CS2102 Analysis and Design of Algorithms

Linear Programming

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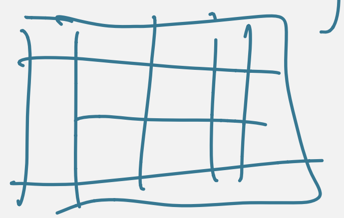
1. Overview + Intuition
2. Correctness + Duality
3. Simplex method
4. Exercises

$\max f(x)$ $\min f'(x)$

$f(x,y) = x^2 + y$.

— constraint



Overview

min z   max z

z = f(x,y)

Linear Programming

Linear	Objective function and linear constraints.
Programming	Common framework to solve optimization problems (maximize or minimize).

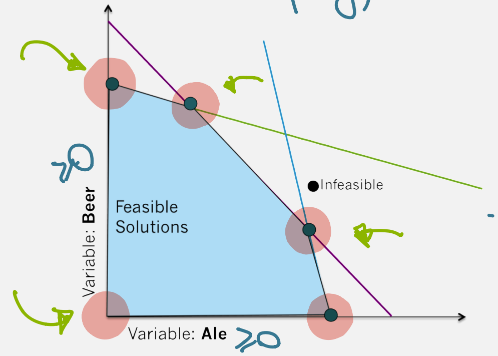
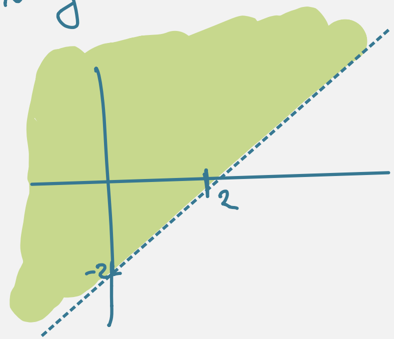


Figure 1: Linear constraints bounding a feasible solutions region

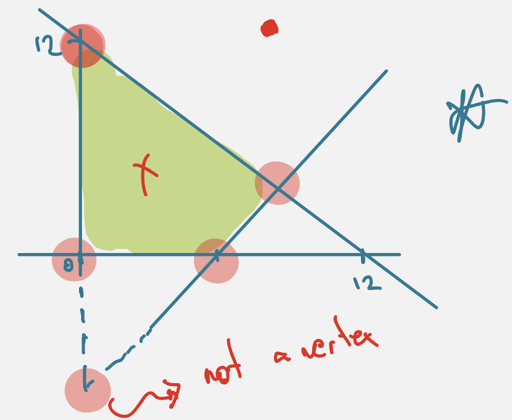
Many practical problems in operations research can be expressed as linear programming problems. Planning, production, transportation and other related fields.

