International Baccalaureate

MATHEMATICS

Analysis and Approaches (SL and HL)

Lecture Notes

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TOPIC 3
GEOMETRY AND TRIGONOMETRY

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3.1 THREE DIMENSIONAL GEOMETRY

♦ 3D COORDINATE GEOMETRY

We know that a point in the Cartesian plane has the form P(x,y). In 3D space we add one more coordinate, thus a point has the form P(x,y,z).

The distance between two points **A**(x_1,y_1,z_1) and **B**(x_2,y_2,z_2) is given by

$$d_{AB}=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$$

while the midpoint of the line segment AB is given by

$$\mathbf{M}\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$

EXAMPLE 1

Let A(1,0,5) and B(2,3,1). Find

(a) the distance between A and B
(b) the distance between O and B
(c) the coordinates of the midpoint M of the line segment [AB]
(d) the coordinates of point C given that B is the midpoint of [AC]

Solution

(a) $d_{AB}=\sqrt{(1-2)^2+(0-3)^2+(5-1)^2}=\sqrt{1+9+16}=\sqrt{26}$

(b) $d_{OB}=\sqrt{2^2+3^2+1^2}=\sqrt{14}$

(c) $\mathbf{M}\left(\frac{1+2}{2},\frac{0+3}{2},\frac{5+1}{2}\right)$ i.e. $\mathbf{M}\left(\frac{3}{2},\frac{3}{2},3\right)$

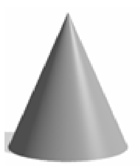
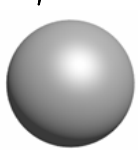
(d) C(3,6,−3)

Notice: the coordinates of A,B,C (B midpoint) form arithmetic sequences

x: 1,2,3
y: 0,3,6
z: 5,1,−3

♦ VOLUMES AND SURFACE AREAS OF KNOWN SOLIDS

The volumes and the surface areas of 5 known solids are given below:

Solid	**Volume**	**Surface area**
Cuboid 	$V = xyz$	$S = 2xy + 2yz + 2zx$
Pyramid 	$V = \frac{1}{3}(\text{area of base}) \times (\text{height})$	$S =$ (sum of areas of the faces)
Cylinder 	$V = \pi r^2 h$	$S = 2\pi rh + 2\pi r^2$
Cone 	$V = \frac{1}{3}\pi r^2 h$	$S = \pi r L + \pi r^2$ where $L = \sqrt{r^2 + h^2}$
Sphere 	$V = \frac{4}{3}\pi r^3$	$S = 4\pi r^2$

Notation

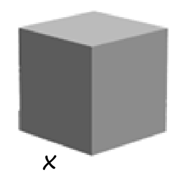
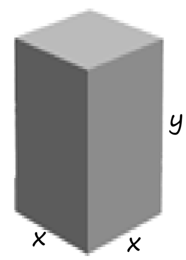
x, y, z : length-width-height

r : radius of circular base

h : vertical height

EXAMPLE 2

The volume and the surface area for the following solids

Cube of side x	Cuboid of square base x
	

Cube: $V = xxx = x^3$ $S = 6x^2$

Cuboid of square base: $V = x^2 y$ $S = 2x^2 + 4xy$

EXAMPLE 3

Given that the volume of a cylinder is 25,

(a) express h in terms of r

(b) **hence** express the surface area in terms of r

Solution

(a) $V = \pi r^2 h \Rightarrow \pi r^2 h = 25 \Rightarrow h = \dfrac{25}{\pi r^2}$

(b) $S = 2\pi r h + 2\pi r^2 = 2\pi r \dfrac{25}{\pi r^2} + 2\pi r^2 = \dfrac{50}{r} + 2\pi r^2$

EXAMPLE 4

Given that the surface area of a cylinder is 100π,

(a) express h in terms of r

(b) **hence** express the volume in terms of r

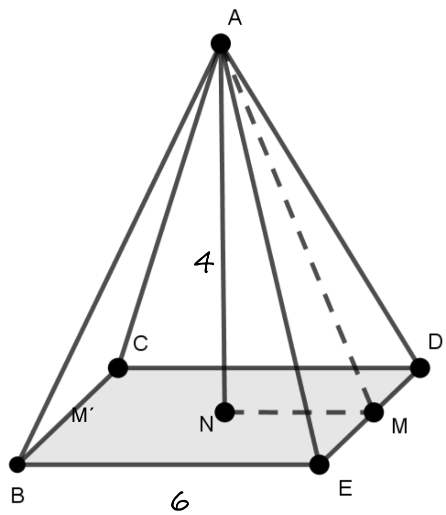
Solution

(a) $S = 2\pi r h + 2\pi r^2 \Rightarrow 2\pi r h + 2\pi r^2 = 100\pi \Rightarrow h = \dfrac{50 - r^2}{r}$

(b) $V = \pi r^2 h = \pi r^2 \dfrac{50 - r^2}{r} = \pi r(50 - r^2) = 50\pi r - 50 r^3$

EXAMPLE 5

Find the volume and the surface area of a right pyramid of square base of side 6 and vertical height 4.



Solution

The vertical height is h=4.

For the slant height AM we use the Pythagoras theorem on ANM.

$$AM^2 = AN^2 + NM^2 \Leftrightarrow AM^2 = 4^2 + 3^2 \Leftrightarrow AM = 5$$

The area of the triangle AED (and any side triangle) is

$$A = \frac{1}{2} \times ED \times AM = \frac{1}{2} \times 6 \times 5 = 15$$

The volume is $V = \frac{1}{3}(\text{area of base}) \times (\text{height}) = \frac{1}{3} \times 6^2 \times 4 = 48$

The surface area is S=(area of square base)+4A = $6^2+4\times(15)$ =96

Notice about the angles between lines and planes:

Angle between **line AM** and **plane BCDE** = angle $A\hat{M}N$

Angle between **line AD** and **plane BCDE** = angle $A\hat{D}N$

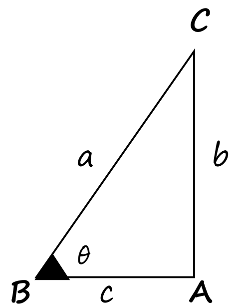
Angle between **the planes ADE and BCDE** = angle $A\hat{M}N$

Angle between **the planes ACB and ADE** = angle $M\hat{A}M'$ **= 2×**$M\hat{A}N$

3.2 TRIANGLES – THE SINE RULE - THE COSINE RULE

♦ BASIC NOTIONS

For any right-angled triangle



we define the **sine**, the **cosine** and the **tangent** of angle θ by:

$$\sin\theta = \frac{b}{a} = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{c}{a} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{b}{c} = \frac{\text{opposite}}{\text{adjacent}}$$

Clearly

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

It also holds

$$\text{Pythagoras' theorem} \qquad a^2 = b^2 + c^2$$

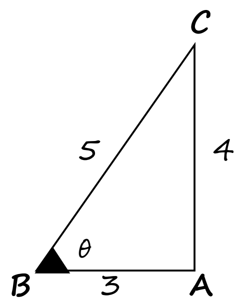
We can easily derive the so-called Pythagorean identity

$$\sin^2\theta + \cos^2\theta = 1$$

Indeed,

$$\sin^2\theta + \cos^2\theta = \left(\frac{b}{a}\right)^2 + \left(\frac{c}{a}\right)^2 = \frac{b^2 + c^2}{a^2} = \frac{a^2}{a^2} = 1$$

EXAMPLE 1



$\sin B = \dfrac{4}{5}$

$\cos B = \dfrac{3}{5}$

$\tan B = \dfrac{4}{3}$

We can also confirm Pythagoras' theorem: $5^2 = 3^2 + 4^2$

Every angle has a fixed sine, cosine and tangent. For example

$$\sin 30^o = \frac{1}{2}, \qquad \cos 30^o = \frac{\sqrt{3}}{2}, \qquad \tan 30^o = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

If we know the sine, the cosine or the tangent of an **acute** angle θ (i.e. $\theta<90$), we can find θ by using the inverse functions of our GDC:

$$\mathbf{sin^{-1}}, \quad \mathbf{cos^{-1}} \quad \text{and} \quad \mathbf{tan^{-1}}$$

For example,

$$\text{if } \sin\theta = \frac{1}{2} \qquad \text{then} \qquad \theta = \mathbf{sin^{-1}}\frac{1}{2} = 30^o$$

Thus, we can find the angle B of the triangle above:

$$B = \sin^{-1}\frac{4}{5} = 53.1°$$

(Notice that $\cos^{-1}\dfrac{3}{5}$ and $\tan^{-1}\dfrac{4}{3}$ give the same result)

Then

$$C = 90° - 53.1° = 36.9°$$

For two angles A and B

A and B are called **complementary** if $\boxed{A + B = 90^o}$

A and B are called **supplementary** if $\boxed{A + B = 180^o}$

♦ **SIN**, **COS**, **TAN** for basic angles: 0^o, 30^o, 45^o, 60^o, 90^o

We mention a practical way to memorise them:

θ	0^o	30^o	45^o	60^o	90^o
$\sin\theta$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$

For **cosθ** we obtain the same values in the opposite order

For **tanθ** we simply divide sinθ by cosθ.

Hence,

θ	0^o	30^o	45^o	60^o	90^o
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–

Notice

$$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

<u>**REMARKS**</u>:

- For an **acute** angle ($\theta < 90^o$) if we know the result sinθ we can find the angle θ itself by using the inverse function $\sin^{-1}$ in our GDC. Similarly, for cosθ and tanθ. For example,

  If $\sin\theta = 0.5$ then $\theta = \sin^{-1} 0.5 = 30^o$

  If $\cos\theta = 0.3$ then $\theta = \cos^{-1} 0.3 = 72.5^o$

- sinθ, cosθ, tanθ, cotθ are also defined for **obtuse** angles ($\theta > 90^o$). At the moment, it is enough to know that

  <u>supplementary angles have **equal sines** but **opposite cosines**</u>:

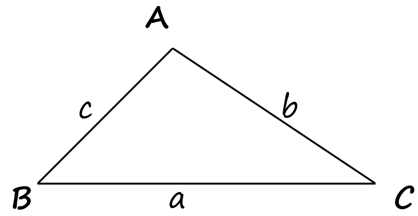
  e.g. $\sin 30^o = 0.5$, $\sin 150^o = 0.5$

  $\cos 30^o = \sqrt{3}/2$ $\cos 150^o = -\sqrt{3}/2$

- The values of sinθ and cosθ range between –1 and 1.

♦ SINE RULE – COSINE RULE

For any triangle two rules always hold:



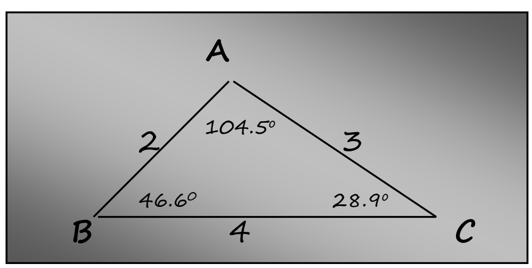
SINE RULE $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

COSINE RULE $a^2=b^2+c^2-2bc\cos A$

Notice: There are two more versions of the cosine rule:

$b^2=c^2+a^2-2ca\cos B$ $c^2=a^2+b^2-2ab\cos C$

Consider, for example, the following triangle



We confirm by GDC that the SINE RULE holds:

$$\frac{4}{\sin 104.5}\cong \mathbf{4.13} \qquad \frac{3}{\sin 46.6}\cong \mathbf{4.13} \qquad \frac{2}{\sin 28.9}\cong \mathbf{4.13}$$

We can also confirm the three versions of the COSINE RULE:

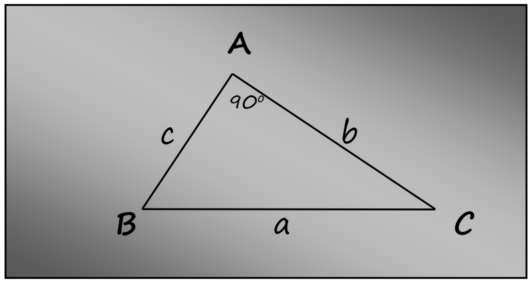
$4^2 = 3^2+2^2-2(3)(2)\cos 104.5$ (LHS = **16** RHS = **16**)

$3^2= 2^2+4^2-2(2)(4)\cos 46.6$ (LHS = **9** RHS = **9**)

$2^2= 4^2+3^2-2(4)(3)\cos 28.9$ (LHS = **4** RHS = **4**)

EXAMPLE 2

Consider the following right-angled triangle



Then

$$\frac{a}{\sin 90^o} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow a = \frac{b}{\sin B} = \frac{c}{\sin C}$$

and so

$$\boxed{\sin B = \frac{b}{a} \quad \text{and} \quad \sin C = \frac{c}{a}}$$

as expected by the definition of $\sin\theta$

Also, $a^2 = b^2 + c^2 - 2bc \cdot \cos 90^o$ implies

$$\boxed{a^2 = b^2 + c^2}$$

that is the Pythagoras' theorem, since $\cos 90^o = 0$.

Moreover

$$b^2 = c^2 + a^2 - 2ca \cdot \cos B \Rightarrow b^2 = c^2 + (b^2 + c^2) - 2ca \cdot \cos B$$
$$\Rightarrow -2c^2 = -2ca \cdot \cos B$$
$$\Rightarrow \boxed{\cos B = \frac{c}{a}}$$

as expected by the definition of $\cos\theta$. Similarly we get $\boxed{\cos C = \frac{b}{a}}$

Consequently,

SINE RULE generalizes the definition of $\sin\theta$

COSINE RULE generalizes of the definition of $\cos\theta$
and Pythagoras' theorem

♦ THE SOLUTION OF A TRIANGLE

Any triangle has 6 basic elements: 3 sides and 3 angles.

If we are given any 3 among those 6 elements (except 3 angles!) we are able to find the remaining 3 elements by using the sine rule or the cosine rule appropriately.

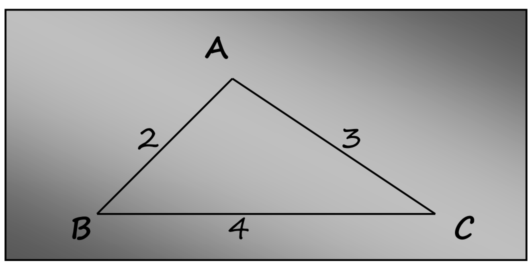
Roughly speaking

If we know	we use
(**three** sides) OR (**two** sides and **an** included angle)	**COSINE RULE**
otherwise	**SINE RULE**

In other words

we use the SINE RULE when we know an **angle-opposite side** pair.

EXAMPLE 3 (given three sides)



We use COSINE RULE

$$4^2 = 2^2 + 3^2 - 12\cos A$$
$$\Rightarrow 3 = -12\cos A$$
$$\Rightarrow \cos A = -0.25$$
$$\Rightarrow \boxed{A = 104.5^o}$$

$$3^2 = 2^2 + 4^2 - 16\cos B$$
$$\Rightarrow -11 = -16\cos B$$
$$\Rightarrow \cos B = 0.6875$$
$$\Rightarrow \boxed{B = 46.6^o}$$

Finally,

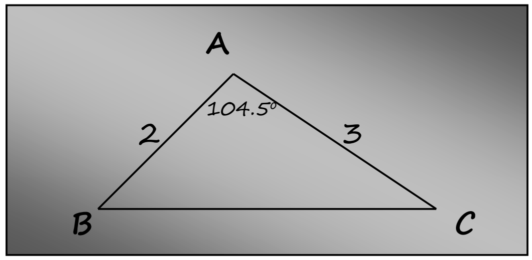
$$C = 180^o - A - B = 180^o - 104.5^o - 46.6^o,$$

Thus

$$\boxed{C = 28.9^o}$$

Notice: We may sometimes have no solutions at all. For example, if a=10, b=3, c=2 it is not possible to construct such a triangle! Indeed, the cosine rule gives us cosA = −7.25 which is not possible!

EXAMPLE 4 (given two sides and an included angle)



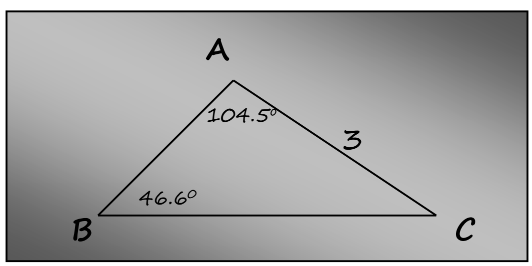
We use COSINE RULE:

$$BC^2 = 2^2 + 3^2 - 12\cos 104.5^o = 16$$

Thus $\boxed{BC = 4}$

Then we know all the three sides and hence B and C can be found as above: $\boxed{B = 46.6^o}$ and $\boxed{C = 28.9^o}$

EXAMPLE 5 (given one side and two angles)



In fact, we know the third angle as well:

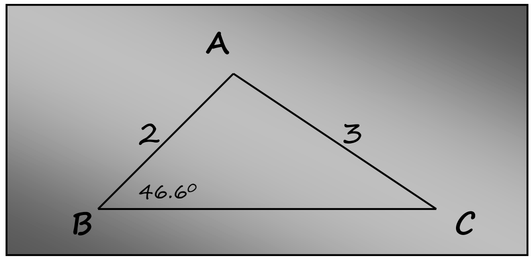
$$C = 180^o - A - B = 180^o - 104.5^o - 46.6^o\text{, thus } \boxed{C = 28.9^o}$$

Now we can use the sine rule twice

$$\frac{3}{\sin 46.6} = \frac{BC}{\sin 104.5} \Rightarrow \boxed{BC = 4}$$

$$\frac{3}{\sin 46.6} = \frac{AB}{\sin 28.9} \Rightarrow \boxed{AB = 2}$$

EXAMPLE 6 (given **two sides** and **a non-included angle**)



We use the sine rule

$$\frac{3}{\sin 46.6} = \frac{2}{\sin C} \Rightarrow \sin C = 0.484$$

Hence, $C = 28.9^o$ (by GDC)

Then

$$A = 180^o - 46.6^o - 28.9^o, \text{ that is } \boxed{A = 104.5^o}$$

The side **BC** can be found either by sine or cosine rule! It is $\boxed{BC=4}$

Notice: In fact, we obtain two values for C.

$C = 28.9^o$ (by GDC)

or $C' = 180^o - 28.9^o = 151.1^o$

(since supplementary angles have equal sines).

But $C' = 151.1^o$ is rejected since

$$B + C' = 46.6^o + 151.1^o > 180^o$$

But this is not always the case!

♦ THE AMBIGUOUS CASE

If we are given **two sides** and **a non-included** angle (as above) we may have as a solution

- Two triangles
- One triangle
- No triangle at all

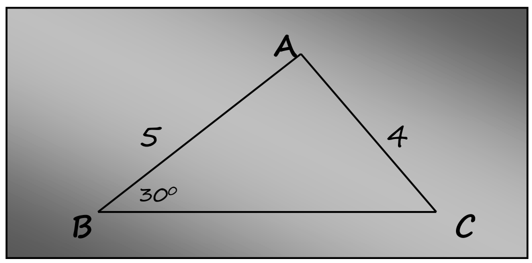
This is because the sine rule provides two values for an unknown angle. For example if we find $\sin C = 0.5$ then

$$C = 30^o \qquad (\text{this is } \sin^{-1}C)$$
$$\text{or} \qquad C' = 180^o - 30^o = 150^o$$

and these two values may result to different solutions.

In example 6 we found only one solution. But in the following example we will find two solutions.

EXAMPLE 7 (given **two sides** and **a non-included angle**)



We use the sine rule:

$$\frac{4}{\sin 30} = \frac{5}{\sin C} \Rightarrow \sin C = 0.625$$

Hence,

$$C = 38.7^o \text{ (by GDC)}$$
$$\text{or} \qquad C' = 180^o - 38.7^o = 141.3^o$$

CASE (1): If $\boxed{C = 38.7^o}$ then

$$A = 180^o - 30^o - 38.7^o, \text{ thus } \boxed{A = 111.3^o}$$

and then

$$BC^2 = 5^2 + 4^2 - 2(5)(4)\cos 111.3^\circ \Rightarrow \boxed{BC = 7.45}$$

CASE (2): If $\boxed{C' = 141.3^o}$ then

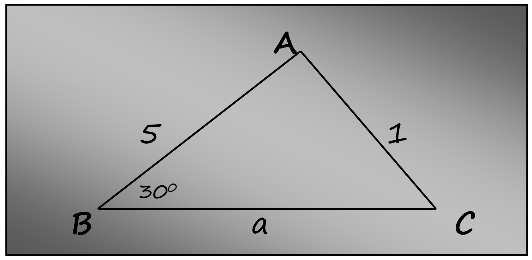
$$A' = 180^o - 30^o - 141.3^o, \text{ thus } \boxed{A' = 8.7^o}$$

and then

$$BC'^2 = 5^2 + 4^2 - 2(5)(4)\cos 8.7^\circ \Rightarrow \boxed{BC' = 1.21}$$

We may, sometimes, obtain no solution at all.

EXAMPLE 8 (given 2 sides and a non-included angle)



We use the sine rule:

$$\frac{1}{\sin 30} = \frac{5}{\sin C} \Rightarrow \sin C = 2.5$$

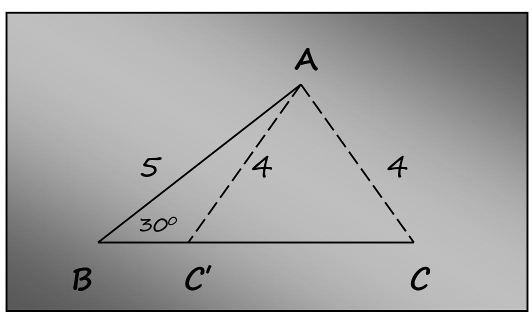
which is impossible!

Hence, there is no such a triangle!

♦ JUSTIFICATION OF THE AMBIGUOUS CASE

In example 7, we were given B=30°, AB=5, AC=4 and we found two solutions for C, and thus two possible triangles: ABC and ABC′.

Indeed, the two triangles satisfying theses conditions are shown below



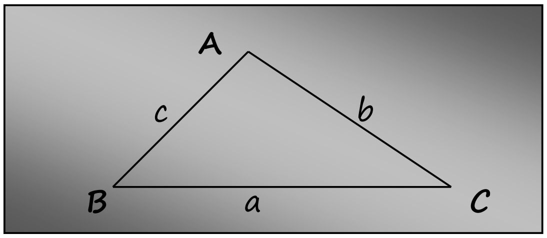
Notice that AC=4 can be placed in two different positions.

For the two possible values of angle C it holds

$$C+C'=180^\circ$$

(can you explain why?)

- THE AREA OF A TRIANGLE



$$\text{Area} = \frac{1}{2} bc \sin A$$

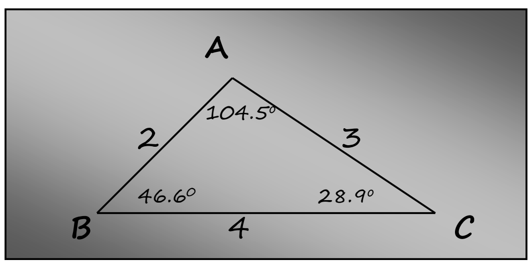
Notice that **two sides** and **an included angle** are involved in the formula!

We can derive two similar versions for this formula:

$$\text{Area} = \frac{1}{2} ab \sin C \qquad \text{Area} = \frac{1}{2} ac \sin B$$

EXAMPLE 9

Look at again the triangle in example 1:



$$\text{Area} = \frac{1}{2} 2 \cdot 3 \cdot \sin 104.5^{o} \cong 2.90$$

The other two versions give the same result:

$$\text{Area} = \frac{1}{2} 2 \cdot 4 \cdot \sin 46.6^{o} \cong 2.90$$

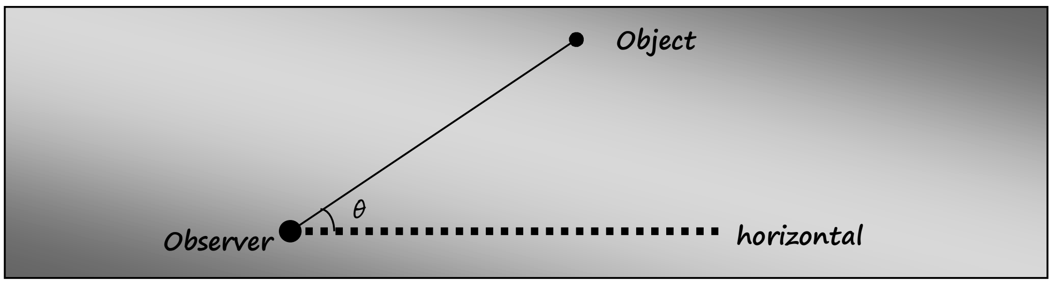
$$\text{Area} = \frac{1}{2} 3 \cdot 4 \cdot \sin 28.9^{o} \cong 2.90$$

(you may notice little deviations on the result due to rounding!)

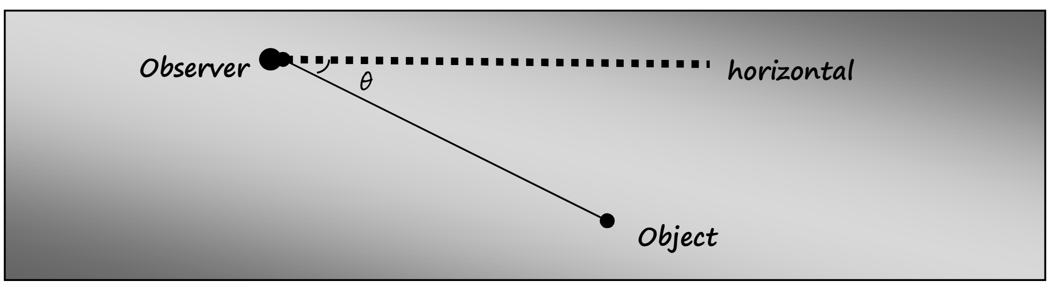
3.3 APPLICATIONS IN 3D GEOMETRY – NAVIGATION

♦ ANGLE OF ELEVATION – ANGLE OF DEPRESSION

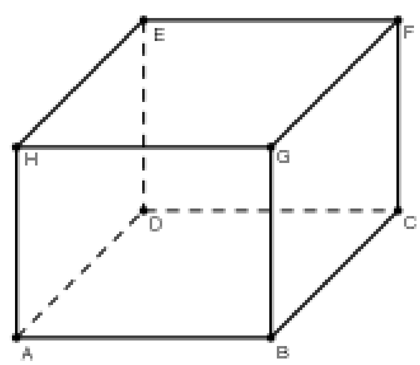
Suppose that an object is above the horizontal level of an observer. The **angle of elevation θ** to the object is shown below:



If the object is below the level of the observer the **angle of depression θ** to the object is shown below:



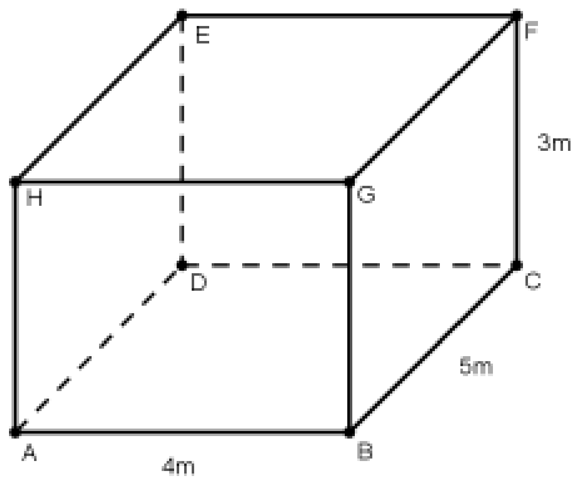
We very often see these notions in 3D shapes. For example,



The angle of elevation from A to G is the angle $B\hat{A}G$.
The angle of elevation from A to F is the angle $C\hat{A}F$ (explain why!)

The angle of depression from H to B is the angle $G\hat{H}B$.
The angle of depression from H to C is the angle $F\hat{H}C$ (explain why!)

EXAMPLE 1



An observer is situated at point A.

(a) Find the distance AG and the angle of elevation of point G.

(b) Find the distance AF and the angle of elevation of point F.

Solution

(a) We consider the triangle AGB.

By Pythagoras' theorem,

$$AG^2 = 4^2 + 3^2 \Rightarrow AG = 5$$

The angle of elevation is $B\hat{A}G$. Hence,

$$\tan B\hat{A}G = \frac{3}{4} \Rightarrow B\hat{A}G = 36.9^o$$

(b) For point F we consider the vertical height FC and thus the triangle AFC.

We firstly need the side AC. By Pythagoras theorem in ABC

$$AC^2 = 4^2 + 5^2 \Rightarrow AC = \sqrt{41}$$

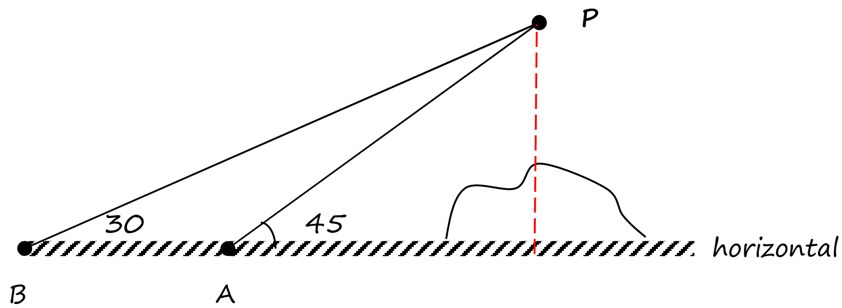
Now, by Pythagoras' theorem in AFC,

$$AF^2 = \sqrt{41}^2 + 3^2 \Rightarrow AF = \sqrt{50}$$

The angle of elevation is $C\hat{A}F$. Hence,

$$\tan C\hat{A}F = \frac{3}{\sqrt{41}} \Rightarrow C\hat{A}F = 25.1^o$$

EXAMPLE 2



An object P is above a hill. Two observers A and B are situated as in the diagram above.

The angle of elevation from A is 45^o.

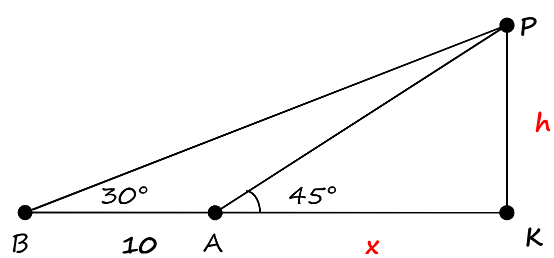
The angle of elevation from B is 30^o.

The distance between A and B is 10m.

Find the vertical height h of the object P above the ground.

