MAE5003 Homework5

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Question 1

Answer 1

Let

$$X_k = x_k - \bar{x} \tag{1}$$

$$Y_k = y_k - \overline{y} \tag{2}$$

Consider original loss function $E(A, B)$, we have

$$E(A, B) = \sum_{k=1}^{N} (y_k - (Ax_k + B))^2$$

Let (1)(2) in, we have

$$E(A, B) = \sum_{k=1}^{N} (\overline{y} + Y_k - (A\bar{x} + AX_k + B))$$

From normal equations, we have $\overline{y} - (A\bar{x} + B) = 0$

Thus $E(A, B) = \sum\limits_{k=1}^{N} (Y_k - AX_k)^2$

Let the loss function $E(A) = \sum\limits_{k=1}^{N} (Y_k - AX_k)^2$

Take the partial derivative with respect to A, we have

$$\frac{\partial E}{\partial A} = \sum_{k=1}^{N} 2(Y_k - AX_k)(-X_k) \tag{3}$$

Let $\frac{\partial E}{\partial A} = 0$, solve A, we have

$$A = \frac{\sum\limits_{k=1}^{N} X_k Y_k}{\sum\limits_{k=1}^{N} X_k^2} \tag{4}$$

Let (1) and (2) in, we have

$$A = \frac{\sum_{k=1}^{N}(y_k - \bar{y})(x_k - \bar{x})}{\sum_{k=1}^{N}(x_k - \bar{x})^2} = \frac{1}{C}\sum_{k=1}^{N}(y_k - \bar{y})(x_k - \bar{x})$$

where $C = \sum_{k=1}^{N}(x_k - \bar{x})^2$

Question 2

Answer 2

(a)

$$E(A) = \frac{1}{2}\sum_{k=1}^{N}(y_k - Ax_k)^2$$

$$\frac{\partial E}{\partial A} = \sum_{k=1}^{N}2(y_k - Ax_k)(-x_k)$$

Let $\frac{\partial E}{\partial A} = 0$, solve A, we have

$$\sum_{k=1}^{N}Ax_k^2 - \sum_{k=1}^{N}x_k y_k = 0 \tag{5}$$

So the normal function is

$$A = \frac{\sum\limits_{k=1}^{N} x_k y_k}{\sum\limits_{k=1}^{N} x_k^2}$$

(b)

$$E(A) = \frac{1}{2}\sum_{k=1}^{N}(y_k - Ax_k^3)^2$$

$$\frac{\partial E}{\partial A} = \sum_{k=1}^{N}(y_k - Ax_k^3)(-x_k^3)$$

Let $\frac{\partial E}{\partial A} = 0$, solve A, we have

$$\sum_{k=1}^{N}Ax_k^6 - \sum_{k=1}^{N}x_k^3 y_k = 0 \tag{6}$$

So the normal function is

$$A = \frac{\sum\limits_{k=1}^{N} x_k^3 y_k}{\sum\limits_{k=1}^{N} x_k^6}$$

(c)

Loss function is

$$E(A, B) = \sum_{k=1}^{N} (y_k - (Ax_k^3 + B))$$

Get the partial derivative, we have

$$\frac{\partial E}{\partial A} = -2 \sum_{k=1}^{n} x_k^3 (y_k - Ax_k^3 - B) = 0$$

$$\frac{\partial S}{\partial B} = -2 \sum_{i=1}^{n} (y_i - Ax_i^3 - B) = 0$$

Solve, we have

$$\begin{cases} A \sum_{i=1}^{n} x_i^6 + B \sum_{i=1}^{n} x_i^3 = \sum_{i=1}^{n} x_i^3 y_i \\ A \sum_{i=1}^{n} x_i^3 + Bn = \sum_{i=1}^{n} y_i \end{cases}$$

Question 3

Answer 3

For function 1, arranged to get

$$\ln y = Ax + \ln C \Rightarrow Y = AX + B$$

where

$$Y = \ln y, X = x, B = \ln C \tag{7}$$

Error function $E(A, B)$ equals to

$$E(A, B) = \sum_{i=1}^{N} (Y_i - (AX_i + B))^2$$

Let $\frac{\partial E}{\partial A} = 0$, we have

$$\sum_{i=1}^{N} (X_i Y_i - AX_i^2 - 2BX_i) = 0$$

$$\Rightarrow A \sum_{i=1}^{n} X_i^2 + B \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} X_i Y_i \tag{8}$$

