Joint Distribution

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Learning goals

1

Understand what is meant by a joint PMF, PDF and CDF of two random variables.

2

Be able to compute probabilities and marginals from a joint pmf or pdf.

3

Be able to test whether two random variables are independent.

Outline

Introduction

Joint Distribution

- Discrete case
- Continues case
- Joint CDF
- Properties of joint CDF
- Marginal distribution
- Marginal PMF
- Marginal PDF
- Marginal CDF

Independence

Conditional

Introduction

- In real life, people often interested in measuring the relationship of 2 or more random variables (R.V.'s) at the same time.
- Eg:
  - ➢ measure the height and weight of giraffes
  - ➢ measure the IQ and birthweight of children
  - ➢ measure the frequency of exercise and the rate of heart disease in adults
  - ➢ measure the level of air pollution and rate of respiratory illness in cities
  - ➢ measure the number of Facebook friends and the age of Facebook members

Introduction

In such situations the R.V.'s have a **joint distribution** that allows us to compute probabilities of events involving both variables and understand the relationship between the variables. This is simplest when the variables are **independent**. When they are not, we use **covariance** and **correlation** as measures of the nature of the dependence between them.

Joint Distribution

X and Y are jointly distributed random variables.

- Discrete: Probability mass function (PMF)

$$p(x_i, y_j)$$

- Continues: Probability density function (PDF)

$$f(x, y)$$

- Both: Cumulative distribution function (CDF)

$$F(x, y) = P(X \leq x, Y \leq y)$$

Joint Distribution
a. Discrete case

Suppose X and Y are two discrete random variables.

- X takes values $\{x_1, x_2, \ldots, x_n\}$
- Y takes values $\{y_1, y_2, \ldots, y_m\}$

The ordered pair (X, Y) take values in the product $\{(x_1, y_1), (x_1, y_2), \ldots (x_n, y_m)\}$

The **joint probability mass function** (**joint PMF**) of X and Y is the function $p(x_i, y_j)$ giving the probability of the joint outcome $X = x_i, Y = y_j$.

$$P_{XY}(x, y) = P(X = x, Y = y)$$

Joint Distribution
a. Discrete case

Joint probability table

$X \backslash Y$	y_1	y_2	$\cdots$	y_j	$\cdots$	y_m
x_1	$p(x_1, y_1)$	$p(x_1, y_2)$	$\cdots$	$p(x_1, y_j)$	$\cdots$	$p(x_1, y_m)$
x_2	$p(x_2, y_1)$	$p(x_2, y_2)$	$\cdots$	$p(x_2, y_j)$	$\cdots$	$p(x_2, y_m)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_i	$p(x_i, y_1)$	$p(x_i, y_2)$	$\cdots$	$p(x_i, y_j)$	$\cdots$	$p(x_i, y_m)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
x_n	$p(x_n, y_1)$	$p(x_n, y_2)$	$\cdots$	$p(x_n, y_j)$	$\cdots$	$p(x_n, y_m)$

Joint Distribution
a. Discrete case

A joint probability mass function must satisfy two **properties**:

1. $0 \le p(x_i, y_j) \le 1$
2. The total probability is 1. We express it as a *double sum*:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) = 1$$

Joint Distribution
a. Discrete case

Example 1.

- Roll 2 dice: X = the value on 1st die, Y = the value on 2nd die
- X takes values in 1, 2, … , 6,
- Y takes values in 1, 2, … , 6.
- Joint probability table →

$X\backslash Y$	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

PMF: $p(x_i, y_j) = 1/36$ for any i and j between 1 and 6.

