International Baccalaureate

MATHEMATICS

Analysis and Approaches (SL and HL)

Lecture Notes

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TOPIC 1
NUMBER AND ALGEBRA

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1.1 NUMBERS – ROUNDING

♦ NOTATION FOR SETS OF NUMBERS

Remember the following known sets of numbers:

N = {0, 1, 2, 3, 4, …}	**natural**	
Z = {0, ±1, ±2, ±3, …}	**integers**	
Q = $\{\frac{a}{b}: a,b\in Z,\ b\neq 0\}$	**rational**	(fractions of integers)
R = rational + irrational	**real**	

Known irrational numbers:

$\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and all $\sqrt{a}$ where a is not a perfect square

$\pi = 3.14159....$

$e = 2.7182818...$

To indicate particular subsets we use the indices +,-, * as follows:

$Z^+ = \{1, 2, 3, \ldots\}$	**positive integers**
$Z^- = \{-1, -2, -3, \ldots\}$	**negative integers**
$Z^* = \{\pm 1, \pm 2, \pm 3, \ldots\}$	**non-zero integers i.e.** $Z^* = Z - \{0\}$

Similar notations apply for the other sets above.

For intervals of real numbers we use the following notations:

$x\in[a,b]$	for $a \le x \le b$
$x\in]a,b[$ or $x\in(a,b)$	for $a < x < b$
$x\in[a,b[$ or $x\in[a,b)$	for $a \le x < b$
$x\in[a,+\infty[$ or $x\in[a,+\infty)$	for $x \ge a$
$x\in]-\infty,a]$ or $x\in(-\infty,a]$	for $x \le a$
$x\in]-\infty,a]\cup[b,+\infty[$	for $x \le a$ or $x \ge b$

I have to continue my notes with a – not so pleasant – discussion about rounding of numbers. The numerical answer to a problem is not always **exact** and we have to use some rounding.

♦ DECIMAL PLACES vs SIGNIFICANT FIGURES

Consider the number

123.4567

There are two ways to round up the number by using fewer digits:

- To a specific number of **decimal places (d.p.)**

to 1 d.p.	123.5
to 2 d.p.	123.46
to 3 d.p.	123.457

We can also round up before the decimal point:

to the nearest integer	123
to the nearest 10	120
to the nearest 100	100

- To a specific number of **significant figures (s.f.)**: for the position of rounding, we start counting from the first non-zero digit:

to 4 s.f.	123.5
to 5 s.f.	123.46
to 6 s.f.	123.457

But also

to 2 s.f.	120
to 1 s.f.	100

Notice that the number at the critical position

remains as it is	if the following digit is 0, 1, 2, 3, 4
Increases by 1	if the following digit is 5, 6, 7, 8, 9

EXAMPLE 1

Consider the number

$$0.04362018$$

to decimal places		to significant figures	
to 2 d.p.	0.04	to 2 s.f.	0.044
to 3 d.p.	0.044	to 3 s.f.	0.0436
to 4 d.p.	0.0436	to 4 s.f.	0.04362
to 6 d.p.	0.043620	to 5 s.f.	0.043620

Important remark: In the final IB exams the requirement is to give the answers either in **exact** form or **to 3 s.f.** . For example

exact form	to 3sf
$\sqrt{2}$	1.41
2π	6.28
12348	12300

♦ THE SCIENTIFIC FORM $a\times10^k$

Any number can be written in the form

$$a\times10^k \quad \text{where} \quad 1 \le a < 10$$

We simply move the decimal point after the first non-zero digit. For example, the number

$$123.4567 \quad \text{can be written as} \quad 1.234567\times10^2$$

Indeed,

$$1.234567\times10^2 = 1.234567\times100 = 123.4567$$

Notice that

we moved the decimal point 2 positions **to the left**

$$\Rightarrow k = 2$$

Even for a "small" number, say

$$0.000012345$$

we can find such an expression:

$$1.2345\times10^{-5}$$

Notice that

we moved the decimal point 5 positions **to the right**

$$\Rightarrow k = -5$$

NOTICE:

- They may ask us to give the number in scientific form but also to 3 s.f. Then

$$1.2345\times10^{2} \cong 1.23\times10^{2}$$
$$1.2345\times10^{-5} \cong 1.23\times10^{-5}$$

- Many calculators use the symbol E±-- for the scientific notation:
  The notation 1.2345E+02 means 1.2345×10^{2}
  The notation 1.2345E-05 means 1.2345×10^{-5}

EXAMPLE 2

(a) Give the scientific form of the numbers

$x = 100000$ $y = 0.00001$ $z = 4057.52$ $w = 0.00107$

(b) Give the standard form of the numbers

$s = 4.501\times10^{7}$ $t = 4.501\times10^{-7}$

Solution

(a) $x = 1\times10^{5}$

$y = 1\times10^{-5}$

$z = 4.05752\times10^{3}$

$w = 1.07\times10^{-3}$

(b) $s = 45010000$

$t = 0.0000004501$

EXAMPLE 3

Consider the numbers

$x = 3\times10^7$ and $y = 4\times10^7$

Give $x+y$ and xy in scientific form.

Solution

$x+y = 7\times10^7$	[add 3+4] [keep the same exponent]
$xy = 12\times10^{14}$	[multiply 3×4] [add exponents]
$= 1.2\times10^{15}$	[modify a so that $1 \le a < 10$]

EXAMPLE 4

Consider the numbers

$x = 3\times10^7$ and $y = 4\times10^9$

Give $x+y$ and xy in scientific form.

Solution

For addition we must modify y (or x) in order to achieve similar forms

$x = 3\times10^7$

$y = 4\times10^9 = 400\times10^7$

$x+y = 403\times10^7$	[add 3+400] [keep the same exponent]
$= 4.03\times10^9$	[modify a so that $1 \le a < 10$]

For multiplication there is no need to modify y:

$xy = 12\times10^{16}$	[multiply 3×4] [add exponents]
$= 1.2\times10^{17}$	[modify a so that $1 \le a < 10$]

1.2 SEQUENCES IN GENERAL – SERIES

♦ SEQUENCE

A ***sequence*** is just an ordered list of numbers (**terms** in a definite order). For example

2,	5,	13,	5,	−4,	…
↑	↑	↑	↑	↑	
1st term	2nd term	3rd term	4th term	5th term	

Usually, the terms of a sequence follow a specific pattern, for example

$0,2,4,6,8,10,\ldots$ (even numbers)

$1,3,5,7,9,11,\ldots$ (odd numbers)

$5,10,15,20,25,\ldots$ (positive multiples of 5)

$2,4,8,16,32,\ldots$ (powers of 2)

We use the notation u_n to describe the n-th term. Thus, the terms of the sequence are denoted by

$$u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ \ldots$$

♦ SERIES

A ***series*** is just a sum of terms:

$S_n = u_1 + u_2 + u_3 + \cdots + u_n$ (the sum of the first n terms)

$S_\infty = u_1 + u_2 + u_3 + \cdots$ (the sum of all terms, ∞ terms)

We say that S_∞ is an infinite series, while the finite sums $S_1, S_2, S_3,\ldots$ are called **partial sums**.

EXAMPLE 1

Consider the sequence

$1, 3, 5, 7, 9, 11, \ldots$ (odd numbers)

Some of the terms are the following

$u_1 = 1$, $u_2 = 3$, $u_3 = 5$, $u_6 = 11$, $u_{10} = 19$

Also,

$S_1 = 1$,

$S_2 = 1+3 = 4$,

$S_3 = 1+3+5 = 9$,

$S_4 = 1+3+5+7 = 16$

Finally,

$S_\infty = 1+3+5+7+\cdots$ (in this case the result is $+\infty$)

♦ SIGMA NOTATION ($\sum_{n=1}^{k}$)

Instead of writing

$$u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8 + u_9$$

we may write

$$\sum_{n=1}^{9} u_n$$

It stands for the sum of all terms u_n, where n ranges from 1 to 9. In general,

$$\boxed{\sum_{n=1}^{k} u_n}$$

expresses the sum of all terms u_n, where n ranges from 1 to k.

We may also start with another value for n, instead of 1, e.g. $\sum_{n=4}^{9} u_n$

EXAMPLE 2

- $\sum_{n=1}^{3} 2^n = 2^1 + 2^2 + 2^3 = 2+4+8 = 14$
- $\sum_{n=1}^{4} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{12+6+4+3}{12} = \frac{25}{12}$
- $\sum_{k=1}^{3} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{4+2+1}{8} = \frac{7}{8}$
- $\sum_{n=3}^{6} (2n+1) = 7+9+11+13 = 40$
- $\sum_{x=3}^{20} \frac{x}{x+2} = \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \cdots + \frac{20}{22} =$ ... *whatever that is, I don't mind!!!*

We can also express an infinite sum as follows

- $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ *(it never finishes!)*

The result is 1. (I know it looks strange, but believe me, it is right!)

♦ *NOTICE*

There are two basic ways to describe a sequence

A) *by a GENERAL FORMULA*

We just describe the general term u_n in terms of n.

For example, $u_n = 2n$ *(It gives $u_1 = 2$, $u_2 = 4$, $u_3 = 6$, ...)*

It is the sequence 2,4,6,8,10,...

EXAMPLE 3

$u_n = n^2$ *is the sequence* $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

that is 1, 4, 9, 16, 25, ...

$u_n = 2^n$ *is the sequence* 2, 4, 8, 16, 32, ...

B) by a RECURSIVE RELATION (mainly for Math HL)

Given: u_1 , the first term

u_{n+1} in terms of u_n

For example,

$u_1 = 10$

$u_{n+1} = u_n + 2$

This says that the first term is 10 and then

$u_2 = u_1 + 2$

$u_3 = u_2 + 2$

$u_4 = u_3 + 2$ and so on.

In simple words, begin with 10 and keep adding 2 in order to find the following term.

It is the sequence 10, 12, 14, 16, 18, …

EXAMPLE 4

$u_1 = 3 \qquad u_{n+1} = 2u_n + 5$

It is the sequence 3, 11, 27, 59, …

EXAMPLE 5

Sometimes, we are given the first two terms u_1, u_2 and then a recursive formula for u_{n+1} in terms of u_n and u_{n-1}.

The most famous sequence of this form is the **Fibonacci sequence**

$u_1 = 1$, $u_2 = 1$

$u_{n+1} = u_n + u_{n-1}$

In other words,

we add u_1, u_2 in order to obtain u_3,

we add u_2, u_3 in order to obtain u_4, and so on.

It is the sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

1.3 ARITHMETIC SEQUENCE (A.S.)

♦ THE DEFINITION

Let's start with an example! I give you the first term of a sequence, say $u_1=5$, and I always ask you to add a fixed value, say $d=3$, in order to find the next term. The following sequence is generated

5, 8, 11, 14, 17, …

Such a sequence is called **arithmetic**. That is, in an arithmetic sequence the difference between any two consecutive terms is constant.

We only need

The **first term**	u_1
The **common difference**	d

EXAMPLE 1

If $u_1=1$, $d=2$ the sequence is 1, 3, 5, 7, 9, …

If $u_1=2$, $d=2$ the sequence is 2, 4, 8, 10, 12, …

If $u_1=-10$, $d=5$ the sequence is −10, −5, 0, 5, 10, …

If $u_1=10$, $d=-3$ the sequence is 10, 7, 4, 1, −2, …

Notice that the common difference d may also be negative!

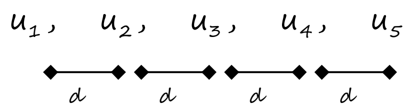
♦ **QUESTION A**: What is the general formula for u_n?

If we know u_1 and d, then

$$\boxed{u_n = u_1 + (n-1)d}$$

Indeed, let us think:

In order to find u_5, we start from u_1 and then add 4 times the difference d



Hence, $u_5 = u_1 + 4d$

Similarly, $u_{10} = u_1 + 9d$

$u_{50} = u_1 + 49d$

In general, $u_n = u_1 + (n-1)d$

EXAMPLE 2

In an arithmetic sequence let u_1=3 and d=5. Find

(a) the first four terms (b) the 100th term

Solution

(a) 3, 8, 13, 18

(b) Now we need the general formula

$$u_{100} = u_1 + 99d = 3 + 99 \cdot 5 = 498$$

EXAMPLE 3

In an arithmetic sequence let u_1=100 and $u_{16} = 145$. Find u_7

Solution

We know u_1, we need d. We exploit the information for u_{16} first.

$u_{16} = u_1 + 15d$

145 = 100 + 15d

45 = 15d

d=3

Therefore, $u_7 = u_1 + 6d = 100 + 6 \cdot 3 = 118$

REMEMBER: Usually, our first task in an A.S. is to find the basic elements, u_1 and d, and then everything else!

EXAMPLE 4

In an arithmetic sequence let $u_{10}=42$ and $u_{19}=87$. Find u_{100}

Solution

The formula for u_{10} and u_{19} takes the form

$$u_{10}=u_1+9d \quad \text{thus} \quad u_1+9d=42 \quad (a)$$
$$u_{19}=u_1+18d \quad\quad u_1+18d=87 \quad (b)$$

Subtract (b)-(a):
$$18d-9d=87-42$$
$$9d=45$$
$$d=5$$

Then, (a) gives
$$u_1=42-9d$$
$$=42-9\cdot 5$$
$$=-3$$

Since we know $u_1=-3$ and $d = 5$ we are able to find any term we like! Thus,

$$u_{100}=u_1+99d=-3+99\cdot 5=492$$

♦ **QUESTION B**: What is the sum S_n of the first n terms?

It is directly given by

$$\boxed{S_n=\frac{n}{2}(u_1+u_n) \qquad (1)}$$

or otherwise by

$$\boxed{S_n=\frac{n}{2}[2u_1+(n-1)d] \qquad (2)}$$

NOTICE: Use (1) if you know u_1 and the last term u_n

Use (2) if you know u_1 and d (the basic elements)

EXAMPLE 5

For the A.S. 3, 5, 7, 9, 11, … find S_3 and S_{101}

Solution

We have u_1=3 and d=2. For S_3 the result is direct:

$$S_3 = 3+5+7 = 15$$

[check though that formulas (1), (2) give the same result for S_3]

For S_{101} we use formula (2)

$$S_{101} = \frac{101}{2}[2u_1 + 100d] = \frac{101}{2}206 = 10403$$

EXAMPLE 6

Find 10 + 20 + 30 + … + 200

Solution

We have an arithmetic sequence with u_1=10 and d=10.

The number of terms is clearly 20 and u_{20}=200

$$S_{20} = \frac{20}{2}(u_1 + u_{20}) = 10\ (10+200) = 2100$$

EXAMPLE 7

Show that

$$\boxed{1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}}$$

Solution

This is the simplest arithmetic series with u_1=1 and d=1.

We ask for S_n

$$S_n = \frac{n}{2}(u_1 + u_n) = \frac{n}{2}(1+n) = \frac{n(n+1)}{2}$$

For example,

$$1+2+3+ \ldots + 100 = \frac{100 \cdot 101}{2} = 5050$$

EXAMPLE 8

The 3rd term of an A.S. is zero while the sum of the first 15 terms is -300. Find the first term and the sum of the first ten terms.

Solution

Well, too much information!!! Let us organize our data:

GIVEN: $u_3=0$ $S_{15}=-300$

ASK FOR: u_1 S_{10}

The formulas for u_3 and S_{15} give

$$u_3 = u_1+2d \quad \Leftrightarrow \quad 0=u_1+2d$$

$$S_{15} = \frac{15}{2}(2u_1+14d) \quad \Leftrightarrow -300=15u_1+105d$$

We solve the system

$$u_1+2d=0$$
$$15u_1+105d=-300$$

And obtain $u_1=8$ and $d=-4$.

Finally,

$$S_{10} = \frac{10}{2}(2u_1+9d)=5(16-36)=-100$$

♦ NOTICE FOR CONSECUTIVE TERMS

Let

$$a, \ x, \ b$$

be consecutive terms of an arithmetic sequence (we don't mind if these are the first three terms or some other three consecutive terms). The common difference is equal to

$$x - a = b - x$$

Hence, $2x=a+b$, that is $x = \frac{a+b}{2}$ (x is the mean of a and b)

EXAMPLE 9

Let $x+1$, $3x$, $6x-5$ be consecutive terms of an A.S. Find x.

Solution

It holds $(3x)-(x+1) = (6x-5)-(3x)$

$\Leftrightarrow \quad 2x-1 = 3x-5$

$\Leftrightarrow \quad x = 4$

(Indeed, the three terns are 5, 12, 19)

EXAMPLE 10

Let a, 10, b, $a+b$ be consecutive terms of an A.S. Find a and b

Solution

Clearly $10-a = b-10 = (a+b)-b$

that is $10-a = b-10 = a$

Hence,

$$10-a = a \Leftrightarrow 2a = 10 \Leftrightarrow a = 5$$

$$b-10 = a \Leftrightarrow b-10 = 5 \Leftrightarrow b = 15$$

EXAMPLE 11

Let 100, a, b, c, 200 be consecutive terms of an A.S. Find the values of a, b and c.

Solution

Notice that 100, b, 200 are also in arithmetic sequence.

Thus b is the mean of 100 and 200, that is $b=150$

Now

a is the mean of 100 and 150, that is $a = 125$

c is the mean of 150 and 200, that is $c = 175$

1.4 GEOMETRIC SEQUENCE (G.S.)

♦ THE DEFINITION

I give you the first term of a sequence, say $u_1=5$ and this time I ask you to multiply by a fixed number, say $r=2$, in order to find the next term. The following sequence is generated

$$5, 10, 20, 40, 80, \ldots$$

Such a sequence is called **geometric**. That is, in a geometric sequence the ratio between any two consecutive terms is constant.

We only need

The **first term**	u_1
The **common ratio**	r

EXAMPLE 1

(a) $u_1=1$, $r=2$ the sequence is $1, 2, 4, 8, 16, 32, 64, \ldots$

(b) $u_1=5$, $r=10$ the sequence is $5, 50, 500, 5000, \ldots$

(c) $u_1=1$, $r=-2$ the sequence is $1, -2, 4, -8, 16, \ldots$

(d) $u_1=1$, $r=\dfrac{1}{2}$ the sequence is $1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \dfrac{1}{32}, \ldots$

(e) $u_1=1$, $r=-\dfrac{1}{2}$ the sequence is $1, -\dfrac{1}{2}, \dfrac{1}{4}, -\dfrac{1}{8}, \dfrac{1}{16}, -\dfrac{1}{32}, \ldots$

NOTICE:

- The common ratio r may also be negative! In this case the signs alternate (+, –, +, –, …) [see (c) and (e) above].
- The common ratio r may be between –1 and 1, that is $|r|<1$. In such a sequence the terms approach 0 [see (d) and (e) above].

♦ **QUESTION A**: What is the general formula for u_n?