Let $\frac{\partial E}{\partial B} = 0$, we have

$$\sum_{i=1}^{N}(Y_i - AX_i - B) = 0$$
$$\Rightarrow A\sum_{i=1}^{n} X_i + Bn = \sum_{i=1}^{n} Y_i$$

Thus the Normal equations is

$$\begin{cases} A\sum_{i=1}^{n} X_i^2 + B\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} X_i Y_i \\ A\sum_{i=1}^{n} X_i + Bn = \sum_{i=1}^{n} Y_i \end{cases}$$

Let data point $A : (x_k, y_k) = \{(1, 0.6), (2, 1.9), (3, 4.3), (4, 7.6), (5, 12.6)\}$ and equation (7) in, we can solve

$$\begin{cases} A = 0.747 \\ C = 0.363 \end{cases}$$

The final loss is

$$E = \sum_{i=1}^{N}(Y_i - (AX_i + B))^2, \ \text{Point set } A$$

Solve to get $E = 8.102$

For function 2, arranged to get

$$\ln y = A\ln x + \ln \rightarrow Y = AX + B$$

where

$$Y = \ln y, \quad X = \ln x, \quad B = \ln C$$

Error function $E(A, B)$ equals to

$$E(A, B) = \sum_{i=1}^{N}(Y_i - (AX_i + B))^2$$

Let $\frac{\partial E}{\partial A} = 0$, we have

$$\sum_{i=1}^{N} X_i Y_i - A\sum_{i=1}^{N} X_i^2 - B\sum_{i=1}^{N} X_i = 0$$

$$\Rightarrow A\sum X_i^2 + B\sum X_i = \sum X_i Y_i$$

Let $\frac{\partial E}{\partial B}$, we have

$$\sum_{i=1}^{N} Y_i - A\sum_{i=1}^{N} X_i - BN = 0$$

$$\Rightarrow A\sum X_i + BN = \sum Y_i$$

Thus the Normal equations are

$$\begin{cases} A\sum_{i=1}^n X_i^2 + B\sum_{i=1}^n X_i = \sum_{i=1}^n X_iY_i \\ A\sum_{i=1}^n X_i + Bn = \sum_{i=1}^n Y_i \end{cases}$$

Let data point

$$A : (x_k, y_k) = \{(1, 0.6), (2, 1.9), (3, 4.3), (4, 7.6), (5, 12.6)\}$$

and apply transformation $X_i = \ln x_i,\ Y_i = \ln y_i$ into equation (8), we solve

$$\begin{cases} A = 1.886 \\ C = 0.562 \end{cases}$$

The final loss is 0.8725938171171367.

Thus , we can have function 2 is better.

Data B

Same as Data A,

Let data points B in, we can calculate for $f(x) = Ce^{Ax}$, A and C is

$$\begin{cases} A = -0.508 \\ C = 3.866 \end{cases}$$

The final loss $E = 0.07$

For function $f(x) = \frac{1}{Ax+B}$, we can calculate for $\frac{1}{g(x)} = Ax + B$, where $g(x) = \frac{1}{f(x)}$, which A and C be

$$\begin{cases} A = 0.24 \\ B = 0.30 \end{cases}$$

The final loss $E = 103$.