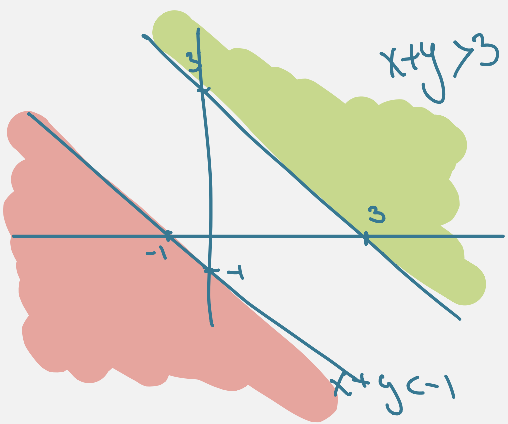
Linear Inequalities

$x - y < 2$

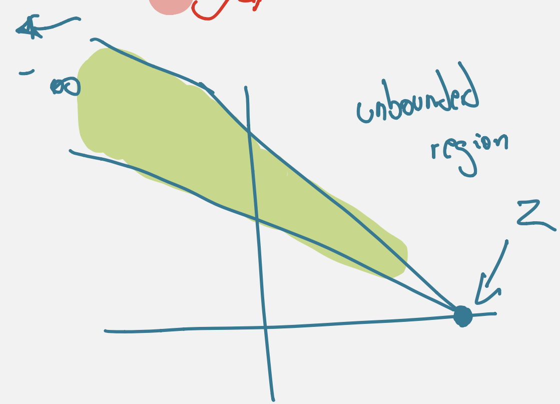


$$\begin{cases} x + y \leq 12 \\ x, y \geq 0 \\ 3x - 4y \leq 15 \end{cases}$$





no solution



unbounded region

1 $f(x,y) = z$ 2

$$\begin{aligned} x + y &\leq 5 \\ 2x + 4y &\geq 12 \end{aligned} \qquad \begin{aligned} x + 2y &= 6 \\ 3x - y &= -3 \end{aligned}$$

Algorithm 1 substitution_method()

1: Use first equation to solve for one variable in terms of the others

2: Substitute into other equations

3: Solve recursively

4: Substitute back in to first equation to get initial variable

Basic row operations:

- Adding ✓ +

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & 4 & 12 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 2 & 2 \end{array}\right]$$

- Scaling ✓ ×

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

- Swapping ✓

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 5 \end{array}\right]$$

Consider the system given by the matrix

$$\left[\begin{array}{cccc|c} 2 & 4 & -1 & 0 & 2 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

After applying row reduction method

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Algorithm 2 row_reduction_method(A)

1: Leftmost non-zero
2: Swap row to top if needed
3: Make entry pivot
4: Scale to make pivot 1
5: Subtract row from others to make other entries in columns 0
6: Repeat

Then, for any value of z:

$x = -1 - z$ $\quad y = 1 + z$ $\quad w = 0$

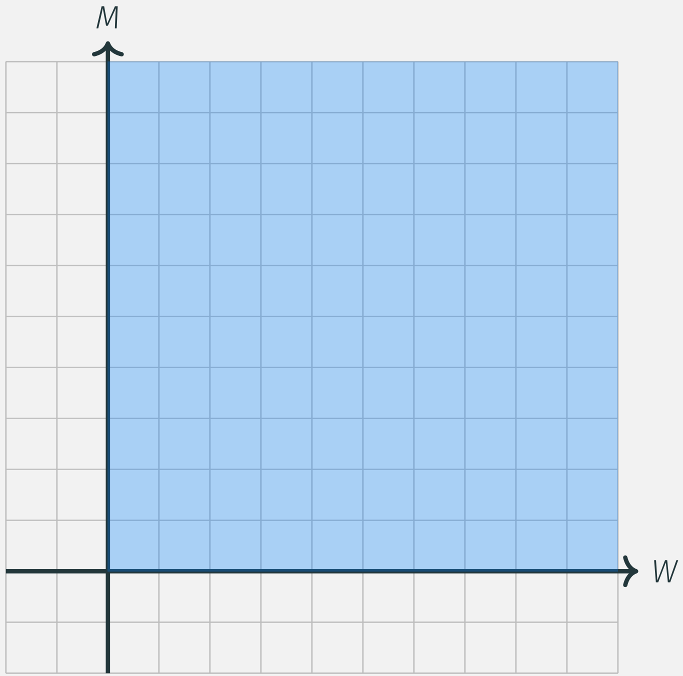
Factory optimization problem

A factory wants to optimize the procedure that uses to build some items. Those items can be produced by workers (*W*) or by machines (*M*).

While a machine can produce 600 items a day a single worker produces 200 items a day, nevertheless a machine needs of 2 workers to operate properly. The total demand for items per day is 100,000.

