International Baccalaureate

MATHEMATICS

Analysis and Approaches (SL and HL)

Lecture Notes

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TOPIC 5
CALCULUS

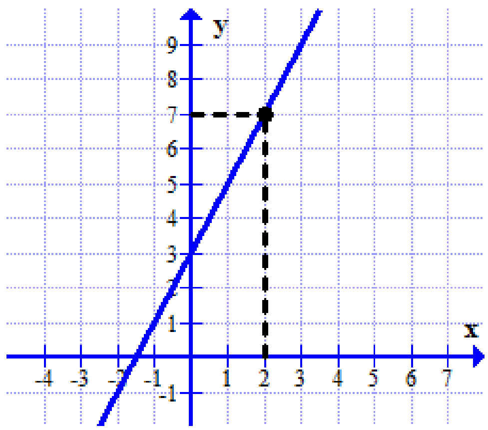
July 2023

5.1 THE LIMIT limf(x) – THE DERIVATIVE f ′(x): A ROUGH IDEA

This paragraph may look very "technical". Do not pay much attention on your first reading. You may skip and proceed to paragraph 5.2; you will realize that the derivative in practice is much easier than it appears here!

♦ THE LIMIT limf(x)

Consider the function f(x) = 2x+3.



Let us investigate how the function behaves at x=2. Clearly f(2)=7. But what happens when x is very close to 2?

x approaches 2^- (from values less than 2)	
x	**f(x)**
1.9	6.8
1.99	6.98
1.999	6.998

x approaches 2+ (from values greater than 2)	
x	**f(x)**
2.1	7.2
2.01	7.02
2.001	7.002

if $x \to 2^-$ then $f(x) \to 7$

if $x \to 2^+$ then $f(x) \to 7$

Thus in general, if x tends to 2, f(x) tends to 7.

In order to express this fact we write

$$\boxed{\lim_{x\to 2} f(x) = 7}$$

and say that: <u>the limit of f(x), as x tends to 2, is 7.</u>

Remark

In fact for the left column we write $\lim_{x\to 2^-} f(x) = 7$

while for the right column we write $\lim_{x\to 2^+} f(x) = 7$

and these are called **side limits**. If the side limits are equal then

$$\lim_{x\to 2} f(x) = 7$$

In this example

$\lim_{x\to 2} f(x) = 7$ which in fact is f(2)

$\lim_{x\to 3} f(x) = 9$ which in fact is f(3)

The situation $\lim_{x\to a} f(x) = f(a)$ occurs very often, however, this is not always the case (otherwise the limit would be nothing more than a simple substitution!).

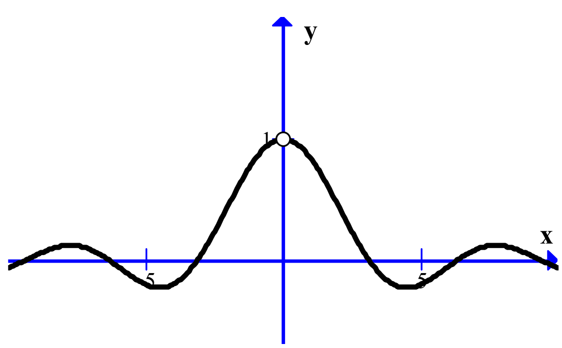
Let' s see a case where the limit is not a simple substitution!

Consider the function f(x) = $\frac{\sin x}{x}$. It is not defined at x=0.

However, we will find (informally) the limit

$$\lim_{x\to 0} \frac{\sin x}{x}.$$

The graph looks like



Although f(0) does not exist, it seems, by observation, that that the limit at x=0 is 1.

Let's approach x=0, by using our GDC:

x approaches 0^- (from values less than 0)	
x	f(x)
-0.1	0.998334
-0.01	0.999983
-0.001	0.999999

x approaches 0^+ (from values greater than 0)	
x	f(x)
0.1	0.998334
0.01	0.999983
0.001	0.999999

if $x \to 0^-$ then $f(x) \to 1$

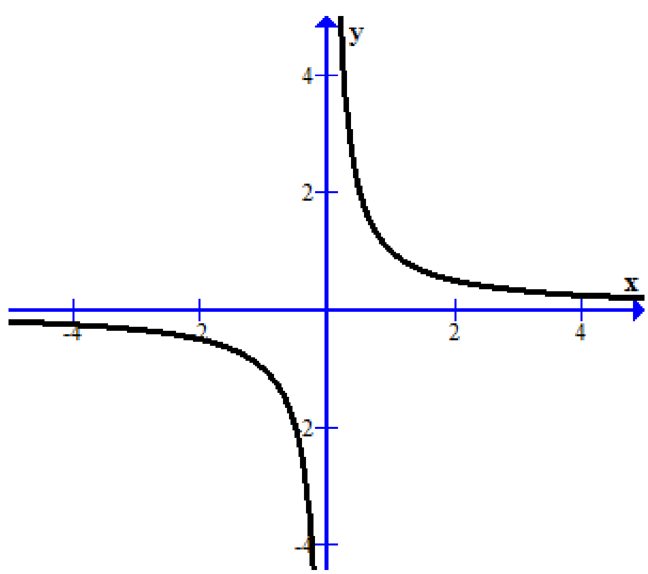
if $x \to 0^+$ then $f(x) \to 1$

The limit when x tends to 0 is 1. We write

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x}=1}$$

The limit can be $+\infty$ or $-\infty$.

Let $f(x)=\dfrac{1}{x}$



At x=0, we only have **side-limits:**

$$\boxed{\lim_{x\to 0^-}\frac{1}{x}=-\infty} \quad \text{and} \quad \boxed{\lim_{x\to 0^+}\frac{1}{x}=+\infty}$$

In fact, these results justify that x=0 is a horizontal asymptote.

In general:

If $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$ (or both) we say that

x=a is a **vertical asymptote**.

We also define limits of the form $\lim_{x\to+\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$.

we observe in fact the behavior of the function

when x approaches $+\infty$ (large positive) or $-\infty$ (large negative).

In our example $f(x)=\dfrac{1}{x}$ (see graph above):

$$\boxed{\lim_{x\to+\infty}\frac{1}{x}=0} \quad \text{and} \quad \boxed{\lim_{x\to-\infty}\frac{1}{x}=0}.$$

For example, if x=1,000,000 or -1,000,000, then y is close to 0.

In fact, these results justify that, y=0 is a horizontal asymptote.

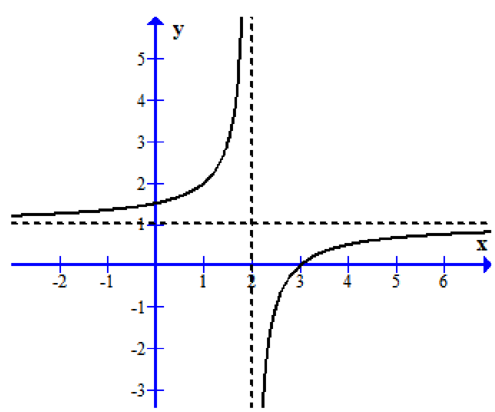
In general:

If $\lim_{x\to+\infty} f(x) = a$ or $\lim_{x\to-\infty} f(x) = a$ (or both), we say that

y=a is a **horizontal asymptote**.

<u>EXAMPLE 1</u>

$$f(x) = \frac{x-3}{x-2}$$



- ***x=2 is a vertical asymptote.***

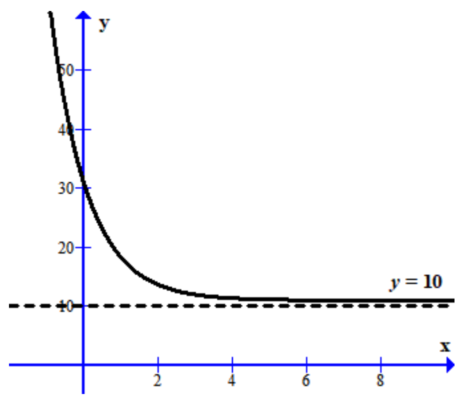
  The formal justification is that $\lim_{x\to2^-} f(x) = +\infty$ and $\lim_{x\to2^+} f(x) = -\infty$

- ***y=1 is a horizontal asymptote.***

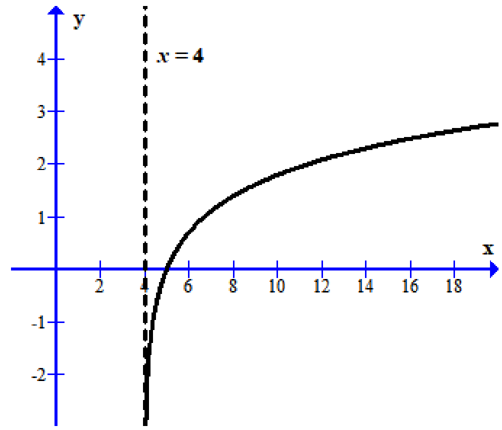
  The formal justification is that $\lim_{x\to+\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = 1$

EXAMPLE 2

$f(x) = 20e^{-x} + 10$



$g(x) = \ln(x-4)$



- ***y=10 is a horizontal asymptote for f(x).***
  The formal justification is that $\lim_{x\to+\infty} f(x) = 10$
- ***x=4 is a vertical asymptote for g(x).***
  The formal justification is that $\lim_{x\to 4^+} f(x) = -\infty$

Look at an interesting limit that provides the irrational number

e = 2.7182818…

EXAMPLE 2

Investigate (informally) the limit: $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x$.

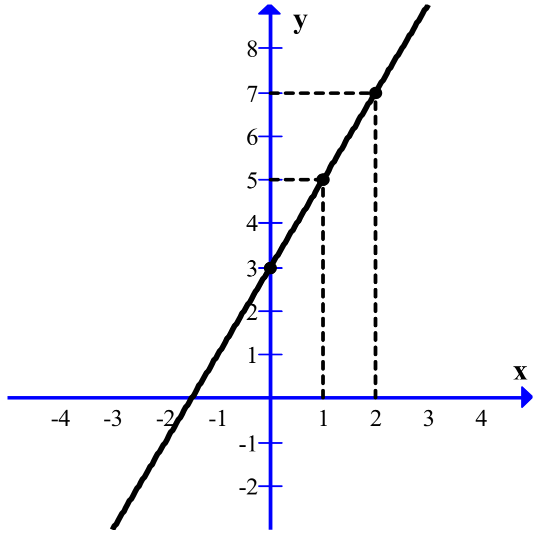
x approaches $+\infty$	
x	f(x)
1000	2.7169239…
1000000	2.7182804…
10^{10}	2.7182818…

The resulting limit is in fact the number **e=2.7182818…** That is,

$$\boxed{\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e\,.}$$

♦ RATE OF CHANGE (OR GRADIENT) IN A STRAIGHT LINE

Consider the line $f(x)=2x+3$.



Let us pick the points (1,5) and (2,7) on the line. Notice:

when x changes from 1 to 2

then y changes from 5 to 7

Hence, the corresponding rate of change is

$$\frac{\Delta y}{\Delta x}=\frac{f(2)-f(1)}{2-1}=\frac{7-5}{2-1}=2$$

We understand that for a straight line, the rate of change between any two points is always the same.

For example, if we pick the points (0,3) and (2,7)

when x changes from 0 to 2

then y changes from 3 to 7

Hence, the corresponding rate of change is still

$$\frac{\Delta y}{\Delta x}=\frac{f(2)-f(0)}{2-0}=\frac{7-3}{2-0}=2$$

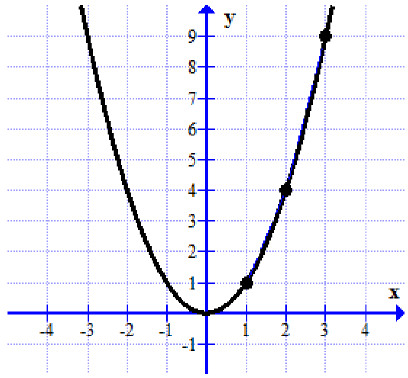
This constant value is the **gradient** of the line.

Next, we will see that the gradient is not only defined for straight lines but also for other curves.

♦ RATE OF CHANGE (OR GRADIENT) IN A CURVE

In a curve which is not a straight line, the rate of change between any two points is not always the same.

For example, in $f(x)=x^2$



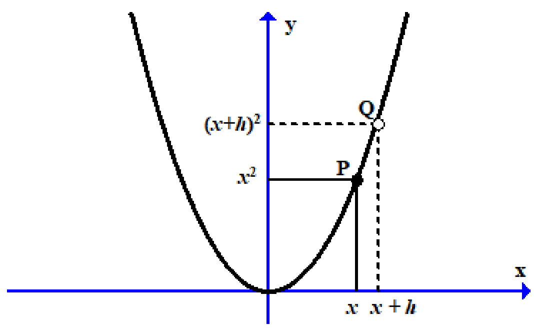
Rate of change from x=1 to x=2: $\frac{\Delta y}{\Delta x}=\frac{f(2)-f(1)}{2-1}=\frac{4-1}{2-1}=3$

Rate of change from x=1 to x=3: $\frac{\Delta y}{\Delta x}=\frac{f(3)-f(1)}{3-1}=\frac{9-1}{3-1}=4$

However, we can measure the "instantaneous" rate of change at any point P(x,y) on the curve. This will be the **gradient** at point P.

The random point P has the form $P(x, x^2)$.

Let Q be a point close to P, say at x+h: $Q(x+h\ ,\ (x+h)^2)$



Rate of change from P to Q:

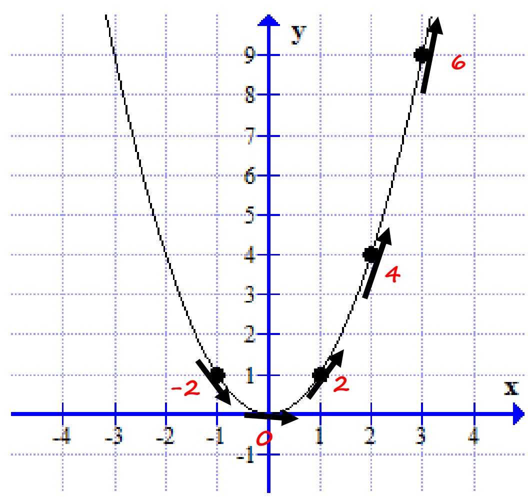
$$\frac{\Delta y}{\Delta x}=\frac{(x+h)^2-x^2}{(x+h)-x}=\frac{x^2+2hx+h^2-x^2}{h}=\frac{2hx+h^2}{h}=2x+h$$

By letting h tend to 0, we find the gradient at point P:

$$\lim_{h\to 0}(2x+h)=2x$$

Hence, the gradient at any point x on the curve is m=2x.

In the following diagram we provide the gradient at some points of the curve $y = x^2$:



gradient at x: m =2x

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gradient at x=0: m =0
gradient at x=1: m =2
gradient at x=2: m =4
gradient at x=3: m =6
gradient at x=–1: m = –2

Thus, for the function

$$f(x) = x^2$$

a new function $f'(x)$ is determined; it gives the gradient at any x:

$$f'(x) = 2x$$

Thus, for example, the gradient at x=3 is: $f'(3) = 6$

The new function $f'(x)$, corresponding to $f(x)$, is called

the ***derivative*** of $f(x)$

Thus, for a function $f(x)$:

$f'(x) =$ **DERIVATIVE = RATE OF CHANGE = GRADIENT at x .**

5.2 DERIVATIVES OF KNOWN FUNCTIONS – RULES

The derivative of a function $f(x)$ is a new function denoted by $f'(x)$. As explained in the preceding section, $f'(x)$ indicates the rate of change, or otherwise the gradient of $f(x)$ at any particular point x.

We have seen that the derivative of the function $f(x)=x^2$ is

$$f'(x)=2x$$

We can similarly show that for $f(x)=x^3$ the derivative is

$$f'(x)=3x^2.$$

In general, the derivative of the power function $f(x)=x^n$ is

$$f'(x)=nx^{n-1}$$

Look at the table below

$f(x)=x^n$	$f'(x)=nx^{n-1}$
x^{10}	$10x^9$
x^4	$4x^3$
x^3	$3x^2$
x^2	$2x$
x	1

The derivatives of the most common functions are shown below

f(x)	**f′(x)**
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
c (constant)	0

Let us especially elaborate on the power formula $(x^n)' = nx^{n-1}$
This formula also applies for –tive values of n:

$f(x)= x^n$	$f'(x) = nx^{n-1}$
x^{-10}	$-10x^{-11}$
x^{-3}	$-3x^{-4}$
x^{-2}	$-2x^{-3}$
x^{-1}	$-x^{-2}$

It also applies for rational values of n:

$f(x)= x^n$	$f'(x) = nx^{n-1}$
$x^{6.4}$	$6.4x^{5.4}$
$x^{3/2}$	$\frac{3}{2}x^{1/2}$
$x^{5/3}$	$\frac{5}{3}x^{2/3}$
$x^{1/2}$	$\frac{1}{2}x^{-1/2}$

EXAMPLE 1

Show that (a) $\left(\frac{1}{x^2}\right)' = \frac{-2}{x^3}$ (b) $\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$

Solution

(a) $\frac{1}{x^2} = x^{-2}$, so the derivative is $-2x^{-3} = \frac{-2}{x^3}$

(b) $\sqrt{x} = x^{1/2}$, so the derivative is $\frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$

EXAMPLE 2

Let $f(x)=x^7$. Find

(a) $f(0)$, $f(1)$, $f(2)$
(b) $f'(x)$
(c) $f'(0)$, $f'(1)$, $f'(2)$
(d) the rate of change of $f(x)$ at $x=2$
(e) the gradient of $f(x)$ at $x=2$

Solution

(a) $f(0)=0$, $f(1)=1$, $f(2)=128$

(b) $f'(x)=7x^6$

(c) $f'(0)=0$, $f'(1)=7\cdot 1^6=7$, $f'(2)=7\cdot 2^6=448$

(d) it is $f'(2)=448$

(e) it is $f'(2)=448$

♦ NOTATION

If $y=f(x)$, the derivative is denoted by the following symbols

$$y' \quad \text{or} \quad f'(x) \quad \text{or} \quad \frac{dy}{dx} \quad \text{or} \quad \frac{d}{dx}f(x)$$

The derivative at some specific value of x, say x=2, is denoted by

$$f'(2) \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=2}$$

For example, if $y=f(x)=x^3$, we can write

$$y'=3x^2 \quad \text{or} \quad (x^3)'=3x^2 \quad \text{or} \quad \frac{dy}{dx}=3x^2 \quad \text{or} \quad \frac{d}{dx}x^3=3x^2$$

Moreover,

$$f'(2)=12 \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=2}=12$$

The procedure of finding the derivative is called **differentiation**.

Notice that the GDC does provide $f'(x)$ but it gives the derivative at a specific point, for example $f'(2)$ (check!)

Finally, the notation $\frac{dy}{dx}$ is more indicative when we use other variables.

For example,

if $s=t^2$, then $\frac{ds}{dt}=2t$.

If $P=\ln Q$, then $\frac{dP}{dQ}=\frac{1}{Q}$. Also $Q=e^P$, so that $\frac{dQ}{dP}=e^P$.

- RULES OF DIFFERENTIATION

Rule (1): $(f+g)' = f' + g'$ $\qquad$ $(f-g)' = f' - g'$

EXAMPLE 3

For $f(x) = x^5+x^3$, $\qquad f'(x) = 5x^4+3x^2$

For $g(x)= x^7-e^x+\sin x-x+5$, $\quad g'(x) = 7x^6-e^x+\cos x-1$

Rule (2): $(af)' = af'$ $\qquad$ (a=constant number)

EXAMPLE 4

For $f(x) = 3\sin x$, $\qquad f'(x) = 3\cos x$

For $g(x) = 7e^x$, $\qquad g'(x) = 7e^x$

For $h(x) = 5x^3$, $\qquad h'(x) = 5(3x^2)=15x^2$

Let us combine Rules (1) and (2):

$$(af + bg)' = af' + bg'$$

EXAMPLE 5

For $f(x) = 2x^3-3x^2+7x+5$, $\qquad f'(x) = 6x^2-6x+7$

For $g(x) = 5x^7+3\ln x-7\cos x$, $\qquad g'(x) = 35x^6+\dfrac{3}{x}+7\sin x$

NOTICE:

The differentiation rules above may also be expressed as follows

$$\frac{d}{dx}[f(x)\pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) \qquad \frac{d}{dx}[af(x)] = a\frac{d}{dx}f(x)$$

$$\frac{d}{dx}[af(x)+bg(x)] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)$$

Rule (3): (product rule)

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

Be careful !!!

If $f(x) = x^5 \sin x$ then $f'(x)$ **is not** $\mathbf{(5x^4)(\cos x)}$

We must follow the product rule above. That is

$$f'(x) = (x^5)' \sin x + x^5 (\sin x)' = 5x^4 \sin x + x^5 \cos x$$

EXAMPLE 6

For $f(x) = x\ln x$, $f'(x) = (x)'\ln x + x(\ln x)' = 1\ln x + x\dfrac{1}{x} = \ln x + 1$

For $g(x) = x^2e^x$, $g'(x) = 2xe^x + x^2e^x$

For $h(x) = 2x^3\cos x$, $h'(x) = 6x^2\cos x - 2x^3\sin x$

Rule (4): (quotient rule)

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

EXAMPLE 7

For $f(x) = \dfrac{x^3}{\sin x}$, $f'(x) = \dfrac{(x^3)'\sin x - x^3(\sin x)'}{(\sin x)^2} = \dfrac{3x^2\sin x - x^3\cos x}{\sin^2 x}$

For $g(x) = \dfrac{3x+5}{4x+1}$, $g'(x) = \dfrac{3(4x+1) - 4(3x+5)}{(4x+1)^2} = \dfrac{-17}{(4x+1)^2}$

For $h(x) = \dfrac{x^3 - 5x}{x^2+1}$,

$$h'(x) = \frac{(3x^2-5)(x^2+1) - (x^3-5x)2x}{(x^2+1)^2}$$

$$= \frac{3x^4 + 3x^2 - 5x^2 - 5 - 2x^4 + 10x^2}{(x^2+1)^2}$$

$$= \frac{x^4 + 8x^2 - 5}{(x^2+1)^2}$$

Sometimes, we can avoid the quotient rule. Look at the following

EXAMPLE 8

For $f(x) = \dfrac{x^3-2x+1}{x}$

method A: The quotient rule gives

$$f'(x) = \frac{(3x^2-2)x-(x^3-2x+1)1}{x^2} = \frac{2x^3-1}{x^2} = 2x-\frac{1}{x^2}$$

method B: we may modify f(x) by splitting into three fractions

$$f(x) = \frac{x^3}{x}-\frac{2x}{x}+\frac{1}{x} = x^2-2+x^{-1}\text{, so that}$$

$$f'(x) = 2x-x^{-2} = 2x-\frac{1}{x^2}$$

♦ HIGHER DERIVATIVES

We can continue differentiating the 1st derivative $f'(x)$ and thus find the 2nd derivative $f''(x)$, the 3rd derivative $f'''(x)$ and so on. The nth derivative is also denoted by $f^{(n)}(x)$.

EXAMPLE 9

- For $f(x) = x^5$, $f'(x) = 5x^4$ $f''(x) = 20x^3$ $f'''(x) = 60x^2$
- For $g(x) = \sin x$ $g'(x) = \cos x$ $g''(x) = -\sin x$ $g'''(x) = -\cos x$
- For $h(x) = e^x$, $h'(x) = e^x$ $h''(x) = e^x$ $h'''(x) = e^x$

  Clearly $h^{(4)}(x) = e^x$ and in general $h^{(n)}(x) = e^x$

Alternative notation:

$f''(x)$ can also be written as $\dfrac{d^2y}{dx^2}$ or $\dfrac{d^2}{dx^2}f(x)$

$f'''(x)$ can also be written as $\dfrac{d^3y}{dx^3}$ or $\dfrac{d^3}{dx^3}f(x)$

5.3 THE CHAIN RULE

Look at the differentiation table again

f(x)	f′(x)
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
x^3	$3x^2$

In simple words the chain rule says:

If we replace x by another function u(x) then the derivative is as above but we also multiply the result by u′(x).

Let us repeat the table above where x is replaced by u(x):

f(x)	f′(x)
$\sin u$	$(\cos u)\times u'$
$\cos u$	$(-\sin u)\times u'$
e^u	$e^u \times u'$
$\ln u$	$\frac{1}{u}\times u'$
$\sqrt{u}$	$\frac{1}{2\sqrt{u}}\times u'$
u^3	$3u^2 \times u'$

EXAMPLE 1

$f(x) = \sin(2x^2+3)$ $[u=2x^2+3]$

$$f'(x) = \cos(2x^2+3)\times(2x^2+3)' = \cos(2x^2+3)\times 4x$$
$$= 4x\cos(2x^2+3)$$

EXAMPLE 2

$$f(x) = e^{5x+3} \qquad [u=5x+3]$$

$$f'(x) = e^{5x+3} \times 5 = 5e^{5x+3}$$

and $f''(x) = 25e^{5x+3}$

EXAMPLE 3

$$f(x) = e^{\sin x} \qquad [u=\sin x]$$

$$f'(x) = e^{\sin x}\cos x$$

EXAMPLE 4

$$f(x) = \ln(x^2+4) \qquad [u= x^2+4]$$

$$f'(x) = \frac{1}{x^2+4}2x$$

$$= \frac{2x}{x^2+4}$$

EXAMPLE 5

$$f(x) = \sqrt{3x^2+5x+2} \qquad [u=3x^2+5x+2]$$

$$f'(x) = \frac{1}{2\sqrt{3x^2+5x+2}}(6x+5)$$

$$= \frac{6x+5}{2\sqrt{3x^2+5x+2}}$$

EXAMPLE 6

$$f(x) = \sqrt{\cos x} \qquad [u=\cos x]$$

$$f'(x) = \frac{-\sin x}{2\sqrt{\cos x}}$$

EXAMPLE 7

Consider the function $f(x) = (2x^2+3)^2$

If we first expand: $f(x)=4x^4+12x^2+9$ then $f'(x) = 16x^3+24x$

If we follow the chain rule [with $u=2x^2+3$]

$$f'(x) = 2(2x^2+3)\times(4x) = 8x(2x^2+3) = 16x^3+24x$$

But what about

$$f(x) = (2x^2+3)^{10} \ ?$$

Do not attempt to expand, it will be laborious! The chain rule gives

$$f'(x) = 10\,(2x^2+3)^9\times(4x) = 40x(2x^2+3)^9$$

NOTICE

In many examples $u=ax+b$ (linear) so that $u'=a$. Hence,

we simply differentiate for u and multiply the result by a.

EXAMPLE 8

Let us consider all the usual functions with $u=3x+7$.

$f(x)$	$f'(x)$
$\sin(3x+7)$	$3\cos(3x+7)$
$\cos(3x+7)$	$-3\sin(3x+7)$
e^{3x+7}	$3e^{3x+7}$
$\ln(3x+7)$	$\dfrac{3}{3x+7}$
$\sqrt{3x+7}$	$\dfrac{3}{2\sqrt{3x+7}}$
$(3x+7)^5$	$15(3x+7)^4$

Next, instead of x we have u=5x.

f(x)	f′(x)
sin(5x)	5cos(5x)
cos(5x)	−5sin(5x)
e^{5x}	$5e^{5x}$
ln(5x)	$\frac{5}{5x}=\frac{1}{x}$

Perhaps the most confusing case of the chain rule is the differentiation of the function $\sin^n x$ (or $\cos^n x$). Remember that

$$\sin^n x \quad \text{means} \quad (\sin x)^n \qquad [\text{so that } u=\sin x]$$

EXAMPLE 9

For $f(x) = \sin^3 x$, $f'(x) = 3\sin^2 x \cos x$

For $f(x) = \sin^2 x$, $f'(x) = 2\sin x \cos x$

For $f(x) = \cos^5 x$, $f'(x) = -5\cos^4 x \sin x$

For $f(x) = \frac{1}{\sin x} = (\sin x)^{-1}$, $f'(x) = -(\sin x)^{-2} \cos x = -\frac{\cos x}{\sin^2 x}$

For $f(x) = \frac{1}{\cos^2 x} = (\cos x)^{-2}$, $f'(x) = -2(\cos x)^{-3}(-\sin x) = \frac{\sin x}{\cos^3 x}$

NOTICE

In fact, the chain rule is the rule of differentiation for the composition of two functions

$$(f \circ g)(x) = f(g(x))$$

It says that

$$\left[f(g(x))\right]' = f'(g(x))g'(x)$$

I admit that this definition is not so "elegant"! The best way to learn the chain rule is to practice with many examples.

It is possible to have a "double chain", that is a chain inside another chain! It is in fact the derivative of the composition of three functions. The following example is indicative.

EXAMPLE 10

a) Let $f(x) = \ln(\sin(3x+1))$

$$f'(x) = \frac{1}{\sin(3x+1)}[\sin(3x+1)]' \qquad [u = \sin(3x+1)]$$

$$= \frac{1}{\sin(3x+1)}[3\cos(3x+1)] \qquad [v = 3x+1]$$

$$= 3\frac{\cos(3x+1)}{\sin(3x+1)}$$

b) Let $f(x) = 3\sin^5(x^2+1)$.

Better to see that as $3[\sin(x^2+1)]^5$

$$f'(x) = 15\sin^4(x^2+1)[\sin(x^2+1)]' \qquad [u = \sin(x^2+1)]$$

$$= 15\sin^4(x^2+1)\cos(x^2+1)(2x) \qquad [v = x^2+1]$$

$$= 30x\sin^4(x^2+1)\cos(x^2+1)$$

c) Let $f(x) = e^{\sin(3x)}$

$$f'(x) = e^{\sin(3x)}[\sin(3x)]' \qquad [u = \sin(3x)]$$

$$= e^{\sin(3x)}[3\cos(3x)] \qquad [v = 3x]$$

$$= 3e^{\sin(3x)}\cos(3x)$$

d) Let $f(x) = \sqrt{\sin^2 x + \sin 2x}$

$$f'(x) = \frac{1}{2\sqrt{\sin^2 x + \sin 2x}}(\sin^2 x + \sin 2x)'$$

$$= \frac{1}{2\sqrt{\sin^2 x + \sin 2x}}(2\sin x\cos x + 2\cos 2x)$$

$$= \frac{\sin x\cos x + \cos 2x}{\sqrt{\sin^2 x + \sin 2x}}$$

We may also have a combination of all the rules we have seen.