Question 4

Let the loss equation be

$$E(A, B, C) = \sum_{k=1}^{N}(Ax_k + By_k + C - z_k)^2$$

We can have

$$\frac{\partial E}{\partial A} = \sum_{k=1}^{N} 2(Ax_k + By_k + C - z_k) \quad (x_k)$$

$$\frac{\partial E}{\partial B} = \sum_{k=1}^{N} 2(Ax_k + By_k + C - z_k) \quad (y_k)$$

$$\frac{\partial E}{\partial C} = \sum_{k=1}^{N} 2(Ax_k + By_k + C - z_k)$$

Let $\frac{\partial E}{\partial A}, \frac{\partial E}{\partial B}, \frac{\partial E}{\partial C}$ be 0, we have equations

$$\sum_{k=1}^{N}(Ax_k^2 + Bx_k y_k + Cx_k - x_k z_k) = 0$$

$$\sum_{k=1}^{N}(Ax_k y_k + By_k^2 + Cy_k - y_k z_k) = 0$$

$$Ax_k + By_k + C - z_k \quad = 0$$

Which is equals to

$$A\sum_{k=1}^{N} x_k^2 + B\sum_{k=1}^{N} x_k y_k + C\sum_{k=1}^{N} x_k - \sum_{k=1}^{N} x_k z_k = 0$$

$$A\sum_{k=1}^{N} x_k y_k + B\sum_{k=1}^{N} y_k^2 + C\sum_{k=1}^{N} y_k - \sum_{k=1}^{N} y_k z_k = 0$$

$$A\sum_{k=1}^{N} x_k + B\sum_{k=1}^{N} y_k + C - \sum_{k=1}^{N} z_k \quad = 0$$

Equals to the given functions.

Question 5

Given the second derivative formulation:

$$\frac{h_{i-1}}{6}M_{i-1} + \frac{h_{i-1}+h_i}{3}M_i + \frac{h_i}{6}M_{i+1} = \frac{y_{i+1}-y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}$$

where $h_{i-1} = x_i - x_{i-1}, h_i = x_{i+1} - x_i$

and equation to calculate $S_i(x)$:

$$S_i(x) = \frac{M_{i+1}}{6h_i}(x-x_i)^3 + \frac{M_i}{6h_i}(x_{i+1}-x)^3 + \left(\frac{y_{i+1}}{h_i} - \frac{M_{i+1}h_i}{6}\right)(x-x_i) + \left(\frac{y_i}{h_i} - \frac{M_i h_i}{6}\right)(x_{i+1}-x) \tag{9}$$

and four points:

$$A_0 = (-3, 2), A_1 = (-2, 0), A_2 = (1, 3), A_3 = (4, 1)$$

Let $i = 1$,we have

$$\frac{1}{6}M_{i-1} + \frac{4}{3}M_i + \frac{3}{6}M_{i+1} = \frac{3}{3} - \frac{-2}{1}$$

Let $i = 2$, we have

$$\frac{3}{6}M_{i-1} + \frac{6}{3}M_i + \frac{3}{6}M_{i+1} = \frac{-2}{3} - \frac{3}{3}$$

Simplify, we have

$$\frac{4M_1}{3} + \frac{M_2}{2} + \frac{M_0}{6} = 3$$

$$2M_2 + \frac{M_3}{2} + \frac{M_1}{2} = -\frac{5}{3}$$

Also, we have $M_0 = 0$, $M_3 = 0$. Solve the linear equation, we have

$$\begin{cases} M_1 = 2.83 \\ M_2 = -1.54 \end{cases}$$

Let M_0, M_1, M_2, M_3 in to equation (9), we have

Segment $\mathbf{1}$: $\mathbf{S}_0(x)$, interval$[-3, -2]$

$$S_0(x) = 2 - 2.471264(x + 3) + 0.471264(x + 3)^3$$

Segment $\mathbf{2}$: $\mathbf{S}_1(x)$, interval$[-2, 1]$

$$S_1(x) = -1.057471(x + 2) + 1.413793(x + 2)^2 - 0.242656(x + 2)^3$$

Segment $\mathbf{3}$: $\mathbf{S}_2(x)$, interval$[1, 4]$

$$S_2(x) = 3 + 0.873563(x - 1) - 0.770115(x - 1)^2 + 0.085568(x - 1)^3$$

Question 6

Derivate the interval into

- $[-3,-2)$:

$$S_1(x) = a_1 + b_1(x+3) + c_1(x+3)^2 + d_1(x+3)^3$$

- $[-2,1)$:

$$S_2(x) = a_2 + b_2(x+2) + c_2(x+2)^2 + d_2(x+2)^3$$

- $[1,4)$:

$$S_3(x) = a_3 + b_3(x-1) + c_3(x-1)^2 + d_3(x-1)^3$$

From the limitation of function value, we have

$$\begin{cases} S_1(-3) = 2 \\ S_1(-2) = S_2(-2) = 0 \\ S_2(1) = S_3(1) = 3 \\ S_3(4) = 1 \end{cases} \tag{10}$$