If we know u_1 and r, then

$$\boxed{u_n = u_1 r^{n-1}}$$

Indeed, let us think:

In order to find u_5, we start from u_1 and then multiply 4 times by the ratio r

$$u_1, \quad u_2, \quad u_3, \quad u_4, \quad u_5$$

$$\xrightarrow{\;r\;}\;\xrightarrow{\;r\;}\;\xrightarrow{\;r\;}\;\xrightarrow{\;r\;}$$

Hence, $u_5 = u_1 r^4$

Similarly, $u_{10} = u_1 r^9$, $u_{50} = u_1 r^{49}$. In general, $u_n = u_1 r^{n-1}$

EXAMPLE 2

In a geometric sequence let u_1=3 and r =2. Find

(a) the first four terms (b) the 100th term

Solution

(a) 3, 6, 12, 24

(b) Now we need the general formula

$$u_{100} = u_1 r^{99} = 3 \cdot 2^{99}$$ (too big! This answer is enough!)

EXAMPLE 3

In a geometric sequence let u_1=10 and $u_{10} = 196830$. Find u_3

Solution

We know u_1, we need r. We exploit the information for u_{10} first.

$$u_{10} = u_1 r^9 \iff 196830 = 10 \cdot r^9$$
$$\iff r^9 = 19683$$
$$\iff r = \sqrt[9]{19683} = 3$$

Therefore, $u_3 = u_1 r^2 = 10 \cdot 3^2 = 90$

REMEMBER: Our first task in a G.S. is to find the basic elements, u_1 and r, and then everything else!

EXAMPLE 4

A geometric sequence has a **fifth** term of 3 and a **seventh** term of 0.75. Find

a) the first term u_1 and the common ratio r

b) the tenth term.

Solution

We know that $u_5 = 3$ and $u_7 = 0.75$.

a) The formula for u_5 and u_7 takes the form

$$u_5 = u_1 r^4$$
$$u_7 = u_1 r^6$$

Divide u_7 by u_5: $\dfrac{u_7}{u_5} = \dfrac{u_1 r^6}{u_1 r^4}$

$$\frac{0.75}{3} = r^2$$

$$r^2 = 0.25$$

$$r = \pm 0.5$$

Then, the first equation gives $u_5 = u_1 r^4$

$$3 = u_1 0.0625$$

$$u_1 = \frac{3}{0.0625} = 48$$

b) It is $u_{10} = u_1 r^9$

If r = 0.5 then $u_{10} = 48(0.5)^9 = 0.09375$

If r = –0.5 then $u_{10} = 48(-0.5)^9 = -0.09375$

Notice: there are two sequences that satisfy our criteria:

1st : 48, 24, 12, 6, $\boxed{3}$, 1.5, $\boxed{0.75}$, ... (r = 0.5)

2nd : 48, –24, 12, –6, $\boxed{3}$, –1.5, $\boxed{0.75}$, ... (r = –0.5)

u_5 u_7

♦ **QUESTION B**: What is the sum S_n of the first n terms?

Given that $r \neq 1$, the result is given by

$$\boxed{S_n = \frac{u_1(r^n - 1)}{r-1}} \quad \text{or} \quad \boxed{S_n = \frac{u_1(1-r^n)}{1-r}}$$

Proof (mainly for Math HL)

By definition

$$S_n = u_1 + u_1 r + u_1 r^2 + \cdots + u_1 r^{n-2} + u_1 r^{n-1} \qquad (1)$$

$$r \cdot S_n = u_1 r + u_1 r^2 + u_1 r^3 + \cdots + u_1 r^{n-1} + u_1 r^n \qquad (2)$$

Hence, (2)-(1) gives

$$r \cdot S_n - S_n = u_1 r^n - u_1$$

$$(r-1)\,S_n = u_1(r^n - 1)$$

and finally, $$S_n = \frac{u_1(r^n-1)}{r-1}$$

EXAMPLE 5

Consider the sum $2 + 2^2 + 2^3 + \cdots + 2^{10}$

It is a geometric series of 10 terms with u_1=2 and r=2

Hence, the sum is

$$S_{10} = \frac{u_1(r^{10}-1)}{r-1} = \frac{2(2^{10}-1)}{2-1} = 2046$$

EXAMPLE 6

Consider the sum $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{10}}$

It is a geometric series of 11 terms with u_1=1 and r=1/2

Hence, the sum is given by (prefer the second version of S_n)

$$S_{11} = \frac{u_1(1-r^{11})}{1-r} = \frac{1\cdot\left(1-\frac{1}{2}^{11}\right)}{1-\frac{1}{2}} = \frac{1-\frac{1}{2048}}{\frac{1}{2}} = \frac{2047}{1024}$$

EXAMPLE 7

In a G.S. $u_2=-30$ and $S_2=-15$. Find u_1 and r.

$$S_2=u_1+u_2 \Leftrightarrow -15 = u_1-30 \Leftrightarrow u_1 = 15$$

Since $u_1 = 15$ and $u_2=-30$, we obtain $r = \frac{u_2}{u_1} = \frac{-30}{15} = -2$

[Indeed, it is the G.S. 15,-30,60,-120,… with $u_2=-30$, $S_2=-15$]

♦ NOTICE FOR CONSECUTIVE TERMS

Let

$$a, \ x, \ b$$

be consecutive terms of a geometric sequence (we don't mind if these are the first three terms or some other three consecutive terms). The common ratio is equal to

$$\frac{x}{a} = \frac{b}{x}$$

For example, if 10, x, 90 are consecutive terms in a G.S.

$$\frac{x}{10} = \frac{90}{x} \Rightarrow x^2 = 900 \Rightarrow x = \pm 30 \quad \text{(two solutions)}$$

♦ THE SUM OF ∞ TERMS IN A G.S

Consider the sum of the infinite geometric sequence

$$S_\infty = u_1 + u_2 + u_3 + \cdots \qquad \text{(it never stops!)}$$

The result exists only if $-1< r <1$. It is given by the formula

$$\boxed{S_\infty = \frac{u_1}{1-r}}$$

In this case we say that the series **converges**.

Otherwise (that is if $|r|>1$) we say that the series ***diverges***.

Three Proofs of the formula (mainly for Math HL)

(a) Consider the formula for S_n:

$$S_n = \frac{u_1(r^n - 1)}{r - 1}$$

If $n \to \infty$ (i.e. n tends to infinity) then $r^n \to 0$ (since $-1 < r < 1$) and

$$S_n \to \frac{u_1(0-1)}{r-1} = \frac{u_1}{1-r}$$

(b) An alternative proof is similar to that for S_n.

$$S_\infty = u_1 + u_1 r + u_1 r^2 + u_1 r^3 + \cdots \qquad (1)$$
$$r \cdot S_\infty = u_1 r + u_1 r^2 + u_1 r^3 + \cdots \qquad (2)$$

Assuming that S_∞ exists, we may subtract (1)-(2)

$$S_\infty - r \cdot S_\infty = u_1$$
$$\Rightarrow \quad (1-r)\, S_\infty = u_1$$

and finally,

$$S_\infty = \frac{u_1}{1-r}$$

(c) A slight modification of proof (b):

$$S_\infty = u_1 + u_1 r + u_1 r^2 + u_1 r^3 + \cdots$$
$$= u_1 + r\,(u_1 + u_1 r + u_1 r^2 + u_1 r^3 + \cdots)$$
$$= u_1 + r\, S_\infty$$

Hence, (assuming again that S_∞ exists)

$$S_\infty = u_1 + r\, S_\infty$$
$$\Rightarrow S_\infty - r \cdot S_\infty = u_1$$
$$\Rightarrow \quad (1-r)\, S_\infty = u_1$$

and finally,

$$S_\infty = \frac{u_1}{1-r}$$

EXAMPLE 8

Show that $$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = 1 \qquad (*)$$

Solution

This is an infinite G.S. with $u_1=\frac{1}{2}$ and $r=\frac{1}{2}$.

Since $|r|<1$ we obtain

$$S_\infty=\frac{u_1}{1-r}=\frac{\frac{1}{2}}{1-\frac{1}{2}}=\frac{\frac{1}{2}}{\frac{1}{2}}=1$$

EXAMPLE 9

Show that $$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=2$$

Solution

Method A: We just add 1 on both sides of the equation (*) above!

Method B: It is an infinite G.S. with $u_1=1$ and $r=1/2$. Hence,

$$S_\infty=\frac{u_1}{1-r}=\frac{1}{1-\frac{1}{2}}=\frac{1}{\frac{1}{2}}=2$$

EXAMPLE 10

We will show that $$\boxed{0.3333\ldots=\frac{1}{3}}$$

We can write

$$0.3333\ldots = 0.3 + 0.03 + 0.003 + \cdots$$

We have in fact an infinite G.S. with $u_1=0.3$ and $r=0.1$. Hence,

$$0.3333\ldots=\frac{u_1}{1-r}=\frac{0.3}{1-0.1}=\frac{0.3}{0.9}=\frac{1}{3}$$

EXAMPLE 11

Did you know that $\boxed{0.9999... = 1}$?

Indeed,

$$0.9999... = 0.9+0.09+0.009+\cdots = \frac{u_1}{1-r} = \frac{0.9}{1-0.1} = \frac{0.9}{0.9} = 1$$

If you are not persuaded, look at two alternative proofs:

- We know that

  $$0.3333... = \frac{1}{3}$$

  If we multiply both sides by 3 then

  $$0.9999... = \frac{3}{3} = 1$$

- Let $x=0.9999...$ (1)

  Then $10x=9.9999...$ (2)

  We subtract (2)-(1)

  $$10x-x = 9$$

  $$9x = 9$$

  $$x = 1 \qquad (3)$$

  Thus, (1) and (3) give $0.9999... = 1.$

Surprising, isn't it?

1.5 PERCENTAGE CHANGE – FINANCIAL APPLICATIONS

In problems with

Interest rates **r%**

Population growth **r%**, etc

where

some initial quantity **(present value PV)**

increases by **r%** (per year, per month or per any period)

the **future value FV** after **n periods** is given by

$$FV=PV\left(1+\frac{r}{100}\right)^n$$

If the present value PV **decreases** by r% per period, the formula takes the form

$$FV=PV\left(1-\frac{r}{100}\right)^n$$

EXAMPLE 1

(a) An amount of 2000 euros is **invested** at 8% per year. What is the amount returned after 10 years?

(b) An amount of 2000 euros is **depreciated** by 8% every year. What is the amount returned after 10 years?

Solution

For both problems PV=2000, r=8, n=10

(a) $FV=PV\left(1+\frac{r}{100}\right)^n=2000\left(1+\frac{8}{100}\right)^{10}=2000(1.08)^{10}\cong 4317.85$

(b) $FV=PV\left(1-\frac{r}{100}\right)^n=2000\left(1-\frac{8}{100}\right)^{10}=2000(0.92)^{10}\cong 868.78$

Remark: In financial mathematics we usually answer in 2dp.

Notice: Using GDC (Casio) for Exercise 1

MENU – Financial – F2: [Compound Interest]

$n = 10$

$I\% = 8$

$PV = -2000$ (use "–" because we **pay**)

[leave PMT=0, P/Y=1, C/Y=1]

Press **F5**: [FV] and you obtain FV=4317.85

For question (b), set I% =–8. You obtain FV=868.78

In fact, this is a geometric sequence with common ratio $R=1+\frac{r}{100}$

Explanation for the common ratio R

Suppose we invest an amount PV for r% per year.

In our example: PV = 2000 r = 8%

	In our example	**In general**
Present Value	2000	PV
Interest	$2000\times\frac{8}{100}$	$PV\times\frac{r}{100}$
After 1 year	**Present value + interest**	
	$2000+2000\times\frac{8}{100}$ $2000\left(1+\frac{8}{100}\right)$	$PV+PV\frac{r}{100}$ $=PV\left(1+\frac{r}{100}\right)$

That is, if an amount **increases** by r%, we multiply by $R=1+\frac{r}{100}$

After two years we multiply again by R. Thus $FV=PV\left(1+\frac{r}{100}\right)^2$

Thus after n years: $FV=PV\left(1+\frac{r}{100}\right)^n$

For example, r% is translated into a common ratio as follows:

r%	(increasing ratio) $R=1+\frac{r}{100}$	(decreasing ratio) $R=1-\frac{r}{100}$
12%	R = 1.12	R = 0.88
20%	R = 1.20	R = 0.80
5%	R = 1.05	R = 0.95
7.2%	R = 1.072	R = 0.928

Mind the following slight difference:

Is the initial amount given in "year 1" or "today"?

PROBLEM 1	PROBLEM 2
Rate of increase 12% Amount in year 1: $u_1=1000$. Find the amount in year 10. Then • In year 2 $u_2=1000\times(1.12)$ • In year 3 $u_3=1000\times(1.12)^2$ • **In year n** $u_n=\mathbf{1000\times(1.12)^{n-1}}$	Rate of increase 12% Present value: PV=1000. Find the amount after 10 years. Then • After 1 year $1000\times(1.12)$ • After 2 years $1000\times(1.12)^2$ • **After n years** $\mathbf{FV=1000\times(1.12)^n}$

(Mind that the exponent in Problem 2 is **n** and not **n-1**)
In both cases the growth is **exponential** (geometric sequence)

EXAMPLE 2

There are ten boxes in a row. The first box contains 100€ and any subsequent box contains 10% more that the previous one. What is the amount in the 10^{th} box?

Solution

Here $u_1=100$ and $r = 1.10$. Thus $u_{10}=100(1.10)^9 \cong 235.8$

This is in fact the FV formula, but say that **after 9 boxes** $FV=100(1.10)^9$

EXAMPLE 3 (if the question is about the number of years n)

An amount of 2000 euros is invested at 8% per year. The investment exceeds 5000 after n **complete** years. Find n.

Solution

$$FV=2000\left(1+\frac{8}{100}\right)^n>5000$$

Method A (trial and error): We check several values for n by GDC and realize that

for $n=11$ $FV = 4663.27$
for $n=12$ $FV = 5036.34$

Therefore, $\boxed{n = 12}$

Method B (by using GDC-Financial mode):

Set I% = 8
PV = -2000
FV = 5000 [keep PMT=0, P/Y=1, C/Y=1]

Press **F1:** [n] and you obtain n=11.9

Since we are looking for complete years we accept the first integer above 11.9, that is $\boxed{n = 12}$

Method C (by using SolveN in the GDC):

We can solve the corresponding equation FV = 5000, that is

$$2000(1.08)^n = 5000$$

The solution is $n \cong 11.9$. Thus $\boxed{n = 12}$

Method D (by using logarithms: come back after you learn logs!)

We solve the exponential equation $2000(1.08)^n = 5000$ by using logs! The solution is $n = \frac{\log 2.5}{\log 1.08} \cong 11.9$.

Thus $\boxed{n = 12}$

EXAMPLE 4

The current population of a city is 800,000. The population increases by 5.2% every year. Find

(a) the population of the city after 7 years;
(b) the population of the city 7 years ago;
(c) after how many complete years the population of the city doubles.

Solution

(a) We have an exponential growth with PV=800,000 and r=8%

The population of the city after 7 years is

$$FV=800{,}000\left(1+\frac{5.2}{100}\right)^{7} \cong 1{,}140{,}775$$

(b) The formula works for the past as well!

The population of the city 7 years ago was

$$FV=800{,}000\left(1+\frac{5.2}{100}\right)^{-7} \cong 561{,}022$$

[short explanation:
for the future we multiply by 1.052 every year
for the past we divide by 1.052 every year, or otherwise we multiply by $\frac{1}{1.052}=1.052^{-1}$ every year]

(c) we solve the equation

$$FV=2\times 800{,}000$$

$$800{,}000\left(1+\frac{5.2}{100}\right)^{n} = 1{,}600{,}000$$

By using a GDC we find n=13.7

Therefore, the population doubles after 14 complete years.

♦ INTEREST COMPOUNDED IN k TIME PERIODS

Suppose that an initial amount PV=1000€ is invested with an interest rate 12% per year. We have seen what happens if the amount is compounded yearly.

However, the interest may be compounded in k periods per year:

Semiannually (half-yearly): k=2
Quarterly: k=4
Monthly: k=12

For example, if the interest is compounded twice per year (semiannually), the interest rate for the 6-month period is 6% and

Amount after 1 year = $1000\times(1.06)^2$ (1 year = 2 time periods)

In general, the FV formula takes the form

$$FV=PV\left(1+\frac{r}{100k}\right)^{kn}$$

For our example of PV = 1000:

Compounded	**After 5 years**
yearly (k=1)	$FV=1000\left(1+\frac{12}{100}\right)^{5}$ = 1762
half-yearly (k=2)	$FV=1000\left(1+\frac{12}{100\times 2}\right)^{2\times 5}$ = 1791
quarterly (k=4)	$FV=1000\left(1+\frac{12}{100\times 4}\right)^{4\times 5}$ = 1806
monthly (k=12)	$FV=1000\left(1+\frac{12}{100\times 12}\right)^{12\times 5}$ = 1817

<u>NOTICE for GDC-Financial mode</u>

C/Y stands for the number of periods per year. Check FV for

n = 5 PMT=0
I% = 12 P/Y =1
PV = -1000 **C/Y = 1, 2, 4, 12 respectively**

- INVESTMENT WITH EXTRA REGULAR PAYMENTS

    PV is invested with an annual interest rate **r%** .
    An extra payment (**PMT**) is **added** at the **end** of each year

In this case we mainly use **GDC-Financial mode**.
    Insert PMT as a negative value [we pay]

EXAMPLE 5

An amount of 1000€ is invested with an interest rate 12% compounded yearly. An extra payment of 300€ is added at the **end** of each year. Find the value of the investment after 7 years.

By using GDC–Financial mode

n = 7	PMT= -300
I% = 12	P/Y =1
PV = -1000	C/Y = 1

FV gives 5237.38.

Remark: if the last payment is not included we subtract one PMT:
Value = FV-PMT = 5237.38-300 = 4937.38

EXAMPLE 6

An initial amount of 1000€ and then an extra amount of 1000€ at the end of **each year** are invested with an interest rate 12% compounded yearly. Find the value of the investment after 7 years.

By using GDC–Financial mode

n = 7	PMT= -1000
I% = 12	P/Y =1
PV = -1000	C/Y = 1

FV gives 12299.69.

Remark: if the last payment is not included we subtract one PMT:
Value = FV-PMT = 12299.69-1000 = 11299.69

EXAMPLE 7

An amount of 1000€ is invested with an interest rate 12% compounded **monthly**. An extra payment of 300€ is added at the **end of each year**. Find the value of the investment after 7 years.

By using GDC–Financial mode

n = 7	PMT= -300
I% = 12	P/Y =1
PV = -1000	C/Y = 12

FV gives 5397.73.