EXAMPLE 11

Find the derivative of $f(x) = e^{2x}\sin 3x$.

Solution

We have a product of two functions e^{2x} and $\sin 3x$, but for each function we must apply the chain rule

$$f'(x) = (e^{2x})'(\sin 3x) + (e^{2x})(\sin 3x)'$$
$$= 2e^{2x}\sin 3x + 3e^{2x}\cos 3x$$

EXAMPLE 12

Find the derivative of $f(x) = e^{x^2\sin x}$.

Solution

This is an example of chain rule where $u = x^2\sin x$ is a product.

$$f'(x) = e^{x^2\sin x}\left(x^2\sin x\right)'$$
$$= e^{x^2\sin x}\left(2x\sin x + x^2\cos x\right)$$

Finally, let us see some slightly more abstract examples:

EXAMPLE 13

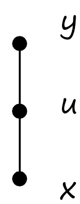
Differentiate the following functions (of the first column):

$y = \sin f(x)$	$\frac{dy}{dx} = \cos f(x) \times f'(x)$
$y = f(\sin x)$	$\frac{dy}{dx} = f'(\sin x) \times \cos x$
$y = f(x)^5$	$\frac{dy}{dx} = 5f(x)^4 \times f'(x)$
$y = f(x^5)$	$\frac{dy}{dx} = f'(x^5) \times 5x^4$

♦ AN ALTERNATIVE WAY TO LOOK AT THE CHAIN RULE

Let y depend on u, and u depend on x:

We have the chain



The chain rule says

$$\boxed{\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}}$$

[Notice: it is easy to remember this formula as it looks like a simplification of fractions!!!]

Look at again

$$y = (2x^2+3)^{10}$$

Then $\quad y = u^{10} \quad$ where $\quad u=2x^2+3$

The chain rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= 10u^9\,(4x)\\ &= 10(2x^2+3)^9(4x) && \text{[replace back } u=2x^2+3]\\ &= 40x(2x^2+3)^9 && \text{[as found earlier]}\end{aligned}$$

EXAMPLE 14

Let $y= e^{\sin x}$.

Then $y =e^u$ were $u=\sin x$.

$$\frac{dy}{dx}= \frac{dy}{du}\frac{du}{dx} = e^u \cos x = e^{\sin x} \cos x$$

[this is what we obtain if we apply the chain rule as usual]

The following example also justifies the term "chain"!!!

EXAMPLE 15 (Mainly for HL)

Let $P = Q^3$ and $Q=\ln R$. Find $\frac{dP}{dR}$ in terms of R

We have the chain

$$\begin{tikzcd} P \arrow[d, dash] \\ Q \arrow[d, dash] \\ R \end{tikzcd}$$

(as P depends on Q and Q depends on R)

The chain rule gives

$$\begin{aligned}\frac{dP}{dR} &= \frac{dP}{dQ}\cdot\frac{dQ}{dR}\\ &= 3Q^2\cdot\frac{1}{R}\\ &= 3(\ln R)^2\frac{1}{R}\end{aligned}$$

5.4 TANGENT LINE – NORMAL LINE AT SOME POINT x_o

Remember: A straight line with

- gradient **m**
- passing through point **(x_o,y_o)**

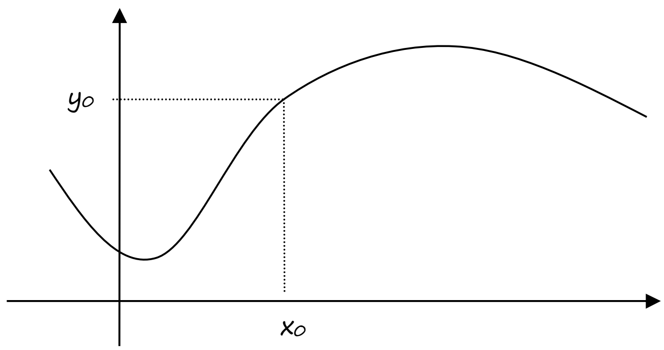
has equation

$$y - y_o = m(x - x_o)$$

For example, the line passing through A(1,2) with gradient m=3 has equation

$$y - 2 = 3(x - 1)$$

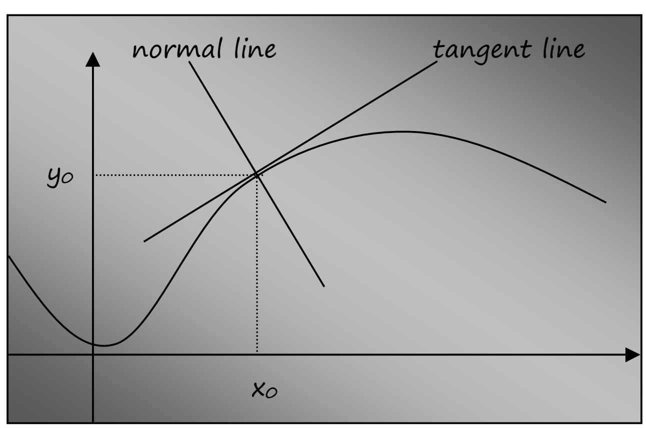
Consider a function $y=f(x)$ and some point x_o. Then we also know $y_o=f(x_o)$



We know that the gradient of the graph at (x_o,y_o) is $m_T=f'(x_o)$.
We define:

TANGENT LINE at x_o:
the line with gradient m_T
which passes through (x_o,y_o)

NORMAL LINE at x_o:
the perpendicular line to the tangent at (x_o,y_o).
Its gradient is $m_N=-\frac{1}{m_T}$



The point (x_o,y_o) is also known as <u>point of contact</u>.

- METHODOLOGY

Given $y=f(x)$
and some point $x=x_O$

We find the point of contact (x_O,y_O) since $y_O=f(x_O)$

$f'(x)$

$m_T=f'(x_O)$ and so $m_N=-\frac{1}{m_T}$

The equations of the two lines are

TANGENT LINE	**NORMAL LINE**
$y-y_O=m_T(x-x_O)$	$y-y_O=m_N(x-x_O)$

EXAMPLE 1

Consider the function

$$f(x)=x^2$$

Find the equations of the tangent line and the normal line at $x=3$.

Solution

The point of contact is $(3,9)$ (since $f(3)=9$)

The gradient function is $f'(x)=2x$. Thus

$$m_T=f'(3)=6 \quad \text{and} \quad m_N=-\frac{1}{6}$$

The tangent line is $y-9=6(x-3)$

The normal line is $y-9=-\frac{1}{6}(x-3)$

If they ask us to express them in the form $y=mx+c$, we obtain

Tangent line: $y-9=6(x-3) \Leftrightarrow y-9=6x-18 \Leftrightarrow \boxed{y=6x-9}$

Normal line: $y-9=-\frac{1}{6}(x-3) \Leftrightarrow y-9=-\frac{1}{6}x+\frac{1}{2} \Leftrightarrow \boxed{y=-\frac{1}{6}x+\frac{19}{2}}$

NOTICE

An alternative way to obtain the tangent and the normal lines is to use the formulas:

TANGENT LINE	NORMAL LINE
$y=m_T x+c$	$y=m_N x+c$

The point (x_0,y_0) helps us to find the constant c.

In the previous example, since m=6 and the point is (3,9):

- Tangent line: $y=6x+c$

  At (3,9) $18+c=9 \Leftrightarrow c=-9,$ thus $\boxed{y=6x-9}$

- Normal line: $y=-\frac{1}{6}x+c$

  At (3,9) $-\frac{1}{6}3+c=9 \Leftrightarrow c=9+\frac{1}{2} \Leftrightarrow c=\frac{19}{2}$ thus $\boxed{y=-\frac{1}{6}x+\frac{19}{2}}$

EXAMPLE 2

Consider the function $f(x)=5x^3-2x+1$

Find the tangent lines to the curve which are parallel to the line

$$L: y=13x+8$$

Solution

A tangent line parallel to L must have gradient $m_T=13$.

But $f'(x) = 15x^2-2$, so

$$15x^2-2=13 \Leftrightarrow 15x^2=15 \Leftrightarrow x^2=1 \Leftrightarrow x=1 \text{ or } x=-1$$

Hence, we will have two parallel lines, at the points x=1 and x=−1, with gradient m=13. The points of contact are (1,4) and (−1,−2)

- At (1,4), $y-4=13(x-1) \Leftrightarrow y-4=13x-13 \Leftrightarrow \boxed{y=13x-9}$
- At (−1,−2) $y+2=13(x+1) \Leftrightarrow y+2=13x+13 \Leftrightarrow \boxed{y=13x+11}$

NOTICE

Suppose that the gradient at (x_0,y_0) is $m_T=0$. Then

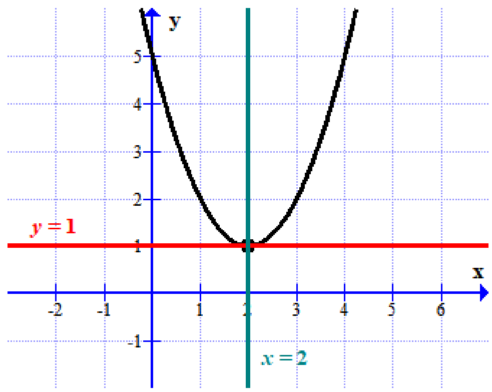
The tangent line is a horizontal line with equation $\boxed{y=y_0}$

The normal line is a vertical line[1] with equation $\boxed{x=x_0}$

EXAMPLE 3

Consider the function $f(x)=x^2-4x+5$

Find the equations of the tangent line and the normal line at $x=2$.



Solution

It is $f'(x)=2x-4$

At $x=2$, $y=1$, thus the point of contact is $(2,1)$

$m_T = 0$ (m_N is not defined)

Tangent line: the horizontal line $\boxed{y=1}$

Normal line: the vertical line $\boxed{x=2}$ (look at the graph above!)

For trickier questions, let's have in mind the following observation.

At the point of contact between **f(x)** and a tangent line **y=mx+c**

functions are equal: $f(x)=y$

derivatives are equal: $f'(x)=m$

[1] A vertical line has no gradient. If it passes through (x_0,y_0), it has equation $x=x_0$.

EXAMPLE 4

The line $y = mx - 3$ is tangent to the curve $f(x)=x^4-x$. Find m

Solution

At the point of contact x:

functions are equal: $x^4-x = mx - 3$

derivatives are equal: $4x^3-1 = m$

Hence,

$$x^4-x = (4x^3 - 1)x - 3 \Leftrightarrow x^4 - x = 4x^4 - x - 3$$
$$\Leftrightarrow 3x^4 = 3$$
$$\Leftrightarrow x^4 = 1 \Leftrightarrow x = \pm 1$$

If $x = 1$ then m=3

If $x = -1$ then m=-5

EXAMPLE 5

Consider the function $f(x)=x^4-x$. Find the tangent lines passing through the point (0,-3) [notice that the point is not on the curve]

Method A:

A line passing through the point (0,-3) has the form

$$y + 3 = m(x - 0) \quad \text{i.e} \quad y = mx - 3$$

This is in fact the example 4 above. We found two solutions:

If $x = 1$ then m=3 and the tangent line is $y = 3x - 3$

If $x = -1$ then m=-5 and the tangent line is $y = -5x - 3$

Method B (first find the tangent line at any point of contact x=a):

At any point (a,f(a)), i.e. (a, a^4-a).

$f'(x)=4x^3-1$, so $m_T = 4a^3-1$.

Therefore, $y-(a^4-a) = (4a^3-1)(x-a)$

$$\Rightarrow y = (4a^3-1)x - 3a^4$$

It passes through (0,-3): $-3a^4 = -3 \Leftrightarrow a^4 = 1 \Leftrightarrow a = \pm 1$.

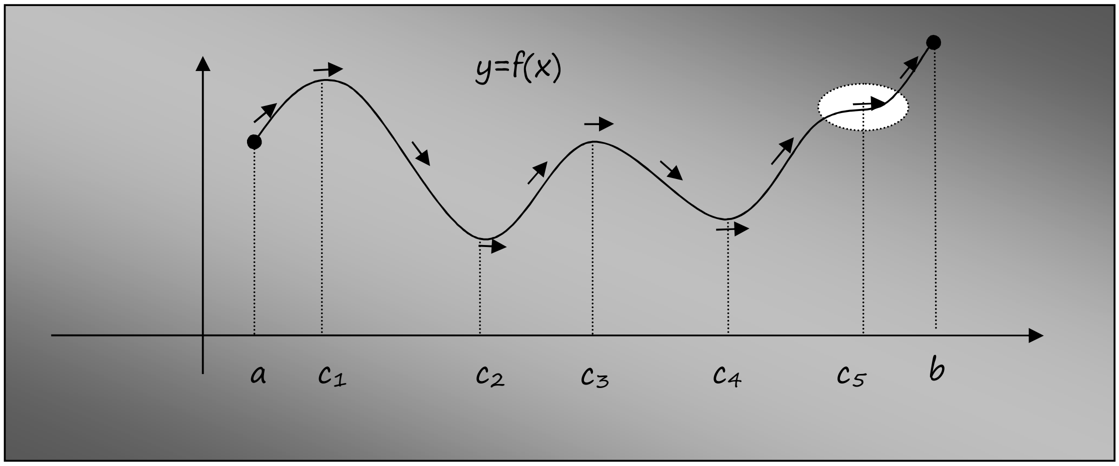
For a =1 we obtain the equation $y = 3x - 3$.

For a =-1 we obtain the equation $y = -5x - 3$.

5.5 MONOTONY – MAX, MIN

♦ INCREASING – DECREASING FUNCTIONS (MONOTONY)

Consider the following graph



Let us make some observations:

- The domain of the graph is the interval $[a,b]$
  The points $x=a$ and $x=b$ are called **endpoints**
- We say that we have a **local max (**or just **max)** at points:
  $$x=c_1,\ x=c_3,\ x=b$$
  (can you explain why?)
- We say that we have a **local min** (or just **min**) an points:
  $$x=a,\ x=c_2,\ x=c_4$$
  (can you explain why?)
  All these points (max and min) are called ***turning points*** or ***extreme values***
- Notice that $x=c_5$ is not a turning point (neither max nor min), as near $f(c_5)$ you can find smaller as well as larger values.
- The function is **increasing** (goes up) in the interval (a,c_1)
- The function is **decreasing** (goes down) in the interval (c_1,c_2)
- The function is **increasing** (goes up) in the interval (c_2,c_3)
- The function is **decreasing** (goes down) in the interval (c_3,c_4)
- The function is **increasing** (goes up) in the interval (c_4,b)

Remember though that

A +tive gradient means that the function is increasing (goes up)
A –tive gradient means that the function is decreasing (goes down).

But we know that

$$\boxed{\text{derivative} \quad = \quad \text{gradient}}$$

In other words

If $f'(x)>0$ then f is increasing ($\nearrow$)
If $f'(x)<0$ then f is decreasing ($\searrow$)

Notice: The increasing or decreasing behavior of a function is also known as **monotony**!

♦ TURNING POINTS: MAX – MIN

How can we find the turning points (max or min) of a function?

First, the end points are extreme values (see a and b above).

As far as the interior points is concerned, observe that the gradient at any turning point is 0 (the tangent lines at those values are horizontal!).

PROPOSITION:

If f(x) has a turning point (max or min) at some interior point c and $f'(c)$ exists, then

$$f'(c)=0$$

Notice that $f'(x)=0$ at c_1, c_2, c_3, c_4 and c_5
We have a local max at c_1, c_3 and a local min at c_2, c_4.
However, c_5 is not a turning point.
Hence the inverse proposition is not true.

Therefore, apart from the endpoints, the possible turning points (max/min) are the following

- points x where f′(x)=0 (called **stationary points**)
- points where f′(x) does not exist

All these points are also called **critical points.**

Here we only deal with **stationary points** (points where f′(x)=0).

To verify whether such a stationary point x=c is a turning point (max or min) we perform the following test

---

FIRST DERIVATIVE TEST for c

Check the sign of f′(x) to verify if the function is increasing or decreasing just before and after c:

x	c	
f ′(x)	+	−
Conclusion for f	↗	↘
	max	

x	c	
f ′(x)	−	+
Conclusion for f	↘	↗
	min	

If the sign does not change, we have neither a max nor a min.

♦ METHODOLOGY

Given y=f(x)

Step 1 we find f′(x)

Step 2 we solve f′(x)=0 (say that roots are a,b,c)

Step 3 we construct a table as follows to perform the first derivative test

x	a	b	c	
f ′(x)	+	−	+	+
Conclusion for f	↗	↘	↗	↗
	max	min	nothing	

EXAMPLE 1

Consider

$$f(x)=\frac{1}{3}x^3-2x^2+3x+5$$

We find

$$f'(x)=x^2-4x+3$$

We solve

$$x^2-4x+3=0$$

The solutions are $x=1$ and $x=3$

We construct the table

x		1		3	
$f'(x)=x^2-4x+3$	+		−		+
Conclusion for f	↗		↘		↗
		max		min	

Therefore,

we have a max at $x=1$ [and the max value of f is $f(1)=6.33$]

we have a min at $x=3$ [and the min value of f is $f(3)=5$]

An alternative way to check if a stationary point is a max or a min is the following:

SECOND DERIVATIVE TEST for c

Find $f''(x)$ (if it exists!)

If $f''(c)>0$ then c is a min

If $f''(c)<0$ then c is a max

If $f''(c)=0$ we don't get an answer. We go back to the first derivative test.

Let us use the same example as above

EXAMPLE 2

Consider

$$f(x)= \frac{1}{3}x^3-2x^2+3x+5$$

We found

$$f'(x)=x^2-4x+3$$

and the stationary points x=1 and x=3

We find

$$f''(x)=2x-4$$

For x=1, $f''(1)=-2<0$, so we have a max at x=1

For x=3, $f''(3) = 2>0$, so we have a min at x=3

EXAMPLE 3

Consider

$$f(x)= (x-1)^4$$

We find

$$f'(x)=4(x-1)^3$$

There is only one stationary point at x=1.

$$f''(x) = 12(x-1)^2$$

For x=1, $f''(1) = 0$ (neither positive nor negative).

Thus, we cannot conclude if it is a max or a min.

The table of signs gives

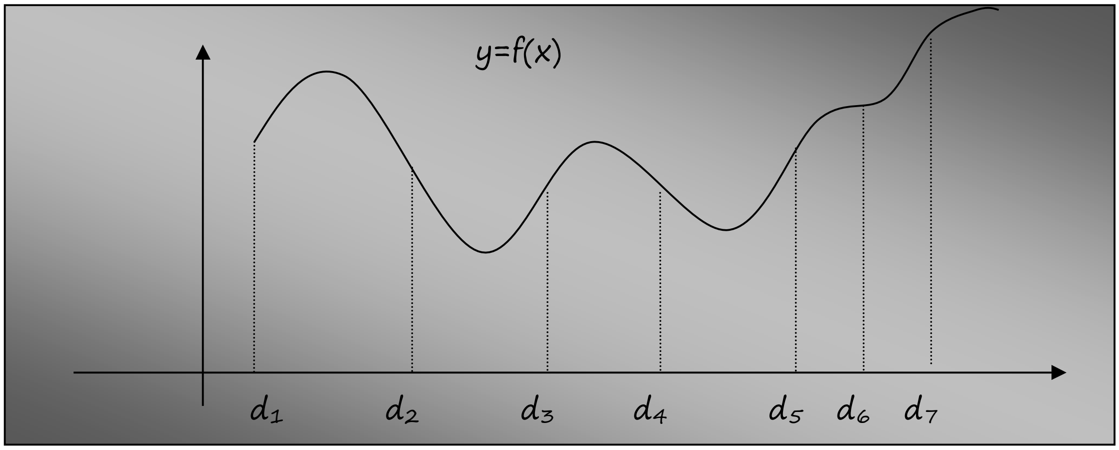
<table>
<tr><td>x</td><td colspan="2">1</td></tr>
<tr><td>$f'(x)=4(x-1)^3$</td><td>−</td><td>+</td></tr>
<tr><td rowspan="2">Conclusion for f(x)</td><td>↘</td><td>↗</td></tr>
<tr><td colspan="2">min</td></tr>
</table>

Therefore, we have a min at x=1.

5.6 CONCAVITY – POINTS OF INFLEXION

♦ CONCAVITY

Consider again the graph of the preceding section



Our concern now is different! It is to investigate the intervals where the curve

looks like (∪) : we say that the function is **concave up**

looks like (∩) : we say that the function is **concave down**[2]

We observe that:

- The function is **concave down** (∩) in the interval (d_1,d_2)
- The function is **concave up** (∪) in the interval (d_2,d_3)
- The function is **concave down** (∩) in the interval (d_3,d_4)
- The function is **concave up** (∪) in the interval (d_4,d_5)
- The function is **concave down** (∩) in the interval (d_5,d_6)
- The function is **concave up** (∪) in the interval (d_6,d_7)

- The concavity changes at the points $x=d_2$, d_3, d_4, d_5, d_6, d_7

These points are called **points of inflexion.**

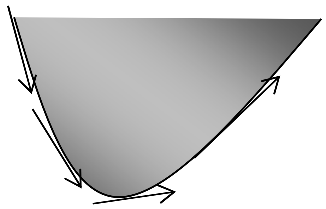
It is easy to verify the concavity by using the second derivative

If $f''(x) > 0$	then	f is concave up (∪)
If $f''(x) < 0$	then	f is concave down (∩)

[2] To be more formal, a function is concave up/down if the tangent line at each point is under/above the curve.

Short explanation (mainly for HL)

Look at the curve of the following concave up function f(x)



The gradient is -tive in the beginning, it is "less" -tive as we move forward, it sometimes becomes 0 and then becomes +tive and "more" +tive as we move forward. In other words the gradient increases. That is, the function of the gradient $f'(x)$ is increasing. But we know that the derivative of an increasing function is +tive. Hence, the derivative of $f'(x)$, that is $f''(x)$ is +tive!

Similarly, if the function f(x) is concave down, the second derivative must be -tive!

♦ POINTS OF INFLEXION

How can we find the points of inflexion?

Since the concavity changes at such a point, the sign of $f''(x)$ changes from + to – or vice-versa. Therefore, the second derivative at any point of inflexion must be 0.

PROPOSITION:

If f(x) has a point of inflexion at some point d and $f''(d)$ exists, then

$$f''(d)=0$$

Notice again that the equation $f''(x)=0$ gives us the possible points of inflexion. To verify if x=d is indeed a point of inflexion we must check the sign of $f''(x)$ just before and after that point.

♦ METHODOLOGY

Given $y=f(x)$

Step 1 we find $f'(x)$ and $f''(x)$

Step 2 we solve $f''(x)=0$ (say that roots are a,b,c)

Step 3 we construct a table as follows

x	a b c			
$f''(x)$	+	−	+	+
Conclusion for f	∪	∩	∪	∪
	poi	poi	nothing	

EXAMPLE 1

Consider again

$$f(x)=\frac{1}{3}x^3-2x^2+3x+5$$

We find

$$f'(x)=x^2-4x+3$$

$$f''(x)=2x-4$$

We solve

$$2x-4=0$$

The solution is $x=2$

We construct the table

x	2	
$f''(x)=2x-4$	−	+
Conclusion for f	∩	∪
	poi	

Therefore, $x=2$ is a point of inflexion.

Let us summarize the information we have for our example

$$f(x)=\frac{1}{3}x^3-2x^2+3x+5$$

in order to sketch the graph of this function.

First, it helps to know the y-intercept and the x-intercepts, that is the roots of f(x) (if possible):

y-intercept: for x=0, y=f(0)=5

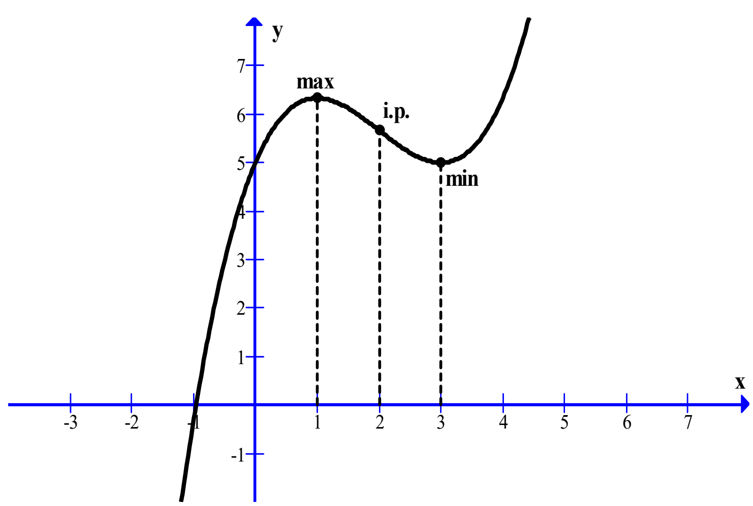
x-intercepts: we must solve f(x)=0 (only by GDC as the degree is 3)

We consider a summary table containing the solutions of both $f'(x)=0$ and $f''(x)=0$:

x		1		2		3	
$f'(x)$	+		−		−		+
$f''(x)$	−		−		+		+
Conclusion for f	↗ ∩		↘ ∩		↘ ∪		↗ ∪
		max		poi		min	

The following table of values will also help

x	0	1	2	3
f(x)	5	6.33	5.66	5



EXAMPLE 2

Consider the function

$$f(x)= xe^x$$

Find possible maximum, minimum values and points of inflexion.

Solution

We have

$$f'(x) = xe^x + e^x$$

Stationary points: $xe^x + e^x = 0 \Leftrightarrow e^x(x+1) = 0 \Leftrightarrow x = -1$

We use table:

x	-1	
$f'(x)$	-	+
Conclusion for f	↘	↗
	min	

Furthermore,

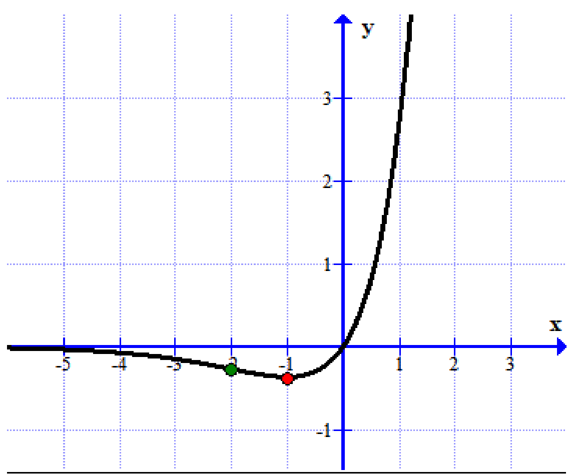
$$f''(x) = xe^x + e^x + e^x = xe^x + 2e^x$$

Then $xe^x + 2e^x = 0 \Leftrightarrow e^x(x+2) = 0 \Leftrightarrow x = -2$

We use table:

x	-2	
$f''(x)$	-	+
Conclusion for f	∩	∪
	poi	

The graph of the function is shown below

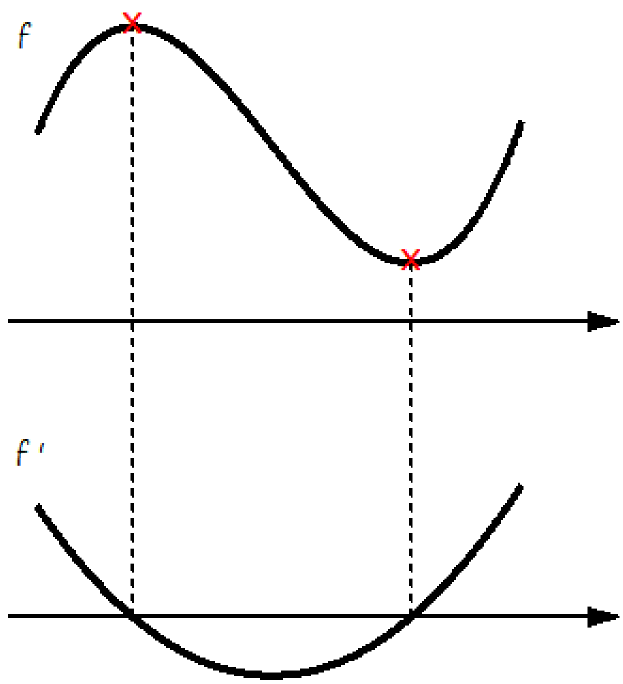


Notice: Look at the graph of this function at your GDC to confirm that $x=-1$ gives a min and observe that at $x=-2$ there is a point of inflexion.

