# Transformation of Two or More Random Variables

TSD 2023/2024

# Transformations of Random Variable Techniques

CDF

Transformation

MGF

# Review

1. **CDF Technique**

The CDF technique is used to find the distribution of a new random variable by deriving the CDF from another random variable.

2. **Transformation technique**

The transformation technique to find the distribution of a new random variable that can be divided into two: the one-to-one transformation and the non one-to-one transformation.

3. **MGF Technique**

The MGF technique uses a moment generating function to find the distribution of a new random variable.

# CDF Method

Given a random variable $X$ with density $f_X$, and a measurable function $g$, we are often interested in the distribution (CDF, PDF, or PMF) of the random variable $\mathrm{Y} = g(X)$.

For the case of a discrete random variable $X$, this is straightforward:

$$P_Y(y) = P\{Y = y\} = \sum_{x|g(x)=y} P_X(x)$$

# CDF Method

- In general, we have

$$P\{Y = y\} = P\{g(X) = y\} = P\{x: g(x) = y\} = \sum_{x: g(x)=y} P\{X = x\}$$

- An analogous formula can be used for functions of two variables, that is, random variables of the form $Z = g(X, Y)$, where the distribution of $Z$ is expressed in terms of the distribution of $X$ and $Y$.

# CDF Method Continues case

**Theorem**:

Let $X = X_1, X_2, \cdots, X_k$ is a vector k-dimensionality of continuous random variable with joint pdf $f(x_1, x_2, \cdots, x_k)$. If $Y$ is a function from $X$, so $Y = g(X)$, then

$$F_Y(y) = P[g(X) \leq y]$$

$$F_Y(y) = \int \cdots \int_{x:g(x) \leq y} f(x_1, x_2, \cdots, x_k)\, dx_1 \cdots dx_k$$

# Transformation One to One Method (Discrete case)

**Theorem**

Suppose $X$ is a discrete random variable with density $f_X(x)$. If $\mathrm{Y} = g(X)$ is an one on one transformation, then the PDF of $Y$ is:

$$f_Y(y) = f_X\left(g^{-1}(y)\right)$$

Note:
The transformation of discrete random variables is carried out as in continuous random variables, but for discrete **Jacobian variables** it is **always equal to one** $(\boldsymbol{J} = \mathbf{1})$.

# Transformation One to One Method (Continuous case)

**Theorem**

If $g$ is a continuous differentiable function with inverse $g^{-1}$ and $X$ is a continuous random variable with density $f_X$, then the density of $\mathrm{Y} = g(X)$ is

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|g^{-1\prime}(y)\right|$$

# Transformation One to One Method (Continuous case)

We begin with the simplest case, when $g$ is a strictly **monotone function**.

- If $g$ is ***increasing***, we write for the CDF of $Y$ :

$$F_Y(y) = P\{Y \leq y\} = P\{g(X) \leq y\} = P\{X \leq g^{-1}(y)\} = F_X\left(g^{-1}(y)\right)$$

The density is obtained by differentiating CDF, and consequently

$$f_Y(y) = \frac{d}{dy} F_X\left(g^{-1}(y)\right) = f_X\left(g^{-1}(y)\right) g^{-1\prime}(y)$$

- If $g$ is monotonically ***decreasing***, $g^{-1}$ must be a decreasing function too:

