Transformation of Two or More Random Variables

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Transformation of Two or More Random Variables

Suppose we know the joint probability density function for random variable X_1 and X_2 is $f_{X_1,X_2}(x_1, x_2)$.

If other random variables are defined, Y_1 and Y_2, where $Y_1 = g_1(x_1, x_2)$ and $Y_2 = g_2(x_1, x_2)$, then we want to know the joint probability density function for the random variables Y_1 and Y_2, $f_{Y_1,Y_2}(y_1, y_2)$.

Bivariate Discrete Random Variable Transformation

Let $X = (x_1, x_2, \cdots, x_n)$ be a discrete random variable with dimension n. Defined, $Y = (y_1, y_2, \cdots, y_k) = (g_1(x), g_2(x), \cdots, g_k(x))$. Then, the probability mass function (pmf) of Y is:

$$f_{Y_1,Y_2}(y_1, y_2) = P_{Y_1,Y_2}(y_1, y_2) = \sum_{x_1,x_2,\cdots,x_n} P_{x_1,x_2,\cdots,x_n}(x_1, x_2, \cdots, x_n)$$

Example 1

X_1 and X_2 are 2 random variables with pmf below in the table. Find the pmf for random variable $Y = |X_1 - X_2|$.

		$\mathbf{X_1}$	
		-1	**1**
$\mathbf{X_2}$	**0**	1/6	1/6
	0.5	1/3	1/3

Solution:

X	$f_{X_1,X_2}(x_1,x_2)$	$Y = \|X_1 - X_2\|$
(-1, 0)	1/6	1
(1, 0)	1/6	1
(-1, 0.5)	1/3	1.5
(1, 0.5)	1/3	0.5

Define the part of Y.

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \frac{2}{6} & ; \quad y = 1 \\ \frac{1}{3} & ; \quad y = 1.5 \\ \frac{1}{3} & ; \quad y = 0.5 \end{cases}$$

Bivariate Discrete Random Variable Transformation

Suppose X_1 and X_2 are 2 discrete random variables with density $f_{X_1,X_2}(x_1, x_2)$. If $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ are one on one transformation, then the joint pmf of Y_1, Y_2:

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} f_{X_1,X_2}\left(g_1{}^{-1}(y_1, y_2), g_2{}^{-1}(y_1, y_2)\right) & ; \quad y_1, y_2 \,\epsilon\, \mathcal{B} \\ 0 \quad ; \quad otherwise & \end{cases}$$

Theorem

Suppose we know the joint probability density function for the random variable X_1 and X_2 is $f_{X_1,X_2}(x_1,x_2)$ which is positive and continuous on the group $\mathcal{A} \subseteq R^2$, and defined function $g_1, g_2 : \mathcal{A} \rightarrow R$ and $\mathcal{B}$ is an image of $\mathcal{A}$ as a one-to-one transformation of (g_1, g_2).

Therefore, if $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ then the inverse, $x_1 = g_1^{-1}(y_1, y_2)$ and $x_2 = g_2^{-1}(y_1, y_2) \in \mathcal{B}$.

Assume that for $(y_1, y_2) \in \mathcal{B}$, dx_1/dy_1 and dx_2/dy_2 exist, continue, and not equal to 0.

Then the joint probability density function for the random variables Y_1 and Y_2 is:

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}\{g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)\} \cdot |J|, \qquad (y_1, y_2) \in \mathcal{B}$$

Joint probability density function for the random variables Y_1 and Y_2

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}\{g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)\} \cdot |J|, (y_1, y_2) \in \mathcal{B}$$

$$J = Jacobian = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

Example 2

Suppose the continuous random variables X_1 and X_2 has a distribution Exponential with $mean = 1$ and its probability density function: $f(x) = e^{-x}, I_{(0,\infty)}(x)$.

If $Y_1 = X_1$ and $Y_2 = X_1 + X_2$ are defined, determine:

a) Joint density function for random variables X_1 and X_2, $f_{X_1,X_2}(x_1, x_2)$.

b) Joint density function for random variables Y_1 and Y_2, $f_{Y_1,Y_2}(y_1, y_2)$.

c) Marginal density function for random variables Y_1, $f_{Y_1}(y_1)$.