Remark: if the last payment is not included we subtract one PMT:

Value = FV-PMT = 5397.73-300 = 5097.73

NOTICE.

If the annual interest rate **r%** is compounded in k periods and the regular payments also take place in k periods then

$$P/Y = k, \quad C/Y = k$$

But now n is the total number of periods, i.e. n=k×(years).

EXAMPLE 8

An amount of 1000€ is invested with an interest rate 12% compounded **monthly**. An extra payment of 300€ is added at the **end of each month**. Find the value of the investment after 7 years.

By using GDC–Financial mode

n = 7×12 = 84	PMT= -300
I% = 12	P/Y =12
PV = -1000	C/Y = 12

FV gives 41508.41.

Remark: if the last payment is not included, we subtract one PMT:

Value = FV-PMT = 41508.41-300 = 41208.41

Explanation (mainly for HL)

It is worth to know how the results are derived.

If the annual interest rate is **r%**, the ratio of the geom. sequence is:

$$R = 1 + \frac{r}{100}$$

The value of the investment after n years is given by

$$FV = \boxed{PV \times R^n} + \boxed{PMT \times \left(\frac{R^n - 1}{R - 1}\right)}$$

Indeed,

- PV is invested for n years: $PV\left(1 + \frac{r}{100}\right)^n = \boxed{PV \times R^n}$ [1st box]

- 1st PMT is invested for n-1 years: $PMT \times R^{n-1}$
- 2nd PMT is invested for n-2 years: $PMT \times R^{n-2}$
  … …
- last but one PMT is invested for 1 year: $PMT \times R$
- the last payment is PMT PMT

The sum of the last n terms is a G.S. with $u_1 = PMT$, ratio=R

Thus $S_n = \boxed{PMT \times \left(\frac{R^n - 1}{R - 1}\right)}$ [2nd box].

If each payment is equal to the present value (PMT=PV) we obtain in fact a G.S. of n+1 terms:

$$FV = \boxed{PMT \times R^n + PMT \times \left(\frac{R^n - 1}{R - 1}\right)} = \boxed{PMT \times \left(\frac{R^{n+1} - 1}{R - 1}\right)}$$

NOTICE

- *If we don't wish to include the last payment in our calculations we just subtract one payment (PMT)*

Let us revisit the examples of this paragraph.

In EXAMPLE 5

PV = 1000€,

PMT = 300€ at the **end** of each year

r=12% compounded yearly (so R=1.12).

The value of the investment after 7 years is

$$FV = 1000 \times 1.12^7 + 300 \times \left(\frac{1.12^7 - 1}{1.12 - 1}\right) = 5237.38$$

In EXAMPLE 6

PV = 1000€,

PMT = 1000€ at the **end** of each year

r=12% compounded yearly (so R=1.12).

The value of the investment after 7 years is

$$FV = 1000 \times \left(\frac{1.12^8 - 1}{1.12 - 1}\right) = 12299.69$$

NOTICE. If

- the annual interest rate **r%** is compounded in k periods
- the payments occur yearly

then the ratio for 1 year is: $R = \left(1 + \frac{r}{100k}\right)^k$

In EXAMPLE 7

PV = 1000€,

PMT = 300€ at the **end** of each year

r=12% compounded **monthly**. Now the ratio for 1 year is

$$R = \left(1 + \frac{12}{100 \times 12}\right)^{12} = 1.01^{12}$$

The value of the investment after 7 years is

$$FV = 1000 \times R^7 + 300 \times \left(\frac{R^7 - 1}{R - 1}\right) = 5397.73$$

NOTICE. If

- the annual interest rate **r%** is compounded in k periods
- the payments also occur in k periods

then the ratio for 1 period is: $R=\left(1+\frac{r}{100k}\right)$

But now n is the total number of periods, i.e. $n=k\times(\text{years})$.

In EXAMPLE 8

PV = 1000€,

PMT = 300€ at the **end** of each month.

r=12% compounded **monthly**. Now the ratio for 1 **month** is

$$R=\left(1+\frac{12}{100\times 12}\right)=1.01$$

The value of the investment after 7 years (so $n=7\times 12=84$) is

$$FV=1000\times 1.01^{84}+300\times\left(\frac{1.01^{84}-1}{1.01-1}\right)=41508.41$$

♦ ANNUITY – AMORTIZATION

We invest PV but now, we regularly **withdraw** an amount PMT.

Notice for GDC–Financial mode

- We insert the payment PMT as a **positive** value [we **receive**]
- Here, the last payment is always included.

EXAMPLE 9

An amount of 1000€ euros is invested with an interest rate 12% compounded **monthly**. A **withdrawal** of 150€ is made at the end of each **year**. Find the value of the investment after 7 years.

By using GDC–Financial mode

n = 7	PMT= 150
I% = 12	P/Y =1
PV = -1000	C/Y = 12

FV gives 761.22.

We can easily check that in our example,

(annual withdrawal) > (the annual interest)

Thus, FV will be zeroed out after a certain time.

Amortization refers to the time needed for this to occur.

By using GDC–Financial mode in our example above:

Set FV =0 and press n.

We obtain n=15.64, so n=16

For n=15, the GDC gives FV=87.11

For n=16, the GDC gives FV=-51.84

Therefore, the amount left for the last withdrawal is

150-51.84=98.16 euros.

Explanation (mainly for HL)

The only difference is that we subtract the part of the payments

Now

$$\mathbf{FV} = \boxed{PV \times R^n} - \boxed{PMT \times \left(\frac{R^n - 1}{R - 1}\right)}$$

<u>In EXAMPLE 9</u>

The ratio for 1 year is

$$R = \left(1 + \frac{12}{100 \times 12}\right)^{12} = 1.01^{12}$$

The value of the investment after 7 years is

$$FV = 1000 \times R^7 - 150\left(\frac{R^7 - 1}{R - 1}\right) = 761.22$$

For Amortization we solve for n the equation FV=0, i.e.

$$1000 \times R^n = 150\left(\frac{R^n - 1}{R - 1}\right)$$

The solution is 15.64 so the last withdrawal will take place after 16 years.

♦ INFLATION – REAL VALUE OF THE INVESTEMENT

In some problems the **inflation** is also taken into account. Apart from the interest rate r% they also give us the inflation rate a%. Then

the **future value** (FV) of the investment
is translated into
the **real value** (RV) of the investment

Method 1 (the simplistic one):

We subtract the rates: $r' = r - a$

Real value: $RV = PV\times\left(1+\frac{r'}{100}\right)^n$

But this is an approximation, (IB sometimes accepts it!!)

Method 2 (the correct one!):

- We find the future value as usual: $FV = PV\times\left(1+\frac{r}{100}\right)^n$
- For the real value RV, we divide the FV by $\left(1+\frac{a}{100}\right)^n$.

EXAMPLE 10

An amount of 1000€ euros is invested with

- an interest rate 5% per year (r = 5%)
- an inflation rate 2% per year. (a = 2%)

Find the **real value** of the investment after 10 years.

Method 1: r' = 5%-2% = 3%

Real value: **RV** = $1000\times\left(1+\frac{3}{100}\right)^{10} = 1343.92$

Method 2: We find FV first

$$FV = 1000\times\left(1+\frac{5}{100}\right)^{10} = 1628.89$$

Real value: **RV** = $\dfrac{FV}{\left(1+\frac{2}{100}\right)^{10}} = \dfrac{1628.89}{(1.02)^{10}} = 1336.26$

1.6 THE BINOMIAL THEOREM – $(a+b)^n$

♦ THE SYMBOL $n!$

A new symbol called "n factorial", is defined by

$$n! = 1\cdot 2\cdot 3\cdots n$$

For example

$$1! = 1$$
$$2! = 1\cdot 2 = 2$$
$$3! = 1\cdot 2\cdot 3 = 6$$
$$4! = 1\cdot 2\cdot 3\cdot 4 = 24$$
$$5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$$

We also agree that $0! = 1$

(it looks peculiar, I know! Please accept it!)

NOTICE: GDC can be used for the calculation of x!

Select RUN in the MENU: OPTN – PROB – x!

♦ THE SYMBOL nCr OR $\binom{n}{r}$

This symbol is read **"n choose r"**; it is given by the formula

$$\binom{n}{r} = \frac{n!}{r!\cdot(n-r)!}$$

For example,

"5 choose 2" or 5C2 is $\binom{5}{2} = \frac{5!}{2!\cdot 3!} = 10$

"10 choose 3" or 10C3 is $\binom{10}{3} = \frac{10!}{3!\cdot 7!} = 120$

NOTICE: GDC can be used for the calculation of nCr.

Select RUN in the MENU: OPTN – PROB – nCr

For example, for 5C2, press 5, then nCr, then 2. The result is 10

NOTICE:

In fact $\binom{5}{2}$ is the number of ways to choose 2 items out of 5:

Consider 5 items

A, B, C, D, E.

How many **combinations** of 2 items are there?

AB AC AD AE BC BD BE CD CD DE

The answer is 10. That is why 5C2=10 (5 choose 2 equals 10)

♦ THE BINOMIAL THEOREM $(\boldsymbol{a}+\boldsymbol{b})^n$ (PRACTICALLY)

We already know

$$
\begin{aligned}
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4
\end{aligned}
$$

The coefficients may be obtained by **Pascal's Triangle** below

1		
1 1	→	coefficients of $(a+b)^1$
1 2 1	→	coefficients of $(a+b)^2$
1 3 3 1	→	coefficients of $(a+b)^3$
1 4 6 4 1	→	coefficients of $(a+b)^4$
1 5 10 10 5 1	→	coefficients of $(a+b)^5$
	etc	

[Start with 1

1 1

the outboard numbers are always 1;

under a pair of numbers x y write down their sum x+y: x y

x+y

So the next line is 1 2 1; Carry on]

Can you now guess the formula for $(a+b)^5$?

STEP 1: Write down the six terms

$a^5b^0 \quad a^4b^1 \quad a^3b^2 \quad a^2b^3 \quad a^1b^4 \quad a^0b^5$

- *exponents of **a** decrease from 5 to 0*
- *exponents of **b** increase from 0 to 5*
- *the sum of the exponents is always 5*

STEP 2: Obtain the coefficients from **Pascal's triangle** above

1 5 10 10 5 1

Therefore, the expansion of $(a+b)^5$ is

$$(a+b)^5 = 1a^5b^0 \quad +5a^4b^1 \quad +10a^3b^2 \quad +10a^2b^3 \quad +5a^1b^4 \quad +1a^0b^5$$

that is

$$\boxed{(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5}$$

♦ THE BINOMIAL THEOREM $(a+b)^n$ (FORMALLY)

Another way to obtain the coefficients is by using the symbol $\binom{n}{r}$

For $(a+b)^5$:

$$\begin{array}{cccccc} \binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 5 & 10 & 10 & 5 & 1 \end{array}$$

In general,

$$\boxed{(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^0b^n}$$

Notice that the general term is

$$\boxed{\binom{n}{r}a^{n-r}b^r}$$

EXAMPLE 1

Find the expansions of $(2x+3)^3$ and $(2x-3)^3$

Solution

At the beginning we apply the binomial theorem to express $(a+b)^3$

$$(a+b)^3 = \binom{3}{0} a^3b^0 + \binom{3}{1} a^2b^1 + \binom{3}{2} a^1b^2 + \binom{3}{3} a^0b^3$$

i.e.

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

Then

$$\begin{aligned}(2x+3)^3 &= (2x)^3+3(2x)^2 3+3(2x)3^2+3^3 \\ &= 8x^3+36x^2+54x+27\end{aligned}$$

For $(2x-3)^3$, we simply use the substitution a=2x and b=-3

$$\begin{aligned}(2x-3)^3 &= (2x)^3+3(2x)^2(-3)+3(2x)(-3)^2+(-3)^3 \\ &= 8x^3 - 36x^2+54x-27\end{aligned}$$

Notice: if we have $(a-b)^n$ the signs +/- alternate. It is more practical then to use the formula

$$(a-b)^3 = a^3-3a^2b+3ab^2-b^3$$

Similarly

$$(a-b)^4 = a^4-4a^3b+6a^2b^2-4ab^3+b^4$$

$$(a-b)^5 = a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$$

EXAMPLE 2

Expand $(2x-3)^4$

Solution

We apply the formula for $(a-b)^4$

Thus

$$\begin{aligned}(2x-3)^4 &= (2x)^4 - 4(2x)^3(3) + 6(2x)^2(3)^2 - 4(2x)(3)^3 + 3^4 \\ &= 16x^4 - 96x^3 + 216x^2 - 216x + 81\end{aligned}$$

They usually ask us to find only a particular term instead of the whole expansion.

EXAMPLE 3

In the expansion of $(2x-3)^4$ find the term of x^3

Solution

The term is

$$\binom{4}{1}(2x)^3(-3)^1$$

We actually follow three steps

STEP 1: Write down the two terms a and b: $(2x)(-3)$

STEP 2: Split the exponent n=4 in two parts appropriately according to question (we expect x^3) $(2x)^3(-3)^1$

STEP 3: Attach the coefficient $\binom{n}{r}$

(n=4 and r=1 is the exponent of b) $\binom{4}{1}(2x)^3(-3)^1$

Hence, the result is

$$\binom{4}{1}(2x)^3(-3)^1 = 4(8x^3)(-3) = -96x^3$$

EXAMPLE 4

Find the term of x^5 in the expansion of $(2x-3)^7$

Solution

The term is

$$\binom{7}{2}(2x)^5(-3)^2$$

That is, $21(2)^5(-3)^2x^5 = 6048x^5$

EXAMPLE 5

In the expansion of $(2x^2+1)^8$ find the coefficient of x^{10}

Solution

The term is

$$\binom{8}{3}(2x^2)^5(1)^3$$

[Remember

Step 1:	the terms:	$(2x^2)(1)$
Step 2:	split 8 appropriately in order to obtain x^{10}:	$(2x^2)^5(1)^3$
Step 3:	the coefficient $\binom{n}{r}$:	$\binom{8}{3}(2x^2)^5(1)^3$]

Thus, the term is

$$56(2^5)(1)^3x^{10} = 1792\ x^{10}$$

The coefficient is 1792.

EXAMPLE 6

In the expansion of $(2x+\frac{1}{x})^6$ find

(a) the coefficient of x^2 (b) the constant term

Solution

(a) The term is

$$\binom{6}{2}(2x)^4(\frac{1}{x})^2 = 15(2^4)\ \frac{x^4}{x^2} = 240x^2$$

Thus the coefficient is 240.

(b) The constant term is in fact the coefficient of x^0.

In order to eliminate x's we must split n=6 into 3 and 3.

$$\binom{6}{3}(2x)^3(\frac{1}{x})^3 = 20(2^3)\ \frac{x^3}{x^3} = 160$$

EXAMPLE 7

Find the constant term in the expansion of $(2x^2 - \frac{3}{x})^{12}$

Solution

Step 1 $(2x^2)(-\frac{3}{x})$

Step 2 $(2x^2)^4(-\frac{3}{x})^8$ [in order to eliminate x's]

Step 3 $\binom{12}{8}(2x^2)^4(-\frac{3}{x})^8$

The constant term is

$$495(2^4)(-3)^8\ \frac{x^8}{x^8} = 51963120$$

EXAMPLE 8

Find the term of x^5 in the expansion of $(2x+3)(4x+1)^7$

Solution

In the expansion of $(4x+1)^7$ we need two terms:

the term of x^4 (to be combined with $2x$)

the term of x^5 (to be combined with 3)

We respectively find

$$\binom{7}{3}(4x)^4(1)^3 = 35(4^4)x^4 = 8960x^4$$

$$\binom{7}{2}(4x)^5(1)^2 = 21(4^5)x^5 = 21504x^5$$

Therefore, the final term of x^5 is

$$2x(8960x^4) + 3(21504x^5) = 17920x^5 + 64512x^5 = 82432x^5$$

EXAMPLE 9 (mainly for HL)

(a) Verify that $2x^2-3x-2=(2x+1)(x-2)$

(b) Find the coefficient of x^2 in the expansion of $(2x^2-3x-2)^5$

Solution

(a) $(2x+1)(x-2) = 2x^2-4x+x-2 = 2x^2-3x-2$

(b) In fact, we will find the coefficient of x^2 in the expansion of

$$(2x+1)^5(x-2)^5$$

We expand each factor up to the term x^2

$$(1+2x)^5 = 1^5 + 5(1)^4(2x) + \binom{5}{2}(1)^3(2x)^2 + \cdots = 1+10x+40x^2+\cdots$$

$$(-2+x)^5 = (-2)^5 + 5(-2)^4x + \binom{5}{2}(-2)^3x^2 + \cdots = -32+80x-80x^2+\cdots$$

Therefore

$$(2x+1)^5(x-2)^5 = (1+10x+40x^2+\cdots)(-32+80x-80x^2+\cdots)$$

The term of x^2 is

$$\underline{1\cdot(-80x^2)} + \underline{(10x)(80x)} + \underline{40x^2\cdot(-32)} = -560x^2$$

Thus, the coefficient of x^2 is -560.

Later on, in another chapter, we will study an extended version of the binomial theorem, that is $(a+b)^n$ for rational exponents n.

1.7 SIMPLE DEDUCTIVE PROOF

♦ THE DIFFERENCE BETWEEN THE SYMBOLS = AND ≡

It is easy to check that

$$\frac{1}{4}+\frac{1}{12}=\frac{1}{3}$$

Indeed,

$$\frac{1}{4}+\frac{1}{12}=\frac{3}{12}+\frac{1}{12}=\frac{4}{12}=\frac{1}{3}$$

However, there is a more general result that explains this particular example as well:

$$\frac{1}{m+1}+\frac{1}{m^2+m}=\frac{1}{m} \quad (*)$$

Indeed, for m=3 we obtain $\frac{1}{4}+\frac{1}{12}=\frac{1}{3}$

We can easily check that (*) is true for any particular value of m.

We say that this general result is an **identity** to emphasize that it holds for <u>any real value of m</u>. In this case we use the notation "≡"

$$\frac{1}{m+1}+\frac{1}{m^2+m}\equiv\frac{1}{m}$$

[of course it is not a mistake to use "=" instead of "≡" since the equality holds anyway, we only emphasize the "stronger" relation].

Let us explain this difference by another couple of relations:

$$3x = 6$$

$$2(x+3) \equiv 2x+6$$

The first is an "equation": it holds for a particular value of x, **x=2**

The second is an "identity": it holds for **any** value of x.

In this paragraph we show mathematical results by using a simple **deductive proof**. That is, we start from some premise (or premises) and follow logical steps to draw a conclusion.