Solution

Consider the triangle



$$\tan 45^\circ = \frac{h}{x} \Leftrightarrow \frac{h}{x} = 1 \Leftrightarrow h = x$$

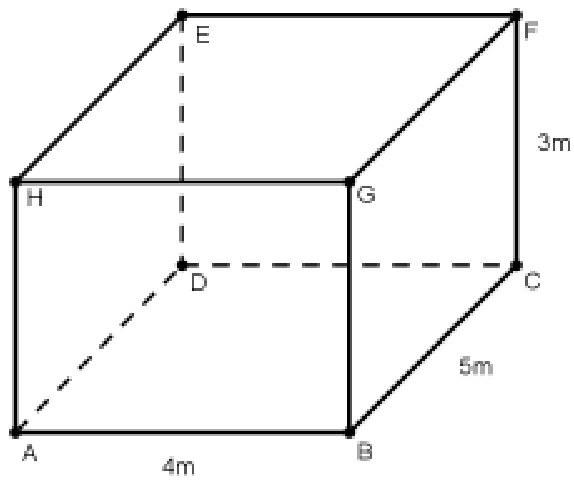
$$\tan 30^\circ = \frac{h}{x+10} \Leftrightarrow \frac{h}{x+10} = \frac{1}{\sqrt{3}} \Leftrightarrow h\sqrt{3} = x+10$$

Therefore,

$$h\sqrt{3} = h+10 \Leftrightarrow h(\sqrt{3}-1) = 10 \Leftrightarrow h = \frac{10}{\sqrt{3}-1} \cong 13.7 \text{ m}$$

Notice: Another approach is to work in triangle ABP first, to find AP=19.318 and then by $\sin 45^\circ = \frac{h}{19.318}$, we find $h \cong 13.7$

EXAMPLE 2



An observer is situated at point A.

(c) Find the distance AG and the angle of elevation of point G.

(d) Find the distance AF and the angle of elevation of point F.

Solution

(a) We consider the triangle AGB.

By Pythagoras' theorem,

$$AG^2 = 4^2 + 3^2 \Rightarrow AG = 5$$

The angle of elevation is $B\hat{A}G$. Hence,

$$\tan B\hat{A}G = \frac{3}{4} \Rightarrow B\hat{A}G = 36.9^o$$

(b) For point F we consider the vertical height FC and thus the triangle AFC.

We firstly need the side AC. By Pythagoras theorem in ABC

$$AC^2 = 4^2 + 5^2 \Rightarrow AC = \sqrt{41}$$

Now, by Pythagoras' theorem in AFC,

$$AF^2 = \sqrt{41}^2 + 3^2 \Rightarrow AF = \sqrt{50}$$

The angle of elevation is $C\hat{A}F$. Hence,

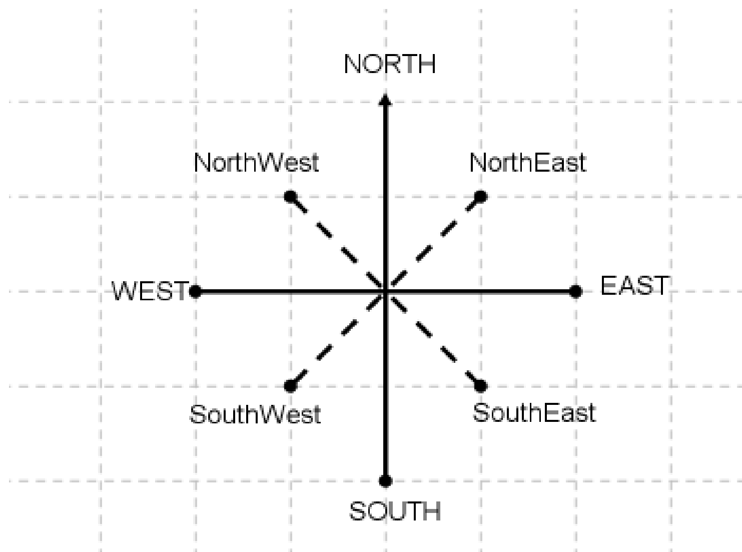
$$\tan C\hat{A}F = \frac{3}{\sqrt{41}} \Rightarrow C\hat{A}F = 25.1^o$$

♦ NAVIGATION – BEARING

When we navigate on a map we should have in mind the four main directions

North, East, South, West

as well as the four intermediate directions as shown below.



The angle between any consecutive directions is 45°.

Thus, for example,

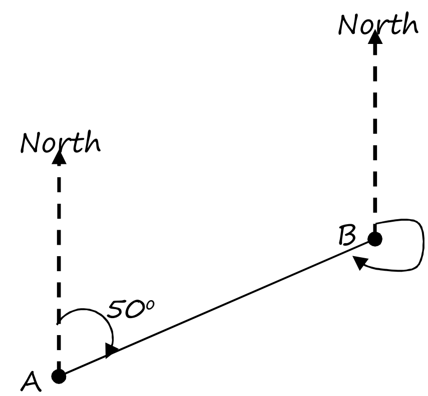
- if two persons walk towards the North and East directions respectively the angle between their directions is 90°.
- if two persons walk towards the North and Southeast directions respectively the angle between their directions is 135°.

Another keyword in navigation is the **bearing**. Suppose that a moving body goes from point A to point B.

> The **bearing** of the course AB is the clockwise angle between the **North direction** and **AB**.

The following diagram will clarify this notion.

According to the diagram:



the bearing of the course AB is 50°

the bearing of the course BA is 230° (explain why!)

EXAMPLE 3

A car travels:

from point A to point B in bearing 50°,

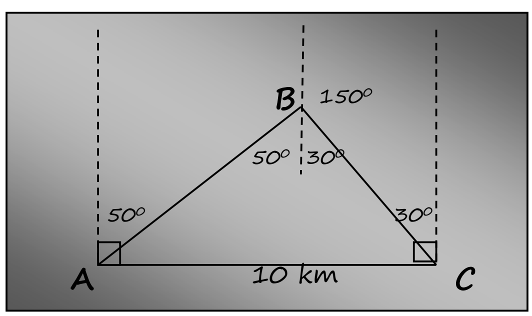
then from point B to point C with bearing 150°,

then goes back to point A with bearing 270°.

The distance AC is 10km.

Draw a diagram to show the details find the distances AB and AC.

Solution



According to the diagram $\hat{A} = 40°$, $\hat{B} = 80°$, $\hat{C} = 60°$

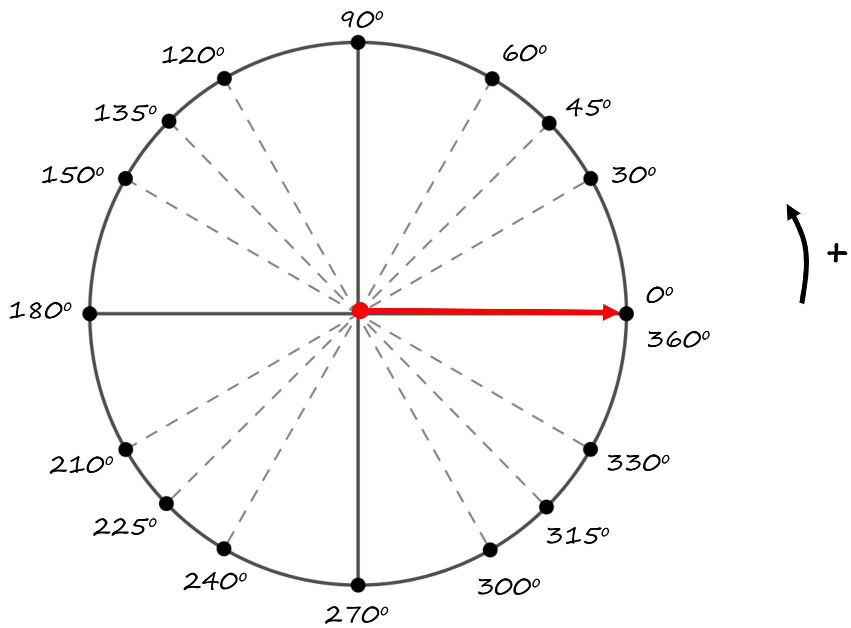
Then, by using the sine rule

$$\frac{10}{\sin 80°} = \frac{AB}{\sin 60°} = \frac{BC}{\sin 40°}$$

we find AB = 8.79km and BC = 6.53km

3.4 THE TRIGONOMETRIC CIRCLE – ARCS AND SECTORS

The values of the angles can be represented well on the following trigonometric circle:



In fact, each value on the circle indicates the angle between the corresponding radius and the positive x-axis radius (red arrow).

The angle formed after a complete circle is 360°.

The angle formed after half a circle is 180°.

However, after completing a full circle (**1st period**) we can continue counting:

361°, 362°, 263° and so on

The next full circle (**2nd period**) finishes at 2×360° = 720°.

Similarly, we can move clockwise, considering **negative** angles:

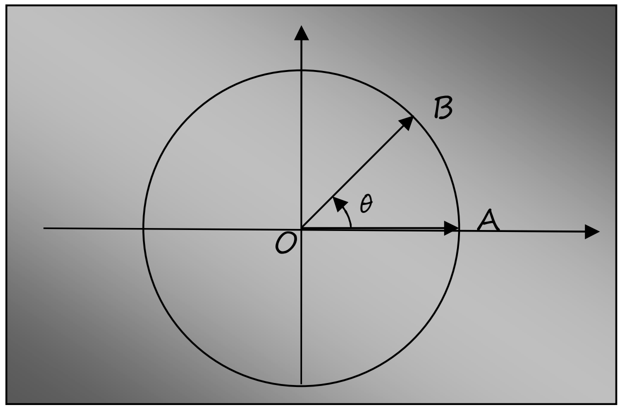
–1°, –2°, –3° and so on

For example, 270° can also be seen as –90°.

Therefore, an angle may have any value from $-\infty$ to $+\infty$.

♦ DEGREES AND RADIANS

Consider the following circle of radius r =1 (unit circle).

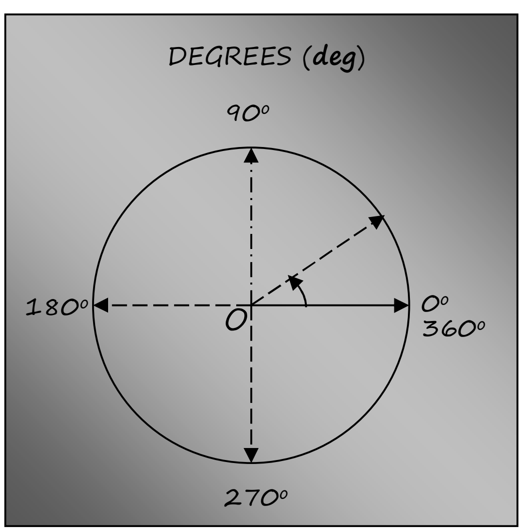


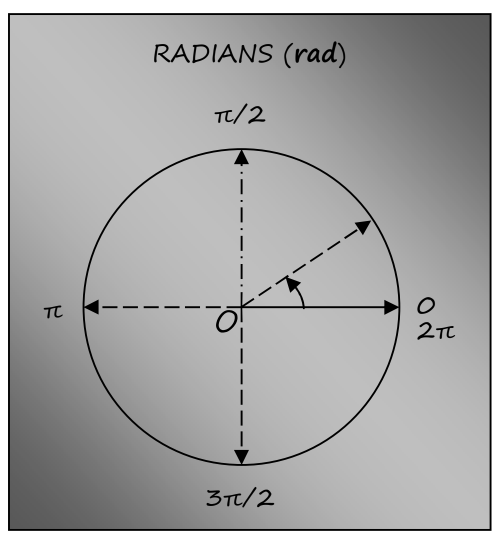
The circumference of the circle is $2\pi r$ = **2π**.

Start from point A and move anticlockwise. What is the length of the arc AB?

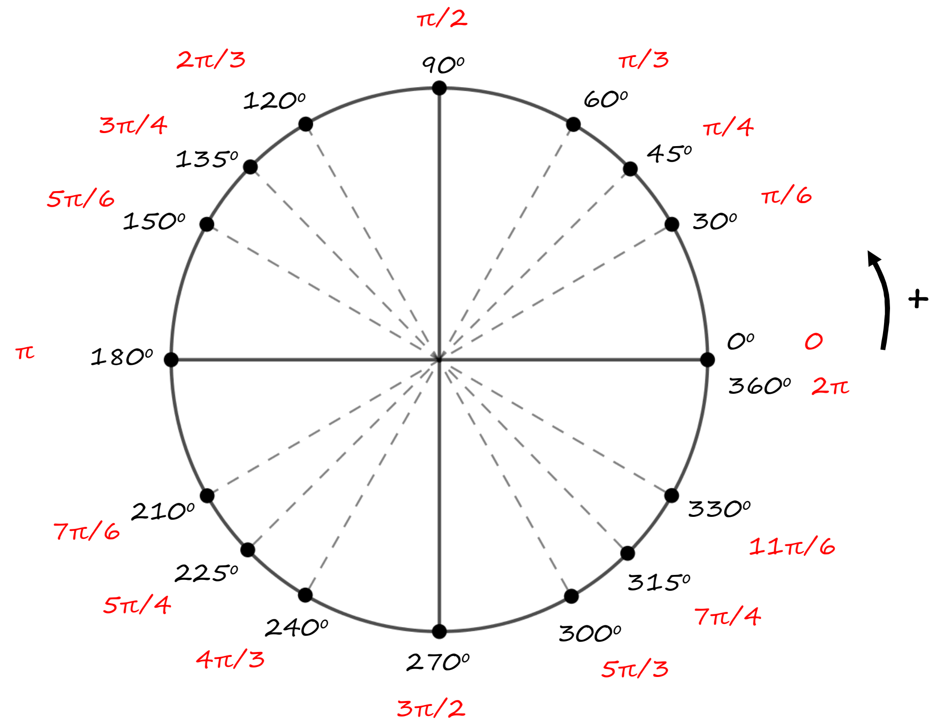
If $\theta = 0^o$	then AB=0	
If $\theta = 360^o$	then AB=2π	(full circle)
If $\theta = 180^o$	then AB=π	(semicircle)
If $\theta = 90^o$	then AB=$\pi/2$	(quarter of a circle)

Thus, an alternative way to measure the angle $\theta = A\hat{O}B$ is to measure the corresponding arc AB. The new unit of measurement is called **radian**.

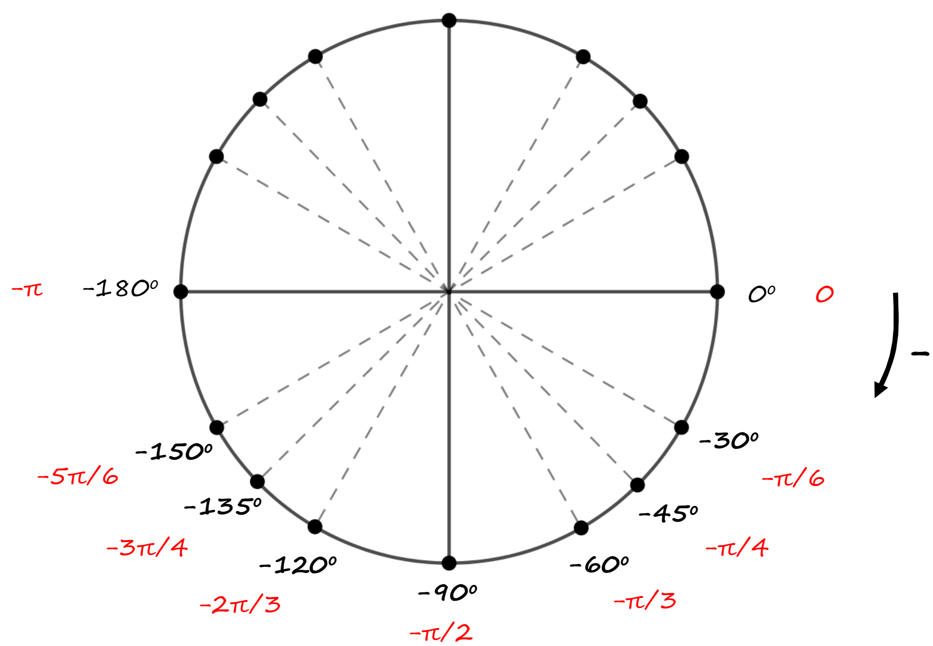




Let us see the basic angles, in degrees and radians, on the trigonometric circle.



We can also move in the opposite direction (clockwise) and consider negative angles:



NOTICE:

The ratio between degrees and radians is given by

$$\boxed{\frac{degrees}{radians} = \frac{180^o}{\pi}}$$

EXAMPLE 1

Let $\theta_1 = 30^o$, $\theta_2 = 80^o$, $\theta_3 = 27^o$. Transform in radians.

We use the ratio $\frac{deg}{rad}$:

For θ_1: $\frac{30^o}{x} = \frac{180^o}{\pi} \Rightarrow 180x = 30\pi \Rightarrow x = \frac{30\pi}{180} = \frac{\pi}{6}rad$

For θ_2: $\frac{80^o}{x} = \frac{180^o}{\pi} \Rightarrow 180x = 80\pi \Rightarrow x = \frac{80\pi}{180} = \frac{4\pi}{9}rad$

For θ_3: $\frac{27^o}{x} = \frac{180^o}{\pi} \Rightarrow 180x = 27\pi \Rightarrow x = \frac{27\pi}{180} = 0.471rad$

EXAMPLE 2

Let $\theta_1 = \frac{\pi}{3}rad$, $\theta_2 = \frac{4\pi}{9}rad$, $\theta_3 = 2$ rad. Transform in degrees.

We use the ratio $\frac{deg}{rad}$:

For θ_1: $\frac{x}{\pi/3} = \frac{180^o}{\pi} \Rightarrow \pi x = \frac{180\pi}{3} \Rightarrow x = 60^o$

For θ_2: $\frac{x}{4\pi/9} = \frac{180^o}{\pi} \Rightarrow \pi x = \frac{4 \cdot 180\pi}{9} \Rightarrow x = 80^o$

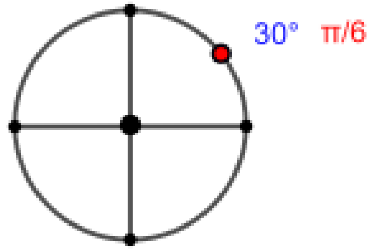
For θ_3: $\frac{x}{2} = \frac{180^o}{\pi} \Rightarrow \pi x = 360 \Rightarrow x = \frac{360}{\pi} = 114.6^o$

NOTICE (not necessary to remember though!!!)

$$\boxed{1\ rad = \frac{180}{\pi} = 57.3^o} \qquad \boxed{1^o = \frac{\pi}{180} = 0.0174\ rad}$$

♦ THE ANGLE VALUES OF A POINT ON THE UNIT CIRCLE

Consider the point on the unit circle corresponding to 30^o.



Let's start from 0^o and move anticlockwise. We pass through 30^o and after completing a full circle, we pass through the same point

at $30^o+360^o=390^o$

and then again at $30^o+360^o\times 2=750^o$

and so on.

In other words, we add (or subtract) multiples of 360^o:

In this way, the same point has infinitely many angle values:

$$\mathbf{30^o+360^o k} \qquad \text{where } k\in Z$$

Working in radians, we add multiples of 2π, so that the point has the infinitely many angle values:

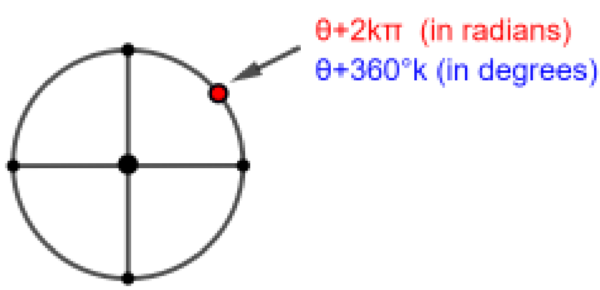
$$\frac{\pi}{6}+2k\pi \qquad \text{where } k\in Z$$

Thus, for $k = \ldots -1, 0, 1, 2, \ldots$ we obtain the values

$$\ldots, -330^o, 30^o, 390^o, 750^o, \ldots \qquad \text{[in degrees]}$$

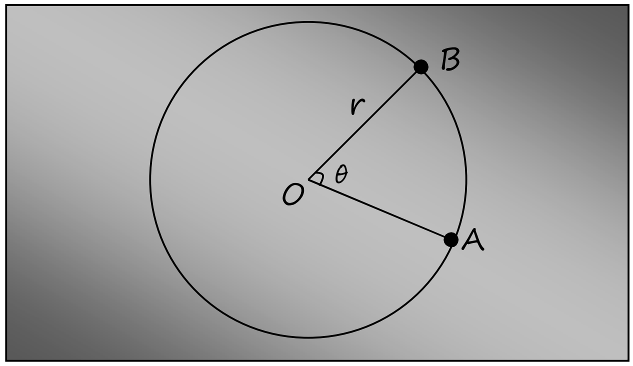
$$\ldots, -\frac{11\pi}{6}, \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}, \ldots \qquad \text{[in radians]}$$

Therefore, any point θ on the circle has infinitely many values:



♦ ARCS AND SECTORS

Consider a circle of radius r. Let θ be the angle shown below **measured in radians!**



The length of the arc AB is given by $\boxed{L=r\theta}$

The area of the sector OAB is given by $\boxed{A=\frac{1}{2}r^2\theta}$

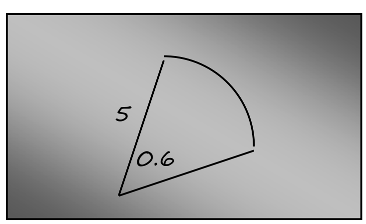
In particular, if we consider $\theta=2\pi$ (complete circle), we obtain

$L = r\theta = 2\pi r$ (the circumference of the circle)

$A = \frac{1}{2}r^2\theta = \pi r^2$ (the area in the circle)

EXAMPLE 3

Consider the following sector of a circle with $r=5$m and $\theta=0.6$rad:



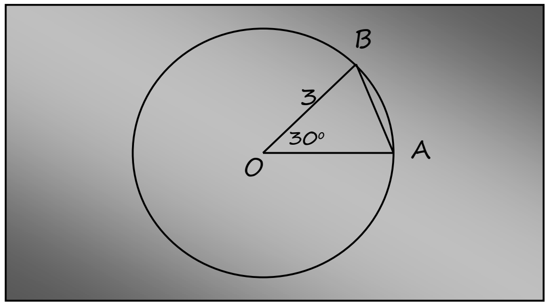
Then

Length of arc: $L = r\theta = 5(0.6) = 3$

Area of sector: $A = \frac{1}{2}r^2\theta = \frac{1}{2}5^2(0.6) = 7.5\ m^2$

Perimeter of sector: $L+r+r = 3+5+5 = 13m$

EXAMPLE 4



Let r =3cm and θ=30°. Find

a) the length of the arc AB **b)** the area of the sector OAB

c) the distance AB **d)** the area of the triangle OAB

Solution

First of all we have to transform θ in radians: $\theta = \dfrac{\pi}{6}$

a) $L = r\theta = 3\dfrac{\pi}{6} = \dfrac{\pi}{2} = 1.57 \text{ cm}$

b) $A_{sector} = \dfrac{1}{2}r^2\theta = \dfrac{1}{2}3^2\dfrac{\pi}{6} = \dfrac{3\pi}{4} = 2.36 \text{ cm}^2$

c) For AB we use COSINE RULE:

$AB^2 = 3^2+3^2-2\cdot3\cdot3\cos\dfrac{\pi}{6} \Rightarrow AB = \sqrt{2.41154} = 1.55 \text{ cm}$

d) $A_{triangle} = \dfrac{1}{2}3\cdot3\sin\dfrac{\pi}{6} = 2.25 \text{ cm}^2$

Notice in the example above

- length of arc AB > length of side AB : 1.57 > 1.55
- area of sector OAB > area of triangle OAB : 2.36 > 2.25

as expected!

Furthermore, the **area of the segment** between side AB and arc AB is the difference $A_{sector}-A_{triangle}$ =2.36-2.25=0.11.

In general,

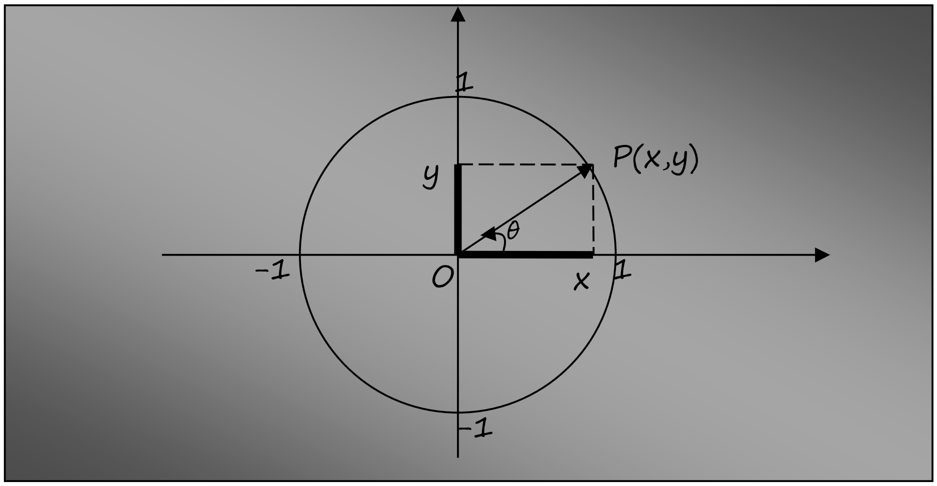
$$A_{segment} = \frac{1}{2}r^2(\theta-\sin\theta)$$

(can you explain why?).

3.5 SIN, COS, TAN ON THE UNIT CIRCLE

- **sinθ AND cosθ**

Consider again the **unit circle** (radius r =1) on the Cartesian plane.



Let P(x,y) be a point on the circle,

OP = r = 1

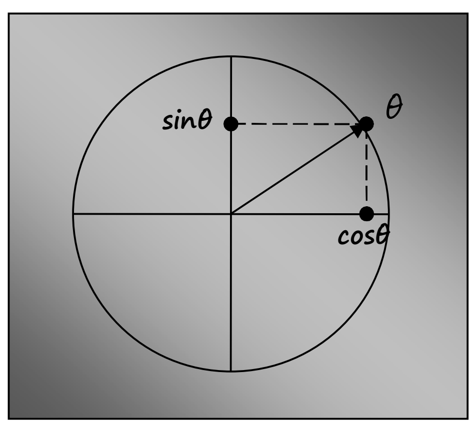
θ = angle between OP and x-axis

Then

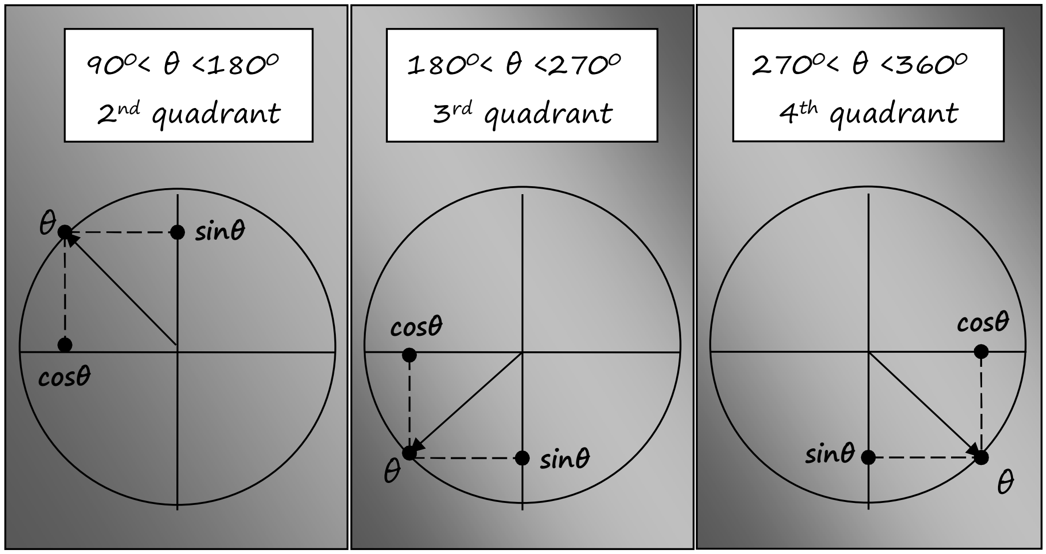
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{1} = y \qquad \text{and} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{1} = x$$

Thus, if we think the angle θ as a point on the circle:

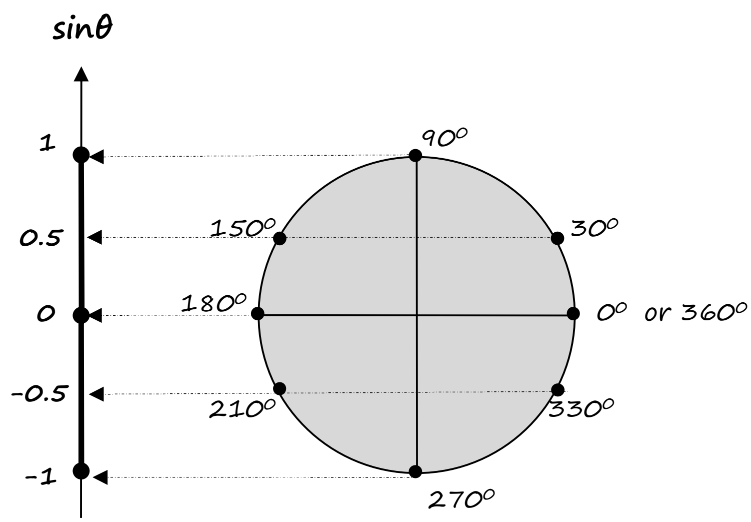
sinθ = y coordinate of θ
cosθ = x coordinate of θ



This description helps us to define sinθ and cosθ not only for angles within $0^o \le \theta \le 90^o$, but for any value of θ on the circumference.



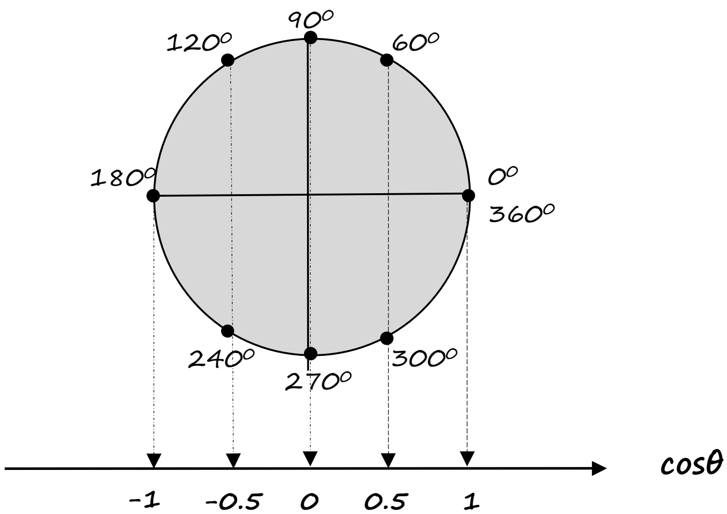
Let us move the y-axis (which shows sinx) to the left of the circle:



sinθ is the y-coordinate of θ									
θ	0^o	30^o	90^o	150^o	180^o	210^o	270^o	330^o	360^o
sinθ	0	0.5	1	0.5	0	– 0.5	–1	– 0.5	0

This diagram explains why **supplementary angles have equal sines**.