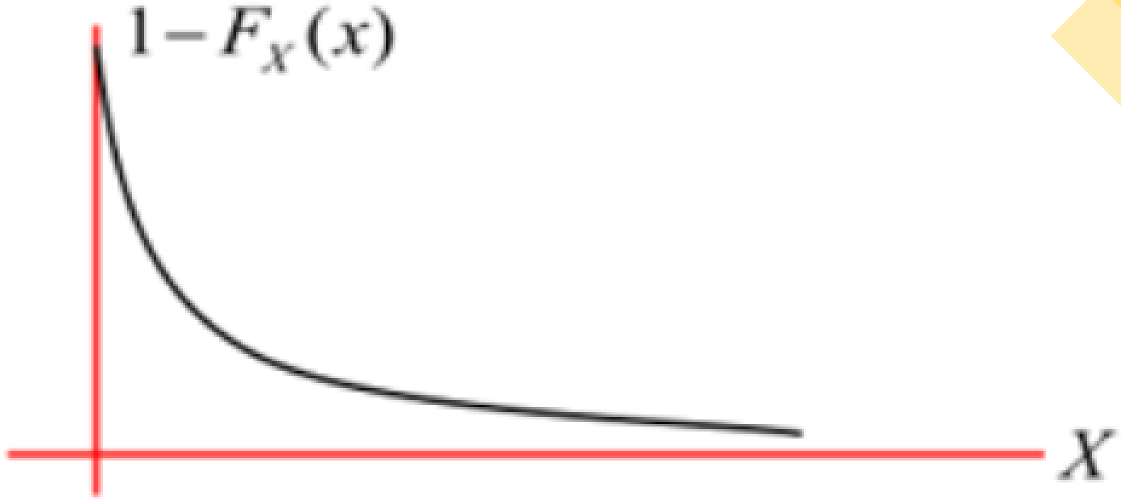
$$F_Y(y) = P\{g(X) \leq y\} = P\{X \geq g^{-1}(y)\} = 1 - F_X\left(g^{-1}(y)\right)$$

Therefore,

$$f_Y(y) = -f_X\left(g^{-1}(y)\right) g^{-1\prime}(y)$$

monotone increasing function


monotone decreasing function



# Sums of Random Variables

**Theorem**

Suppose X and Y are independent r.v. Let W = X + Y. Then,

- $f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\, dx$ if continuous
- $p_W(w) = \sum_{all\ x} p_X(x) p_Y(w-x)$ if discrete

# Moment Generating Function (MGF) Technique

- Theorem 1:

If x is a random variable whose moment generating function (MGF) is $M_X(t)$ and $Y$ is a function of $X, Y = g(X)$ then the moment generation function (MGF) of the random variable $Y$ can be expressed as:

$$M_Y(t) = E(e^{tY})$$

- Theorem 2:

If $X_1, X_2, \cdots, X_n$ is an independent random variable with moment generating function (MGF) is $M_{X_i}(t)$, then the moment generating function (MGF) of $Y = \sum_{i=1}^{n} X_i$ can be determined as follows:

$$M_Y(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_n}(t)$$

- Theorem 3:

If $X_1, X_2, \cdots, X_n$ is a sample that comes from a population with the same probability density function (pdf) and moment generating function (MGF), $f(x)$ with $M_X(t)$, then:

$$M_Y(t) = [M(t)]^n$$

# Transformation of Two or More Random Variables

Suppose we know the joint probability density function for random variable $X_1$ and $X_2$ is $f_{X_1,X_2}(x_1, x_2)$.

If other random variables are defined, $Y_1$ and $Y_2$, where $Y_1 = g_1(x_1, x_2)$ and $Y_2 = g_2(x_1, x_2)$, then we want to know the joint probability density function for the random variables $Y_1$ and $Y_2$, $f_{Y_1,Y_2}(y_1, y_2)$.

# Theorem

Suppose we know the joint probability density function for the random variable $X_1$ and $X_2$ is $f_{X_1,X_2}(x_1, x_2)$ which is positive and continuous on the group $S \subseteq R^2$, and defined function $g_1, g_2: S \to R$ and $T$ is an image of $S$ as a one-to-one transformation of $(g_1, g_2)$.

Therefore, if $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ then the inverse, $x_1 = g_1^{-1}(y_1, y_2)$ and $x_2 = g_2^{-1}(y_1, y_2) \in T$.

Assume that for $(y_1, y_2) \in T$, $dx_1/dy_1$ and $dx_2/dy_2$ exist, continue, and not equal to 0.