♦ PROOF OF AN EQUALITY (OR IDENTITY)

Suppose we have to prove an equality [or identity] of the form

$$A = B \qquad or \qquad A \equiv B$$

We will show 3 different techniques

Method A: From LHS to RHS or From RHS to LHS

$$\mathbf{A = \ldots = \ldots = \ldots = B}$$

Let us prove the identity

$$(a+b)^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3$$

Here it is more convenient to start from the LHS and expand. The aim is to obtain the RHS:

$$\begin{aligned}(a+b)^3 &= (a+b)^2(a+b)\\ &= (a^2+2ab+b^2)(a+b)\\ &= a^3+a^2b+2a^2b+2ab^2+ab^2+b^3\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

Method B: From LHS to C – From RHS to C [work separately]

$$\mathbf{A = \ldots = \ldots = \ldots = C}$$
$$\mathbf{B = \ldots = \ldots = \ldots = C}$$

Let us prove the identity

$$(2a+b)^2 - 4a^2 - b^2 \equiv (a+b)^2 - (a-b)^2$$

It is more convenient to work independently on each side.

LHS = $4a^2 + 4ab + b^2 - 4a^2 - b^2 = 4ab$

RHS = $(a^2 + 2ab + b^2) - (a^2 - 2ab + b^2) = 4ab$

Therefore, **LHS = RHS**

Method C: From LHS = RHS to an equivalent true statement

$$A = B \Leftrightarrow \ldots \Leftrightarrow \ldots \Leftrightarrow \text{something true}$$

Given that $a \neq \pm b$, show that

$$\frac{a-2b}{a-b} \equiv \frac{a^2-ab-2b^2}{a^2-b^2}$$

Here, we cannot easily elaborate on any side, neither LHS nor RHS. Thus, we work as follows

$$\frac{a-2b}{a-b} = \frac{a^2-ab-2b^2}{a^2-b^2}$$

$$\Leftrightarrow \quad (a-2b)(a^2-b^2) = (a-b)(a^2-ab-2b^2)$$

$$\Leftrightarrow \quad a^3-ab^2-2a^2b+2b^3 = a^3-a^2b-2ab^2-a^2b+ab^2+2b^3$$

$$\Leftrightarrow \quad 0=0$$

which is true.

Hence, the original relation is true.

Let us prove the identity (*) in the introduction of this paragraph.

EXAMPLE 1

Prove the identity

$$\frac{1}{m+1} + \frac{1}{m^2+m} \equiv \frac{1}{m}$$

Solution

We follow the LHS to RHS proof:

$$\frac{1}{m+1} + \frac{1}{m^2+m} = \frac{1}{m+1} + \frac{1}{m(m+1)}$$

$$= \frac{m}{m(m+1)} + \frac{1}{m(m+1)}$$

$$= \frac{m+1}{m(m+1)}$$

$$= \frac{1}{m}$$

[provided of course that m≠0 and m+1≠0]

EXAMPLE 2

Show that

$$2x^2 - 12x + 19 \equiv 2(x-3)^2 + 1$$

Solution

It is more convenient to work from RHS to LHS

$$2(x-3)^2 + 1 = 2(x^2 - 6x + 9) + 1 = 2x^2 - 12x + 19$$

Let us see another version of the same example:

EXAMPLE 3

Given that

$$2x^2 - 12x + 19 \equiv a(x-b)^2 + c$$

is an identity, find the values of a, b and c.

[Remark: Later on, when studying quadratics, we will show how we can express any quadratic in the RHS form. But here we will show a different technique to see how the strong relation "≡" works]

Solution

We expand the RHS

$$a(x-b)^2 + c = a(x^2 - 2bx + b^2) + c = ax^2 - 2abx + ab^2 + c$$

Since

$$2x^2 - 12x + 19 \equiv ax^2 - 2abx + ab^2 + c$$

the coefficients of the two expressions must be equal:

$$a=2$$
$$-2ab = -12$$
$$ab^2+c = 19$$

Thus a=2.

The second relation gives b=3.

The third relation gives c=1.

ONLY FOR

HL

1.8 METHODS OF PROOF (for HL)

Before we present other methods of proof let us explain the terms

CONVERSE and CONTRAPOSITIVE

of a given statement.

For a statement of the form	if A then B
the **converse** statement is	if B then A
the **contrapositive** statement is	if not B then not A

EXAMPLE 1

For the original statement:	**if $x=0$ then $x^2=0$**
the converse statement is:	**if $x^2=0$ then $x=0$**
the contrapositive statement is:	**if $x^2\neq 0$ then $x\neq 0$**

Remark: Here all the three statements are true.

Notice though that

> The contrapositive of a true statement is always true.
> The converse of a true statement is not necessarily true.

EXAMPLE 2

For the original statement:	**if $x=1$ then $x^2=1$**	**[true]**
the converse statement is:	**if $x^2=1$ then $x=1$**	**[false]**
the contrapositive statement is:	**if $x^2\neq 1$ then $x\neq 1$**	**[true]**

For the following examples, notice that

> the negation of "**and**" is "**or**"
> the negation of "**or**" is "**and**".

EXAMPLE 3

For the original statement: **if ($x=0$ or $y=0$) then $xy=0$**

the converse statement is: **if $xy=0$ then ($x=0$ or $y=0$)**

the contrapositive statement is: **if $xy\neq 0$ then ($x\neq 0$ and $y\neq 0$)**

Remark: Here all the three statements are true.

EXAMPLE 4

For the original statement: **if ($x\geq 0$ and $y\geq 0$) then $xy\geq 0$**

the converse statement is: **if $xy\geq 0$ then ($x\geq 0$ and $y\geq 0$)**

the contrapositive statement is: **if $xy<0$ then ($x<0$ or $y<0$)**

Remark: In this example, the original statement is true, as well as the contrapositive, but the converse statement is false (why?).

It is not an accident that the contrapositive statement is always true. In fact, the statements

if A then B

if not B then not A

are equivalent.

For example,

$$x=1 \Rightarrow x^2=1$$

is equivalent to

$$x^2\neq 1 \Rightarrow x\neq 1$$

Can you think why? If yes, you understood the method of contradiction!

But let us use a non-mathematical example to explain three methods of proof:

DEDUCTION – COUNTEREXAMPLE - CONTRADICTION

♦ THREE METHODS OF PROOF

We will use a simple example to demonstrate three kinds of proof.

Let X be the name of a city

- ***Deductive proof***

The usual process of reasoning is to start from the hypothesis and reach the result by using logical steps. We have already discussed this method in the previous paragraph.

Show that:

If X is a Greek city, then X is a European city

Proof. Assume that X is a Greek city.
But Greece is part of Europe.
Then X is a Europe city.

- ***Proof by a counterexample***

We use a counterexample to establish that a statement **is not true** in general. The converse of the statement above is

If X is a European city, then X is a Greek city.

Show that this statement is not true.

Proof. Select Rome. It is a European city but not in Greece! Hence the statement is not true in general.

- ***Proof by contradiction***

The contrapositive of the original statement is true:

If X is a non-European city, then it is not a Greek city.

Proof. Let X be a non-European city.
Assume that the result is false, i.e. X is a Greek city.
But then X would be a European city! Contradiction.
Thus, the result is true: X **is not a Greek city**.

The principle of contradiction is based on the fact that any statement is equivalent to its contrapositive statement.

Let us see some more mathematical examples.

As we know, integers are either

	even	i.e. of the form $2n$ (multiples of 2)
or	**odd**	i.e. of the form $2n+1$

EXAMPLE 5

if a is even then a^2 is also even.

Deductive proof

$$\begin{aligned} a \text{ is even} &\Rightarrow a=2n && \text{for some } n\in Z \\ &\Rightarrow a^2=4n^2=2(2n^2) && \text{where } 2n^2\in Z \\ &\Rightarrow a^2 \text{ is even.} \end{aligned}$$

We will prove that the converse is also true.

EXAMPLE 6

if a^2 is even then a is even.

Proof by contradiction

Suppose that a^2 is even.

Assume that a is not even (for contradiction) then $a=2n+1$ for some $n\in Z$. Then

$$a^2=(2n+1)^2=4n^2+4n+1=2(2n^2+2n)+1$$

Hence a^2 is odd, contradiction.

Therefore, a is even.

NOTICE.

In fact, in the last example, instead of the statement

if a^2 is even then a is even.

we have shown the equivalent contrapositive statement

if a is odd then a^2 is odd

Let us see what happens with multiples of 4.

EXAMPLE 7

if a is a multiple of 4 then a^2 is also a multiple of 4.

Deductive proof

a is a multiple of 4 $\Rightarrow a = 4n$ for some $n \in Z$

$\Rightarrow a^2 = 16n^2 = 4(4n^2)$ where $4n^2 \in Z$

$\Rightarrow a^2$ is a multiple of 4.

We will prove that the converse is not true.

EXAMPLE 8

a^2 is a multiple of 4 does not imply that a is a multiple of 4.

Proof by counterexample

Choose $a=6$. Then $6^2=36$ is a multiple of 4 while 6 is not.

The following is a typical example of contradiction.

EXAMPLE 9

Show that $\sqrt{2}$ is irrational.

Proof by contradiction.

Assume that $\sqrt{2}$ is rational, that is $\sqrt{2} = \frac{a}{b}$

where a,b are coprime integers [i.e. the fraction is simplified]

Then

$$a = \sqrt{2}\,b \Rightarrow a^2 = 2b^2$$

Thus a^2 is even and so **a is also even**, say $a=2c$.

Then

$$(2c)^2 = 2b^2 \Rightarrow 4c^2 = 2b^2 \Rightarrow b^2 = 2c^2$$

Thus b^2 is even and so **b is also even**.

But a, b cannot be both even, as they are coprime. Contradiction!

An equation of the form

$$ax + by = c$$

has infinitely many solutions.

An interesting question is if it has integer solutions.

For example

$$3x+5y = 19$$

has the integer solutions (x,y)=(3,2) or (x,y)=(-2,5) etc.

EXAMPLE 10

The equation

$$6x + 15y = 100$$

has no integer solutions.

Proof by contradiction.

Assume it has an integer solution (x,y). Then the LHS is a multiple of 3 but the RHS is not, contradiction.

The search of integer solutions in general is of particular interest in real life problems.

EXAMPLE 11

Chris claims that the equation

$$x^2 + y^2 = 100$$

has no integer solutions. Investigate his claim.

Proof by counterexample that the statement is false.

The integer values x=6 and y=8 satisfy the equation. Indeed,

$$x^2 + y^2 = 6^2 + 8^2 = 36 + 64 = 100$$

If we add two rational numbers (i.e. fractions of integers) we get a rational number (a fraction of integers again). We can easily show that by using a deductive proof.

Is this property true for irrational numbers as well?

EXAMPLE 12

If we add two irrational numbers the result is not necessarily irrational.

Proof by counterexample.

$\sqrt{2}$ and $-\sqrt{2}$ are irrational but their sum is 0 which is rational.

- **THE PIGEONHOLE PRINCIPLE**

A classic example of contradiction is the pigeonhole principle:

Suppose that n+1 pigeons are placed in n pigeonholes
Then, there exists a pigeonhole with at least 2 pigeons

Indeed, assuming that all pigeonholes had at most 1 pigeon each, we would have at most n pigeons, contradiction.

EXAMPLE 13

There are 400 people in a club. At least two of them have their birthday on the same day.

Indeed, assume that all of them have their birthday on different days. We would have at most 366 people, contradiction.

A more advanced version of this principle is presented below

EXAMPLE 14

Suppose that 64 pigeons are placed in 7 pigeonholes. Show that some pigeonhole contains at least 10 pigeons.

Assume that all the pigeonholes have at most 9 pigeons. We would have at most 7x9=63 pigeons, contradiction.

1.9 MATHEMATICAL INDUCTION (for HL)

♦ DISCUSSION

Induction is a smart technique to prove propositions of the form

"For any $n \in N$, it holds … P(n)"

or "For any $n \geq 1$, it holds … P(n)"

where P(n) is a statement depending on a natural number n.

Let us work on the following example

For any $n \geq 1$, it holds

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

There are several ways to prove this relation (for example, this is the sum S_n for the arithmetic sequence with $u_1=1$ and $d=1$). However, this is a good example to explain induction.

One way to persuade ourselves about the validity of this relation is to check for several values of n:

For n=1, LHS = 1 RHS=$\frac{1(1+1)}{2}$=1, the result is true!

For n=2, LHS = 1+2=3 RHS=$\frac{2(2+1)}{2}$=3, the result is true!

For n=3, LHS = 1+2+3=6 RHS=$\frac{3(3+1)}{2}$=6, the result is true!

For n=4, LHS = 10 RHS=$\frac{4(5+1)}{2}$=10, the result is true!

But this is not a proof. It is just an indication that the statement is true for any n.

Instead of that, we use the induction technique which consists of three steps:

Induction:

1. We **show** that the statement is true for n=1
2. We **assume** that the statement is true for n=k (some k)
3. We **prove** that the statement is true for n=k+1
   based on the assumption of step 2

NOTICE

Roughly speaking, we construct a mechanism

$$\boxed{k} \longrightarrow \boxed{k+1}$$

which uses the outcome for k, to prove the next outcome for k+1. Step 1 is necessary to switch on the mechanism!

Hence, we have the initial outcome for n=1

Based on that we obtain the next outcome for n=2

Based on that we obtain the next outcome for n=3

and so on (we automatically obtain the result for any n)

Let us present a complete proof for our first example!

EXAMPLE 1

Prove by mathematical induction that

$$1+2+3+\cdots+n = \frac{n(n+1)}{2} \quad \text{for any } n \geq 1$$

Proof

- For n=1 the statement is true. Indeed,

$$LHS = 1, \quad RHS = \frac{1(1+1)}{2} = 1$$

- We **assume** that the statement is true for n=k, that is

$$1+2+3+\cdots+k = \frac{k(k+1)}{2}$$

- We **prove** that the statement is true for n=k+1, that is

$$1+2+3+\cdots+k+(k+1) = \frac{(k+1)(k+2)}{2}$$

Indeed,

$$1+2+3+\cdots+k+(k+1) = (1+2+3+\cdots+k) + (k+1)\ ^{*}$$

$$= \frac{k(k+1)}{2} + (k+1) \qquad \text{[by assumption]}$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2} \qquad \text{QED}$$

Therefore, by mathematical induction, the statement is true for any $n\geq1$.

Notice: The IB examiners love the following blabbering ending:

The statement is true for n=1 and assuming it is true for n=k, it also true for n=k+1. Therefore, by mathematical induction, the statement is true for any $n\geq1$.

EXAMPLE 2

The number 6^n-1 is divisible by 5, for any $n\geq1$.

Proof

- For n=1 the statement is true. Indeed,

$6^1-1=5$ is divisible by 5.

- We **assume** that the statement is true for n=k, that is

6^k-1 is divisible by 5;

---

[*] The main task in step 3 is to embed the assumption inside the proof.

Say $6^k - 1 = 5m$, for some $m \in Z$

- We **prove** that the statement is true for n=k+1, that is

$$6^{k+1} - 1 \text{ is divisible by } 5.$$

Indeed,

$$\begin{aligned} 6^{k+1} - 1 &= 6 \cdot 6^k - 1 \\ &= 6 \cdot (5m+1) - 1 \quad \text{[by assumption]} \\ &= 30m+5 \\ &= 5(6m+1) \quad \text{which is divisible by 5 (QED).} \end{aligned}$$

Therefore, by mathematical induction, the statement is true for any $n \geq 1$.

Sometimes the induction does not begin from n=1.

In general, inequalities are trickier! I suggest the following scheme:

EXAMPLE 3

Prove by mathematical induction that

$$n! > 2^n \quad \text{for any } n \geq 4$$

[we may easily verify that the result is not true for n=1,2,3]

Proof

- For n=4, the statement is true. Indeed,

$$\text{LHS} = 4! = 24, \qquad \text{RHS} = 2^4 = 16 \qquad \text{and } 24 > 16.$$

- We **assume** that the statement is true for n=k, that is

$$k! > 2^k$$

- We **prove** that the statement is true for n=k+1, that is

$$(k+1)! > 2^{k+1}$$

Indeed,

$$\begin{aligned} (k+1)! &= k!\,(k+1) \quad &\text{[by the definition of } n!] \\ &> 2^k\,(k+1) \quad &\text{[since } k! > 2^k \text{ by assumption]} \end{aligned}$$

It **suffices** to show that $2^k (k+1) \geq 2^{k+1}$:

$$2^k (k+1) \geq 2^{k+1} \Leftrightarrow k+1 \geq 2 \Leftrightarrow k \geq 1 \qquad \text{which is true.}$$

Therefore, by induction, the statement is true for any $n \geq 4$.

EXAMPLE 4

Consider the sequence $u_1=0$, $u_{n+1}=2u_n+2$

a) Find the first six terms of the sequence. Is it an arithmetic or a geometric sequence?

b) Compare the results with the first powers of 2. What do you notice? Can you guess a general formula for u_n in terms of n?

c) Prove that your guess is true by mathematical induction.

Solution

a) the first 5 terms of the sequence are the following

$$\underline{0,\ 2,\ 6,\ 14,\ 30,\ 62}$$

[the recursive formula says, begin with 0 and then multiply by 2 and add 2 for each subsequent term]

This sequence is neither arithmetic nor geometric.

b) Look at the first 6 powers of two: 2, 4, 8, 16, 32, 64

It seems that the terms of the sequence can be obtained by the powers of 2 if we subtract 2. That is

$$u_n = 2^n-2$$

c) We will use mathematical induction to prove our guess:

Given $u_1=0$, $u_{n+1}=2u_n+2$, it holds

$$\boxed{u_n = 2^n-2}$$

- For n=1, the statement is true. Indeed,

$$LHS = u_1 = 0 \qquad RHS = 2^1-2 = 0$$

- We **assume** that the statement is true for n=k, that is

$$u_k = 2^k-2$$

- We **prove** that the statement is true for n=k+1, that is

$$u_{k+1} = 2^{k+1} - 2$$

Indeed,

$$\begin{aligned} u_{k+1} &= 2u_k + 2 && \text{[by definition]} \\ &= 2(2^k - 2) + 2 && \text{[by assumption]} \\ &= 2^{k+1} - 4 + 2 \\ &= 2^{k+1} - 2 && \text{QED} \end{aligned}$$

Therefore, by mathematical induction, the statement is true for any $n \geq 1$.

NOTICE

Sometimes, we must assume two preceding steps in order to prove to the next step. These proofs look like

> Induction:
> 1. We **show** that the statement is true for n=1 and n=2
> 2. We **assume** that the statement is true for n=k and n=k+1
> 3. We **prove** that the statement is true for n=k+2 based on the assumptions of step 2.

It will be very clear whether we have to follow this proof.