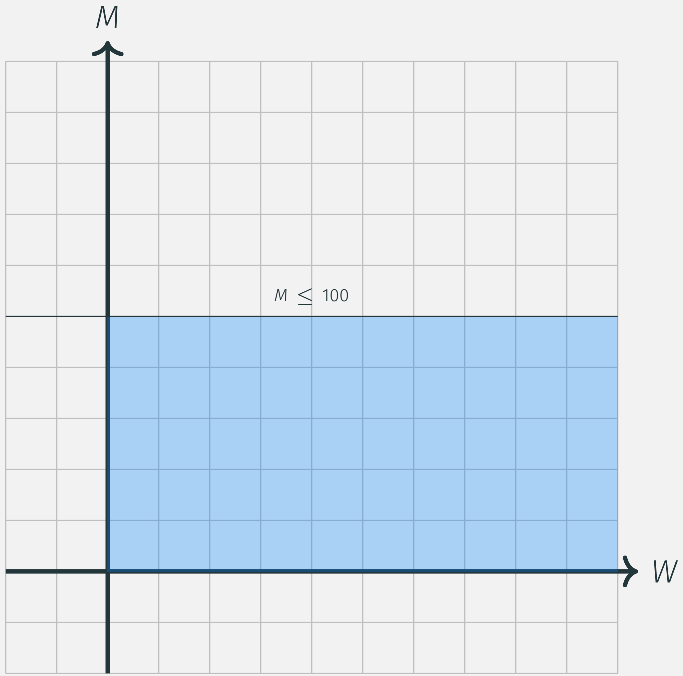
- Each worker costs $100.00 per day
- Each item generates a profit of $1.00

