Homework 6

12432670

Zitong Huang

Problem 1 Solution

Part (a): $f(x, y) = x^2 + y^2 + 2xy$ at $(3, 4)$

1. Exact Partial Derivatives:

- $f_x(x, y) = \frac{\partial}{\partial x}(x^2 + y^2 + 2xy)$
- $f_y(x, y) = \frac{\partial}{\partial y}(x^2 + y^2 + 2xy)$
- Exact $f_x(x, y) = 2 * x + 2 * y$, $f_x(3, 4)$ = 14.
- Exact $f_y(x, y) = 2 * x + 2 * y$, $f_y(3, 4)$ = 14

2. Approximations using Formula (2):

- **For $f_x(3, 4)$:**
  - $f(x, y) = (x + y)^2$. At $(3, 4)$, $f_x(3, 4) \approx \frac{f(3+h,4)-f(3-h,4)}{2h} = \frac{(7+h)^2-(7-h)^2}{2h}$
  - $h = 0.1$: $f_x(3, 4) \approx 13.999999999999986$
  - $h = 0.01$: $f_x(3, 4) \approx 13.999999999999702$
  - $h = 0.0001$: $f_x(3, 4) \approx 13.999999999995794$
- **For $f_y(3, 4)$:**
  - At $(3, 4)$, $f_y(3, 4) \approx \frac{f(3,4+h)-f(3,4-h)}{2h} = \frac{(7+h)^2-(7-h)^2}{2h}$
  - $h = 0.1$: $f_y(3, 4) \approx 13.999999999999986$
  - $h = 0.01$: $f_y(3, 4) \approx 13.999999999999702$
  - $h = 0.0001$: $f_y(3, 4) \approx 14.000000000002899$

3. Comparison:

For $f(x, y) = x^2 + y^2 + 2xy$ at $(3, 4)$:

- The exact values are $f_x(3, 4) = f_y(3, 4) = 14$
- For $f_x(3, 4)$:
  - $h = 0.1$: error ≈ 1.4e-14
  - $h = 0.01$: error ≈ 2.98e-13
  - $h = 0.0001$: error ≈ 4.206e-12
- For $f_y(3, 4)$:
  - $h = 0.1$: error ≈ 1.4e-14
  - $h = 0.01$: error ≈ 2.98e-13
  - $h = 0.0001$: error ≈ 2.899e-12

Part (b): $f(x, y) = \frac{x^2y^2}{x+y}$ at $(2, 3)$

1. Exact Partial Derivatives:

- $f_x(x, y) = \frac{\partial}{\partial x}\left(\frac{x^2y^2}{x+y}\right)$
- $f_y(x, y) = \frac{\partial}{\partial y}\left(\frac{x^2y^2}{x+y}\right)$
- Exact $f_x(x, y) = -x^2 * y^2/(x+y)^2 + 2 * x * y^2/(x+y)$, $f_x(2, 3)$ = 5.76.
- Exact $f_y(x, y) = -x^2 * y^2/(x+y)^2 + 2 * x * y^2/(x+y)$, $f_y(2, 3)$ = 3.36

2. Approximations using Formula (2):

- **For $f_x(2, 3)$:**
  - $f_x(2, 3) \approx \frac{f(2+h,3)-f(2-h,3)}{2h}$
  - $h = 0.1$: $f_x(2, 3) \approx 5.7587034813925575$
  - $h = 0.01$: $f_x(2, 3) \approx 5.759987039948067$
  - $h = 0.0001$: $f_x(2, 3) \approx 5.759999870400101$
- **For $f_y(2, 3)$:**
  - $f_y(2, 3) \approx \frac{f(2,3+h)-f(2,3-h)}{2h}$
  - $h = 0.1$: $f_y(2, 3) \approx 3.3597438975590332$
  - $h = 0.01$: $f_y(2, 3) \approx 3.3599974399896926$
  - $h = 0.0001$: $f_y(2, 3) \approx 3.359999974400285$

3. Comparison:

For $f(x, y) = \frac{x^2y^2}{x+y}$ at $(2, 3)$:

- The exact values are $f_x(2, 3) = 5.76$ and $f_y(2, 3) = 3.36$
- For $f_x(2, 3)$:
  - $h = 0.1$: error ≈ 1.296e-3
  - $h = 0.01$: error ≈ 1.296e-5
  - $h = 0.0001$: error ≈ 1.296e-7
- For $f_y(2, 3)$:
  - $h = 0.1$: error ≈ 2.56e-4
  - $h = 0.01$: error ≈ 2.56e-6
  - $h = 0.0001$: error ≈ 2.56e-8

Observations:

1. For the first function (simple polynomial), the numerical method achieves very high accuracy with errors in the range of 10^-12 to 10^-14.
2. For the second function (rational function), the errors are relatively larger, ranging from 10^-3 to 10^-8.
3. In both cases, accuracy generally improves as h decreases, but for the first function, there is a slight loss of precision when h becomes too small, likely due to rounding errors.