Let us move now the x-axis (which shows cosx) under the circle:



cosθ is the **x**-coordinate of θ									
θ	0^o	60^o	90^o	120^o	180^o	240^o	270^o	300^o	360^o
cosθ	1	0.5	0	– 0.5	–1	– 0.5	0	0.5	1

This diagram explains why **opposite angles have equal cosines**.

NOTICE

As we have said, any point on the circle represents infinitely many angle values. In that sense, all these angles have the same sine and the same cosine.

For example, the point of 30^o also represents the values

30^o+360^ok: **…, –330°, 30°, 390°, 750°, …**

Check by your GDC:

$\sin 30^o = 0.5$ $\cos 30^o = \sqrt{3}/2$

$\sin 390^o = 0.5$ $\cos 390^o = \sqrt{3}/2$

$\sin 750^o = 0.5$ etc $\cos 750^o = \sqrt{3}/2$ etc

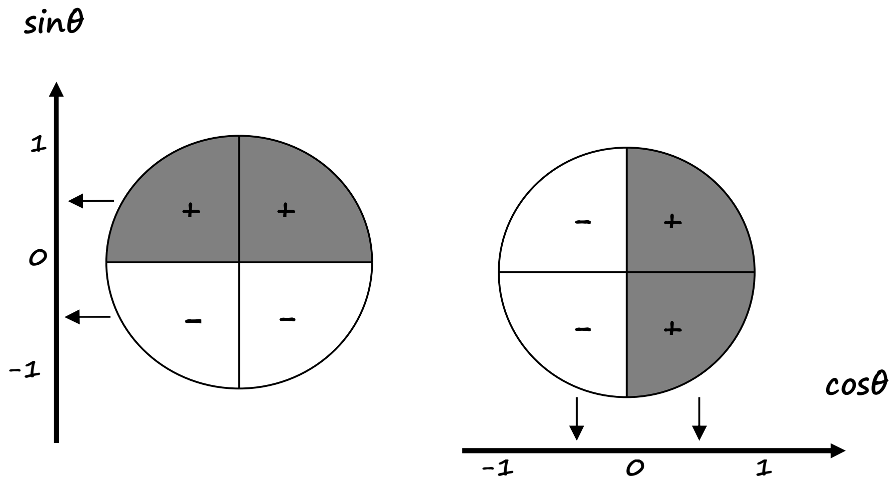
A similar observation applies when θ is in **radians**!

All values $\frac{\pi}{6}+2k\pi$: have the same sine and cosine

---

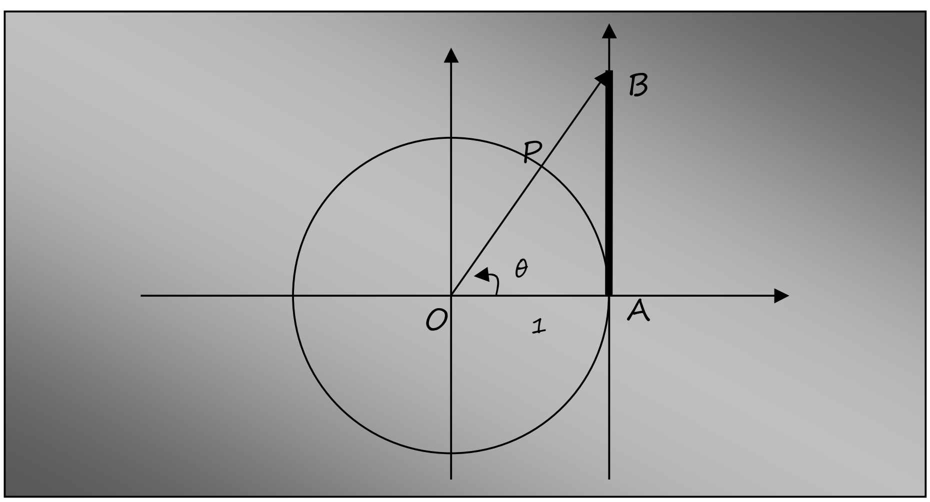
We understand that

$$\boxed{-1 \le \sin\theta \le 1} \qquad\qquad \boxed{-1 \le \cos\theta \le 1}$$



♦ **tanθ**

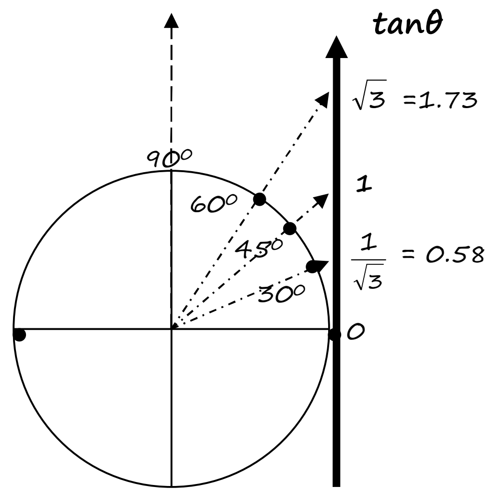
Consider now the **unit circle** below and an additional vertical axis passing through point A (it is tangent to the circle!)



Then

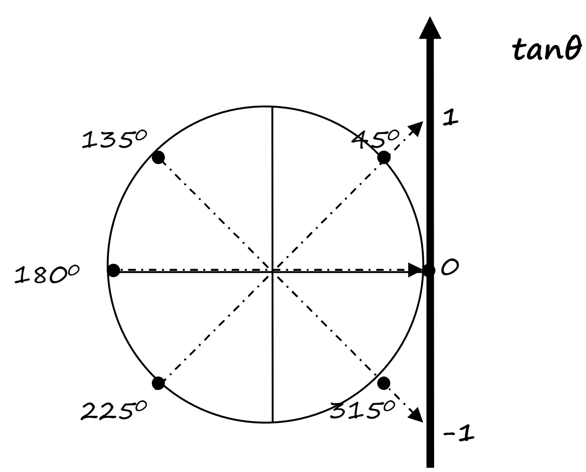
$$\boxed{\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{AB}{1} = AB}$$

Thus, the value on this axis indicates the value of the tangent:



θ	0^o	30^o	45^o	60^o	90^o
tanθ	0	$1/\sqrt{3}$	1	$\sqrt{3}$	$+\infty$

Again, this description helps us to define tanθ not only for angles θ within $0^o \le \theta \le 90^o$.



x	0^o	45^o	135^o	180^o	225^o	315^o	360^o
tanθ	0	1	−1	0	1	− 1	0

It is clear that **diametrically opposite angles have equal tangents**.

NOTICE

- Not only θ, but all values

$$\theta + 180k^{o} \quad \text{(in degrees)}$$
$$\theta + k\pi \quad \text{(in radians)}$$

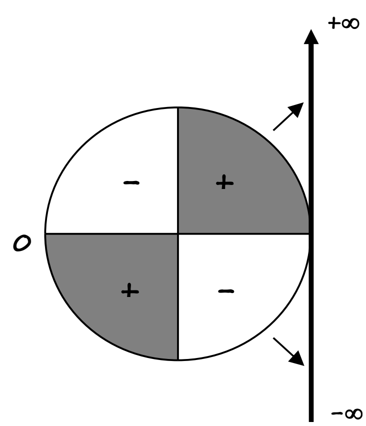
  have equal tangents (we just add or subtract semicircles).

- It is obvious that $\tan\theta$ is not defined for $\theta=90^{o}$ or $\theta=-90^{o}$. In fact, $\tan\theta$ is not defined for

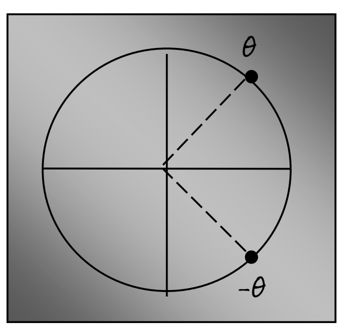
$$90^{o} + 180k^{o} \quad \text{(in degrees)} \qquad \frac{\pi}{2} + k\pi \quad \text{(in radians)}$$

  For any other value of θ,

$$\boxed{-\infty < \tan\theta < +\infty}$$



It is worthwhile to notice that for opposite angles, θ and $-\theta$



$\cos(-\theta) = \cos\theta$	[even function]
$\sin(-\theta) = -\sin\theta$	[odd function]
$\tan(-\theta) = -\tan\theta$	[odd function]

- **TRIGONOMETRIC IDENTITIES**

The trigonometric numbers are interconnected by some relations known as trigonometric identities.

We have already seen the fundamental Pythagorean identity

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

Thus, for example. if we know $\sin\theta$ we can find $\cos\theta$ (provided we know the quadrant of θ).

The following identities are known as **double angle identities.**

They connect the trigonometric numbers of 2θ with those of θ.

$\sin 2\theta = 2\sin\theta\cos\theta$	$\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\cos 2\theta = 2\cos^2\theta - 1$ $\cos 2\theta = 1 - 2\sin^2\theta$	$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

[only for HL]

EXAMPLE 1

Let $\sin\theta = \frac{3}{5}$. Find

$$\cos\theta,\ \tan\theta,\ \sin 2\theta,\ \cos 2\theta,\ \tan 2\theta$$

if

(a) $\theta < 90^o$ (acute)

(b) $90^o < \theta < 180^o$ (obtuse)

Solution

By the fundamental identity $\sin^2\theta + \cos^2\theta = 1$, we obtain

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25},$$

thus

$$\cos\theta = \pm\frac{4}{5}$$

If θ is acute (1st quadrant) $\boxed{\cos\theta = \frac{4}{5}}$, if θ is obtuse $\boxed{\cos\theta = -\frac{4}{5}}$

(a) Since $\theta < 90^o$

$$\cos\theta = \frac{4}{5}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3}{4},$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\,\frac{3}{5}\,\frac{4}{5} = \frac{24}{25}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{24}{7}.$$

(b) Since $90^o < \theta < 180^o$

$$\cos\theta = -\frac{4}{5}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{3}{4},$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(-\frac{3}{5}\right)\frac{4}{5} = -\frac{24}{25}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = -\frac{24}{7}.$$

NOTICE

Consider the double angle identity

$$\sin 2\theta = 2\sin\theta\cos\theta$$

That means

$$\sin 30^o = 2\sin 15^o\cos 15^o$$

$$\sin 100^o = 2\sin 50^o\cos 50^o$$

or

$$\sin 4\theta = 2\sin 2\theta\cos 2\theta$$

$$\sin 10\theta = 2\sin 5\theta\cos 5\theta$$

Similar variations can be obtained by the other identities, e.g.

$$\cos 30^o = 1 - 2\sin^2 15^o$$

$$\cos 4\theta = 1 - 2\sin^2 2\theta$$

NOTICE

If we divide the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$, we obtain

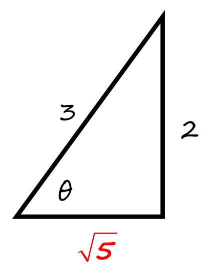
$$\tan^2\theta + 1 = \frac{1}{\cos^2\theta}$$

This identity connects $\tan\theta$ with $\cos\theta$. Thus, all three trigonometric numbers $\sin\theta$, $\cos\theta$, $\tan\theta$ are interconnected.

However, we can easily obtain this interconnection by using the **right-angled triangle method**:

For example, let θ be an angle in the first quadrant with $\sin\theta = \frac{2}{3}$.

We construct a right-angled triangle to represent this information.



By using the Pythagorean identity, we find the third side: $\sqrt{5}$

Thus, we know all three trigonometric numbers:

$$\sin\theta = \frac{2}{3}. \qquad \cos\theta = \frac{\sqrt{5}}{3}. \qquad \tan\theta = \frac{2}{\sqrt{5}}.$$

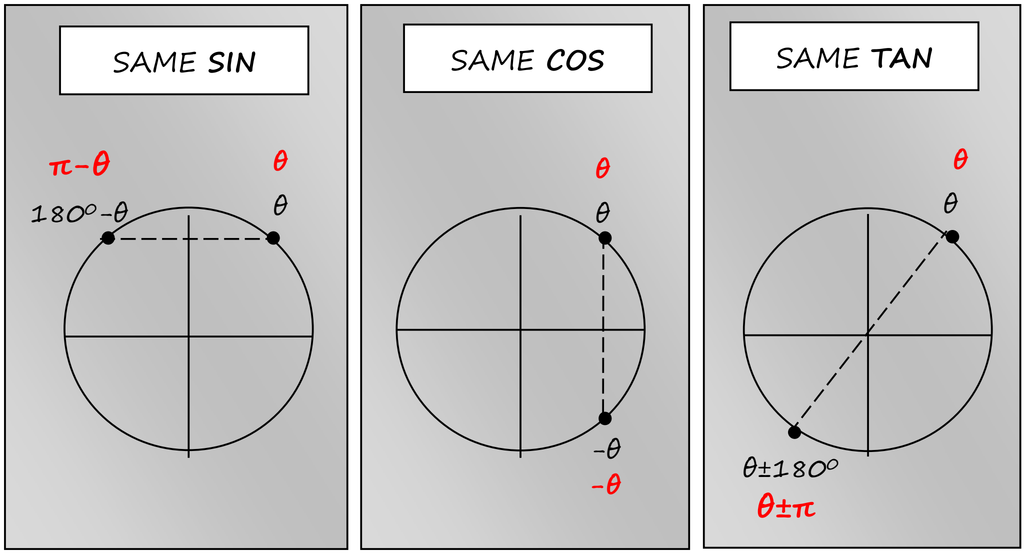
In fact, this method provides the absolute value of the trigonometric numbers. If θ belongs to another quadrant we must take care of the sign ($\pm$) as well.

For example, in the example above, if θ is in the 2^{nd} quadrant, it is still $\sin\theta = \frac{2}{3}$, but we know that $\cos\theta$ and $\sin\theta$ are negative. So

$$\cos\theta = -\frac{\sqrt{5}}{3}. \qquad \tan\theta = -\frac{2}{\sqrt{5}}$$

3.6 TRIGONOMETRIC EQUATIONS

- Remember that



Consider for example the equation

$$\sin x = \frac{1}{2}$$

We know that $\sin 30^o = \frac{1}{2}$. But also $\sin 150^o = \frac{1}{2}$

These two values give the complete set of solutions:

$$x = 30^o + 360k,$$
$$x = 150^o + 360k, \quad (k \in Z)$$

(the two formulas above are known as **general solution**).

We usually ask for the solutions within a particular domain

Methodology:

Equation:	$\sin x = \frac{1}{2}$	
Think:	$\sin x = \sin 30^o$	[you may skip this step]
General solution:	$x=30^o+360k$,	
	$x=150^o+360k$,	
Particular solutions:	Try for $k = 0, \pm1, \pm2, \ldots$ to find solutions within the given domain	

In general, for the following equations,

$$\sin x = a$$

$$\cos x = a$$

$$\tan x = a$$

we find the principal solution θ and express them in the form

$$\sin x = \sin\theta$$

$$\cos x = \cos\theta$$

$$\tan x = \tan\theta$$

The general solutions are given below:

EQUATION	IN DEGREES	IN RADIANS
$\sin x=\sin\theta$	$x=\theta+360^\circ k$ $x=(180^\circ-\theta)+360^\circ k$	$x=\theta+2k\pi$ $x=(\pi-\theta)+2k\pi$
$\cos x=\cos\theta$	$x=\theta+360^\circ k$ $x=-\theta+360^\circ k$	$x=\theta+2k\pi$ $x=-\theta+2k\pi$
$\tan x=\tan\theta$	$x=\theta+180^\circ k$	$x=\theta+k\pi$

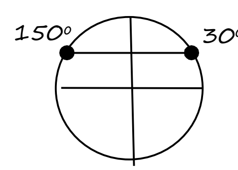
EXAMPLE 2

Solve the equations: (a) $\sin x=\frac{1}{2}$, $0^\circ \leq x \leq 360^\circ$

(b) $\cos x=\frac{1}{2}$, $0^\circ \leq x \leq 360^\circ$

Solution

(a) $\sin x=\frac{1}{2} \Leftrightarrow \sin x=\sin 30^\circ$



General solution: $x=30^\circ+360^\circ k$

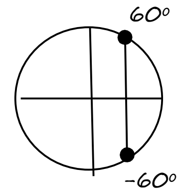
$x=150^\circ+360^\circ k$

Within the given domain:

The 1st equation gives only $\boxed{x=30^\circ}$ (for $k=0$)

The 2nd equation gives only $\boxed{x=150^\circ}$ (for $k=0$)

(b) $\cos x=\frac{1}{2} \Leftrightarrow \cos x=\cos 60^\circ$



General solution: $x=60^\circ + 360^\circ k$

$x=-60^\circ + 360^\circ k$

Within the given domain:

The 1st equation gives only $\boxed{x=60^\circ}$ (for k=0)

The 2nd equation gives only $\boxed{x=300^\circ}$ (for k=1)

In the following example, let us see an equation under three different domains.

EXAMPLE 3

Solve the equation tanx=1

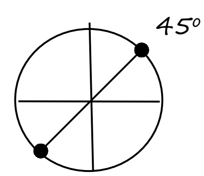
(a) for $0^\circ \le x \le 360^\circ$

(b) for $-180^\circ \le x \le 180^\circ$

(c) for $-180^\circ \le x \le 450^\circ$

Solution

$\tan x=1 \Leftrightarrow \tan x=45^\circ$



General solution: $x=45^\circ + 180^\circ k$

(a) Within $0^\circ \le x \le 360^\circ$ we obtain

$\boxed{x=45^\circ}$ $\boxed{x=225^\circ}$

(b) Within $-180^\circ \le x \le 180^\circ$

$\boxed{x=-135^\circ}$ $\boxed{x=45^\circ}$

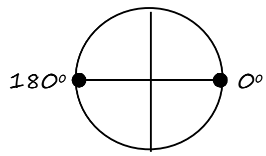
(c) Within $-180^\circ \le x \le 450^\circ$

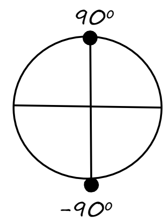
$\boxed{x=-135^\circ}$ $\boxed{x=45^\circ}$ $\boxed{x=225^\circ}$ $\boxed{x=405^\circ}$

NOTICE

Particularly for the equations

$\sin x=0$ and $\cos x=0$





there exist two different approaches for the general formula.

(a) For $\sin x=0$

$\sin x=0$	$x = 0°+360°k$ $x = 180°+360°k$	$x = 0+2k\pi$ $x = \pi+2k\pi$

Since the two basic solutions are diametrically opposite, we may merge the two general solutions into one:

$\sin x=0$	$x = 180°k$	$x = k\pi$

(b) For $\cos x=0$

$\cos x=0$	$x = 90°+360°k$ $x = -90°+360°k$	$x = \frac{\pi}{2}+2k\pi$ $x = -\frac{\pi}{2}+2k\pi$

Since the two basic solutions are diametrically opposite, we may merge the two general solutions into one:

$\cos x=0$	$x = 90°+180°k$	$x = \frac{\pi}{2}+k\pi$

(c) Practically,

solutions of sinx=0	**solutions of cosx=0**
…, -180°, 0°, 180°, 360° , …	…, -90°, 90°, 270° , 450° …
…, $-\pi$, 0, π, 2π, 3π, …	…, $-\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, …

The general solutions of the basic trigonometric equations are presented below.

Equation	Basic solution	General solution in degrees	General solution in radians
$\sin x=\frac{1}{2}$	$30° = \frac{\pi}{6}$	$x=30°+360°k$ $x=150°+360°k$	$x=\frac{\pi}{6}+2k\pi$ $x=\frac{5\pi}{6}+2k\pi$
$\sin x=\frac{\sqrt{2}}{2}$	$45° = \frac{\pi}{4}$	$x=45°+360°k$ $x=135°+360°k$	$x=\frac{\pi}{4}+2k\pi$ $x=\frac{3\pi}{4}+2k\pi$
$\sin x=\frac{\sqrt{3}}{2}$	$60° = \frac{\pi}{3}$	$x=60°+360°k$ $x=120°+360°k$	$x=\frac{\pi}{3}+2k\pi$ $x=\frac{2\pi}{3}+2k\pi$
$\sin x=-\frac{1}{2}$	$-30° = -\frac{\pi}{6}$	$x=-30°+360°k$ $x=-150°+360°k$	$x=-\frac{\pi}{6}+2k\pi$ $x=-\frac{5\pi}{6}+2k\pi$
$\sin x=-\frac{\sqrt{2}}{2}$	$-45° = -\frac{\pi}{4}$	$x=-45°+360°k$ $x=-135°+360°k$	$x=-\frac{\pi}{4}+2k\pi$ $x=-\frac{3\pi}{4}+2k\pi$
$\sin x=-\frac{\sqrt{3}}{2}$	$-60° = -\frac{\pi}{3}$	$x=-60°+360°k$ $x=-120°+360°k$	$x=-\frac{\pi}{3}+2k\pi$ $x=-\frac{2\pi}{3}+2k\pi$
Extreme cases			
$\sin x=1$	$90° = \frac{\pi}{2}$	$x=90°+360°k$	$x=\frac{\pi}{2}+2k\pi$
$\sin x=-1$	$-90° = -\frac{\pi}{2}$	$x=-90°+360°k$	$x=-\frac{\pi}{2}+2k\pi$
For $\sin x=0$			
$\sin x=0$	0	$x=180°k$	$x=k\pi$

Equation	Basic solution	General solution in degrees	General solution in radians
$\cos x = \frac{1}{2}$	$60° = \frac{\pi}{3}$	$x = \pm 60° + 360°k$	$x = \pm\frac{\pi}{3} + 2k\pi$
$\cos x = \frac{\sqrt{2}}{2}$	$45° = \frac{\pi}{4}$	$x = \pm 45° + 360°k$	$x = \pm\frac{\pi}{4} + 2k\pi$
$\cos x = \frac{\sqrt{3}}{2}$	$30° = \frac{\pi}{6}$	$x = \pm 30° + 360°k$	$x = \pm\frac{\pi}{6} + 2k\pi$
$\cos x = -\frac{1}{2}$	$120° = \frac{2\pi}{3}$	$x = \pm 120° + 360°k$	$x = \pm\frac{2\pi}{3} + 2k\pi$
$\cos x = -\frac{\sqrt{2}}{2}$	$135° = \frac{3\pi}{4}$	$x = \pm 135° + 360°k$	$x = \pm\frac{3\pi}{4} + 2k\pi$
$\cos x = -\frac{\sqrt{3}}{2}$	$150° = \frac{5\pi}{6}$	$x = \pm 150° + 360°k$	$x = \pm\frac{5\pi}{6} + 2k\pi$
Extreme cases			
$\cos x = 1$	$0° = 0$	$x = 360°k$	$x = 2k\pi$
$\cos x = -1$	$180° = \pi$	$x = 180° + 360°k$	$x = \pi + 2k\pi$
For $\cos x = 0$			
$\cos x = 0$	$90° = \frac{\pi}{2}$	$x = 90° + 180°k$	$x = \frac{\pi}{2} + k\pi$

Equation	Basic solution	General solution in degrees	General solution in radians
$\tan x = \sqrt{3}$	$60° = \frac{\pi}{3}$	$x = 60° + 180°k$	$x = \frac{\pi}{3} + k\pi$
$\tan x = 1$	$45° = \frac{\pi}{4}$	$x = 45° + 180°k$	$x = \frac{\pi}{4} + k\pi$
$\tan x = 1/\sqrt{3}$	$30° = \frac{\pi}{6}$	$x = 30° + 180°k$	$x = \frac{\pi}{6} + k\pi$
$\tan x = 0$	$0° = 0$	$x = 180°k$	$x = k\pi$
$\tan x = -1/\sqrt{3}$	$-30° = -\frac{\pi}{6}$	$x = -30° + 180°k$	$x = -\frac{\pi}{6} + k\pi$
$\tan x = -1$	$-45° = -\frac{\pi}{4}$	$x = -45° + 180°k$	$x = -\frac{\pi}{4} + k\pi$
$\tan x = -\sqrt{3}$	$-60° = -\frac{\pi}{3}$	$x = -60° + 180°k$	$x = -\frac{\pi}{3} + k\pi$

Let us see some slightly different equations of this form.

EXAMPLE 4

Solve the equation $\sin 2x = \frac{\sqrt{3}}{2}$,

(a) in the domain $0^o \le x \le 360^o$ (in degrees)

(b) in the domain $0 \le x \le 2\pi$ (in radians)

Solution

(a) $\sin 2x = \sin 60^o$

The general solution is

$$2x = 60^o + 360^o k \Rightarrow x = 30^o + 180^o k \quad (1)$$
$$2x = 120^o + 360^o k \Rightarrow x = 60^o + 180^o k \quad (2)$$

(1) gives $\boxed{x=30^o}$ $\boxed{x=210^o}$ (2) gives $\boxed{x=60^o}$ $\boxed{x=240^o}$

(b) $\sin 2x = \sin\frac{\pi}{3}$.

The general solution is

$$2x = \frac{\pi}{3} + 2k\pi \Rightarrow x = \frac{\pi}{6} + k\pi = \frac{\pi + 6k\pi}{6} \quad (1)$$
$$2x = \frac{2\pi}{3} + 2k\pi \Rightarrow x = \frac{\pi}{3} + k\pi = \frac{\pi + 4k\pi}{3} \quad (2)$$

(1) gives $\boxed{x=\frac{\pi}{6}}$ $\boxed{x=\frac{7\pi}{6}}$ (2) gives $\boxed{x=\frac{\pi}{3}}$ $\boxed{x=\frac{4\pi}{3}}$

EXAMPLE 5

Solve the equation $\cos 3x = 0$, $-180^o \le x \le 180^o$

Solution

$\cos 3x = \cos 90^o$.

The general solution is

$$3x = 90^o + 180^o k \Rightarrow x = 30^o + 60^o k$$

Therefore $\boxed{x=30^o}$ $\boxed{x=90^o}$ $\boxed{x=150^o}$ $\boxed{x=-30^o}$ $\boxed{x=-90^o}$ $\boxed{x=-150^o}$

EXAMPLE 6

Solve the equation $\cos 2x=\frac{\sqrt{2}}{2}$ $0\le x\le 2\pi$

Solution

$\cos 2x=\cos\frac{\pi}{4}$

Hence $2x=\frac{\pi}{4}+2k\pi \Rightarrow x=\frac{\pi}{8}+k\pi=\frac{\pi+8\pi}{8}$ (1)

$2x=-\frac{\pi}{4}+2k\pi \Rightarrow x=-\frac{\pi}{8}+k\pi=\frac{-\pi+8\pi}{8}$ (2)

(1) gives $\boxed{x=\frac{\pi}{8}}$, $\boxed{x=\frac{9\pi}{8}}$ (2) gives $\boxed{x=\frac{7\pi}{8}}$, $\boxed{x=\frac{15\pi}{8}}$.

EXAMPLE 7

Solve the equation $\tan 3x=\sqrt{3}$

(a) in the domain $-180^o\le x\le 180^o$ (in degrees)

(b) in the domain $-\pi\le x\le \pi$ (in radians)

Solution

(a) $\tan 3x=\tan 60^o$

Hence $3x=60^o+180^o k \Leftrightarrow x=20^o+60^o k$

For k=0,1,2, … we obtain $\boxed{x=20^o}$, $\boxed{x=80^o}$, $\boxed{x=140^o}$

For k=-1,-2, … we obtain $\boxed{x=-40^o}$, $\boxed{x=-100^o}$, $\boxed{x=-160^o}$

(c) $\tan 3x=\tan 60^o$

Hence $3x=\frac{\pi}{3}+k\pi \Leftrightarrow x=\frac{\pi}{9}+\frac{k\pi}{3}=\frac{\pi+3k\pi}{9}$

For k=0,1,2, … we obtain $\boxed{x=\frac{\pi}{9}}$, $\boxed{x=\frac{4\pi}{9}}$, $\boxed{x=\frac{7\pi}{9}}$

For k=-1,-2, … we obtain $\boxed{x=-\frac{2\pi}{9}}$, $\boxed{x=-\frac{5\pi}{9}}$, $\boxed{x=-\frac{8\pi}{9}}$

More complicated trigonometric equations usually reduce to simple equations as above.

EXAMPLE 8

Solve the equation $\sin 2x=\sin x$, $0^\circ \le x \le 360^\circ$

LHS, sin2x, can be simplified by using the double angle formula.

$$\begin{aligned} \sin 2x=\sin x &\Leftrightarrow 2\sin x\cos x=\sin x \\ &\Leftrightarrow 2\sin x\cos x-\sin x=0 \\ &\Leftrightarrow \sin x(2\cos x-1)=0 \\ &\Leftrightarrow \sin x=0 \text{ or } 2\cos x-1=0 \\ &\Leftrightarrow \sin x=0 \text{ or } \cos x=1/2 \end{aligned}$$

We solve the two simple equations

- $\sin x=0 \Leftrightarrow \boxed{x=0^\circ}$, or $\boxed{x=180^\circ}$ or $\boxed{x=360^\circ}$
- $\cos x=1/2 \Leftrightarrow \boxed{x=60^\circ}$ or $\boxed{x=300^\circ}$

Hence, the equation has five solutions $\boxed{0^\circ, 60^\circ, 180^\circ, 300^\circ, 360^\circ}$.

REMARKS:

- If the equation is given in radians under the restriction $0 \le x \le 2\pi$ we obtain

  $\sin x=0 \Leftrightarrow \boxed{x=0}$, or $\boxed{x=\pi}$ or $\boxed{x=2\pi}$

  $\cos x=1/2 \Leftrightarrow \boxed{x=\pi/3}$ or $\boxed{x=5\pi/3}$

  Thus the five solutions are $\boxed{0, \pi/3, \pi, 5\pi/3 \text{ and } 2\pi}$

- If $180^\circ \le x \le 180^\circ$ we obtain $\boxed{x=0^\circ}$, $\boxed{x=\pm 60^\circ}$, $\boxed{x=\pm 180^\circ}$

- If $-\pi \le x \le \pi$ we obtain $\boxed{x=0}$, $\boxed{x=\pm\pi/3}$, $\boxed{x=\pm\pi}$

An interesting case is the following where the trigonometric equation has a quadratic form.

