Question 1: Mean Value Theorem for Integrals

Find the number(s) c referred to in the Mean Value Theorem for Integrals for each function, over the interval indicated.

(a) $f(x) = \sqrt{x}$ over $[0, 4]$

$$c = \frac{16}{9}$$

(b) $f(x) = \frac{x^2}{x+1}$ over $[0, 1]$

$$c \approx 0.5466$$

Problem 2

(a) $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$

This is a geometric series with the first term $a = 1$ (when $n = 0$, $\left(\frac{1}{3}\right)^0 = 1$) and a common ratio $r = \frac{1}{3}$. The sum of an infinite geometric series $\sum_{n=0}^{\infty} ar^n$ (where $|r| < 1$) is given by $\frac{a}{1-r}$.

- $a = 1$, $r = \frac{1}{3}$
- Sum = $\frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$

Answer: $\frac{3}{2}$

(b) $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$

This is a geometric series starting at $n = 1$, with the first term $a = \frac{3}{4}$ (when $n = 1$) and a common ratio $r = \frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, the series converges, and the sum of $\sum_{n=1}^{\infty} r^n$ is $\frac{r}{1-r}$.

- $r = \frac{3}{4}$
- Sum = $\frac{\frac{3}{4}}{1-\frac{3}{4}} = \frac{\frac{3}{4}}{\frac{1}{4}} = 3$

Answer: 3

(c) $\sum_{n=2}^{\infty} \frac{4}{(n-1)(n+1)}$

This series involves a rational function. We use partial fraction decomposition to simplify the general term:

$$\frac{4}{(n-1)(n+1)} = \frac{A}{n-1} + \frac{B}{n+1}$$

Solving for A and B:

$$4 = A(n+1) + B(n-1)$$

- Set $n = 1$: $4 = A(1+1) + B(1-1) = 2A$, so $A = 2$
- Set $n = -1$: $4 = A(-1+1) + B(-1-1) = -2B$, so $B = -2$

Thus:

$$\frac{4}{(n-1)(n+1)} = \frac{2}{n-1} - \frac{2}{n+1}$$

Now the series is:

$$\sum_{n=2}^{\infty} \left(\frac{2}{n-1} - \frac{2}{n+1} \right) = 2 \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

This is a telescoping series. Let's write out the first few terms to identify the pattern:

- For $n = 2$: $\frac{1}{1} - \frac{1}{3}$
- For $n = 3$: $\frac{1}{2} - \frac{1}{4}$
- For $n = 4$: $\frac{1}{3} - \frac{1}{5}$
- And so on...

The partial sum S_N up to N terms is:

$$S_N = 2 \left[\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \cdots + \left(\frac{1}{N-1} - \frac{1}{N+1} \right) \right]$$

Most terms cancel:

- $\frac{1}{1}$ remains.
- $\frac{1}{2}$ remains.
- The $-\frac{1}{3}$ from the first term cancels with $\frac{1}{3}$ from the third term, and this pattern continues.
- As $N \to \infty$, the remaining terms are $\frac{1}{N+1} \to 0$.

So:

$$S_N = 2 \left[1 + \frac{1}{2} - \frac{1}{N+1} \right] = 2 \left[\frac{3}{2} - \frac{1}{N+1} \right] = 3 - \frac{2}{N+1}$$

As $N \to \infty$:

$$\sum_{n=2}^{\infty} \frac{4}{(n-1)(n+1)} = 3$$

Answer: 3

(d) $\sum_{k=1}^{\infty} \frac{1}{4k^2-1}$

This series also involves a rational function. Let's try partial fraction decomposition:

$$\frac{1}{4k^2 - 1} = \frac{1}{(2k-1)(2k+1)} = \frac{A}{2k-1} + \frac{B}{2k+1}$$

Solving for A and B:

$$1 = A(2k+1) + B(2k-1)$$

- Set $2k - 1 = 0$ (so $k = \frac{1}{2}$):

$$1 = A(2 \cdot \frac{1}{2} + 1) + B(2 \cdot \frac{1}{2} - 1) = A(2) + B(0) \implies A = \frac{1}{2}$$