4. The second function shows significant improvement in accuracy as h decreases, indicating that choosing smaller step sizes is beneficial for such complex functions.

Problem 2

Let $\epsilon = 5 \cdot 10^{-4}$ be the round-off error in y_k.
Given $|f^{(3)}(c)| \leq M_3 = 1.5$ and $|f^{(5)}(c)| \leq M_5 = 1.5$.
The best step size h minimizes total error $E(h) \approx E_t(h) + E_r(h)$.

(a) Formula: $f'(x_0) \approx \frac{y_1 - y_{-1}}{2h}$

1. **Truncation Error (E_t):**
   $E_t(h) \approx \frac{h^2}{6} M_3 = \frac{1.5}{6} h^2 = 0.25h^2$.
2. **Round-off Error (E_r):**
   The error in $y_1 - y_{-1}$ is bounded by 2ϵ.
   $E_r(h) \approx \frac{2\epsilon}{2h} = \frac{\epsilon}{h} = \frac{5 \cdot 10^{-4}}{h}$.
3. **Total Error & Minimization:**
   $E(h) = 0.25h^2 + \frac{5 \cdot 10^{-4}}{h}$.
   $\frac{dE}{dh} = 0.5h - \frac{5 \cdot 10^{-4}}{h^2} = 0$.
   $0.5h^3 = 5 \cdot 10^{-4} \Rightarrow h^3 = 10^{-3}$.
   $h = (10^{-3})^{1/3} = 0.1$.

Best step size for (a): $h = 0.1$.

(b) Formula: $f'(x_0) \approx \frac{-y_2 + 8y_1 - 8y_{-1} + y_{-2}}{12h}$

1. **Truncation Error (E_t):**
   $E_t(h) \approx \frac{h^4}{30} M_5 = \frac{1.5}{30} h^4 = 0.05h^4$.
2. **Round-off Error (E_r):**
   The error in the numerator $(-y_2 + 8y_1 - 8y_{-1} + y_{-2})$ is bounded by $(1 + 8 + 8 + 1)\epsilon = 18\epsilon$.
   $E_r(h) \approx \frac{18\epsilon}{12h} = \frac{3\epsilon}{2h} = \frac{3 \cdot (5 \cdot 10^{-4})}{2h} = \frac{7.5 \cdot 10^{-4}}{h}$.
3. **Total Error & Minimization:**
   $E(h) = 0.05h^4 + \frac{7.5 \cdot 10^{-4}}{h}$.
   $\frac{dE}{dh} = 0.2h^3 - \frac{7.5 \cdot 10^{-4}}{h^2} = 0$.
   $0.2h^5 = 7.5 \cdot 10^{-4} \Rightarrow h^5 = \frac{7.5 \cdot 10^{-4}}{0.2} = 37.5 \cdot 10^{-4} = 3.75 \cdot 10^{-3}$.
   $h = (3.75 \cdot 10^{-3})^{1/5}$.

Best step size for (b): $h = (0.00375)^{1/5} \approx 0.3268$.

Problem 3

Let $f(x) = \cos(x)$ and $x_0 = 1.2$. The approximation formula is $f'(x_0) \approx \frac{-y_2 + 8y_1 - 8y_{-1} + y_{-2}}{12h}$.
The inherent round-off error for $y_k = f(x_0 + kh)$ is $|e_k| \leq 5 \cdot 10^{-6}$.

Part (a)

1. **Case 1:** $h = 0.1$
   - $y_2 = f(1.4) = 0.16997$
   - $y_1 = f(1.3) = 0.26750$
   - $y_{-1} = f(1.1) = 0.45360$

- $y_{-2} = f(1.0) = 0.54030$
- $f'(1.2) \approx \frac{-0.16997+8(0.26750)-8(0.45360)+0.54030}{12(0.1)}$
- $f'(1.2) \approx \frac{-0.16997+2.14000-3.62880+0.54030}{1.2} = \frac{-1.11847}{1.2} \approx -0.932058$

2. **Case 2:** $h = 0.001$
   - $y_2 = f(1.202) = 0.36049$
   - $y_1 = f(1.201) = 0.36143$
   - $y_{-1} = f(1.199) = 0.36329$
   - $y_{-2} = f(1.198) = 0.36422$
   - $f'(1.2) \approx \frac{-0.36049+8(0.36143)-8(0.36329)+0.36422}{12(0.001)}$
   - $f'(1.2) \approx \frac{-0.36049+2.89144-2.90632+0.36422}{0.012} = \frac{-0.01115}{0.012} \approx -0.929167$

Part (b)

The total error bound is $B(h) \approx |E_{round}(f,h)|_{max} + |E_{trunc}(f,h)|_{max}$.

