Homework 6

DDL: May 11th

Problem 1 (10 pts)

Question: The partial derivative $f_x(x,y)$ of $f(x,y)$ with respect to x is obtained by holding y fixed and differentiating with respect to x. Similarly, $f_y(x,y)$ is found by holding x fixed and differentiating with respect to y. Formula (1) can be adapted to partial derivatives

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} \tag{1}$$

$$\begin{aligned} f_x(x,y) &= \frac{f(x+h,y) - f(x-h,y)}{2h} + O(h^2) \\ f_y(x,y) &= \frac{f(x,y+h) - f(x,y-h)}{2h} + O(h^2) \end{aligned} \tag{2}$$

(a) Let $f(x,y) = x^2 + y^2 + 2xy$. Calculate approximations to $f_x(3,4)$ and $f_y(3,4)$ using the formulas in (2) with $h = 0.1, 0.01, 0.0001$. Compare with the values obtained by differentiating $f(x,y)$ partially.

(b) Let $f(x,y) = x^2y^2/(x+y)$. Calculate approximations to $f_x(2,3)$ and $f_y(2,3)$ using the formulas in (2) with $h = 0.1, 0.01, 0.0001$. Compare with the values obtained by differentiating $f(x,y)$ partially.

Problem 2 (10 pts)

Question: Suppose that a table of the function $f(x_k)$ is computed where the values are rounded off to three decimal places and the inherent round-off error is $5 * 10^{-4}$. Also, assume that $|f^{(3)}(c)| \leq 1.5$ and $|f^{(5)}(c)| \leq 1.5$.

(a) Find the best step size h for formula $f'(x_0) \approx (y_1 - y_{-1})/2h$.

(b) Find the best step size h for formula $f'(x_0) \approx (-y_2 + 8y_1 - 8y_{-1} + y_{-2})/12h$.

Problem 3 (10 pts)

Question: Let $f(x)$ be given by the table following. The inherent round-off error has the bound $|e_k| \leq 5 * 10^{-6}$. Use the rounded values in your calculations.

x	$f(x) = \cos(x)$
1.000	0.54030
1.100	0.45360
1.198	0.36422
1.199	0.36329
1.200	0.36236
1.201	0.36143
1.202	0.36049
1.300	0.26750
1.400	0.16997

(a) Approximate $f'(1.2)$ using $f'(x_0) \approx (-y_2 + 8y_1 - 8y_{-1} + y_{-2})/12h$ with $h = 0.1$ and $h = 0.001$.

(b) Find the total error bound $E(f,h) = E_{round}(f,h) + E_{trunc}(f,h) = \frac{-e_2+8e_1-8e_{-1}+e_{-2}}{12h} + \frac{h^4 f^{(5)}(c)}{30}$ for the two cases in part (a)

Problem 4 (15 pts)

Question: Consider the table for $f(x) = ln(x)$ rounded to four decimal places.

x	$f(x) = \ln(x)$
4.90	1.5892
4.95	1.5994
5.00	1.6094
5.05	1.6194
5.10	1.6292

$$f''(x_0) \approx \frac{f_1 - 2f_0 + f_{-1}}{h^2} \tag{3}$$

$$f''(x_0) \approx \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2} \tag{4}$$

(a) Use formula (3) with $h = 0.05$ to approximate $f''(5)$.

(b) Use formula (3) with $h = 0.1$ to approximate $f''(5)$.

(c) Use formula (4) with $h = 0.05$ to approximate $f''(5)$.

(d) Which answer, (a), (b), or (c), is the most accurate?

Problem 5 (15 pts)

Question: The numerical solution of a certain differential equation requires an approximation to $f''(x) + f'(x)$ of order $O(h^2)$.

(a) Find the central-difference formula for $f''(x) + f'(x)$ by adding the formulas for $f'(x)$ and $f''(x)$ of order $O(h^2)$.

(b) Find the backward-difference formula for $f''(x) + f'(x)$ by adding the formulas for $f'(x)$ and $f''(x)$ of order $O(h^2)$.

(c) What would happen if a formula for $f'(x)$ of order $O(h^4)$ were added to a formula for $f''(x)$ of order $O(h^2)$?

Problem 6 (Programming 20 pts)

[ATTENTION] : Existing python/matlab packages (i.e., sympy.solve,numpy.solve, scipy.fsolve, gekko.solve, matlab fsolve, solve, roots...) are NOT ALLOWED in programming question(s) for all Homeworks. Direct use of such "solve" functions will result in the loss of ALL POINTS for programming problems.

Question: Use Program 6.1 (Textbook *Numerical methods Using MATLAB*) to approximate the derivatives of each of the following functions at the given value of x. Approximations should be accurate to **13** decimal places. [Note]. It may be necessary to change the values of $max1$ and the initial value of h in the program.

(a) $f(x) = sin(cos(1/x)); x = 1/\sqrt{2}$

(b) $f(x) = x^{x^x}; x = 0.0001$

Problem 7 (Programming 20 pts)

[ATTENTION] : Existing python/matlab packages (i.e., sympy.solve,numpy.solve, scipy.fsolve, gekko.solve, matlab fsolve, solve, roots...) are NOT ALLOWED in programming question(s) for all Homeworks. Direct use of such "solve" functions will result in the loss of ALL POINTS for programming problems.

Question: Modify Program 6.1 (Textbook *Numerical methods Using MATLAB*) to implement the centered formula (formula 5 as follows) of order $O(h^4)$. Use this program to approximate the derivatives of the functions given in Problem 6. Again, approximations should be accurate to **13** decimal places.

$$f'(x) \approx \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} \tag{5}$$