Joint Distribution
a. Discrete case

Example 2

- Roll two dice. Let X be the value on the first die and let T be the total on both dice.
- Here is the joint probability table:

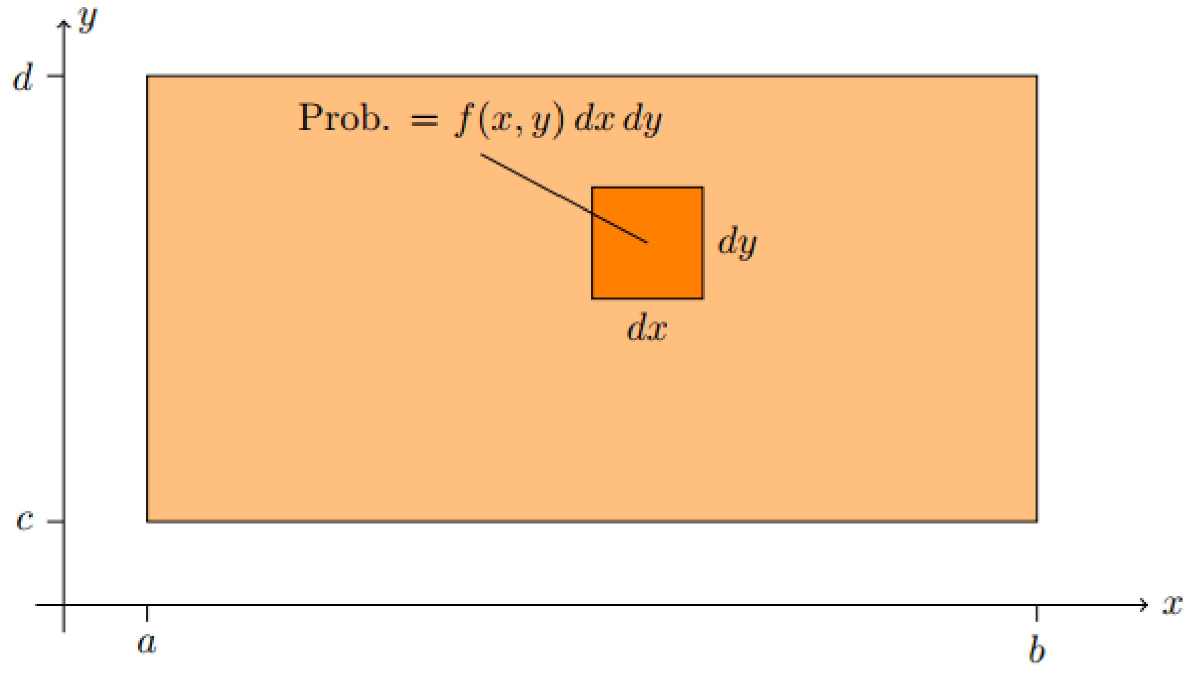
$X\backslash T$	2	3	4	5	6	7	8	9	10	11	12
1	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	0
2	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0
3	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0
4	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0
5	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0
6	0	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36

Joint Distribution
b. Continues case

- The continuous case is essentially the same as the discrete case:
  a) Replace discrete sets of values by continuous intervals,
  b) Replace the joint probability mass function by a joint probability density function,
  c) Replace the sums by integrals.

Joint Distribution
b. Continues case

- If X takes values in [a, b] and Y takes values in [c, d] then the pair (X, Y) takes values in the product [a, b] × [c, d].
- The joint probability density function (joint PDF) of X and Y is a function f(x, y) giving the probability density at (x, y).
- That is, the probability that (X, Y) is in a small rectangle of width dx and height dy around (x, y) is $f(x, y)\ dx\ dy$.



Joint Distribution
b. Continues case

A joint probability density function must satisfy two **properties**:

1. $0 \leq f(x, y)$
2. The total probability is 1. We express it as a *double integral*:

$$\int_{c}^{d} \int_{a}^{b} f(x, y)\, dx\, dy = 1$$

Note: as with the PDF of a single random variable, the joint PDF $f(x, y)$ can take values greater than 1. It is a probability *density, not a probability*