1. $|E_{round}(f,h)|_{max}$:
   The formula for the round-off error contribution is $E_{round}(f,h) = \frac{-e_2+8e_1-8e_{-1}+e_{-2}}{12h}$.
   $|E_{round}(f,h)|_{max} \leq \frac{|e_2|+8|e_1|+8|e_{-1}|+|e_{-2}|}{12h} \leq \frac{(1+8+8+1)\cdot 5\cdot 10^{-6}}{12h} = \frac{18\cdot 5\cdot 10^{-6}}{12h} = \frac{7.5\cdot 10^{-6}}{h}$.
2. $|E_{trunc}(f,h)|_{max}$:
   The formula for the truncation error is $E_{trunc}(f,h) = \frac{h^4 f^{(5)}(c)}{30}$.
   $f(x) = \cos(x)$, so $f^{(5)}(x) = -\sin(x)$. Thus, $|f^{(5)}(c)| = |-\sin(c)| \leq 1$.
   $|E_{trunc}(f,h)|_{max} \leq \frac{h^4 \cdot 1}{30} = \frac{h^4}{30}$.
3. **Total Error Bound Calculation:**
   - **For** $h = 0.1$**:**
     - $|E_{round}(f,0.1)|_{max} \leq \frac{7.5\cdot 10^{-6}}{0.1} = 7.5 \cdot 10^{-5}$
     - $|E_{trunc}(f,0.1)|_{max} \leq \frac{(0.1)^4}{30} = \frac{10^{-4}}{30} \approx 0.03333 \cdot 10^{-4} = 3.333 \cdot 10^{-6}$
     - Total Error Bound
       $B(0.1) \leq 7.5 \cdot 10^{-5} + 3.333 \cdot 10^{-6} = 75 \cdot 10^{-6} + 3.333 \cdot 10^{-6} = 78.333 \cdot 10^{-6} \approx 7.83 \cdot 10^{-5}$
   - **For** $h = 0.001$**:**
     - $|E_{round}(f,0.001)|_{max} \leq \frac{7.5\cdot 10^{-6}}{0.001} = 7.5 \cdot 10^{-3}$
     - $|E_{trunc}(f,0.001)|_{max} \leq \frac{(0.001)^4}{30} = \frac{10^{-12}}{30} \approx 3.333 \cdot 10^{-14}$
     - Total Error Bound $B(0.001) \leq 7.5 \cdot 10^{-3} + 3.333 \cdot 10^{-14} \approx 7.5 \cdot 10^{-3}$

Thus:

- For $h = 0.1$: $f'(1.2) \approx -0.932058$; Total Error Bound $\approx 7.83 \cdot 10^{-5}$.
- For $h = 0.001$: $f'(1.2) \approx -0.929167$; Total Error Bound $\approx 7.5 \cdot 10^{-3}$.

Problem 4

True Value:
For $f(x) = \ln(x)$:
$f'(x) = \frac{1}{x}$
$f''(x) = -\frac{1}{x^2}$

So, $f''(5) = -\frac{1}{5^2} = -\frac{1}{25} = -0.04$.

Given data values from the table for $x_0 = 5$:
$f(4.90) = 1.5892$
$f(4.95) = 1.5994$
$f(5.00) = 1.6094$ (this is f_0 or $f(x_0)$)
$f(5.05) = 1.6194$
$f(5.10) = 1.6292$

(a) Using formula (3) with $h = 0.05$:
Formula (3): $f''(x_0) \approx \frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2}$
Here, $x_0 = 5$ and $h = 0.05$.
$f(x_0 + h) = f(5.05) = 1.6194$
$f(x_0) = f(5.00) = 1.6094$
$f(x_0 - h) = f(4.95) = 1.5994$
$h^2 = (0.05)^2 = 0.0025$
$f''(5) \approx \frac{1.6194-2(1.6094)+1.5994}{0.0025} = \frac{1.6194-3.2188+1.5994}{0.0025} = \frac{0.0000}{0.0025} = 0.0000$

(b) Using formula (3) with $h = 0.1$:
Here, $x_0 = 5$ and $h = 0.1$.
$f(x_0 + h) = f(5.10) = 1.6292$
$f(x_0) = f(5.00) = 1.6094$
$f(x_0 - h) = f(4.90) = 1.5892$
$h^2 = (0.1)^2 = 0.01$
$f''(5) \approx \frac{1.6292-2(1.6094)+1.5892}{0.01} = \frac{1.6292-3.2188+1.5892}{0.01} = \frac{-0.0004}{0.01} = -0.0400$

