Practice of Analysis of Algorithms

1) Solve the following expressions using Master Method, if possible:

a) $T(n) = 4T(n/3) + n$ $\Theta(n^{\log_3 4})$

b) $T(n) = 4T(n/2) + n^2$ $\Theta(n^2 \log n)$

c) $T(n) = 3T(\sqrt{n}) + \log n$ s.m.t.d.a.

d) $T(n) = 3T(n/5) + \log^2 n$ s.m.t.d.a.

e) $T(n) = T(4n/5) + T(n/5) + \Theta(n)$ smtda.

f) $T(n) = T(n/2) + \sqrt{n}$ $\Theta(n^{1/2})$

g) $T(n) = 7T(\lceil n/2 \rceil) + \Theta(n^2)$ $\approx \Theta(n^{\log_2 7})$

h) $T(n) = 7T(n/2) + n^2$ $\Theta(n^{\log_2 7})$

i) $T(n) = 2T(n/3) + n^3$ $\Theta(n^3)$

j) $T(n) = 3T(n/2) + n\log n$ $\Theta(n^{\log_2 3})$

k) $T(n) = T(\sqrt{n} + n/2) + \sqrt{6046}$ s.m.t.d.a.

l) $T(n) = \sqrt{n}\, T(\sqrt{n}) + 100n$ s.m.t.d.a.

m) $T(n) = 3T(\lceil n/2 \rceil) + \Theta(n)$ $\approx \Theta(n^{\log_2 3})$

n) $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \Theta(n \log n)$

smtda $\approx \Theta(n \log^2 n)$ ↳ expansion

smtda = standard Master Method doesn't apply

2) $T(n) = T(\lceil n/2 \rceil) + 1$, $T(n) = \Theta(\log n)$

$T(k) \leq c \log k$

$$\begin{aligned}
T(n) &\leq c\log(\lceil n/2 \rceil) + 1 \\
&\leq c\log(n/2 + 1) + 1 \\
&\leq c\log(n+2/2) + 1 \\
&\leq c\log(n+2) - c\log 2 + 1 \\
&\leq c\log(n+2) - c + 1
\end{aligned}$$

↳ doesn't close the induction

$T(k) \leq c\log(k-2)$

$$\begin{aligned}
T(n) &\leq c\log(\lceil n/2 \rceil - 2) + 1 \\
&\leq c\log(n/2 + 1 - 2) + 1 \\
&\leq c\log(n-2/2) + 1 \\
&\leq c\log(n-2) - c\log 2 + 1 \\
&\leq \underline{c\log(n-2)} - \underbrace{(c-1)}_{\geq 0,\ c \geq 1}
\end{aligned}$$

4) $T(n) = T(n-1) + n$

$T(n) = \Theta(n^2)$

$T(k) \leq ck^2, k < n$

$$\begin{aligned}
T(n) &\leq c(n-1)^2 + n \\
&\leq c[n^2 - 2n + 1] + n \\
&\leq cn^2 - 2cn + c + n \\
&\leq \underline{cn^2} - \underbrace{[2cn - c - n]}_{\geq 0 \Rightarrow c \geq 1}
\end{aligned}$$

3) HINT:

- Approximate a lower / upper bound or
- solve a variation like $T(n-1) + \log n$

5) $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$, $T(n) = O(n\log n)$

$T(k) \leq ck\log k$, $k < n$

$$\begin{aligned}
T(n) &\leq 2c(\lfloor n/2 \rfloor + 17)\log(\lfloor n/2 \rfloor + 17) + n \\
&\leq 2c(n/2 + 17)\log(n/2 + 17) + n \\
&\leq \cancel{2}c\left(\frac{n+34}{\cancel{2}}\right)\log(n/2 + 17) + n \\
&\leq (cn + 34c)\log(n/2 + n/4) + n, \quad \forall\, n/4 \geq 17 \\
&\leq (cn + 34c)\log(3n/4) + n \\
&\leq (cn + 34c)\log(n/4/3) + n \\
&\leq cn\log n - cn\log 4/3 + 34c\log n - 34c\log 4/3 + n \\
&\leq \underline{cn\log n}, \text{ since } \log n = O(n),\ 34c\log n \leq kn
\end{aligned}$$