EXAMPLE 5

Consider the Fibonacci sequence

$$u_1 = 1,\ u_2 = 1, \qquad u_{n+2} = u_n + u_{n+1}$$

[it is the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, …]

Prove by induction that

$$\boxed{u_n < 2^n} \quad \text{for any } n \geq 1$$

Solution

- For n=1 and n=2 the statement is true. Indeed,

$$u_1 = 1 < 2^1 \text{ and } u_2 = 1 < 2^2$$

- We **assume** that the statement is true for n=k and n=k+1, i.e.

$$u_k < 2^k \qquad u_{k+1} < 2^{k+1}$$

- We **prove** that the statement is true for n=k+2, i.e.

$$u_{k+2} < 2^{k+2}$$

Indeed,

$$\begin{aligned} u_{k+2} &= u_k + u_{k+1} && \text{[by definition]} \\ &< 2^k + 2^{k+1} && \text{[by assumption]} \end{aligned}$$

It **suffices** to show that $2^k + 2^{k+1} \leq 2^{k+2}$:

$$\begin{aligned} 2^k+2^{k+1} \leq 2^{k+2} &\Leftrightarrow 1+2 \leq 4 && \text{[just dividing by } 2^k\text{]} \\ &\Leftrightarrow 3 \leq 4 && \text{which is true!} \end{aligned}$$

Therefore, by math induction, the statement is true for any $n \geq 1$.

NOTICE

The critical step is the connection between

the (k+1)-statement and the k-statement,

in order to embed the assumption into the proof. We may use the following table as a guide:

If the statement involves		have in mind that
power of a number	a^n	$a^{k+1} = a^k \cdot a$
	a^{2n}	$a^{2(k+1)} = a^{2k} \cdot a^2$
n factorial	$n!$	$(k+1)! = k! \cdot (k+1)$
Sum of n terms	$u_1+u_2+\cdots+u_n$	$u_1+u_2+\cdots+u_{k+1}=(u_1+u_2+\cdots+u_k)+u_{k+1}$
	$\sum_{r=1}^{n} u_r$	$\sum_{r=1}^{k+1} u_r = \left(\sum_{r=1}^{k} u_r\right) + u_{k+1}$
n^{th} derivative	$f^{(n)}(x)$	$f^{(k+1)}(x) = [f^{(k)}(x)]'$
	$\frac{d^n y}{dx^n}$	$\frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$
$(2n)^{th}$ derivative	$f^{(2n)}(x)$	$f^{(2k+2)}(x) = [f^{(2k)}(x)]''$

1.10 SYSTEMS OF SIMULTANEOUS LINEAR EQUATIONS

In this section we will study systems of equations of the form

2x2 system	3x3 system
$a_1x+b_1y=c_1$ $a_2x+b_2y=c_2$	$a_1x+b_1y+c_1z=d_1$ $a_2x+b_2y+c_2z=d_2$ $a_3x+b_3y+c_3z=d_3$

For example,

$$\begin{aligned} 2x+3y&=9 \\ 4x+7y&=19 \end{aligned} \qquad \begin{aligned} 5x+11y-21z &= -22 \\ x+2y-4z &= -4 \\ 3x-2y+3z &= 11 \end{aligned}$$

For the number of solutions of any nxn system we have the following three cases:

- a unique solution
- no solution
- infinitely many solutions

A system which has solutions (either unique or infinitely many) is said to be **consistent**. Otherwise it is **inconsistent**.

<u>**REMARK**</u>. Even for a single linear equation of the form

$$ax=b$$

(this is a trivial 1x1 system) we have only those three cases:

$a \neq 0$	**Unique solution**	$x=\frac{b}{a}$.
$a=0$ and $b \neq 0$	**No solution**	$0x=5$ has no solution
$a=0$ and $b=0$	∞ **solutions**	$0x=0$, true for any $x \in R$

The equations $2x=3$, $0x=0$ are consistent. The equations $0x=5$ is not

---

♦ 2x2 SYSTEMS

Let us see an example for each case.

EXAMPLE 1

Consider the systems

(a) $2x+3y=9$, $4x+7y=19$ (b) $x+2y=1$, $2x+4y=10$ (c) $x+2y=1$, $2x+4y=2$

- The first system has a unique solution. We can easily obtain

$$(x,y)=(3,1).$$

The two systems (b) and (c) can take the equivalent forms

(b) $x+2y=1$, $x+2y=5$ (c) $x+2y=1$, $x+2y=1$

- System (b) has **no solution** (it is inconsistent)
- System (c) reduces to one equation only: $x+2y=1$.
  The system is consistent with **infinitely many solutions**.
  Indeed, (1,0), (-1,1), (-3,2), (3,-1) are some of the solutions.
  If we set $y=\lambda$ (a real parameter), the set of solutions is given by

$$(x,y) = (1-2\lambda,\lambda) \text{ where } \lambda\in R.$$

Notice. For practice, please use your GDC to obtain the answer for each case above.

♦ 3x3 SYSTEMS

Again, for such a system there are exactly three cases

- a unique solution
- no solution
- infinitely many solutions

Let us see an example for each case and what the GDC gives.

EXAMPLE 2

Consider the systems

(a) $5x+11y-21z=-22$ $x+2y-4z=-4$ $3x-2y+3z=11$

(b) $2x+3y+3z=3$ $x+y-2z=4$ $5x+7y+4z=5$

(c) $2x+3y+3z=3$ $x+y-2z=4$ $5x+7y+4z=10$

For system (a) the GDC gives a unique solution

$$(x,y,z)=(2,-1,1).$$

For system (b) the GDC gives no solution.

For system (c) the GDC gives an infinite number of solutions:

$$x = 9+9\lambda$$
$$y = -5-7\lambda$$
$$z = \lambda \qquad (\lambda\in R \text{ free variable})$$

The **general solution** of the system is

$$(x,y,z)=(9+9\lambda,-5-7\lambda,\lambda).$$

We are going to explain how we obtain the solution for each case by using the so-called Gaussian elimination. We refer only to the case of 3x3 systems (the process for any number of equations is similar).

♦ GAUSSIAN ELIMINATION

Consider the system (a) above, that is

$$\begin{aligned} 5x+11y-21z &= -22 \\ x+2y-4z &= -4 \\ 3x-2y+3z &= 11 \end{aligned}$$

We interchange the first two equations
(it helps to have a leading coefficient 1 in the first equation)

$$\begin{aligned} x + 2y - 4z &= -4 \\ 5x + 11y - 21z &= -22 \\ 3x - 2y + 3z &= 11 \end{aligned}$$

STEP 1: Use equation 1 to eliminate x from equations 2 and 3:

$$\begin{aligned} x + 2y - 4z &= -4 && \\ y - z &= -2 && [\textbf{Equ2} - 5 \times \textbf{Equ1}] \\ -8y + 15z &= 23 && [\textbf{Equ3} - 3 \times \textbf{Equ1}] \end{aligned}$$

STEP 2: Use equation 2 to eliminate y from equation 3:

$$\begin{aligned} x + 2y - 4z &= -4 && \\ y - z &= -2 && \\ 7z &= 7 && [\textbf{Equ3} + 8 \times \textbf{Equ2}] \end{aligned}$$

In fact, we can repeat this process by working on the <u>**augmented matrix**</u> below (where we keep only the coefficients):

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 5 & 11 & -21 & -22 \\ 3 & -2 & 3 & 11 \end{array}\right)$$

Our target is to eliminate the elements below the main diagonal (shown in the triangle below)

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 5 & 11 & -21 & -22 \\ 3 & -2 & 3 & 11 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

We proceed, step by step, to equivalent matrices by performing appropriate row operations. The equivalence between two matrices is denoted by the symbol ~ :

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 0 & 1 & -1 & -2 \\ 0 & -8 & 15 & 23 \end{array}\right) \begin{array}{l} R_1 \\ R_2 - 5R_1 \\ R_3 - 3R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 7 & 7 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 + 8R_2 \end{array}$$

Then we unfold the equations using the so-called **back substitution**:

The 3rd equation gives: $7z=7 \Rightarrow z=1$

The 2nd equation gives $y-z=-2 \Rightarrow y-1=-2 \Rightarrow y=-1$

The 1st: equation gives $x+2y-4z=-4 \Rightarrow x-2-4=-4 \Rightarrow x=2$

Therefore, the unique solution is $(x,y,z)=(2,-1,1)$.

NOTICE

- If the last row was $(0 \quad 0 \quad 0 \,|\, 7)$, the corresponding equation would be $0x+0y+0z=7$ which is impossible (there is no solution)
- If the last row was $(0 \quad 0 \quad 0 \,|\, 0)$, the corresponding equation would be $0x+0y+0z=0$. This equation can be eliminated and the first two equations provide ∞-ly many solutions.

In general, the row operations we may perform in order to obtain equivalent matrices are the following

- Interchange rows (e.g. $R_1 \leftrightarrow R_2$)
- Multiply a row by a scalar (e.g. $R_1 \to 5R_1$)
- Add to a row the multiple of another row (e.g. $R_1 \to R_1 \pm 3R_2$)

♦ METHODOLOGY OF AUGMENTED MATRIX FOR 3X3 SYSTEMS

1) We consider the augmented matrix of the system

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right)$$

2) We transform to equivalent matrices of the form

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right)$$ by using row R_1 to clear out 1st column

then $$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right)$$ by using row R_2 to clear out 2nd column

3) a) If the system has a UNIQUE SOLUTION we expect an equivalent matrix of the form

$$\left(\begin{array}{ccc|c} \otimes & * & * & * \\ 0 & \otimes & * & * \\ 0 & 0 & \otimes & * \end{array}\right)$$

(all the elements in the main diagonal are nonzero)
Back substitution will provide the unique solution.

Otherwise, we expect an equivalent matrix of the form

$$\left(\begin{array}{ccc|c} \otimes & * & * & * \\ 0 & \otimes & * & * \\ 0 & 0 & 0 & d \end{array}\right)$$

b) if $d \neq 0$ the system has NO SOLUTION

c) if $d=0$, the system has INFINITELY MANY SOLUTIONS:
We set $z=\lambda$ (free variable) and back substitution will provide the **general solution**.
(if the second row also contains zeros, we have in fact only one equation: we set $z=\lambda, y=\mu$ (two free variables) and then express x in terms of λ and μ)

REMARK

Ideally, in step 2 we attempt to have matrices of the form

$$\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right) \text{ and } \left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & * & * \end{array}\right)$$

where the leading coefficient of the selected row is 1.

Let us work now on the system (b) (which had no solutions)

$$\begin{aligned} 2x + 3y + 3z &= 3 \\ x + y - 2z &= 4 \\ 5x + 7y + 4z &= 5 \end{aligned}$$

Working on the augmented matrix we obtain

$$\left(\begin{array}{ccc|c} 2 & 3 & 3 & 3 \\ 1 & 1 & -2 & 4 \\ 5 & 7 & 4 & 5 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 2 & 3 & 3 & 3 \\ 5 & 7 & 4 & 5 \end{array}\right) \begin{array}{l} R_2 \\ R_1 \\ \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & 2 & 14 & -15 \end{array}\right) \begin{array}{l} R_1 \\ R_2 - 2R_1 \\ R_3 - 5R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & 0 & 0 & -5 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 - 2R_2 \end{array}$$

Hence the system has **no solution**.

Finally, let us work on the system (c) (which had ∞ solutions)

$$\begin{aligned} 2x + 3y + 3z &= 3 \\ x + y - 2z &= 4 \\ 5x + 7y + 4z &= 10 \end{aligned}$$

Working on the augmented matrix we obtain

$$\left(\begin{array}{ccc|c} 2 & 3 & 3 & 3 \\ 1 & 1 & -2 & 4 \\ 5 & 7 & 4 & 10 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 2 & 3 & 3 & 3 \\ 5 & 7 & 4 & 10 \end{array}\right) \begin{array}{l} R_2 \\ R_1 \\ \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & 2 & 14 & -10 \end{array}\right) \begin{array}{l} R_1 \\ R_2 - 2R_1 \\ R_3 - 5R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 - 2R_2 \end{array}$$

Hence the system has **infinitely many solutions**. We eliminate the last row

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 7 & -5 \\ & & & \end{array}\right)$$

Back substitution gives

Set $z = \lambda$

R_2: $y+7z = -5 \Rightarrow y = -5 - 7z \Rightarrow y = -5-7\lambda$

R_1: $x+y-2z = 4 \Rightarrow x = 4-y+2z = 4-(-5-7\lambda)+2\lambda \Rightarrow x = 9+9\lambda$

The general solution is

$$x = 9+9\lambda$$
$$y = -5-7\lambda$$
$$z = \lambda \in R \text{ (free variable)}$$

NOTICE (Geometrical Intepretation)

- An equation of the form

$$\mathbf{ax+by=c}$$

represents a line in the Cartesian plane.

When we solve a 2x2 system, we are looking in fact for the intersection of two lines.

Unique solution implies a unique intersection point.

No solution implies that the lines are parallel.

∞ solutions imply that the two lines coincide.

- We will see later (in Vectors) that an equation of the form

$$\mathbf{ax+by+cz=d}$$

represents a plane in 3D space,

When we solve a 3x3 system we are looking in fact for the intersection of three planes.

Analogous interpretations apply here but let us wait until Vectors.

EXAMPLE 3

Suppose that the final step after Gaussian elimination is

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & a & b \end{array}\right)$$

where a and b are parameters to be determined.

The system may have a unique, none or infinitely many solutions.

- If **$a \neq 0$** there is a unique solution.

  Back substitution gives

  $az=b \Rightarrow z=\frac{b}{a}$

  $y+5z=6 \Rightarrow y=6-5z \Rightarrow y=6-5\frac{b}{a}$

  $x+2y+3z=4 \Rightarrow x=4-2y-3z \Rightarrow x=4-2(6-5\frac{b}{a})-3\frac{b}{a} \Rightarrow x=-8+7\frac{b}{a}$

- If **$a=0$ and $b \neq 0$** there is no solution.
- If **$a=0$ and $b=0$** there are infinitely many solutions. The system becomes

  $$\begin{aligned} x+2y+3z&=4 \\ y+5z&=6 \end{aligned}$$

  We set $z=\lambda$ and back substitution gives

  $$\begin{aligned} &y=6-5z=6-5\lambda \\ &x=4-2y-3z=4-2(6-5\lambda)-3\lambda = -8+7\lambda \end{aligned}$$

  The general solution of the system is $(x,y,z)=(-8+7\lambda, 6-5\lambda, \lambda)$.

1.11 COMPLEX NUMBERS – BASIC OPERATIONS (for HL)

As we know, there are no real numbers of the form

$$\sqrt{-1}\ \sqrt{-4}\ ,\ \sqrt{-9}\ ,\ \sqrt{-5}$$

However, we agree to accept an **imaginary** number i such that

$$\boxed{i^2 = -1}$$

(so, in some way, the definition of i is: $i = \sqrt{-1}$)

The imaginary numbers mentioned above can be written as follows:

instead of $\sqrt{-4}$ we write $2i$
instead of $\sqrt{-9}$ we write $3i$
instead of $\sqrt{-5}$ we write $\sqrt{5}\,i$

Consider now the equation

$$x^2 - 4x + 13 = 0$$

Since $\Delta=-36$, there are no **real** solutions. However, if we accept that $\sqrt{\Delta} = i\sqrt{36} = 6i$ we obtain solutions of the following form

$$x = \frac{4 \pm \sqrt{\Delta}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

These "new" numbers are known as **complex numbers**.

In general, if $\Delta<0$ the complex roots of the quadratic are given by

$$x = \frac{-b \pm i\sqrt{|\Delta|}}{2a}$$

NOTICE for the GDC - For Casio:

- use **shift-0** for i
- for a quadratic with $\Delta<0$ you may obtain the complex roots if you use **SET UP – Complex mode: a+bi**

This gives rise to the following definition.

♦ THE DEFINITION

A number z of the form $z = x + yi$

where x, y ∈R, is called a **complex number**. We also say,

the **real part** of z is x: Re(z)=x
the **imaginary part** of z is y: Im(z)=y

The set of all complex numbers is denoted by C. A real number x is also complex of the form x+0i (it has no imaginary part).

♦ THE CONJUGATE $\bar{z}$

The **conjugate** complex number of $z = x+yi$ is given by $\bar{z}=x-yi$

(Sometimes, the conjugate number of z is denoted by z^*)

EXAMPLE 1

For z=3+4i, we write Re(z)=3, Im(z)=4, $\bar{z}$=3−4i.
Similarly

Complex number z	Real part Re(z)	Imaginary part Im(z)	Conjugate $\bar{z}$
2+3i	2	3	2−3i
2−3i	2	−3	2+3i
−2+3i	−2	3	−2−3i
−2−3i	−2	−3	−2+3i
1+i	1	1	1−i
3i	0	3	−3i
2	2	0	2
0	0	0	0
i	0	1	−i
$\frac{2+3i}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{2-3i}{4}$

- THE MODULUS $|z|$

The **modulus** of z=x+yi is defined by: $|z| = \sqrt{x^2+y^2}$

For example, if z=2+3i, then $|z|=|2+3i|=\sqrt{2^2+3^2}=\sqrt{13}$

Notice

z	$x+yi$
$\bar{z}$	$x-yi$
$-z$	$-x-yi$
$-\bar{z}$	$-x+yi$

all have the same modulus $\sqrt{x^2+y^2}$

Thus 3+4i, 3−4i, −3−4i, −3+4i have the same modulus

$$\sqrt{3^2+4^2}=\sqrt{25}=5$$

Finally, observe that $|3|=3$ and $|-3|=3$. That is, the modulus generalizes the notion of the absolute value for real numbers.

- EQUALITY: $z_1 = z_2$

Two complex numbers are equal if they have equal real parts and equal imaginary parts: Let $z_1=x_1+y_1i$ and $z_2=x_2+y_2i$

$$z_1 = z_2 \quad \Leftrightarrow \quad \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases}$$

Thus, the equation of complex number must be thought as a system of two simultaneous equations.

EXAMPLE 2

Let $z_1=3+4i$ and $z_2=a+(3b-2)i$. Find a,b if $z_1 = z_2$.

$$z_1 = z_2 \Leftrightarrow \begin{cases} 3 = a \\ 4 = 3b-2 \end{cases} \Leftrightarrow \begin{cases} a = 3 \\ b = 2 \end{cases}$$

♦ ADDITION–SUBTRACTION–MULTIPLICATION–DIVISION

The four operations for complex numbers follow the usual laws of algebra. We only have in mind that $i^2=-1$.

EXAMPLE 3

Consider the two complex numbers $z=7+4i$ and $w=2+3i$

- $z+w=(7+4i)+(2+3i)=9+7i$ [add real parts; add imaginary parts]
- $z-w=(7+4i)-(2+3i)=5+i$ [similarly]

For multiplication we need some extra work:

- $zw=(7+4i)(2+3i)=14+21i+8i+12i^2=14+21i+8i-12=2+29i$

What about the division?

The fraction $\frac{z}{w}=\frac{7+4i}{2+3i}$ is also a complex number!