(c) Using formula (4) with $h = 0.05$:
Formula (4): $f''(x_0) \approx \frac{-f(x_0+2h)+16f(x_0+h)-30f(x_0)+16f(x_0-h)-f(x_0-2h)}{12h^2}$
Here, $x_0 = 5$ and $h = 0.05$.
$f(x_0 + 2h) = f(5.10) = 1.6292$
$f(x_0 + h) = f(5.05) = 1.6194$
$f(x_0) = f(5.00) = 1.6094$
$f(x_0 - h) = f(4.95) = 1.5994$
$f(x_0 - 2h) = f(4.90) = 1.5892$
$12h^2 = 12(0.05)^2 = 12(0.0025) = 0.03$
Numerator:
$-1.6292 + 16(1.6194) - 30(1.6094) + 16(1.5994) - 1.5892$
$= -1.6292 + 25.9104 - 48.2820 + 25.5904 - 1.5892$
$= (25.9104 + 25.5904) - (1.6292 + 48.2820 + 1.5892)$
$= 51.5008 - 51.5004 = 0.0004$
$f''(5) \approx \frac{0.0004}{0.03} = \frac{4}{300} = \frac{1}{75} \approx 0.0133$

(d) Comparation
True value $f''(5) = -0.04$.
Comparing the approximations:
(a) Approximation = 0.0000. Absolute error = $|0.0000 - (-0.04)| = 0.0400$.
(b) Approximation = -0.0400. Absolute error = $|-0.0400 - (-0.04)| = 0.0000$.
(c) Approximation ≈ 0.0133. Absolute error = $|0.013333... - (-0.04)| = |0.053333...| \approx 0.0533$.

The approximation from **(b) is the most accurate**, as its absolute error is the smallest (in this case, 0 due to the rounding of the provided table values).

Problem 5

(a) Central-difference formula for $f''(x) + f'(x)$ of order $O(h^2)$:

Let $f_c'(x) = \frac{f(x+h)-f(x-h)}{2h}$ (which is $O(h^2)$) and $f_c''(x) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$ (which is $O(h^2)$).

Then, $f''(x) + f'(x) \approx f_c''(x) + f_c'(x) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2} + \frac{f(x+h)-f(x-h)}{2h}$.
This simplifies to:
$f''(x) + f'(x) \approx \frac{2(f(x+h)-2f(x)+f(x-h))+h(f(x+h)-f(x-h))}{2h^2}$
$f''(x) + f'(x) \approx \frac{(2+h)f(x+h)-4f(x)+(2-h)f(x-h)}{2h^2}$.
The order of this combined formula is $O(h^2)$.

(b) Backward-difference formula for $f''(x) + f'(x)$ of order $O(h^2)$:

Let $f_b'(x) = \frac{3f(x)-4f(x-h)+f(x-2h)}{2h}$ (which is $O(h^2)$) and $f_b''(x) = \frac{2f(x)-5f(x-h)+4f(x-2h)-f(x-3h)}{h^2}$ (which is $O(h^2)$).

Then, $f''(x) + f'(x) \approx f_b''(x) + f_b'(x) = \frac{2f(x)-5f(x-h)+4f(x-2h)-f(x-3h)}{h^2} + \frac{3f(x)-4f(x-h)+f(x-2h)}{2h}$.
This simplifies to:
$f''(x) + f'(x) \approx \frac{2(2f(x)-5f(x-h)+4f(x-2h)-f(x-3h))+h(3f(x)-4f(x-h)+f(x-2h))}{2h^2}$
$f''(x) + f'(x) \approx \frac{(4+3h)f(x)-(10+4h)f(x-h)+(8+h)f(x-2h)-2f(x-3h)}{2h^2}$.
The order of this combined formula is $O(h^2)$.

(c) What would happen if a formula for $f'(x)$ of order $O(h^4)$ were added to a formula for $f''(x)$ of order $O(h^2)$?

If a formula for $f'(x)$ with error $E_1 = C_1h^4 + \ldots$ (order $O(h^4)$) is added to a formula for $f''(x)$ with error $E_2 = C_2h^2 + \ldots$ (order $O(h^2)$), the total error is $E_{total} = E_1 + E_2 = C_2h^2 + C_1h^4 + \ldots$.
The dominant term in the error for small h is C_2h^2.
Therefore, the resulting formula would have an order of accuracy of $O(h^2)$.

Problem 6

```
● (base) ➜ Homework6 git:(main) ✗ python Problem6.py
  f(x)                 x      approx f'(x)    analytic f'(x)    rel. err.    h_used
  -------------   --------   --------------   ----------------   -----------   --------
  sin(cos(1/x))   0.707107          1.95156            1.95156     3.14e-11      1e-06
  x^x^x           0.0001            1.01521           -8.20278     1.12          1e-09
```

Problem 7

```
● (base) ➜ Homework6 git:(main) ✗ python Problem7.py
  0.9999999999994271 0.0001
```