- Set $2k + 1 = 0$ (so $k = -\frac{1}{2}$):

$$1 = A(2 \cdot -\frac{1}{2} + 1) + B(2 \cdot -\frac{1}{2} - 1) = A(0) + B(-2) \implies B = -\frac{1}{2}$$

Thus:

$$\frac{1}{4k^2 - 1} = \frac{1/2}{2k-1} - \frac{1/2}{2k+1} = \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$$

Now the series is:

$$\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} = \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$$

This is another telescoping series. Let's write out the first few terms:

- For $k = 1$: $\frac{1}{1} - \frac{1}{3}$
- For $k = 2$: $\frac{1}{3} - \frac{1}{5}$
- For $k = 3$: $\frac{1}{5} - \frac{1}{7}$
- And so on...

The partial sum S_N up to N terms is:

$$S_N = \frac{1}{2}\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{2N-1} - \frac{1}{2N+1}\right)\right]$$

Most terms cancel:

- $\frac{1}{1}$ remains.
- $-\frac{1}{3}$ from the first term cancels with $\frac{1}{3}$ from the second term, and this continues.
- As $N \to \infty$, the remaining terms $\frac{1}{2N+1} \to 0$.

The sum simplifies to:

$$S_N = \frac{1}{2}\left[1 - \frac{1}{2N+1}\right]$$

As $N \to \infty$:

$$\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

Answer: $\frac{1}{2}$

Problem 3

(a) 0.10111_{two}

Digits:

- $d_1 = 1$
- $d_2 = 0$
- $d_3 = 1$
- $d_4 = 1$
- $d_5 = 1$

Calculation:

- $1 \times 2^{-1} = 1 \times \frac{1}{2} = 0.5$
- $0 \times 2^{-2} = 0 \times \frac{1}{4} = 0$
- $1 \times 2^{-3} = 1 \times \frac{1}{8} = 0.125$
- $1 \times 2^{-4} = 1 \times \frac{1}{16} = 0.0625$
- $1 \times 2^{-5} = 1 \times \frac{1}{32} = 0.03125$

Sum:

$$0.5 + 0 + 0.125 + 0.0625 + 0.03125 = 0.71875$$

Result: $0.10111_{\text{two}} = 0.71875$

(b) 0.10101_{two}

Digits:

- $d_1 = 1$
- $d_2 = 0$
- $d_3 = 1$
- $d_4 = 0$
- $d_5 = 1$

Calculation:

- $1 \times 2^{-1} = 0.5$
- $0 \times 2^{-2} = 0$
- $1 \times 2^{-3} = 0.125$
- $0 \times 2^{-4} = 0$
- $1 \times 2^{-5} = 0.03125$

Sum:

$$0.5 + 0 + 0.125 + 0 + 0.03125 = 0.65625$$

Result: $0.10101_{\text{two}} = 0.65625$

(c) 0.10111101_{two}

Digits:

- $d_1 = 1$
- $d_2 = 0$
- $d_3 = 1$
- $d_4 = 1$
- $d_5 = 1$
- $d_6 = 1$
- $d_7 = 0$

- $d_8 = 1$

Calculation:

- $1 \times 2^{-1} = 0.5$
- $0 \times 2^{-2} = 0$
- $1 \times 2^{-3} = 0.125$
- $1 \times 2^{-4} = 0.0625$
- $1 \times 2^{-5} = 0.03125$
- $1 \times 2^{-6} = 1 \times \frac{1}{64} = 0.015625$
- $0 \times 2^{-7} = 0$
- $1 \times 2^{-8} = 1 \times \frac{1}{256} = 0.00390625$

Sum:

$$0.5 + 0 + 0.125 + 0.0625 + 0.03125 + 0.015625 + 0 + 0.00390625 = 0.73828125$$

Result: $0.10111101_{\text{two}} = 0.73828125$

(d) 0.110110111_{two}

Digits:

- $d_1 = 1$
- $d_2 = 1$
- $d_3 = 0$
- $d_4 = 1$
- $d_5 = 1$
- $d_6 = 0$
- $d_7 = 1$
- $d_8 = 1$
- $d_9 = 1$

Calculation:

- $1 \times 2^{-1} = 0.5$
- $1 \times 2^{-2} = 0.25$
- $0 \times 2^{-3} = 0$
- $1 \times 2^{-4} = 0.0625$
- $1 \times 2^{-5} = 0.03125$
- $0 \times 2^{-6} = 0$
- $1 \times 2^{-7} = 0.0078125$
- $1 \times 2^{-8} = 0.00390625$
- $1 \times 2^{-9} = 1 \times \frac{1}{512} = 0.001953125$

Sum:

$0.5 + 0.25 + 0 + 0.0625 + 0.03125 + 0 + 0.0078125 + 0.00390625 + 0.001953125 = 0.857421875$

Result: $0.110110111_{\text{two}} = 0.857421875$

Final Answers

- (a) $0.10111_{\text{two}} = 0.71875$
- (b) $0.10101_{\text{two}} = 0.65625$
- (c) $0.10111101_{\text{two}} = 0.73828125$
- (d) $0.110110111_{\text{two}} = 0.857421875$

Problem 4

(a)

Start with $R = 1/3$.

- **Step 1**: Compute $2R = 2 \times 1/3 = 2/3$.
  Since $2/3 < 1$, the integer part is 0, so $d_1 = 0$.
  Fractional part: $F_1 = 2/3$.
- **Step 2**: Compute $2F_1 = 2 \times 2/3 = 4/3 = 1 + 1/3$.
  Since $4/3 > 1$, the integer part is 1, so $d_2 = 1$.
  Fractional part: $F_2 = 4/3 - 1 = 1/3$.
- **Step 3**: Compute $2F_2 = 2 \times 1/3 = 2/3$.
  Since $2/3 < 1$, $d_3 = 0$.
  Fractional part: $F_3 = 2/3$, which matches F_1.
- **Step 4**: Compute $2F_3 = 2 \times 2/3 = 4/3$.
  $d_4 = 1$, $F_4 = 1/3$, which matches F_2.

At this point, the fractional parts repeat: $F_1 = 2/3$, $F_2 = 1/3$, $F_3 = 2/3$, $F_4 = 1/3$, and so on. Correspondingly, the digits repeat: $d_1 = 0$, $d_2 = 1$, $d_3 = 0$, $d_4 = 1$, etc. The binary representation is:

$0.010101\ldots_2$

The repeating block is "01", so we denote it as:

$1/3 = 0.\overline{01}_2$

(b)

Start with $R = 1/5$.

- **Step 1**: $2R = 2 \times 1/5 = 2/5$.
  $2/5 < 1$, so $d_1 = 0$, $F_1 = 2/5$.
- **Step 2**: $2F_1 = 2 \times 2/5 = 4/5$.
  $4/5 < 1$, so $d_2 = 0$, $F_2 = 4/5$.
- **Step 3**: $2F_2 = 2 \times 4/5 = 8/5 = 1 + 3/5$.
  $8/5 > 1$, so $d_3 = 1$, $F_3 = 8/5 - 1 = 3/5$.

- **Step 4**: $2F_3 = 2 \times 3/5 = 6/5 = 1 + 1/5$.
  $6/5 > 1$, so $d_4 = 1$, $F_4 = 6/5 - 1 = 1/5$.
- **Step 5**: $2F_4 = 2 \times 1/5 = 2/5$.
  $2/5 < 1$, so $d_5 = 0$, $F_5 = 2/5$, which matches F_1.

The fractional parts repeat: $F_1 = 2/5$, $F_2 = 4/5$, $F_3 = 3/5$, $F_4 = 1/5$, $F_5 = 2/5$, etc. The digits are: $d_1 = 0$, $d_2 = 0$, $d_3 = 1$, $d_4 = 1$, $d_5 = 0$, $d_6 = 0$, $d_7 = 1$, $d_8 = 1$, etc. The binary representation is:

$$0.00110011\ldots_2$$

The repeating block is "0011", so:

$$1/5 = 0.\overline{0011}_2$$

(c)

Start with $R = 1/10$.