5.7 THE GRAPH OF f′(x)

If we know the graph of $f(x)$ we can roughly sketch the graph of $f'(x)$. What we need is

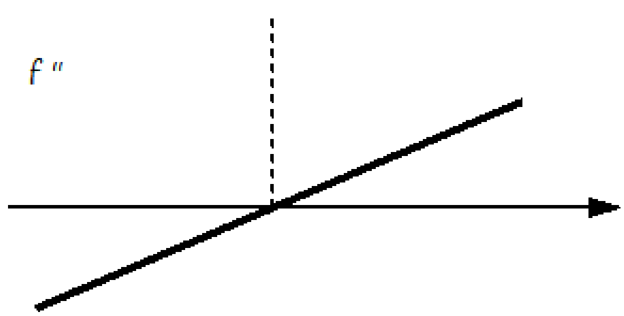
- Spot the **stationary points**; they become **roots**
- **Increasing** sections have positive **f′(x)** (i.e. **above** axis)
- **Decreasing** sections have negative **f′(x)** (i.e. **below** axis)



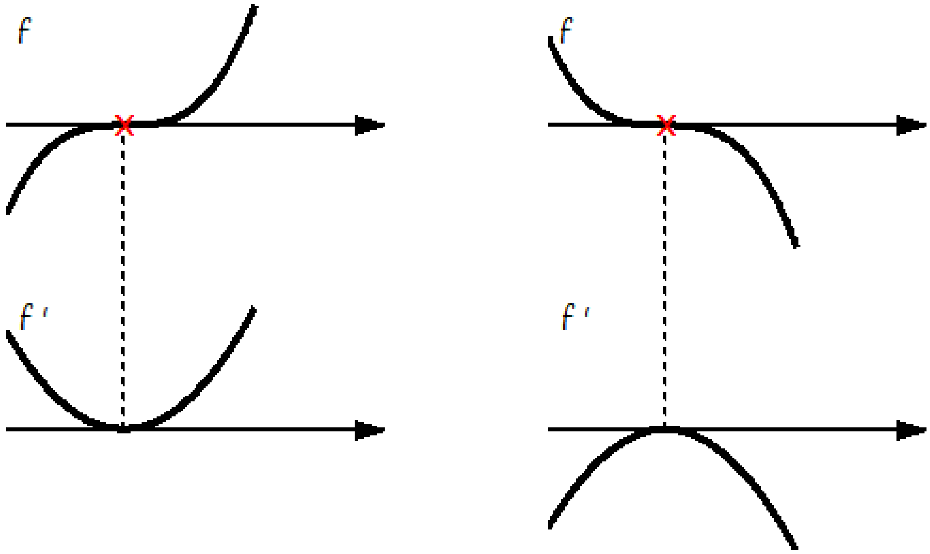
Furthermore

- POI become max or min

In the same way we can derive the graph of $f''(x)$ from $f'(x)$

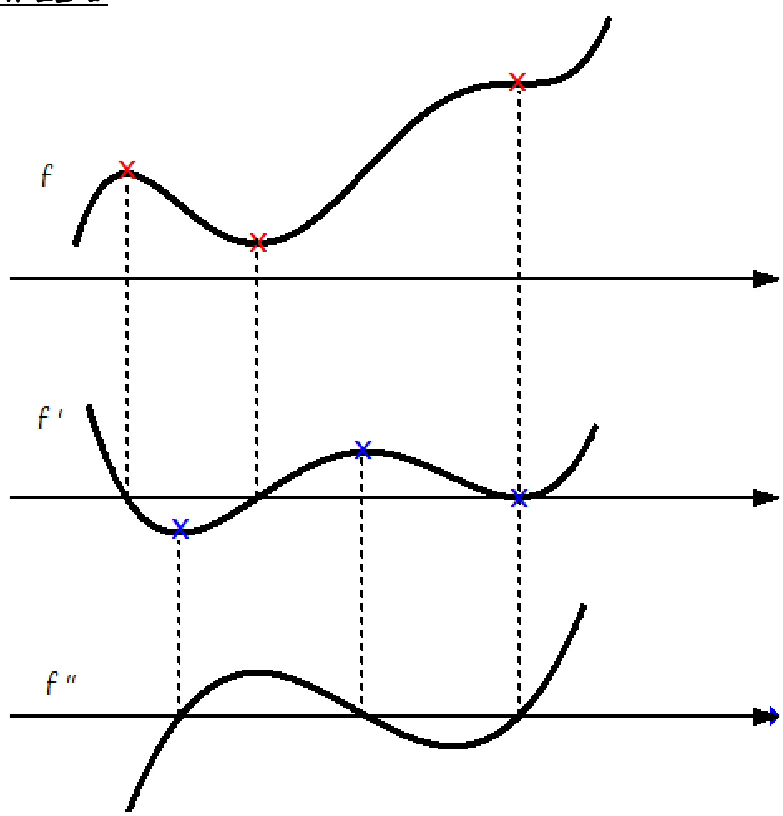


Notice also the behavior of a **stationary POI** ($f' = 0$ and $f'' = 0$)



The following example contains 3 stationary points. The third one is a stationary POI.

EXAMPLE 1

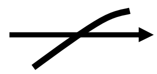


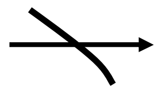
The opposite direction is slightly more difficult. That is, given the graph of **f′(x)** they may ask us to sketch the graph of $f'(x)$.

Now

- Spot the **roots**; they become **stationary points**

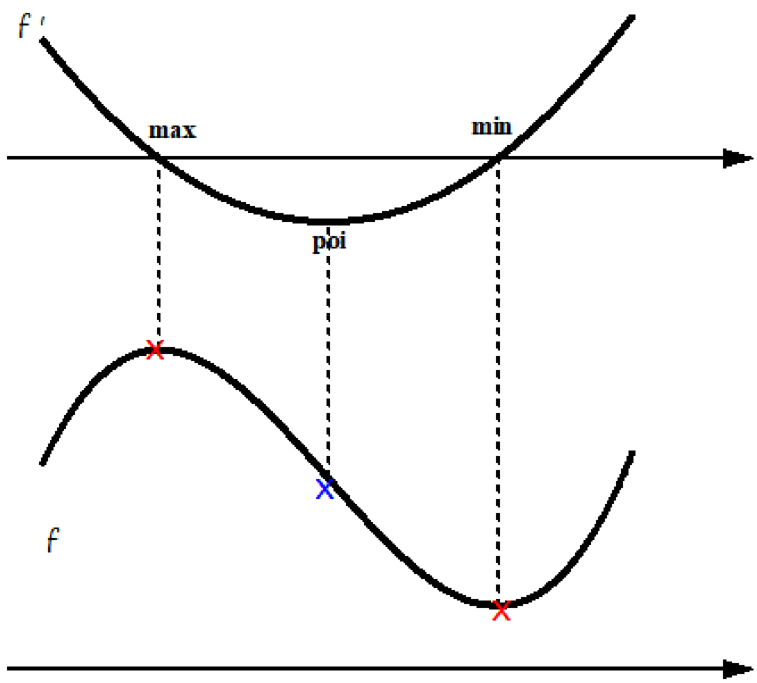
In particular,

a root like  corresponds to a **min**;

a root like  corresponds to a **max**;

The problem is that we don't know the position of max or min. We place the max and the min at random positions.

EXAMPLE 2



In fact, any vertical translation of the second graph is also a possible solution, so the graph could be lower or higher than above. They often give us some extra information (e.g. starting point) to have an idea how to place the max/min before sketching the graph.

5.8 OPTIMIZATION

In problems of optimization, we have to construct a function in terms of some variable x, and then we use derivatives to find the "**optimum**" solution, that is the maximum or the minimum value of the function.

EXAMPLE 1

Among all the rectangles of perimeter 20, find the one of the maximum area.

Discussion

A rectangle of perimeter 20 may have dimensions

1×9	**2×8**	**3×7**	**4×6**	**etc**

The corresponding areas are

9	**16**	**21**	**24**	**etc**

Which is the one of the maximum area?

Solution

y

x

Let x be one of the sides (this will be our main variable).

If the other side is y, then

Perimeter = 20 $\Rightarrow$ $2x + 2y = 20$ $\Rightarrow$ $y = 10-x$

The function of optimisation is

Area: $A = xy = x(10-x) = 10x - x^2$

We find $\frac{dA}{dx} = 10 - 2x$

Stationary points: $\frac{dA}{dx} = 0 \Leftrightarrow 10-2x = 0 \Leftrightarrow x = 5$

The 2nd derivative test is easier here: $A'' = -5$.

At $x=5$ $A'' < 0$, thus we have a maximum value there.

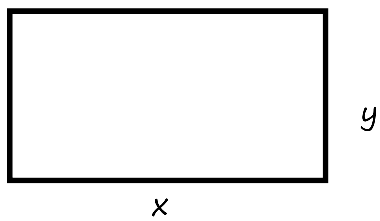
Therefore, the rectangle of maximum area is the square (that is when $x = 5$), and the maximum area is $A_{max} = 25$.

Let's reverse the role of the perimeter and the area. Next we know the area of the rectangle and we are looking for the minimum perimeter.

EXAMPLE 2

Among all the rectangles of area 25, find the one of the minimum perimeter.

Solution



Again, let x be one of the sides (this will be our main variable).

If the other side is y, then

$$\text{Area} = 25 \Rightarrow xy=25 \Rightarrow y = \frac{25}{x}$$

The function of optimisation is

$$\text{Perimeter:} \qquad P = 2x+2y = 2x+\frac{50}{x}$$

We find $\frac{dP}{dx} = 2-\frac{50}{x^2}$

Stationary points: $\frac{dP}{dx} = 0 \Leftrightarrow 2-\frac{50}{x^2} = 0 \Leftrightarrow x^2 = 25 \Leftrightarrow x = 5$

The 2nd derivative test gives: $P'' = \frac{100}{x^3}$.

For $x=5$, $P'' > 0$, thus we have a minimum value there.

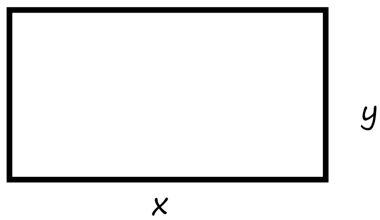
Therefore, the rectangle of minimum perimeter is the square, (that is when $x = 5$), and the minimum perimeter is $P_{min} = 20$.

Sometimes, there is a different "cost" for each side.

EXAMPLE 3

We want to construct a rectangle fence for an area of $24m^2$, but the cost for the material of the front side is 10$ per meter while the cost for the material of the other 3 sides is 5$ per meter. Find the cheapest solution!

Solution



Let x be the front side (this will be our main variable).

If the other side is y, then

$$\text{Area} = 24 \Rightarrow xy=24 \Rightarrow y = \frac{24}{x}$$

The function of optimisation is

$$\text{Cost:} \qquad C = 10x+5x+2(5y) = 15x+10y = 15x+\frac{240}{x}$$

We find $\frac{dC}{dx} = 15-\frac{240}{x^2}$

Stationary points: $\frac{dC}{dx}=0 \Leftrightarrow 15-\frac{240}{x^2} = 0 \Leftrightarrow x^2 = 16 \Leftrightarrow x = 4$

The 2^{nd} derivative test gives: $C'' = \frac{480}{x^3}$.

For $x=4$, $C'' > 0$, thus we have a minimum value there.

Therefore, the best rectangle has dimensions 4×6 and the minimum cost is $C_{min} = 120$$

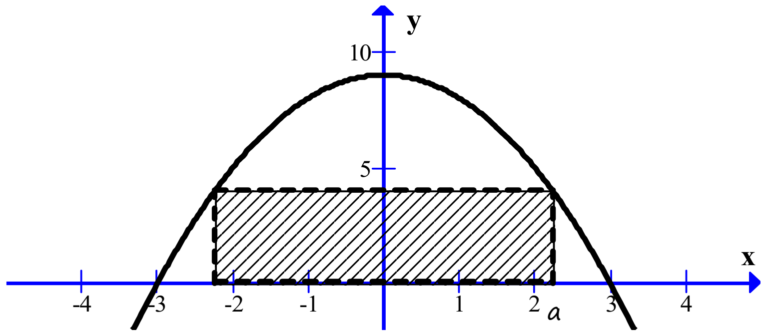
This rationale applies to 3D shapes as well. For example, they may give us a rectangular prism or a cylinder of a given volume and we are looking for the optimum surface area, or vice-versa.

Let's see an example of a shape determined by the boundaries of a given function!

EXAMPLE 4

Consider the region enclosed by $y = 9 - x^2$ and x-axis.

Find the rectangle of largest area inscribed within that region.



Discussion

There are two extreme cases:

- the height of the rectangle is 0, the width is 6. The area is 0.
- the height of the rectangle is 9, the width is 0. The area is 0.

Somewhere in between there is a rectangle of maximum area.

Solution

Key point: Let a be the x-coordinate of the bottom right corner.

Then

Width = $2a$

Height = $y = 9 - a^2$ [it is f(a) !]

Thus, the function of optimisation is

Area: $A = 2a(9-a^2) = 18a - 2a^3$

We find $\frac{dA}{da} = 18 - 6a^2$

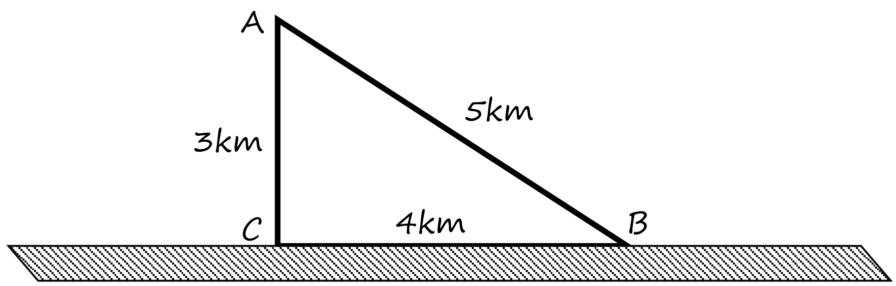
Stationary points: $\frac{dA}{dx} = 0 \Leftrightarrow 18 - 6a^2 = 0 \Leftrightarrow a = \sqrt{3}$

The 2nd derivative test gives: $A'' = -12a$.

At $a = \sqrt{3}$, $A'' < 0$, thus we have a maximum value there.

Therefore, the rectangle of maximum area has dimensions $2\sqrt{3} \times 6$ and the maximum area is $A_{max} = 12\sqrt{3}$.

EXAMPLE 5



A swimmer is at point A inside the sea, 3km away from the beach. She wants to go to point B at the beach, which is 5 km away.
When she swims she covers 1 km in 30 minutes
When she runs she covers 1km in 15 minutes. Find

(a) the time she spends when she swims directly to B;

(b) the time she spends when she swims to C and then runs to B;

(c) the minimum time she can achieve if she swims first to some point D between B and C and then runs to B

Solution

(a) $T=T_{AB} = 5\times 30 =$ **150 min**

(b) $T= T_{AC} + T_{CB} = 3\times 30 + 4\times 15 =$ **150 min**

(c) Let D be between C and B and CD=x km. Then DB = 4-x km

Also $AD^2 = AC^2 + CD^2 \Rightarrow AD = \sqrt{9+x^2}$

Therefore, the function of optimization is the total time

$$T= T_{AD}+T_{DB} = 30\sqrt{9+x^2} +15(4-x) = 30\sqrt{9+x^2} -15x+60 \text{ min}$$

We find $\dfrac{dT}{dx}=\dfrac{30x}{\sqrt{9+x^2}}-15$

Stationary points: $\dfrac{30x}{\sqrt{9+x^2}}-15=0 \Leftrightarrow 30x=15\sqrt{9+x^2}$

$$\Leftrightarrow 2x=\sqrt{9+x^2} \Leftrightarrow 4x^2=9+x^2 \Leftrightarrow x^2=3 \Leftrightarrow x=\sqrt{3}\cong 1.73 \text{ km}$$

We can easily verify that this is a minimum (2^{nd} derivative test).

Thus, the point D is 1.73km from C.

The minimum time is $\mathbf{T_{min}}=45\sqrt{3}+60 \cong$ **138** minutes.

5.9 THE INDEFINITE INTEGRAL $\int f(x)dx$

♦ THE INDEFINITE INTEGRAL

Consider

$$F(x)=x^2$$

The derivative is

$$F'(x)=2x$$

<u>The reverse problem</u>: If they give us the result

$$f(x)=2x$$

can we find a function F(x), such that $F'(x)=f(x)$?

Of course, one answer is $F(x)=x^2$. We say that $F(x)=x^2$ is **an antiderivative** of $f(x)=2x$. But it is not the only one!

Notice that

$$(x^2)' = 2x$$
$$(x^2+1)' = 2x$$
$$(x^2+2)' = 2x$$
$$(x^2+5)' = 2x$$

in general

$$(x^2+c)' = 2x \qquad \text{for any constant } c$$

Therefore, the functions x^2+c are also antiderivatives of $f(x)=2x$.

We say that x^2+c is the ***indefinite integral*** of $f(x)=2x$ and we use the notation

$$\int 2x\,dx = x^2+c$$

Hence,

$$\boxed{\text{if} \quad F'(x) = f(x) \quad \text{then} \int f(x)dx = F(x)+c}$$

For example,

since $(x^5)'=5x^4$ we obtain $\int 5x^4dx = x^5 + c$

We also deduce that $\int x^4dx = \dfrac{x^5}{5} + c$ (why?)

Therefore, we can easily obtain the following results

$f(x)$	$\int f(x)dx$
1	$x + c$
x	$\frac{x^2}{2} + c$
x^2	$\frac{x^3}{3} + c$
x^3	$\frac{x^4}{4} + c$
x^{10}	$\frac{x^{11}}{11} + c$

In general,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \qquad (\text{if } n \neq -1)$$

Notice also,

$$\int 5dx = 5x + c, \quad \text{since} \quad (5x)' = 5$$
$$\int adx = ax + c, \quad \text{since} \quad (ax)' = a \qquad (a = \text{constant})$$
$$\int \cos x dx = \sin x + c, \quad \text{since} \quad (\sin x)' = \cos x$$

If we remember the derivatives of the basic functions we obtain the following results:

$f(x)$	$\int f(x)dx$	
x^n	$\frac{x^{n+1}}{n+1}$	+ c
a	ax	
e^x	e^x	
$\sin x$	$-\cos x$	
$\cos x$	$\sin x$	
$\frac{1}{x}$	$\ln x$	

NOTICE: The operation of finding the integral is called ***integration***.

- REMARK FOR $\boxed{\int x^n dx = \dfrac{x^{n+1}}{n+1} + c}$

The same formula applies for negative values of n. For example,

$$\int x^{-5} dx = \frac{x^{-4}}{-4} + c = \frac{1}{-4x^4} + c$$

What about $\int \frac{1}{x^2} dx$? We know that $\frac{1}{x^2} = x^{-2}$, so

$$\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{x^{-1}}{-1} + c = \frac{1}{-x} + c$$

Also, the same formula applies for rational values of n:

$$\int x^{3/5} dx = \frac{x^{8/5}}{8/5} + c = \frac{5}{8} x^{8/5} + c$$

What about $\int \sqrt{x} dx$? We know that $\sqrt{x} = x^{1/2}$, so

$$\int \sqrt{x} dx = \int x^{1/2} dx = \frac{x^{3/2}}{3/2} + c = \frac{2}{3} x^{3/2} + c$$

Notice that this formula does not apply for $\int \frac{1}{x} dx = \int x^{-1} dx$.

Only for this particular power we have the formula $\int \frac{1}{x} dx = \ln x + c$

Thus, for example

$f(x)$	$\int f(x)dx$	
$\frac{1}{x^2} = x^{-2}$	$\frac{x^{-1}}{-1} + c$	$-\frac{1}{x} + c$
$\frac{1}{x^3} = x^{-3}$	$\frac{x^{-2}}{-2} + c$	$-\frac{1}{2x^2} + c$
$\frac{1}{x^4} = x^{-4}$	$\frac{x^{-3}}{-3} + c$	$-\frac{1}{3x^3} + c$
$\sqrt[3]{x^2} = x^{\frac{2}{3}}$	$\frac{3}{5} x^{\frac{5}{3}} + c$	$\frac{3}{5} \sqrt[3]{x^5} + c$

♦ REMARK FOR $\int \frac{1}{x}dx$ (**only for HL**)

In fact $\boxed{\int \frac{1}{x}dx = \ln|x| + c}$.

Indeed: if x>0, then $[\ln|x|]' = [\ln x]' = \frac{1}{x}$,

if x<0, then $[\ln|x|]' = [\ [\ln(-x)]' = \frac{1}{-x}(-1) = \frac{1}{x}$.

That is why the antiderivative of $\frac{1}{x}$ is $\ln|x| + c$.

♦ TWO BASIC RULES OF INTEGRATION

Rule 1: $\boxed{\int [f(x) + g(x)]dx \ = \ \int f(x)dx + \int g(x)dx}$

For example,

$$\int (x^4 + x^2)dx = \int x^4 dx + \int x^2 dx = \frac{x^5}{5} + \frac{x^3}{3} + c$$

Rule 2: $\boxed{\int af(x)dx \ = \ a\int f(x)dx}$

For example,

$$\int 10x^4 dx = 10\int x^4 dx = 10\frac{x^5}{5} + c = 2x^5 + c$$

In practice, if we have a "long" expression like

$$\int [af(x) + bg(x) + ch(x) + dk(x)]dx$$

where a, b, c, d are constant coefficients, we keep a, b, c, d and integrate only f(x), g(x), h(x), k(x).

EXAMPLE 1

$$\begin{aligned}\int [3x^2 + 5e^x - 2\cos x]dx &= 3\int x^2 dx + 5\int e^x dx - 2\int \cos x dx \\ &= 3(\frac{x^3}{3}) + 5e^x - 2\sin x + c \\ &= x^3 + 5e^x - 2\sin x + c\end{aligned}$$

In fact, the shaded step above is not necessary! We can proceed to the next step by estimating directly the integrals of x^2, e^x, $\cos x$

EXAMPLE 2

$$\int[2x^4+8x^3-5x^2+7x+2]dx = 2\frac{x^5}{5} +8\frac{x^4}{4} -5\frac{x^3}{3} -7\frac{x^2}{2} +2x + c$$

$$= \frac{2}{5}x^5 +2x^4 -\frac{5}{3}x^3 -\frac{7}{2}x^2 + 2x + c$$

EXAMPLE 3

$$\int[\frac{2}{x^4}+\frac{8}{x^3}-\frac{5}{x^2}+2]dx = \int[2x^{-4}+8x^{-3}-5x^{-2}+2]dx$$

$$= 2\frac{x^{-3}}{-3} +8\frac{x^{-2}}{-2} -5\frac{x^{-1}}{-1} +2x + c$$

$$= -\frac{2}{3x^3}-\frac{4}{x^2}+\frac{5}{x} + 2x + c$$

- FIND the constant c

Sometimes, we are given an extra condition of the form $f(a)=b$ in order to find the value of the constant c. We find in fact, the specific antiderivative which satisfies this condition.

EXAMPLE 4

Let $f'(x)=6x^2-4x+5$. Find $f(x)$ given that $f(1)=8$.

Clearly, $f(x)$ is the integral of $f'(x)=6x^2-4x+5$. That is,

$$f(x)= \int[6x^2-4x+5]dx = 6\frac{x^3}{3} - 4\frac{x^2}{2}+5x+c = 2x^3-2x^2+5x+c$$

Next, we have to find the value of c:

$f(1)=8$, so $2(1)^3-2(1)^2+5(1)+c=8$

so $5+c=8$

so $c=3$

Therefore,

$$f(x) = 2x^3-2x^2+5x+3$$

5.10 INTEGRATION BY SUBSTITUTION

Before we present this method of integration, let us see a simple case where we have (ax) or (ax+b) instead of x in our function.

- THE LINEAR CASE: $\int f(ax)dx$ or $\int f(ax+b)dx$

Consider the integrals

$$\int \cos(3x)dx$$
$$\int \cos(3x+5)dx$$

They look like $\int \cos x dx$ but instead of x we have u=3x and u=3x+5 respectively.

> If we have **ax** or **ax+b** instead of **x**, then at the result
>
> we also write ax+b instead of x,
>
> but we divide the result by a

Hence,

$$\int \cos(3x)dx = \frac{\sin(3x)}{3} + c \qquad [u=3x]$$

$$\int \cos(3x+5)dx = \frac{\sin(3x+5)}{3} + c \qquad [u=3x+5]$$

Indeed, if we differentiate the results

the derivative of $\frac{\sin(3x)}{3}$ is $\frac{\cos(3x)}{3} \times 3 = \cos(3x)$

the derivative of $\frac{\sin(3x+5)}{3}$ is $\cos(3x+5)$

Notice that,

$$\int \cos(x+5)dx = \sin(x+5) + c$$

since the coefficient of x is 1.

Indeed,

the derivative of $\sin(x+5)$ is $\cos(x+5)$.

EXAMPLE 1

$$\int e^{3x+2}dx = \frac{e^{3x+2}}{3} + c$$

$$\int e^{-5x+2}dx = \frac{e^{-5x+2}}{-5} + c$$

$$\int e^{10x}dx = \frac{e^{10x}}{10} + c$$

$$\int e^{x+8}dx = e^{x+8} + c$$

$$\int \sin(2x+1)dx = -\frac{\cos(2x+1)}{2} + c$$

$$\int \sin 2x dx = -\frac{\cos 2x}{2} + c$$

$$\int \cos(5-x)dx = \frac{\sin(5-x)}{-1} + c = -\sin(5-x) + c$$

$$\int \frac{1}{7x+3}dx = \frac{\ln(7x+3)}{7} + c$$

$$\int \frac{1}{3-7x}dx = \frac{\ln(3-7x)}{-7} + c$$

$$\int \frac{1}{x-3}dx = \ln(x-3) + c$$

$$\int (3x+5)^5 dx = \frac{(3x+5)^6}{3\cdot 6} + c$$

$$\int \frac{1}{(3x+5)^5}dx = \int (3x+5)^{-5}dx = \frac{(3x+5)^{-4}}{-4\cdot 3} + c = \frac{(3x+5)^{-4}}{-12} + c$$

$$\int \sqrt{3x+5}\,dx = \int (3x+5)^{1/2}dx = \frac{\frac{2}{3}(3x+5)^{3/2}}{3} + c = \frac{2}{9}(3x+5)^{3/2} + c$$

EXAMPLE 2

Find $I = \int \cos^2 x dx$ by using the double angle formula $\cos 2x = 2\cos^2 x - 1$

Solution

If we solve for $\cos^2 x$ we get $\cos^2 x = \frac{1+\cos 2x}{2}$. Thus

$$I = \frac{1}{2}\int (1+\cos 2x)dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + c = \frac{x}{2} + \frac{\sin 2x}{4} + c$$

♦ SOME DISCUSSION ABOUT THE DIFFERENTIAL dx (you may skip this page, it is extra information)

The quantity dx is called the **differential** of x.

It indicates a little change in x. (Compare with $\Delta x = x_2 - x_1$)

For a function y=f(x), we know that

$$\frac{dy}{dx} = f'(x)$$

Although $\frac{dy}{dx}$ is a symbol and not a fraction, it behaves as a fraction. Besides, its definition comes from a fraction $\frac{\Delta y}{\Delta x}$!

Thus we can solve for dy and write

$$\boxed{dy = f'(x)dx}$$

This gives a relation between the differentials dy and dx.

In fact, this relation indicates in what extend a little change in x causes a little change in y.

For example, consider the function $y=x^2$. Then

$$\frac{dy}{dx}=2x \quad \Rightarrow \quad dy=2xdx$$

We can also solve for dx and write

$$dx=\frac{dy}{2x}.$$

Similarly, for y=sinx

$$\frac{dy}{dx}=\cos x \quad \Rightarrow \quad dy=\cos x dx$$

$$\text{or} \quad dx=\frac{dy}{\cos x}$$

♦ THE METHOD OF SUBSTITUTION (THE SIMPLE CASE)

In fact, we have already seen a simple version of substitution, where an expression of the form u=ax+b appears in the integral.

For example, we have seen that

$$I=\int \cos(3x+5)dx = \frac{1}{3}\sin(3x+5)+c$$

Let us use this example to demonstrate analytically the method of substitution.

- Let $u=3x+5$
- Then $\frac{du}{dx}=3 \Rightarrow dx=\frac{du}{3}$
- The integral becomes

$$I=\int \cos u\frac{du}{3}=\frac{1}{3}\int \cos u du=\frac{1}{3}\sin u+c$$

Thus, $I=\frac{1}{3}\sin(3x+5)+c$

If u=ax+b we can directly write down the result as explained. But for more difficult cases we follow this process.

In general, our target is to eradicate x and dx from the original integral and obtain a simpler integral in terms of u and du.

METHODOLOGY:

We have to imagine an appropriate substitution $u=g(x)$

[criterion: $g'(x)$ must also exist in the integral as a factor]

- Let $u=g(x)$
- Then $\frac{du}{dx}=g'(x) \Rightarrow dx=\frac{du}{g'(x)}$
- Express the initial integral in terms of u and du
- Calculate the new integral
- Replace $u=g(x)$ back in the result

Consider

$$I=\int 3x^2(x^3+5)^7dx$$

We let $u=x^3+5$ [since the derivative $3x^2$ exists inside the integral]

Let $u=x^3+5$

Then $\frac{du}{dx}=3x^2 \Rightarrow dx=\frac{du}{3x^2}$

The integral obtains the convenient form

$$I=\int 3x^2u^7\frac{du}{3x^2}=\int u^7du$$

Hence,

$$I=\frac{u^8}{8}+c$$

Finally, we replace back $u=x^3+5$ to get

$$I=\frac{(x^3+5)^8}{8}+c$$

Notice that

$$I=\int x^2(x^3+5)^7dx$$

can be treated in the same way. The derivative of $u=x^3+5$ is $3x^2$. We are happy that x^2 exists inside the integral (we don't mind for the constant 3). Thus,

let $u=x^3+5$

then $\frac{du}{dx}=3x^2 \Rightarrow dx=\frac{du}{3x^2}$

The result is

$$I=\int x^2u^7\frac{du}{3x^2}=\frac{1}{3}\int u^7du=\frac{1}{3}\frac{u^8}{8}+c=\frac{(x^3+5)^8}{24}+c$$

Sometimes, the substitution is given as a hint! See next example.

EXAMPLE 3

Find $I=\int x\sqrt{x^2+3}dx$ by using the substitution $u=x^2+3$.

Solution

Let $u=x^2+3$, then $\frac{du}{dx}=2x \Rightarrow dx=\frac{du}{2x}$

Thus,

$$I=\int x\sqrt{u}\ \frac{du}{2x}=\frac{1}{2}\int\sqrt{u}du=\frac{1}{2}\ \frac{2}{3}\ u^{\frac{3}{2}}+c=\frac{1}{3}\ u^{3/2}+c=\frac{1}{3}\ (x^2+3)^{3/2}+c$$

EXAMPLE 4

Find $I=\int\frac{2x+\cos x}{x^2+\sin x}dx$.

Solution

Notice that the derivative of the denominator is the numerator.