EXAMPLE 9 (Quadratic form)

Solve the equation $2\cos^2x-3\cos x+1=0, \quad 0\le x\le\pi$

If you let $y=\cos x$, the given equation has the form $2y^2-3y+1=0$

The roots of this equation are $y=1$ and $y=1/2$

Thus

- $\cos x=1$. This equation has only one solution: $x=0$
- $\cos x=1/2$. This equation has only one solution: $x=\pi/3$

Therefore, there are two solutions: $\boxed{x=0}$, $\boxed{x=\pi/3}$

EXAMPLE 10

Solve the equation $3(1-\cos x)=2\sin^2x, \quad 0\le x\le\pi$

(We wish to have only cosx or only sinx. Hence, we use the Pythagorean identity to substitute $\sin^2x$ by $1-\cos^2x$)

$$3(1-\cos x)=2(1-\cos^2x) \Leftrightarrow 3-3\cos x=2-2\cos^2x$$
$$\Leftrightarrow 2\cos^2x-3\cos x+1=0$$

This is in fact the equation in example 8 above.

Therefore, there are two solutions: $\boxed{x=0}$, $\boxed{x=\pi/3}$

Equations of the form $A\sin x=B\cos x$, take the form $\tan x=\frac{B}{A}$

EXAMPLE 11

Solve the equation $\sqrt{3}\sin x=\cos x, \quad 0\le x\le 2\pi$

It takes the form $\tan x=\frac{1}{\sqrt{3}}$.

The general solution is $x=\frac{\pi}{6}+\kappa\pi$

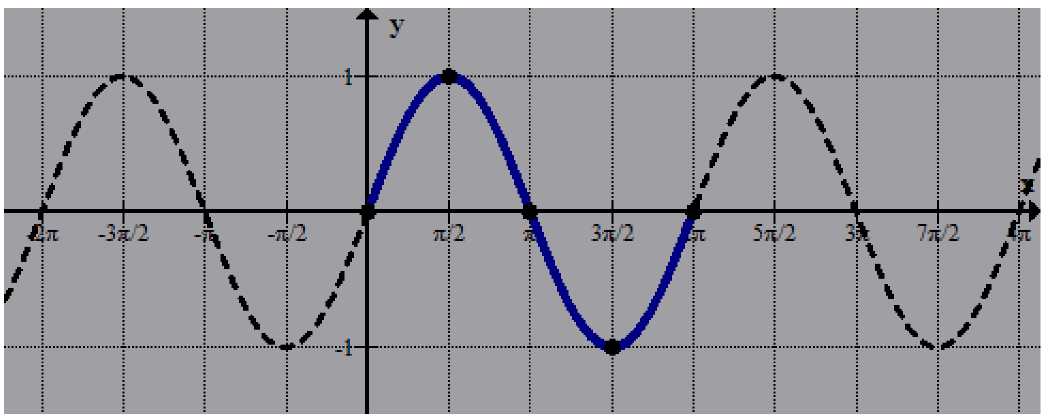
There are two solutions: $\boxed{x=\pi/6}$, and $\boxed{x=7\pi/6}$.

3.7 TRIGONOMETRIC FUNCTIONS

♦ **f(x) = sinx**

Let us construct the graph of this function in the traditional way, that is in the Cartesian plane Oxy.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	…
f(x)	0	1	0	-1	0	



We have:

Domain: $x \in R$

Range: $y \in [-1,1]$ [since $y_{min}=-1$ and $y_{max}=1$]

For functions of this form we also define

Central line: y = 0

Amplitude = 1 (distance between max and central line)

Period: T= 2π (the length of a complete cycle)

Notice that

<u>Amplitude</u> = y_{max}-central value (it is also $\frac{y_{max}-y_{min}}{2}$)

<u>Period</u>=2π means that the curve is repeated every 2π units

Use your GDC to see the graph and compare with the curve above

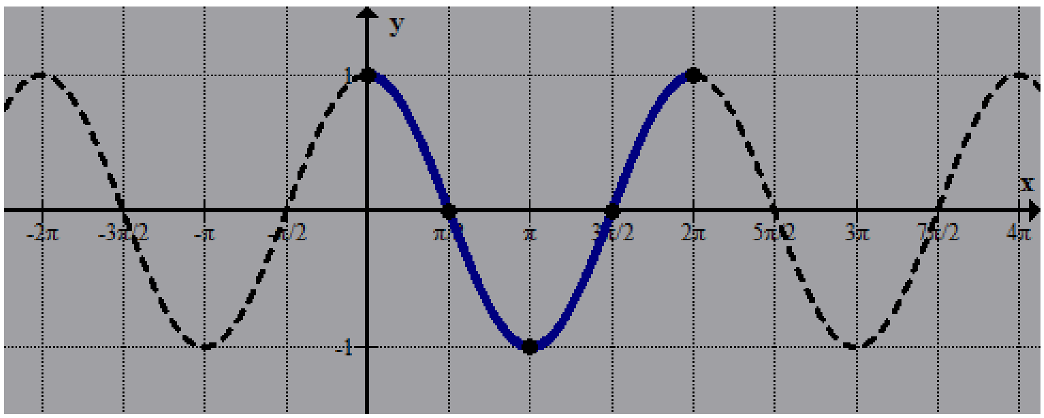
Set V-Window: x **from -2π to 2π**

y **from -2 to 2**

- **$f(x) = \cos x$**

Let us now construct the graph of this function.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	...
$f(x)$	1	0	-1	0	1	



Again **Domain:** $x \in R$

Range: $y \in [-1,1]$ $[y_{min}=-1$ and $y_{max}=1]$

Central line: $y = 0$

Amplitude $= 1$

Period: $T = 2\pi$

NOTICE:

- For both functions $y=\sin x$ and $y=\cos x$, the horizontal distance

  between two consecutive max $= 2\pi$ (one period)

  between two consecutive min $= 2\pi$ (one period)

  between consecutive max and min $= \pi$ (half a period)

- Use your GDC to see the graph in **degrees.** Period $= 360°$

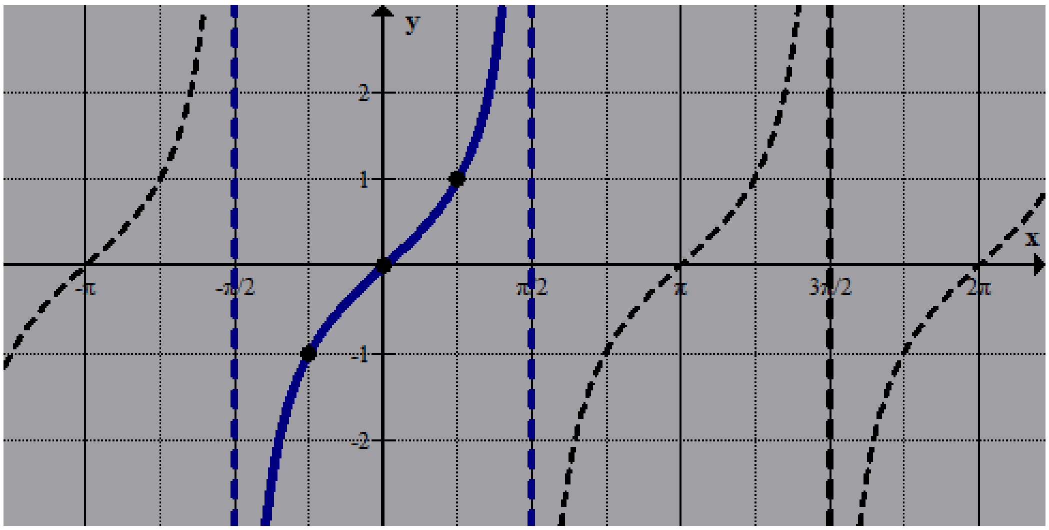
  Set V-Window: x **from $-360°$ to $360°$**

  y **from -2 to 2**

- **$f(x) = \tan x$**

Next, we construct the graph of $y=\tan x$.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	...
$f(x)$	–	-1	0	1	–	



We have:

Domain: $x \in R - \{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...\}$

Range: $y \in R$ [there is no min, no max, no amplitude]

Central line: $y=0$

Period: $T = \pi$

Vertical asymptotes: $x=\frac{\pi}{2}$, $x=-\frac{\pi}{2}$, etc

---

NOTICE

Remember our discussion about asymptotes in Topic 2.

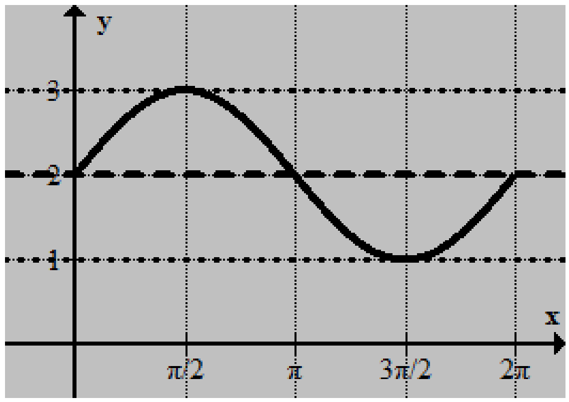
A vertical line $x=a$ is an asymptote of the graph $y=f(x)$ if

- the function is not defined at $x=a$, and
- for values of x very close to a the value of $y=f(x)$ approaches $+\infty$ or $-\infty$

♦ TRANSFORMATIONS OF sinx AND cosx

Consider the function $f(x) = \sin x + 2$.

Its graph is a vertical translation of sinx, 2 units up:



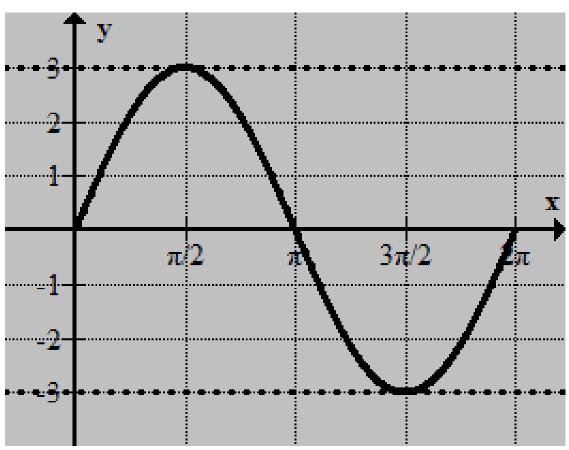
Clearly, **Central line: y=2**

Range: $y \in [1,3]$ $[y_{min}=1$ and $y_{max}=3]$

Amplitude=1 and Period: $T=2\pi$ remain the same.

Consider the function $f(x) = 3\sin x$.

Its graph is a vertical stretch of sinx with scale factor 3:



Clearly, **Amplitude = 3**

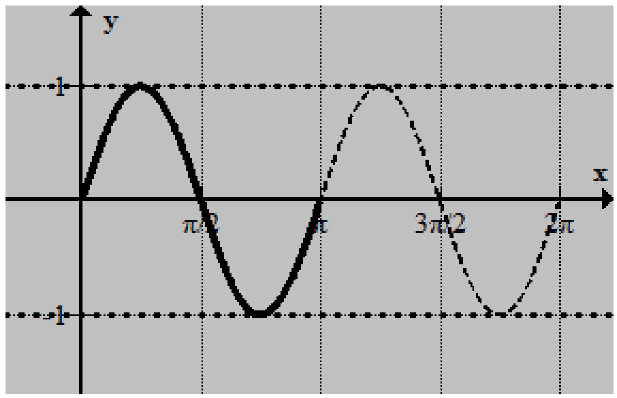
Range: $y \in [-3,3]$ $[y_{min}=-3$ and $y_{max}=3]$

Central line: y=0 and Period: $T=2\pi$ remain the same

Notice: the amplitude of $f(x)=-3\sin x$ is still 3

Consider the function f(x) = sin2x.

Its graph is a horizontal stretch of sinx with scale factor 1/2:



Now, **Period: T = π**

Central line: y=0, Amplitude=1, Range: $y\in[-1,1]$ remain the same

In general, the function

$$f(x) = A\sin Bx + C$$

with A>0, is obtained by three transformations on sinx:

a vertical stretch with scale factor **A**,

a horizontal stretch with scale factor 1/**B**,

a vertical translation by **C** units (up or down),

(If **A**<0 we also have a reflection in x-axis at the beginning)

Consequently, for the new function f(x):

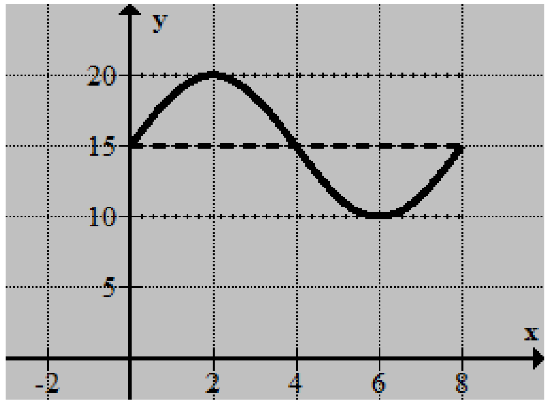
C	is the **central value**. The central line is **y=C**
$\|A\|$	is the **amplitude**
$T=\dfrac{2\pi}{B}$	Is the **period**. Hence $B=\dfrac{2\pi}{T}$

Notice:

- f(x) ranges between the values C±A
- Similar observations apply for **f(x) = AcosBx +C**

EXAMPLE 1

The graph of f(x) = AsinBx + C, is given below (A>0). Find A,B,C.

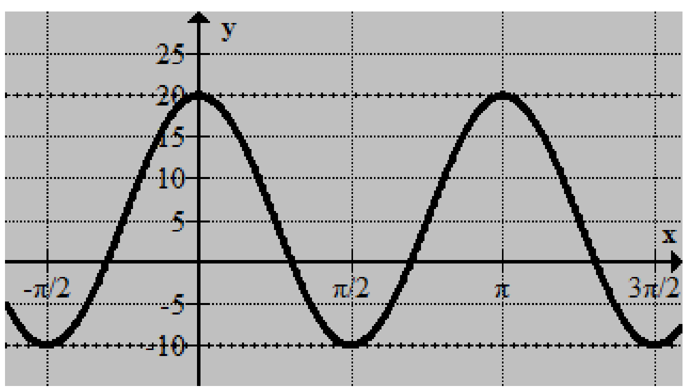


- Central line at y=15, so C=15
- Amplitude = 5, so A=5
- Period T= 8, hence $B=\frac{2\pi}{T}=\frac{2\pi}{8}=\frac{\pi}{4}$

Therefore, the equation of the function is **$f(x) = 5\sin(\frac{\pi}{4}x) + 15$**

EXAMPLE 2

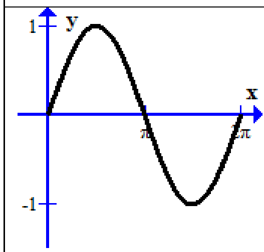
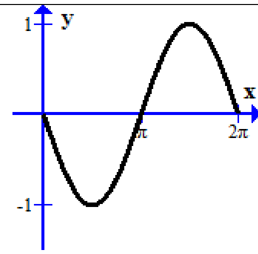
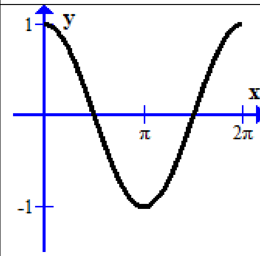
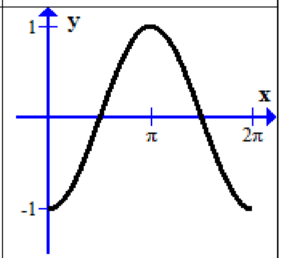
The graph of f(x) = AcosBx +C is given below (A>0). Find A,B,C.



- Central line at $\frac{y_{max}+y_{min}}{2}$=5, so C=5
- Amplitude = y_{max}-C =15, so A=15
- Period T= π, hence $B=\frac{2\pi}{T}=\frac{2\pi}{\pi}=2$

Therefore, the equation of the function is **$f(x) = 15\cos(2x) + 5$**

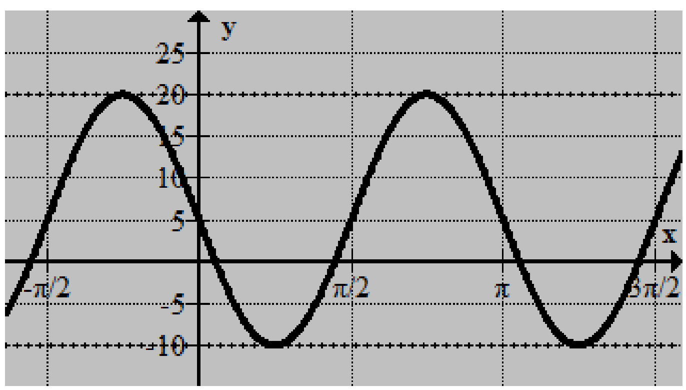
We distinguish **four basic types** of trigonometric functions:

sinx	-sinx	cosx	-cosx
			
y-intercept central/going up	y-intercept central/going down	y-intercept max	y-intercept min

Notice that the amplitude is always positive but the coefficient A of sinx or cosx can be **positive or negative.**

EXAMPLE 3

Express the following graph as a trigonometric function.



We can easily find that

Central line: $y=5$ hence $\mathbf{C=5}$

Amplitude = 15

Period: $T=\pi$ hence $\mathbf{B}=\frac{2\pi}{\pi}=2$

The function of type $-\sin x$ (y-int central/going down), so $\mathbf{A=-15}$

Therefore, the equation of the function is

$$f(x) = -15\sin(2x) + 5$$

Conversely, if we are given a trigonometric function we can easily draw the graph

---

EXAMPLE 4

Draw the graph of the function $f(x) = 5\sin 2x+7, \quad 0 \le x \le 2\pi$

Solution

Central value = 7

Amplitude = 5

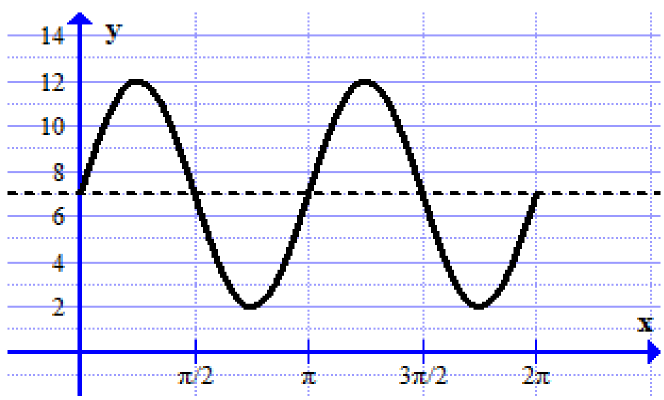
max=12, min=2 (since f(x) ranges between 7 ± 5)

Period $T = \dfrac{2\pi}{2} = \pi$

Thus, we have to draw two periods.

The function is of type **sinx** (y-intercept central/going up).

The graph is



Finally, remember the horizontal transformations f(x-a):

sin(x-D) cos(x-D)	translation D units **to the right**

Therefore, for the functions

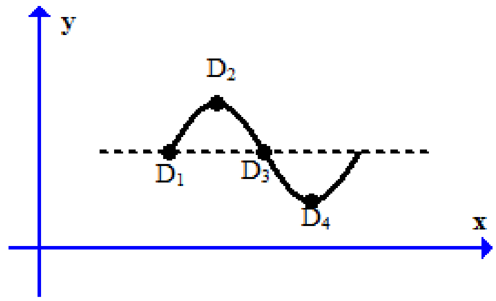
$f(x) = A\sin[B(x-D)] + C$ $\qquad$ $f(x) = A\cos[B(x-D)] + C$

A,B,C are determined as above.

D shows a horizontal translation.

Practically, the value of D depends on the type of the function.

When we see a graph we spot the points shown below!



D_1 for type sinx

D_2 for type cosx

D_3 for type -sinx

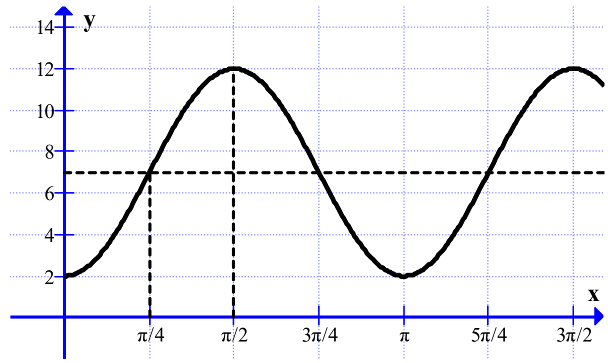
D_4 for -cosx

The value of D is the x-coordinate of the corresponding point above

---

EXAMPLE 5

Consider the graph of a trigonometric function given below.



Central value = 7,	thus C =7
Amplitude = 5,	thus $\lvert A\rvert=5$
Period T = π ,	thus B=2π/π = 2

The most appropriate form is of type -cosx (y-intercept min).

$$f(x)= -5\cos 2x+7$$

However, the same function can be expressed as

[type sinx] $f(x)= 5\sin[2(x-\frac{\pi}{4})]+7$

[type cosx] $f(x)= 5\cos[2(x-\frac{\pi}{2})]+7$

[type -sinx] $f(x)= -5\sin[2(x-\frac{3\pi}{4})]+7$

♦ TRANSFORMATIONS OF tanx

In a similar way,

$$f(x) = A\tan Bx + C$$

is a transformation of tanx. For the new function

$\lvert A \rvert$	is the scale factor of a vertical stretch
$T=\dfrac{\pi}{B}$	is the period
C	is the new central value (units up or down)

EXAMPLE 6

$$f(x)= 10\tan 4x + 30$$

central value= 30

NO min, NO max (A=10 simply shows a vertical stretch of tanx)

Period = $\dfrac{\pi}{4}$

Finally,

tan(x-D)	translation D units **to the right** or **to the left**

Therefore, for a function of the form

$$f(x) = A\tan[B(x-D)] + C$$

A,B,C are determined as above,

D shows a horizontal translation of tanx

ONLY FOR

HL

3.8 MORE TRIGONOMETRIC IDENTITIES AND EQUATIONS

♦ SEC, COSEC, COT

In the HL course we consider three more trigonometric numbers.

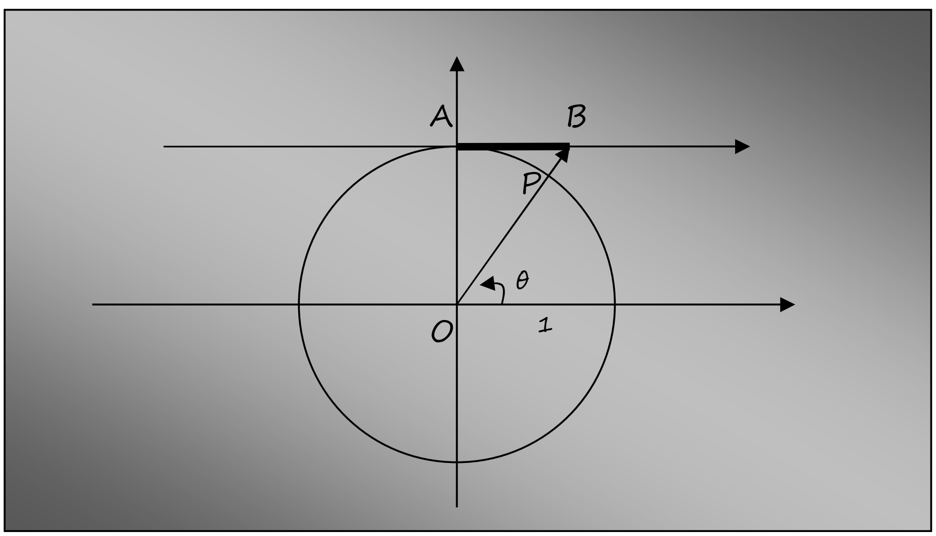
secant: $\sec\theta = \dfrac{1}{\cos\theta}$

cosecant: $\csc\theta = \dfrac{1}{\sin\theta}$

cotangent: $\cot\theta = \dfrac{1}{\tan\theta}$

Particularly for $\cot\theta$, there is a geometrical representation, very similar to that of $\tan\theta$.

Consider the **unit circle** below and an additional horizontal axis passing through point A.



As θ is moving around the circle

$$\cot\theta = AB$$

Thus, for example,

$$\cot 90° = 0, \quad \cot 45° = 1, \quad \cot 30° = \sqrt{3}, \quad \cot 60° = \frac{1}{\sqrt{3}},$$

- **MORE TRIGONOMETRIC IDENTITIES**

If we divide the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$

by $\cos^2\theta$ we obtain $\tan^2\theta + 1 = \dfrac{1}{\cos^2\theta} = \sec^2\theta$

by $\sin^2\theta$ we obtain $\cot^2\theta + 1 = \dfrac{1}{\sin^2\theta} = \csc^2\theta$

For the sum **A+B** and the difference **A–B** of two angles it holds

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

Notice if we set A=B=θ to the formulas for **A+B** we obtain the double-angle identities.

EXAMPLE 2

Find $\sin 75^o$ and $\tan 15^o$ by using appropriate identities.

Solution

$$\sin 75^o = \sin(45^o+30^o) = \sin 45^o\cos 30^o + \cos 45^o\sin 30^o$$

$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

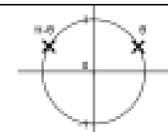
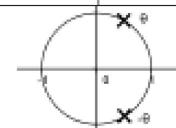
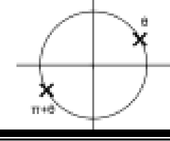
$$\tan 15^o = \tan(60^o-45^o) = \frac{\tan 60^o - \tan 45^o}{1+\tan 60^o\tan 45^o}$$

$$= \frac{\sqrt{3}-1}{1+\sqrt{3}\cdot 1} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

(if we rationalise the last result we obtain $2-\sqrt{3}$)

♦ MORE GENERAL TRIGONOMETRIC EQUATIONS

The following results will help:

If **sinA=sinB**	$A=B+\mathbf{2k\pi}$ $A=(\pi-B)+\mathbf{2k\pi}$	
If **cosA=cosB**	$A=B+\mathbf{2k\pi}$ $A=-B+\mathbf{2k\pi}$	
If **tanA=tanB**	$A=B+\mathbf{k\pi}$	

REMARK If the equations are given in degrees:
$2k\pi$ becomes $360^o k$ $k\pi$ becomes $180^o k$

The basic trigonometric equations we have seen so far, can take the form of the equations above:

$\sin x=a \Leftrightarrow \sin x=\sin\theta$

$\cos x=a \Leftrightarrow \cos x=\cos\theta$ (where θ is the principal solution)

$\tan x=a \Leftrightarrow \tan x=\tan\theta$

EXAMPLE 2

The trigonometric equation

$$\sin(10x)=1/2$$

can take the form

(in radians)	(in degrees)
$\sin(10x)=\sin\frac{\pi}{6}$	$\sin(10x)=\sin 30^o$
$10x=\frac{\pi}{6}+2k\pi \Rightarrow \boxed{x=\frac{\pi+12k\pi}{60}}$ $10x=\frac{5\pi}{6}+2k\pi \Rightarrow \boxed{x=\frac{5\pi+12k\pi}{60}}$	$10x=30^o+360^o k \Rightarrow \boxed{x=3^o+36^o k}$ $10x=150^o+360^o k \Rightarrow \boxed{x=15^o+36^o k}$

However, we can also solve equations where
both A and B are in terms of x.

EXAMPLE 3

Solve the equation $\sin 3x=\sin x$, $0 \le x \le 2\pi$

$$\sin 3x=\sin x \Rightarrow \begin{cases} 3x = x + 2k\pi \\ 3x = \pi - x + 2k\pi \end{cases}$$

$$\Rightarrow \begin{cases} 2x = 2k\pi \\ 4x = \pi + 2k\pi \end{cases}$$

$$\Rightarrow x=k\pi \quad \text{or} \quad x=\frac{\pi+2k\pi}{4}$$

Hence, the solutions are

$$x=0,\ x=\pi,\ x=2\pi,\ x=\frac{\pi}{4},\ x=\frac{3\pi}{4},\ x=\frac{5\pi}{4},\ x=\frac{7\pi}{4}.$$

EXAMPLE 4

Solve the equation $\cos 3x=\cos x$, $0 \le x \le 2\pi$

$$\cos 3x=\cos x \Rightarrow \begin{cases} 3x = x + 2k\pi \\ 3x = -x + 2k\pi \end{cases}$$

$$\Rightarrow \begin{cases} 2x = 2k\pi \\ 4x = 2k\pi \end{cases}$$

$$\Rightarrow x=k\pi \quad \text{or} \quad x=\frac{k\pi}{2}$$

Hence, the solutions are

$$x=0,\ x=\pi,\ x=2\pi,\ x=\frac{\pi}{2},\ x=\frac{3\pi}{2}.$$

EXAMPLE 5

Solve the equation $\tan 3x=\tan x$, $0 \le x \le 2\pi$

$$\tan 3x=\tan x \Rightarrow 3x=x+k\pi \Rightarrow 2x=k\pi \Rightarrow x=\frac{k\pi}{2}$$

Hence, the solutions are

$$x=0,\ x=\frac{\pi}{2},\ x=\pi,\ x=\frac{3\pi}{2},\ x=2\pi.$$

Remember that

$$\sin\frac{\pi}{3}=\cos\frac{\pi}{6} \quad \text{and} \quad \cos\frac{\pi}{3} = \sin\frac{\pi}{6}$$

This is true for any pair of complementary angles. Therefore,

cosA=sinB can take the form **$\cos A=\cos(\frac{\pi}{2}-B)$**

EXAMPLE 6

Solve the equation $\cos 3x=\sin x$, $0 \le x \le 2\pi$

We can write sinx as $\cos(\frac{\pi}{2}-x)$. Thus

$$\cos 3x= \cos(\frac{\pi}{2}-x) \Rightarrow \begin{cases} 3x = \frac{\pi}{2} - x + 2k\pi \\ 3x = -\frac{\pi}{2} + x + 2k\pi \end{cases}$$

$$\Rightarrow \begin{cases} 4x = \frac{\pi}{2} + 2k\pi \\ 2x = -\frac{\pi}{2} + 2k\pi \end{cases}$$

$$\Rightarrow \quad x = \frac{\pi}{8} + \frac{k\pi}{2} \quad \text{or} \quad x = -\frac{\pi}{4} + k\pi$$

Hence, the solutions are

$$x=\frac{\pi}{8},\ x=\frac{5\pi}{8},\ x=\frac{9\pi}{8},\ x=\frac{13\pi}{8} \quad \text{and} \quad x=\frac{3\pi}{4},\ x=\frac{7\pi}{4}.$$

Finally, sinx and tanx are **odd** functions, i.e.

$$-\sin x = \sin(-x) \qquad -\tan x = \tan(-x)$$

Hence,

sinA=−sinB can take the form **sinA=sin(−B)**

tanA=−tanB can take the form **tanA=tan(−B)**

However,

cosA=−cosB takes the form **cosA=cos(π−B)**

since **cos(π−x)=−cosx (prove it!)**

EXAMPLE 7

Solve the equation sin3x=−sinx, $0 \le x \le 2\pi$

We can write −sinx as sin(−x). Thus

$$\sin 3x = \sin(-x) \quad \Rightarrow \begin{cases} 3x = -x + 2k\pi \\ 3x = \pi + x + 2k\pi \end{cases}$$

$$\Rightarrow \begin{cases} 4x = 2k\pi \\ 2x = \pi + 2k\pi \end{cases}$$

$$\Rightarrow \ x = \frac{k\pi}{2} \quad \text{or} \quad x = \frac{\pi}{2} + k\pi$$

Hence, the solutions are

$$x=0, \ x=\frac{\pi}{2}, \ x=\pi, \ x=\frac{3\pi}{2}, \ x=2\pi$$

For the reciprocal trigonometric functions, we observe that

- secx=a is equivalent to the equation cosx=$\frac{1}{a}$
- cscx=a is equivalent to the equation sinx=$\frac{1}{a}$
- cotx=a is equivalent to the equation tanx=$\frac{1}{a}$

EXAMPLE 8

Solve the equation secx=2.