Then the joint probability density function for the random variables $Y_1$ and $Y_2$ is:

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}\{g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)\} \cdot |J|, \qquad (y_1, y_2) \in T$$

# Joint probability density function for the random variables $Y_1$ and $Y_2$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}\{g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)\} \cdot |J|, \qquad (y_1, y_2) \in T$$

$$J = Jacobian = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix}$$

# Example 1

Suppose the continuous random variable $X$ has a distribution $U(0,1)$, while $X_1$ and $X_2$ are independent random variable examples of this distribution. If $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$ are defined, determine:

a) Joint density function for random variables $Y_1$ and $Y_2$, $f_{Y_1,Y_2}(y_1, y_2)$.

b) Marginal density function for random variables $Y_1$ and $Y_2$, $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$.

# Example 1 solution (a)

$X \sim U(0,1)$ and $X_1, X_2$ are independent and identical random sample of this distribution. Then the joint probability density function for $X_1$ and $X_2$ is:

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) = 1; 0 < x_1 < 1 \; and \; 0 < x_2 < 1$$

then it is defined that:

$$y_1 = g_1(x_1, x_2) = x_1 + x_2$$
$$y_2 = g_2(x_1, x_2) = x_1 - x_2$$

# Example 1 solution (a)

Through the substitution or elimination method from the above equation, the following equation will be obtained:

$$x_1 = g_1^{-1}(y_1, y_2) = (y_1 + y_2)/2$$

$$x_2 = g_2^{-1}(y_1, y_2) = (y_1 - y_2)/2$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

# Example 1 solution (a)

So, the common density for the random variables $Y_1$ and $Y_2$ is

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}\{g_1^{-1}(y_1,y_2), g_2^{-1}(y_1,y_2)\} \cdot |J|$$

$$f_{Y_1,Y_2}(y_1,y_2) = \{(y_1+y_2)/2\,,(y_1-y_2)/2\} \cdot \left|-\frac{1}{2}\right|$$

$$f_{Y_1,Y_2}(y_1,y_2) = (1) \cdot \frac{1}{2}$$

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2}, (y_1,y_2) \ \in T$$

# Example 1 solution (a)

determine the limit value for $y_1$ and $y_2$ that is $T$,

For $0 < x_1 < 1$

$$0 < x_1 < 1 \rightarrow 0 < (y_1 + y_2)/2 < 1 \rightarrow 0 < y_1 + y_2 < 2$$
$$0 < y_1 + y_2 \; and \; y_1 + y_2 < 2$$
$$y_2 > y_1 \; and \; y_2 < 2 - y_1$$

For $0 < x_2 < 1$

$$0 < x_2 < 1 \rightarrow 0 < (y_1 - y_2)/2 < 1 \rightarrow 0 < y_1 - y_2 < 2$$
$$0 < y_1 - y_2 \; and \; y_1 - y_2 < 2$$
$$y_2 < y_1 \; and \; y_2 > y_1 - 2$$

# Example 1 solution (b)

The marginal distribution for $y_1$ is

For $0 < y_1 \leq 1$

$$f_{Y_1}(y_1) = \int_{-y_1}^{y_1} f_{Y_1Y_2}(y_1, y_2)dy_2 = \int_{-y_1}^{y_1} \frac{1}{2}dy_2 = y_1$$

For $1 < y_1 < 2$

$$f_{Y_1}(y_1) = \int_{y_1-2}^{2-y_1} f_{Y_1Y_2}(y_1, y_2)dy_2 = \int_{y_1-2}^{2-y_1} \frac{1}{2}dy_2 = 2 - y_1$$

Therefore,

$$f_{Y_1}(y_1) = \begin{cases} y_1 & ; \quad 0 < y_1 \leq 1 \\ 2 - y_1 & ; \quad 1 < y_1 < 2 \\ 0 & ; \quad otherwise \end{cases}$$

# Example 1 solution (b)

The marginal distribution for $y_2$ is

For $-1 < y_2 \leq 0$

$$f_{Y_2}(y_2) = \int_{-y_2}^{y_2+2} f_{Y_1Y_2}(y_1, y_2)dy_1 = \int_{-y_2}^{y_2+2} \frac{1}{2}dy_1 = y_2 + 1$$