Let $u=x^2+\sin x$, then $\frac{du}{dx}=2x+\cos x \Rightarrow dx=\frac{du}{2x+\cos x}$

Thus,

$$I=\int\frac{2x+\cos x}{u}\ \frac{du}{2x+\cos x}=\int\frac{1}{u}du=\ln u+c=\ln(x^2+\sin x)+c$$

EXAMPLE 5

Find $I_1=\int\frac{(\ln x)^2}{x}dx$, $I_2=\int\frac{1}{x(\ln x)^2}dx$, $I_3=\int\frac{\sqrt{\ln x}}{x}dx$

Solution

For all three of them we let $u=\ln x$ [Why?].

Then $\frac{du}{dx}=\frac{1}{x}\Rightarrow dx=xdu$

Thus

$$I_1=\int\frac{u^2}{x}xdu=\int u^2du=\frac{u^3}{3}+c=\frac{(\ln x)^3}{3}+c,$$

$$I_2=\int\frac{1}{xu^2}xdu=\int\frac{1}{u^2}du=\frac{u^{-1}}{-1}+c=-\frac{1}{u}+c=-\frac{1}{\ln x}+c,$$

$$I_3=\int\frac{\sqrt{u}}{x}xdu=\int\sqrt{u}du=\frac{2}{3}u^{3/2}+c=\frac{2}{3}(\ln x)^{3/2}+c$$

- **CALCULATION BY INSPECTION**

If you get used to this simple case of substitution you may directly write down the result as we are going to explain below. But unless you feel confident enough follow the whole process of substitution!

Most of the integrals that require substitution have one of the following forms:

integral	result	
$\int u'e^u dx$	$= e^u + c$	where u is a function of x.
$\int u'\sin u dx$	$= -\cos u + c$	
$\int u'\cos u dx$	$= \sin u + c$	
$\int \frac{u'}{u} dx$	$= \ln u + c$	
$\int u' \cdot u^n dx$	$= \frac{u^{n+1}}{n+1} + c$	

Indeed, the derivatives of the functions in the 2nd column are the expressions in the 1st column (chain rule). That is why the substitution method is also known as ANTI-CHAIN rule.

For example,

$$\int \frac{2x + \cos x}{x^2 + \sin x} dx \qquad \text{has the form } \int \frac{u'}{u} dx$$

Hence the result is

$$\ln u + c = \ln(x^2 + \sin x) + c$$

Also

$$\int \frac{(\ln x)^2}{x} dx \qquad \text{has the form } \int u' \cdot u^2 dx$$

Hence the result is

$$\frac{u^3}{3} + c = \frac{(\ln x)^3}{3} + c$$

Sometimes a slight modification is needed to have the required form. For example,

the integral $\int \frac{x}{x^2+4}dx$ is "almost" of the form $\int \frac{u'}{u}dx$

(since the derivative of x^2+4 is $2x$ and not x).

But we can write

$$\int \frac{x}{x^2+4}dx = \frac{1}{2}\int \frac{2x}{x^2+4}dx \quad \text{[now it has exactly the form } \int \frac{u'}{u}dx\text{]}$$

$$= \frac{1}{2}\ln(x^2+4)+c$$

EXAMPLE 6

Find $\int \sin x e^{\cos x}dx$,

Solution

The integral is "almost" of the form $\int u'e^u dx$, since $u' = -\sin x$.

Thus

$$\int \sin x e^{\cos x}dx = -\int -\sin x e^{\cos x}dx = e^{\cos x}+c$$

EXAMPLE 7

Find $\int x\cos(3x^2+1)dx$,

Solution

The integral is "almost" of the form $\int u'\cos u dx$, since $u' = 6x$. Thus

$$\int x\cos(3x^2+1)dx = \frac{1}{6}\int 6x\cos(3x^2+1)dx = \frac{1}{6}\sin(3x^2+1)+c$$

EXAMPLE 8

Find $I = \int \frac{5x^2}{x^3+1}dx$,

Solution

The integral is "almost" of the form $\int \frac{u'}{u}dx$, since $u' = 3x^2$.

Thus

$$\int \frac{5x^2}{x^3+1}dx = \frac{5}{3}\int \frac{3x^2}{x^3+1}dx = \frac{5}{3}\ln(x^3+1)+c$$

5.11 THE DEFINITE INTEGRAL $\int_a^b f(x)dx$ –AREA BETWEEN CURVES

The definite integral is denoted by

$$\int_a^b \mathbf{f(x)dx}$$

where a and b are real numbers within the domain of f.

The calculation is easy:

If	$\int f(x)dx = F(x) + c$,
then	$\int_a^b f(x)dx = [F(x)]_a^b$ which means $F(b)-F(a)$

For example,

Indefinite: $\int 2xdx = x^2 + c$

Definite: $\int_1^3 2xdx = \left[x^2\right]_1^3 = (3^2)-(1^2) = 8$

Indefinite: $\int (2x+3)dx = x^2 + 3x + c$

Definite: $\int_0^4 (2x+3)dx = \left[x^2+3x\right]_0^4 = (28)-(0) = 28$

Notice: the constant c is omitted in the definite integral! (why?)

EXAMPLE 1

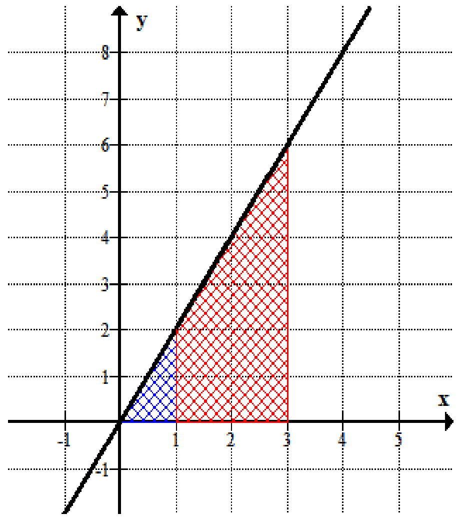
$$\int_{-1}^{1}(8x^3+12x^2-6x+3)dx = \left[2x^4+4x^3-3x^2+3x\right]_{-1}^{1}$$
$$=(2+4-3+3)-(2-4-3-3)$$
$$=8-(-8)=16$$

EXAMPLE 2

$$\int_2^3 \frac{8}{x^3}dx = \int_2^3 8x^{-3}dx = \left[8\frac{x^{-2}}{-2}\right]_2^3 = \left[-\frac{4}{x^2}\right]_2^3 = (-\frac{4}{9})-(-1) = -\frac{4}{9}+1 = \frac{5}{9}$$

- GEOMETRICAL INTERPRETATION

Consider the graph of the straight line f(x)=2x



The area under the line between 0 and 1 (blue triangle) is $\frac{1 \cdot 2}{2} = 1$.

but also $\int_0^1 2x dx = \left[x^2\right]_0^1 = (1)-(0) = 1.$

The area under the line between 0 and 3 (big triangle) is $\frac{3 \cdot 6}{2} = 9$.

but also $\int_0^3 2x dx = \left[x^2\right]_0^3 = (9)-(0) = 9.$

The area under the line between 1 and 3 (red region) is 9−1=8.

but also $\int_1^3 2x dx = \left[x^2\right]_1^3 = (9)-(1) = 8.$

This is not an accident!

In general, if y=f(x) is a "continuous" curve above the x-axis, then the area between the curve and the x-axis, from x=a to x=b is equal to the definite integral

$$\boxed{\int_a^b f(x)dx}$$

We are going to discuss the areas in more detail in the following paragraph. Now, let us concentrate on the algebraic part of the definite integral.

♦ USE OF GDC

We can use the GDC to find the definite integral.

For Casio CFX we select

- MENU
- Run-Matrix
- MATH [F4]
- [F6] and then [F1]

We can also see the result graphically

- MENU
- Graph (insert the function)
- G-Solv [F5]
- [F6] and then [F3]
- At the moment select $\int dx$ **[F1]**
- Select lower and upper limit values (just enter the values)

♦ PROPERTIES OF THE DEFINITE INTEGRAL

The rules of indefinite integrals are still valid here

- $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
- $\int_a^b af(x)dx = a\int_a^b f(x)dx$

Moreover,

- $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$ (use areas to explain why!)

  [notice that the limits a,b,c are consecutive]
- $\int_b^a f(x)dx = -\int_a^b f(x)dx$ (accepted by definition)
- $\int_a^b f\,'(x)dx = [f(x)]_a^b$ (derivative and integral are inverse to each other)
- $\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(t)dt = \ldots$

  [in other words, the variable x is irrelevant!!]

EXAMPLE 3

Suppose that $\int_0^5 f(x)dx = 10$. It is also given that f(0)=15, f(5)=3.

Then we can also find the following integrals:

- $\int_0^5 2f(x)dx = 20$
- $\int_5^0 f(x)dx = -10$
- $\int_5^0 \frac{1}{2}f(x)dx = -5$
- $\int_0^5 (f(x)+4x)dx = \int_0^5 f(x)dx + \int_0^5 4xdx = 10 + \left[2x^2\right]_0^5 = 10+50 = 60$
- $\int_0^5 (2f(x)+1)dx = \int_0^5 2f(x)dx + \int_0^5 1dx = 20 + [x]_0^5 = 20+5 = 25$
- $\int_0^2 f(x)dx + \int_2^5 f(x)dx = \int_0^5 f(x)dx = 10$
- $\int_0^5 f'(x)dx = [f(x)]_0^5 = f(5)-f(0) = 15-3 = 12$
- $\int_0^5 f(t)dt = \int_0^5 f(x)dx = 10$

- **SUBSTITUTION FOR DEFINITE INTEGRALS**

Consider an integral

$\int_a^b f(x)dx$ that requires a substitution u=g(x).

The limits a and b refer to x (we say dx).

After substitution dx becomes du, thus the limits a, b are not valid.

In such a case, it would be safe to

- find the indefinite integral first (expressed in terms of x)
- then evaluate the definite integral.

Let us explain.

EXAMPLE 4

Find $I=\int_0^2 \frac{x}{x^2+4}dx$,

Solution

Method A: we find the indefinite integral first

We use $u=x^2+4$, thus $\frac{du}{dx}=2xdx \Rightarrow dx=\frac{du}{2x}$, so that

$$\int \frac{x}{x^2+4}dx=\int \frac{x}{u}\frac{du}{2x}=\frac{1}{2}\ln u+c=\frac{1}{2}\ln(x^2+4)+c$$

Therefore,

$$I=\left[\frac{1}{2}\ln(x^2+4)\right]_0^2=\frac{1}{2}\ln 8-\frac{1}{2}\ln 4 = \frac{1}{2}\ln\frac{8}{4}=\frac{1}{2}\ln 2$$

Method B: we change the limits

Again we use $u=x^2+4$, thus $\frac{du}{dx}=2xdx \Rightarrow dx=\frac{du}{2x}$

We also change the limits of the definite integral using $u=x^2+4$

x	u
0	4
2	8

In this case we do not go back to x. Namely,

$$I=\int_0^2 \frac{x}{x^2+4}dx=\int_4^8 \frac{x}{u}\frac{du}{2x}=\left[\frac{1}{2}\ln u\right]_4^8=\frac{1}{2}\ln 8-\frac{1}{2}\ln 4=\frac{1}{2}\ln 2$$

EXAMPLE 5

Suppose that $\int_0^5 f(x)dx=10$.

Then we can also find the following integrals:

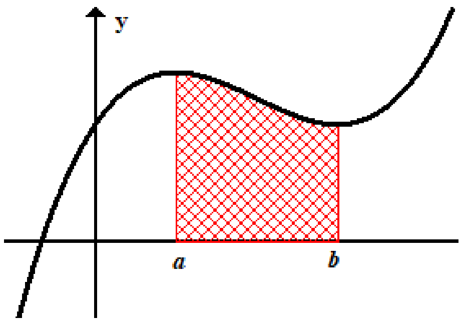
- $\int_3^8 f(x-3)dx = \int_0^5 f(y)dy=10$ by using $y=x-3$, $\frac{dy}{dx}=1 \Rightarrow dx=dy$
- $\int_0^{2.5} f(2x)dx = \frac{1}{2}\int_0^5 f(y)dy=5$ by using $y=2x$, $\frac{dy}{dx}=2 \Rightarrow dx=\frac{dy}{2}$

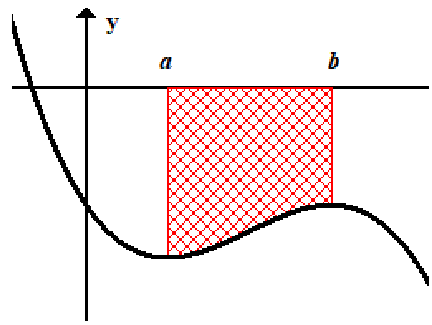
♦ AREA BETWEEN THE CURVE AND x-AXIS

In fact, the definite integral $\int_a^b f(x)dx$ is not always positive.

If y=f(x) is **above** the x-axis the result is **positive**.

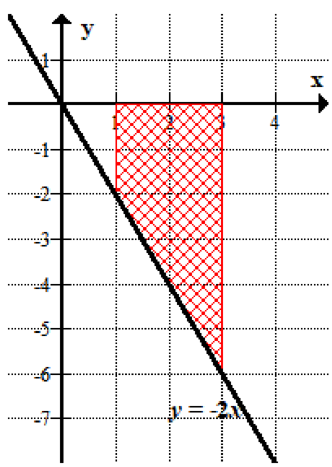
If y=f(x) is **below** the x-axis the result is **negative**





$\int_a^b f(x)dx$ = **(red area)** $\int_a^b f(x)dx$ = **– (red area)**

Indeed, if you look at the curve y=-2x from x=1 to x=3 then



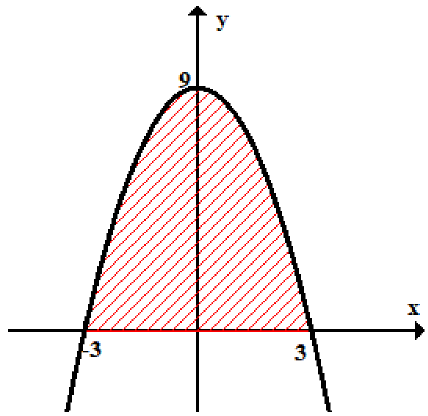
$$\int_1^3 -2x dx = \left[-x^2\right]_1^3 = -9+1 = -8$$

But of course, we say that the area between the curve and x-axis from x=1 to x=3 is equal to 8.

Sometimes we need some extra work to find the limits of the integral.

EXAMPLE 6

Find the area of the region between the curve $y=9-x^2$ and x-axis



Firstly, we need the roots of the function:

$$9-x^2=0 \Leftrightarrow x=-3,\ x=3.$$

Hence,

$$\text{Area} = \int_{-3}^{3}(9-x^2)dx$$

$$=\left[9x-\frac{x^3}{3}\right]_{-3}^{3}$$

$$=(27-9)-(-27+9)=18+18=36$$

Notice that the curve is symmetric about the y-axis so we can find the area between x=0 and x=3 and multiply by 2.

$$\text{Area} = 2\int_{0}^{3}(9-x^2)dx=2\left[9x-\frac{x^3}{3}\right]_{0}^{3}=2(27-9)=36$$

NOTICE for the GDC – graph mode

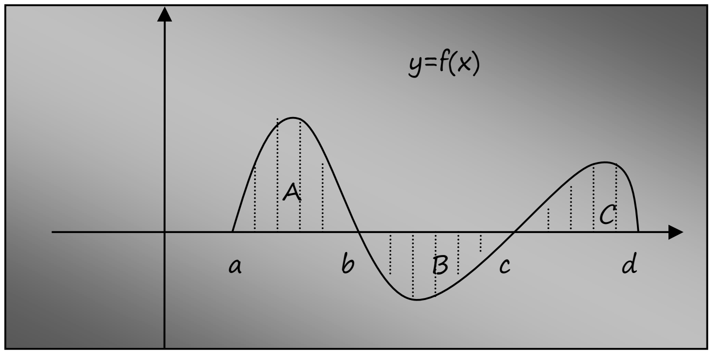
After G-solv and integral, select the option MIXED – [F4].

Then you may

either enter the limits on yourself

or use the arrows to enter the roots directly

Suppose that f(x) has both positive and negative values, say



where A,B,C are the areas of the regions shown above.
Then

$$\boxed{\int_a^d f(x)dx = A-B+C}$$

Hence,

the integral $\int_a^d f(x)dx$ **does not** give
the total area between the curve and the x-axis.

This is given by the formula

$$\boxed{\text{AREA} = \int_a^d |f(x)|\,dx}$$

In our example the area is A+B+C. In practice, we have to split this formula into three integrals

$$\boxed{\int_a^b f(x)dx = A \qquad \int_b^c f(x)dx = -B \qquad \int_c^d f(x)dx = C}$$

and then

$$\text{AREA} = A+B+C$$

NOTICE The GDC – graph mode gives the AREA as well.
After G-solv and integral, select the option MIXED – [F4].

You obtain both,

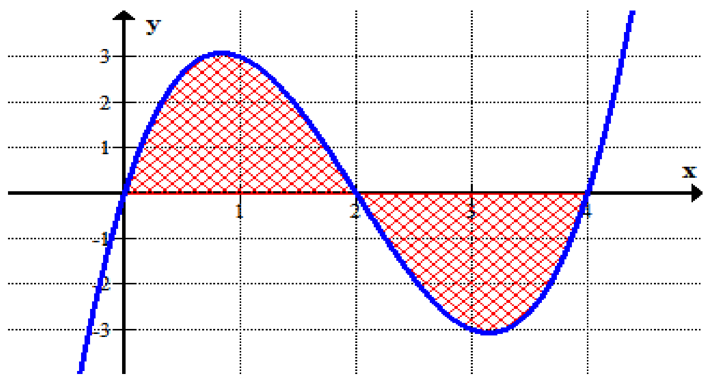
the value of $\boxed{\int_a^d f(x)dx}$

and the AREA $\boxed{\int_a^d |f(x)|\,dx}$

EXAMPLE 7

Consider the function

$$f(x) = x^3 - 6x^2 + 8x$$



Find

(a) $\int_0^2 f(x)dx$ (b) $\int_2^4 f(x)dx$ (c) $\int_0^4 f(x)dx$

(d) the total area between the curve and the x-axis within [0,4]

Solution

(a) $\int_0^2 f(x)dx = \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2 = 4-0 = 4$

(b) $\int_2^4 f(x)dx = \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_2^4 = 0-4 = -4$

(c) $\int_0^4 f(x)dx = \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^4 = 0-0 = 0$

[it is in fact the sum of the first two integrals above]

(d) The total area is given by

$$A = \int_0^4 \left|x^3 - 6x^2 + 8x\right| dx$$

In practice, we find (a) and (b) and add the absolute values

$$A = 4 + 4 = 8$$

Notice: your GDC in **graph mode** finds both

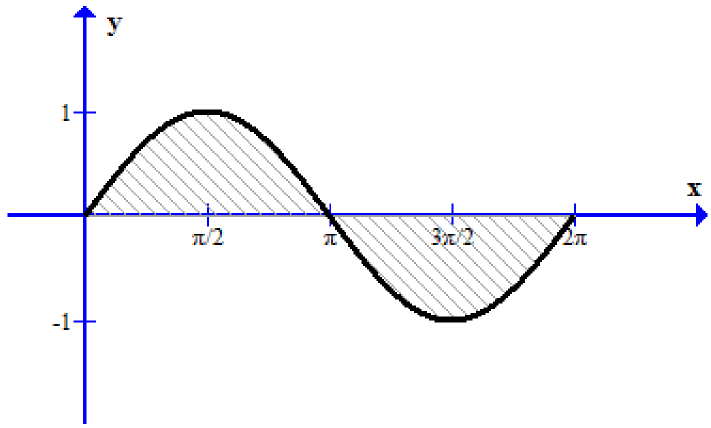
(c) the definite integral = 0 and

(d) the total area = 4

EXAMPLE 8

Consider the function

$$f(x)=\sin x, \qquad 0\le x\le 2\pi$$



Find

(a) $\int_0^{\pi} f(x)dx$ (b) $\int_{\pi}^{2\pi} f(x)dx$ (c) $\int_0^{2\pi} f(x)dx$

(d) the total area between the curve and the x-axis within $[0,2\pi]$

Solution

(a) $\int_0^{\pi} \sin x dx = [-\cos x]_0^{\pi} = -\cos\pi+\cos 0=1+1=2$

(b) $\int_{\pi}^{2\pi} \sin x dx = [-\cos x]_{\pi}^{2\pi} = -\cos 2\pi+\cos\pi=-1-1=-2$

(c) $\int_0^{2\pi} \sin x dx = [-\cos x]_0^{2\pi} = -\cos 2\pi+\cos 0=-1+1=0$

[it is in fact the sum of the first two integrals above]

(d) The total area is given by

$$A= \int_0^{2\pi} |\sin x| dx$$

In practice, we find (a) and (b) and add the absolute values

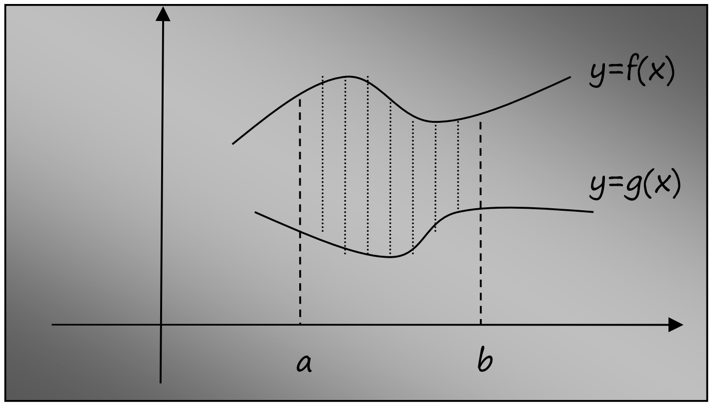
$$A = 2 + 2 = 4$$

Notice: your GDC in **graph mode**, estimates directly both

(c) the definite integral = 0 and (d) the total area = 4

- **AREA BETWEEN TWO CURVES**

Consider the graphs of two functions y=f(x) and y=g(x), such that $f(x) \geq g(x)$ for any x



The area between the two curves from x=a to x=b is given by

$$\int_a^b [f(x) - g(x)]dx$$

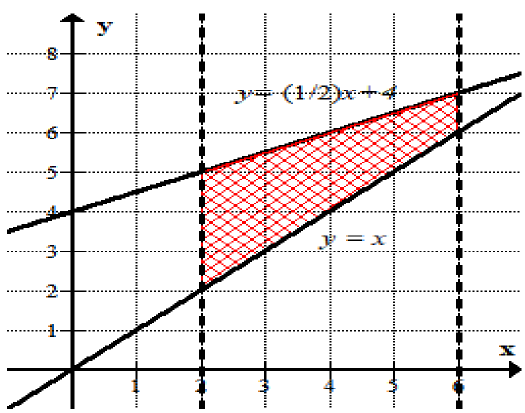
Indeed, $\int_a^b f(x)dx$ gives the area between y=f(x) and x-axis

while $\int_a^b g(x)dx$ gives the area between y=g(x) and x-axis

hence their difference gives the shaded area requested!

EXAMPLE 9

Find the shaded area below:



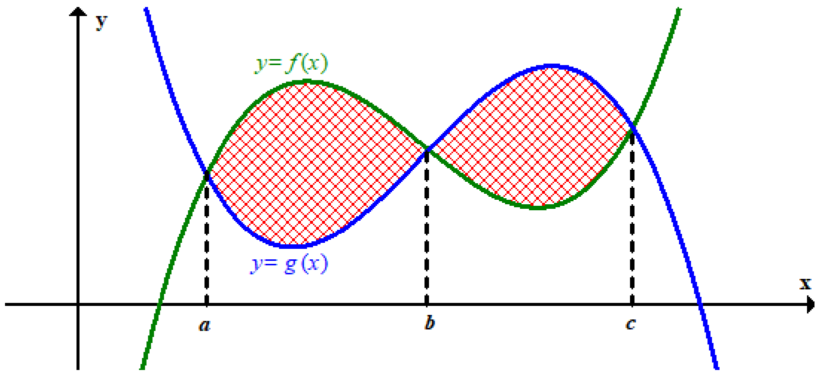
$$\text{Area } = \int_2^6 [(\frac{1}{2}x + 4) - x]dx = \int_2^6 (4 - \frac{1}{2}x)dx$$

$$= \left[4x - \frac{x^2}{4}\right]_2^6 = (24-9)-(8-1) = 15-7 = 8$$

In general, the area between two curves is given by

$$\int_a^b |f(x) - g(x)|\, dx$$

Consider the following situation

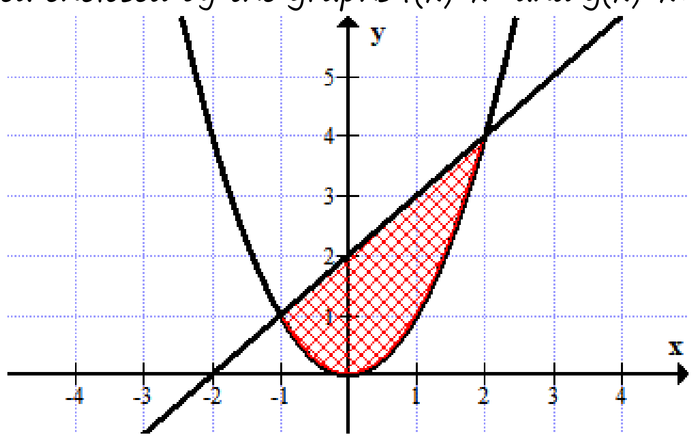


♦ **METHODOLOGY**

- Find the intersection points by solving the equation f(x)=g(x) (in our example: x=a, x=b, x=c)
- Split the integral appropriately: for each interval, determine which curve is "above" and which curve is "below" (in the example: $\int_a^b [f(x)-g(x)]dx$ + $\int_b^c [g(x)-f(x)]dx$)

EXAMPLE 10

Find the area enclosed by the graphs $f(x)=x^2$ and $g(x)=x+2$



- Intersection points:

$$f(x)=g(x) \Leftrightarrow x^2=x+2$$

$$\Leftrightarrow x^2-x-2=0$$

- The shaded area is

$$\int_{-1}^{2}[(x+2)-x^2]dx = \int_{-1}^{2}(x+2-x^2)dx = \left[\frac{x^2}{2}+2x-\frac{x^3}{3}\right]_{-1}^{2}$$

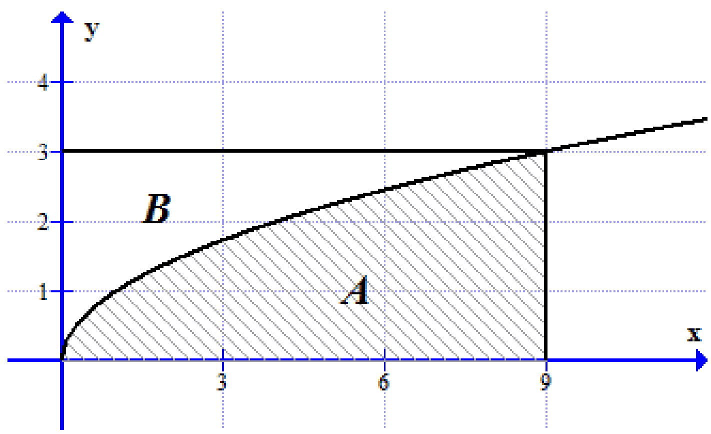
$$= (2+4-\frac{8}{3}) - (\frac{1}{2}-2+\frac{1}{3}) = \frac{10}{3}+\frac{7}{6} = \frac{9}{2}$$

Mind carefully the region required.

EXAMPLE 11

Consider the curve

$$y=\sqrt{x}$$



We define two regions

A: among the curve $y=\sqrt{x}$, x-axis and the line x=9.

B: among the curve $y=\sqrt{x}$, y-axis and the line y=3.

The corresponding areas are given by

$$A=\int_0^9\sqrt{x}dx=\int_0^9 x^{1/2}dx=\left[\frac{2}{3}x^{3/2}\right]_0^9=\frac{2}{3}9^{3/2}-0=18$$

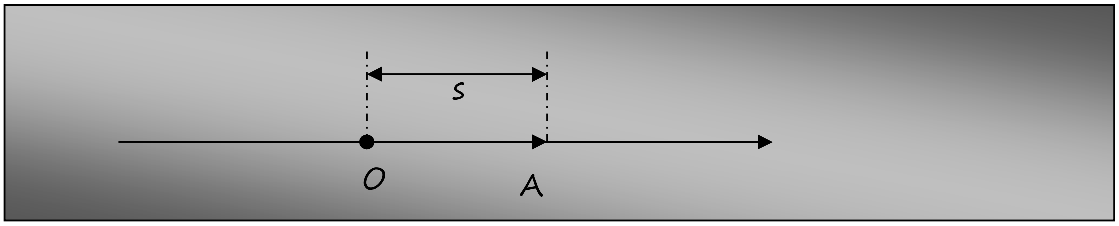
$$B=\int_0^9\left(3-\sqrt{x}\right)dx=\int_0^9\left(3-x^{1/2}\right)dx=\left[3x-\frac{2}{3}x^{3/2}\right]_0^9=\left(27-\frac{2}{3}9^{3/2}\right)-0=9$$

or

$$B=(\text{Area of rectangle})-A=27-18=9$$

5.12 KINEMATICS (DISPLACEMENT-VELOCITY-ACCELERATION)

Consider a straight line and a fixed point O on it. A body is moving on the line (forwards or backwards). The **displacement s** is ± the distance $|OA|$ from the fixed point O.



The displacement s (in meters) is given as a function of time.
For example,

$$s=t^2-4t+3$$

which means that

at time t=0 the displacement is 3 m from the fixed point O
at time t=1 the displacement is 0 m (it goes back to point O)
at time t=2 the displacement is -1 m, (the body is before point O)
at time t=3 the displacement is 0 m, (at point O again)
at time t=4 the displacement is 3 m, (it is moving forward)

Then

Velocity = rate of change of displacement: $\boxed{v=\dfrac{ds}{dt}}$

Acceleration = rate of change of velocity: $\boxed{a=\dfrac{dv}{dt}}$

Notice also that **a** is the second derivative of s. $\boxed{a=\dfrac{d^2s}{dt^2}}$

Displacement	Velocity	Acceleration
s	**v**	**a**

$\xrightarrow{\textbf{derivative}}$

EXAMPLE 1

Consider

$$s= t^3-12t+15$$

representing the motion of a particle along a straight line, where

the displacement s is given in m (meters),

the time t is given in sec (seconds).