$n(1 - c\log 4/3) + kn$

$n(k + 1 - c\log 4/3)$ $\rightarrow$ $c \geq (k+1)/\log 4/3$

~~2)~~ Prove that $T(n) = T(\lceil n/2 \rceil) + 1$ is $O(\log n)$

3) Prove that $T(n) = T(n-2) + \log n$ is $\Theta(n\log n)$:)

~~4)~~ Prove that $T(n) = T(n-1) + n$ is $\Theta(n^2)$

~~5)~~ Prove that $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$ is $O(n\log n)$

6) Prove that $T(n) = T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n)$ is $\Theta(n\log n)$:)

7) Reduce the following expression to show that:

$$\sum_{k=1}^{n-1} k\log k \leq \frac{1}{2}n^2\log n - \frac{1}{8}n^2$$

$\rightarrow$ HINT: used in Quicksort analysis

(Extra points),

$$\sum_{k=1}^{n-1} k\log k = \sum_{k=1}^{\lceil n/2 \rceil - 1} k\log k + \sum_{k=\lceil n/2 \rceil}^{n-1} k\log k$$

$$= \log(n-1)\sum_{k=1}^{\lceil n/2 \rceil - 1} k + \log(n)\sum_{k=\lceil n/2 \rceil}^{n-1} k \quad \ldots$$

8) Count the number of times each line is being executed and write the total execution time:

a) for (i=1; i< n; ++i) $\Theta(n)$

<statement>

b) for (i=1; i<n; i=i+2) $\Theta(n)$

<statement>

c) for (i=1; i<n; ++i)

for (j=1; j<n; ++j) $\Theta(n^3)$

<statement>

for (k=1; k<n; ++k)

<statement>

d) for (i=1; i<n; i=i*2) $\Theta(\log n)$

<statement>

e) for(i=~~1~~ n; i>=1; i=i/2) $\Theta(\log n)$

<statement>

f) p=0

for (i=1; ~~i~~ p <= n; ++i) $\Theta(\sqrt{n})$

p = p+i

<statement>

9) Write the recursion tree for the following expressions and find the intuition of the execution time:

a) $T(n) = T(n/2) + T(n/4) + \Theta(n^2)$

b) $T(n) = 2T(n/2) + n^2$

c) $T(n) = 4T(n/2) + n^3$

10) Show if the following conjectures are always, never or sometimes true.

a) $f(n) \neq O(g(n))$ and $g(n) \neq O(f(n))$ sometimes true

b) $f(n) = \Omega(g(n))$ and $f(n) = o(g(n))$ never true

c) $f(n) = \Theta(f(n/2))$ never true

d) $f(n) = O(g(n))$ implies $g(n) = \Omega(f(n))$ always true

Apply definition by Limits.

$\lim \frac{T(n)}{f(n)} \begin{cases} = 0 & , O, o \\ < c > 0 & , \Omega, O, \Theta \\ \to \infty & , \Omega, \omega \end{cases}$

9.a) $T(n) = T(n/2) + T(n/4) + \Theta(n^2)$

n^2 — n^2

$(n/2)^2 \quad (n/4)^2$ — $5n^2/16$

$(n/4)^2 \quad (n/4)^2 \quad (n/8)^2 \quad (n/16)^2$ — $25n^2/16$

$\ldots$

$T(n) \doteq \Theta(n^2)$

9.b) $T(n) = 2T(n/2) + n^2$

n^2 — n^2

$(n/2)^2 \quad (n/2)^2$ — $n^2/2$

$(n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2$ — $n^2/4$

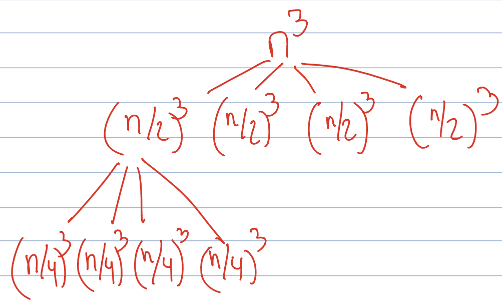
$\ldots$

$T(n) = \Theta(n^2)$

Good luck!

B

9.c) $T(n) = 4T(n/2) + n^3$



n^3

$n^3/2$

$n^3/4$

$T(n) = \Theta(n^3)$