Factory problem



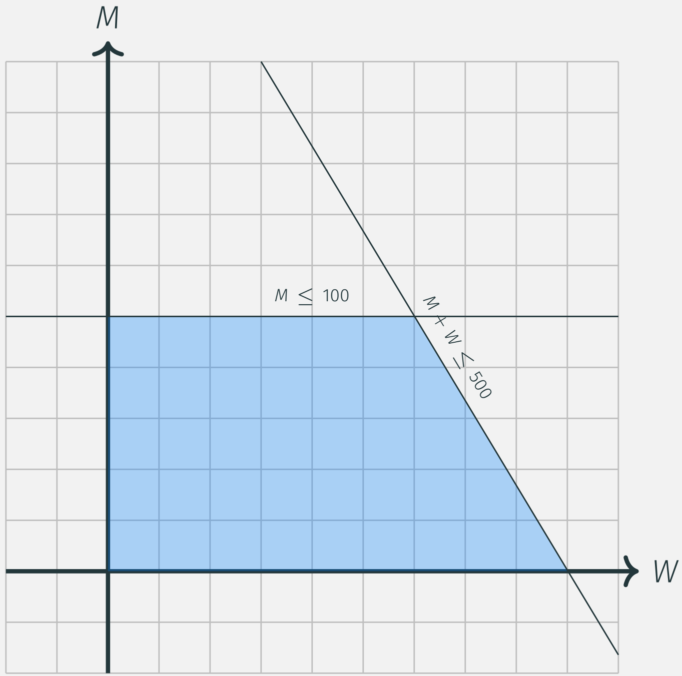
Problem constraints

- $W, M \geq 0$



Problem constraints

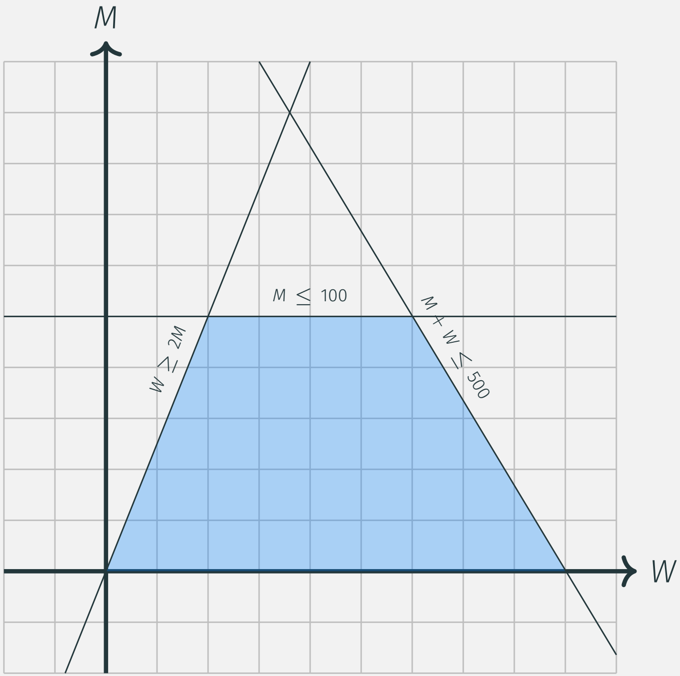
- $W, M \geq 0$
- $M \leq 100$



Problem constraints

- $W, M \geq 0$
- $M \leq 100$
- $M + W \leq 500$

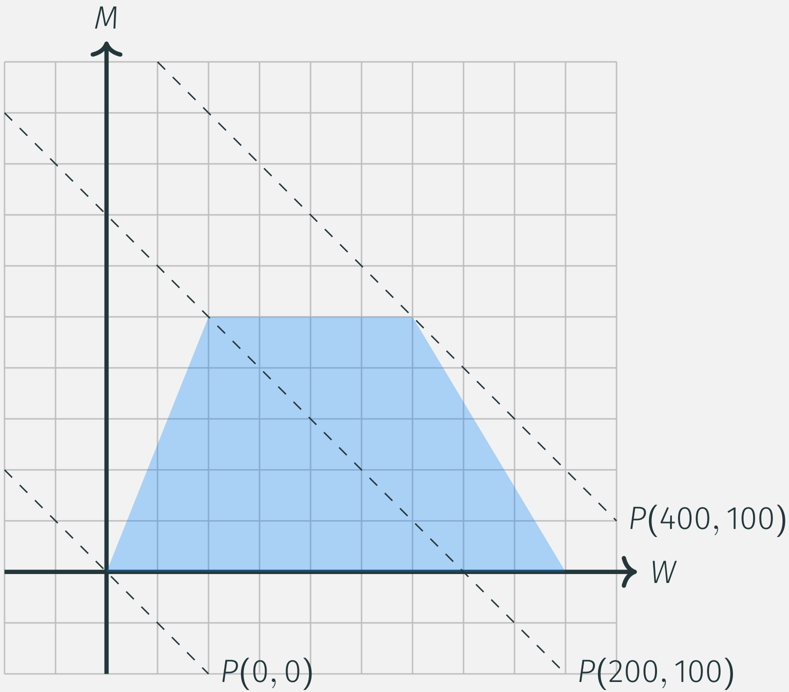
Factory problem



Problem constraints

- $W, M \geq 0$
- $M \leq 100$
- $M + W \leq 500$
- $W \geq 2M$

Factory problem



Problem constraints

- $W, M \geq 0$
- $M \leq 100$
- $M + W \leq 500$
- $W \geq 2M$

Bounded feasible region (convex)

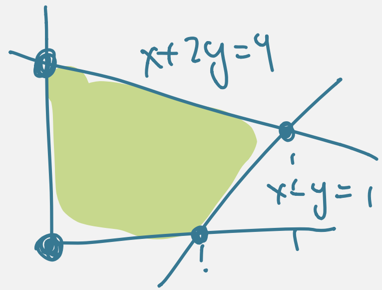
$P = 200(W - 2M) + 600M - 100W$
$P = 100W + 200M$