Joint Distribution
c. Joint CDF

- Suppose X and Y are jointly-distributed random variables.
- We will use the notation "X ≤ x, Y ≤ y" to mean the event "X ≤ x and Y ≤ y".
- The joint cumulative distribution function (joint CDF) is defined as

$$F(x, y) = P(X \leq x, Y \leq y)$$

Joint Distribution
c. Joint CDF

- **Continues case**

If X and Y are continuous random variables with joint density $f(x, y)$ over the range [a, b] × [c, d] then the joint CDF is given by the *double integral*

$$F(x,y) = \int_c^y \int_a^x f(u,v)\, du\, dv$$

To recover the joint PDF, we differentiate the joint CDF. Because there are two variables we need to use partial derivatives:

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}(x,y)$$

Joint Distribution
c. Joint CDF

- **Discrete case**

If X and Y are discrete random variables with joint PMF $p(x_i, y_j)$ then the joint CDF gives by the *double sum*

$$F(x, y) = \sum_{x_i \leq x} \sum_{y_j \leq y} p(x_i, y_j)$$

Joint Distribution
d. Properties of Joint CDF

The joint CDF F(x, y) of X and Y must satisfy several properties:

1. F(x, y) is non-decreasing: i.e. if x or y increase then F(x, y) must stay constant or increase.

2. F(x, y) = 0 at the lower-left of the joint range.

If the lower left is $(-\infty, -\infty)$ then this means $\lim_{(x,y)\to(-\infty,-\infty)} F(x, y) = 0.$

3. F(x; y) = 1 at the upper-right of the joint range.

If the upper-right is (∞, ∞) then this means $\lim_{(x,y)\to(\infty,\infty)} F(x, y) = 1.$

Joint Distribution
e. Marginal distributions

When X and Y are jointly-distributed random variables, we may want to consider **only one of them**, say X.

In that case we need to find the PMF (or PDF or CDF) of X without Y .

This is called a **marginal** PMF (or PDF or CDF).

Joint Distribution
f. Marginal PMF

- Example: rolled two dice and let X be the value on the first die and T be the total on both dice. Compute the marginal PMF of X and of T .
- Solution: In the table each row represents a single value of X. So the event “X = 3” is the third row of the table. To find P (X = 3) we simply have to sum up the probabilities in this row. We put the sum in the right-hand margin of the table. Likewise P (T = 5) is just the sum of the column with T = 5. We put the sum in the bottom margin of the table.

$X \backslash T$	2	3	4	5	6	7	8	9	10	11	12	$p(x_i)$
1	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	0	1/6
2	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	1/6
3	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	1/6
4	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	1/6
5	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	1/6
6	0	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	1/6
$p(t_j)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36	1

Computing the marginal probabilities $P(X = 3) = 1/6$ and $P(T = 5) = 4/36$.

Joint Distribution
g. Marginal PDF

- For a continous joint density $f(x, y)$ with range [a, b] × [c, d], the marginal PDF's are:

$$f_X(x) = \int_c^d f(x,y)dy$$

$$f_Y(y) = \int_a^b f(x,y)dx$$

Joint Distribution
h. Marginal CDF

- Finding the marginal CDF from the joint CDF is easy. If X and Y jointly take values on [a, b] × [c, d] then

$$F_X(x) = F(x, d)$$
$$F_Y(y) = F(b, y)$$

- If d is ∞ then this becomes a limit $F_X(x) = \lim_{y \to \infty} F(x, y)$. Likewise, for $F_Y(y)$.

Independence

- We are now ready to give a careful mathematical definition of independence. Of course, it will simply capture the notion of independence we have been using up to now. But it is nice to finally have a solid definition that can support complicated probabilistic and statistical investigations.
- Recall that events A and B are independent if

$$P(A \cap B) = P(A)P(B)$$

Independence

- Jointly-distributed random variables X and Y are independent if their joint CDF is the product of the marginal CDF’s:

$$F(X,Y) = F_X(x)F_Y(y)$$

- For discrete variables this is equivalent to the joint pmf being the product of the marginal PMF’s:

$$p(x_i, y_j) = p_X(x_i)p_Y(y_j)$$

- For continous variables this is equivalent to the joint pdf being the product of the marginal PDF’s:

$$f(x,y) = f_X(x)f_Y(y)$$

Conditional

- Recall for events A dan B

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- We now apply this condition to random variables (R.V.'s) X and Y.
- Given R.V.'s X and Y with joint probability $f_{XY}(x, y)$, the conditional probability distribution of Y given $X = x$ is

  $f_{Y|x}(y) = \frac{f_{XY}(x,y)}{f_X(x)}$ for $f_X(x) > 0$

- The conditional probability can be stated as the joint probability over the marginal probability.