Then

$$v=\frac{ds}{dt}=3t^2-12 \qquad a=\frac{dv}{dt}=6t$$

Notice that v is measured in m/sec while a is measured in m/sec^2.

For example,

at time $t=1$, $s=4m$

$v=-9m/sec$

$a=6m/sec^2$

at time $t=3$, $s=6m$

$v=15m/sec$

$a=18m/sec^2$

Notice also,

$\boxed{\text{The body is stationary when the velocity is 0}}$

Hence, let us solve

$$\begin{aligned} & v=0 \\ \Leftrightarrow\ & 3t^2-12=0 \\ \Leftrightarrow\ & t^2=4 \\ \Leftrightarrow\ & t=2sec \end{aligned}$$

(notice that time is always positive, thus we reject $t=-2$)

At that time ($t=2$), $s=-1m$

$v=0m/sec$

$a=12m/sec^2$

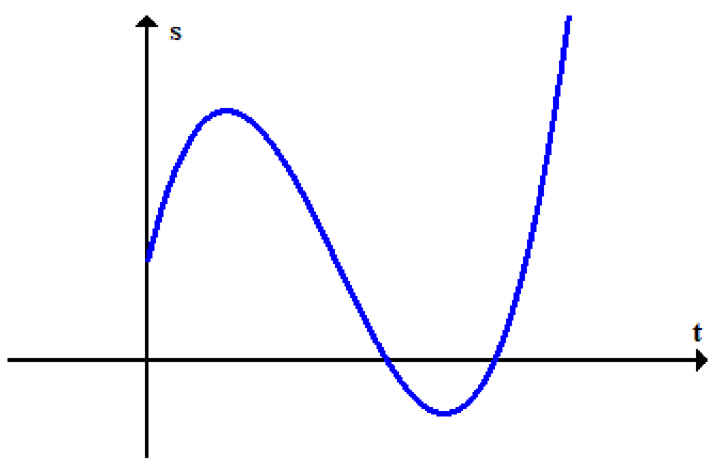
NOTICE

If $s>0$, the body is to the right of the fixed point O.
If $s<0$, the body is to the left of the fixed point O.
If $s=0$, the body is at the fixed point O.

If $v>0$, the body is moving to the right
If $v<0$, the body is moving to the left
If $v=0$, the body is stationary (it changes direction)

If $a>0$, the body accelerates
If $a<0$, the body decelerates
If $a=0$, the velocity is stationary (probably maximum speed)

The motion may be given as a graph of s with respect to t

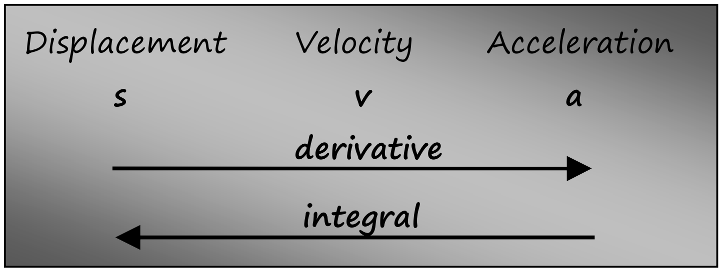


The displacement s from the fixed point O is the y-coordinate for each t-value (time); It can be

positive (after the fixed point O);
negative (before the fixed point O); or
zero (at the fixed point O).

♦ GOING BACKWARDS FROM a TO s (BY USING INTEGRATION)

If we are given the acceleration of a moving body we can find the velocity and the displacement by using integration.



However, in this opposite direction we must be given some extra information in order to evaluate any emerging constant c.

EXAMPLE 2

Let $v=12t^2-2t$. Find the acceleration and the displacement, given that the initial displacement is 5m.

Solution

$a=\frac{dv}{dt}=24t-2$ ms^{-1}

$s=\int vdt=\int(12t^2-2t)dt=4t^3-t^2+c$

but $s=5$, when $t=0$, thus $c=5$. Hence, $s=4t^3-t^2+5$ m

EXAMPLE 3

Let $a=12t$. Find the displacement, given that the moving body starts from rest; the initial displacement is 5m.

Solution

$v=\int adt=\int 12tdt=6t^2+c$

but when $t=0$, $v=0$ (at rest!), thus $c=0$. Hence, $v=6t^2$

$s=\int vdt=\int 6t^2dt=2t^3+c$

but when $t=0$, $s=5$, thus $c=5$. Hence, $s=2t^3+5$ m

Mind the difference between the displacement and the distance travelled of a moving body.

♦ **DISPLACEMENT vs DISTANCE TRAVELLED**

Suppose that the velocity v is given in terms of t. Then

Displacement from O	$s = \int v dt$
Displacement from t_1 to t_2	$S = \int_{t_1}^{t_2} v dt$
Distance travelled from t_1 to t_2	$d = \int_{t_1}^{t_2} \lvert v \rvert dt$

EXAMPLE 4

The velocity of a moving body is given in ms^{-1} by $v = 12 - 3t^2$
The initial displacement from a fixed point O is 1m.
Then

- The displacement of the body from O is given by

$$s=\int vdt=12t-t^3+c$$

  Since s=1 when t=0, we obtain c=1 and so $s=12t-t^3+1$

  Thus, after 2 seconds, s = 17m. After 3 seconds s = 10m

- The displacement in the first 2 seconds is

$$S=\int_0^2 vdt=\int_0^2 (12-3t^2+1)dt=\left[12t-t^3+t\right]_0^2=16m$$

  while the displacement in the first 3 seconds is

$$S=\int_0^3 vdt=\int_0^3 (12-3t^2)dt=\left[12t-t^3\right]_0^3=9m$$

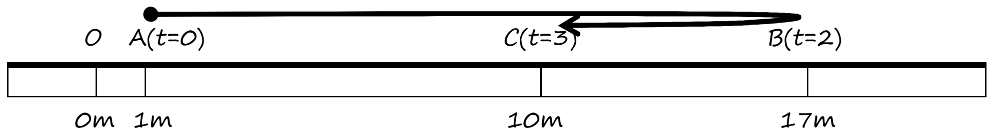
- The distance travelled in the first 3 seconds is

$$d=\int_0^2 |v|dt=\int_0^2 |12-3t^2|dt=\int_0^2 (12-3t^2)dt=\left[12t-t^3\right]_0^2=16m$$

  while the distance travelled in the first 3 seconds is

$$d=\int_0^3 |v|dt=\int_0^3 |12-3t^2|dt=\int_0^2 (12-3t^2)dt-\int_2^3 (12-3t^2)dt=16+7=23m$$

Explanation

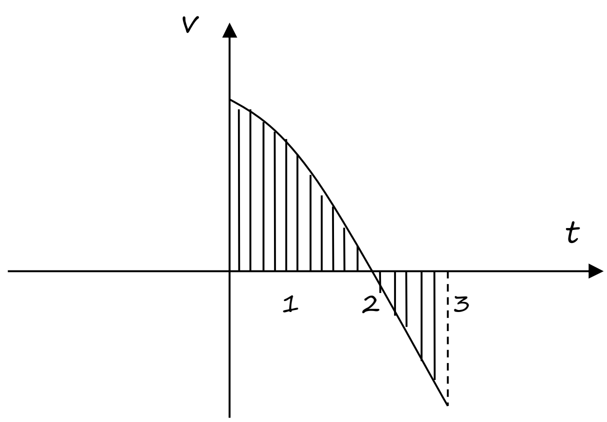


Time t	Displacement from O	Displacement from t=0 to t	Distance travelled	Velocity v
Initially **t=0** (Position A)	s = 1m	S = 0m	d=0m	v=12
after **t=2** sec (Position B)	s = 17m	S = 16m	d=16m	v=0
after **t=3** sec (Position C)	s = 10m	S = 9m	d=23m	v=-15

In other words, the moving body

starts from A
is moving forward in the first two seconds (to position B)
changes direction at B (v=0)
goes back to position C.

If we draw the graph of v against t,



The displacement S from t=0 to t=3 is the definite integral from t=0 to t=3 (area above the t-axis minus the area below the t-axis).

The distance travelled from t=0 to t=3 is given by the **total area** from t=0 to t=3 between the curve and t-axis (shaded area above)

EXAMPLE 5

Let

$$v = 4t - t^2$$

Given that the initial displacement is 10m find the displacement and the distance travelled

a) during the first 3 sec

b) during the first 6 sec.

Solution

Let us solve first the equation v=0

$$v = 0 \Leftrightarrow 4t - t^2 = 0 \Leftrightarrow t = 0 \text{ or } t = 4$$

So the body changes direction after 4 sec.

(a) During the first 3 seconds the displacement and the distance travelled coincide.

$$S = d = \int_0^3 (4t - t^3)dt = \left[2t^2 - \frac{t^3}{3}\right]_0^3 = 9$$

(b) During the first 6 seconds the body changes direction.

The displacement is

$$S = \int_0^6 (4t - t^3)dt = \left[2t^2 - \frac{t^3}{3}\right]_0^6 = 0$$

so it goes back to the initial position.

For the distance travelled, we have to split the integral:

$$\int_0^4 (4t - t^2)dt = \left[2t^2 - \frac{t^3}{3}\right]_0^4 = \frac{32}{3}$$

$$\int_4^6 (4t - t^2)dt = \left[2t^2 - \frac{t^3}{3}\right]_4^6 = -\frac{32}{3}$$

Hence,

$$d = \frac{32}{3} + \frac{32}{3} = \frac{64}{3} = 21.3m$$

Notice The GDC directly gives the results. For example, for (b)

$$S = \int_0^6 (4t - t^3)dt = 0\,m \qquad d = \int_0^6 \left|4t - t^2\right|dt = 21.3\,m$$

ONLY FOR

HL

5.13 MORE DERIVATIVES (for HL)

The formula booklet contains the derivatives of a few more functions, for the HL part.

The derivatives of any exponential and any logarithmic function:

$f(x)$	$f'(x)$
a^x	$a^x \cdot \ln a$
$\log_a x$	$\dfrac{1}{x\ln a}$

Notice: for a=e we obtain the particular cases, $(e^x)'=e^x$ and $(\ln x)'=\dfrac{1}{x}$

EXAMPLE 1

For $f(x) = 3^x$, $f'(x)= 3^x\ln 3$

For $g(x) = \log_5 x$, $g'(x) = \dfrac{1}{x\ln 5}$

The derivatives of the following trigonometric and inverse trigonometric functions:

$f(x)$	$f'(x)$
$\tan x$	$\dfrac{1}{\cos^2 x} = \sec^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$

The first four trigonometric cases can be proved by using the known rules. Let's prove, for example, the 2nd and the 4th formulas.

EXAMPLE 2

Show that the derivative of $f(x)=\sec x$ is $f'(x)=\sec x\tan x$

Proof

We know that $f(x)=\sec x=\dfrac{1}{\cos x}=(\cos x)^{-1}$. The chain rule gives

$$f'(x)=-(\cos x)^{-2}(-\sin x)=\frac{\sin x}{\cos^2 x}=\tan x\frac{1}{\cos x}=\sec x\tan x$$

EXAMPLE 3

Show that the derivative of $f(x)=\cot x$ is $f'(x)=-\operatorname{cosec}^2 x$

Proof

We know that $f(x)=\cot x=\dfrac{\cos x}{\sin x}$. The quotient rule gives

$$f'(x)=\frac{(-\sin x)(\sin x)-(\cos x)(\cos x)}{\sin^2 x}=-\frac{\sin^2 x+\cos^2 x}{\sin^2 x}=-\frac{1}{\sin^2 x}=-\operatorname{cosec}^2 x$$

Let us apply the known rules (product, quotient, chain) to find derivatives that involve functions of this paragraph..

EXAMPLE 4

For $f(x) = 3^x\tan x$, $\quad f'(x)= 3^x\ln 3\tan x+3^x\sec^2 x$

For $g(x) = \dfrac{\arcsin x}{2x+3}$, $\quad g'(x) = \dfrac{\dfrac{2x+3}{\sqrt{1-x^2}}-2\arcsin x}{(2x+3)^2}$

For $h(x) = \sec 2x$, $\quad h'(x) = 2\sec 2x\tan 2x$

For $k(x) = \arctan x^2$, $\quad k'(x) = \dfrac{2x}{1+x^4}$

We know that

$$(x^2)' = 2x$$

$$(2^x)' = 2^x \ln 2$$

But what about the function

$$f(x) = x^x$$

This new function has neither of the two forms above. We modify the original function using logarithms.

EXAMPLE 5

Show that the derivative of $f(x)=x^x$ is

$$f'(x)=x^x(\ln x+1)$$

Proof

$$f(x) = x^x = e^{\ln x^x} = e^{x\ln x}$$

Then

$$f(x) = e^{x\ln x}(x\ln x)'$$

$$= e^{x\ln x}(\ln x+1)$$

$$= x^x(\ln x+1)$$

Finally, we can now use all these new functions in applications of tangent lines, extreme values, etc. Let's see a simple example.

EXAMPLE 6

Find the equation of the tangent line to the curve of $f(x)=x\arctan x$ at the point $(1, \frac{\pi}{4})$. Express the answer in the form $y=mx+c$.

Solution

$f'(x) = \arctan x+\dfrac{x}{1+x^2}$, $\qquad m_T = f'(1) = \dfrac{\pi}{4}+\dfrac{1}{2}=\dfrac{\pi+2}{4}$

Tangent line: $\qquad y-\dfrac{\pi}{4}=\dfrac{\pi+2}{4}(x-1) \Rightarrow y=\dfrac{\pi+2}{4}x-\dfrac{1}{2}$

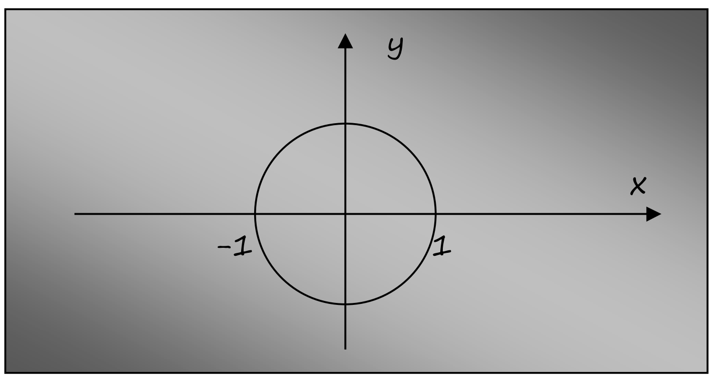
5.14 IMPLICIT DIFFERENTIATION – MORE KINEMATICS (for HL)

We are very familiar with functions of the form $y=f(x)$. However, in many real applications, we are not given a clear function where y is expressed in terms of x, but a more complicated relation which involves x and y.

Consider for example the relation

$$x^2+y^2=1$$

Actually, this relation represents a circle of radius 1 on the Cartesian plane:



(i.e. the pairs (x,y) that satisfy this relation form a unit circle)

Obviously this is not the graph of a function (as a vertical line may cross the graph at two points). However, if we solve for y we obtain

$$y^2=1-x^2 \Leftrightarrow y=\pm\sqrt{1-x^2}$$

that is, two different functions together:

$y=\sqrt{1-x^2}$ is the semicircle above the x-axis

$y=-\sqrt{1-x^2}$ is the semicircle under the x-axis

The question is

What is the derivative $y'=\dfrac{dy}{dx}$?

Case 1: If $y=\sqrt{1-x^2}$ the derivative is $\frac{dy}{dx} = \frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$

Case 2: If $y=-\sqrt{1-x^2}$ the derivative is $\frac{dy}{dx} = -\frac{-2x}{2\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}}$

We can observe that in both cases the result is equal to

$$\frac{dy}{dx} = -\frac{x}{y}$$

An alternative (and elegant) way to obtain the same result is the following: Consider again

$$x^2+y^2=1$$

Differentiate both sides with respect to x. Do have in mind though that y is a function of x, so

the derivative of y^2 is not $2y$ but $2y\frac{dy}{dx}$ (chain rule).

Hence, the differentiation gives

$$2x+2y\frac{dy}{dx} = 0 \Leftrightarrow 2y\frac{dy}{dx} = -2x \Leftrightarrow \frac{dy}{dx} = -\frac{x}{y}$$

This process is known as **implicit differentiation**.

In practice, whenever we differentiate y's, we multiply by $\frac{dy}{dx}$.

EXAMPLE 1

Find $y'=\frac{dy}{dx}$ given that $2x^2+x^2y^3 = x+y^2+3$

Solution

The implicit differentiation gives

$$4x+2xy^3+x^2 3y^2\frac{dy}{dx} = 1+2y\frac{dy}{dx}$$

We now solve for $\frac{dy}{dx}$: $\frac{dy}{dx}(3x^2y^2-2y) = 1-4x-2xy^3$

$$\Leftrightarrow \quad \boxed{\frac{dy}{dx} = \frac{1-4x-2xy^3}{3x^2y^2-2y}}$$

Well, you may complain that the result is not a clear expression in terms of x as usual; It involves x and y as well! However, this is not a problem in general. Look at the following application.

EXAMPLE 2

Consider again the equation of the circle

$$x^2+y^2=1$$

Find the tangent lines

(a) at the point $(x,y)=(\frac{1}{2},\frac{\sqrt{3}}{2})$ (confirm it lies on the circle)

(b) at the points of the circle with x=0

Solution

We have seen that

$$\frac{dy}{dx}=-\frac{x}{y}$$

(a) At $(x,y)=(\frac{1}{2},\frac{\sqrt{3}}{2})$

$$m_T=-\frac{x}{y}=-\frac{1/2}{\sqrt{3}/2}=-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$$

Hence, the equation of the tangent line is

$$y-\frac{\sqrt{3}}{2}=-\frac{\sqrt{3}}{3}(x-\frac{1}{2})$$

That is

$$y-\frac{\sqrt{3}}{2}=-\frac{\sqrt{3}}{3}x+\frac{\sqrt{3}}{6}$$

and finally

$$y=-\frac{\sqrt{3}}{3}x+\frac{4\sqrt{3}}{3}$$

(b) For x=0 we have to find the value(s) of y:

$$x^2+y^2=1 \Leftrightarrow y^2=1 \Leftrightarrow y=\pm 1$$

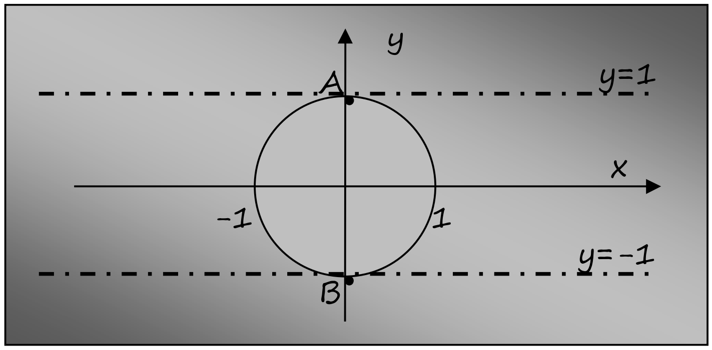
- At (x,y)=(0,1), the gradient is m=0 and the tangent line is

$$\boxed{y=1}$$

- At (x,y)=(0,−1), the gradient is m=0 and the tangent line is

$$\boxed{y=-1}$$

[Indeed, notice that A(0,1) and B(0,-1) are the highest and the lowest points of the circle respectively



Apparently, the tangent lines at those points are the horizontal lines y=1 and y=-1]

EXAMPLE 3

Let

$$x^2+y+x\cos y=\pi.$$

Find the tangent and the normal lines at x=0.

Solution

Firstly, for x=0 we obtain y=π, so the given point is (0,π).

Implicit differentiation gives

$$2x+\frac{dy}{dx}+\cos y-x\sin y\frac{dy}{dx}=0$$

$$\Leftrightarrow \frac{dy}{dx}(1-x\sin y)=-2x-\cos y$$

$$\Leftrightarrow \frac{dy}{dx}=\frac{2x+\cos y}{x\sin y-1}$$

At (x,y)=(0,π),

$$m_T=\frac{0+\cos\pi}{0-1}=1 \quad \text{and} \quad m_N=-1$$

The tangent line is

$$y-\pi = 1(x-0) \quad \Leftrightarrow \quad \boxed{y=x+\pi}$$

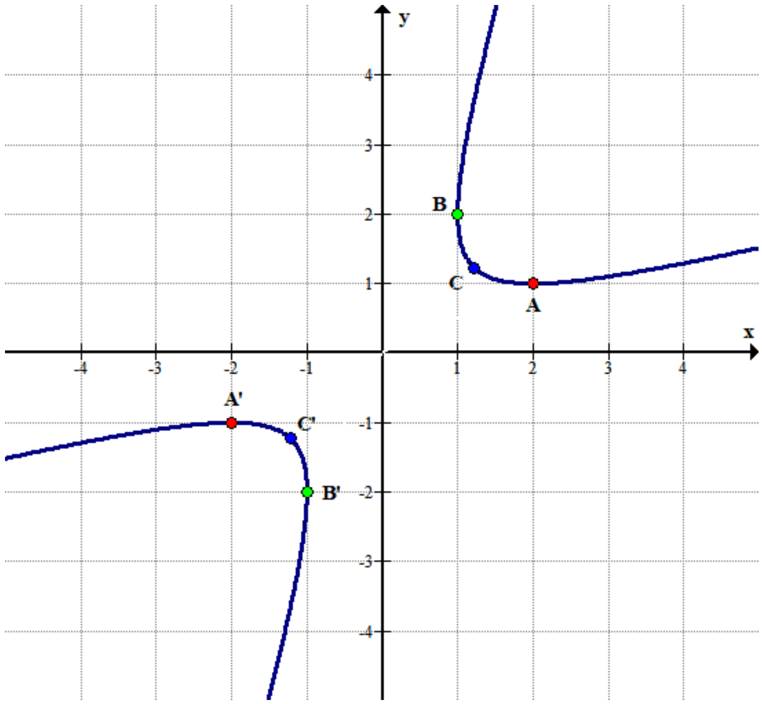
The normal line is

$$y-\pi = -1(x-0) \quad \Leftrightarrow \quad \boxed{y=-x+\pi}$$

EXAMPLE 4

The diagram below shows the graph of the relation

$$x^2+y^2 = 4xy-3.$$



Find the points on the curve where the tangent line is parallel to

(a) parallel to x-axis

(b) parallel to y-axis

(c) parallel to the line $y=-x$

Solution

Implicit differentiation gives

$$2x+2y\frac{dy}{dx}=4y+4x\frac{dy}{dx}$$

$$\Leftrightarrow x+y\frac{dy}{dx}=2y+2x\frac{dy}{dx}$$

$$\Leftrightarrow \frac{dy}{dx}(y-2x)=2y-x$$

$$\Leftrightarrow \frac{dy}{dx}=\frac{2y-x}{y-2x}$$

We solve each of the equations obtained from (a), (b) and (c) together with the original equation (as simulation equations).

(a) $\frac{dy}{dx}=0 \Leftrightarrow 2y-x=0 \Leftrightarrow x=2y$

The original equation gives

$$4y^2+y^2 = 8y^2-3 \Leftrightarrow y^2=1 \Leftrightarrow y=\pm 1$$

The points are A(2,1) and A′(-2,-1) (see diagram above).

(b) $\frac{dy}{dx}=\text{not defined} \Leftrightarrow y-2x=0 \Leftrightarrow y=2x$

The original equation gives

$$x^2+4x^2 = 8x^2-3 \Leftrightarrow x^2=1 \Leftrightarrow x=\pm 1$$

The points are B(1,2) and B′(-1,-2) (see diagram above).

(c) $\frac{dy}{dx}=-1 \Leftrightarrow \frac{2y-x}{y-2x}=-1 \Leftrightarrow 2y-x=-y+2x \Leftrightarrow 3y=3x \Leftrightarrow y=x$

The original equation gives $x=\pm\sqrt{\frac{3}{2}}$ (check!)

The points are $C(\sqrt{3/2},\sqrt{3/2})$ and $C'(-\sqrt{3/2},-\sqrt{3/2})$

♦ THE DERIVATIVE OF AN INVERSE FUNCTION

Consider again the relation $x^2+y^2 = 4xy-3$.

The differentiation in terms of x (i.e. $\frac{d}{dx}$) resulted to $\frac{dy}{dx}=\frac{2y-x}{y-2x}$

The differentiation in terms of y (i.e. $\frac{d}{dy}$) will give $\frac{dx}{dy}=\frac{y-2x}{2y-x}$

This is not an accident! In general

$$\frac{dx}{dy}=\left(\frac{dy}{dx}\right)^{-1}$$

In fact, this result gives the derivative of an inverse function $y=f^{-1}(x)$, given that we know the derivative of f(x).

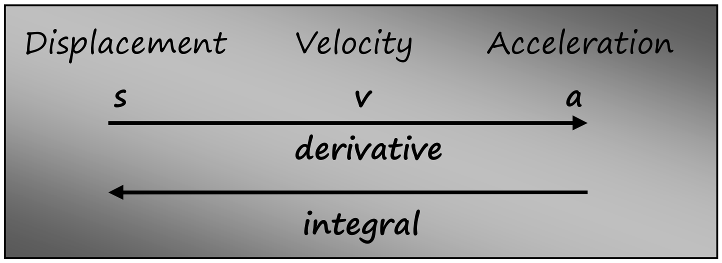
For example, if we know the derivative of $f(x)=e^x$ we can derive the derivative of $f^{-1}(x)=\ln x$. Indeed

Let $y=\ln x$. Then $x=e^y$.

$$\frac{dx}{dy}=e^y \Rightarrow \frac{dy}{dx}=\frac{1}{e^y} \Rightarrow \frac{dy}{dx}=\frac{1}{x}$$

- MORE ON KINEMATICS

Remember the following scheme



In this situation, s,v,a are all given in terms of time t.

For example, given that

$$v=6t^2$$

then

$$a=\frac{dv}{dt}=12t$$

However, the velocity (v) is sometimes given as a function of s, instead of t. For example, $v=3s^2$

The following formula for acceleration may be derived (using the chain rule)

$$\boxed{a=\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}=\frac{dv}{ds}v}$$

Concentrate on

$$\boxed{a=v\frac{dv}{ds}}$$

EXAMPLE 5

Let

$$v=3s^2$$

Then

$$a=v\frac{dv}{ds}=(3s^2)(6s)=18s^3$$

In fact, we use implicit differentiation on $v=3s^2$ with respect to t:

$$a=\frac{dv}{dt}=(6s)\ \frac{ds}{dt}=(6s)\ v=(6s)\ (3s^2)=18s^3$$

5.15 RELATED RATES (for HL)

In this paragraph we deal with problems that involve two or more quantities depending on time t. If A is a function of t we know that

$$\frac{dA}{dt} = \text{rate of change of } A$$

The rate of change of $y=A^2$ is given by

$$\frac{dy}{dt}=\frac{dy}{dA}\frac{dA}{dt}=2A\frac{dA}{dt}$$

The rate of change of $y=A^3$ is given by

$$\frac{dy}{dt}=3A^2\frac{dA}{dt}$$

In general, the rate of change of any function of A, say $y=f(A)$ is given by

$$\frac{dy}{dt}=f'(A)\frac{dA}{dt}$$

a) We are given a relation between two quantities A and B. Any change according to time in one of them implies a change in the other one as well. The relation between $\frac{dA}{dt}$ and $\frac{dB}{dt}$, the rates of change of A and B, can be found by implicit differentiation with respect to time t:

 We differentiate both parts of the relation with respect to t;
 whenever we differentiate A we multiply by $\frac{dA}{dt}$;
 whenever we differentiate B we multiply by $\frac{dB}{dt}$.

 For example

If	$A=2B^3$	then	$\frac{dA}{dt}=6B^2\frac{dB}{dt}$
If	$A=2B+\ln B$	then	$\frac{dA}{dt}=2\frac{dB}{dt}+\frac{1}{B}\frac{dB}{dt}$
If	$A^2=2B^3$	then	$2A\frac{dA}{dt}=6B^2\frac{dB}{dt}$
If	$\sin A=\frac{3}{B}$	then	$\cos A\frac{dA}{dt}=-\frac{3}{B^2}\frac{dB}{dt}$

b) We are given a relation between three quantities A,B and C. The relation between the rates of change $\frac{dA}{dt}, \frac{dB}{dt}$ and $\frac{dC}{dt}$, can be found by implicit differentiation with respect to time t.

For example

If $A=2B^3+4C^2$ then $\frac{dA}{dt}=6B^2\frac{dB}{dt}+8C\frac{dC}{dt}$

If $A=B^2C^3$ then $\frac{dA}{dt}=2BC^3\frac{dB}{dt}+3B^2C^2\frac{dC}{dt}$

If $A^3+B^2=5$ then $3A^2\frac{dA}{dt}+2B\frac{dB}{dt}=0$

If $\cos A=B+5BC$ then $-\sin A\frac{dA}{dt}=\frac{dB}{dt}+5C\frac{dB}{dt}+5B\frac{dC}{dt}$

In problems involving rates of change we work as follows

Methodology

The problem usually refers to the rates of change of two quantities. One rate is given, one rate is required (usually at some instant).

1. Determine the two quantities A and B
   (Say that $\frac{dA}{dt}$ is given and $\frac{dB}{dt}$ is required)
2. Find the general relation between A and B (*)
3. Find the relation between $\frac{dA}{dt}$ and $\frac{dB}{dt}$ (**)
4. If the question mentions a specific instant (say for a particular value of B)
   - We use (*) to find the value of A (if necessary).
   - We substitute all known values in (**) to find $\frac{dB}{dt}$

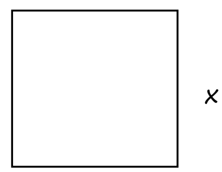
Notice: Be careful, the values of A or B at some particular instant are used only in the final step 4 of substitution

Work similarly if three or more quantities are involved.

