Homework 5

DDL: April 11th

Problem 1

Question: Show that the coefficients A and B for the lease-squares line can be computed as follows. First compute the means $\overline{x}$ and $\overline{y}$, and then perform the calculations. (***Hint***, Use $X_k = x_k - \overline{x}, Y_K = y_k - \overline{y}$ and first find the line $Y = AX$)

$$\begin{gathered} \overline{x} = \frac{1}{N}\sum_{k=1}^{N} x_k \quad and \quad \overline{y} = \frac{1}{N}\sum_{k=1}^{N} y_k \quad for \ \{(x_k, y_k)\}_{k=1}^{N} \\ C = \sum_{k=1}^{N}(x_k - \overline{x})^2, \ \ A = \frac{1}{C}\sum_{k=1}^{N}(x_k - \overline{x})(y_k - \overline{y}), \ \ B = \overline{y} - A\overline{x} \end{gathered} \tag{1}$$

Problem 2

Question:

(a) Derive the normal equation for finding the least-squares linear fit through the origin $y = Ax$.

(b) Derive the normal equation for finding the least-squares power fit $y = Ax^3$.

b Derive the normal equation for finding the least-squares fit $y = Ax^3 + B$

Problem 3

Question: For each set of data, find the least-squares curve:

(a.1) $f(x) = Ce^{Ax}$, by using the change of variables $X = x, Y = ln(y)$ and $C = e^B$ from Table 5.6 (Textbook Page 269) to linearize the data points A.

(a.2) $f(x) = Cx^A$, by using the change of variables $X = ln(x), Y = ln(y)$ and $C = e^B$ from Table 5.6 (Textbook Page 269) to linearize the data points A.

(a.3) Use $E_2(f)$ to determine which curve (a.1, a.2) gives the best fit.

$$data\ points\ A : (x_k, y_k) = \{(1, 0.6), (2, 1.9), (3, 4.3), (4, 7.6), (5, 12.6)\} \tag{2}$$

(b.1) $f(x) = Ce^{Ax}$, by using the change of variables $X = x, Y = ln(y)$ and $C = e^B$ from Table 5.6 to linearize the data points B.

(b.2) $f(x) = 1/(Ax + B)$, by using the change of variables $X = x, Y = 1/y$ from Table 5.6 to linearize the data points B.

(b.3) Use $E_2(f)$ to determine which curve (b.1, b.2) gives the best fit.

$$data\ points\ B: (x_k, y_k) = \{(-1, 6.62), (0, 3.94), (1, 2.17), (2, 1.35), (3, 0.89)\} \tag{3}$$

Problem 4

Question: The least-squares plane $z = Ax + By + C$ for the N points $(x_1, y_1, z_1), \ldots, (x_N, y_N, z_N)$) is obtained by minimizing $E(A, B, C) = \sum_{k=1}^{N}(Ax_k + By_k + C - z_k)^2$. Derive the normal equations:

$$\begin{gathered}
(\sum_{k=1}^{N} x_k^2)A + (\sum_{k=1}^{N} x_k y_k)B + (\sum_{k=1}^{N} x_k)C = \sum_{k=1}^{N} z_k x_k, \\
(\sum_{k=1}^{N} x_k y_k)A + (\sum_{k=1}^{N} y_k^2)B + (\sum_{k=1}^{N} y_k)C = \sum_{k=1}^{N} z_k y_k, \\
(\sum_{k=1}^{N} x_k)A + (\sum_{k=1}^{N} y_k)B + NC = \sum_{k=1}^{N} z_k.
\end{gathered} \tag{4}$$

Problem 5

Question: Find the natural cubic spline that passes through the points $(-3, 2), (-2, 0), (1, 3), (4, 1)$ with the free boundary conditions $S''(-3) = 0$ and $S''(4) = 0$.

Problem 6

Question: Find the parabolically terminated cubic spline that passes through the points $(-3, 2)$,$(-2, 0)$, $(1, 3), (4, 1)$.

Problem 7

Question:

(a) Using the nodes $x_0 = -2$ and $x_1 = 0$, determine whether $f(x) = 2x^3 - x$ is is own clamped cubic spline on the interval $[-2, 0]$.

(b) Using the nodes $x_0 = -2, x_1 = 0$ and $x_2 = 2$, show that $f(x) = x^3 - x$ is is own clamped cubic spline on the interval $[-2, 2]$. *NOTE*: f has an inflextion point at x_1.

(c) Use the results from part (b) to show that any third-degree polynomial, $f(x) = a_0 + a_1 + a_2x^2 + a_3x^3$ is its own clamped cubic spline on any cloased interval $[a, b]$.

(b) What, if anything, can be said about the other four types of cubic splines described in Lemmas 5.2-5.5 from Textbook Page 285-287.

Problem 8

$$a_0 = \frac{1}{P}\int_{-P}^{P} f(x)dx$$
$$a_j = \frac{1}{P}\int_{-P}^{P} f(x)cos(\frac{j\pi x}{P})dx \quad for\ j = 1, 2, \dots \tag{5}$$
$$b_j = \frac{1}{P}\int_{-P}^{P} f(x)sin(\frac{j\pi x}{P})dx \quad for\ j = 1, 2, \dots$$

Question: For a periodic function $f(x)$ with period $2P$, $f(x+2P) = f(x)$, we have Euler's formulas (5) and (6) for f as illustrated above. Please use the above equations to find the Fourier series representations of the given function. Graph $f(x), S_4(x)$ and $S_6(x)$ on the same coordinate system.

$$f(x) = \begin{cases} -1 & for\ -3 \le x < -1 \\ x & for\ -1 \le x < 1 \\ 1 & for\ 1 \le x < 3 \end{cases} \tag{6}$$

Problem 9

Question: Show that

(a) $P''(0) = N(N-1)(P_2 - 2P_1 + P_0)$

(b) $P''(1) = N(N-1)(P_N - 2P_{N-1} + P_{N-2})$

Problem 10 (Programming)

Please submit executable files, and append your programming result in your homework.

ATTENTION : Existing python/matlab packages (i.e., sympy.solve, matlab fsolve...) are NOT ALLOWED

Question: The distance d_k that a car traveled at time t_k is given in the following table. Use Program 5.3 with the first derivative boundary conditions $S'(0) = 0$ and $S'(8) = 98$, and find the clamped cubic spline for the points.

$$(t_k, d_k) = \{(0,0), (2,40), (4,160), (6,300), (8,480)\} \tag{7}$$

Problem 11 (Programming)

Please submit executable files, and append your programming result in your homework.

ATTENTION : Existing python/matlab packages (i.e., sympy.solve, matlab fsolve...) are NOT ALLOWED

Question: Modify Program 5.3 to find the (a) natural, (b) extrapolated, (c) parabolically terminated, or (d) endpoint curvature-adjusted cubic splines for a given set of points.