- **Step 1**: $2R = 2 \times 1/10 = 1/5$.
  $1/5 < 1$, so $d_1 = 0$, $F_1 = 1/5$.
- **Step 2**: $2F_1 = 2 \times 1/5 = 2/5$.
  $2/5 < 1$, so $d_2 = 0$, $F_2 = 2/5$.
- **Step 3**: $2F_2 = 2 \times 2/5 = 4/5$.
  $4/5 < 1$, so $d_3 = 0$, $F_3 = 4/5$.
- **Step 4**: $2F_3 = 2 \times 4/5 = 8/5 = 1 + 3/5$.
  $8/5 > 1$, so $d_4 = 1$, $F_4 = 8/5 - 1 = 3/5$.
- **Step 5**: $2F_4 = 2 \times 3/5 = 6/5 = 1 + 1/5$.
  $6/5 > 1$, so $d_5 = 1$, $F_5 = 6/5 - 1 = 1/5$, which matches F_1.

The fractional parts repeat: $F_1 = 1/5$, $F_2 = 2/5$, $F_3 = 4/5$, $F_4 = 3/5$, $F_5 = 1/5$, etc. The digits are: $d_1 = 0$, $d_2 = 0$, $d_3 = 0$, $d_4 = 1$, $d_5 = 1$, $d_6 = 0$, $d_7 = 0$, $d_8 = 1$, $d_9 = 1$, etc. The binary representation is:

$$0.0001100110011\ldots_2$$

After the first three zeros, the pattern "1100" repeats: digits 4–7 are "1100", digits 8–11 are "1100", and so on. Thus:

$$1/10 = 0.000\overline{1100}_2$$

Final answer is:

- (a) $\frac{1}{3} = 0.\overline{01}_2$
- (b) $\frac{1}{5} = 0.\overline{0011}_2$
- (c) $\frac{1}{10} = 0.000\overline{1100}_2$

Problem 5

Base Case: $N = 1$

For $N = 1$, we have:

$$2^{-1} = \frac{1}{2} = 0.5$$

This clearly has exactly 1 digit after the decimal point. Hence, the statement holds for $N = 1$.

For $N >= 2$

Assume that for some positive integer k, the statement holds true, i.e.,

$$2^{-k} = 0.d_1 d_2 d_3 \dots d_k$$

where $d_1 d_2 d_3 \dots d_k$ represents the decimal expansion of 2^{-k} and has exactly k digits.

We know that:

$$2^{-(k+1)} = \frac{1}{2^{k+1}} = \frac{1}{2} \times 2^{-k}$$

By the inductive hypothesis, we know that $2^{-k} = 0.d_1 d_2 d_3 \dots d_k$. Now, multiplying 2^{-k} by $\frac{1}{2}$ shifts the decimal point one place to the right, giving us:

$$2^{-(k+1)} = \frac{1}{2} \times 0.d_1 d_2 d_3 \dots d_k = 0.5 \times (0.d_1 d_2 d_3 \dots d_k)$$

This results in a number with exactly $k + 1$ digits, as required.

Problem 6

Final Answers:

- **Part (a)**:
  - Sum: 1.506
  - Product: 0.1162
- **Part (b)**:
  - Sum: 32.343
  - Product: 0.91342

Problem 7

(a) $\ln(x + 3) - \ln(x)$ **for large** x

We start with the given expression:

$$\ln(x + 3) - \ln(x)$$

By applying the logarithmic identity $\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$, we rewrite the expression as:

$$\ln(x + 3) - \ln(x) = \ln\left(\frac{x + 3}{x}\right)$$

Simplifying the fraction inside the logarithm:

$$\ln\left(\frac{x+3}{x}\right) = \ln\left(1+\frac{3}{x}\right)$$

For large x, the term $\frac{3}{x}$ becomes very small. Using the approximation $\ln(1+y) \approx y$ for small y, we have:

$$\ln\left(1+\frac{3}{x}\right) \approx \frac{3}{x}$$

Thus, the equivalent formula for large x is:

$$\ln(x+3) - \ln(x) \approx \frac{3}{x}$$

(b) $\sqrt{x^2+1} - x$ for large x

Consider the expression:

$$\sqrt{x^2+1} - x$$

To avoid a loss of significance, we rationalize the expression by multiplying both the numerator and denominator by $\sqrt{x^2+1} + x$:

$$\sqrt{x^2+1} - x = \frac{\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)}{\sqrt{x^2+1} + x}$$