EXAMPLE 1

Consider a square object which is expanding. If the side of the object increases in a constant rate of $2ms^{-1}$ find the rate of change of its area, at the instant when the side is 10m.

Solution



x= side, $\frac{dx}{dt}$=2m/sec

A= area, $\frac{dA}{dt}$=?

The relation between A and x is: $A=x^2$

Hence,

$$\frac{dA}{dt}=2x\frac{dx}{dt}$$

Therefore, when x=10m

$$\frac{dA}{dt}=2\times10\times2=40m^2/sec$$

EXAMPLE 2

Consider an expanding sphere. If the volume increases in rate $5cm^3/sec$ find the rate of change of its radius r,

(i) when r = 3 cm (ii) when the volume reaches 36π cm^3

Solution

$\frac{dV}{dt}=5cm^3/sec$, $\frac{dr}{dt}=?$.

The relation between V and r is given by $V=\frac{4}{3}\pi r^3$. Hence,

$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$$

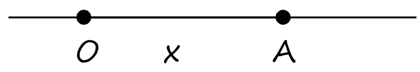
(i) when r = 3,

$$5=4\pi 3^2\frac{dr}{dt} \Leftrightarrow \frac{dr}{dt}=\frac{5}{36\pi} \text{ m/sec.}$$

(ii) when $V=36\pi$, the original relation gives $36\pi=\frac{4}{3}\pi r^3 \Leftrightarrow r = 3cm$.

Therefore, the answer is the same as above.

Have in mind that speed is also a rate of change. When a moving body A has speed 5m/sec, that means that the distance x from some fixed point O changes in rate 5m/sec

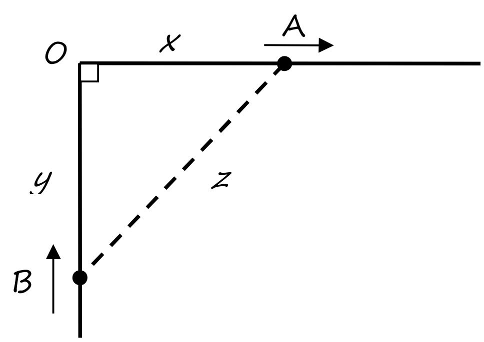


If A is moving to the right, x increases so $\frac{dx}{dt}=5m/sec$

If A is moving to the left, x decreases so $\frac{dx}{dt}=-5m/sec$

EXAMPLE 3

Two cars, A and B, are traveling at 50 km/h and 70 km/h respectively, on straight roads, as shown in the diagram below.



At a given instant both cars are 5 km away from O. Find, at that instant, the rates of change

(a) of the distance between the cars.

(b) of the angle $\theta = O\hat{A}B$ in radians per minute.

Solution

$\frac{dx}{dt}=50km/h$ $\frac{dy}{dt}=-70km/h$ $\frac{dz}{dt}=?$ $\frac{d\theta}{dt}=?$

When $x=y=5$, then $z=5\sqrt{2}$ and $\theta=\frac{\pi}{4}$

(a) Relation between x, y, z: $z^2=x^2+y^2$

Hence $2z\frac{dz}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt} \Rightarrow z\frac{dz}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt}$

$\Rightarrow 5\sqrt{2}\frac{dz}{dt}=5(50)-5(70) \Rightarrow \frac{dz}{dt}=-10\sqrt{2}km/h$

(b) Relation between x, y, θ : $\tan\theta = \frac{y}{x} \Rightarrow y = x\tan\theta$

Hence $\frac{dy}{dt} = \frac{dx}{dt}\tan\theta + x\sec^2\theta\frac{d\theta}{dt}$

$\Rightarrow -70 = 50\tan\frac{\pi}{4} + 5\left(\sec^2\frac{\pi}{4}\right)\frac{d\theta}{dt}$

$\Rightarrow \frac{d\theta}{dt} = -12$ rad/h = -0.2 rad/min

Let us see a final example with three variables.

EXAMPLE 4

It is given that

$$A = \frac{1}{3}r^2h + 2r^3$$

Find the rate of change of h when r=3 and h=6, under two circumstances:

(a) when h is always double of r and $\frac{dA}{dt} = 30$

(b) when $\frac{dA}{dt} = 30$ and $\frac{dh}{dt} = 8$

Solution

(a) Since h=2r, the original relation becomes $A = \frac{2}{3}r^3 + 2r^3 = \frac{8}{3}r^3$

Hence

$$\frac{dA}{dt} = 8r^2\frac{dr}{dt}$$

Therefore, when r=3

$$30 = 72\frac{dr}{dt} \Leftrightarrow \frac{dr}{dt} = \frac{5}{12}$$

(b) By implicit differentiation on the original relation we obtain

$$\frac{dA}{dt} = \frac{2}{3}rh\frac{dr}{dt} + \frac{1}{3}r^2\frac{dh}{dt} + 6r^2\frac{dr}{dt}$$

Therefore, when r=3 and h=6,

$$30 = 12\frac{dr}{dt} + 24 + 54\frac{dr}{dt} \Leftrightarrow 6 = 66\frac{dr}{dt} \Leftrightarrow \frac{dr}{dt} = \frac{1}{11}$$

5.16 CONTINUITY AND DIFFERENTIABILITY

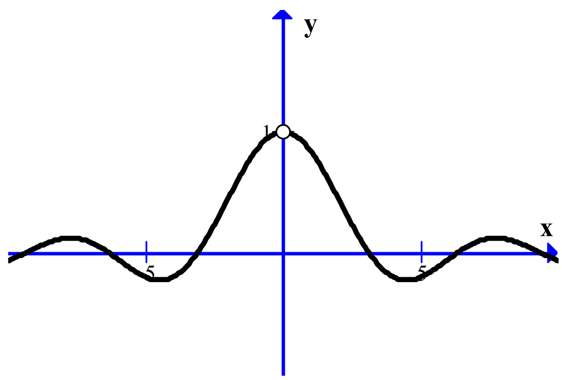
- CONTINUITY

In paragraph 5.1 we saw that

for $f(x) = 2x+3$, $\lim_{x\to 2} f(x) = 7$ and $f(2) = 7$

for $f(x) = \frac{\sin x}{x}$ $\lim_{x\to 0} f(x) = 1$, but $f(0)$ is not defined.

If we observe the graph of $f(x) = \frac{\sin x}{x}$



we notice a "discontinuity" at x=0.

It's worth it to see another similar situation.

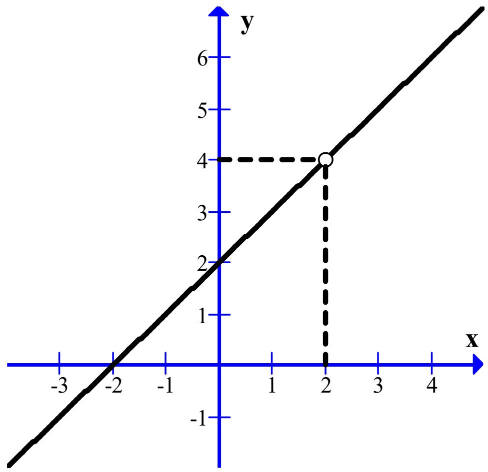
$$f(x) = \frac{x^2 - 4}{x - 2}$$

The function is not defined at x=2. However x=2 is not a vertical asymptote. It is interesting to see what happens to f(x) as x →2.

x	f(x)
1.999	3.999
2.001	4.001

It seems that $\lim_{x\to 2} f(x) = 4$.

Indeed, look at the graph of f(x).



There is an "empty" point on it!

When x approaches 2, the value of y approaches 4. Thus

$f(2)$ is not defined

but $\lim_{x\to 2} f(x) = 4$.

In fact, we can simplify the function as

$$f(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2, \quad \text{where } x \neq 2$$

That is why we obtain the graph of the straight line y=x+2 with some "discontinuity" at x=2. Moreover,

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2}(x+2) = 4$$

We say that a function is **continuous** at x=a, when

- The value f(a) exists;
- The limit $\lim_{x\to a} f(x)$ exists;
- $\lim_{x\to a} f(x) = f(a)$

For the continuity at any particular point we must check all three presuppositions.

Most of the known functions are continuous everywhere (since they look like uninterrupted curves!). For example, lines, quadratics, polynomials in general, exponentials are continuous functions.

Remember that at some x=a we may have different side limits

$$\lim_{x\to a^-} f(x) \text{ and } \lim_{x\to a^+} f(x)$$

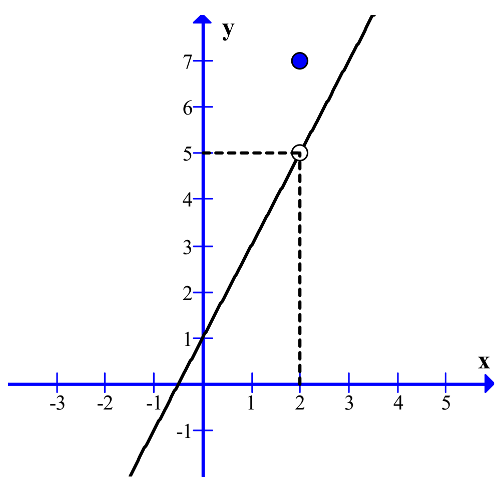
If they are equal, say $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = b$, then we can say that

$$\lim_{x\to a} f(x) = b$$

It is worthwhile to see the following examples of "step" functions to further clarify the notions of limit and continuity.

EXAMPLE 1

Let $f(x) = \begin{cases} 2x+1, & \text{if } x \neq 2 \\ 7, & \text{if } x = 2 \end{cases}$



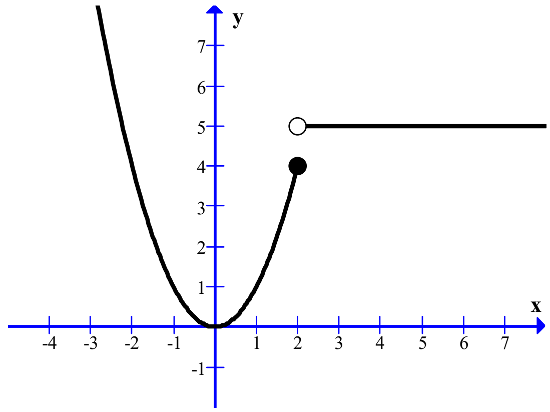
- $\lim_{x\to 2} f(x) = 5$ [when x approaches 2,the value f(x) approaches 5];
- $f(2) = 7$.
- But $\lim_{x\to 2} f(x) \neq f(2)$

Thus the function is not continuous at x=2.

EXAMPLE 2

Consider the function

$$f(x)=\begin{cases} x^2, & \text{if} \quad x\le 2 \\ 5, & \text{if} \quad x>2 \end{cases}$$



We can see that f(2)=4 but $\lim\limits_{x\to 2} f(x)$ does not exist.

[In fact, only side limits exist: $\lim\limits_{x\to 2^-} f(x)=4$ and $\lim\limits_{x\to 2^+} f(x)=5$]

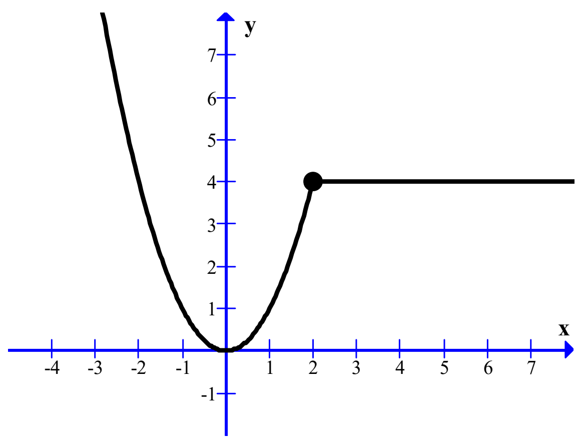
Therefore, the function is **not continuous at x=2**.

(We also say that the function is not continuous in its domain R).

EXAMPLE 3

Consider the function

$$f(x)=\begin{cases} x^2, & \text{if } x\le 2 \\ 4, & \text{if } x>2 \end{cases}$$



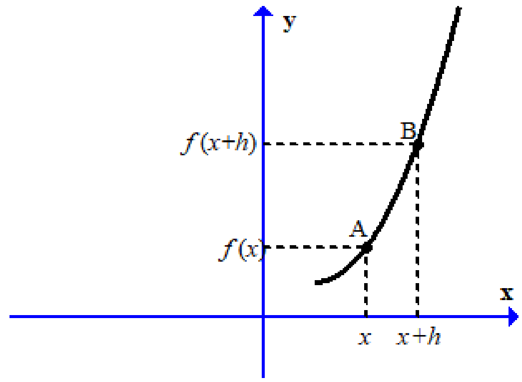
We can see that f(2)=4 and also $\lim\limits_{x\to 2} f(x)=4$.

Since $\lim\limits_{x\to 2} f(x)=f(2)$ the function is **continuous at x=2**.

(in fact the function is continuous everywhere).

♦ THE FORMAL DEFINITION OF THE DERIVATIVE

Let y=f(x) be a continuous curve and A(x,f(x)) some point on it:



We select a neighboring point B with

x-coordinate = x+h (where h is very small)

y-coordinate =f(x+h)

As we move from point A to point B, the rate of change is

$$\frac{\Delta y}{\Delta x}=\frac{f(x+h)-f(x)}{h}$$

If we let h become very small, that is h→0, the result will be the rate of change at point A, that is the derivative f′(x).

$$\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

Let us apply this formula to the function f(x) = x^2.

$$\begin{aligned} f'(x) &=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} \\ &=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} \\ &=\lim_{h\to 0}\frac{h(2x+h)}{h} \\ &=\lim_{h\to 0}(2x+h)=2x \end{aligned}$$

Therefore,

$$f'(x) = 2x$$

If we apply the definition for $f(x)=x^n$ we will find that

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = nx^{n-1} \quad \text{(exercise!)}$$

EXAMPLE 4

Show <u>from first principles</u> (that is by using the formal definition), that the derivative of the function

$$f(x) = x^3 + 2x$$

is

$$f'(x) = 3x^2 + 2\,.$$

Solution

$$\begin{aligned}
f'(x) \quad &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{[(x+h)^3 + 2(x+h)] - [x^3 + 2x]}{h} \\
&= \lim_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h - x^3 - 2x}{h} \\
&= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3 + 2h}{h} \\
&= \lim_{h\to 0}\frac{h(3x^2 + 3xh + h^2 + 2)}{h} \\
&= \lim_{h\to 0}(3x^2 + 3xh + h^2 + 2)
\end{aligned}$$

Now, we are able to set $h=0$ and obtain,

$$f'(x) = 3x^2 + 2$$

If $f'(x)$ exists we say that $f(x)$ is **differentiable at x**.

We also say that $f(x)$ **differentiable** if it $f'(x)$ exists for any point x of the domain.

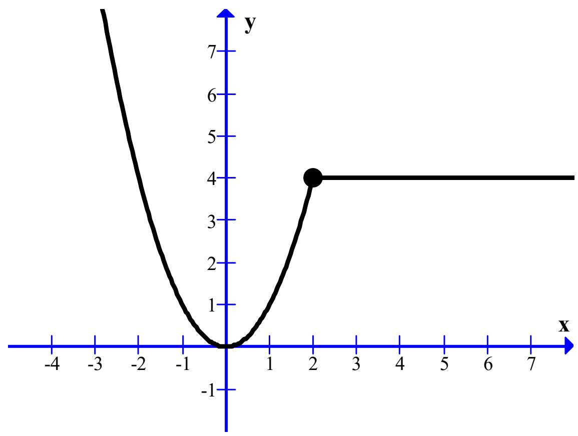
Notice

$$f \text{ differentiable at } x \quad \Rightarrow \quad f \text{ continuous at } x$$

The opposite is not necessarily true: that is, the function may be continuous at some point x but not differentiable at this point.

Indeed, let us see again the step function of example 3

$$f(x)=\begin{cases} x^2, & \text{if } x\le 2 \\ 4, & \text{if } x>2 \end{cases}$$



The function **is continuous** at x=2 but it **is not differentiable** at x=2.

Informally speaking, this is because there is a "corner" at x=2:

we cannot draw a tangent line at x=2

In other words, the curve is continuous but not "smooth" at x=2

Actually, at x=2 we find the "side" derivatives $f'_-(2)$ and $f'_+(2)$:

For $f'_-(2)$ we differentiate the expression $f(x)=x^2$ (before x=2):

$$f'(x)=2x\,, \quad \text{thus} \quad f'_-(2)=4$$

For $f'_+(2)$ we differentiate the expression $f(x)=4$ (after x=2):

$$f'(x)=0\,, \quad \text{thus} \quad f'_+(2)=0$$

Since $f'_-(2)=4$ and $f'_+(2)=0$ (they differ), $f'(2)$ does not exist.

Let us slightly modify this function:

EXAMPLE 5

$$f(x)=\begin{cases} x^2, & \text{if } x\le 2 \\ 4x-4, & \text{if } x>2 \end{cases}$$

We firstly show that f is continuous at x=2:

- $\lim_{x\to 2^-} f(x)=2^2=4$
- $\lim_{x\to 2^+} f(x)=8-4=4$
- f(2)=4

Therefore $\lim_{x\to 2} f(x)=4=f(2)$ and the function is continuous at x=2.

We next show that f is differentiable at x=2:

- For $f(x)=x^2$, $f'(x)=2x$, thus $f'_-(2)=4$
- For $f(x)=4x-4$, $f'(x)=4$, thus $f'_+(2)=4$

Therefore $f'(2)=4$ and the function is differentiable at x=2.

We will see the same example in a different version:

EXAMPLE 5

$$f(x)=\begin{cases} x^2, & \text{if } x<2 \\ a, & \text{if } x=2 \\ bx+c, & \text{if } x>2 \end{cases}$$

Find a, b and c, given that the function is continuous and differentiable.

Solution

We check continuity and differentiability at x=2.

Continuity:

- $\lim_{x\to 2^-} f(x)=2^2=4$
- $\lim_{x\to 2^+} f(x)=2b+c$
- f(2)=a

Since f is continuous at x=2: $4=2b+c=a$ (1)

Differentiability:

- For $f(x)=x^2$, $f'(x)=2x$, thus $f'_-(2)=4$
- For $f(x)=bx+c$, $f'(x)=b$, thus $f'_+(2)=b$

Since f is differentiable at x=2: $b=4$ (2)

(1) and (2) give

$$a=4,\ b=4,\ c=-4$$

5.17 L'HÔPITAL's RULE (for HL)

♦ A FIRST DISCUSSION

We know that

$$\frac{0}{5}=0 \quad \text{and} \quad \frac{5}{0} \text{ is not defined}$$

However, when x tends to 0,

$$\frac{x}{5} \text{ tends to } 0 \quad \text{and} \quad \frac{5}{x} \text{ tends either to } +\infty \text{ or } -\infty$$

But what about

$$\frac{0}{0}=?$$

Consider a function of the form $\frac{f(x)}{g(x)}$

if $f(x)\to 0$, $g(x)\to 5$	then $\frac{f(x)}{g(x)}$ tends to 0	e.g $\lim_{x\to 5}\frac{x-5}{x}=\frac{0}{5}=0$

if $f(x)\to 5$, $g(x)\to 0$	then $\frac{f(x)}{g(x)}$ tends to $+\infty$ or $-\infty$	e.g $\lim_{x\to 5}\frac{x}{(x-5)^2}=+\infty$

But again, what happens when both expressions tend to 0:

$$f(x)\to 0$$
$$g(x)\to 0$$

The result could be anything:

$$0 \quad \text{or} \quad +\infty \quad \text{or} \quad -\infty \quad \text{or} \quad \text{any real number!}$$

For example, when $x\to 0$, all the functions below have the form $\frac{0}{0}$, however

$$\lim_{x\to 0}\frac{x^3}{x}=0, \quad \lim_{x\to 0}\frac{x}{x^3}=+\infty, \quad \lim_{x\to 0}\frac{\sin x}{x}=1, \quad \lim_{x\to 0}\frac{2\sin x}{x}=2 \quad \text{etc}$$

(check these functions on your GDC near x=0).

That is why we say that: $\frac{0}{0}$ is an **indeterminate form**.

In the same sense: $\frac{\infty}{\infty}$ is also an **indeterminate form**.

- **$\lim \dfrac{f(x)}{g(x)}$ IN GENERAL**

Here we deal with limits of the form

$$\lim_{x\to a}\frac{f(x)}{g(x)},\qquad \lim_{x\to+\infty}\frac{f(x)}{g(x)},\qquad \lim_{x\to-\infty}\frac{f(x)}{g(x)}$$

If the fraction is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ we can easily find the result. For example,

$$\lim_{x\to1}\frac{2x+3}{3x+5}=\frac{5}{8}\qquad \lim_{x\to+\infty}\frac{5+\frac{1}{x}}{7-\frac{1}{x}}=\frac{5}{7}\qquad \lim_{x\to3}\frac{x-3}{x^2+3}=\frac{0}{9}=0$$

$$\lim_{x\to+\infty}\frac{2}{3x+5}=0\qquad \lim_{x\to-\infty}\frac{e^x}{x+3}=0$$

$$\lim_{x\to+\infty}\frac{2x+7}{3}=+\infty\qquad \lim_{x\to-\infty}\frac{2x+7}{e^x}=-\infty$$

$$\lim_{x\to3}\frac{2}{(x-3)^2}=+\infty\qquad \lim_{x\to3}\frac{-2}{(x-3)^2}=-\infty$$

$$\lim_{x\to3^+}\frac{2}{x-3}=+\infty\qquad \lim_{x\to3^-}\frac{2}{x-3}=-\infty$$

- **THE INDETERMINATE FORMS $\frac{0}{0}$ or $\frac{\infty}{\infty}$**

In some cases the answer is easy. For example

$\frac{\infty}{\infty}$: $\quad \lim_{x\to+\infty}\dfrac{4x+7}{2x+3}=2$, $\quad \lim_{x\to+\infty}\dfrac{4x+7}{2x^2+3}=0$, $\quad \lim_{x\to+\infty}\dfrac{4x^2+7}{2x+3}=+\infty$

[remember the discussion about horizontal asymptotes]

$\frac{0}{0}$: $\quad \lim_{x\to3}\dfrac{x^2-3x}{x-3}=\lim_{x\to3}\dfrac{x(x-3)}{x-3}=\lim_{x\to3}x=3$

It is also known that

$$\lim_{x\to0}\frac{\sin x}{x}=1\qquad (\text{it is } \tfrac{0}{0})$$

For more complicated cases the following theorem helps; we simply need the derivatives of $f(x)$ and $g(x)$.

L' Hôpital's rule:

If $\begin{matrix} f(x)\to 0 \\ g(x)\to 0 \end{matrix}$ or $\begin{matrix} f(x)\to \pm\infty \\ g(x)\to \pm\infty \end{matrix}$	$\lim \dfrac{f(x)}{g(x)} = \lim \dfrac{f'(x)}{g'(x)}$

provided that $\lim \dfrac{f'(x)}{g'(x)}$ exists.

EXAMPLE 1

$$\lim_{x\to+\infty} \frac{4x+7}{2x+3} \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{x\to+\infty} \frac{(4x+7)'}{(2x+3)'} = \lim_{x\to+\infty} \frac{4}{2} = 2$$

Remember to indicate the form above = : $\overset{\left(\frac{0}{0}\right)}{=}$ or $\overset{\left(\frac{\infty}{\infty}\right)}{=}$.

Sometimes we need to apply L'Hôpital's rule more than once.

EXAMPLE 2

$$\lim_{x\to+\infty} \frac{2x^2+4x-7}{5x^2-3x+2} \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{x\to+\infty} \frac{4x+4}{10x-3} \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{x\to+\infty} \frac{4}{10} = \frac{2}{5}$$

EXAMPLE 3

$$\lim_{x\to 0} \frac{e^x-x-1}{x^2} \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0} \frac{e^x-1}{2x} \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to+\infty} \frac{e^x}{2} = \frac{1}{2}$$

EXAMPLE 4

Find the horizontal asymptotes of $f(x) = \dfrac{3e^{2x}+2}{e^{2x}-1}$

- when $x\to+\infty$

$$\lim_{x\to+\infty} \frac{3e^{2x}-2}{e^{2x}+1} \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{x\to+\infty} \frac{6e^{2x}}{2e^{2x}} = \lim_{x\to+\infty} \frac{6}{2} = 3 \qquad \text{H.A. } y=3$$

- when $x\to-\infty$, the limit is not of the form $\dfrac{\infty}{\infty}$ since $e^{2x}\to 0$.

$$\lim_{x\to-\infty} \frac{3e^{2x}-2}{e^{2x}+1} = \frac{-2}{1} = -2 \qquad \text{H.A. } y=-2$$

♦ OTHER INDETERMINATE FORMS

The following are also indeterminate forms

$$0\cdot\infty, \quad \infty-\infty, \quad 1^{\infty}$$

For example, look at the following $0\cdot\infty$ forms:

$$\lim_{x\to 0}\left(x\cdot\frac{1}{x}\right)=\lim_{x\to 0}1=1$$

$$\lim_{x\to 0}\left(x^2\cdot\frac{1}{x}\right)=\lim_{x\to 0}x=0$$

$$\lim_{x\to 0}\left(x\cdot\frac{1}{x^3}\right)=\lim_{x\to 0}\frac{1}{x^2}=+\infty$$

More complicated forms can be transformed to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and thus answered by using L'Hôpital's rule.

EXAMPLE 5

When x→0 then lnx→-∞. Thus $\lim_{x\to 0}(x\ln x)$ is of the form $0\cdot\infty$.

But

$$\lim_{x\to 0}(x\ln x)=\lim_{x\to 0}\frac{\ln x}{\frac{1}{x}}\overset{\left(\frac{\infty}{\infty}\right)}{=}\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0}(-x)=0$$

EXAMPLE 6

The limit $\lim_{x\to+\infty}\left(\sqrt{x+1}-\sqrt{2x}\right)$ is of the form $\infty-\infty$. But

$$\lim_{x\to+\infty}\left(\sqrt{x+1}-\sqrt{2x}\right)=\lim_{x\to+\infty}\frac{\left(\sqrt{x+1}-\sqrt{2x}\right)\left(\sqrt{x+1}+\sqrt{2x}\right)}{\sqrt{x+1}+\sqrt{2x}}$$

$$=\lim_{x\to+\infty}\frac{x+1-2x}{\sqrt{x+1}+\sqrt{2x}}=\lim_{x\to+\infty}\frac{1-x}{\sqrt{x+1}+\sqrt{2x}}$$

$$\overset{\left(\frac{\infty}{\infty}\right)}{=}\lim_{x\to+\infty}\frac{-1}{\frac{1}{2\sqrt{x+1}}+\frac{1}{\sqrt{2x}}}=-\infty$$

EXAMPLE 7

We will show that

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e.$$

It has the indeterminate form 1^∞. But

$$\left(1+\frac{1}{x}\right)^x = e^{\ln\left(1+\frac{1}{x}\right)^x} = e^{x\ln\left(1+\frac{1}{x}\right)}$$

Let's find the limit of the exponent $x\ln\left(1+\frac{1}{x}\right)$; it has the form $0\cdot\infty$:

$$\lim_{x\to+\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to+\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to+\infty}\frac{\frac{1}{\left(1+\frac{1}{x}\right)}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to+\infty}\frac{1}{\left(1+\frac{1}{x}\right)} = 1$$

Hence

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to+\infty} e^{x\ln\left(1+\frac{1}{x}\right)} = e^1 = e$$

NOTICE

- In a similar way we can show that $\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^x = e^a$.
- As we said earlier we accept the limit $\lim_{x\to0}\frac{\sin x}{x} = 1$ as known.

  Although it is of the form $\frac{0}{0}$ and the rule would give

  $$\lim_{x\to0}\frac{\sin x}{x} \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to0}\frac{(\sin x)'}{x'} = \lim_{x\to0}\frac{\cos x}{1} = 1$$

  we **cannot** use L'Hôpital ! This is because the proof of the fact

  $$(\sin x)' = \cos x \qquad \text{(by first principles)}$$

  already uses the limit $\lim_{x\to0}\frac{\sin x}{x} = 1$.

  The proof of this limit in particular uses other trigonometric techniques and it is beyond the scope of this course.

5.18 MORE INTEGRALS (for HL)

In the IB Math HL formula booklet you will also find the formulas

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin \frac{x}{a} + c$$

We can easily verify that the derivative of each RHS is the LHS.

For example, the derivative of f(x) = $\frac{1}{a} \arctan \frac{x}{a}$ is

$$f'(x) = \frac{1}{a} \frac{1}{1+\left(\frac{x}{a}\right)^2} \times \frac{1}{a} = \frac{1}{a^2} \frac{1}{\frac{a^2+x^2}{a^2}} = \frac{1}{a^2+x^2}$$

Later on, we will present a different proof for the last two formulas.

Let's find some integrals of these three forms.

EXAMPLE 5

$$\int (2^x + 3^x) dx = \frac{2^x}{\ln 2} + \frac{3^x}{\ln 3} + c$$

$$\int (x^5 + 5^x) dx = \frac{x^6}{6} + \frac{5^x}{\ln 5} + c$$

$$\int \frac{2^x + 4^x + 1}{2^x} dx = \int 1 + 2^x + 2^{-x} dx = x + \frac{2^x}{\ln 2} - \frac{2^{-x}}{\ln 2} + c$$

EXAMPLE 6

$$\int \frac{1}{1+x^2}dx = \arctan x + c$$

$$\int \frac{1}{4+x^2}dx = \frac{1}{2}\arctan\frac{x}{2} + c$$

$$\int \frac{5}{13+x^2}dx = \frac{5}{\sqrt{13}}\arctan\frac{x}{\sqrt{13}} + c$$

$$\int \frac{5}{9+4x^2}dx = \frac{5}{4}\int \frac{1}{\frac{9}{4}+x^2}dx = \frac{5}{4}\frac{2}{3}\arctan\frac{2x}{3} + c = \frac{5}{6}\arctan\frac{2x}{3} + c$$

In general,

$$\int \frac{a}{b+cx^2}dx = \frac{a}{c}\int \frac{1}{\frac{b}{c}+x^2}dx = \frac{a}{c}\frac{\sqrt{c}}{\sqrt{b}}\arctan\frac{\sqrt{c}x}{\sqrt{b}} + c = \frac{a}{\sqrt{bc}}\arctan\frac{\sqrt{c}x}{\sqrt{b}} + c$$

Similarly

EXAMPLE 7

$$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + c$$

$$\int \frac{1}{\sqrt{4-x^2}}dx = \arcsin\frac{x}{2} + c$$

$$\int \frac{5}{\sqrt{13-x^2}}dx = 5\arcsin\frac{x}{\sqrt{13}} + c$$

$$\int \frac{5}{\sqrt{9-4x^2}}dx = \frac{5}{2}\int \frac{1}{\sqrt{\frac{9}{4}-x^2}}dx = \frac{5}{2}\arcsin\frac{2x}{3} + c$$

In general,

$$\int \frac{a}{\sqrt{b-cx^2}}dx = \frac{a}{\sqrt{c}}\int \frac{1}{\sqrt{\frac{b}{c}-x^2}}dx = \frac{a}{\sqrt{c}}\arcsin\frac{\sqrt{c}x}{\sqrt{b}} + c$$

- **Integrals derived by the list of derivatives.**

The list of the derivatives provides some extra integrals as well.