In order to obtain the usual form $x+yi$, we multiply both terms by the conjugate of the denominator i.e. by $\overline{w}=2-3i$

- $\frac{z}{w}=\frac{7+4i}{2+3i}=\frac{7+4i}{2+3i}\frac{\times(2-3i)}{\times(2-3i)}=\frac{14-21i+8i+12}{13}=\frac{26-13i}{13}=2-i$

Thus

$$\frac{7+4i}{2+3i}=2-i$$

(Confirm the result by multiplying $(2+3i)(2-i)$; you must find $7+4i$)

NOTICE

$$\boxed{|z|^2=z\cdot\overline{z}}$$

Indeed, both sides are equal to x^2+y^2: For $z=x+yi$

$$|z|^2=x^2+y^2$$

$$z\overline{z}=(x+yi)(x-yi)=x^2-y^2i^2=x^2+y^2$$

Get used to the multiplication by a conjugate:

EXAMPLE 4

$(3+4i)(3-4i) = 9+16 = 25$

$(1+i)(1-i) = 1+1 = 2$

$(2-i)(2+i) = 4+1 = 5$

The result is always a real number (the square of the modulus).

EXAMPLE 5

Let us estimate the powers of i:

$i^0=1$	$i^1=i$	$i^2=-1$	$i^3=-i$
$i^4=1$	$i^5=i$	$i^6=-1$	$i^7=-i$
$i^8=1$	$i^9=i$	$i^{10}=-1$	$i^{11}=-i$
… and so on			

Thus, for example

$$i^{35}=i^{32+3}=i^3=-i$$

(since 32 is a multiple of 4).

EXAMPLE 6

Calculate

(a) $z = (2+i)^3$ (b) $w = \dfrac{(2+i)^3}{1-i}$

Solution

(a) $z = (2+i)^3 = 2^3+3{\cdot}2^2i+3{\cdot}2i^2+i^3$ or $z = (2+i)^2(2+i)$

$= 8+12i+6i^2+i^2i$ $\quad$ $= (4+4i+i^2)(2+i)$

$= 8+12i-6-i$ $\quad$ $= (3+4i)(2+i) = 6+3i+8i+4i^2$

$= 2+11i$ $\quad$ $= 2+11i$

(b) $w = \dfrac{(2+i)^3}{1-i} = \dfrac{2+11i}{1-i} = \dfrac{2+11i}{1-i}\,\dfrac{1+i}{1+i} = \dfrac{2+2i+11i+11i^2}{2}$

$= \dfrac{-9+13i}{2} = \dfrac{-9}{2} + \dfrac{13}{2}i$

EXAMPLE 7

Find z if

$$z(1-i) = 2+11i$$

Solution

Method A (Analytical and safe but laborious):

Let $z = x+yi$. Then

$$\begin{aligned} z(1-i) = 2+11i \quad &\Leftrightarrow (x+yi)(1-i) = 2+11i \\ &\Leftrightarrow x-xi+yi-yi^2 = 2+11i \\ &\Leftrightarrow (x+y)+(y-x)i = 2+11i \\ &\Leftrightarrow \begin{cases} x+y=2 \\ y-x=11 \end{cases} \end{aligned}$$

The solution of the system is $x = -9/2$ and $y = 13/2$

Hence, $z = \dfrac{-9}{2}+\dfrac{13}{2}i$

Method B (quicker): think as in the equation $ax=b$

$$\begin{aligned} z(1-i) = 2+11i \Leftrightarrow z &= \frac{2+11i}{1-i} \\ &= \dots \\ &= \frac{-9}{2}+\frac{13}{2}i \quad \text{[look at Exercise 6(b)]} \end{aligned}$$

1.12 POLYNOMIALS OVER THE COMPLEX FIELD (for HL)

♦ THE FUNDAMENTAL THEOREM OF ALGEBRA

As we know a quadratic may have

- two different real roots: $f(x)=a(x-r_1)(x-r_2)$
- two equal real roots: $f(x)=a(x-r_1)^2$
- two non-real complex roots: $f(x)$=irreducible quadratic

We can say in general that

a quadratic has <u>always two roots (in C)</u>.

(having in mind that a real number is also complex, and allowing repetition of a root)

This is in fact a particular case of the so called

Fundamental theorem of algebra

A polynomial of degree $n>1$ has exactly n roots (in C)

<u>NOTICE</u>

In fact, a first version of this theorem says that

A polynomial of degree $n>1$ has always a root (in C)

However, if $f(x)$ is a polynomial of degree n and $x=r_1$ is a complex root, the long division of $f(x)$ by $(x-r_1)$ gives

$$f(x)=(x-r_1)q(x)$$

where $q(x)$ is a polynomial of degree $n-1$.
But $q(x)$ has also a complex root r_2, and so on. By repeating long divisions we find exactly n roots of $f(x)$.

♦ FACTORIZATION AND ROOTS OF A POLYNOMIAL

Remember the equation $x^2-4x+13=0$ has two complex roots

$$x=\frac{4\pm\sqrt{\Delta}}{2}=\frac{4\pm 6i}{2}=2\pm 3i$$

Thus, a quadratic with $\Delta<0$ has always two complex roots which are conjugate to each other! This is not accidental. For any polynomial of any degree,

if $z=a+bi$ is a root then $\bar{z}=a-bi$ is also a root! (*)

Lemma

For the conjugates of complex numbers we can easily verify that

$$\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$$

$$\overline{z_1 z_2}=\overline{z_1}\,\overline{z_2}$$

$$\overline{z^n}=\bar{z}^n$$

Proof of (*)

Consider the polynomial

$$p(x)=a_n x^n+\cdots+a_2x^2+a_1x+a_0$$

with real coefficients, that is $a_0, a_1, \ldots a_n \in R$

If z is a root then $\bar{z}$ is also a root! Indeed,

$$\begin{aligned}
z \text{ is a root} \quad &\Rightarrow a_n z^n+\cdots+a_2z^2+a_1z+a_0=0\\
&\Rightarrow \overline{a_n z^n+\cdots+a_2z^2+a_1z+a_0}=\overline{0}\\
&\Rightarrow \overline{a_n z^n}+\cdots+\overline{a_2z^2}+\overline{a_1z}+\overline{a_0}=0\\
&\Rightarrow \overline{a_n}\,\overline{z^n}+\cdots+\overline{a_2}\,\overline{z^2}+\overline{a_1}\,\overline{z}+\overline{a_0}=0\\
&\Rightarrow a_n \bar{z}^n+\cdots+a_2\bar{z}^2+a_1\bar{z}+a_0=0\\
&\Rightarrow \bar{z} \text{ is a root}
\end{aligned}$$

Therefore, a polynomial has either real roots or non-real complex roots which always come in pairs of conjugate numbers.

NOTICE:

Usually, when we consider complex roots a polynomial is written as

$$p(z)= a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0$$

If $a+bi$ and $a-bi$ are two conjugate roots then

$$\boxed{(z-a-bi)} \text{ and } \boxed{(z-a+bi)}$$

are two factors of the polynomial.

Then

$$(z-a-bi)(z-a+bi)=(z-a)^2-(bi)^2=(z-a)^2+b^2=z^2-2az+(a^2+b^2)$$

That is, $p(z)$ has an irreducible quadratic factor of the form

$$z^2+pz+q \qquad \text{where } p,q\in R$$

Let us collect our results: for a polynomial of degree n

- the factorization over the set C consists of n linear factors

  $$p(z)=a_n(z-r_1)(z-r_2)\ldots(z-r_n)$$

  where $r_1,r_2,\ldots,r_n\in C$ (the n complex roots of the polynomial)
- the factorization over the set R consists of
  - ✓ linear factors of the form $(z-r)$
  - ✓ irreducible quadratic factors of the form z^2+pz+q

  where $r,p,q\in R$

NOTICE:

A cubic function may have one of the following factorizations in R

- three real roots: $f(x)=a(x-r_1)(x-r_2)(x-r_3)$

  or $f(x)=a(x-r_1)^2(x-r_2)$

  or $f(x)=a(x-r_1)^3$
- one real, two non-real roots $f(x)=a(x-r_1)(x^2+px+q)$

EXAMPLE 1

Version 1. Find all the three roots of the cubic function

$$f(z)=z^3-5z^2+9z-5$$

given that z=1 is a real root.

Solution

If we divide f(z) by z−1 we obtain

$$f(z)=(z-1)(z^2-4z+5)$$

The quadratic factor z^2-4z+5 has no real roots. Consequently, this is the finest factorization over R.

However, if we extend to the complex field C, we find two extra roots of the quadratic z^2-4z+5. Namely

$$2+i,\ 2-i$$

Notice that the factorization over C is

$$f(z)=(z-1)(z-2+i)(z-2-i)$$

Version 2. Find all the three roots of the cubic function

$$f(z)=z^3-5z^2+9z-5$$

given that z=2+i is a complex root.

Solution

We immediately know the conjugate root z=2−i.

If we combine the two complex roots we obtain the quadratic factor

$$(z-2+i)(z-2-i)=(z-2)^2-i^2=z^2-4z+5$$

If we divide f(x) by z^2-4z+5 we obtain the factor

$$z-1$$

Hence, z=1 is the third root of the polynomial.

The factorization of f(z) over C is

$$f(z)=(z-1)(z-2+i)(z-2-i)$$

Version 3. Consider the cubic function

$$f(z)=z^3+az^2+bz+c$$

Given the roots $z=1$, $z=2+i$, find the coefficients a,b,c.

Solution

The third root is $z=2-i$. Hence,

$$\begin{aligned} f(z) &= (z-1)(z-2+i)(z-2-i) \\ &= (z-1)[(z-2)^2-i^2] \\ &= (z-1)(z^2-4z+5) \end{aligned}$$

This is the finest factorization over R. Finally

$$f(z) = z^3-5z^2+9z-5$$

Therefore, $a=-5$, $b=9$, $c=-5$.

Notice

In the last version, we could also consider the equations

$$f(1) = 0$$

$$f(2+i)=0$$

which imply three simultaneous equations in a,b,c. But this approach is much more time-consuming!

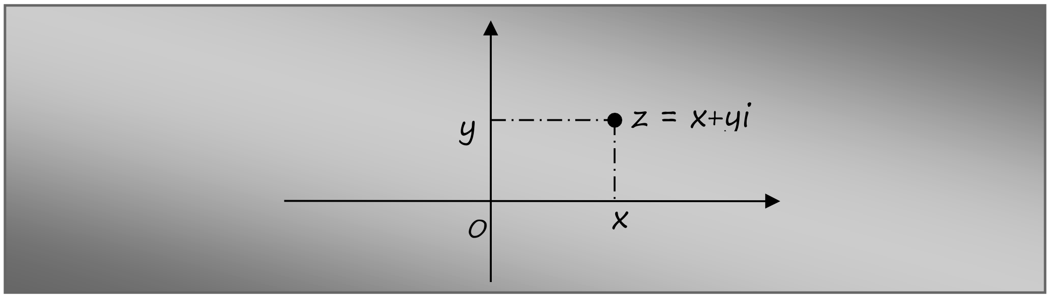
NOTICE:

In an analogue way, a polynomial of degree 4 may have

- four real roots (not necessarily distinct)
- two real roots and two non-real roots;
  it will have the factorization
  $$f(x)=a(x-r_1)(x-r_2)(x^2+px+q)$$
  where the quadratic has two conjugate complex roots
- four non-real roots: it will have the form
  $$f(x)=a(x^2+px+q)(x^2+rx+s)$$
  where each quadratic has two conjugate complex roots

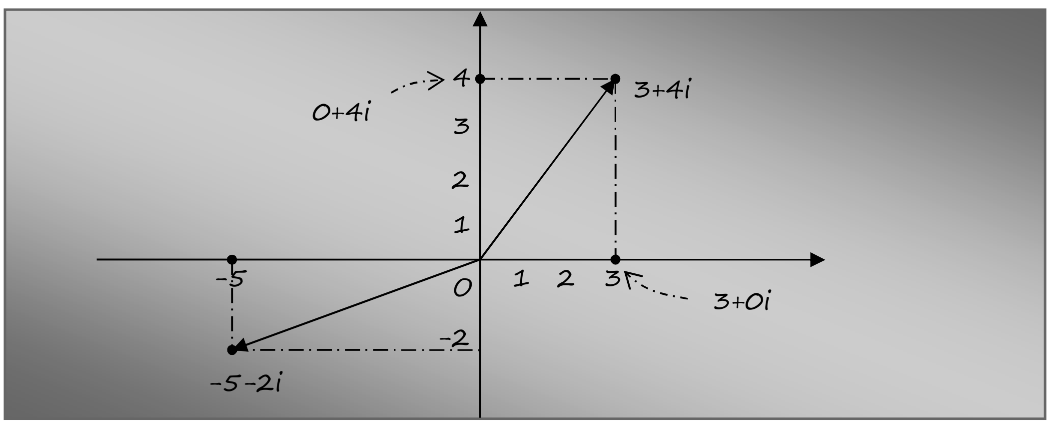
1.13 THE COMPLEX PLANE (for HL)

The complex number $z=x+yi$ can be represented on the Cartesian plane as follows



- $z=x+yi$ is the point (x,y) [†]
- Real part = x-coordinate, Imaginary part = y-coordinate
- The modulus $|z|=\sqrt{x^2+y^2}$ is in fact the distance from the origin.

EXAMPLE 1



Notice

$|3+4i|=\sqrt{25}=5$, $|3|=3$, $|4i|=4$, $|-5|=5$, $|-5-2i|=\sqrt{29}$

The modulus is always the distance from the origin.

[†] We may also think of z as a vector from the origin to the point (x,y). Compare with vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ in paragraph 3.11

NOTICE

We already know that the sets

N = natural numbers

Z = integers

Q = rational numbers

R = real numbers

can be represented on the real axis. We extend this representation here to the complex plane (considering an imaginary y-axis).

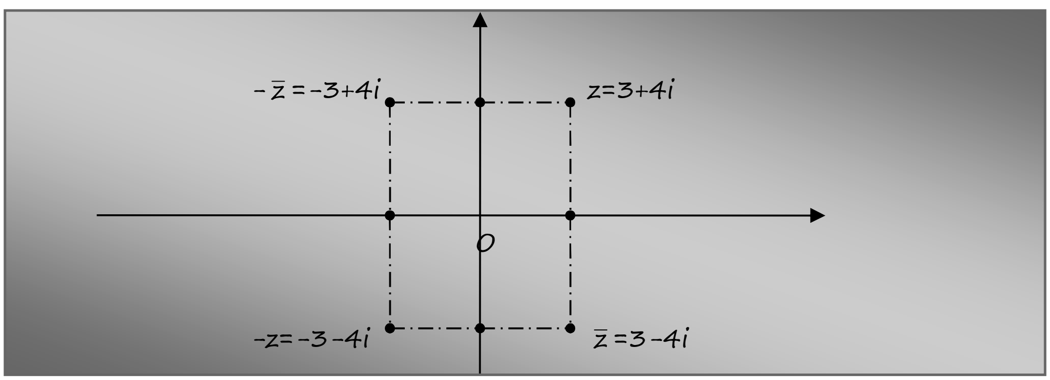
It also holds

$$N \subset Z \subset Q \subset R \subset C$$

EXAMPLE 2

It is interesting to see the representations of $z, \bar{z}, -z, -\bar{z}$. For example,

$z=3+4i$ $\bar{z}=3-4i$ $z=-3+4i$ $-\bar{z}=-3-4i$



- The modulus of all those is 5 (distance from the origin).
- $\bar{z}$ is symmetric to z about the x axis
- $-z$ is symmetric to z about the origin

Think that these observations hold for real numbers as well:

- The absolute value of 5 and -5 is 5 (distance from origin)
- The conjugate of 5 is 5 itself (symmetric about x-axis)
- The opposite of 5 is -5 (symmetric about the origin)

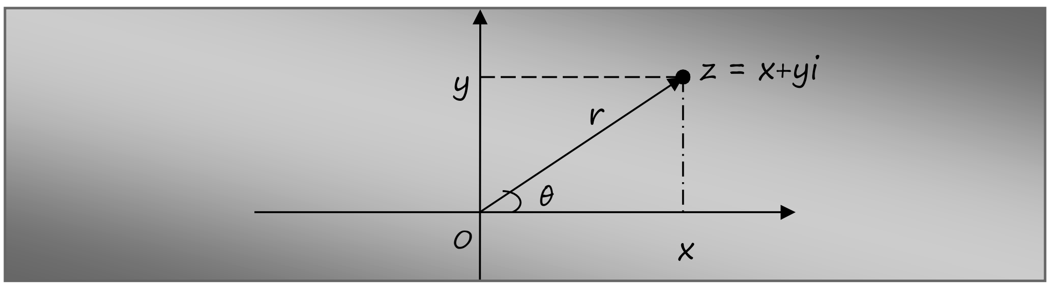
♦ THE POLAR FORM (MODULUS-ARGUMENT FORM)

We have just seen that a complex number $z=x+yi$ is represented on the plane by a pair of **Cartesian coordinates** (**x**,**y**).

An alternative way to describe a point on the plane (and thus the position of z) is the so-called **Polar coordinates** (**r**,**θ**):

We draw a vector (an arrow) from O to the point and consider

r = the length of the vector
θ = the angle between the x-axis and the vector



Notice:

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}, \quad \tan\theta = \frac{y}{x} \qquad (*)$$

For a complex number $z=x+yi$

r is in fact the **modulus** $|z|$

θ is called **argument** of z. We write **arg(z)=θ**

REMARK

Of course, the argument θ is not unique. For example, if $\theta=30^o$ is an argument of z then

$360^o+30^o=$ **390^o**, $720^o+30^o=$ **750^o**, etc

are also arguments of z.

For the **principal argument** θ, we agree

$$-180^o<\theta\leq180^o \quad \text{or} \quad -\pi<\theta\leq\pi$$

but in our applications, we may consider any equivalent argument.

The relations (*) above give

$$\mathbf{x=r\cos\theta} \quad \text{and} \quad \mathbf{y=r\sin\theta}$$

Thus, a complex number can be also written as

$$z = x+yi = (r\cos\theta)+(r\sin\theta)i = r\,(\cos\theta+i\sin\theta)$$

The form

$$\boxed{z = r\,(\cos\theta+i\sin\theta)}$$

is known as the **polar form** of the complex number z
(or otherwise **modulus-argument form** or **trigonometric form**)

♦ TRANSFORMATION FROM **z = x+yi** TO **z = r (cosθ+isinθ)**

Given: $z = x+yi$. We find r and θ by

- $\boxed{r = |z| = \sqrt{x^2+y^2}}$
- $\boxed{\tan\theta = \frac{y}{x}}$, having in mind the quadrant of $x+yi$

EXAMPLE 3

Find the polar form of $z = 1+\sqrt{3}\,i$ and $w = 3+4i$.