Graphical Method

Maximize $3x + 2y = z$

Subject to

$$x + 2y \le 4$$
$$x - y \le 1$$
$$x \ge 0$$
$$y \ge 0$$



$(0,0)$: $z = 3(0) + 2(0) = 0$ ✓

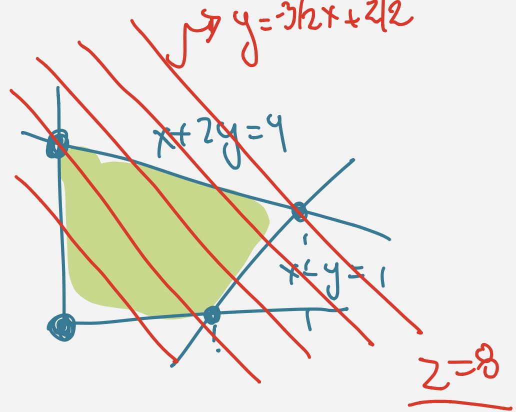
$(1,0)$: $z = 3(1) + 2(0) = 3$ ✓

$(2,1)$ $z = 3(2) + 2(1) = \boxed{8}$ ✓

$(0,2)$ $z = 3(0) + 2(2) = 4$ ✓

$x = 2,\ y = 1 \rightarrow \max z$

$$y = -3/2\,x + 2/2$$



Correctness

Convexity

A region $\mathcal{C} \subset \mathbb{R}^n$ is convex if for any $x, y \in \mathcal{C}$ the line segment connecting x and y is contained in $\mathcal{C}$ defined by the system $Ax \geq b$. Left side shows a convex shape and right side shows a non-convex shape

- An inequality defines a halfspace[1].
- A polytope is a region in $\mathbb{R}^n$ bounded by finitely many flat surfaces

Lemmas

- An intersection of halfspaces is convex
- A linear function on a polytope takes its minimum/maximum values on vertices

[1] An equality defines an hyperplane

Linear program (primal)

Minimize $v \cdot x$

Subject to $Ax \geq b$

Dual linear program

Maximize $y \cdot b$

Subject to $y^T A = v$ and $y \geq 0$

This means that one can instead solve the dual problem because sometimes its easier and often provides insight into the solution

Theorems

- A linear program and its dual always have the same value
- The dual of the dual is the primal

Minimize $7x_1 + x_2 + 5x_3$

Subject to constraints:

- $x_1 - x_2 + 3x_3 \geq 10$
- $5x_1 + 2x_2 - x_3 \geq 6$
- $x_1, x_2, x_3 \geq 0$

- How can we certify that the optimal value is at least 10? (lower bound) Is there any better?

- What is the best lower bound we can obtain?

Simplex method

Converting linear programs [Cormen et al., 2009]

A linear program might not be in standard form for any of four possible reasons:

- The objective function might be a minimization rather than a maximization.
- There might be variables without non-negativity constraints.
- There might be equality constraints, which have an equal sign rather than a less-than-or-equal-to sign.
- There might be inequality constraints, but instead of having a less-than-or-equal-to sign, they have a greater-than-or-equal-to sign.

transpose dual trans

primal maxim,

Minimize

Algorithm 3 simplex

Start at vertex p

loop

  for all equation through p **do**

    Relax equation to get edge

    if edge improves objective **then**

      replace p by other end

    end if

  end for

end loop

Maximize $8x_1 + 10x_2 + 7x_3$

Subject to constraints:

- $x_1 + 3x_2 + 2x_3 \leq 10$
- $x_1 + 5x_2 + x_3 \leq 8$
- $x_1, x_2, x_3 \geq 0$

Simplex Method

Standard form

$$\left[\begin{array}{ccc|c} a_{11}x_1 & a_{12}x_2 \ldots & a_{1n}x_n & b_1 \\ & & & b_2 \\ a_{m1}x_1 & & a_{mn}x_n & \vdots \end{array}\right]$$

Maximize:

$Z = 4x_1 + 6x_2$ // objective function

subject to:

$$\left.\begin{aligned} -x_1 + x_2 &\le 11 \\ x_1 + x_2 &\le 27 \\ 2x_1 + 5x_2 &\le 90 \end{aligned}\right\} \longrightarrow \left\{\begin{aligned} -x_1 + x_2 + S_1 &= 11 \\ x_1 + x_2 + S_2 &= 27 \\ 2x_1 + 5x_2 + S_3 &= 90 \end{aligned}\right.$$

slack variables

$$\begin{array}{cc|cccc} x_1 & x_2 & S_1 & S_2 & S_3 & b \\ \hline -1 & 1 & 1 & 0 & 0 & 11 \\ 1 & 1 & 0 & 1 & 0 & 27 \\ 2 & 5 & 0 & 0 & 1 & 90 \\ \hline -4 & \boxed{-6} & 0 & 0 & 0 & 0 \end{array}$$

a_{ij}

$11/1 = \boxed{11}$

$27/1 = 27$

$90/5 = 18$

ratio b_i/a_{ij}

least non-negative $\underline{a_{ij} > 0}$

$(0, 0, 11, 27, 90)$

Simplex Method

Standard form

Maximize:

$Z = 4x_1 + 6x_2$ // jective function

subject to:

$-x_1 + x_2 \leq 11$

$x_1 + x_2 \leq 27$

$2x_1 + 5x_2 \leq 90$

slack variables

$$\begin{cases} -x_1 + x_2 + S_1 = 11 \\ x_1 + x_2 + S_2 = 27 \\ 2x_1 + 5x_2 - S_3 = 90 \end{cases}$$

$$\begin{array}{cccccc} x_1 & x_2 & S_1 & S_2 & S_3 & b \\ \end{array}$$

$$\left[\begin{array}{cc|cccc} -1 & 1 & 1 & 0 & 0 & 11 \\ 1 & 1 & 0 & 1 & 0 & 27 \\ 2 & 5 & 0 & 0 & 1 & 90 \\ \hline -4 & -6 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ -R_1 + R_2 \\ -5R_1 + R_3 \\ 6R_1 + R_4 \end{array} \Rightarrow$$

$$\begin{array}{cccccc} x_1 & x_2 & S_1 & S_2 & S_3 & b \\ \end{array}$$

$$\left[\begin{array}{cccccc} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & \boxed{66} \end{array}\right] \to Z$$

$$\begin{cases} x_1 = 0 \quad x_2 = 11 \quad S_1 = 0 \\ S_2 = 16 \quad S_3 = 35 \end{cases}$$

✓

$4(0) + 6(11) = 66$

Simplex Method

$$\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & b \end{array}$$

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix} \begin{array}{c} X \\ 16/2 \\ 35/7 \\ \end{array}$$

$$\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & b \end{array}$$

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix} \quad \tfrac{1}{7}R_3$$

$$\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & b \end{array}$$

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 1 & 0 & -5/7 & 0 & 1/7 & 5 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix}$$

Simplex Method

$$\begin{array}{cccccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & b & \\
\end{array}$$

$$\left[\begin{array}{cccccc}
-1 & 1 & 1 & 0 & 0 & 11 \\
2 & 0 & -1 & 1 & 0 & 16 \\
1 & 0 & -5/7 & 0 & 1/7 & 5 \\
-10 & 0 & 6 & 0 & 0 & 66
\end{array}\right]
\begin{array}{l}
R_1 + R_3 \\
R_2 - 2R_3 \\
\checkmark \\
R_4 + 10R_3
\end{array}$$

$$\begin{array}{cccccc}
x_1 & x_2 & s_1 & s_2 & s_3 & b
\end{array}$$

$$\left[\begin{array}{cccccc}
0 & 1 & 2/7 & 0 & 1/7 & 16 \\
0 & 0 & 3/7 & 1 & 2/7 & 6 \\
1 & 0 & -5/7 & 0 & 1/7 & 5 \\
0 & 0 & -8/7 & 0 & 10/7 & 116
\end{array}\right]$$

$$\begin{array}{cccccc}
x_1 & x_2 & s_1 & s_2 & s_3 & b
\end{array}$$

$$\left[\begin{array}{cccccc}
0 & 1 & 0 & -2/3 & 1/3 & 12 \\
0 & 0 & 1 & 7/3 & -2/3 & 14 \\
1 & 0 & 0 & 5/3 & -1/3 & 15 \\
0 & 0 & 0 & 8/3 & 2/3 & \boxed{132}
\end{array}\right]$$

$$\left.\begin{array}{l} x_1 = 15 \\ x_2 = 12 \end{array}\right\} \quad 4(15) + 6(12) = 132$$