For example, since $(tanx)' = sec^2x$, we can derive that

$$\boxed{\int sec^2xdx = tanx + c}$$

Similarly,

$$\boxed{\int cosec^2xdx = -cotx + c}$$

EXAMPLE 8

Find the integrals

$$A = \int tanxdx \qquad B = \int tan^2xdx \qquad C = \int tan^3xdx$$

Solution

$$A = \int tanxdx = \int \frac{sinx}{cosx}dx = -ln(cosx) + c$$

For B, we use the identity $tan^2x + 1 = sec^2x$:

$$B = \int tan^2xdx = \int (sec^2x - 1)dx = tanx - x + c$$

For C, we use the same identity:

$$C = \int tan^3xdx = \int tanx(sec^2x - 1)dx$$

$$= \int tanxsec^2xdx - \int tanxdx$$

For the first integral we use the substitution u= tanx. Thus,

$$C = \int \frac{tan^2x}{2}dx + ln(cosx) + c$$

- **Integrals of the form**

$$\int \frac{A}{ax^2 + bx + c}dx \quad \text{or} \quad \int \frac{A}{\sqrt{ax^2 + bx + c}}dx$$

the methodology depends on the discriminant **Δ** of the quadratic.

Look at the following examples.

The three integrals below look very similar; however the results are completely different. The difference lies in the fact that the three quadratics involved have two, exactly one, or no real roots respectively.

EXAMPLE 9

Consider

$$I_1 = \int \frac{2}{x^2-4x+3}dx \qquad I_2 = \int \frac{2}{x^2-4x+4}dx \qquad I_3 = \int \frac{2}{x^2-4x+5}dx$$

Hints:

For I_1, use partial fractions: $\frac{2}{x^2-4x+3} = \frac{1}{x-3} - \frac{1}{x-1}$.

For I_2, use the fact that the quadratic is a perfect square.

For I_3, use the vertex form of the quadratic: $x^2-4x+5=(x-2)^2+1$

Solution

$$I_1 = \int\left(\frac{1}{x-3} - \frac{1}{x-1}\right)dx = \ln|x-3| - \ln|x-1| + c$$

$$I_2 = \int \frac{2}{(x-2)^2}dx = \int 2(x-2)^{-2}dx = -\frac{2}{x-2} + c$$

$$I_3 = \int \frac{2}{(x-2)^2+1}dx = 2\arctan(x-2) + c$$

EXAMPLE 10

Use the vertex form of $-x^2+4x-3$ to find

$$I = \int \frac{2}{\sqrt{-x^2+4x-3}}dx$$

Solution

The vertex form is $-(x-2)^2+1$.

Thus

$$I = \int \frac{2}{\sqrt{1-(x-2)^2}}dx = 2\arcsin(x-2) + c$$

5.19 FURTHER INTEGRATION BY SUBSTITUTION (for HL)

We have seen the simple case of substitution u=f(x), when the derivative of f(x) is part of the integrand.

For example,

$$I=\int \frac{f'(x)}{f(x)}dx = \ln(f(x)) + c$$

Indeed,

Let $u = f(x)$

Then $\int \frac{du}{dx} = f'(x) \Rightarrow dx = \frac{du}{f'(x)}$

and

$$I=\int \frac{f'(x)}{u}\frac{du}{f'(x)} = \int \frac{1}{u}du = \ln u + c = \ln(f(x)) + c$$

In this case we can omit the whole process and give directly the result.

EXAMPLE 1

Find

$$A=\int \frac{\cos x}{\sin x+1}dx, \qquad B=\int \frac{x}{x^2+1}dx$$

Solution

Since the derivative of $u=\sin x+1$ is $\cos x$

$$A=\int \frac{\cos x}{\sin x+1}dx = \ln(\sin x+1)$$

For B, the derivative of $u=x^2+1$ is $2x$.

We slightly modify the integral to obtain the form $\int \frac{f'(x)}{f(x)}dx$

$$B=\int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{2x}{x^2+1}dx = \frac{1}{2}\ln(x^2+1)+c$$

Similarly,

$$\int e^{f(x)} \times f'(x)dx = e^{f(x)} + c$$

$$\int \sin f(x) \times f'(x)\, dx = -\cos f(x) + c$$

$$\int \cos f(x) \times f'(x)\, dx = \sin f(x) + c$$

$$\int f(x)^n \times f'(x)\, dx = \frac{f(x)^{n+1}}{n+1} + c$$

etc.

EXAMPLE 2

$$\int \sin x\, e^{\cos x} dx = e^{\cos x} + c$$

$$\int x^2 e^{x^3+5} dx = \frac{1}{3}\int 3x^2 e^{x^3+5} dx = \frac{1}{3} e^{x^3+5} + c$$

$$\int x\sin(7x^2+3)dx = \frac{1}{14}\int 14x\sin(7x^2+3)dx = -\frac{1}{14}\cos(7x^2+3) + c$$

$$\int 5x(x^2+3)^5 dx = \frac{5}{2}\int 2x(x^2+3)^5 dx = \frac{5}{2}\frac{(x^2+3)^6}{6} + c = \frac{5}{12}(x^2+3)^6 + c$$

$$\int 3x\sqrt{x^2+3}dx = \frac{3}{2}\int 2x(x^2+3)^{\frac{1}{2}} dx = \frac{3}{2}\frac{2}{3}(x^2+3)^{3/2} + c = (x^2+3)^{3/2} + c$$

$$\int \frac{(\ln x)^3}{2x} dx = \frac{1}{2}\int \frac{1}{x}(\ln x)^3 dx = \frac{1}{2}\frac{(\ln x)^4}{4} + c = \frac{(\ln x)^4}{8} + c$$

Remark.

If you don't feel confident to find directly the result, you may always follow the detailed procedure of substitution.

We are going to see two more cases of substitution.

CASE 1: We reuse u=f(x) to get rid of all remaining x's.

Consider

$$I=\int x^3\sqrt{x^2+3}dx$$

Let's use the substitution

$$u=x^2+3$$

(despite the fact that x^3 is not the derivative of u!)

Then

$$\frac{du}{dx}=2x \Rightarrow dx=\frac{du}{2x}$$

Thus,

$$I=\int x^3\sqrt{u}\,\frac{du}{2x}=\frac{1}{2}\int x^2\sqrt{u}du$$

Well, the result still contains x^2, but

$$u=x^2+3 \Rightarrow x^2=u-3$$

and thus

$$I=\frac{1}{2}\int (u-3)\sqrt{u}du$$

Therefore,

$$\begin{aligned} I &= \frac{1}{2}\int (u\sqrt{u}-3\sqrt{u})du \\ &= \frac{1}{2}\int (u^{3/2}-3u^{1/2})du \\ &= \frac{1}{2}\left(\frac{2}{5}u^{5/2}-3\frac{2}{3}u^{3/2}\right)+c \\ &= \frac{1}{5}u^{5/2}-u^{3/2}+c \\ &= \frac{1}{5}(x^2+3)^{5/2}-(x^2+3)^{3/2}+c \end{aligned}$$

Characteristic examples of this case are integrals of the form

$$\int\frac{polynomial}{ax+b}dx\,,\ \int\frac{polynomial}{(ax+b)^n}dx$$

where we let $u=ax+b$

EXAMPLE 3

Find $I=\int \frac{x^2}{x+2}dx$

Solution

Let $u=x+2$ (so that $x=u-2$)

Then $\frac{du}{dx}=1 \Rightarrow dx=du$

Thus,

$$I=\int \frac{x^2}{u}du=\int \frac{(u-2)^2}{u}du=\int \frac{u^2-4u+4}{u}du$$

$$=\int (u-4+\frac{4}{u})du=\frac{u^2}{2}-4u+4\ln u+c$$

$$=\frac{(x+2)^2}{2}-4(x+2)+4\ln(x+2)+c$$

Another popular substitution of this kind is $u=e^x$ due to its simple derivative. When you see rational expressions containing e^x, think of this substitution!

EXAMPLE 4

Find $I_1=\int \frac{e^x}{e^{2x}+4}dx$ $I_2=\int \frac{e^{2x}}{e^x+4}dx$

Solution

For both integrals:

let $u=e^x$, then $\frac{du}{dx}=e^x \Rightarrow dx=\frac{du}{e^x}$

Then

$$I_1=\int \frac{u}{u^2+4}\frac{dx}{e^x}=\int \frac{du}{u^2+4}=\frac{1}{2}\arctan\frac{u}{2}+c=\frac{1}{2}\arctan\frac{e^x}{2}+c$$

while

$$I_2=\int \frac{u^2}{u+4}\frac{dx}{e^x}=\int \frac{u}{u+4}du=\int \frac{u+4-4}{u+4}du=\int 1-\frac{4}{u+4}du$$

$$=u-4\ln\left|u+4\right|+c=e^x-4\ln\left|e^x+4\right|+c$$

- **CASE 2: Let x=(expression of u) [instead of u=(expression of x)]**

In this case, the substitution is not very obvious and it is usually given in an exam!

We will see two characteristic substitutions of this kind.

EXAMPLE 5

For $I=\int \frac{dx}{x^2+4}$, we use the substitution $x=2\tan u$ (so $u=\arctan\frac{x}{2}$)

We have, $\frac{dx}{du}=2\sec^2 u \Rightarrow dx=2\sec^2 u\,du$

Thus,

$$I=\int \frac{2\sec^2 u\,du}{4\tan^2 u+4}=\frac{1}{2}\int \frac{\sec^2 u\,du}{\tan^2 u+1}=\frac{1}{2}\int \frac{\sec^2 u\,du}{\sec^2 u}=\frac{1}{2}\int du$$

$$=\frac{1}{2}u+c=\frac{1}{2}\arctan\frac{x}{2}+c$$

EXAMPLE 6

For $I=\int \frac{dx}{\sqrt{4-x^2}}$, we use the substitution $x=2\sin u$ (so $u=\arcsin\frac{x}{2}$)

We have, $\frac{dx}{du}=2\cos u \Rightarrow dx=2\cos u\,du$

Thus,

$$I=\int \frac{2\cos u\,du}{\sqrt{4-4\sin^2 u}}=\frac{2}{2}\int \frac{\cos u\,du}{\sqrt{1-\sin^2 u}}=\int \frac{\cos u}{\cos u}=\int du$$

$$=u+c=\arcsin\frac{x}{2}+c$$

In general, the two formulas (of the formula booklet)

$$\int \frac{1}{a^2+x^2}dx=\frac{1}{a}\arctan\frac{x}{a}+c \qquad \int \frac{1}{\sqrt{a^2-x^2}}dx=\arcsin\frac{x}{a}+c$$

can be shown by suing the substitutions $x=a\tan u$ and $x=a\sin u$ respectively.

We can see these two substitutions in similar cases:

see expression	use substitution
a^2+x^2	$x=a\tan\theta$
$\sqrt{a^2-x^2}$	$x=a\sin\theta$

EXAMPLE 7

Find $I=\int\sqrt{16-x^2}dx$, by using the substitution $x=4\sin\theta$

Solution.

We have,

$$\frac{dx}{du}=4\cos\theta \Rightarrow dx=4\cos\theta d\theta$$

Thus,

$$I=\int\sqrt{16-16\sin^2\theta}\ 4\cos\theta d\theta=16\int\sqrt{1-\sin^2\theta}\cos\theta d\theta=16\int\cos^2\theta d\theta$$

We use the double angle identity

$$\cos^2\theta=\frac{\cos2\theta+1}{2}$$

Thus

$$I=8\int(\cos2\theta+1)d\theta=8\left(\frac{\sin2\theta}{2}+\theta\right)+c=4\sin2\theta+8\theta+c$$

But

$$\theta=\arcsin\frac{x}{4}$$

and

$$4\sin2\theta=8\sin\theta\cos\theta=2x\cos\theta$$

$$=2x\sqrt{1-\sin^2\theta}=2x\sqrt{1-\frac{x^2}{16}}=\frac{x}{2}\sqrt{16-x^2}$$

Therefore,

$$I=\frac{x}{2}\sqrt{16-x^2}+8\arcsin\frac{x}{4}+c$$

5.20 INTEGRATION BY PARTS (for HL)

♦ DISCUSSION

In this paragraph we study integrals of the form

$$I=\int (f\cdot g)dx$$

Let's make it clear from the very beginning that there is no "product rule" for integrals. It is sometimes very difficult, and more often impossible, to find the indefinite integral of a product. However, we exploit the product rule for differentiation

$$(u\cdot v)' = u'\cdot v + u\cdot v'$$

which gives

$$u\cdot v' = (u\cdot v)' - u'\cdot v$$

If we apply integration on both sides we obtain the so-called **Integration by parts formula**

$$\boxed{\int u\cdot v' dx = uv - \int u'\cdot v dx}$$

This formula does not give an answer for any product but in some cases the integral in the RHS is much easier than the original and thus we obtain a result.

♦ THE METHOD

Consider

$$I=\int xe^x dx$$

Since $(e^x)'=e^x$ this integral can be expressed as

$$I=\int x(e^x)' dx$$

The integration by parts formula gives

$$I = xe^x - \int x'e^x dx \quad = xe^x - \int e^x dx = xe^x - e^x + c$$

[you may easily confirm that the derivative of the result gives xe^x]

In fact, the process is even quicker. We integrate one of the factors and then we differentiate the other factor as follows

$$\int \underbrace{u\cdot v'}_{\text{integral}}dx = u\cdot v - \int u'\cdot v\,dx$$

derivative

For example, in I=$\int xe^x dx$ we integrate $\underline{\underline{e^x}}$ and then differentiate $\underline{x}$

$$\int x\underline{\underline{e^x}}dx = \underline{x}e^x - \int e^x dx = xe^x - e^x + c$$

NOTICE

- Many students find it helpful to determine explicitly u and v ′.

  In the example above: $u = x \quad\longrightarrow\quad u' = 1$

  $v' = e^x \qquad v = e^x$

  and then find $u\cdot v - \int u'\cdot v\,dx = \underline{x}e^x - \int e^x dx$
- If we try to integrate the other factor, that is $\underline{\underline{x}}$, we obtain

  $$\int \underline{\underline{x}}e^x dx = \frac{x^2}{2}\underline{e^x} - \int \frac{x^2}{2}e^x dx$$

  The result of course is not wrong, but it is not practical!

  The second integral is worse than the original!

But how do we choose the factor v′ that we integrate first?
We follow the priority list below

Priority list for v' (i.e. for integration)

1. e^x, sinx, cosx
2. x^n (or polynomials)
3. lnx, arctanx, arcsinx, arccosx

In the following easy examples we indicate by a double underscore the factor we integrate and by a single underscore the function we differentiate.

EXAMPLE 1

(a) $\int x\cos x dx = ?$ [**cosx** has a priority for integration against **x**]

$$\int x\underline{\underline{\cos x}}dx = \underline{x}\sin x - \int \sin x dx = x\sin x + \cos x + c$$

(b) $\int \underline{\underline{e^x}}(2x+5)dx = e^x(\underline{2x+5}) - \int 2e^x dx$

$= e^x(2x+5) - 2e^x + c$

$= 2xe^x + 3e^x + c$

(c) $\int x^2\underline{\underline{e^x}}dx = \underline{x^2}\, e^x - \int 2xe^x dx$

$= x^2 e^x - 2\int x\underline{\underline{e^x}}dx$ [we repeat the rule]

$= x^2 e^x - 2[\underline{x}e^x - \int e^x dx]$

$= x^2 e^x - 2xe^x + 2e^x + c$

(d) $\int \underline{\underline{x^2}}\ln x dx = \frac{1}{3}x^3\underline{\ln x} - \frac{1}{3}\int x^3\frac{1}{x}dx$

$= \frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2 dx$

$= \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + c$

(e) $I = \int \ln x dx = ?$ Well, we do not see any product here!

But this can be written as

$$\int 1\cdot\ln x dx$$

and $1 = x^0$ has a priority for v' against $\ln x$. Then

$$I = \int \underline{\underline{1}}\cdot\ln x dx = x\underline{\ln x} - \int x\cdot\frac{1}{x}dx = x\ln x - \int dx = x\ln x - x + c$$

In the following example we calculate the integral

$$I = \int e^x \sin x dx.$$

Notice that both factors lie in the first priority. You may choose any factor you like for v' (I bet you will choose e^x !!)

The result is quite interesting!

EXAMPLE 2

Find $I = \int e^x sinx dx$

Solution

We choose e^z for integration:

$$I = \int e^x sinxdx = e^x sinx - \int e^x cosxdx \quad \text{[we carry on; choose again } e^x\text{]}$$

$$= e^x sinx - \left[e^x cosx + \int e^x sinxdx\right]$$

$$= e^x sinx - e^x cosx - \int e^x sinxdx$$

In the final expression we obtain again the original integral I, which seems to lead to a dead-end. But in fact we have

$$I = e^x sinx + e^x cosx - I$$

We solve for I and obtain

$$2I = e^x sinx + e^x cosx$$

and finally

$$I = \frac{e^x sinx + e^x cosx}{2} + c$$

♦ **FURTHER OBSERVATIONS**

In the priority list above we may have some variations of the factors

- Instead of e^x you may have e^{ax}, or any exponential a^x
- Instead of sinx, cosx you may have sin(ax), cos(bx)
- Instead of lnx you may have any logarithm $log_a x$

EXAMPLE 3

$$I = \int x^2 e^{3x} dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x e^{3x} dx$$

$$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left[\frac{1}{3}x e^{3x} - \frac{1}{3}\int e^{3x} dx\right]$$

$$= \frac{1}{3}x^2 e^{3x} - \frac{2}{9}x e^{3x} + \frac{2}{27}e^{3x} + c$$

♦ USUAL CASES OF INTEGRATION BY PARTS

General Form	Examples	Theoretical Questions
$I_n = \int x^n e^x dx$ $I_{n,m} = \int x^n e^{mx} dx$	$\int x^3 e^x dx$, $\int x^2 e^{3x} dx$	Express I_n in terms of I_{n-1} Hence find $I_0, I_1, I_2, \ldots$
$I_n = \int x^n \cos x dx$ $I_n = \int x^n \sin x dx$	$\int x^2 \cos x dx$	Express I_n in terms of I_{n-2}
$I_{n,m} = \int x^n \cos(mx) dx$ $I_{n,m} = \int x^n \sin(mx) dx$	$\int x^2 \cos 3x dx$	
$I_n = \int x^n \ln x dx$	$\int x^5 \ln x dx$, $\int \frac{\ln x}{x^5} dx$ $\int \sqrt{x} \ln x dx$	Find a general formula for I_n
$I_{n,m} = \int e^{nx} \sin(mx) dx$ $I_{n,m} = \int e^{nx} \cos(mx) dx$	$\int e^{3x} \sin 2x dx$ $\int e^{-x} \sin 2x dx$	Find a general formula for $I_{n,m}$
$I_n = \int \cos^n x dx$ $I_n = \int \sin^n x dx$	$\int \cos^2 x dx$, $\int \cos^3 x dx$	Express I_n in terms of I_{n-2} Hence find I_2, I_4 and I_3, I_5
$I_{n,m} = \int \sin(nx)\cos(mx) dx$	$\int \sin 2x \cos 3x dx$	Find a general formula for $I_{n,m}$
$I_n = \int x^n \arctan x dx$ $I_n = \int x^n \arcsin x dx$ $I_n = \int x^n \arccos x dx$	$\int \arctan x dx$, $\int x \arctan x dx$, $\int x^2 \arctan x dx$ $\int \arcsin x dx$, $\int x^2 \arcsin x dx$	
$I_n = \int (\ln x)^n dx$	$\int (\ln x)^2 dx$, $\int (\ln x)^3 dx$	

EXAMPLE 4

If $I_n = \int \cos^n x dx$, express I_n in terms of I_{n-2}

$$I_n = \int \cos x \cos^{n-1} x dx = \sin x \cos^{n-1} x + (n-1)\int \sin^2 x \cos^{n-2} x dx$$
$$= \sin x \cos^{n-1} x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x dx$$
$$= \sin x \cos^{n-1} x + (n-1)\int (\cos^{n-2} x - \cos^n x) dx$$
$$= \sin x \cos^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow \quad I_n + (n-1) I_n = \sin x \cos^{n-1} x + (n-1) I_{n-2}$$

$$\Rightarrow \quad n I_n = \sin x \cos^{n-1} x + (n-1) I_{n-2}$$

Thus

$$I_n = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}$$

♦ INTEGRATION BY PARTS FOR DEFINITE INTEGRALS

Again, as in the case of substitution, it would be safe to find the indefinite integral first and then proceed to the definite integral! Otherwise the integration by parts for definite integrals takes the form

$$\boxed{\int_a^b f'\cdot g\,dx = [f\cdot g]_a^b - \int_a^b f\cdot g'\,dx}$$

EXAMPLE 5

Find

$$I=\int_0^2 e^x(2x+5)dx$$

Solution

Method A: we find the indefinite integral first

$$\int e^x(2x+5)dx = 2xe^x+3e^x+c \qquad \text{[example 1(b) above]}$$

Therefore,

$$I=\left[2xe^x+3e^x\right]_0^2=(4e^2+3e^2)-(0+3)=7e^2-3$$

Method B: we keep the boundaries

$$I=\int_0^2 e^x(2x+5)dx = \left[e^x\cdot(2x+5)\right]_0^2-\int_0^2 e^x\cdot 2dx$$

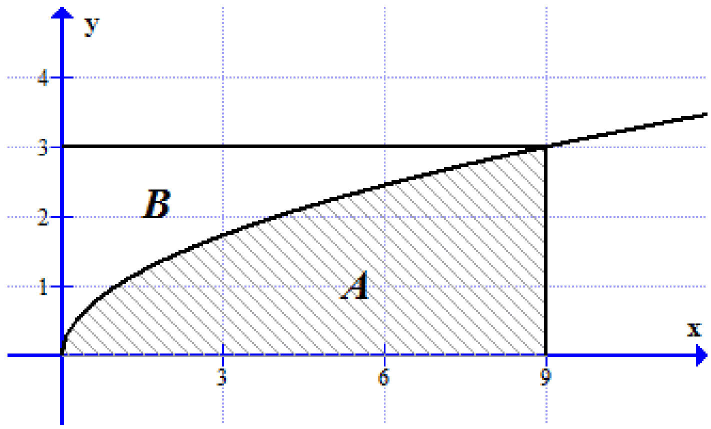
$$=(9e^2-5)-\left[2e^x\right]_0^2$$

$$=(9e^2-5)-(2e^2-2)=7e^2-3$$

5.21 FURTHER AREAS BETWEEN CURVES – VOLUMES (for HL)

Consider the curve

$$y=\sqrt{x}$$



We found in paragraph 5.10 that

$$A=\int_0^9 \sqrt{x}dx=18 \qquad \text{and} \qquad B=\int_0^9 (3-\sqrt{x})dx=9$$

An alternative way to measure the area is to move in y-axis:

- consider the functions as x in terms of y
- let y move in y-axis (instead of x in x-axis).

Thus we have two formulas for the area

about x-axis	about y-axis
Area= $\int_a^b ydx$	Area= $\int_a^b xdy$

In our example above,

$$y=\sqrt{x} \Rightarrow x=y^2$$

and y ranges between 0 and 3:

$$B=\int_0^3 y^2 dy=\left[\frac{y^3}{3}\right]_0^3=9-0=9$$

$$A=\int_0^3 (9-y^2)dy==\left[9y-\frac{y^3}{3}\right]_0^3=(27-9)-0=18$$

NOTICE

If $y=f(x) \Rightarrow x=f^{-1}(y)$ the alternative formula

$$\text{Area}=\int_a^b x\,dy = \int_a^b f^{-1}(y)\,dy$$

can be written as

$$\text{Area}=\int_a^b f^{-1}(x)\,dx$$

In other words, we can write the inverse function in terms of y (followed by dy) or in terms of x as usual (followed by dx).

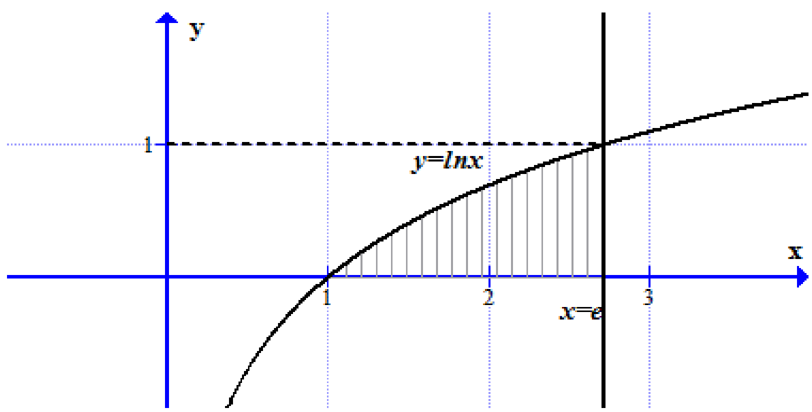
In our example above

$$\int_0^3 y^2 dy \quad \text{and} \quad \int_0^3 x^2 dx$$

are exactly the same.

EXAMPLE 1

Find the area among y=lnx, x-axis and the line x=e.



About x-axis:

$$A=\int_1^e \ln x\,dx=\left[x\ln x-x\right]_1^e=(e\ln e-e)-(1\ln 1-1)=1$$

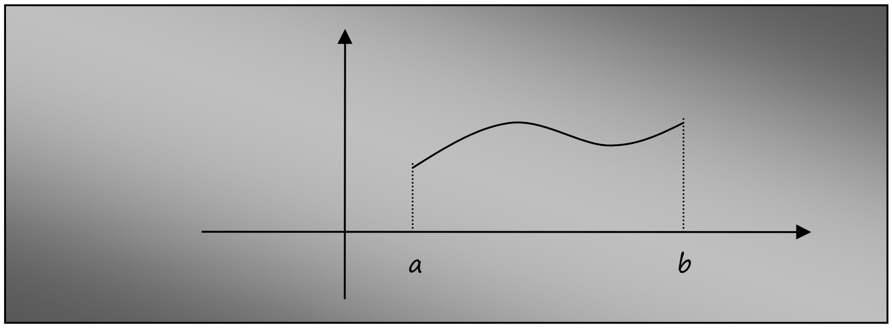
About y-axis:

$$y=\ln x \Rightarrow x=e^y$$

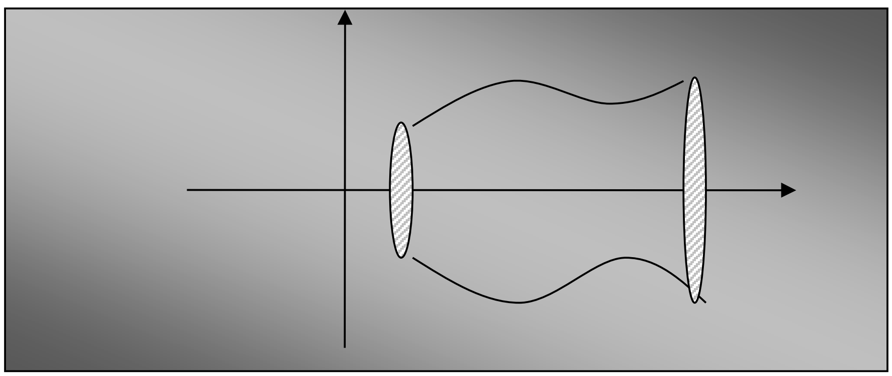
Thus

$$A=\int_0^1 (e-e^y)dy=\left[ey-e^y\right]_0^1=(e-e)-(0-1)=1$$

- **VOLUME OF REVOLUTION**



If we rotate the graph of $y=f(x)$, 360^o about the x-axis, we will obtain a 3-D shape as follows:



The volume of this shape is directly given by (the proof is omitted!)

$$V=\pi\int_a^b y^2 dx$$

If we have two curves

$$y_1=f_1(x) \quad \text{and} \quad y_2=f_2(x)$$

with $f_1(x) \geq f_2(x) \geq 0$

the solid generated when the region between the two curves is revolved 360^o about x-axis is given by

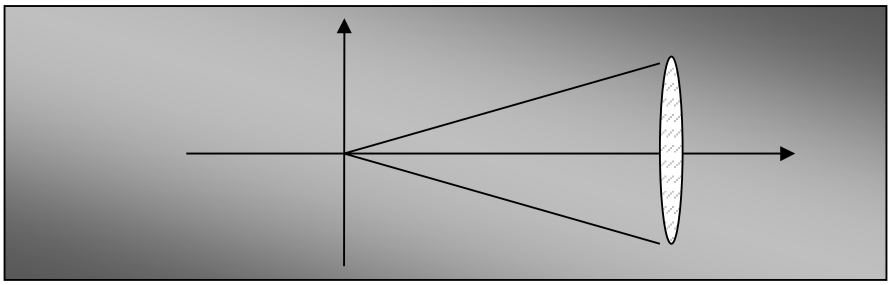
$$V=\pi\int_a^b (y_1{}^2 - y_2{}^2)dy$$

EXAMPLE 2

Consider the segment of the straight line

$$y=\frac{1}{4}x\text{, where } 0\le x\le 4.$$

Find the volume of the cone generated by a 360^o rotation of this segment.



$$V = \pi\int_a^b y^2dx = \pi\int_0^4 (\frac{1}{4}x)^2 dx$$

$$=\frac{\pi}{16}\int_0^4 x^2dx=\frac{\pi}{16}\left[\frac{x^3}{3}\right]_0^4$$

$$=\frac{\pi}{16}\frac{4^3}{3}=\frac{4\pi}{3}$$

[Notice that the known formula for the volume of the cone of height h=4 and radius r=1 gives:

$$V=\frac{1}{3}\pi r^2h=\frac{4\pi}{3}$$

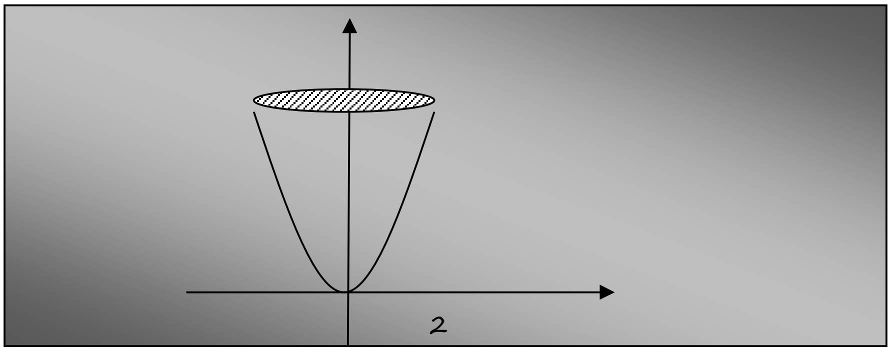
as expected!