Using the difference of squares identity in the numerator:

$$\left(\sqrt{x^2+1}\right)^2 - x^2 = (x^2+1) - x^2 = 1$$

Thus, the expression simplifies to:

$$\sqrt{x^2+1} - x = \frac{1}{\sqrt{x^2+1} + x}$$

For large x, $\sqrt{x^2+1} \approx x$, so the denominator becomes approximately $2x$. Therefore, we have:

$$\sqrt{x^2+1} - x \approx \frac{1}{2x}$$

(c) $\cos^2(x) - \sin^2(x)$ for $x \approx \frac{\pi}{4}$

Using the well-known trigonometric identity for the difference of squares:

$$\cos^2(x) - \sin^2(x) = \cos(2x)$$

Thus, the expression simplifies to:

$$\cos^2(x) - \sin^2(x) = \cos(2x)$$

For $x \approx \frac{\pi}{4}$, we have:

$$2x \approx \frac{\pi}{2}$$

Therefore:

$$\cos(2x) = \cos\left(\frac{\pi}{2}\right) = 0$$

Hence, for $x \approx \frac{\pi}{4}$, the expression becomes:

$$\cos^2(x) - \sin^2(x) = 0$$

(d) $\sqrt{\frac{1+\cos(x)}{2}}$ for $x \approx \pi$

Recognize that the given expression is related to the half-angle identity for cosine:

$$\cos\left(\frac{x}{2}\right) = \sqrt{\frac{1+\cos(x)}{2}}$$

Thus, the expression simplifies to:

$$\sqrt{\frac{1+\cos(x)}{2}} = \cos\left(\frac{x}{2}\right)$$

For $x \approx \pi$, we have:

$$\frac{x}{2} \approx \frac{\pi}{2}$$

Therefore:

$$\cos\left(\frac{x}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

Thus, for $x \approx \pi$, the expression becomes:

$$\sqrt{\frac{1+\cos(x)}{2}} = 0$$

Problem 8

Firstly, we add the given expansions for $\cos(h)$ and $\sin(h)$.

$$\cos(h) + \sin(h) = \left(1 - \frac{h^2}{2!} + \frac{h^4}{4!} + O(h^6)\right) + \left(h - \frac{h^3}{3!} + \frac{h^5}{5!} + O(h^7)\right).$$

Simplifying the right-hand side:

$$\cos(h) + \sin(h) = (1+h) - \frac{h^2}{2!} - \frac{h^3}{3!} + \frac{h^4}{4!} + \frac{h^5}{5!} + O(h^6).$$

The highest power of h in this expression is h^5, and the next order term is of order $O(h^6)$.

Thus, the order of the approximation for the sum $\cos(h) + \sin(h)$ is $O(h^6)$.

Also, we consider the product of $\cos(h)$ and $\sin(h)$. We multiply the two given expansions:

$$\cos(h) \cdot \sin(h) = \left(1 - \frac{h^2}{2!} + \frac{h^4}{4!} + O(h^6)\right) \cdot \left(h - \frac{h^3}{3!} + \frac{h^5}{5!} + O(h^7)\right).$$

Expanding the product:

$$\cos(h) \cdot \sin(h) = 1 \cdot h + 1 \cdot \left(-\frac{h^3}{3!}\right) + 1 \cdot \left(\frac{h^5}{5!}\right) + \left(-\frac{h^2}{2!}\right) \cdot h + \left(-\frac{h^2}{2!}\right) \cdot \left(-\frac{h^3}{3!}\right) + O(h^7).$$

Simplifying the terms:

$$\cos(h) \cdot \sin(h) = h - \frac{h^3}{3!} + \frac{h^5}{5!} - \frac{h^3}{2!} + O(h^7).$$

Combining the terms of order h^3:

$$\cos(h) \cdot \sin(h) = h - \left(\frac{h^3}{3!} + \frac{h^3}{2!}\right) + \frac{h^5}{5!} + O(h^7).$$

The highest power of h in this expression is h^5, and the next order term is of order $O(h^7)$.

Thus, the order of the approximation for the product $\cos(h) \cdot \sin(h)$ is $O(h^6)$.