This is equivalent to cosx=1/2

Thus, the general solutions are

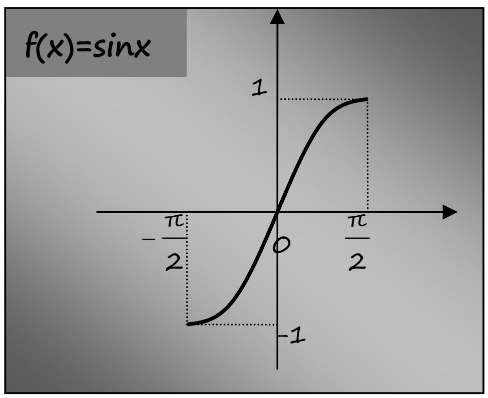
in radians	in degrees
$x=\frac{\pi}{3}+2k\pi$, $x=-\frac{\pi}{3}+2k\pi$	$x=60^o+360^ok$, $x=-60^o+360^ok$

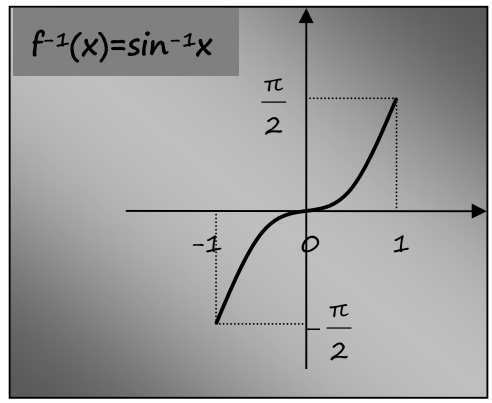
3.9 INVERSE TRIGONOMETRIC FUNCTIONS (for HL)

- **$\sin^{-1}x$**

Look at the graph of f(x)=sinx. This is not a "1-1" function (horizontal line test!)

However, if we restrict f(x)=sinx to $[-\frac{\pi}{2},\frac{\pi}{2}]$, we may define f^{-1}.



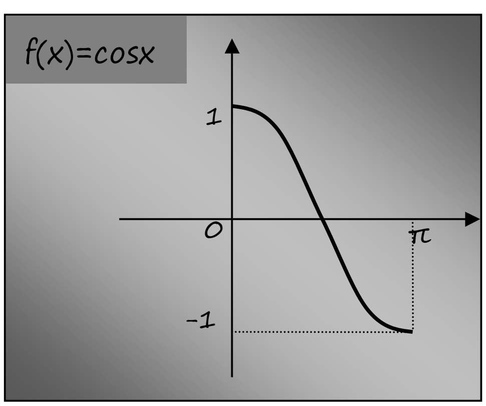


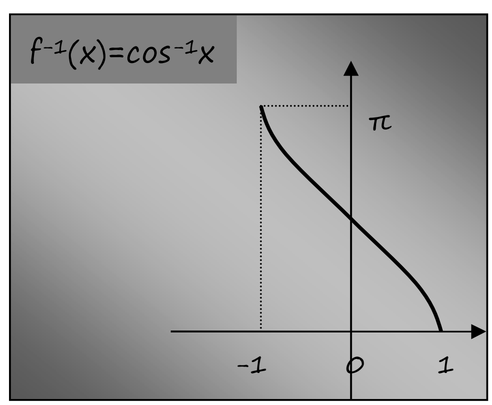
DOMAIN f: $x \in [-\frac{\pi}{2},\frac{\pi}{2}]$ DOMAIN f^{-1}: $x \in [-1,1]$

RANGE f: $y \in [-1,1]$ RANGE f^{-1}: $y \in [-\frac{\pi}{2},\frac{\pi}{2}]$

- **$\cos^{-1}x$**

Similarly, if we restrict f(x)=cosx to $[0,\pi]$, we may define f^{-1}.



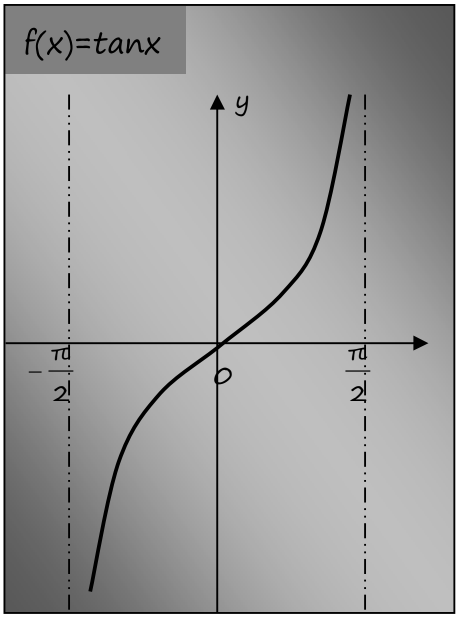


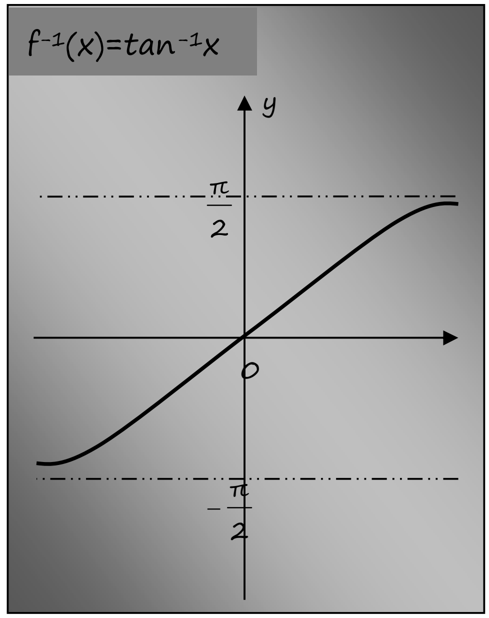
DOMAIN f: $x \in [0,\pi]$ DOMAIN f^{-1}: $x \in [-1,1]$

RANGE f: $y \in [-1,1]$ RANGE f^{-1}: $y \in [0,\pi]$

♦ **$\tan^{-1}x$**

Similarly, if we restrict f(x)=tanx to $(-\frac{\pi}{2},\frac{\pi}{2})$, we may define f^{-1}.





DOMAIN f: $x\in(-\frac{\pi}{2},\frac{\pi}{2})$

RANGE f: $y\in R$

DOMAIN f^{-1}: $x\in R$

RANGE f^{-1}: $y\in(-\frac{\pi}{2},\frac{\pi}{2})$

We also use the notation

arcsinx	for	**$\sin^{-1}x$**
arccosx	for	**$\cos^{-1}x$**
arctanx	for	**$\tan^{-1}x$**

Therefore,

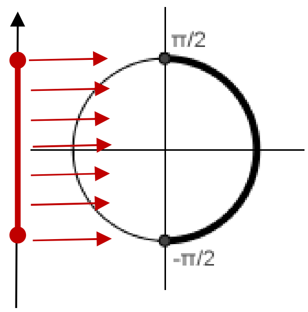
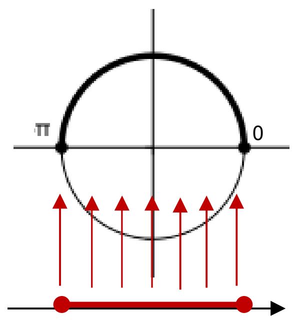
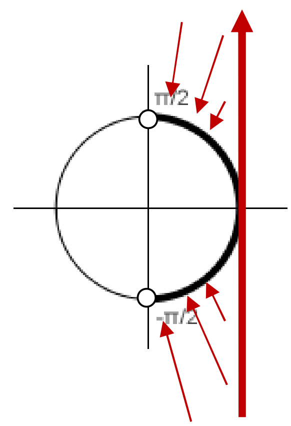
arcsinx=y	⇒	siny=x
arccosx=y	⇒	cosy=x
arctanx=y	⇒	tany=x

Notice

Mind the difference between

$\sin^{-1}x$	which is the inverse function **(arcsinx)**
$\frac{1}{\sin x}$	which is the reciprocal **(secx)**

Notice also that

for $x \in [-1,1]$ **arcsinx** lies in the semicircle $[-\frac{\pi}{2}, \frac{\pi}{2}]$	for $x \in [-1,1]$ **arccosx** lies in the semicircle $[0, \pi]$	for $x \in R$ **arctanx** lies in the semicircle $]-\frac{\pi}{2}, \frac{\pi}{2}[$
		

In fact, arcsin(**a**) is the **principal** solution of the equation sinx=**a**

EXAMPLE 1

By using the trigonometric tables of known angles

$\arcsin 0.5 = \frac{\pi}{6}$ $\arcsin(-0.5) = -\frac{\pi}{6}$

$\arccos 0.5 = \frac{\pi}{3}$ $\arccos(-0.5) = \frac{2\pi}{3}$

$\arctan 1 = \frac{\pi}{4}$ $\arctan(-1) = -\frac{\pi}{4}$

For $x=0$

$\arcsin 0 = 0$ $\arccos 0 = \frac{\pi}{2}$ $\arctan 0 = 0$

For a non-basic angle we use the GDC, e.g. $\arctan 5 = 1.37$

NOTICE:

Since sinx and arcsinx are inverse to each other (and similarly for the other functions) it clearly holds

$$\sin(\arcsin x)=x$$
$$\cos(\arccos x)=x$$
$$\tan(\arctan x)=x$$

Notice however that arcsin(sinx)=x holds only for $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$)

Indeed,

$$\arcsin(\sin\frac{\pi}{6}) = \arcsin\frac{1}{2}=\frac{\pi}{6}$$

but $$\arcsin(\sin\frac{5\pi}{6}) = \arcsin\frac{1}{2} = \frac{\pi}{6} \neq \frac{5\pi}{6}$$

Similarly, arccos(cosx)=x and arctan(tanx)=x hold only in a restricted domain.

EXAMPLE 2

Show that $\arctan 3-\arctan 0.5=\frac{\pi}{4}$

Let A= arctan3, B= arctan0.5.

$$\tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B} = \frac{3-0.5}{1+3(0.5)} = 1$$

Thus A-B is either $\pi/4$ or $-3\pi/4$.

But arctan3 > arctan0.5, so A-B > 0

Hence,

$$\arctan 3-\arctan 0.5 = \frac{\pi}{4}$$

EXAMPLE 3

Find

$$A=\tan(\arctan\frac{2}{3})$$

$$B=\sin(\arctan\frac{2}{3})$$

$$C=\cos(\arctan\frac{2}{3})$$

The first result is immediate

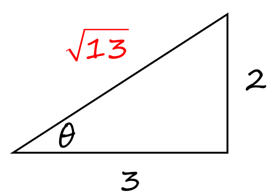
$$A=\frac{2}{3}$$

For the other two results we use the "triangle method":

Let $\theta=\arctan\frac{2}{3}$. Then

$$\tan\theta=\frac{2}{3}$$

We represent this information on a right-angled triangle:



We find the third side by using Pythagoras theorem.

Then

$$B=\sin\theta=\frac{2}{\sqrt{13}}$$

$$C=\cos\theta=\frac{3}{\sqrt{13}}$$

The following paragraphs

are on

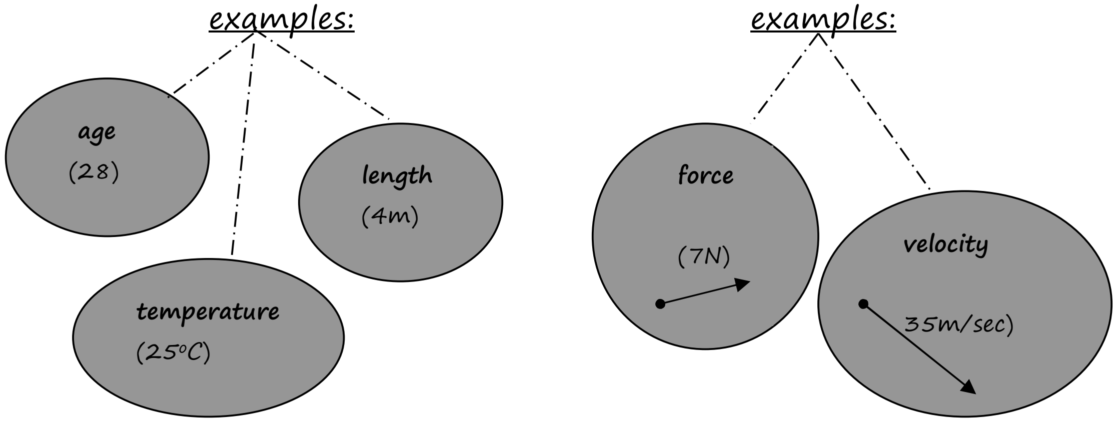
VECTORS

3.10 VECTORS: GEOMETRIC REPRESENTATION (for HL)

♦ DEFINITION

We distinguish two kinds of quantities in nature:

SCALARS	vs	**VECTORS**
(magnitude)		(magnitude and direction)
examples:		examples:

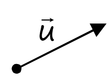


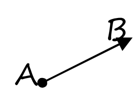
Thus, for a **vector** it is not enough to know the magnitude. We also need to know its direction (eg 35m/sec towards southeast)

Geometrically, a vector is represented by an arrow and denoted by

a letter: $\vec{u}$ or two letters: $\overrightarrow{AB}$

[A=tail, B=head]



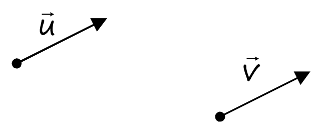


The length of the vector is called **magnitude** (it is a scalar).
It is denoted by $|\vec{u}|$ or $|\overrightarrow{AB}|$

Until now, we used to play with numbers: add numbers, multiply numbers etc. In this topic, we will "play with vectors: we will add vectors, multiply vectors etc.

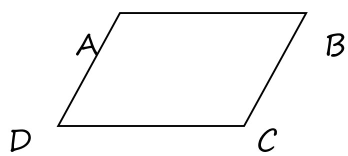
♦ EQUAL VECTORS

Two vectors are equal if they have the same magnitude and the same direction. Thus, two equal vectors must be parallel.



$$\boxed{\vec{u}=\vec{v}}$$

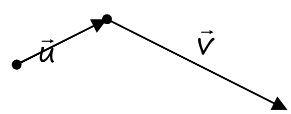
In other words, a vector does not have a specific position; it is exactly the same as long as it is translated in a parallel position. Hence, in a parallelogram



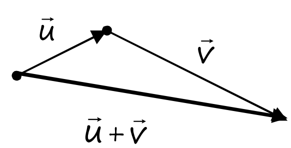
$$\overrightarrow{AB}=\overrightarrow{DC} \qquad \overrightarrow{AD}=\overrightarrow{BC}$$
$$\overrightarrow{BA}=\overrightarrow{CD} \qquad \overrightarrow{DA}=\overrightarrow{CB}$$

♦ ADDITION OF VECTORS: $\vec{u}+\vec{v}$

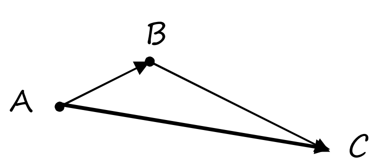
In order to add two vectors we must place them one after the other (head to tail)



Then the sum $\vec{u}+\vec{v}$ is given by the following shape



Here, it is more convenient to use the head and tail notation

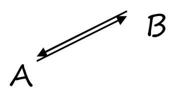


$$\boxed{\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}}$$

[think as follows: If you go from A to B (vector AB) and then from B to C (vector BC), the result is that you go from A to C (vector AC)]

♦ THE OPPOSITE VECTOR: $-\vec{u}$

It has the same magnitude but the opposite direction. Again, the two vectors are parallel. It is more convenient to use the head and tail notation



$$\boxed{\overrightarrow{AB} = -\overrightarrow{BA}}$$

NOTICE:

- From now on in the head and tail notation we will be writing AB instead of $\overrightarrow{AB}$ as the direction from A to B is obvious.
- It is easy to verify that

  $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ (commutative law)

  $\vec{u}+(\vec{v}+\vec{w}) = (\vec{v}+\vec{u})+\vec{w}$ (associative law)

- A vector AB can be written as a sum of consecutive vectors in several ways. For example:

  AB=AC+CB

  AB=AC+CD+DE+EB etc

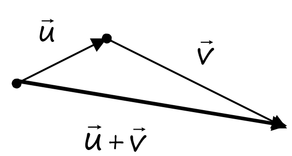
  The only thing we preserve is to start from A and finish at B.

- Head and tail notation helps to add several vectors even without drawing them:

  AB+CD+BC = AB+BC+CD = AD

  AB−AC=AB+CA=CA+AB=CB

- If $|\vec{u}|=5$ and $|\vec{v}|=3$ then $|\vec{u}+\vec{v}|$ is not necessarily 8. It is expected to be less than 8. Indeed, the triangle inequality gives



$$\boxed{|\vec{u}+\vec{v}| \le |\vec{u}|+|\vec{v}|}$$

Only if $\vec{u}$ and $\vec{v}$ have the same direction it holds: $|\vec{u}+\vec{v}|=|\vec{u}|+|\vec{v}|$

♦ THE ZERO VECTOR: $\vec{0}$

It is a vector of zero magnitude and no direction!

Notice that

$$\vec{u}-\vec{u}=\vec{\mathbf{0}} \qquad \text{or} \qquad AB-AB = AB+BA = AA = \vec{\mathbf{0}}$$

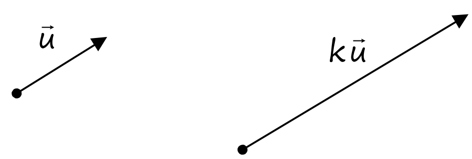
♦ MULTIPLICATION BY A SCALAR: $k\vec{u}$

Instead of $\vec{u}+\vec{u}$ we can write $2\vec{u}$.

Similarly, if k is a natural number

$$k\vec{u}=\vec{u}+\vec{u}+\cdots+\vec{u} \quad (k \text{ times})$$

In general, if k is any positive scalar ($k\in R^{+}$) the product $k\vec{u}$ is defined as a new vector of the same direction and magnitude $k|\vec{u}|$.



For k<0 the vector $k\vec{u}$ simply has the opposite direction.

Thus for two vectors $\vec{u}$ and $\vec{v}$

$$\vec{u} \,//\, \vec{v} \Leftrightarrow \vec{u} = k\vec{v} \quad \text{for some } k\in R$$

NOTICE:

It is easy to verify that

$k(\vec{u}+\vec{v}) = k\vec{v}+k\vec{u}$ (distributive law)

$(k+m)\,\vec{u} = k\vec{u}+m\vec{u}$ (distributive law)

$k(m\vec{u}) = (km)\,\vec{u}$

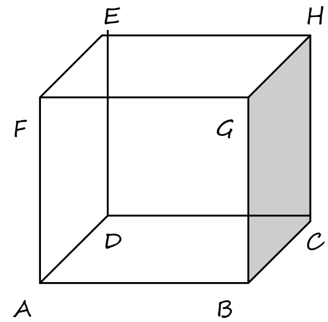
$1\vec{u} = \vec{u}$

$0\vec{u} = \vec{\mathbf{0}}$

$k\vec{\mathbf{0}} = \vec{\mathbf{0}}$

EXAMPLE 1

Consider the following cube



Let $\vec{a}$=AB,$\vec{b}$ =AD $\vec{c}$=AF.

Any other edge can be corresponded to $\vec{a}$,$\vec{b}$,$\vec{c}$.

Namely,

$\vec{a}$=AB=DC=FG=EH.

$\vec{b}$=AD=BC=FE=GH

$\vec{c}$=AF=BG=DE=CH

Can you express FC in terms of $\vec{a}$,$\vec{b}$ and $\vec{c}$?

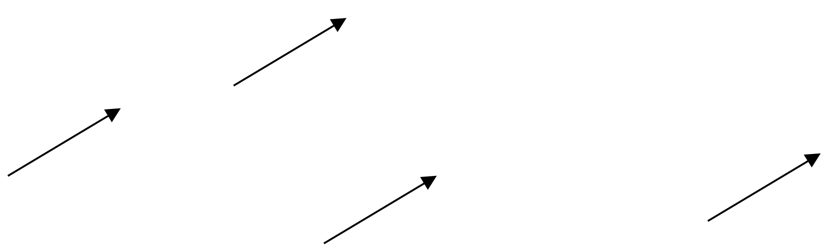
First of all we must find a path from F to C: FC=FG+GB+BC

Then we observe

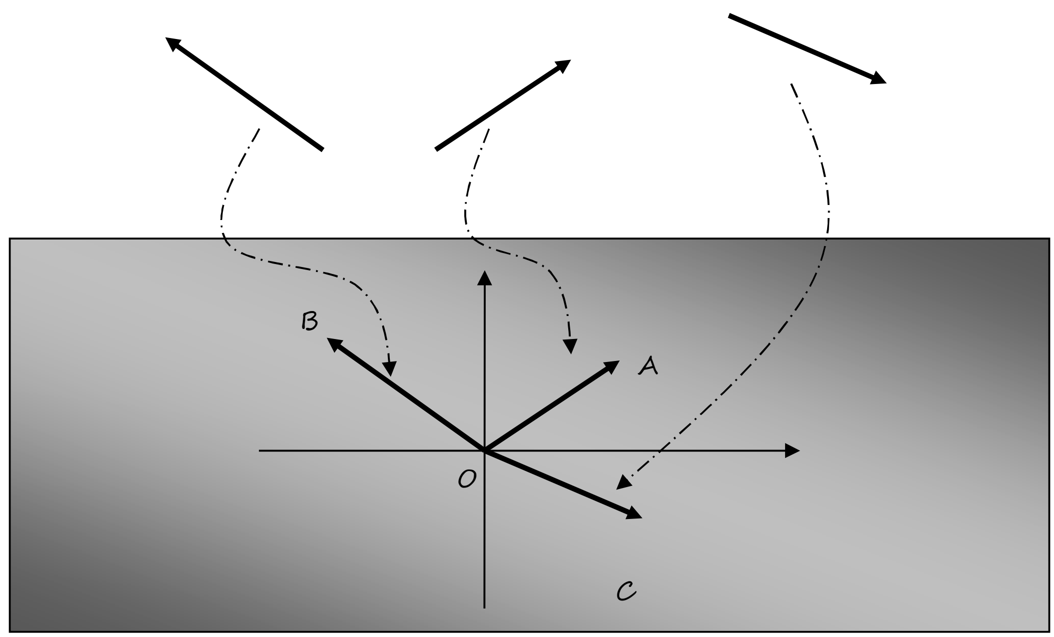
$$FC=\vec{a}-\vec{c}+\vec{b}$$

- VECTORS ON THE CARTESIAN PLANE

Remember that a parallel translation of any vector results to an equal vector:

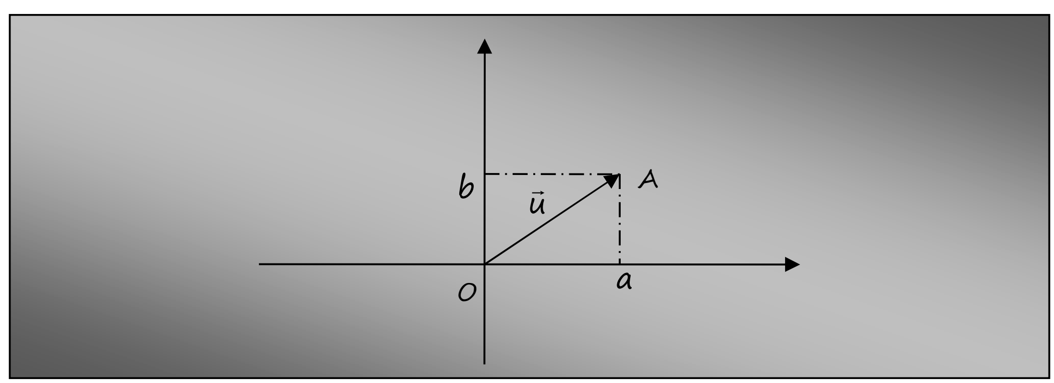


If we can consider the Cartesian Plane, any vector on the plane can be moved so as to start from the origin O.



Thus any vector on the plane can be written in the form OA.

Suppose that the coordinates of the point A are: A(a,b)



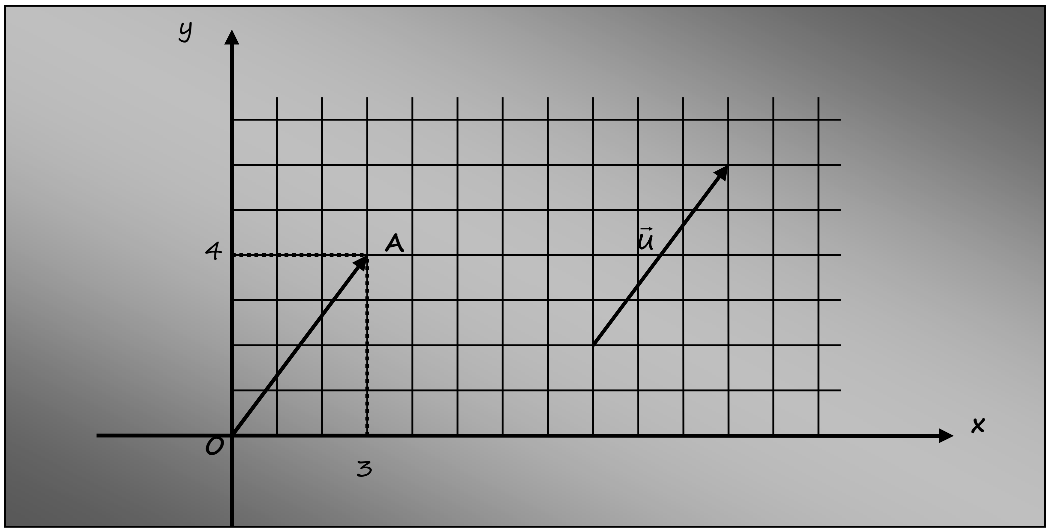
We agree to denote the vector $\vec{u}$=OA by $\begin{pmatrix} a \\ b \end{pmatrix}$

We say that $\vec{u}$=OA=$\begin{pmatrix} a \\ b \end{pmatrix}$ is the **position vector** of the point A(a,b).

Notice also that the length of OA is $\sqrt{a^2+b^2}$ (according to the Pythagoras' Theorem). This is in fact the magnitude of $\vec{u}$=OA.

EXAMPLE 2

Consider the following vector $\vec{u}$ and its equivalent vector OA.



Notice that the point A has coordinates (3,4)

The position vector of the point A(3,4) is OA=$\begin{pmatrix}3\\4\end{pmatrix}$

In practice, by a vector $\vec{u}=\begin{pmatrix}3\\4\end{pmatrix}$ we imply that we are moving

3 units in the x-direction and

4 units in the y-direction

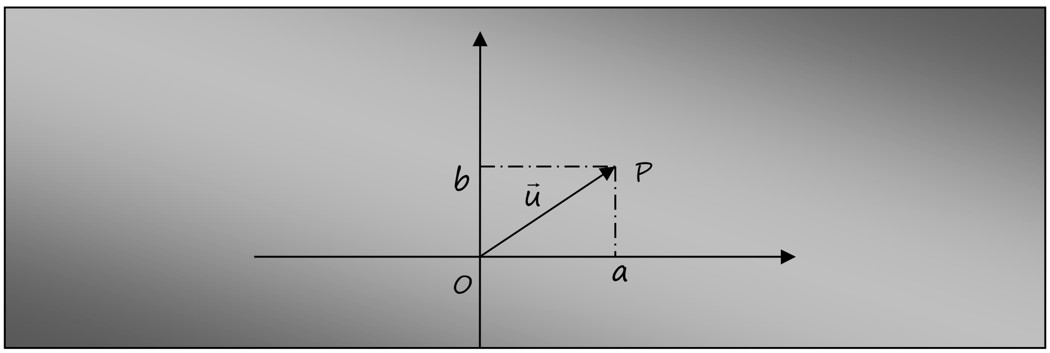
This description introduces an alternative approach for vectors in the following paragraph.

3.11 VECTORS: ALGEBRAIC REPRESENTATION (for HL)

♦ 2- DIMENSIONAL VECTORS

A **vector** $\vec{u}$ is a pair of numbers in column form: $\begin{pmatrix} a \\ b \end{pmatrix}$.

A vector $\vec{u}=\begin{pmatrix} a \\ b \end{pmatrix}$ is represented on the Cartesian plane as an arrow from the origin O to the point P(a,b). We say that $\vec{u}$=OP is the **position vector** of the point P.



The **magnitude** of a vector $\vec{u}$ is defined by $\boxed{|\vec{u}|=\sqrt{a^2+b^2}}$

EXAMPLE 1

Consider the vectors $\vec{u}=\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\vec{v}=\begin{pmatrix} -3 \\ 4 \end{pmatrix}$, $\vec{w}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{r}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The corresponding magnitudes are

$|\vec{u}|=\sqrt{3^2+4^2}=\sqrt{25}=5$ $\qquad$ $|\vec{v}|=\sqrt{(-3)^2+4^2}=\sqrt{25}=5$

$|\vec{w}|=\sqrt{1^2+1^2}=\sqrt{2}$ $\qquad$ $|\vec{r}|=\sqrt{0^2+0^2}=0$

♦ ADDITION OF VECTORS: $\vec{u}+\vec{v}$

If $\vec{u}=\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $\vec{v}=\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ then $\vec{u}+\vec{v}=\begin{pmatrix} a_1+a_2 \\ b_1+b_2 \end{pmatrix}$

- THE OPPOSITE VECTOR: $-\vec{u}$

If $\vec{u}=\begin{pmatrix}a\\b\end{pmatrix}$ then $-\vec{u}=\begin{pmatrix}-a\\-b\end{pmatrix}$

- THE ZERO VECTOR: $\vec{o}$

$\vec{o}=\begin{pmatrix}0\\0\end{pmatrix}$

- MULTIPLICATION BY A SCALAR: $k\vec{u}$

If $k\in R$ (scalar) and $\vec{u}=\begin{pmatrix}a\\b\end{pmatrix}$ then $k\vec{u}=\begin{pmatrix}ka\\kb\end{pmatrix}$

If $k>0$, we say that $\vec{u}$ and $k\vec{u}$ have the same direction
If $k<0$, we say that $\vec{u}$ and $k\vec{u}$ have the opposite direction

---

EXAMPLE 2

Consider the vectors $\vec{u}=\begin{pmatrix}3\\4\end{pmatrix}$, $\vec{v}=\begin{pmatrix}2\\5\end{pmatrix}$. Then

$$\vec{u}+\vec{v}=\begin{pmatrix}3\\4\end{pmatrix}+\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}5\\9\end{pmatrix},\quad \vec{u}-\vec{v}=\begin{pmatrix}3\\4\end{pmatrix}+\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix},\quad \vec{v}-\vec{u}=\begin{pmatrix}2\\5\end{pmatrix}-\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}$$

$$2\vec{u}=2\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}6\\8\end{pmatrix},\quad 5\vec{u}=\begin{pmatrix}15\\20\end{pmatrix},\quad -3\vec{u}=\begin{pmatrix}-9\\-12\end{pmatrix},\qquad -1\vec{u}=-\vec{u}=\begin{pmatrix}-3\\-4\end{pmatrix}$$

$$2\vec{u}+3\vec{v}=2\begin{pmatrix}3\\4\end{pmatrix}+3\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}6\\8\end{pmatrix}+\begin{pmatrix}6\\15\end{pmatrix}=\begin{pmatrix}12\\23\end{pmatrix}$$

NOTICE

For $\vec{u}=\begin{pmatrix}3\\4\end{pmatrix}$, it is $|\vec{u}|=5$. For $2\vec{u}=\begin{pmatrix}6\\8\end{pmatrix}$, $|2\vec{u}|=\sqrt{6^2+8^2}=\sqrt{100}=10$.