From the derivation limitation, we have

$$\begin{cases} S_1'(-2) = S_2'(-2) \\ \\ S_2'(1) = S_3'(1) \end{cases} \tag{11}$$

From the second derivation limitation, we have

$$\begin{cases} S_1''(-2) = S_2''(-2) \\ \\ S_2''(1) = S_3''(1) \end{cases} \tag{12}$$

From the Parabolic Termination, we have

$$\begin{cases} S_1''(-3) = S_2''(-2) \\ \\ S_2''(1) = S_3''(4) \end{cases} \tag{13}$$

Solve the equation (10)(11)(12)(13), we have

$$S_1(x) = 2 - \frac{67}{21}(x+3) + \frac{25}{21}(x+3)^2$$

$$S_2(x) = -\frac{17}{21}(x+2) + \frac{25}{21}(x+2)^2 - \frac{37}{189}(x+2)^3$$

$$S_3(x) = 3 + \frac{22}{21}(x-1) - \frac{4}{7}(x-1)^2$$

Question 7

(a) The derivation of $f(x)$ is

$$f'(x) = 6x^2 - 1$$

Thus

$$f'(-2) = 6(-2)^2 - 1 = 24 - 1 = 23$$

$$f'(0) = -1$$

So it is a Clamped Cubic Spline in $[-2, 0]$

(b) For Function Continuity we have

$$f'(x) = 3x^2 - 1$$

$$f''(x) = 6x$$

For $x = 0$ we have

$$f'(0^-) = f'(0^+) = -1$$

$$f''(0^-) = f''(0^+) = 0$$

For function bounds data we have

$$f'(-2) = 3 \cdot 4 - 1 = 11$$

$$f'(2) = 3 \cdot 4 - 1 = 11$$

So it is a Clamped Cubic Spline in $[-2, 2]$

(c)

Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ be any third-degree polynomial defined on a closed interval $[a, b]$.

A clamped cubic spline $S(x)$ on $[a, b]$ satisfies the following properties:

1. $S(x)$ is a piecewise cubic polynomial defined on subintervals of $[a, b]$,
2. $S(x)$, $S'(x)$, and $S''(x)$ are continuous on $[a, b]$,
3. The first derivatives at the endpoints satisfy the clamped (also known as Hermite) boundary conditions:

$$S'(a) = f'(a), S'(b) = f'(b) \tag{14}$$

.

In this case, since $f(x)$ is already a single cubic polynomial, it is trivially a cubic spline over any subinterval partition of $[a, b]$, with no need for piecewise definitions. Additionally, all derivatives of $f(x)$ are continuous everywhere, and particularly on $[a, b]$.

Moreover, because $S(x) = f(x)$, the endpoint derivatives of the spline naturally satisfy equation (14)

which meets the clamped boundary conditions.

Therefore, all requirements of a clamped cubic spline are satisfied by f(x) itself.

Question 8

For a_0, we have

$$a_0 = \frac{1}{P}\int_{-3}^{3} f(x)\,dx = \frac{1}{3}\left[\int_{-3}^{-1}(-1)\,dx + \int_{-1}^{1} x\,dx + \int_{1}^{3} 1\,dx\right] \Rightarrow a_0 = 0$$

For a_j, we have

$$a_j = \frac{1}{3}\left[\int_{-3}^{-1}(-1)\cos\left(\frac{j\pi x}{3}\right)dx + \int_{-1}^{1} x\cos\left(\frac{j\pi x}{3}\right)dx + \int_{1}^{3} 1\cos\left(\frac{j\pi x}{3}\right)dx\right]$$

For b_j, we have

$$b_j = \frac{1}{3}\left[\int_{-3}^{-1}(-1)\sin\left(\frac{j\pi x}{3}\right)dx + \int_{-1}^{1} x\sin\left(\frac{j\pi x}{3}\right)dx + \int_{1}^{3} 1\sin\left(\frac{j\pi x}{3}\right)dx\right]$$