Solution

- For $z = 1+\sqrt{3}\,i$:

  $r = \sqrt{1+3} = 2$,

  $\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$, [1st quadrant] $\Rightarrow$ $\theta = \frac{\pi}{3}$

  Therefore, $z = 2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$ [or $2\,(\cos60°+i\sin60°)$]

- For $w = 3+4i$:

  $r = \sqrt{3^2+4^2} = 5$,

  $\tan\theta = \frac{4}{3}$, [1st quadrant] $\Rightarrow$ $\theta = 0.927$ (by GDC)

Therefore, $w = 5\,[\cos(0.927)+i\sin(0.927)]$

EXAMPLE 4

Find the polar form of $z_1 = 1+i$, $z_2 = -1-i$, $z_3 = 1-i$, $z_4 = -1+i$

Solution

For all of them the modulus is $r = \sqrt{1+1} = \sqrt{2}$

- For $z_1 = 1+i$: $\tan\theta = \frac{1}{1} = 1$, [1st quadrant] $\Rightarrow \theta = \frac{\pi}{4}$

  Therefore, $z_1 = \sqrt{2}\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$
- For $z_2 = -1-i$: $\tan\theta = \frac{-1}{-1} = 1$, [3rd quadrant] $\Rightarrow \theta = \frac{5\pi}{4}$

  Therefore, $z_2 = \sqrt{2}\,(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4})$
- For $z_3 = 1-i$: $\tan\theta = \frac{1}{-1} = -1$, [4th quadrant] $\Rightarrow \theta = -\frac{\pi}{4}$

  Therefore, $z_3 = \sqrt{2}\,(\cos-\frac{\pi}{4} + i\sin-\frac{\pi}{4})$
- For $z_4 = -1+i$: $\tan\theta = \frac{-1}{1} = -1$, [2nd quadrant] $\Rightarrow \theta = \frac{3\pi}{4}$

  Therefore, $z_4 = \sqrt{2}\,(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$

♦ TRANSFORMATION FROM $\mathbf{z = r\,(\cos\theta + i\sin\theta)}$ TO $\mathbf{z = x+yi}$

This is much easier! We just perform the operations! In fact, the polar form is also Cartesian:

$$z = r(\cos\theta + i\sin\theta) = r\cos\theta + ir\sin\theta \qquad x = r\cos\theta,\ y = r\sin\theta$$

For example,

the Cartesian form of $z = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ is $2(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 1+\sqrt{3}\,i$

NOTICE for the GDC

GDC transforms one form to another. **For Casio** use:

Run-Matrix – OPTN – COMPLEX - ▶r<θ or ▶a+bi

♦ CIS FORM: $z=r\text{cis}\theta$

There is an abbreviation for the polar form

$$z = r(\cos\theta+i\sin\theta).$$

It is sometimes written as

$$z = r\text{cis}\theta.$$

For example, $z = 2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}) = 2\text{cis}\frac{\pi}{3}$.

♦ EULER'S FORM: $z=re^{i\theta}$

Another abbreviation is due to Euler.

We define‡

$$\boxed{e^{i\theta}=\cos\theta+i\sin\theta}$$

Consequently, the trigonometric form $z=r(\cos\theta+i\sin\theta)$ obtains the form

$$\boxed{z=re^{i\theta}}$$

For example, $z = 2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}) = 2e^{i\frac{\pi}{3}}$.

EXAMPLE 5

Write down all the possible forms of $z_1=1+i$, $z_2 =3+4i$, $z_3 =3+4i$

Cartesian	Polar form	cis form	Euler form
$1+i$	$2(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$	$\sqrt{2}\,\text{cis}\frac{\pi}{4}$	$\sqrt{2}\,e^{i\frac{\pi}{4}}$
$3+4i$	$5[\cos(0.927)+i\sin(0.927)]$	$5\text{cis}(0.927)$	$5e^{0.927i}$
$3-4i$	$5[\cos(-0.927)+i\sin(-0.927)]$	$5\text{cis}(-0.927)$	$5e^{-0.927i}$

---

‡ This is not accidental! It can be shown that $e^{i\theta}$ follows all known exponential properties (we will verify that later on)

NOTICE

- Any complex number with modulus 1 has polar form

$$z = cis\theta = cos\theta + isin\theta$$

(Indeed, $|z| = \sqrt{cos^2\theta + sin^2\theta} = 1$)

For any z given in polar form we know $|z|$. For example,

if $z = 23cis3\pi = 23(cos3\pi + isin3\pi)$ then $|z| = 23$

- For real numbers $\pm a$ (on the horizontal, real axis) the polar form is immediate, since
  - the argument of a positive real number is 0
  - the argument of a negative real number is π.

1	$cis0$	e^{i0}
2	$2cis0$	$2e^{i0}$
3	$3cis0$	$3e^{i0}$
…		

-1	$cis\pi$	$e^{i\pi}$
-2	$2cis\pi$	$2e^{i\pi}$
-3	$3cis\pi$	$3e^{i\pi}$
…		

- For imaginary numbers of the form $\pm ai$: (on the imaginary axis) the argument is either π/2 or −π/2.

i	$cis\frac{\pi}{2}$	$e^{i\pi/2}$
$2i$	$2cis\frac{\pi}{2}$	$2e^{i\pi/2}$
$3i$	$3cis\frac{\pi}{2}$	$3e^{i\pi/2}$
…		

$-i$	$cis\left(-\frac{\pi}{2}\right)$	$e^{-i\pi/2}$
$-2i$	$2cis\left(-\frac{\pi}{2}\right)$	$2e^{-i\pi/2}$
$-3i$	$3cis\left(-\frac{\pi}{2}\right)$	$3e^{-i\pi/2}$
…		

- The conjugate of $z = r\,(cos\theta + isin\theta)$ is $\bar{z} = r\,(cos\theta - isin\theta)$

Attention!! The latter <u>is not</u> in polar form. We must have a (+) sign in front of i. However, we know that

$$cos(-\theta) = cos\theta \qquad \text{and} \qquad sin(-\theta) = -sin\theta$$

Hence $$\bar{z} = r\,[cos(-\theta) + isin(-\theta)]$$

Indeed, $$\boxed{\text{if } arg(z) = \theta \text{ then } arg(\bar{z}) = -\theta}$$

1.14 DE MOIVRE'S THEOREM (for HL)

♦ **PROPOSITION 1**

Let $z = r\,cis\theta$. Then

$$\mathbf{z^{-1}=r^{-1}cis(-\theta)} \qquad i.e. \qquad \mathbf{\frac{1}{z}=\frac{1}{r}cis(-\theta)}$$

Proof.

Remember that $z\bar{z}=|z|^2=r^2$ and $\bar{z}= r\,cis(-\theta)$

Then

$$z^{-1}=\frac{1}{z}=\frac{\bar{z}}{z\bar{z}}=\frac{rcis(-\theta)}{r^2}=\frac{1}{r}cis(-\theta)$$

♦ **PROPOSITION 2**

Let $z_1 = r_1cis\theta_1$ and $z_2 = r_2cis\theta_2$

Then $$\mathbf{z_1z_2 = r_1r_2cis(\theta_1+\theta_2)}$$

Proof.

$$\begin{aligned} z_1\,z_2 &= r_1cis\theta_1\; r_2cis\theta_2 \\ &= r_1r_2(cos\theta_1+isin\theta_1)(cos\theta_2+i\;sin\theta_2) \\ &= r_1r_2[cos\theta_1cos\theta_2-sin\theta_1sin\theta_2+i(sin\theta_1cos\theta_2+sin\theta_2cos\theta_1)] \\ &= r_1r_2[cos(\theta_1+\theta_2)+isin(\theta_1+\theta_2)] \\ &= r_1r_2\,cis(\theta_1+\theta_2) \end{aligned}$$

♦ **PROPOSITION 3**

Let $z_1 = r_1cis\theta_1$ and $z_2 = r_2cis\theta_2$

Then $$\mathbf{\frac{z_1}{z_2}=\frac{r_1}{r_2}cis(\theta_1-\theta_2)}$$

Proof.

$$\frac{z_1}{z_2} = z_1 z_2^{-1}$$

$= r_1 cis\theta_1\, r_2^{-1} cis(-\theta_2)$ [by Proposition1]

$= r_1 r_2^{-1} cis(\theta_1-\theta_2)$ [by Proposition2]

$= \frac{r_1}{r_2}\, cis(\theta_1-\theta_2)$

NOTICE:

By Proposition 2, the modulus of z_1z_2 is r_1r_2 : $|z_1z_2| = r_1r_2$

By Proposition 3, the modulus of $\frac{z_1}{z_2}$ is $\frac{r_1}{r_2}$: $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}$

In other words,

$$\boxed{|z_1z_2| = |z_1||z_2|} \qquad \boxed{\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}}$$

Furthermore,

By Proposition 2, the argument of z_1z_2 is $\theta_1+\theta_2$: $arg(z_1z_2) = \theta_1+\theta_2$

By Proposition 3, the argument of $\frac{z_1}{z_2}$ is $\theta_1-\theta_2$: $arg(\frac{z_1}{z_2}) = \theta_1-\theta_2$

In other words,

$$\boxed{arg(z_1z_2)=arg(z_1)+arg(z_2)} \qquad \boxed{arg(\frac{z_1}{z_2})=arg(z_1)-arg(z_2)}$$

In simple words,

the modulus $|z|$ preserves the operations

the argument $arg(z)$ behaves like $log(z)$

EXAMPLE 1

Let $z = 2cis\frac{\pi}{6}$ and $w = cis\frac{\pi}{3}$

(In fact, $z=\sqrt{3}+i$ and $w=\frac{1}{2}+\frac{\sqrt{3}}{2}i$)

Then

- $zw = 2\cdot 1 cis(\frac{\pi}{6}+\frac{\pi}{3}) = 2cis(\frac{3\pi}{6}) = 2cis(\frac{\pi}{2})$

  $=2[cos(\frac{\pi}{2})+isin(\frac{\pi}{2})]=2(0+i)=2i$

- $\frac{z}{w}=\frac{2}{1}cis(\frac{\pi}{6}-\frac{\pi}{3}) = 2cis(-\frac{\pi}{6}) = 2[cos(-\frac{\pi}{6})+isin(-\frac{\pi}{6})]$

  $=2(\frac{\sqrt{3}}{2}-\frac{1}{2}i)= \sqrt{3}-i$

- $z^2=zz=2\cdot 2\ cis(\frac{\pi}{6}+\frac{\pi}{6})=4cis(\frac{\pi}{3}) =4(cos\frac{\pi}{3}+isin\frac{\pi}{3})$

  $=4(\frac{1}{2}+\frac{\sqrt{3}}{2}i)=2+2\sqrt{3}\,i$

- $\frac{z^2}{w}=\frac{zz}{w}=\frac{2\cdot 2}{1}cis(\frac{\pi}{6}+\frac{\pi}{6}-\frac{\pi}{3}) = 4cis(0) = 4$

- $\frac{1}{z}=\frac{1}{2}cis(-\frac{\pi}{6})=\frac{1}{2}[cos(-\frac{\pi}{6})+isin(-\frac{\pi}{6})]=\frac{1}{2}(\frac{\sqrt{3}}{2}-\frac{1}{2}i)= \frac{\sqrt{3}}{4}-\frac{1}{4}i$

- $\frac{1}{w}= cis(-\frac{\pi}{3}) = [cos(-\frac{\pi}{3})+isin(-\frac{\pi}{3})]= \frac{1}{2}-\frac{\sqrt{3}}{2}i$

Notice that for $z = r\ cis\theta$,

$$z^2 = z\,z = r\,r\ cis(\theta+\theta) = r^2\ cis(2\theta)$$
$$z^3 = z^2z = r^2r\ cis(2\theta+\theta) = r^3\ cis(3\theta)$$
and so on.

By using mathematical induction, we can easily derive the following theorem.

♦ DE MOIVRE'S THEOREM

Let $z = r\text{cis}\theta$. For any $n \in Z^+$, it holds

$$z^n = r^n \text{cis}(n\theta)$$

Proof (by mathematical induction)

- For $n=1$ the statement is trivially true, since $z^1=z=r(\text{cis}\theta)$
- We **assume** that the statement is true for $n=k$, i.e. $z^k=r^k\ \text{cis}(k\theta)$
- We **claim** that the statement is true for $n=k+1$, i.e.

$$z^{k+1}=r^{k+1}\text{cis}[(k+1)\theta]$$

Indeed,

$z^{k+1} = z^k z = r^k\text{cis}(k\theta)\ r\text{cis}\theta$ [by hypothesis]

$= r^k r\ [\cos(k\theta)+i\sin(k\theta)]\ [\cos\theta+i\sin\theta]$

$= r^{k+1}[\cos(k\theta+\theta)+i\sin(k\theta+\theta)]$ [By Prop.2]

$= r^{k+1}[\cos(k+1)\theta+i\sin(k+1)\theta]$

$= r^{k+1}\ \text{cis}[(k+1)\theta]$ Q.E.D.

Therefore, by mathematical induction the statement is true for any $n \in Z^+$

NOTICE:

In fact, De Moivre's theorem is true for any integer exponent $n \in Z$.

- $z^5 = r^5\text{cis}(5\theta)$
- $z^0 = r^0\text{cis}(0\theta) = \text{cis}0 = 1$ as expected!
- $z^{-1}= r^{-1}\text{cis}(-\theta)$ as in Proposition 1

  This is also equal to $r^{-1}[\cos(-\theta)+i\sin(-\theta)]=\dfrac{1}{r}(\cos\theta-i\sin\theta)$
- $z^{-5}=r^{-5}\text{cis}(-5\theta)$.

  This is $\dfrac{1}{r^5}(\cos5\theta-i\sin5\theta)$

EXAMPLE 2

Let $z=2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$. This is in fact the number $z=\sqrt{3}+i$.

Then

$$z^2=2^2(\cos\frac{2\pi}{6}+i\sin\frac{2\pi}{6})=4(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})=2+2\sqrt{3}\,i$$

$$z^{-1}=2^{-1}[(\cos(-\frac{\pi}{6})+i\sin(-\frac{\pi}{6})]=\frac{1}{2}(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6})=\frac{\sqrt{3}}{4}-\frac{1}{4}i$$

$$z^6=2^6(\cos\frac{6\pi}{6}+i\sin\frac{6\pi}{6})=64(\cos\pi+i\sin\pi)=-64$$

Notice that $z^6=(\sqrt{3}+i)^6$ may also be found by using the binomial theorem but De Moivre's Theorem gives directly the same result!

For example, $(\sqrt{3}+i)^{60}$ is almost impossible to be estimated by using the binomial theorem, but De Moivre's Theorem gives

$$z^{60} = 2^{60}(\cos\frac{60\pi}{6}+i\sin\frac{60\pi}{6}) =2^{60} (\cos10\pi+i\sin10\pi) =2^{60}$$

EXAMPLE 3

Find $(1+i)^{10}$

It is much more convenient to use the polar form of $1+i$:

$$r=\sqrt{2} \quad \text{and} \quad \tan\theta=1 \ (1^{st} \text{ quadrant}) \Rightarrow \theta=\frac{\pi}{4}$$

$$\text{thus} \qquad 1+i=\sqrt{2}\,cis\frac{\pi}{4}$$

Then $(1+i)^{10}=(\sqrt{2})^{10}cis\frac{10\pi}{4}=2^5cis\frac{10\pi}{4}$

But $\frac{10\pi}{4}=2\pi+\frac{\pi}{2}$.

Then

$$(1+i)^{10}=32cis\frac{\pi}{2}=32i$$

NOTICE:

Remember Euler's notation

$$z=re^{i\theta} = r(\cos\theta+i\sin\theta).$$

De Moivre's Theorem is in accordance to this notation. Indeed

$$z^n= (re^{i\theta})^n = r^n e^{in\theta} = r^n[\cos(n\theta)+i\sin(n\theta)]$$

which is De Moivre's statement.

Remember that $z=\cos\theta+i\sin\theta$ is at the same time in polar and in Cartesian form.

Dealing in parallel with both these forms gives interesting results.

EXAMPLE 4

Let $z=\cos\theta+i\sin\theta$. We calculate z^3 in two different ways:

- De Moivre's theorem gives

$$z^3=\cos3\theta+i\sin3\theta$$

- Binomial theorem gives

$$\begin{aligned} z^3 &=(\cos\theta+i\sin\theta)^3 \\ &=\cos^3\theta+3\cos^2\theta(i\sin\theta)+3\cos\theta(i\sin\theta)^2+(i\sin\theta)^3 \\ &=\cos^3\theta+3i\cos^2\theta\sin\theta-3\cos\theta\sin^2\theta-i\sin^3\theta \\ &=(\cos^3\theta-3\cos\theta\sin^2\theta) +i(3\cos^2\theta\sin\theta-\sin^3\theta) \end{aligned}$$

By comparing the two results

$$\cos3\theta= \cos^3\theta-3\cos\theta\sin^2\theta \quad \text{and} \quad \sin3\theta=3\cos^2\theta\sin\theta-\sin^3\theta$$

We can express $\cos3\theta$ in terms of cosines only:

$$\cos3\theta= \cos^3\theta-3\cos\theta(1-\cos^2\theta) = 4\cos^3\theta-3\cos\theta$$

Similarly we can obtain

$$\sin3\theta=3\sin\theta-4\sin^3\theta$$

Remark: Working with z^4, z^5, etc we obtain similar results for

$\cos4\theta$, $\sin4\theta$, $\cos5\theta$, $\sin5\theta$, etc.