Conditional

Example: Measurements for the length and width of a rectangular plastic covers for CDs are rounded to the nearest mm.

Let X denote the length and Y denote the width.

The possible values of X are 129, 130, and 131 mm. The possible values of Y are 15 and 16 mm.

We show the probability for each pair in the following table:

		X = Length		
		129	**130**	**131**
Y = Width	**15**	0.12	0.42	0.06
	16	0.08	0.28	0.04

The sum of all the probabilities is 1.0.

Conditional

Question:

a) Find the probability that a CD cover has a length of 130mm GIVEN the width is 15mm.

b) Find the conditional distribution of X given Y = 15.

		X = Length			Row total
		129	130	131	
Y = Width	15	0.12	0.42	0.06	0.60
	16	0.08	0.28	0.04	0.40
Column total		0.20	0.70	0.10	1

Conditional

Answer (Contd.)

a) $P(X = 130|Y = 15) = \frac{P(X=130, Y=15)}{P(Y=15)} = \frac{0.42}{0.60} = 0.70$

b) The conditional distribution of X given Y =15, or $f_{X|Y=15}(x)$:

$$P(X = 129|Y = 15) = 0.12/0.60 = 0.20$$
$$P(X = 130|Y = 15) = 0.42/0.60 = 0.70$$
$$P(X = 131|Y = 15) = 0.06/0.60 = 0.10$$

x	**129**	**130**	**131**
$f_{X\|Y=15}(x)$	0.20	0.70	0.10

The sum of these probabilities is 1, and this is a legitimate probability distribution

Conditional

A conditional probability distribution $f_{Y|x}(y)$ has the following properties are satisfied:

- For **discrete** R.V.'s (X, Y)

1. $f_{Y|x}(y) \geq 0$

2. $\sum_y f_{Y|x}(y) = 1$

3. $f_{Y|x}(y) = P(Y = y|X = x)$

Conditional

A conditional probability distribution $f_{Y|x}(y)$ has the following properties are satisfied:

- For **continues** R.V.'s (X, Y)

1. $f_{Y|x}(y) \geq 0$
2. $\int_{-\infty}^{\infty} f_{Y|x}(y)\, dy = 1$
3. $P(Y \in B|X = x) = \int_B \; f_{Y|x}(y)\, dy$ for any set B in the range of Y

Exercise 1

Consider two random variables X and Y with joint PMF given in below Table.

	Y = 0	Y = 1	Y = 2
X = 0	1/6	1/4	1/8
X = 1	1/8	1/6	1/6

a. Find $P(X = 0, Y \leq 1)$
b. Find the marginal PMF's of X and Y
c. Find $P(Y = 1 \mid X = 0)$
d. Are X and Y independent?

Exercise 2

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} x + cy^2 & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the constant c
b) Find $P\left(0 \le X \le \frac{1}{2}, 0 \le Y \le \frac{1}{2}\right)$
c) Find the marginal PDF's $f_X(x)$ and $f_Y(y)$
d) Find the CDF $F(x, y)$

Exercise 3

From the measurements for the length and width of a rectangular plastic covers for CD. Find the $E(Y|X = 129)$ and $\mathrm{Var}(Y|X = 129)$

Exercise 4

The joint CDF

$F(x, y) = \frac{1}{2}(x^3 y + x y^3)$ on $[0,1] \times [0,1]$

Find the marginal CDF's and use $F_X(x)$ to compute $P(X < 0.5)$

Thank You