For $0 < y_2 < 1$

$$f_{Y_2}(y_2) = \int_{y_2}^{2-y_2} f_{Y_1Y_2}(y_1, y_2)dy_1 = \int_{y_2}^{2-y_2} \frac{1}{2}dy_1 = 1 - y_2$$

Therefore,

$$f_{Y_2}(y_2) = \begin{cases} y_2 + 1 & ; \quad -1 < y_2 \leq 0 \\ 1 - y_2 & ; \quad 0 < y_2 < 1 \\ 0 & ; \quad otherwise \end{cases}$$

# Example 2

Suppose the continuous random variable $X$ has the following probability density function

$$f_X(x) = e^{-x}, x \geq 0$$

$X_1$ and $X_2$ is an independent and identic random variable of this probability density function. Determine the probability density function of random variable $Y = X_1/(X_1 + X_2)$.

# Example 2 solution

$X_1$ and $X_2$ is an independent and identic random variable for this probability density function (pdf), then the joint pdf for $X_1$ and $X_2$:

$$f_{X_1,X_2}(x_1, x_2) = e^{-x_1}e^{-x_2} = e^{-(x_1+x_2)}; x_1 \geq 0 \; and \; x_2 \geq 0$$

# Example 2 solution

It is necessary to define another random variable for the transformation. occurs from two-dimensional space to two-dimensional space. Suppose $Z = X_1 + X_2$, so that we get a pair of transformations i.e. $y = x_1/(x_1 + x_2)$ and $z = x_1 + x_2$. This transformation is one-to-one for all functional areas.

# Example 2 solution

Through the substitution or elimination method from the above equation, the following equation will be obtained:

$$x_1 = yz$$
$$x_2 = (1 - y)z$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial z} \end{vmatrix} = \begin{vmatrix} z & y \\ -z & 1 - y \end{vmatrix} = z$$

# Example 2 solution

So, the joint density for the random variables $Y$ and $Z$ is

$$f_{Y,Z}(y,z) = f_{X_1,X_2}(x_1,x_2) \cdot |J|$$
$$= e^{-(yz+(1-y)z)} \cdot |J|$$
$$= ze^{-z}, (y,z) \in T$$

# Example 2 solution

Next determine the limit value for $y$ and $z$, namely $T$.

Pay attention, because $x_1 \geq 0$ and $x_2 \geq 0$, then

$$0 \leq y = x_1/(x_1 + x_2) \leq 1 \quad \rightarrow \quad 0 \leq y \leq 1$$

$$z = x_1 + x_2 \geq 0 \quad \rightarrow \quad z \geq 0$$

Therefore,

$$f_{Y,Z}(y, z) = ze^{-z} \quad , 0 \leq y \leq 1 \quad and\ z \geq 0$$

# Example 2 solution

Marginal distribution for random variables $Y = X_1/(X_1 + X_2)$:

$$\int_0^\infty ze^{-z}dz = 1$$

Hence, the pdf for random variable $Y = X_1/(X_1 + X_2)$:

$$f_Y(y) = \begin{cases} 1 & ; \quad 0 \leq y \leq 1 \\ 0 & ; \quad otherwise \end{cases}$$

# Exercise 1

Suppose the random variables $X$ and $Y$ are independent and have a Negative Exponential probability density function with $\lambda = 1$, and it is defined that the random variable $U = (X + Y)/2$ and $V = (X - Y)/2$. Determine:

a) Joint pdf $f_{U,V}(u, v)$.

b) Marginal pdf $f_U(u)$ and $f_V\,(v)$.

Note:

$$f_X\,(x) = \lambda e^{-\lambda x}, x > 0, \lambda > 0$$

# Exercise 2

Suppose the random variables $X$ and $Y$ are independent and have pdf $Normal(0,1)$and defined $U = (X+Y)$ and $V = (X-Y)$. Determine:

a) Joint pdf $f_{U,V}(u,v)$.

b) Marginal pdf $f_U(u)$ and $f_V\,(v)$.

# Thank you