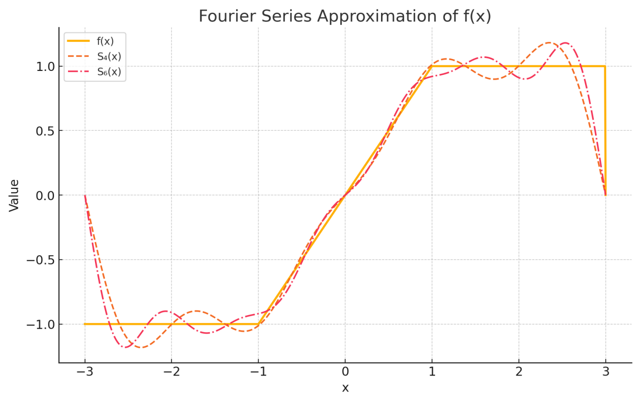
Solve, we have

$$a_j = 0$$

and for b_j, we have

$$b_1 \approx 1.6310$$

$b_2 \approx -0.18669$
$b_3 \approx 0.21221$
$b_4 \approx -0.19206$
$b_5 \approx 0.10626$
$b_6 \approx -0.10610$ The graph of S_4 and S_6 is



Question 9

(a) $P\prime\prime(0)$

Let us express the interpolating polynomial using the Lagrange form:

$P(x) = \sum_{i=0}^{N} P_i \cdot \ell_i(x)$, where $\ell_i(x)$ is the Lagrange basis polynomial defined by:

$$\ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{N} \frac{x - x_j}{x_i - x_j}.$$

Taking the second derivative:

$$P\prime\prime(x) = \sum_{i=0}^{N} P_i \cdot \ell_i\prime\prime(x)$$

.

To evaluate $P\prime\prime(0)$, we compute:

$$P\prime\prime(0) = \sum_{i=0}^{N} P_i \cdot \ell_i\prime\prime(0)$$

.

In the case of equally spaced nodes $x_i = \frac{i}{N}$, it is known that:

$$\ell_0\prime\prime(0) = N(N-1), \quad \ell_1\prime\prime(0) = -2N(N-1), \quad \ell_2\prime\prime(0) = N(N-1)$$

, and

$$\ell_i\prime\prime(0) = 0 for i \geq 3$$

.

Thus,

$$P\prime\prime(0) = P_0 \cdot N(N-1) + P_1 \cdot (-2N(N-1)) + P_2 \cdot N(N-1)$$

,

$$P''(0) = N(N-1)(P_2 - 2P_1 + P_0)$$

.

(b) $P''(1)$
Similarly, we compute:

$$P''(1) = \sum_{i=0}^{N} P_i \cdot \ell_i''(1)$$

.

In the case of equally spaced nodes, we have:

$$\ell_N''(1) = N(N-1), \quad \ell_{N-1}''(1) = -2N(N-1), \quad \ell_{N-2}''(1) = N(N-1)$$

, and

$$\ell_i''(1) = 0 \textbf{ for } i \leq N-3$$

.

Thus, we have

$$P''(1) = P_N \cdot N(N-1) + P_{N-1} \cdot (-2N(N-1)) + P_{N-2} \cdot N(N-1)$$

,

$$P''(1) = N(N-1)(P_N - 2P_{N-1} + P_{N-2})$$

.

Question 10

```
● (base) ➜  Homework5 git:(main) ✗ python problem10.py
  S_0(t) = 0.00 + 0.00(t-0.0) + 8.38(t-0.0)^2 + 0.81(t-0.0)^3
  S_1(t) = 40.00 + 43.25(t-2.0) + 13.25(t-2.0)^2 + -2.44(t-2.0)^3
  S_2(t) = 160.00 + 67.00(t-4.0) + -1.37(t-4.0)^2 + 1.44(t-4.0)^3
  S_3(t) = 300.00 + 78.75(t-6.0) + 7.25(t-6.0)^2 + -0.81(t-6.0)^3
```