frac{\pi}{2})(-\sin\phi'\hat{r} + \cos\phi'\hat{\phi})}{(r^2 + r'^2 - 2rr'(\cos\theta\cos\theta' + \cos\phi'\sin\theta\sin\theta'))^{\frac{1}{2}}}\,. \tag{42}$$

Given the symmetry of $\sin\phi'$, the only non-vanishing component is

$$A_\phi(\vec{r}) = \frac{\mu_0 J R^2}{4\pi}\int_0^{2\pi} d\phi' \frac{\cos\phi'}{(r^2 + R^2 - 2rR\sin\theta\cos\phi')^{\frac{1}{2}}}\,. \tag{43}$$

By setting

$$a = r^2 + R^2\,; \qquad b = 2rR\sin\theta\,; \qquad c = \frac{\mu_0 J R^2}{4\pi}\,; \tag{44}$$

The integral can be rewritten as

$$A_\phi(\vec{x}) = c\int_0^{2\pi} d\phi' \frac{\cos\phi'}{(a - b\cos\phi')^{\frac{1}{2}}}\,, \tag{45}$$

whose solution can be expressed in terms of the elliptic integrals:

$$A_\phi(\vec{x}) = \frac{4c}{b\sqrt{a+b}}\left(a\,K\left(\frac{2b}{a+b}\right) - (a+b)\,E\left(\frac{2b}{a+b}\right)\right)\,, \tag{46}$$

where

$$K(k) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,k\right)\,, \qquad E(k) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},-\frac{1}{2};1,k\right)\,. \tag{47}$$

Finally, from Eq. 2 we get

$$\vec{B}(\vec{x}) = -\partial_z A_\phi \hat{x} + \partial_x A_\phi \hat{k} = B_x(\vec{x})\hat{x} + B_z(\vec{x})\hat{z}\,, \tag{48}$$

whit

$$B_x(\vec{x}) = -2z\partial_a A_\phi\,, \qquad B_z(\vec{x}) = \left(2x\partial_a A_\phi + 2R\partial_b A_\phi + \frac{A_\phi}{x}\right)\,, \tag{49}$$

where

$$\begin{aligned}
\partial_a A_\phi &= \frac{2c}{b(a-b)\sqrt{a+b}}\left((a-b)\,K\left(\frac{2b}{a+b}\right) - a\,E\left(\frac{2b}{a+b}\right)\right)\,,\\
\partial_b A_\phi &= \frac{2c}{b^2(a-b)\sqrt{a+b}}\left((2a^2-b^2)\,E\left(\frac{2b}{a+b}\right) - 2a(a-b)\,K\left(\frac{2b}{a+b}\right)\right)\,.
\end{aligned} \tag{50}$$