Similarly, the magnitude of $10\vec{u}$ is 50, the magnitude of $\frac{1}{5}\vec{u}$ is 1.

In general, the magnitude of $k\vec{u}$ is $|k|$ times the magnitude of $\vec{u}$, that is

$$\boxed{|k\vec{u}|=|k||\vec{u}|}$$

♦ THE UNIT VECTOR

The **unit vector** corresponding to $\vec{u}$ is defined by

$$\hat{u} = \frac{1}{|\vec{u}|}\vec{u}$$

It is in fact a vector in the same direction with magnitude 1.

EXAMPLE 3

Consider the vectors $\vec{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. Then

$$|\vec{u}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \qquad |\vec{v}| = \sqrt{2^2 + 5^2} = \sqrt{29}$$

The corresponding unit vectors are

$$\hat{u} = \frac{1}{5}\vec{u} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}, \qquad \hat{v} = \frac{1}{\sqrt{29}}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{29} \\ 5/\sqrt{29} \end{pmatrix}$$

We can easily confirm that the magnitudes of the unit vectors $\hat{u}$ and $\hat{v}$ are both equal to 1.

Question: Can you find a vector $\vec{a}$ parallel to $\vec{u}$ and a vector $\vec{b}$ parallel to $\vec{v}$, both of them having magnitude 20?

Since $|\vec{u}| = 5$, the vector $\vec{a} = 4\vec{u}$ has magnitude 20

Since $|\vec{v}| = \sqrt{29}$, it is less obvious to find $\vec{b}$. The general method is to find the unit vector $\hat{v}$ first (which has magnitude 1) and then we multiply by the required legnth: $\vec{b} = 20\hat{v} = \frac{20}{\sqrt{29}}\vec{v}$

♦ THE NOTATION $\vec{u} = a\vec{i} + b\vec{j}$

A vector $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ may also be written in the form $\boxed{\vec{u} = a\vec{i} + b\vec{j}}$

where $\vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Notice: In practice, whenever we see an expression like that, say $\vec{u} = 3\vec{i} + 4\vec{j}$, we will be using the column vector form, that is $\vec{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, to make our life easier!!!

- EXPLANATION FOR $\vec{u}=a\vec{i}+b\vec{j}$

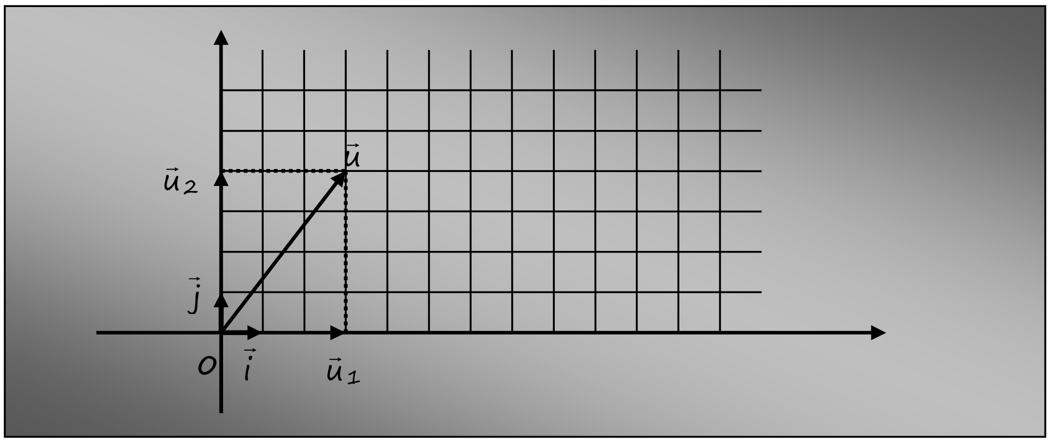
Notice that any vector on the x-axis has the form $\begin{pmatrix} a \\ 0 \end{pmatrix}$

any vector on the y-axis has the form $\begin{pmatrix} 0 \\ b \end{pmatrix}$

Especially, the unit vector on the x-axis is $\vec{i}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

the unit vector on the y-axis is $\vec{j}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Consider for example the vectors $\vec{u}_1=\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\vec{u}_2=\begin{pmatrix} 0 \\ 4 \end{pmatrix}$



Then $\vec{u}=\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ can be written as $\vec{u}=\vec{u}_1+\vec{u}_2$ [indeed, $\begin{pmatrix} 3 \\ 4 \end{pmatrix}=\begin{pmatrix} 3 \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ 4 \end{pmatrix}$]

But $\vec{u}_1=3\vec{i}$ and $\vec{u}_2=4\vec{j}$, thus the vector $\vec{u}$ can be expressed as

$$\vec{u}=3\vec{i}+4\vec{j}$$

In general,

$$\boxed{\begin{pmatrix} a \\ b \end{pmatrix}= a\vec{i}+b\vec{j}}$$

- CONNECTION BETWEEN GEOMETRIC AND ALGEBRAIC REPRESENTATION

In paragraph 4.1 we gave a geometric description of a vector and the operations $\vec{u}+\vec{v}$ and $k\vec{u}$.

In this paragraph we have presented an algebraic description of these notions. Do these descriptions agree?

We will make use of some examples to demonstrate this connection.

- For $\vec{u}+\vec{v}$

Let $\vec{u}=\begin{pmatrix}3\\4\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}5\\-2\end{pmatrix}$. Then $\vec{u}+\vec{v}=\begin{pmatrix}8\\2\end{pmatrix}$

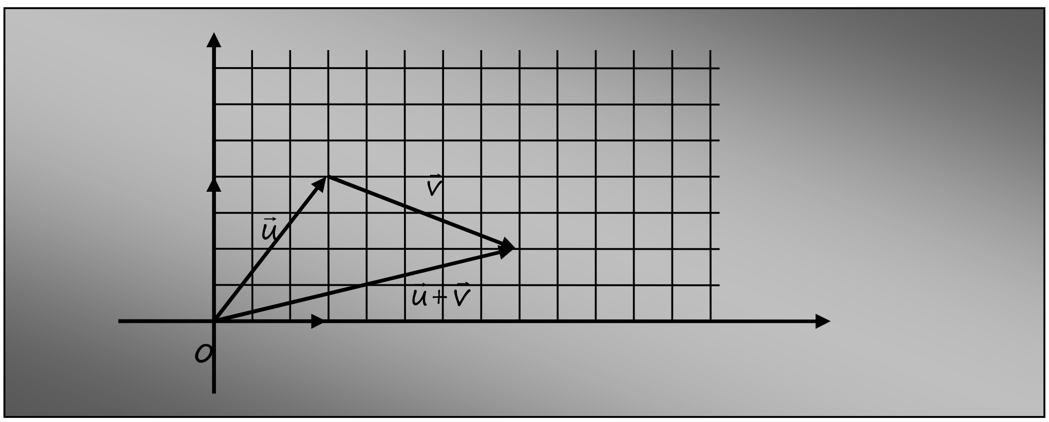
Geometrically, let us draw $\vec{u}$ and $\vec{v}$, starting from the origin, so that $\vec{u}$ and $\vec{v}$ are consecutive:

for $\vec{u}$ we are moving 3 units horizontally, 4 units vertically

for $\vec{v}$ we are moving 5 units horizontally, -2 units vertically

then we observe that

for $\vec{u}+\vec{v}$ we are moving 8 units horizontally 2 units vertically



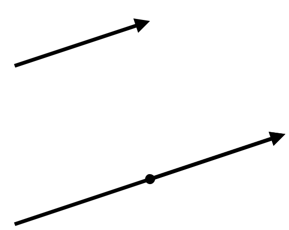
Hence, the geometric description of $\vec{u}+\vec{v}$ that we have seen in paragraph 4.1 keeps up with the algebraic description in this paragraph.

- For $k\vec{u}$

If $\vec{u}=\begin{pmatrix}a\\b\end{pmatrix}$ is a vector of magnitude m

then clearly $2\vec{u}=\begin{pmatrix}2a\\2b\end{pmatrix}$ is a vector in

the same direction with magnitude 2m



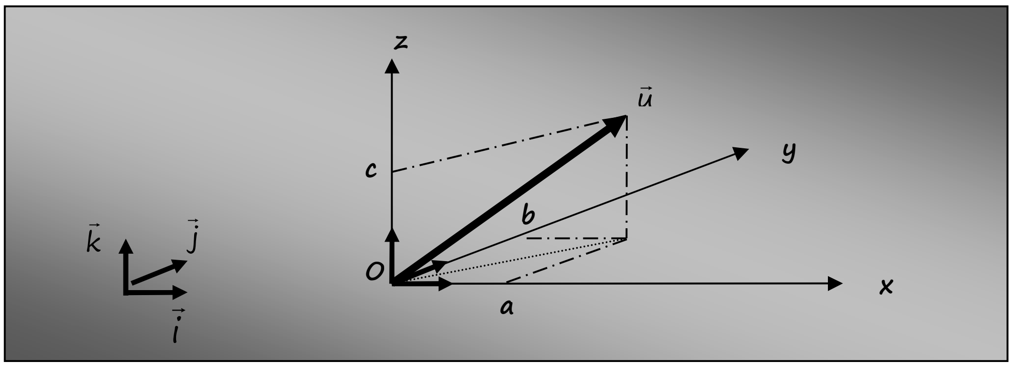
Again, the geometric and the algebraic definitions of $2\vec{u}$ (and $k\vec{u}$ in general) coincide!

♦ 3- DIMENSIONAL VECTORS

In the 3-dimensional space a **vector** has the following form

$$\vec{u}=\begin{pmatrix}a\\b\\c\end{pmatrix} \quad \text{or equivalently} \quad \vec{u}=a\vec{i}+b\vec{j}+c\vec{k}$$

where $\vec{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\vec{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\vec{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$



The **magnitude** of $\vec{u}$ is defined by $\boxed{|\vec{u}|=\sqrt{a^2+b^2+c^2}}$

All the other notions (eg $\vec{u}+\vec{v}$, $k\vec{u}$, unit vector) are defined in an analogue way!

EXAMPLE 4

Consider the vectors $\vec{u}=\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\vec{v}=\begin{pmatrix}2\\5\\-4\end{pmatrix}$. Then

- $3\vec{u}+2\vec{v}=3\begin{pmatrix}1\\2\\3\end{pmatrix}+2\begin{pmatrix}2\\5\\-4\end{pmatrix}=\begin{pmatrix}3\\6\\9\end{pmatrix}+\begin{pmatrix}4\\10\\-8\end{pmatrix}=\begin{pmatrix}7\\16\\1\end{pmatrix}$
- $|\vec{u}|=\sqrt{1^2+2^2+3^2}=\sqrt{14}$
- The unit vector corresponding to $\vec{u}$ is $\hat{u}=\frac{1}{|\vec{u}|}\vec{u}=\begin{pmatrix}1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14}\end{pmatrix}$

♦ POINTS AND VECTORS

	2D	3D
Points	**A**(x_1,y_1) and **B**(x_2,y_2)	**A**(x_1,y_1,z_1) and **B**(x_2,y_2,z_2)
Mid-point	$M(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2})$	$M(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2})$
position vectors of **A** and **B**	$OA=\begin{pmatrix}x_1\\y_1\end{pmatrix}$, $OB=\begin{pmatrix}x_2\\y_2\end{pmatrix}$	$OA=\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}$, $OB=\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}$
vector **AB**	$\begin{pmatrix}x_2\\y_2\end{pmatrix}-\begin{pmatrix}x_1\\y_1\end{pmatrix}=\begin{pmatrix}x_2-x_1\\y_2-y_1\end{pmatrix}$	$\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}-\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}=\begin{pmatrix}x_2-x_1\\y_2-y_1\\z_2-z_1\end{pmatrix}$
distance (A,B) *it is in fact the magnitude* $\lvert AB\rvert$	$\sqrt{(x_1-x_1)^2+(y_1-y_2)^2}$	$\sqrt{(x_1-x_1)^2+(y_1-y_2)^2+(z_1-z_2)^2}$

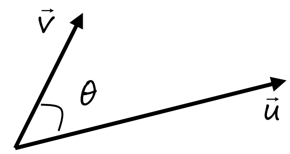
EXAMPLE 5

	2D	3D
Points	**A**(1,2) and **B**(3,4)	**A**(1,2,3) and **B**(4,5,6)
Mid-point	**M**(2,3)	$M(\frac{5}{2},\frac{7}{2},\frac{9}{2})$
position vectors of **A** and **B**	$OA=\begin{pmatrix}1\\2\end{pmatrix}$, $OB=\begin{pmatrix}3\\4\end{pmatrix}$	$OA=\begin{pmatrix}1\\2\\3\end{pmatrix}$, $OB=\begin{pmatrix}4\\5\\6\end{pmatrix}$
vector **AB**	$\begin{pmatrix}3\\4\end{pmatrix}-\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix}$	$\begin{pmatrix}4\\5\\6\end{pmatrix}-\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}3\\3\\3\end{pmatrix}$
distance (A,B) *it is in fact the magnitude* $\lvert AB\rvert$	$\sqrt{2^2+2^2}=\sqrt{8}$	$\sqrt{3^2+3^2+3^2}=\sqrt{27}$

3.12 DOT PRODUCT – ANGLE BETWEEN VECTORS (for HL)

♦ THE GEOMETRIC DEFINITION (the ugly one!)

Let $\vec{u}$ and $\vec{v}$ be two vectors and θ be the angle between those two vectors



The **dot product** (or **scalar product**) of $\vec{u}$ and $\vec{v}$ is defined to be a number given by

$$\boxed{\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cos\theta}$$

For example, if $\vec{u}$ and $\vec{v}$ are vectors of magnitudes 5 and 4 respectively and the angle between them is $\theta=60^o$ then

$$\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cos\theta=(5)\cdot(4)\cdot(0.5) = 10$$

Mind that the result is a number (scalar) and not a vector.

Notice that

If $\theta=90^o$ then $\vec{u}\cdot\vec{v}=0$ [$\vec{u}\perp\vec{v}$, perpendicular vectors]

If $\theta=0^o$ then $\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|$ [$\vec{u}\uparrow\uparrow\vec{v}$, parallel-same direction]

If $\theta=180^o$ then $\vec{u}\cdot\vec{v}=-|\vec{u}||\vec{v}|$ [$\vec{u}\uparrow\downarrow\vec{v}$, parallel-opposite direction]

Thus, the dot product can take any value between the minimum value $-|\vec{u}||\vec{v}|$ and the maximum value $|\vec{u}||\vec{v}|$

In particular, the product $\vec{u}\cdot\vec{u}$ is denoted by $\vec{u}^2$. Since the angle between $\vec{u}$ and itself is 0, $\vec{u}^2$ is equal to $|\vec{u}||\vec{u}|$, hence

$$\boxed{\vec{u}^2=|\vec{u}|^2}$$

Notice that for the unit vectors $\boldsymbol{i}=\begin{pmatrix}1\\0\end{pmatrix}$ and $\boldsymbol{j}=\begin{pmatrix}0\\1\end{pmatrix}$ it holds

$$\boldsymbol{i}^2= 1\ ,\ \boldsymbol{j}^2=1,\ \ \boldsymbol{i}\cdot\boldsymbol{j}= 0 \text{ and } \boldsymbol{j}\cdot\boldsymbol{i}=0$$

♦ THE ALGEBRAIC DEFINITION (the pretty one!)

Let $\vec{u}=\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $\vec{v}=\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ be two vectors. The **scalar product** (or **dot product**) of $\vec{u}$ and $\vec{v}$ is given by

$$\boxed{\vec{u}\cdot\vec{v}=a_1a_2+b_1b_2}$$

For example, if $\vec{u}=\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{v}=\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ then $\vec{u}\cdot\vec{v}=2\cdot5+3\cdot4=22$

♦ BASIC PROPERTIES

It can be shown that the dot product satisfies the following basic rules:

- $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$ (commutative law)
- $\vec{u}\cdot(\vec{v}_1+\vec{v}_2)=\vec{u}\cdot\vec{v}_1+\vec{u}\cdot\vec{v}_2$ (distributive law)
- $k(\vec{u}\cdot\vec{v})=(k\vec{u})\cdot\vec{v}=\vec{u}\cdot(k\vec{v})$

♦ SHORT EXPLANATION FOR THE "PRETTY" DEFINITION

Let $\vec{u}=\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}=a_1\mathbf{i}+b_1\mathbf{j}$ and $\vec{v}=\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}=a_2\mathbf{i}+b_2\mathbf{j}$. Then

$$\vec{u}\cdot\vec{v}=(a_1\mathbf{i}+b_1\mathbf{j})\cdot(a_2\mathbf{i}+b_2\mathbf{j})=a_1a_2\mathbf{i}^2+a_1b_2\mathbf{i}\cdot\mathbf{j}+b_1a_2\mathbf{j}\cdot\mathbf{i}+b_1b_2\mathbf{j}^2=a_1a_2+b_1b_2$$

(since $\mathbf{i}^2=\mathbf{j}^2=1$ and $\mathbf{ij}=\mathbf{ji}=0$)

♦ THE ANGLE BETWEEN TWO VECTORS

If we combine the "ugly" and the "pretty" definitions we obtain a nice way to calculate the angle between two vectors. The "ugly" definition gives

$$\boxed{\cos\theta=\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}}$$

Hence, if we are given two vectors $\vec{u}$ and $\vec{v}$, we can easily calculate $|\vec{u}|$, $|\vec{v}|$ and the dot product $\vec{u}\cdot\vec{v}$ by using the "pretty" definition and the formula above gives the angle θ.

EXAMPLE 1

Consider the vectors $\vec{u}=\begin{pmatrix}3\\4\end{pmatrix}$, $\vec{v}=\begin{pmatrix}1\\-2\end{pmatrix}$.

Find

a) their magnitudes

b) their dot product $\vec{u}\cdot\vec{v}$

c) the angle θ between them

We have

a) $|\vec{u}|=5$, $|\vec{v}|=\sqrt{5}$

b) $\vec{u}\cdot\vec{v}=3\cdot1+4\cdot(-2)=-5$

c) $\cos\theta=\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}=\dfrac{-5}{5\sqrt{5}}=-\dfrac{1}{\sqrt{5}}$, and the GDC gives $\theta=116.56^o$

The dot product is a nice tool to verify whether two vectors are perpendicular or not:

♦ PERPENDICULAR VECTORS AND PARALLEL VECTORS

Recall two basic properties: For two non-zero vectors $\vec{u}$ and $\vec{v}$:

$\boxed{\vec{u}\perp\vec{v} \Leftrightarrow \vec{u}\cdot\vec{v}=0}$ (perpendicular vectors)

$\boxed{\vec{u}//\vec{v} \Leftrightarrow \vec{u}=k\vec{v} \text{ for some } k\in R}$ (parallel vectors)

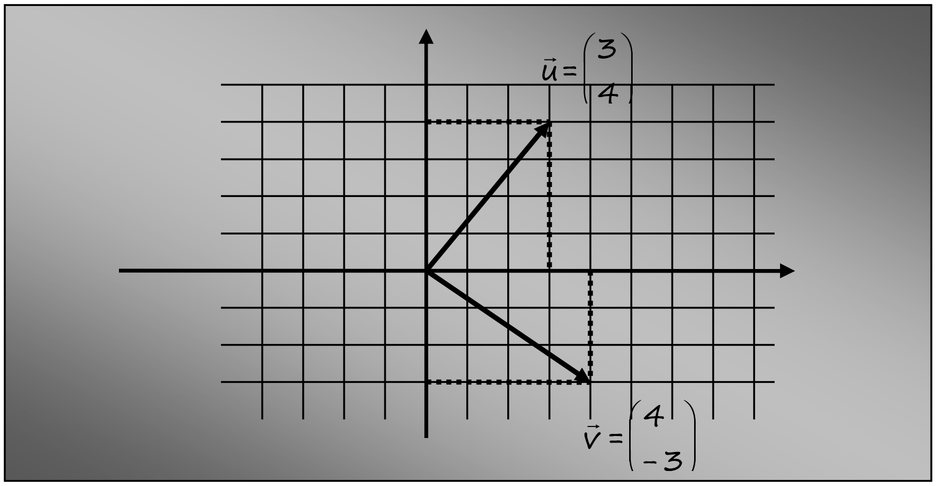
EXAMPLE 2

a) Show that $\vec{u}=\begin{pmatrix}3\\4\end{pmatrix}$, $\vec{v}=\begin{pmatrix}4\\-3\end{pmatrix}$ are perpendicular

b) Find some perpendicular vectors to the vector $\vec{u}=\begin{pmatrix}2\\5\end{pmatrix}$

Solution

a) $\vec{u}\cdot\vec{v}=3\cdot4+4\cdot(-3)=0$, hence $\vec{u}\perp\vec{v}$. Indeed, look at the following diagram



b) In general both $\begin{pmatrix}b\\-a\end{pmatrix}$ and $\begin{pmatrix}-b\\a\end{pmatrix}$ are perpendicular to $\vec{u}=\begin{pmatrix}a\\b\end{pmatrix}$ since the dot product for both pairs is $a{\cdot}b-ab=0$.

Thus, some perpendicular vectors to $\vec{u}=\begin{pmatrix}2\\5\end{pmatrix}$ are the following

$$\begin{pmatrix}5\\-2\end{pmatrix},\begin{pmatrix}10\\-4\end{pmatrix},\begin{pmatrix}15\\-6\end{pmatrix} \text{ and } \begin{pmatrix}-5\\2\end{pmatrix},\begin{pmatrix}-10\\4\end{pmatrix},\begin{pmatrix}-15\\6\end{pmatrix}$$

EXAMPLE 3

Let $\vec{u}=\begin{pmatrix}3\\4\end{pmatrix}$. Find the value of x if $\vec{v}=\begin{pmatrix}x\\-6\end{pmatrix}$ is

a) perpendicular to $\vec{u}$ b) parallel to $\vec{u}$

Solution

a) $\vec{u}\perp\vec{v} \Leftrightarrow \vec{u}{\cdot}\vec{v}=0 \Leftrightarrow 3x+4(-6)=0 \Leftrightarrow 3x=24 \Leftrightarrow \boxed{x=8}$

b) $\vec{u}//\vec{v} \Leftrightarrow \begin{pmatrix}3\\4\end{pmatrix}=\lambda\begin{pmatrix}x\\-6\end{pmatrix}$ for some λ.

But it is more practical to say that the ratios of the corresponding coordinates are equal:

$$\frac{x}{3}=\frac{-6}{4}$$

Therefore, $\boxed{x=-\frac{9}{2}}$

- THE PROPERTY $|\vec{u}|^2=\vec{u}^2$

This property very often helps us to get rid of magnitudes. Look at the following example!

EXAMPLE 4

For two non-zero vectors $\vec{u}$ and $\vec{v}$ it holds $|\vec{u}+\vec{v}|=|\vec{u}-\vec{v}|$. Show that $\vec{u}$ and $\vec{v}$ are perpendicular.

$$|\vec{u}+\vec{v}|=|\vec{u}-\vec{v}| \Rightarrow |\vec{u}+\vec{v}|^2=|\vec{u}-\vec{v}|^2 \qquad \text{[just squaring]}$$

$$\Rightarrow (\vec{u}+\vec{v})^2=(\vec{u}-\vec{v})^2 \qquad \text{[property } |\vec{u}|^2=\vec{u}^2\text{]}$$

$$\Rightarrow \vec{u}^2+2\vec{u}\cdot\vec{v}+\vec{v}^2=\vec{u}^2-2\vec{u}\cdot\vec{v}+\vec{v}^2$$

$$\Rightarrow 4\vec{u}\cdot\vec{v}=0 \Rightarrow \vec{u}\cdot\vec{v}=0 \Rightarrow \vec{u}\perp\vec{v}$$

- 3D VECTORS

For two vectors $\vec{u}=\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}$ the **dot product** is given by

$$\boxed{\vec{u}\cdot\vec{v}=a_1a_2+b_1b_2+c_1c_2}$$

Whatever we said about 2D vectors also applies here!

EXAMPLE 5

Show that $\vec{u}=\begin{pmatrix}4\\2\\-1\end{pmatrix}$, $\vec{v}=\begin{pmatrix}5\\-3\\14\end{pmatrix}$ are perpendicular. Indeed

$$\vec{u}\cdot\vec{v} = 4\cdot5+2(-3)+(-1)(14) = 0, \text{ thus } \vec{u}\perp\vec{v}$$

EXAMPLE 6

Find the angle between $\vec{u}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}1\\2\\3\end{pmatrix}$. We have

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}=\frac{6}{\sqrt{3}\cdot\sqrt{14}}=0.926, \text{ hence } \theta=\cos^{-1}(0.926)= 22.2^{o}$$

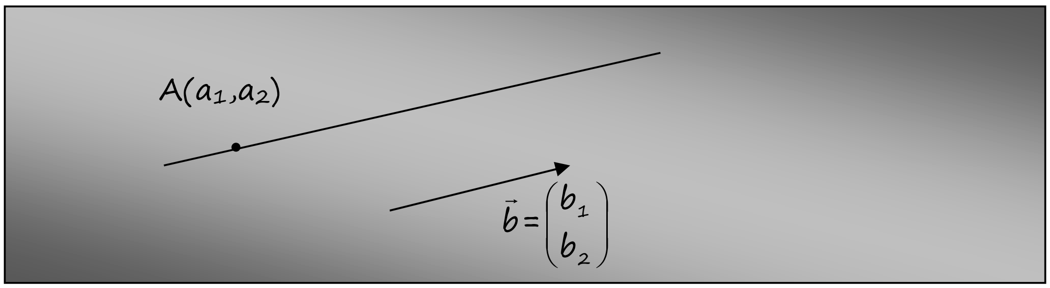
3.13 VECTOR EQUATION OF A LINE IN 2D (for HL)

♦ VECTOR EQUATION

Let $A(a_1,a_2)$ be a point with position vector $\vec{a}=\begin{pmatrix}a_1\\a_2\end{pmatrix}$

$\vec{b}=\begin{pmatrix}b_1\\b_2\end{pmatrix}$ be a vector

There is a unique line passing through A which is parallel to $\vec{b}$.

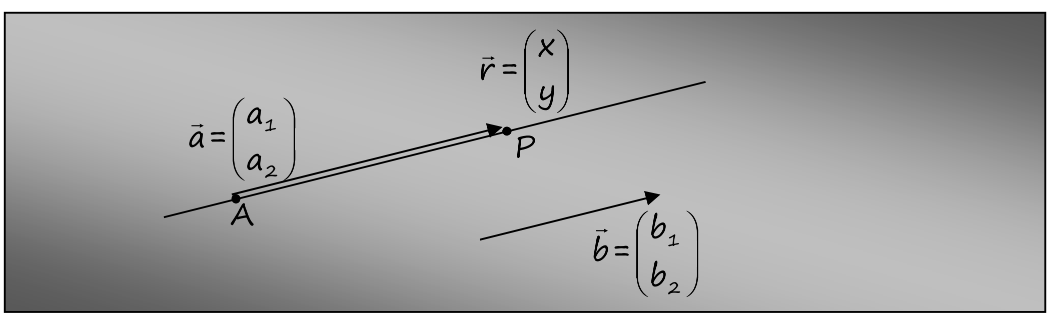


The position vector $\vec{r}=\begin{pmatrix}x\\y\end{pmatrix}$ of the random point P(x,y) in this line is given by

$$\boxed{\vec{r}=\vec{a}+\lambda\vec{b}} \quad \text{or} \quad \boxed{\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a_1\\a_2\end{pmatrix}+\lambda\begin{pmatrix}b_1\\b_2\end{pmatrix}}$$

where λ is a parameter.

♦ SHORT EXPLANATION



The position vector of the point P(x,y) is

$$\vec{r}=OP=OA+AP$$

But $OA=\vec{a}$ and $AP\,//\,\vec{b} \Rightarrow AP=\lambda\vec{b}$ for some $\lambda\in R$. Thus

$$\boxed{\vec{r}=\vec{a}+\lambda\vec{b}}$$

- PARAMETRIC EQUATIONS

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ gives $\boxed{\begin{matrix} x=a_1+\lambda b_1 \\ y=a_2+\lambda b_2 \end{matrix}}$

- CARTESIAN EQUATION

If we solve both equations for λ we get $\lambda=\frac{x-a_1}{b_1}$ and $\lambda=\frac{y-a_2}{b_2}$.

Therefore, the relation between the parameters x,y is

$$\boxed{\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}}$$

EXAMPLE 1

Let A(1,2) be the given **point** and $\vec{b}=\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ be the **direction vector**.

Then the line passing through A, parallel to $\vec{b}$ is

Vector equation: $\vec{r}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ or $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Parametric equations: $x=1+3\lambda$

$y=2+4\lambda$

Now solve for λ and get

Cartesian equation: $\frac{x-1}{3}=\frac{y-2}{4}$

The Cartesian equation may be written in more traditional forms:

$\boxed{ax+by=c}$ or $\boxed{y=mx+c}$

Indeed, the last equation gives

$\frac{x-1}{3}=\frac{y-2}{4} \Leftrightarrow 4x-4=3y-6 \Leftrightarrow \boxed{4x-3y=-2}$

If we solve for y we obtain the traditional form: $\boxed{y=\frac{4}{3}x+\frac{2}{3}}$

NOTICE

Let us consider again the equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

It is the line which is parallel to $\vec{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and passes through A(1,2).

As λ changes we obtain several points of the line.