Example 2 solution (a)

$$f_1(x_1) = e^{-x_1};\ 0 < x_1 < \infty$$
$$f_2(x_2) = e^{-x_2};\ 0 < x_2 < \infty$$
$$f_{X_1,X_2} = f_1(x_1)f_2(x_2) = e^{-x_1}e^{-x_2} = e^{-(x_1+x_2)}; 0 < x_1 < \infty, 0 < x_1 < \infty$$

Example 2 solution (b)

Transformation one-to-one:

$$y_1 = x_1 \rightarrow x_1 = y_1$$
$$y_2 = x_1 + x_2 \rightarrow x_2 = y_2 - x_1 = y_2 - y_1$$

Limit:

$$\mathcal{A} = \{(x_1, x_2); 0 < x_1 < \infty, 0 < x_2 < \infty\}$$
$$\mathcal{B} = \{(y_1, y_2); 0 < y_1 < y_2 < \infty\}$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}\{g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)\} \cdot |J|$$
$$= e^{-y_1} e^{-(y_2 - y_1)} = e^{-y_2}; \ (y_1, y_2) \ \in \mathcal{B}$$

Example 2 solution (c)

$$f_{Y_1}(y_1) = \int_{y_1}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_2$$

$$= \int_{y_1}^{\infty} e^{-y_2}\, dy_2$$

$$= -e^{-y_2}\Big|_{y_1}^{\infty}$$

$$= 0 + e^{-y_1}$$

$$= e^{-y_1}; 0 < y_1 < \infty$$

$$y_1 \sim exponential(1)$$

Example 3

Suppose the continuous random variable X has a distribution $U(0,1)$, while X_1 and X_2 are independent random variable examples of this distribution. If $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$ are defined, determine:

a) Joint density function for random variables Y_1 and Y_2, $f_{Y_1,Y_2}(y_1, y_2)$.

b) Marginal density function for random variables Y_1 and Y_2, $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$.

Example 3 solution (a)

$X \sim U(0,1)$ and X_1, X_2 are independent and identical random sample of this distribution. Then the joint probability density function for X_1 and X_2 is:

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) = 1; 0 < x_1 < 1 \; and \; 0 < x_2 < 1$$

then it is defined that:

$$y_1 = g_1(x_1, x_2) = x_1 + x_2$$
$$y_2 = g_2(x_1, x_2) = x_1 - x_2$$

Example 3 solution (a)

Through the substitution or elimination method from the above equation, the following equation will be obtained:

$$x_1 = g_1^{-1}(y_1, y_2) = (y_1 + y_2)/2$$

$$x_2 = g_2^{-1}(y_1, y_2) = (y_1 - y_2)/2$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

Example 3 solution (a)

So, the common density for the random variables Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}\{g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)\} \cdot |J|$$

$$f_{Y_1,Y_2}(y_1, y_2) = \{(y_1 + y_2)/2\,, (y_1 - y_2)/2\} \cdot \left|-\frac{1}{2}\right|$$

$$f_{Y_1,Y_2}(y_1, y_2) = (1) \cdot \frac{1}{2}$$

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2}, (y_1, y_2) \ \in T$$

Example 3 solution (a)

determine the limit value for y_1 and y_2 that is T,

For $0 < x_1 < 1$

$$0 < x_1 < 1 \rightarrow 0 < (y_1 + y_2)/2 < 1 \rightarrow 0 < y_1 + y_2 < 2$$
$$0 < y_1 + y_2 \; and \; y_1 + y_2 < 2$$
$$y_2 > y_1 \; and \; y_2 < 2 - y_1$$

For $0 < x_2 < 1$

$$0 < x_2 < 1 \rightarrow 0 < (y_1 - y_2)/2 < 1 \rightarrow 0 < y_1 - y_2 < 2$$
$$0 < y_1 - y_2 \; and \; y_1 - y_2 < 2$$
$$y_2 < y_1 \; and \; y_2 > y_1 - 2$$

Example 3 solution (b)

The marginal distribution for y_1 is

For $0 < y_1 \leq 1$

$$f_{Y_1}(y_1) = \int_{-y_1}^{y_1} f_{Y_1Y_2}(y_1, y_2)dy_2 = \int_{-y_1}^{y_1} \frac{1}{2} dy_2 = y_1$$

For $1 < y_1 < 2$

$$f_{Y_1}(y_1) = \int_{y_1-2}^{2-y_1} f_{Y_1Y_2}(y_1, y_2)dy_2 = \int_{y_1-2}^{2-y_1} \frac{1}{2} dy_2 = 2 - y_1$$

Therefore,

$$f_{Y_1}(y_1) = \begin{cases} y_1 & ; \quad 0 < y_1 \leq 1 \\ 2 - y_1 & ; \quad 1 < y_1 < 2 \\ 0 & ; \quad otherwise \end{cases}$$