EXAMPLE 5

Let $z=\cos\theta+i\sin\theta$. Then, by De Moivre's theorem

$$z^n=\cos n\theta+i\sin n\theta \text{ and } z^{-n}=\cos n\theta-i\sin n\theta$$

Then

$$z^n+z^{-n}=2\cos n\theta \qquad (*)$$
$$z^n-z^{-n}=2i\sin n\theta \qquad (**)$$

Let us expand $(z+z^{-1})^3$ in two different ways:

- Relation (*) for n=1 gives

$$(z+z^{-1})^3=(2\cos\theta)^3 = 8\cos^3\theta$$

- Binomial theorem gives

$$\begin{aligned}(z+z^{-1})^3 &= z^3+3z+3z^{-1}+z^{-3}\\ &= (z^3+z^{-3})+3(z+z^{-1})\\ &= 2\cos3\theta+6\cos\theta \qquad [\,(*) \text{ for } n=3 \text{ and } n=1]\end{aligned}$$

By comparing the two results

$$8\cos^3\theta=2\cos3\theta+6\cos\theta$$

and finally

$$\cos^3\theta=\frac{1}{4}\cos3\theta+\frac{3}{4}\cos\theta$$

Remarks

- The expansions of $(z+z^{-1})^4$, $(z+z^{-1})^5$, etc

  give similar results for $\cos^4\theta$, $\cos^4\theta$, etc.
- Working similarly with (**),

  the expansions of $(z-z^{-1})^3$, $(z-z^{-1})^4$, $(z-z^{-1})^5$, etc

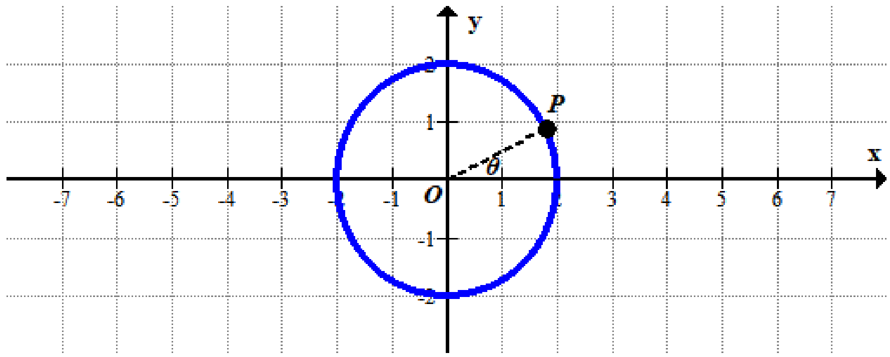
  give similar results for $\sin^3\theta$, $\sin^4\theta$, $\sin^5\theta$, etc

- GEOMETRICAL INTERPRETATION OF MULTIPLICATION

Let P be the point on the Complex plane which represents

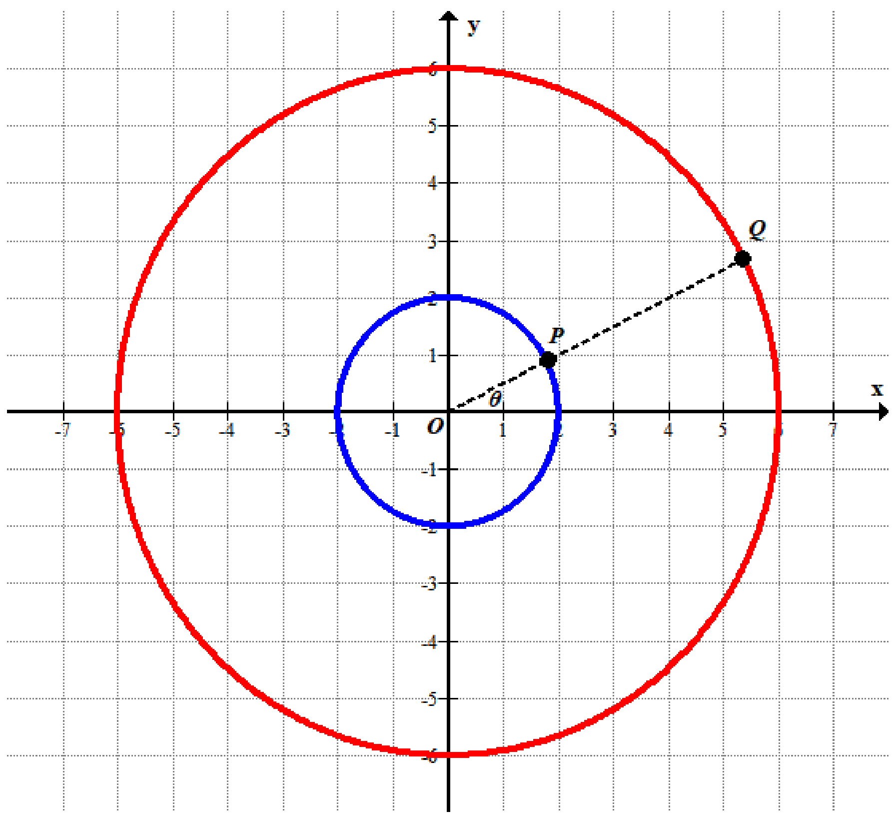
$$w=2cis\theta$$

- P is on a circle of radius 2 (centre at the origin)
- The line OP forms an angle θ with x-axis.



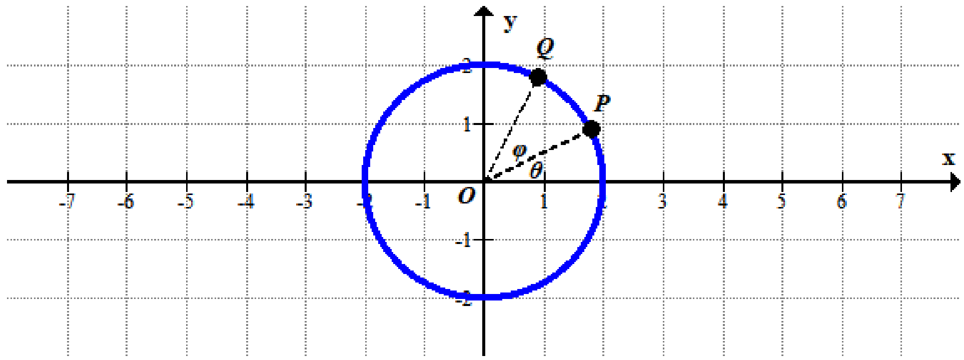
- If we multiply w by **z=3**, the result is $zw=6cis\theta$.

  The image point Q is on a circle of radius 6:



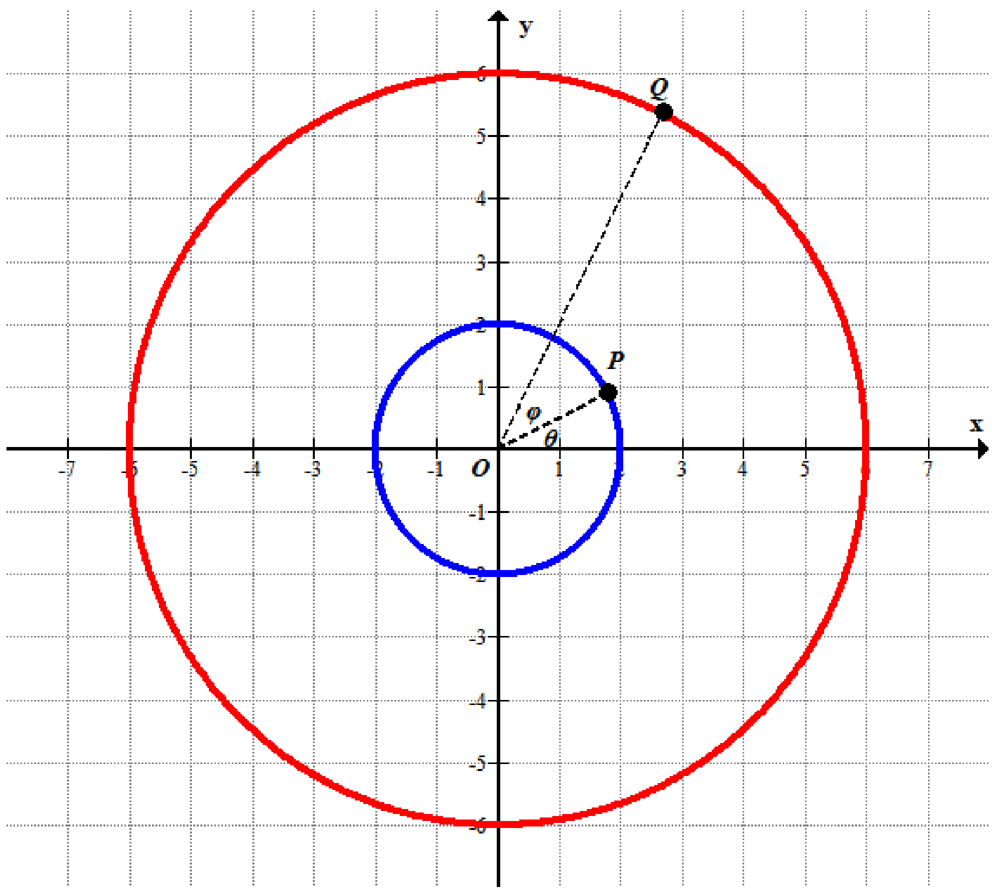
(enlargement by scale factor 3)

- If we multiply w by **z= cisϕ**, the result is zw=2cis(θ+ϕ).
  The image line OQ forms an angle θ+ϕ with x-axis



(rotation by angle ϕ)

- If we multiply w by **z=3cisϕ**, the result is zw=6cis(θ+ϕ).
  Q is on a circle of radius 6; OQ forms an angle θ+ϕ with x-axis



(rotation by angle ϕ and enlargement by scale factor 3)

1.15 ROOTS OF $z^n=a$ (for HL)

Let us start with an observation. Equality in complex numbers is a strong relation: it gives a system of two equations.

For Cartesian forms: $x+yi = a+bi \Leftrightarrow \begin{cases} x=a \\ y=b \end{cases}$

For polar forms $r\text{cis}\theta = \rho\text{cis}\phi \Leftrightarrow \begin{cases} r=\rho \\ \theta=\phi+2k\pi \end{cases}$

♦ n-th ROOTS OF 1

The solutions of $z^2=1$, otherwise the roots of the quadratic z^2-1, are 1 and -1. They are also known as the 2nd roots of 1

In general, the roots of the equation

$$\boxed{z^n=1}$$

that is of the polynomial z^n-1, are known as **n-th roots of 1**.

Let $z=r\text{cis}\theta$ be a root. Then

$$z^n=1 \Leftrightarrow r^n\text{cis}(n\theta) = 1\text{cis}0 \Leftrightarrow \begin{cases} r^n=1 \Leftrightarrow r=1 \\ n\theta=0+2k\pi \Leftrightarrow \theta=\dfrac{2k\pi}{n} \end{cases}$$

Hence, the roots have the form

$$\boxed{z_k = \text{cis}\frac{2k\pi}{n} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}}$$

For $k=0,1,2,\ldots,n-1$ we obtain the n distinct roots of 1.

EXAMPLE 1

The 3rd roots of 1, that is the solutions of $z^3=1$, are the following

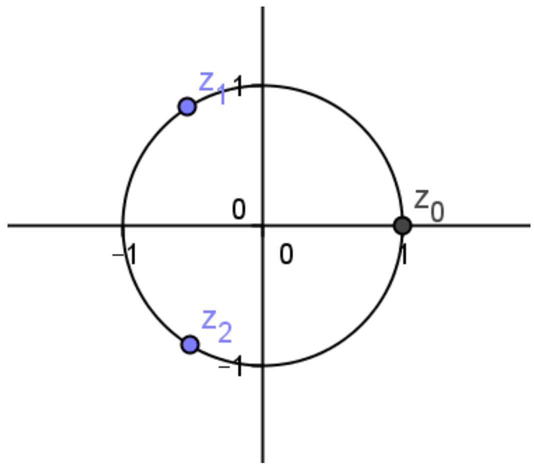
$$z_k = \text{cis}\frac{2k\pi}{3} = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}, \text{ for } k=0,1,2$$

Namely,

$$z_0=1, \qquad z_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}, \qquad z_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

Geometric interpretation:

The modulus of each root is 1, thus the three roots lie on the unit circle. Since their arguments are 0, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, they divide the unit circle in three equal parts.



EXAMPLE 2

The 4^{th} roots of 1, that is the solutions of $z^4=1$, are the following

$$z_k = \cos\frac{2k\pi}{4}, \qquad \text{for } k=0,1,2,3$$

Namely,

$$z_0=1,$$
$$z_1= cis\frac{2\pi}{4} = cis\frac{\pi}{2} = i,$$
$$z_2= cis\frac{4\pi}{4} = cis\pi = -1$$
$$z_3= cis\frac{6\pi}{4} = cis\frac{3\pi}{2} = -i$$

Geometric interpretation:

The four solutions 1, i, −1, −i divide the unit circle in 4 equal parts.

In general,

the n-th roots of 1 divide the unit circle in n equal parts.

NOTICE:

- The solutions of $z^n=1$ are

$$z_0=1,$$

$$z_1=cis\frac{2\pi}{n},$$

$$z_2=cis\frac{4\pi}{n},$$

$$z_3=cis\frac{6\pi}{n},$$

...

Let us denote by **w** the first non-real root z_1, i.e.

$$\mathbf{w = cis\frac{2\pi}{n} = z_1}$$

Then, by using De Moivre, we obtain

$$w^2 = cis\frac{4\pi}{n} = z_2,$$

$$w^3 = cis\frac{6\pi}{n} = z_3,$$

...

In other words, the n-th roots of 1 can be expressed as

$$\boxed{1, w, w^2, ..., w^{n-1}}$$

- We can derive the following important result:

> The sum of the n-th roots of 1 is always 0

Indeed[§],

$$z_0 + z_1 + z_2 + \cdots + z_{n-1} = 1 + w + w^2 + \cdots + w^{n-1} \quad [G.S.]$$

$$= \frac{w^n - 1}{w - 1} = 0 \qquad \text{(since } w^n=1\text{)}$$

[§] You may also use the identity $a^n - 1 = (a-1)(1 + a + a^2 + \cdots + a^{n-1})$

EXAMPLE 3

The 3rd roots of 1 are 1, ω and ω^2, where

$$\omega = cis\frac{2\pi}{3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i\ ,$$

$$\omega^2 = cis\frac{4\pi}{3} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

We can easily verify that $1 + \omega + \omega^2 = 0$

EXAMPLE 4

(a) Write down the 5th roots of 1.

(b) Factorize z^5-1

(c) Use the sum of the roots to show that $\cos\frac{2\pi}{5} + \frac{4\pi}{5} = -\frac{1}{2}$

Solution

(a) The 5th roots of 1 are

$$z_0=1,$$

$$z_1= cis\frac{2\pi}{5} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5},$$

$$z_2= cis\frac{4\pi}{5} = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5},$$

$$z_3= cis\frac{6\pi}{5} = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5},$$

$$z_4= cis\frac{8\pi}{5} = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5},$$

Notice that z_3 may also be written as

$$z_3= cis\frac{-4\pi}{5} = \cos\frac{-4\pi}{5} + i\sin\frac{-4\pi}{5} = \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5},$$

This is the conjugate of z_2.

Similarly

$$z_4= \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} \quad \text{is the conjugate of } z_1.$$

(b) The factorization of z^5-1 contains

one linear factor, namely $(z-1)$

two quadratics with complex roots

The first quadratic factor is obtained by

$$(z-z_1)(z-z_4) = (z-\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5})\ (z-\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5})$$

$$=(z-\cos\frac{2\pi}{5})^2 - (i\sin\frac{2\pi}{5})^2$$

$$=z^2-2z\cos\frac{2\pi}{5} + \cos^2\frac{2\pi}{5}\sin^2\frac{2\pi}{5}$$

$$=z^2-2z\cos\frac{2\pi}{5} + 1$$

Similarly we obtain the second quadratic factor and finally,

$$z^5-1 = (z-1)\ (z^2-2z\cos\frac{2\pi}{5} + 1)\ (z^2-2z\cos\frac{4\pi}{5} + 1)$$

(c) We know that

$$z_0 + z_1 + z_2 + z_3 + z_4 = 0$$

Hence, the sum of the real parts of the roots is also 0:

$$1+\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{2\pi}{5} = 0$$

$$\Rightarrow\ 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = -1$$

$$\Rightarrow\ \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$$

Remark for the factorization of z^n-1

Working as in EXAMPLE 4, (b) above

If n is odd

- By using the 7^{th} roots of 1, we can show that

  $$z^7-1 = (z-1)(z^2-2z\cos\frac{2\pi}{7}+1)(z^2-2z\cos\frac{4\pi}{7}+1)(z^2-2z\cos\frac{6\pi}{7}+1)$$

- Can you guess a similar factorization for z^9-1 ?

If n is even

- By using the 6^{th} roots of 1 (there are two real roots: ±1) we get

$$z^6-1 = (z-1)(z+1)(z^2-2z\cos\frac{2\pi}{6}+1)\ (z^2-2z\cos\frac{4\pi}{6}+1)$$
$$= (z-1)(z+1)(z^2-2z\cos\frac{\pi}{3}+1)\ (z^2-2z\cos\frac{2\pi}{3}+1)$$
$$= (z-1)(z+1)(z^2-z+1)\ (z^2+z+1)$$

- Can you find a similar factorization for z^8-1 ?

Remark for the sum of the n-th roots (only when n is odd)

Working as in EXAMPLE 4, (c) above

- By using the sum of the 7^{th} roots of 1, we can show that

$$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=-\frac{1}{2}$$

- Can you guess a similar formula by using the 9^{th} roots of 1?

♦ n-th ROOTS OF A COMPLEX NUMBER a

Consider now the equation

$$\boxed{z^n=a}$$

where a is a complex number.

Let $z=r\text{cis}\theta$ be a root. Then $z^n = r^n\text{cis}(n\theta)$

We also express the complex number a in polar form: $a=\rho\text{cis}\phi$.

Then

$$z^n=a \Leftrightarrow r^n\text{cis}(n\theta) = \rho\text{cis}\phi \Leftrightarrow \begin{cases} r^n=\rho \Leftrightarrow r=\sqrt[n]{\rho} \\ n\theta=\phi+2k\pi \Leftrightarrow \theta=\dfrac{\phi+2k\pi}{n} \end{cases}$$

For k=0,1,2,…,n-1 we obtain the following n roots of a

$$\boxed{z_k=\sqrt[n]{\rho}\,\text{cis}\left(\frac{\phi+2k\pi}{n}\right)}$$

EXAMPLE 5

Solve the equation $z^3=8i$.

Solution

Let $z=r(\cos\theta+i\sin\theta)$.

The polar form of $8i$ is $8cis\frac{\pi}{2}$. Then

$$z^3=8i \Leftrightarrow r^3cis(3\theta) = 8cis\frac{\pi}{2} \Leftrightarrow \begin{cases} r^3=8 \Leftrightarrow r=2 \\ 3\theta=\frac{\pi}{2}+2k\pi \Leftrightarrow \theta=\frac{\pi+4k\pi}{6} \end{cases}$$

We obtain the solutions

$$z_k=2cis\left(\frac{\pi+4k\pi}{6}\right) \quad \text{where } k=0,1,2.$$

Namely,

$$\boxed{z_0=2cis\frac{\pi}{6}} \qquad \boxed{z_1=2cis\frac{5\pi}{6}} \qquad \boxed{z_2=2cis\frac{9\pi}{6}=2cis\frac{3\pi}{2}}$$

Geometric interpretation:

The modulus of each root is 2, thus the three roots lie on the circle of radius 2. Since their arguments are $\frac{\pi}{6},\frac{5\pi}{6},\frac{9\pi}{6}$ they divide the circle in three equal arcs but now, the first root is at $\theta=\frac{\pi}{6}$.