Some points of the line	Some direction vectors
$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (the given one)	$\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ (the given one)
$\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ (for λ=1)	$\begin{pmatrix} 6 \\ 8 \end{pmatrix}$
$\begin{pmatrix} 7 \\ 10 \end{pmatrix}$ (for λ=2)	$\begin{pmatrix} 9 \\ 12 \end{pmatrix}$ multiples of $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$
$\begin{pmatrix} -2 \\ -2 \end{pmatrix}$ (for λ=−1)	$\begin{pmatrix} -3 \\ -4 \end{pmatrix}$

Hence, if we consider another point from the first column and another vector from the second column the resulting line is still the same! For example the vector equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

describes the same line!

(Confirm that the Cartesian equation derived is exactly the same)

- GIVEN TWO POINTS $A(a_1,a_2)$ AND $B(b_1,b_2)$

What is the equation of the line passing through A and B?

For the first bracket we choose one of the points: say $A(a_1,a_2)$

As a direction vector we consider the vector $\vec{b} = AB = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

Then

$$\vec{r} = \vec{a} + \lambda \vec{b}.$$

EXAMPLE 2

Find the line which passes through A(1,2) and B(4,7)

We consider $\vec{a}=\begin{pmatrix}1\\2\end{pmatrix}$ and $\vec{b}=AB=\begin{pmatrix}4-1\\7-2\end{pmatrix}=\begin{pmatrix}3\\5\end{pmatrix}$

Hence, the line is

$$\boxed{\vec{r}=\begin{pmatrix}1\\2\end{pmatrix}+\lambda\begin{pmatrix}3\\5\end{pmatrix}}$$

EXAMPLE 3

Consider the line y=3x+2. Find a vector equation of the line.

Firstly, let us find two points of this line:

For x=0, y=2 and for x=1, y=5

Hence, we are looking for the line which passes through A(0,2) and B(1,5).

$$\boxed{\vec{r}=\begin{pmatrix}0\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\3\end{pmatrix}}$$

(since $AB=\begin{pmatrix}1\\3\end{pmatrix}$)

EXAMPLE 4 (the inverse of EXAMPLE 3)

Let

$$\vec{r}=\begin{pmatrix}0\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\3\end{pmatrix}$$

Find the Cartesian equation.

The parametric equations are

$$x=0+1\lambda$$

$$y=2+3\lambda$$

Solving both equations for λ we find

$$(\lambda=)\ \frac{x-0}{1}=\frac{y-2}{3}$$

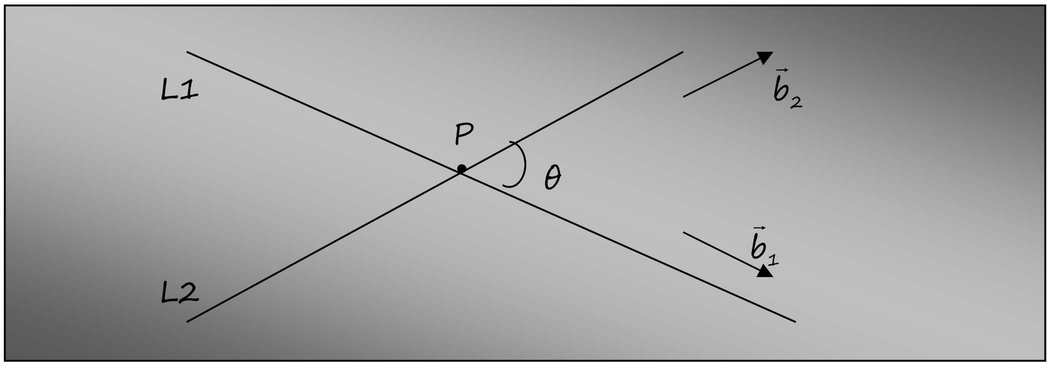
That is, 3x=y-2, or $\boxed{y=3x+2}$ as expected!

If we are given two lines

$$\vec{r}_1 = \vec{a}_1 + \lambda \vec{b}_1 \qquad (L1)$$
$$\vec{r}_2 = \vec{a}_2 + \mu \vec{b}_2 \qquad (L2)$$

two questions arise:

1) Find the intersection point P of these lines
2) Find the angle θ between these lines (usually the acute angle)



♦ INTERSECTION POINT OF TWO LINES

Methodology:

- set $\vec{r}_1 = \vec{r}_2$,
- find λ (or μ),

Substitute to L1 (or L2) to find the point.

EXAMPLE 5

Find the intersection point of the lines

$$\vec{r}_1 = \begin{pmatrix}1\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\4\end{pmatrix} \text{ and } \vec{r}_2 = \begin{pmatrix}2\\-2\end{pmatrix} + \mu\begin{pmatrix}1\\4\end{pmatrix}$$

$$\vec{r}_1 = \vec{r}_2 \Leftrightarrow \begin{pmatrix}1+3\lambda\\2+4\lambda\end{pmatrix} = \begin{pmatrix}2+\mu\\-2+4\mu\end{pmatrix} \Leftrightarrow \begin{cases}3\lambda-\mu=1\\4\lambda-4\mu=-4\end{cases} \Leftrightarrow \begin{cases}3\lambda-\mu=1\\\lambda-\mu=-1\end{cases}$$

The solution of this system is λ=1 and μ=2

For λ=1 the first vector equation gives $\vec{r}_1 = \begin{pmatrix}4\\6\end{pmatrix}$

Therefore, the intersection point is (4,6).

- ANGLE BETWEEN TWO LINES

It is enough to find the angle between $\vec{b}_1$ and $\vec{b}_2$
(since L1// $\vec{b}_1$ and L2// $\vec{b}_2$)

Notice: We usually consider the acute angle between the lines. Hence, if θ>90 we consider the angle $180^o-\theta$.

EXAMPLE 6

Find the angle between the lines

$$\vec{r}_1=\begin{pmatrix}1\\2\end{pmatrix}+\lambda\begin{pmatrix}3\\4\end{pmatrix} \text{ and } \vec{r}_2=\begin{pmatrix}2\\-2\end{pmatrix}+\mu\begin{pmatrix}1\\4\end{pmatrix}$$

It suffices to find the angle between $\vec{b}_1=\begin{pmatrix}3\\4\end{pmatrix}$, and $\vec{b}_2=\begin{pmatrix}1\\4\end{pmatrix}$

We have

$$\vec{b}_1\cdot\vec{b}_2 = 3\cdot1+4\cdot4=19 \quad \text{and} \quad |\vec{b}_1|= 5,\ |\vec{b}_2|= \sqrt{17}$$

so

$$\cos\theta = \frac{19}{5\sqrt{17}}=0.922, \text{ and the GDC gives } \theta=22.8^o$$

3.14 VECTOR EQUATION OF A LINE IN 3D (for HL)

Working in a similar way:

The line passing through A(1,2,3) and parallel to $\vec{b}=\begin{pmatrix}4\\5\\6\end{pmatrix}$ has

vector equation:

$$\vec{r}=\vec{a}+\lambda\vec{b} \quad \text{or} \quad \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}4\\5\\6\end{pmatrix}$$

parametric equations:

$$x=1+4\lambda$$
$$y=2+5\lambda$$
$$z=3+6\lambda$$

Cartesian equations*:

$$\frac{x-1}{4}=\frac{y-2}{5}=\frac{z-3}{6}$$

Notice that any point on this line has the form

$$P(1+4\lambda,\ 2+5\lambda,\ 3+6\lambda)$$

The rest analysis is similar! Let us find for example the line which passes through two given points:

EXAMPLE 1

(a) Find the line which passes through A(1,2,3) and B(5,2,-1)

(b) Does the point C(21,2,-17) lie on the line?

Solution

(a) We consider $\vec{a}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\vec{b}=AB=\begin{pmatrix}5-1\\2-2\\-1-3\end{pmatrix}=\begin{pmatrix}4\\0\\-4\end{pmatrix}$

Hence, the line is $\vec{r}=\vec{a}+\lambda\vec{b}$, that is

---

* We just solve the parametric equations for λ and equate the results

$$\vec{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix}$$

(b) The point C lies on the line if

$$\begin{pmatrix} 21 \\ 2 \\ -17 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix} \text{ for some } \lambda$$

We obtain three equations:

$21=1+4\lambda \Rightarrow \lambda=5$

$2=2$ (which is true anyway)

$-17=3-4\lambda \Rightarrow \lambda=5$

Therefore, yes C lies on the line (we obtain C for $\lambda=5$).

Let us see what happens when we are given two lines

$$\vec{r}_1 = \vec{a}_1 + \lambda \vec{b}_1 \qquad (L1)$$
$$\vec{r}_2 = \vec{a}_2 + \mu \vec{b}_2 \qquad (L2)$$

♦ INTERSECTION POINT OF TWO LINES

In the 3D space, three cases may occur:

- The lines are parallel (special case: they coincide)
- The lines intersect
- The lines are **skew** (neither parallel nor intersecting)

Methodology:

- If $\vec{b}_1 // \vec{b}_2$ the lines are parallel (moreover, if they have a common point they coincide); otherwise
- Set $\vec{r}_1 = \vec{r}_2$. We obtain a system of 3 equations for λ and μ
- Consider the first two equations and find λ and μ,
- If λ and μ satisfy the third equation the lines intersect; substitute λ to L1 (or μ to L2) to find the point
- If λ and μ do not satisfy the third equation the lines are skew (nether parallel nor intersecting)

EXAMPLE 2

Find the intersection point of the lines:

a) $\vec{r}_1=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\4\\5\end{pmatrix}$ and $\vec{r}_2=\begin{pmatrix}1\\1\\1\end{pmatrix}+\mu\begin{pmatrix}6\\8\\10\end{pmatrix}$

b) $\vec{r}_1=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\4\\5\end{pmatrix}$ and $\vec{r}_2=\begin{pmatrix}7\\10\\13\end{pmatrix}+\mu\begin{pmatrix}6\\8\\10\end{pmatrix}$

c) $\vec{r}_1=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\4\\5\end{pmatrix}$ and $\vec{r}_2=\begin{pmatrix}1\\4\\4\end{pmatrix}+\mu\begin{pmatrix}2\\2\\3\end{pmatrix}$

d) $\vec{r}_1=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\4\\5\end{pmatrix}$ and $\vec{r}_2=\begin{pmatrix}1\\4\\4\end{pmatrix}+\mu\begin{pmatrix}2\\2\\2\end{pmatrix}$

We have

a) the lines are parallel since $\begin{pmatrix}6\\8\\10\end{pmatrix}=2\begin{pmatrix}3\\4\\5\end{pmatrix}$. Since (1,2,3) does not lie on the second line (it does not satisfy the equation) the lines are not identical.

b) the lines are parallel since $\begin{pmatrix}6\\8\\10\end{pmatrix}=2\begin{pmatrix}3\\4\\5\end{pmatrix}$. Since (1,2,3) lies on the second line (it satisfies the equation for $\mu=-1$) the lines coincide.

c) $\vec{r}_1=\vec{r}_2 \Leftrightarrow \begin{pmatrix}1+3\lambda\\2+4\lambda\\3+5\lambda\end{pmatrix}=\begin{pmatrix}1+2\mu\\4+2\mu\\4+3\mu\end{pmatrix} \Leftrightarrow \begin{cases}3\lambda-2\mu=0\\4\lambda-2\mu=2\\5\lambda-3\mu=1\end{cases}$

The first two equations give $\lambda=2$, $\mu=3$. These values satisfy the third equation $5\lambda-3\mu=1$, so the lines intersect. For $\lambda=2$ the first equation line gives $\vec{r}_1=\begin{pmatrix}7\\10\\13\end{pmatrix}$. Point of intersection: P(7,10,13).

d) $\vec{r}_1=\vec{r}_2 \Leftrightarrow \begin{pmatrix}1+3\lambda\\2+4\lambda\\3+5\lambda\end{pmatrix}=\begin{pmatrix}1+2\mu\\4+2\mu\\4+2\mu\end{pmatrix} \Leftrightarrow \begin{cases}3\lambda-2\mu=0\\4\lambda-2\mu=2\\5\lambda-2\mu=1\end{cases}$

The first two equations give $\lambda=2$, $\mu=3$. These values do not satisfy the third equation $5\lambda-2\mu=1$, so the lines are skew.

♦ ANGLE BETWEEN TWO LINES

The angle between two lines is the angle between the direction vectors $\vec{b}_1$ and $\vec{b}_2$ (since L1//$\vec{b}_1$ and L2//$\vec{b}_2$). In fact, there are two supplementary angles. We usually ask for the acute one. Thus, if we find $\theta > 90^o$ we consider $180^o-\theta$.

EXAMPLE 3

Find the angle between the two intersecting lines (see Exercise 2(c))

$$\vec{r}_1=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\4\\5\end{pmatrix} \text{ and } \vec{r}_2=\begin{pmatrix}1\\4\\4\end{pmatrix}+\mu\begin{pmatrix}2\\2\\3\end{pmatrix}$$

We find the angle between the direction vectors $\vec{b}_1=\begin{pmatrix}3\\4\\5\end{pmatrix}$, $\vec{b}_2=\begin{pmatrix}2\\2\\3\end{pmatrix}$.

We have

$$\vec{b}_1\cdot\vec{b}_2 = 3\cdot2+4\cdot2+5\cdot3 = 29 \quad \text{and} \quad |\vec{b}_1|=\sqrt{50},\ |\vec{b}_2|=\sqrt{17}$$

so

$$\cos\theta = \frac{29}{\sqrt{50}\sqrt{17}} = 0.995, \text{ and the GDC gives } \theta=5.73°$$

EXAMPLE 4

Show that the angle between the following lines is 90°:

$$\vec{r}_1=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\4\\5\end{pmatrix} \text{ and } \vec{r}_2=\begin{pmatrix}1\\4\\4\end{pmatrix}+\mu\begin{pmatrix}-4\\3\\0\end{pmatrix}$$

The dot product of the direction vectors is $3(-4)+4\cdot3+5\cdot0 = 0$.

♦ DISTANCES

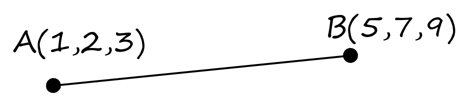
In this paragraph we will study the distance between

- two **points**
- a **point** and a **line**
- two **lines**

Let us present our methodology by considering particular examples:

- **Distance between Points**

Consider A(1,2,3) and B(5,7,9)



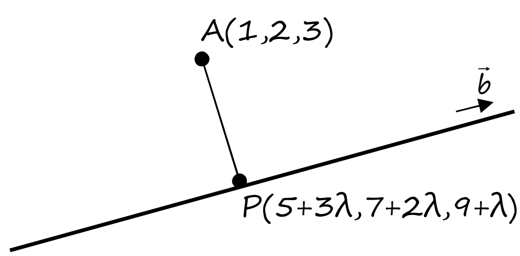
The well-known formula gives

$$d=\sqrt{(5-1)^2+(7-2)^2+(9-3)^2}=\sqrt{77}$$

- **Distance between Point and Line**

Consider

point A(1,2,3) and line L: $\vec{r}=\begin{pmatrix}5\\7\\9\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\1\end{pmatrix}$



<u>Key point: Vector AP is perpendicular to line L.</u>

We first find the foot P(5+3λ,7+2λ,9+λ) on the line L.

$$AP\perp L \Leftrightarrow AP\perp\begin{pmatrix}3\\2\\1\end{pmatrix} \Leftrightarrow \begin{pmatrix}4+3\lambda\\5+2\lambda\\6+\lambda\end{pmatrix}\perp\begin{pmatrix}3\\2\\1\end{pmatrix}$$

Thus

$$\begin{pmatrix} 4+3\lambda \\ 5+2\lambda \\ 6+\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$\Leftrightarrow 3(4+3\lambda)+2(5+2\lambda)+(6+\lambda)=0$$

$$\Leftrightarrow 14\lambda=-28$$

$$\Leftrightarrow \lambda=-2$$

Hence, the foot of the distance is P(–1,3,7)

The distance between the point and the line is

$$d(A,P)= \sqrt{(1+1)^2+(2-3)^2+(3-7)^2} = \sqrt{21}$$

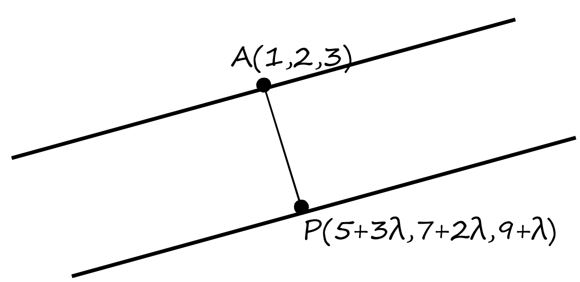
- **Distance between Lines**

(A) If the lines are parallel:

Consider

Line L_1: $\vec{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and line L_2: $\vec{r} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

The lines are clearly parallel (equal direction vectors).



<u>Key point: We select a point in line L_1 and find the distance from line L_2.</u>

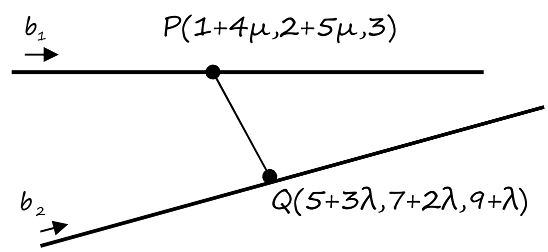
Here, the distance of point (1,2,3) of Line L_1 from line L_2 is exactly the case B above.

(B) If the lines are skew:

Consider

Line L_1: $\vec{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\mu\begin{pmatrix}4\\5\\0\end{pmatrix}$ and line L_2: $\vec{r}=\begin{pmatrix}5\\7\\9\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\1\end{pmatrix}$

It is given that the lines are skew



Key point: Vector PQ is perpendicular to both lines L_1 and L_2.

We first find foots P and Q. Notice that PQ= $\begin{pmatrix}4+3\lambda-4\mu\\5+2\lambda-5\mu\\6+\lambda\end{pmatrix}$

a) $PQ \perp L1 \Leftrightarrow \begin{pmatrix}4+3\lambda-4\mu\\5+2\lambda-5\mu\\6+\lambda\end{pmatrix} \perp \begin{pmatrix}4\\5\\0\end{pmatrix}$

$\Leftrightarrow 4(4+3\lambda-4\mu)+5(5+2\lambda-5\mu)+0(6+\lambda)=0$

$\Leftrightarrow 22\lambda-41\mu=-41$

b) $PQ \perp L2 \Leftrightarrow \begin{pmatrix}4+3\lambda-4\mu\\5+2\lambda-5\mu\\6+\lambda\end{pmatrix} \perp \begin{pmatrix}3\\2\\1\end{pmatrix}$

$\Leftrightarrow 3(4+3\lambda-4\mu)+2(5+2\lambda-5\mu)+(6+\lambda)=0$

$\Leftrightarrow 14\lambda-22\mu=-28$

The system gives $\lambda=-\dfrac{41}{15}$, and $\mu=-\dfrac{7}{15}$

Hence we find $P(-\dfrac{13}{15},-\dfrac{1}{3},3)$ and $Q(-\dfrac{16}{5},\dfrac{23}{15},\dfrac{94}{15})$

and hence we can find the distance $|PQ|$.

3.15 KINEMATICS

A nice application of the vector equation of line is the following:

- VELOCITY AND SPEED

Suppose that a body is moving along a straight line with a constant velocity and its position at time t is given by

$$\boxed{\vec{r}=\vec{a}+t\vec{b}}$$

Then

$\vec{a}$ is the position of the body at time t=0

$\vec{b}$ is the **velocity** vector of the body (usually $\vec{v}$)

$|\vec{b}|$ is the **speed** of the body (usually $|\vec{v}|$)

The vectors (and thus the motion) can be in either 2D or 3D space.

EXAMPLE 1

Suppose that a body is moving according to the equation

$$\vec{r}=\begin{pmatrix}1\\2\end{pmatrix}+t\begin{pmatrix}3\\4\end{pmatrix}$$

where time is measured in seconds and distance in meters.

The initial position (at t=0) of the body is (1,2).

So it is $\sqrt{1^2+2^2}=\sqrt{5}=2.23$m far from the origin.

The position at time t=1sec is (4,6)

So it is $\sqrt{4^2+6^2}=\sqrt{52}=7.21$m far from the origin.

The velocity vector is $\boxed{\vec{v}=\begin{pmatrix}3\\4\end{pmatrix}}$

The speed is $\boxed{|\vec{v}|=\sqrt{3^2+4^2}=5 \text{ m/sec}}$

NOTICE

If $\vec{r}=\vec{a}+\lambda\vec{b}$ is an equation of line, the direction vector $\vec{b}$ can be substituted by any multiple of $\vec{b}$.

If $\vec{r}=\vec{a}+t\vec{b}$ is an equation of motion, the velocity vector $\vec{b}$ CANNOT be substituted by a multiple of $\vec{b}$.

This is because the velocity vector corresponds to one unit of time t. To explain the difference, consider the following situations:

- Suppose that a body is initially at position A(1,2) and after 1 second at position B(5,8). Then

  velocity vector: $\vec{v}$=AB=$\begin{pmatrix}4\\6\end{pmatrix}$, **equation of motion**: $\vec{r}=\begin{pmatrix}1\\2\end{pmatrix}+t\begin{pmatrix}4\\6\end{pmatrix}$

- Suppose that a body is initially at position A(1,2) and after 2 seconds at position B(5,8). Then the direction vector $\vec{b}$=AB=$\begin{pmatrix}4\\6\end{pmatrix}$ corresponds to 2 seconds, hence

  Velocity vector: $\vec{v}=\frac{1}{2}\vec{b}=\begin{pmatrix}2\\3\end{pmatrix}$, **equation of motion**: $\vec{r}=\begin{pmatrix}1\\2\end{pmatrix}+t\begin{pmatrix}2\\3\end{pmatrix}$

EXAMPLE 2

Suppose that a body is moving on a straight line (in 3D space) in the direction of the vector $\vec{b}=\begin{pmatrix}1\\2\\2\end{pmatrix}$ with speed 15 ms^{-1}. Its initial position is A(1,1,1). Find the equation of the motion of the body.

Since

$$|\vec{b}|=\sqrt{1^2+2^2+2^2}=3$$

The unit vector if $\vec{b}$ is

$$\hat{b}=\frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}$$

and since the speed is 15

$$\vec{v}=15\hat{b}=5\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}5\\10\\10\end{pmatrix}$$

Therefore, the equation of the motion is

$$\vec{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}+t\begin{pmatrix}5\\10\\10\end{pmatrix}$$

In paragraph 3.14 we found that the point of intersection of the lines

$$\vec{r}_1=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\4\\5\end{pmatrix} \text{ and } \vec{r}_2=\begin{pmatrix}1\\4\\4\end{pmatrix}+\mu\begin{pmatrix}2\\2\\3\end{pmatrix}$$

is the point (7,10,13) (i.e. with position vector $\begin{pmatrix}7\\10\\13\end{pmatrix}$)

Let us see the same question in terms of Kinematics.

EXAMPLE 3

Two bodies are moving in 3D space according to the equations

$$\vec{r}_1=\begin{pmatrix}1\\2\\3\end{pmatrix}+t\begin{pmatrix}3\\4\\5\end{pmatrix} \text{ and } \vec{r}_2=\begin{pmatrix}1\\4\\4\end{pmatrix}+t\begin{pmatrix}2\\2\\3\end{pmatrix}$$

respectively.

(notice that here we use the same parameter t for time).

(a) Do their paths meet?

(b) Do the two bodies collide?

Solution

We have to solve the equation $\vec{r}_1=\vec{r}_2$.

If we use the equations of $\vec{r}_1$, $\vec{r}_2$ as they are (with t) we will answer only question (b) (the two bodies do not collide).

It helps to call the time parameters t_1 and t_2 respectively.

$$\vec{r}_1=\vec{r}_2 \Leftrightarrow \begin{pmatrix}1\\2\\3\end{pmatrix}+t_1\begin{pmatrix}3\\4\\5\end{pmatrix}=\begin{pmatrix}1\\4\\4\end{pmatrix}+t_2\begin{pmatrix}2\\2\\3\end{pmatrix} \Leftrightarrow \begin{cases}3t_1-2t_2=0\\4t_1-2t_2=2\\5t_1-3t_2=1\end{cases}$$

The first two equations give $t_1=2$, $t_2=3$. These values satisfy the third equation $5t_1-3t_2=1$. Hence

(a) The two paths intersect at $\begin{pmatrix}7\\10\\13\end{pmatrix}$ (use t_1 in equation $\vec{r}_1$)

(b) The two bodies do not collide since $t_1\neq t_2$.

3.16 CROSS PRODUCT (for HL)

This definition applies only for 3D vectors.

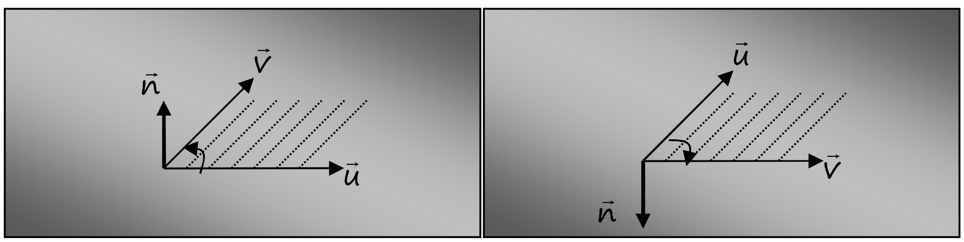
♦ THE GEOMETRIC DEFINITION (the "ugly" one)

Let $\vec{u}$ and $\vec{v}$ be two vectors and θ be the angle between those two vectors (where $0 \le \theta \le \pi$).

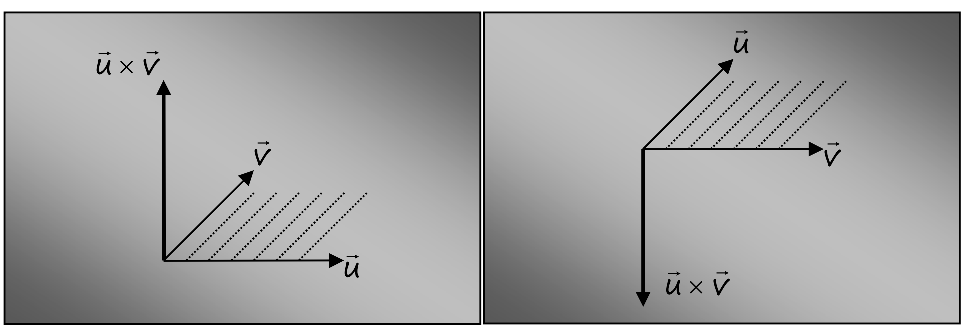
The **cross product** (or **vector product**) of $\vec{u}$ and $\vec{v}$ is defined to be a vector given by

$$\boxed{\vec{u} \times \vec{v} = (|\vec{u}||\vec{v}|\sin\theta)\vec{n}}$$

where $\vec{n}$ is the unit vector which is perpendicular to both $\vec{u}$ and $\vec{v}$ and follows the "screw rule"[†]:



That is, $\vec{u} \times \vec{v}$ is a new vector perpendicular to both $\vec{u}$ and $\vec{v}$ (and so to the plane determined by $\vec{u}$ and $\vec{v}$) with magnitude $|\vec{u}||\vec{v}|\sin\theta$ and direction $\vec{n}$.



Notice that the commutative law does not hold. However,

$$\boxed{\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}}$$

[†] If we place a screw at the common starting point of $\vec{u}$ and $\vec{v}$ and rotate it form $\vec{u}$ to $\vec{v}$, then the screw will move in the direction of $\vec{n}$.

♦ THE ALGEBRAIC DEFINITION (the "pretty" one)

Let $\vec{u}=\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$ and $\vec{v}=\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$ be two vectors. The **cross product** (or **vector product**) of $\vec{u}$ and $\vec{v}$ is given by

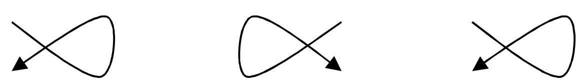
$$\boxed{\vec{u}\times\vec{v} = \begin{pmatrix} b_1c_2-b_2c_1 \\ c_1a_2-c_2a_1 \\ a_1b_2-a_2b_1 \end{pmatrix}}$$

Well, it doesn't look as pretty as the title promised! But the is a kind of symmetry in it!

For the first row of the result, you forget the first rows of $\vec{u}$ and $\vec{v}$ and you move along the arrow below

$$\begin{pmatrix} \cancel{a_1} \\ b_1 \\ c_1 \end{pmatrix} \times \begin{pmatrix} \cancel{a_2} \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1c_2-b_2c_1 \\ \\ \\ \end{pmatrix}$$

Then you carry on in a similar way for the 2nd and the 3rd row. Mind though the order of the operations for the three rows:



NOTICE

For those who know determinants, the definition can be given in the form

$$\boxed{\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}$$

expanded in terms of the first row vectors, i.e.

$$\vec{u}\times\vec{v} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\vec{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}\vec{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\vec{k}$$

EXAMPLE 1

Let $\vec{u}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}4\\5\\6\end{pmatrix}$. Then

a) find $\vec{u}\cdot\vec{v}$,

b) find $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$ (by using the "pretty" definition)

c) verify that $\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$

a) $\vec{u}\cdot\vec{v}=4+10+18=32$

b) $\vec{u}\times\vec{v} = \begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}4\\5\\6\end{pmatrix}=\begin{pmatrix}12-15\\12-6\\5-8\end{pmatrix}=\begin{pmatrix}-3\\6\\-3\end{pmatrix}$

$\vec{v}\times\vec{u} = \begin{pmatrix}4\\5\\6\end{pmatrix}\times\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}15-12\\6-12\\8-5\end{pmatrix}=\begin{pmatrix}3\\-6\\3\end{pmatrix}$. That is $\vec{v}\times\vec{u} = -\vec{u}\times\vec{v}$

c) $\vec{u}\times\vec{v}\perp\vec{u}$ and $\vec{u}\times\vec{v}\perp\vec{v}$

since $\begin{pmatrix}-3\\6\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix}=-3+12-9=0$, $\begin{pmatrix}-3\\6\\-3\end{pmatrix}\cdot\begin{pmatrix}4\\5\\6\end{pmatrix}=-12+30-18=0$

Notice that the "ugly" definition cannot be applied directly as we need the unit vector $\vec{n}$. Let us choose below two more convenient vectors $\vec{u}$ and $\vec{v}$ in order to compare the two definitions.