Example 3 solution (b)

The marginal distribution for y_2 is

For $-1 < y_2 \leq 0$

$$f_{Y_2}(y_2) = \int_{-y_2}^{y_2+2} f_{Y_1Y_2}(y_1, y_2)dy_1 = \int_{-y_2}^{y_2+2} \frac{1}{2}dy_1 = y_2 + 1$$

For $0 < y_2 < 1$

$$f_{Y_2}(y_2) = \int_{y_2}^{2-y_2} f_{Y_1Y_2}(y_1, y_2)dy_1 = \int_{y_2}^{2-y_2} \frac{1}{2}dy_1 = 1 - y_2$$

Therefore,

$$f_{Y_2}(y_2) = \begin{cases} y_2 + 1 & ; \quad -1 < y_2 \leq 0 \\ 1 - y_2 & ; \quad 0 < y_2 < 1 \\ 0 & ; \quad otherwise \end{cases}$$

Example 4

Suppose the continuous random variable X has the following probability density function

$$f_X(x) = e^{-x}, x \geq 0$$

X_1 and X_2 is an independent and identic random variable of this probability density function. Determine the probability density function of random variable $Y = X_1/(X_1 + X_2)$.

Example 4 solution

X_1 and X_2 is an independent and identic random variable for this probability density function (pdf), then the joint pdf for X_1 and X_2:

$$f_{X_1,X_2}(x_1,x_2) = e^{-x_1}e^{-x_2} = e^{-(x_1+x_2)}; x_1 \geq 0\ and\ x_2 \geq 0$$

Example 4 solution

It is necessary to define another random variable for the transformation. occurs from two-dimensional space to two-dimensional space. Suppose $Z = X_1 + X_2$, so that we get a pair of transformations i.e. $y = x_1/(x_1 + x_2)$ and $z = x_1 + x_2$. This transformation is one-to-one for all functional areas.

Example 4 solution

Through the substitution or elimination method from the above equation, the following equation will be obtained:

$$x_1 = yz$$
$$x_2 = (1 - y)z$$

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial y} & \dfrac{\partial x_1}{\partial z} \\ \dfrac{\partial x_2}{\partial y} & \dfrac{\partial x_2}{\partial z} \end{vmatrix} = \begin{vmatrix} z & y \\ -z & 1 - y \end{vmatrix} = z$$

Example 4 solution

So, the joint density for the random variables Y and Z is

$$\begin{aligned} f_{Y,Z}(y,z) &= f_{X_1,X_2}(x_1,x_2) \cdot |J| \\ &= e^{-(yz+(1-y)z)} \cdot |J| \\ &= ze^{-z}, (y,z) \in T \end{aligned}$$

Example 4 solution

Next determine the limit value for y and z, namely T.

Pay attention, because $x_1 \geq 0$ and $x_2 \geq 0$, then

$$0 \leq y = x_1/(x_1 + x_2) \leq 1 \quad \rightarrow \quad 0 \leq y \leq 1$$

$$z = x_1 + x_2 \geq 0 \quad \rightarrow \quad z \geq 0$$

Therefore,

$$f_{Y,Z}(y,z) = ze^{-z} \quad , 0 \leq y \leq 1 \quad and\ z \geq 0$$

Example 4 solution

Marginal distribution for random variables $Y = X_1/(X_1 + X_2)$:

$$\int_0^\infty ze^{-z}dz = 1$$

Hence, the pdf for random variable $Y = X_1/(X_1 + X_2)$:

$$f_Y(y) = \begin{cases} 1 & ; \quad 0 \leq y \leq 1 \\ 0 & ; \quad otherwise \end{cases}$$

Exercise 1

Suppose the random variables X and Y are independent and have a Negative Exponential probability density function with $\lambda = 1$, and it is defined that the random variable $U = (X + Y)/2$ and $V = (X - Y)/2$. Determine:

a) Joint pdf $f_{U,V}(u, v)$.

b) Marginal pdf $f_U(u)$ and $f_V\,(v)$.

Note:

$$f_X\,(x) = \lambda e^{-\lambda x}, x > 0, \lambda > 0$$

Exercise 2

Suppose the random variables X and Y are independent and have pdf $Normal(0,1)$and defined $U = (X + Y)$ and $V = (X - Y)$. Determine:

a) Joint pdf $f_{U,V}(u, v)$.

b) Marginal pdf $f_U(u)$ and $f_V\,(v)$.

Thank you