If the rotation takes place about the vertical axis (y-axis), then we solve $y=f(x)$ for x, (hence $x=f^{-1}(y)$), and the formula now is

$$\boxed{V=\pi\int_a^b x^2dy}$$

Notice that the solids generated when a curve is rotated in x-axis or in y-axis are completely different.

EXAMPLE 3



If we rotate the parabola $y=x^2$ about the y-axis we obtain the 3D shape above! Find the volume if the height is 4.

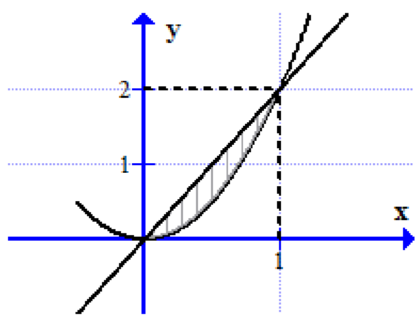
We solve for x, so that $x=\sqrt{y}$.

The volume is

$$V=\pi\int_a^b x^2 dy=\pi\int_0^4 x^2 dy=\pi\int_0^4 y dy=\pi\left[\frac{y^2}{2}\right]_0^4=8\pi$$

EXAMPLE 4

Consider the region between the curves $y=2x^2$ and $y=2x$. The two curves intersect at $x=0$ and $x=1$.



If the region is rotated 2π about x-axis, the volume generated is

$$V=\pi\int_0^1(4x^2-4x^4)dy=4\pi\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_0^1=4\pi\left[\frac{1}{3}-\frac{1}{5}\right]=\frac{8\pi}{15}$$

If the region is rotated 2π about y-axis, the volume generated is

$$V=\pi\int_0^2\left(\frac{y}{2}-\frac{y^2}{4}\right)dy=\pi\left[\frac{y^2}{4}-\frac{y^3}{12}\right]_0^2=\pi\left[1-\frac{2}{3}\right]=\frac{\pi}{3}$$

5.22 DIFFERENTIAL EQUATIONS (for HL)

You already know some simple differential equations:

Find f(x) given that

$$f'(x) = 8x^3, \qquad \text{with } f(1) = 5$$

The problem is usually expressed as: Solve the differential equation

$$\frac{dy}{dx} = 8x^3, \qquad \text{with } y=5 \text{ when } x=1$$

The solution is an expression of y in terms of x.

Clearly, by simple integration we find

$$y = 2x^4 + c$$

The condition y=5 when x=1 (or otherwise y(1)=5) gives

$$c=3$$

Thus the final answer is

$$y = 2x^4 + 3$$

The terminology we use is

$\frac{dy}{dx} = 8x^3$	**differential equation**
$y(1)=5$	**boundary condition**
$y = 2x^4 + c$	**general solution**
$y = 2x^4 + 3$	**particular solution**

Well, a **differential equation** in general relates a function y=f(x) with its derivatives, i.e. it involves

$$x,\ y,\ y',\ y'',\ \text{etc.}$$

Our task is to find the functions y=f(x) that satisfy this relation.

For example,

$$2y''+y'-3y=3x-1$$

is a differential equation and

$$y=e^{x}-x$$

is a particular solution. Indeed,

$$y'=e^{x}-1$$
$$y''=e^{x}$$

and

$$\begin{aligned} LHS &= 2e^{x}+(e^{x}-1)-3(e^{x}-x) \\ &= 2e^{x}+e^{x}-1-3e^{x}+3x \\ &= 3x-1 \\ &= RHS \end{aligned}$$

Perhaps there are more solutions! Our task is to find the whole family of functions y=f(x) that satisfy the differential equation.

This is a 2nd order differential equation as it involves the derivatives up to y''. In general,

the **order of a D.E.** is the order of the highest derivative.

Here we only deal with **1st order D.E.** that is our equations involve only

$$x,\ y \text{ and } \frac{dy}{dx}$$

We will investigate only 3 particular cases of **1st order D.E.**

- D.E. of separable variables
- Homogeneous D.E.
- Linear D.E. (with integrating factor)

We will also present a numerical method (approximation) for D.E.

- Euler's method

♦ D.E. OF SEPARABLE VARIABLES

A differential equation is of **separable variables** if we can separate x's from y's and express the equation in the form

$$f(y)dy=g(x)dx$$

Then we integrate both parts and get the result

$$\int f(y)dy = \int g(x)x$$

EXAMPLE 1

Solve the D.E. $\frac{dy}{dx}=4xy^2$, given that $y(1)=2$.

Solution

We separate the variables:

$$\frac{dy}{dx}=4xy^2 \Rightarrow \frac{dy}{y^2}=4xdx$$

$$\Rightarrow \int\frac{dy}{y^2}=\int 4xdx$$

$$\Rightarrow -\frac{1}{y}= 2x^2+c$$

$$\Rightarrow y=-\frac{1}{2x^2+c} \qquad \text{[general solution]}$$

The boundary condition gives

$$y(1)=2 \Rightarrow -\frac{1}{2+c}=2 \Rightarrow -1=4+2c \Rightarrow c=-\frac{5}{2}$$

Therefore,

$$y=-\frac{1}{2x^2-\frac{5}{2}}$$

or equivalently

$$y=\frac{2}{5-4x^2} \qquad \text{[particular solution]}$$

Even the simple example $\frac{dy}{dx}=8x^3$ in the introduction can be seen as a D.E. of separable variables:

$$\frac{dy}{dx}=8x^3 \Rightarrow dy=8x^3dx \Rightarrow \int dy=\int 8x^3dx \Rightarrow y=2x^4+c$$

The evaluation of the constant c can be done in an earlier step.

EXAMPLE 2

Solve the D.E. $\frac{dy}{dx} = xy^2 + x$, given that y(0)=1.

Solution

$$\frac{dy}{dx} = xy^2 + x \Rightarrow \frac{dy}{dx} = x(y^2 + 1)$$

$$\Rightarrow \frac{dy}{y^2+1} = xdx$$

$$\Rightarrow \int \frac{dy}{y^2+1} = \int xdx$$

$$\Rightarrow \arctan y = \frac{x^2}{2} + c \qquad \text{[general solution]}$$

Since y(0)=1,

$$\arctan 1 = c \Rightarrow c = \frac{\pi}{4}$$

Therefore,

$$\arctan y = \frac{x^2}{2} + \frac{\pi}{4}$$

$$\Rightarrow y = \tan\left(\frac{x^2}{2} + \frac{\pi}{4}\right) \qquad \text{[particular solution]}$$

A common problem in this category is to find a population P where the rate of increase (or decrease) is proportional to the population itself:

$$\frac{dP}{dt} = kP$$

By separating variables

$$\frac{dP}{P} = kdt \Rightarrow \int \frac{dP}{P} = \int kdt \Rightarrow \ln P = kt + c \Rightarrow P = e^{kt+c} = e^{kt}e^{c}$$

By setting $P_o = e^c$ (initial population) we obtain

$$P = P_o e^{kt}$$

that is an exponential growth for the population in terms of time.

♦ HOMOGENEOUS D.E.

A differential equation is called **homogeneous** if it can take the form

$$\frac{dy}{dx}=F\left(\frac{y}{x}\right)$$

[I.e. the RHS is a function of $\frac{y}{x}$]

It is not so difficult to recognize the homogeneous D.E.

For example,

$$\frac{dy}{dx}=\frac{x^2-xy+y^2}{x^2} \quad \text{becomes} \quad \frac{dy}{dx}=1-\frac{y}{x}+\left(\frac{y}{x}\right)^2$$

$$\frac{dy}{dx}=\frac{x^2y+y^3}{x^3} \quad \text{becomes} \quad \frac{dy}{dx}=\frac{y}{x}+\left(\frac{y}{x}\right)^3$$

$$\frac{dy}{dx}=\frac{x^2+y^2}{xy} \quad \text{becomes} \quad \frac{dy}{dx}=\frac{\frac{x^2+y^2}{x^2}}{\frac{xy}{x^2}} \quad \Rightarrow \quad \frac{dy}{dx}=\frac{1+\left(\frac{y}{x}\right)^2}{\frac{y}{x}}$$

$$\frac{dy}{dx}=\frac{x^3+2y^3}{x^2y-3x^3} \quad \text{becomes} \quad \frac{dy}{dx}=\frac{\frac{x^3+2y^3}{x^3}}{\frac{x^2y-3x^3}{x^3}} \quad \Rightarrow \quad \frac{dy}{dx}=\frac{1+2\left(\frac{y}{x}\right)^3}{\frac{y}{x}-3}$$

Then we use the substitution

$$v=\frac{y}{x}\Rightarrow y=xv$$

$$\frac{dy}{dx}=v+x\frac{dv}{dx} \qquad \text{[by using product rule]}$$

and the D.E. $\frac{dy}{dx}=F\left(\frac{y}{x}\right)$ takes the form

$$v+x\frac{dv}{dx}=F(v)$$

which is always a D.E. of separable variables (v in terms of x)

[can you see why?].

EXAMPLE 3

Find the general solution of the D.E.

$$x^2\frac{dy}{dx}=x^2-xy+y^2.$$

Solution

$$x^2\frac{dy}{dx}=x^2-xy+y^2 \Rightarrow \frac{dy}{dx}=\frac{x^2-xy+y^2}{x^2} \Rightarrow \frac{dy}{dx}=1-\frac{y}{x}+\left(\frac{y}{x}\right)^2$$

Let $v=\frac{y}{x} \Rightarrow y=vx$

Then $\frac{dy}{dx}=v+x\frac{dv}{dx}$

Thus

$$v+x\frac{dv}{dx}=1-v+v^2 \quad \Rightarrow x\frac{dv}{dx}=1-2v+v^2$$

$$\Rightarrow x\frac{dv}{dx}=(v-1)^2$$

$$\Rightarrow \frac{dv}{(v-1)^2}=\frac{dx}{x}$$

$$\Rightarrow \int\frac{dv}{(v-1)^2}=\int\frac{dx}{x}$$

$$\Rightarrow -\frac{1}{v-1}=\ln|x|+c$$

$$\Rightarrow v-1=-\frac{1}{\ln|x|+c}$$

$$\Rightarrow v=1-\frac{1}{\ln|x|+c}$$

Finally, since $v=\frac{y}{x}$,

$$\frac{y}{x}=1-\frac{1}{\ln|x|+c} \Rightarrow y=x\left(1-\frac{1}{\ln|x|+c}\right)$$

♦ 1st ORDER LINEAR D.E. (WITH INTEGRATING FACTOR)

The last category contains the D.E. of the form

$$\frac{dy}{dx}+P(x)y=Q(x)$$

where P(x) and Q(x) are functions of x only.

Methodology:

- we spot P(x) and Q(x)
- we find the so called **integrating factor** $\mathrm{I}=e^{\int P(x)dx}$ (ignore +c)
- It holds $\mathrm{I}y=\int \mathrm{I}Q dx$
- We calculate the integral in the RHS and solve for y

We will demonstrate the procedure by using an easy example and then we will explain why this method works!

EXAMPLE 4

Find the general solution of the D.E.

$$\frac{dy}{dx}+\frac{2}{x}y=5x^2.$$

Solution

$$P(x)=\frac{2}{x} \quad \text{and} \quad Q(x)=5x^2.$$

The integrating factor is

ignore +c

$$\mathrm{I}=e^{\int P(x)dx}=e^{\int \frac{2}{x}dx}=e^{2\ln x}=x^2$$

Then

$$\mathrm{I}y=\int \mathrm{I}Q dx \Rightarrow x^2y=\int 5x^4dx=x^5+c$$

Therefore

$$y=x^3+\frac{c}{x^2}$$

Explanation:

Differential equation: $\frac{dy}{dx}+P(x)y=Q(x)$

Integrating factor: $\mathrm{I}=e^{\int P(x)dx}$

Notice that $\frac{d\mathrm{I}}{dx}=\frac{d}{dx}\left(e^{\int P(x)dx}\right)=\left(e^{\int P(x)dx}\right)P(x)=\mathrm{I}P(x)$

If we multiply the D.E. by I we obtain

$$\mathrm{I}\frac{dy}{dx}+\mathrm{I}P(x)y=\mathrm{I}Q(x)$$

But the LHS is the derivative of the product $\mathrm{I}y$ [can you see why?]

Thus

$$\frac{d}{dx}(\mathrm{I}y)=\mathrm{I}Q(x)$$

$$\Rightarrow \mathrm{I}y=\int \mathrm{I}Q(x)dx$$

Provided that the last integral is easy to find we can solve for y and obtain the result.

Based on this explanation, let us provide a more analytical solution for the D.E. in **Example 4**:

$$\frac{dy}{dx}+\frac{2}{x}y=5x^2.$$

The integrating factor is

$$\mathrm{I}=e^{\int P(x)dx}=e^{\int \frac{2}{x}dx}=e^{2\ln x}=x^2$$

We multiply the equation by x^2:

$$x^2\frac{dy}{dx}+2xy=5x^4 \;\Rightarrow\; \frac{d}{dx}(x^2y)=5x^4$$

Thus

$$x^2y=\int 5x^4dx=x^5+c \quad\Rightarrow\quad y=x^3+\frac{c}{x^2}$$

♦ EULER's METHOD: A NUMERICAL SOLUTION

Consider the D.E.

$$\frac{dy}{dx}=f(x,y), \qquad y(x_0)=y_0$$

where $f(x,y)$ is an expression in terms of x and y.

By using a step h, we find consecutive points

$$(x_0,y_0),\ (x_1,y_1),\ (x_2,y_2),\ \dots$$

that approximate the solution.

We complete a table of the form

n	x_n	y_n
0	x_0	y_0
1		
2		

by using the recursive relations

$$x_{n+1} = x_n + h$$
$$y_{n+1} = y_n + hf(x_n,y_n)$$

We will demonstrate the procedure by using the D.E. of **example 2** again and then we will explain the method!

EXAMPLE 5

$$\frac{dy}{dx}=xy^2+x, \quad \text{with } y(0)=1.$$

Find an approximation of $y(1)$ using step $h=0.2$

Solution

The relation $x_{n+1} = x_n + h = x_n + 0.2$ gives directly the column of x_n

n	x_n	y_n
0	0	1
1	0.2	1
2	0.4	
2	0.6	
4	0.8	
5	1	

The relation

$$y_{n+1} = y_n + hf(x_n,y_n) = y_n + 0.2(x_n y_n^2 + x_n)$$

gives

$$y_1 = y_0 + 0.2(x_0 y_0^2 + x_0) = 1$$
$$y_2 = y_1 + 0.2(x_1 y_1^2 + x_1) = 1.08$$
$$y_3 = y_2 + 0.2(x_2 y_2^2 + x_2) =$$

etc

We finally obtain

n	x_n	y_n
0	0	1
1	0.2	1
2	0.4	1.08
2	0.6	1.25331
4	0.8	1.56181
5	1	2.11209

Hence

for x=1, $y \cong 2.11209$

Notice

- We can easily obtain the table above by using **recursion** in your GDC.

  For Casio FX we use

  MENU

  RECURSION

  SET[F5]

  Start=**0**,

  End =**100** (or more),

  a_0=**0**,

  b_0=**1**

  EXIT

  $a_{n+1} = a_n + 0.2$

  $b_{n+1} = b_n + 0.2(a_n b_n^2 + a_n)$

  EXE

- The exact solution of the D.E. is $y=\tan\left(\frac{x^2}{2}+\frac{\pi}{4}\right)$ [see example 2]

  Thus for x=1, the actual value of y is

  $$y=\tan\left(\frac{1}{2}+\frac{\pi}{4}\right)=3.40822$$

  The approximation $y\cong 2.11209$ we found above is not great!

  However, we can improve the result by reducing the step value.

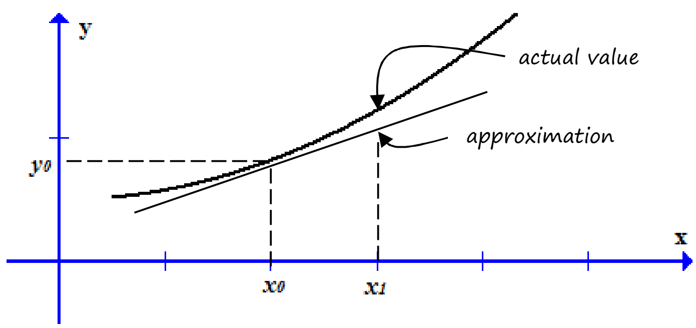
  For h=0.01, (thus we need 100 steps) the GDC gives

  $$\text{for } x=1,\ y\cong 3.26945$$

  which is much better.

Explanation of the method:

$$\frac{dy}{dx}=f(x,y), \qquad y(x_0)=y_0$$



The tangent line at (x_0,y_0) of the unknown solution curve is

$$y-y_0=m(x-x_0)$$

But $m=\left.\frac{dy}{dx}\right|_{(x_0,y_0)}=f(x_0,y_0)$, so

$$y-y_0=f(x_0,y_0)(x-x_0) \quad\Rightarrow y=y_0+(x-x_0)f(x_0,y_0)$$

For $x=x_1$, we let $h=x_1-x_0$ and approximate the actual value of y by

$$y_1=y_0+hf(x_0,y_0)$$

Similarly we get

$$y_2=y_1+hf(x_1,y_1)$$

and so on.

5.23 MACLAURIN SERIES – EXTN OF BINOMIAL THM (for HL)

Consider the infinite geometric series

$$1+x+x^2+x^3+\dots$$

We know that the series converges for $-1<x<1$ and the result is

$$S_\infty=\frac{1}{1-x}$$

In this paragraph we have the opposite task:

We are given a function, say $f(x)=\dfrac{1}{1-x}$, and we wish to express it as an infinite series (which is called power series) of the form

$$a_0+a_1x+a_2x^2+a_3x^3+\dots$$

(the power series looks like a polynomial of "infinite degree").

♦ **THE MACLAURIN SERIES**

Suppose that a function f(x) has derivatives of every order near 0. Then f(x) can be expressed as a power series as follows

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\dots$$

This is known as **Maclaurin series** of the function.

For example, if $f(x)=\dfrac{1}{1-x}$ then

$f^{(n)}(x)$	$f^{(n)}(0)$
$f(x)=(1-x)^{-1}$	1
$f'(x)=(1-x)^{-2}$	1
$f''(x)=2(1-x)^{-3}$	2
$f'''(x)=6(1-x)^{-4}$	3!
etc	

and thus

$$f(x)=1+x+x^2+x^3+\dots$$

It is in fact the geometric series in the beginning. Amazing, isn't it?

Explanation of the Maclaurin series:

We wish to express f(x) as

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

Then

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \dots$$

$$f''(x) = 2a_2 + 3!\,a_3 x + (3)(4)a_4 x^2 + \dots$$

$$f'''(x) = 3!\,a_3 + 4!\,a_4 x + \dots \qquad \text{etc}$$

Therefore,

$$f(0) = a_0 \quad \Rightarrow \quad a_0 = f(0)$$

$$f'(0) = a_1 \quad \Rightarrow \quad a_1 = f'(0)$$

$$f''(0) = 2a_2 \quad \Rightarrow \quad a_2 = \frac{f''(0)}{2}$$

$$f'''(0) = 3!a_2 \quad \Rightarrow \quad a_3 = \frac{f'''(0)}{3!} \qquad \text{etc}$$

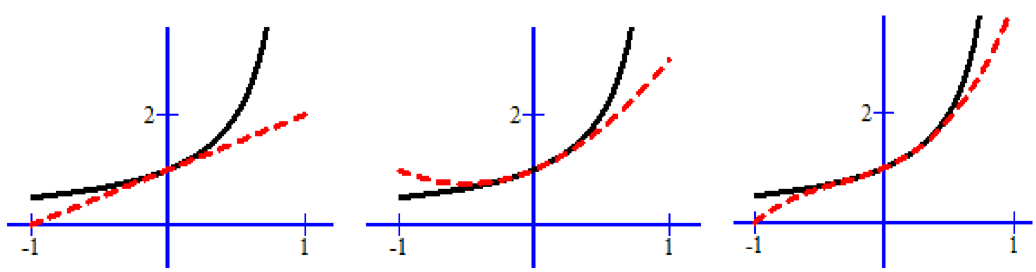
In general

$$a_n = \frac{f^{(n)}(0)}{n!}$$

In fact, the partial sums of the Maclaurin series give good approximations of f(x) near x=0:

$a_0 + a_1 x$	is the tangent line of f(x) at x=0
$a_0 + a_1 x + a_2 x^2$	is the "best" quadratic that approximates f(x)
$a_0 + a_1 x + a_2 x^2 + a_3 x^3$	is the "best" cubic that approximates f(x)

For $f(x) = \dfrac{1}{1-x}$ (black curve) look at the approximations below:



$y = 1 + x$ [tangent] $\qquad y = 1 + x + x^2 \qquad y = 1 + x + x^2 + x^3$

EXAMPLE 1

Find the Maclaurin series of the function f(x)=sinx up to the term in x^5

Solution

$f^{(n)}(x)$	$f^{(n)}(0)$
$f(x)=\sin x$	0
$f'(x)=\cos x$	1
$f''(x)=-\sin x$	0
$f'''(x)=-\cos x$	−1
$f^{(4)}(x)=\sin x$	0
$f^{(5)}(x)=\cos x$	1

and thus

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots$$

gives

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$$

Notice: Use your GDC to compare the graph of sinx with the graphs of the partial sums

$$x-\frac{x^3}{3!}\ ,\ x-\frac{x^3}{3!}+\frac{x^5}{5!}\ ,\quad \text{etc}$$

The result is amazing!

Similarly, we can obtain

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$$

$$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots$$

An alternative way to obtain a Maclaurin series is to modify or combine appropriately the already known series. We can add, multiply, differentiate or integrate series and generate new series.

EXAMPLE 2

Find the Maclaurin series of the function $f(x)=e^{x^2}$

Solution

Since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

we can substitute x by x^2 and obtain

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots$$

EXAMPLE 3

Find the Maclaurin series of the function $f(x)=e^x \sin x$

Solution

We can multiply the series for e^x and $\sin x$.

$$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$

We can find gradually the constant term, the terms in x, the terms in x^2 etc:

$$e^x \sin x = x + x^2 - \frac{x^3}{3!} + \frac{x^3}{2!} - \frac{x^4}{3!} + \frac{x^4}{3!} + \dots$$

$$= x + x^2 - \frac{x^3}{3} + \dots \qquad \text{[there is no } x^4\text{]}$$

Notice also

- if we differentiate the series for $\sin x$ (term by term) we obtain the series of $\cos x$.
- If we differentiate the series for e^x we obtain e^x itself.

In general, we are able to differentiate a series term by term.

EXAMPLE 4

Find the Maclaurin series of the function $f(x)=\dfrac{1}{x^2+1}$

Solution

$$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\dots$$

Since $(\arctan x)'=\dfrac{1}{x^2+1}$

$$\frac{1}{x^2+1}=\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\dots\right)'=1-x^2+x^4-\dots$$

We can also integrate term by term.

EXAMPLE 5

Find the Maclaurin series of cosx by integrating the series of

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots$$

Solution

(a) By using indefinite integrals:

$$\int \sin x dx=\int\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right)dx \Rightarrow -\cos x=\left(\frac{x^2}{2!}-\frac{x^4}{4!}-\dots\right)+c$$

For x=0 we obtain −1=c. Thus

$$-\cos x=\left(\frac{x^2}{2!}-\frac{x^4}{4!}-\dots\right)-1 \Rightarrow \cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots$$

(b) By using definite integrals from 0 to x:

$$\int_0^x \sin x dx=\int_0^x\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right)dx \Rightarrow [-\cos x]_0^x=\left[\frac{x^2}{2!}-\frac{x^4}{4!}-\dots\right]_0^x$$

$$\Rightarrow -\cos x+1=\left(\frac{x^2}{2!}-\frac{x^4}{4!}-\dots\right)-0 \Rightarrow \cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots$$

♦ DIFFERENTIAL EQUATIONS AND MACLAURIN SERIES

Consider a differential equation of the form

$$\frac{dy}{dx}=F(x,y) \quad \text{with boundary condition} \quad y(0)=y_0$$

The analytical solution is not always easy or sometimes not possible at all. However, we can easily find the Maclaurin series of the solution y=f(x) (and thus a good approximation of the solution).

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\dots$$

But, $f(0)=y_0$ [by the boundary condition]

$$f'(0)=\left.\frac{dy}{dx}\right|_{\substack{x=0\\y=a}}=F(0,a) \quad \text{[by the D.E. itself]}$$

Implicit differentiation on $\frac{dy}{dx}$ gives $\frac{d^2y}{dx^2}$ and thus $f''(0)$, and so on.

EXAMPLE 6

Find the Maclaurin series up to x^2 for the solution of the D.E.

$$\frac{dy}{dx}=x^2+y^2 \quad \text{with } y=3 \text{ when } x=0$$

Solution

Clearly $f(0)=3$

$$f'(0)=\left.\frac{dy}{dx}\right|_{\substack{x=0\\y=3}}=0^2+3^2=9$$

$$\frac{d^2y}{dx^2}=2x+2y\frac{dy}{dx}, \text{ thus } \quad f''(0)=\left.\frac{d^2y}{dx^2}\right|_{\substack{x=0\\y=a}}=2\times0+2\times3\times9=54$$

Therefore

$$y\cong3+9x+\frac{54}{2!}x^2+\dots=3+9x+27x^2+\dots$$

Notice. We can also express $\frac{d^2y}{dx^2}$ in terms of x and y only:

$$\frac{d^2y}{dx^2}=2x+2y\frac{dy}{dx}=2x+2y(x^2+y^2), \quad \text{thus } f''(0)=54$$

♦ THE EXTENTION OF THE BINOMIAL THEOREM

Remember that the binomial theorem gives

$$(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n$$

If we expand more the coefficients we obtain

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots + x^n$$

This version allows us to use **negative** or even **rational** values for index n.

Thus for example,

$$(1+x)^{-n} = 1 - nx + \frac{-n(-n-1)}{2} x^2 + \frac{-n(-n-1)(-n-2)}{3!} x^3 + \cdots$$

but now we have infinitely many terms.

This is an alternative way to obtain some infinite power series.

For example

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$$

Similarly,

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

EXAMPLE 7

Find the Maclaurin series up to x^4 for $f(x) = \dfrac{x}{(1+x)^3}$

Solution

$$f(x) = x(1+x)^{-3} = x\left(1 - 3x + \frac{(-3)(-4)}{2} x^2 + \frac{(-3)(-4)(-5)}{3!} x^3 + \cdots\right)$$

and finally

$$f(x) = x - 3x^2 + 6x^3 - 10x^4 + \cdots$$

EXAMPLE 8

Find the Maclaurin series up to x^3 for $f(x)=\sqrt{1+x}$ by using the extension of the binomial theorem.

Solution

The formula

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

gives

$$f(x)=(1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{6}x^3 + \cdots$$

and finally

$$f(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\cdots$$

In general, for the extension of the binomial theorem we modify $(a+b)^n$ as follows

$$(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n$$

and the formula takes the form

$$\boxed{(a+b)^n = a^n\left(1+n\left(\frac{b}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{b}{a}\right)^3+\cdots\right)}$$

For non-integer values of n, we obtain an infinite series which converges (that is, the expansion holds) when

$$\boxed{\left|\frac{b}{a}\right|<1}$$

In the last example above, the expansion holds for $|x|<1$.

EXAMPLE 9

Find the Maclaurin series up to x^2 for $f(x)=(2x+3)^{-2}$ by using the extension of the binomial theorem.

Solution

$$f(x)=(3+2x)^{-2}=\left[3\left(1+\frac{2x}{3}\right)\right]^{-2}=\frac{1}{9}\left(1+\frac{2x}{3}\right)^{-2}$$

The formula

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2}x^2+\cdots$$

gives

$$f(x)=\frac{1}{9}\left(1-2\frac{2x}{3}+\frac{(-2)(-3)}{2}\left(\frac{2x}{3}\right)^2+\cdots\right)$$

and finally

$$f(x)=\frac{1}{9}-\frac{4}{27}x+\frac{4}{27}x^2+\cdots$$

The series converges to $f(x)=(3+2x)^{-2}$ when $\left|\frac{2x}{3}\right|<1$, that is when

$$|x|<\frac{3}{2}$$

or otherwise, when

$$-\frac{3}{2}<x<\frac{3}{2}$$