$s_1 = 14$

$s_2 = 0$

$s_3 = 0$

max Z

Maximize:

$$Z = 4x_1 + 6x_2$$

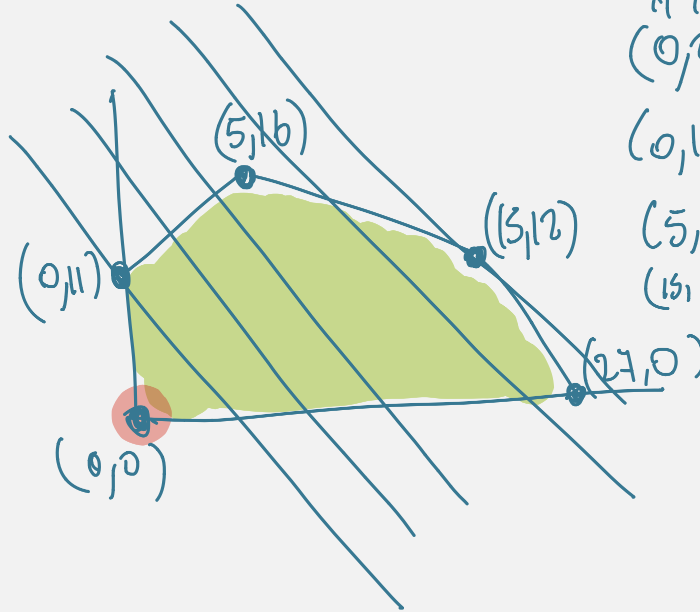
subject to:

$$-x_1 + x_2 \le 11$$

$$x_1 + x_2 \le 27$$

$$2x_1 + 5x_2 \le 90$$

$$x_1 \geq 0, \quad x_2 \geq 0$$



x_1, x_2	
(0,0)	$z = 0$ ✓
(0,11)	$z = 66$ ✓
(5,16)	$z = 116$ ✓
(15,12)	$z = 132$ ✓

Dual of a minimization problem: (Simplex)

$$W = 0.12x_1 + 0.15x_2$$

subject to:

$$60x_1 + 60x_2 \geq 300, \quad x, y \geq 0$$

$$12x_1 + 6x_2 \geq 36$$

$$10x_1 + 30x_2 \geq 90$$

$$\begin{bmatrix} 60 & 60 & 300 \\ 12 & 6 & 36 \\ 10 & 30 & 90 \\ 0.12 & 0.15 & 0 \end{bmatrix} \xrightarrow{\text{transpose}} \begin{array}{cccc} y_1 & y_2 & y_3 & b' \end{array}$$

$$\begin{bmatrix} 60 & 12 & 10 & 0.12 \\ 60 & 6 & 30 & 0.15 \\ 300 & 36 & 90 & 0 \end{bmatrix}$$

$$Z = 300y_1 + 36y_2 + 90y_3$$

→ Dual objective function (maximize)

subject to:

standard form

$$60y_1 + 12y_2 + 10y_3 \leq 0.12$$

$$60y_1 + 6y_2 + 30y_3 \leq 0.15$$

$$y_1, y_2, y_3 \geq 0$$

Exercises

Exercises

Maximize $40x_1 + 88x_2$

Subject to constraints:

- $2x_1 + 8x_2 \leq 60$
- $5x_1 + 2x_2 \leq 60$
- $x_1, x_2 \geq 0$

Minimize $5x_1 - 20x_2$

Subject to constraints:

- $-2x_1 + 10x_2 \leq 5$
- $2x_1 + 5x_2 \leq 10$
- $x_1, x_2 \geq 0$

- Factory optimization problem
- Healthy Diet problem
- Vertex cover problem
- Max flow network problem

Questions?

References i

Cormen, T. H., Leiserson, C. E., Rivest, R. L., and Stein, C. (2009).
Introduction to algorithms.
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Cont’d