EXAMPLE 2

Let $\vec{u}=\begin{pmatrix}3\\2\\0\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}1\\4\\0\end{pmatrix}$. Then

a) find $\vec{u}\times\vec{v}$ by using the "pretty" definition

b) find the angle θ between $\vec{u}$ and $\vec{v}$

c) find the unit vector $\vec{n}$.

d) find $\vec{u}\times\vec{v}$ by using the "ugly" definition

e) verify that $\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$

a) $\vec{u}\times\vec{v} = \begin{pmatrix}3\\2\\0\end{pmatrix}\times\begin{pmatrix}1\\4\\0\end{pmatrix}=\begin{pmatrix}0\\0\\10\end{pmatrix}$

b) $\cos\theta=\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}=\dfrac{11}{\sqrt{13}\sqrt{17}}=0.74$, hence $\theta = 42.27^o$

c) both vectors $\vec{u}$ and $\vec{v}$ are on the plane Oxy so the unit vector $\vec{n}$ is parallel to axis Oz (if we draw $\vec{u}$ and $\vec{v}$ we will realize $\vec{n}$ is in the positive direction so

$$\vec{n}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

d) $\vec{u}\times\vec{v}=(|\vec{u}||\vec{v}|\sin\theta)\vec{n}=(\sqrt{13}\sqrt{17}\sin 42.3^o)\vec{n}=10\vec{n}=\begin{pmatrix}0\\0\\10\end{pmatrix}$

e) clearly $\vec{u}\times\vec{v}$ is parallel to $\vec{n}$ and thus perpendicular to both $\vec{u}$ and $\vec{v}$.

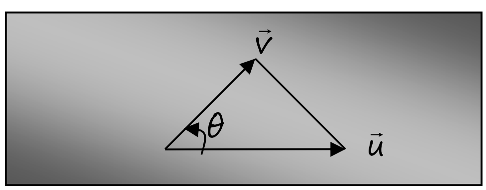
- THE MAGNITUDE $|\vec{u}\times\vec{v}|$

Notice that the ugly definition $\vec{u}\times\vec{v}=(|\vec{u}||\vec{v}|\sin\theta)\vec{n}$ implies

$$\boxed{|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta}$$

since $\vec{n}$ is a unit vector.

But, if we consider the triangle determined by $\vec{u}$ and $\vec{v}$

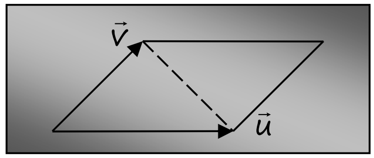


we know that its area is given by $\frac{1}{2}|\vec{u}||\vec{v}|\sin\theta$.

Therefore, the area of this triangle is given by

$$\boxed{\text{Area of triangle} = \frac{1}{2}|\vec{u}\times\vec{v}|}$$

In other words, the magnitude of the cross product $\vec{u}\times\vec{v}$ gives directly the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$



Area of parallelogram = $\left|\vec{u}\times\vec{v}\right|$

EXAMPLE 3

For $\vec{u}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}4\\5\\6\end{pmatrix}$, we have seen that $\vec{u}\times\vec{v}$ = $\begin{pmatrix}-3\\6\\-3\end{pmatrix}$

Therefore, the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$ is given by

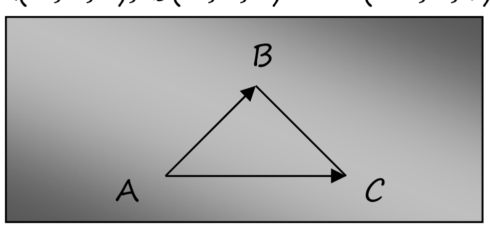
$$\text{Area} = \left|\vec{u}\times\vec{v}\right| = \sqrt{9+36+9} = 7.35$$

Also, the area of the corresponding triangle is $\frac{1}{2}(7.35)=3.67$

EXAMPLE 4

Find the area of the triangle determined by the three points

A(1,1,1), B(1,3,1) and (-3,3,4)



It suffices to find the area of the triangle determined by any two vectors; let's choose the vectors AB and AC.

$$AB = \begin{pmatrix}0\\2\\0\end{pmatrix},\quad AC = \begin{pmatrix}-4\\2\\3\end{pmatrix} \text{ and so } AB\times AC=\begin{pmatrix}0\\2\\0\end{pmatrix}\times\begin{pmatrix}-4\\2\\3\end{pmatrix}=\begin{pmatrix}6\\0\\8\end{pmatrix}$$

Hence,

$$\text{Area of triangle} = \frac{1}{2}\left|AB\times AC\right| = \frac{1}{2}\sqrt{36+0+64} = 5$$

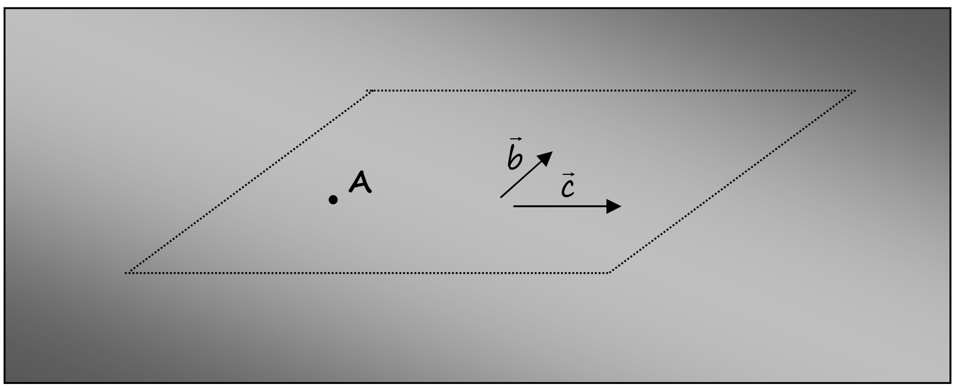
3.17 PLANES (for HL)

- VECTOR EQUATION

Given: **Point A**(a_1,a_2,a_3) (the position vector is $\vec{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$)

Two vectors $\vec{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$, $\vec{c}=\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$ (which are non-parallel)

There is a unique **plane** passing through **A**, parallel to both $\vec{b}$ and $\vec{c}$



The position vector $\vec{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ of any point **P**(x,y,z) of this plane is given by

$$\boxed{\vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c}}$$

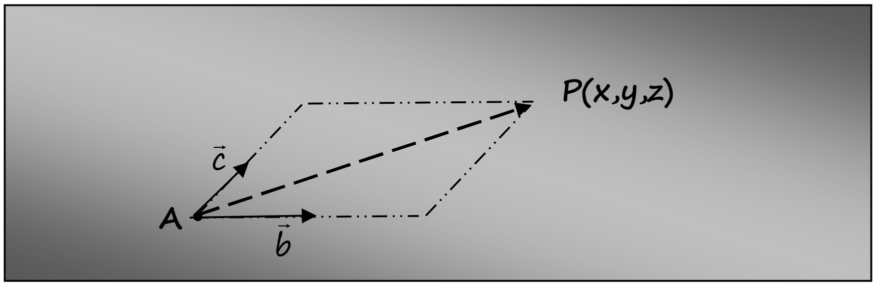
or

$$\boxed{\vec{r}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+\lambda\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}+\mu\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}}$$

where λ,μ are two parameters.

♦ SHORT EXPLANATION

If **P**(x,y,z) is any point on the plane then **AP** lies in fact on the plane determined by $\vec{b}$ and $\vec{c}$.



Hence

$$\mathbf{AP} = \lambda\vec{b} + \mu\vec{c} \quad \text{(for some } \lambda,\mu\text{)}.$$

Then, the position vector of P is given by

$$\vec{r} = OP = \mathbf{OA} + \mathbf{AP} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$$

♦ PARAMETRIC EQUATIONS

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \text{gives} \quad \begin{aligned} x &= a_1 + \lambda b_1 + \mu c_1 \\ y &= a_2 + \lambda b_2 + \mu c_2 \\ z &= a_3 + \lambda b_3 + \mu c_3 \end{aligned}$$

♦ CARTESIAN EQUATION

If we eliminate λ and μ we will obtain an equation of the form

$$\boxed{Ax + By + Cz = D}$$

Remark: Although the method of eliminating λ and μ is not necessary (a much easier method will be given in a while!) we will demonstrate the procedure by using the example below, just to persuade ourselves. The steps are as follows

- Eliminate λ from the first two equations;
- Eliminate λ from the last two equations;
- Eliminate μ from the two resulting equations.

EXAMPLE 1

Let A(1,2,3) be the given point

$\vec{b}=\begin{pmatrix}4\\5\\6\end{pmatrix}$ and $\vec{c}=\begin{pmatrix}7\\8\\8\end{pmatrix}$ be the parallel vectors

Then the plane passing through A, parallel to $\vec{b}$ and $\vec{c}$ is

Vector equation:

$$\vec{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}4\\5\\6\end{pmatrix}+\mu\begin{pmatrix}7\\8\\8\end{pmatrix}$$

Parametric equations:

$$x=1+4\lambda+7\mu \qquad (1)$$
$$y=2+5\lambda+8\mu \qquad (2)$$
$$z=3+6\lambda+8\mu \qquad (3)$$

We eliminate λ from equations (1) and (2)

$5\times(1)-4\times(2)$: $\quad 5x-4y=-3+3\mu \quad (4)$

We eliminate λ from equations (2) and (3)

$6\times(2)-5\times(3)$: $\quad 6y-5z=-3+8\mu \quad (5)$

Next, we eliminate μ from (4) and (5)

$8\times(4)-3\times(5)$: $\quad 40x-32y-18y-15z = -24+9$

$40x-50y+15z=-15$

We simplify the equation by dividing by -5 and we obtain

Cartesian equation: $\boxed{-8x+10y-3z = 3}$

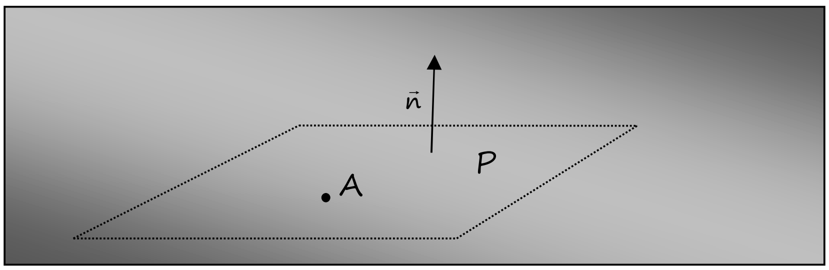
As we said, a much quicker process will give the same result!

♦ VECTOR EQUATION IN NORMAL FORM

Given: **Point A**(a_1,a_2,a_3) (the position vector is $\vec{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$)

Normal vector $\vec{n}=\begin{pmatrix}A\\B\\C\end{pmatrix}$

There is a unique **plane** passing through **A**, perpendicular to $\vec{n}$.



The equation of the plane is

$$\boxed{\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}}$$

Indeed, if **P**(x,y,z) is a random point of the plane then **AP** $\perp\vec{n}$

But **AP** =**OP**−**OA**=$\vec{r}-\vec{a}$, and so

$$\mathbf{AP}\cdot\vec{n}=0\Rightarrow(\vec{r}-\vec{a})\cdot\vec{n}=0\Rightarrow\vec{r}\cdot\vec{n}-\vec{a}\cdot\vec{n}=0\Rightarrow\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$$

NOTICE

The equation $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$ derives immediately the Cartesian form

$$Ax+By+Cz=D$$

Indeed, $\vec{r}\,\vec{n}=\begin{pmatrix}x\\y\\z\end{pmatrix}\begin{pmatrix}A\\B\\C\end{pmatrix}=Ax+By+Cz$

while $\vec{a}\,\vec{n}$ is a constant scalar, say D

In fact, given point **A** and **n**, we directly find the Cartesian form:

write down the LHS using **n** $Ax+By+Cz$
plug in **A** to find the RHS D

EXAMPLE 2

Find the equation of the plane passing through A(1,2,3) which is perpendicular to $\vec{n}=\begin{pmatrix}-8\\10\\-3\end{pmatrix}$ (normal vector)

The equation $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$ implies

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\begin{pmatrix}-8\\10\\-3\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}-8\\10\\-3\end{pmatrix}$$

or directly

$-8x+10y-3z=-8+20-9$ and finally $\boxed{-8x+10y-3z=3}$

NOTICE

In examples 1 and 2 we obtained the same plane: $-8x+10y-3z=3$

We had:

EXAMPLE 1: Point: $\vec{a}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ Parallel vectors: $\vec{b}=\begin{pmatrix}4\\5\\6\end{pmatrix}$ and $\vec{c}=\begin{pmatrix}7\\8\\8\end{pmatrix}$

EXAMPLE 2: Point: $\vec{a}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ Normal vector: $\vec{n}=\begin{pmatrix}-8\\10\\-3\end{pmatrix}$

Indeed, if we consider as $\vec{n}$ the cross product $\vec{b}\times\vec{c}$
(which is $\perp$ to both $\vec{b},\vec{c}$ and hence perpendicular to the plane)
we obtain

$$\vec{n}=\begin{pmatrix}4\\5\\6\end{pmatrix}\times\begin{pmatrix}7\\8\\8\end{pmatrix}=\begin{pmatrix}-8\\10\\-3\end{pmatrix}$$

Thus, given the vector equation of the plane, the Cartesian equation can be easily derived in this way instead of following the elimination process of λ and μ.

NOTICE

If we know the Cartesian form $\boxed{Ax+By+Cz=D}$

we also know a normal vector of the equation. It is $\boxed{\vec{n}=\begin{pmatrix}A\\B\\C\end{pmatrix}}$

EXAMPLE 3

Consider the plane $3x-2y+z = 6$

a) Find a normal vector $\vec{n}$
b) Find three points on the plane
c) Find two vectors $\vec{b}$ and $\vec{c}$ parallel to the plane
d) Confirm that $\vec{n}\perp\vec{b}$ and $\vec{n}\perp\vec{c}$
e) Write down all the forms of equation for this plane

Solution

a) $\vec{n}=\begin{pmatrix}3\\-2\\1\end{pmatrix}$

b) For $y=z=0$ it is $x=2$, thus we obtain the point $A(2,0,0)$.
Similarly we obtain the points $B(0,-3,0)$ and $C(0,0,6)$

c) Let $\vec{b}=AB=\begin{pmatrix}0\\-3\\0\end{pmatrix}-\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}-2\\-3\\0\end{pmatrix}$ and $\vec{c}=AC=\begin{pmatrix}0\\0\\6\end{pmatrix}-\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}-2\\0\\6\end{pmatrix}$

d) We can easily see that $\vec{n}\cdot\vec{b}=-6+6 = 0$ and $\vec{n}\cdot\vec{c}=-6+6 = 0$

e) **Vector form**: $\vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c}$ or $\vec{r}=\begin{pmatrix}2\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}-2\\-3\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\0\\6\end{pmatrix}$

Parametric form: $x=2-2\lambda-2\mu$, $y=-3\lambda$, $z=6\mu$

Normal form: $\vec{r}\cdot\begin{pmatrix}3\\-2\\1\end{pmatrix}=6$ [since $\vec{a}\cdot\vec{n}=6$]

Cartesian form: $3x-2y+z = 6$

EXAMPLE 4

Consider the plane

$$\vec{r}=\begin{pmatrix}3\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\3\end{pmatrix}+\mu\begin{pmatrix}5\\0\\2\end{pmatrix}$$

a) Find two parallel vectors $\vec{b}$ and $\vec{c}$
b) Find three points on the plane
c) Find a normal vector $\vec{n}$
d) Write down all the forms of equation for this plane

Solution

a) $\vec{b}=\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\vec{c}=\begin{pmatrix}5\\0\\2\end{pmatrix}$

b) A(3,1,2) (the obvious one)
B(4,3,5) for $\lambda=1,\mu=0$
C(8,1,4) for $\lambda=0$, $\mu=1$

c) Let $\vec{n}=\vec{b}\times\vec{c}=\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}5\\0\\2\end{pmatrix}=\begin{pmatrix}4\\13\\-10\end{pmatrix}$

d) **Vector form**: $\vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c}$ or $\vec{r}=\begin{pmatrix}3\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\3\end{pmatrix}+\mu\begin{pmatrix}5\\0\\2\end{pmatrix}$

Parametric form: $x=3+\lambda+5\mu$, $y=1+2\lambda$, $z=2+3\lambda+2\mu$

Normal form: $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$

Cartesian form: $4x+13y-10z = 5$

since $\begin{pmatrix}4\\13\\-10\end{pmatrix}$ is a normal vector and $\vec{a}\cdot\vec{n}=12+13-20 = 5$

3.18 INTERSECTIONS AMONG LINES AND PLANES (for HL)

In this section we will study the relative position between

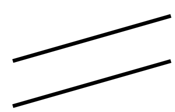
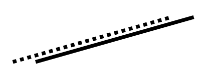
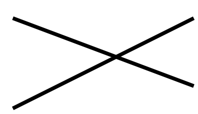
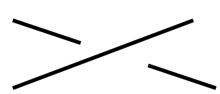
- two lines
- a line and a plane
- two planes
- three planes

♦ TWO LINES

Given: **Lines** L_1: $\vec{r}_1 = \vec{a}_1 + \lambda \vec{b}_1$

L_2: $\vec{r}_2 = \vec{a}_2 + \mu \vec{b}_2$

We have already seen this study in paragraph 3.14. Let us remember all possible cases.

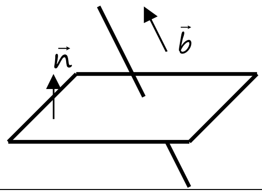
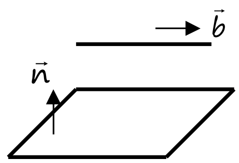
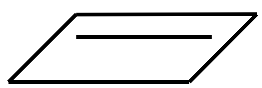
Lines	**Look like**	**Method**
parallel		Check if $\vec{b}_1 // \vec{b}_2$
coincide		Check if $\vec{b}_1 // \vec{b}_2$ + a common point
Intersect at some point		$\vec{r}_1 = \vec{r}_2$ has a solution
skew		$\vec{r}_1 = \vec{r}_2$ has no solution

θ = angle between the two lines	
θ = angle between $\vec{b}_1$ and $\vec{b}_2$	$\cos\theta = \dfrac{\vec{b}_1 \cdot \vec{b}_2}{\lvert\vec{b}_1\rvert \lvert\vec{b}_2\rvert}$

- A LINE AND A PLANE

Given: **Line** **L:** $\vec{r}_1 = \vec{a} + \lambda\vec{b}$

Plane **Π**: $Ax+By+Cz=D$ (so $\vec{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$)

Line and Plane	Look like	Method
Intersect at some point		plug $\vec{r}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ into $Ax+By+Cz=D$ to find λ
parallel		Check if $\vec{b} \perp \vec{n}$ or no intersection point
Line lies on Plane		Check if $\vec{b} \perp \vec{n}$ + a common point or ∞ intersection points

$\boldsymbol{\theta}$ = angle between line and plane	
If ϕ = angle between $\vec{b}$ and $\vec{n}$ then $\theta = 90^o - \phi$	$\sin\theta = \dfrac{\vec{b}\cdot\vec{n}}{\lvert\vec{b}\rvert\lvert\vec{n}\rvert}$

Notice: if the line and the plane are given in other forms, we transform them into the forms **L:** $\vec{r}_1 = \vec{a} + \lambda\vec{b}$ and **Π**: $Ax+By+Cz=D$

EXAMPLE 1

Consider the line **L:** $\vec{r}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ and the plane **Π**: $2x+5y-3z=18$

Find the the angle between **L** and **Π** and the point of intersection.

For the angle between L and P we have

$$\sin\theta = \frac{\vec{b}\cdot\vec{n}}{\lvert\vec{b}\rvert\lvert\vec{n}\rvert} = \frac{15}{\sqrt{77}\sqrt{38}} = 0.277, \text{ hence } \theta = 16.1^o$$

The point of intersection lies on **L**, so it has the form

$$(x,y,z)=(1+4\lambda,2+5\lambda,3+6\lambda)$$

We plug it into the equation of the plane $2x+5y-3z=18$:

$$2(1+4\lambda)+5(2+5\lambda)-3(3+6\lambda)=18 \Leftrightarrow 15\lambda+3=18 \Leftrightarrow \lambda=1$$

Hence, $x=5$, $y=7$, $z=9$ and the intersection point is $(x,y,z)=(5,7,9)$.

EXAMPLE 2

Show that line **L**: $\vec{r}_1=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}4\\5\\6\end{pmatrix}$ is parallel to plane **Π**: $2x+2y-3z=1$

Method A: If $\vec{b}=\begin{pmatrix}4\\5\\6\end{pmatrix}$ and $\vec{n}=\begin{pmatrix}2\\2\\-3\end{pmatrix}$, then

$$\vec{b}\cdot\vec{n}=0 \Rightarrow \vec{b}\perp\vec{n} \Rightarrow \mathbf{L}\,//\,\mathbf{\Pi}$$

The point $(1,2,3)$ of the line does not satisfy $2x+2y-3z=1$, hence the line does not lie on the plane.

Method B: A point on **L** has the form $(x,y,z)=(1+4\lambda,2+5\lambda,3+6\lambda)$

We plug it into the equation of the plane $2x+2y-3z=1$:

$$2(1+4\lambda)+2(2+5\lambda)-3(3+6\lambda)=1 \Leftrightarrow 0\lambda=4$$

The last equation is impossible, thus there is no intersection point.

EXAMPLE 3

Show that line **L**: $\vec{r}_1=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}4\\5\\6\end{pmatrix}$ lies on plane **Π**: $2x+2y-3z=-3$

Method A: Again $\vec{b}\cdot\vec{n}=0 \Rightarrow \vec{b}\perp\vec{n} \Rightarrow \mathbf{L}\,//\,\mathbf{\Pi}$

But this time, the point $(1,2,3)$ of the line satisfies the equation $2x+2y-3z=-3$, hence the line lies on the plane.

Method B: A point on **L** has the form $(x,y,z)=(1+4\lambda,2+5\lambda,3+6\lambda)$

We plug it into the equation of the plane $2x+2y-3z=-3$:

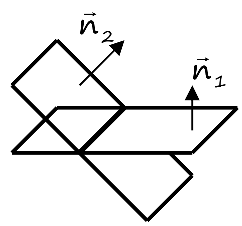
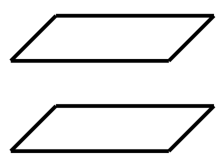
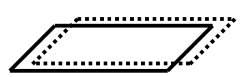
$$2(1+4\lambda)+2(2+5\lambda)-3(3+6\lambda)=-3 \Leftrightarrow 0\lambda=0$$

The last equation is true for any λ, so the line lies on the plane.

♦ TWO PLANES

Given: **Planes** Π_1: $A_1x+B_1y+C_1z=D_1$ so $\vec{n}_1=\begin{pmatrix}A_1\\B_1\\C_1\end{pmatrix}$

Π_2: $A_2x+B_2y+C_2z=D_2$ so $\vec{n}_2=\begin{pmatrix}A_2\\B_2\\C_2\end{pmatrix}$

Planes	Look like	Method
intersecting into a **line** $\vec{r}=\vec{a}+\lambda\vec{b}$		Find two common points and thus the line or one common point $\vec{a}$ and direction vector $\vec{b}=\vec{n}_1\times\vec{n}_2$ or solve simultaneous equations
parallel		Check if $\vec{n}_1 // \vec{n}_2$
coincide		Check if $\vec{n}_1 // \vec{n}_2$ + The equations are multiple to each other

θ = angle between the two planes	
θ = angle between $\vec{n}_1$ and $\vec{n}_2$	$\cos\theta=\dfrac{\vec{n}_1\cdot\vec{n}_2}{\lvert\vec{n}_1\rvert\lvert\vec{n}_2\rvert}$

EXAMPLE 4

Consider the planes

$$x+2y+3z=6$$

$$4x+5y+6z=15$$

Find the angle between the two planes and the line of intersection.

For the angle: $\cos\theta=\dfrac{\vec{n}_1\cdot\vec{n}_2}{\lvert\vec{n}_1\rvert\lvert\vec{n}_2\rvert}=\dfrac{32}{\sqrt{14}\sqrt{77}}\Rightarrow \theta=12.93^o$

For the line of intersection

Method A: Let us first find two common points

For z=0 the equations become

$$x+2y=6 \quad \text{and} \quad 4x+5y=15$$

which give x=0, y=3. Hence, a common point is A(0,3,0).

For z=1 the equations become

$$x+2y=3 \quad \text{and} \quad 4z+5y=9$$

which give x=1, y=1. Hence, a common point is B(1,1,1).

The two points A,B determine the equation of the intersecting line

$$\vec{r}=\begin{pmatrix}0\\3\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

Method B: We find only one common point, say A(0,3,0) and as a direction vector we consider $\vec{n}_1\times\vec{n}_2=\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}4\\5\\6\end{pmatrix}=\begin{pmatrix}-3\\6\\-3\end{pmatrix}$ which is // $\begin{pmatrix}1\\-2\\1\end{pmatrix}$

Method C: We solve the system of the two linear equations (GDC or Gauss elimination). The general solution is $x=\lambda$, $y=3-2\lambda$, $z=\lambda$.

These are the parametric equations of the same line.

EXAMPLE 5

Consider the planes $x+2y+3z=10$

$2x+4y+6z=30$

Their normal vectors $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}2\\4\\6\end{pmatrix}$ are clearly parallel.

Moreover, the two planes do not have a common point (since one equation is not a multiple of the other).

EXAMPLE 6

Consider the planes $x+2y+3z=10$ and $2x+4y+6z=20$

The two planes coincide (one is a multiple of the other)

♦ THREE PLANES

Given: **Planes** $A_1x+B_1y+C_1z=D_1$

$A_2x+B_2y+C_2z=D_2$

$A_3x+B_3y+C_3z=D_3$

The problem reduces to the solution of a 3x3 system of simultaneous equations (see paragraph 1.9). Then

SYSTEM	CONCLUSION
Unique solution (x,y,z)	The three planes have one common point (x,y,z)
No solution	No common point: The planes form a triangular prism or 2 of the planes are parallel
Infinitely many solutions	Planes intersect into a line or at least 2 planes coincide

EXAMPLE 7

Consider the planes

$$\begin{aligned} 2x + 3y + 3z &= 3 \\ x + y - 2z &= 4 \\ 5x + 7y + 4z &= 10 \end{aligned}$$

We may see (either by Gauss elimination or by a GDC) that this system has infinitely many solutions:

$$x = 14 + 16\lambda$$

$$y = -5 + 7\lambda$$

$$z = \lambda \in R \text{ (free variable)}$$

The solution represents the line $\vec{r} = \begin{pmatrix} 14 \\ -5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 16 \\ 7 \\ 1 \end{pmatrix}$.

3.19 DISTANCES (for HL)

We have already studied the distance between

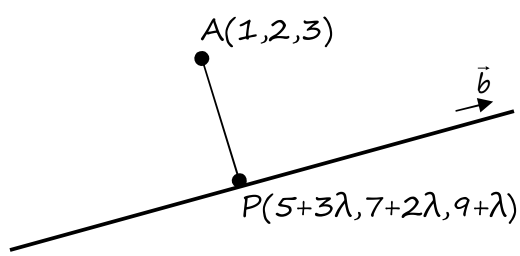
- two **points**
- a **point** and a **line**
- two **lines**

In this section we will also study the distance between

- a **point** and a **plane**
- a **line** and a **plane**
- two **planes**

Let us remember the distance between a point and a line:

point A(1,2,3) and line L: $\vec{r}=\begin{pmatrix}5\\7\\9\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\1\end{pmatrix}$



The advantage here is that the foot P(x,y,z) has the form

$$P(5+3\lambda,7+2\lambda,9+\lambda)$$

thus, it is enough to find the parameter λ.

$$AP\perp L \Leftrightarrow AP\perp\begin{pmatrix}3\\2\\1\end{pmatrix} \Leftrightarrow \begin{pmatrix}4+3\lambda\\5+2\lambda\\6+\lambda\end{pmatrix}\perp\begin{pmatrix}3\\2\\1\end{pmatrix}$$

$$\Leftrightarrow 3(4+3\lambda)+2(5+2\lambda)+(6+\lambda)=0$$

$$\Leftrightarrow 14\lambda=-28$$

$$\Leftrightarrow \lambda=-2$$

Hence, the foot of the distance is P(-1,3,7)

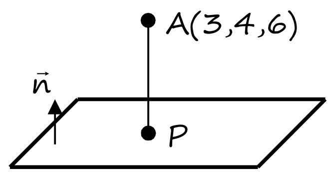
The distance between the point and the line is d(A,P)=$\sqrt{21}$

For the distance between a Point and a Plane, again the first task is to find the corresponding foot P(x,y,z) on the plane.

- **Distance between Point and Plane**

  Consider

  point A(3,4,6) and Plane Π: 2x+3y+5z=10



  Key point: Line AP is parallel to the normal vector n.

  The equation of line AP is:

$$\vec{r}=\begin{pmatrix}3\\4\\6\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\5\end{pmatrix}$$

  Thus the foot has coordinates P(3+2λ,4+3λ,6+5λ).

  But it also lies on Π, so that

$$2(3+2\lambda)+3(4+3\lambda)+5(6+5\lambda)=10 \Leftrightarrow 38\lambda=-38 \Leftrightarrow \lambda=-1$$

  Hence the foot is P(1,1,1).

  The distance is $|AP|$, that is

$$d(A,P)=\sqrt{(3-1)^2+(4-1)^2+(6-1)^2}=\sqrt{38}$$

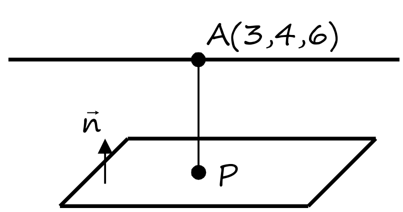
- **Distance between Line and Plane**

  This case occurs only if the line is parallel to the plane.

  Consider

$$\text{Line L: } \vec{r}=\begin{pmatrix}3\\4\\6\end{pmatrix}+\lambda\begin{pmatrix}3\\-2\\0\end{pmatrix} \quad \text{and} \quad \text{Plane } \Pi: 2x+3y+5z=10$$

  It is given that the line is parallel to the plane.



We just find the distance of point A of line L from plane Π

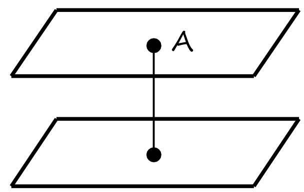
- **Distance between Planes**

This case occurs only if the planes are parallel.

Consider

Plane Π_1: $2x+3y+5z=10$ and Plane Π_2: $2x+3y+5z=48$

Clearly planes Π_1 and Π_2 are parallel.



We just find a point of plane Π_1:

for $x=y=0$, we obtain $z=2$, and thus $A(0,0,2)$

Then we find the distance between point A and plane Π_2 (as above).

We first find the equation of line AP:

$$\vec{r}=\begin{pmatrix}0\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\5\end{pmatrix}$$

The foot has coordinates $P(2\lambda,3\lambda,2+5\lambda)$ and lies on Π_2,

so that

$$2(2\lambda)+3(3\lambda)+5(2+5\lambda)=48 \Leftrightarrow 38\lambda=38 \Leftrightarrow \lambda=1$$

Hence the foot is $P(2,3,7)$.

The distance is

$$d(A,P)=\sqrt{(2-0)^2+(3-0)^2+(7-2)^2}=\sqrt{38}$$