International Baccalaureate

MATHEMATICS

Analysis and Approaches (SL and HL)

Lecture Notes

Christos Nikolaidis

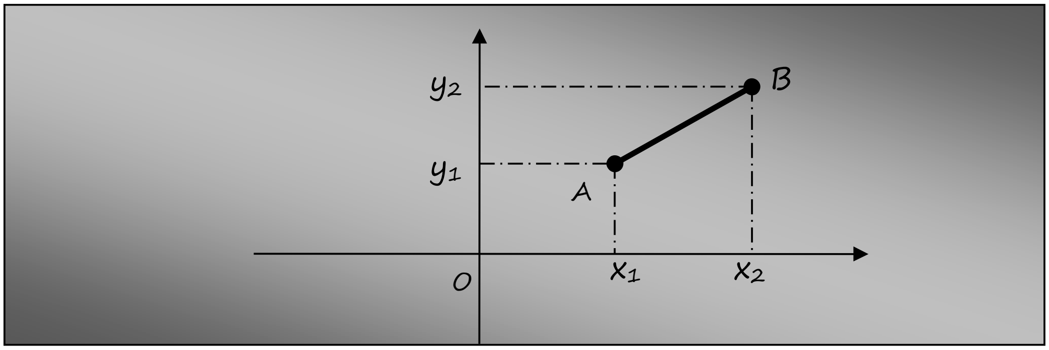
TOPIC 2
FUNCTIONS

September 2020

2.1 LINES (or LINEAR FUNCTIONS)

♦ BASIC NOTIONS ON COORDINATE GEOMETRY

Given two points $A(x_1,y_1)$ and $B(x_2,y_2)$



- The **gradient** or **slope** of line segment AB is given by

$$m=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}$$

This indicates the inclination of the line segment AB. As we are moving along the positive direction of the x-axis, if the line segment is

increasing	(/)	then	**m>0**
decreasing	(\ )	then	**m<0**
horizontal	(—)	then	**m=0**
vertical	(|)	then	**m is not defined**

- The distance between A and B is given by

$$d_{AB}=\sqrt{(\Delta x)^2+(\Delta y)^2}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

- The coordinates of the **midpoint** $M(x,y)$ of the line segment AB are given by

$$x=\frac{x_1+x_2}{2} \qquad y=\frac{y_1+y_2}{2}$$

EXAMPLE 1

a) Given two points A(1,4) and B(7,12)

The slope of the line segment AB is $m=\frac{\Delta y}{\Delta x}=\frac{12-4}{7-1}=\frac{4}{3}$

The distance between them is $d=\sqrt{(7-1)^2+(12-4)^2}=10$

The midpoint is $M(\frac{1+7}{2},\frac{4+12}{2})$ that is M(4,8)

b) Given two points A(1,8) and B(5,8)

It is not necessary to use the formulas. Since A and B have the same y-coordinate:

The slope of the line segment AB is m=0 (horizontal)

The distance between them is d=5-1=4

The midpoint is M(3,8)

c) Given two points A(1,5) and B(1,7)

It is not necessary to use the formulas. Since A and B have the same x-coordinate:

The slope m of the line segment AB is not defined (vertical)

The distance between them is d=7-5=2

The midpoint is M(1,6)

The notion of the **function** will be formally introduced later on, in paragraph 2.3. However, we will start by presenting two families of already known functions

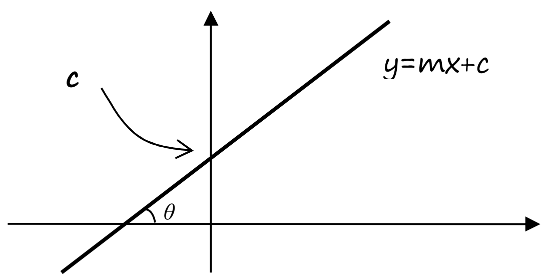
Linear functions: $y=mx+c$ or $f(x) = mx+c$

Quadratic functions: $y=ax^2+bx+c$ or $f(x) = ax^2+bx+c$

- THE EQUATION OF A LINE

Equation of a (straight) line:	**y=mx+c**

m = **gradient** or **slope** c = **y-intercept**

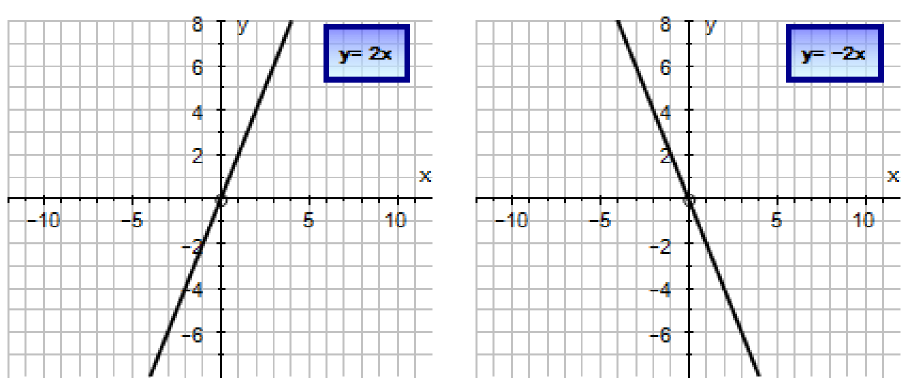


NOTICE:

- A horizontal line has equation **y=c** (slope m=0)
- A vertical line has equation **x=c** (there is no slope)
  (in fact, a vertical line is not a function, that is why the equation x=0 is not a particular case of y=mx+c)
- m=tanθ, where θ is the angle between the line and x-axis

EXAMPLE 2

Look at the graphs of two lines: L_1: **y=2x** and L_2: **y=-2x**



In fact, the slope shows the rise of the line per each unit

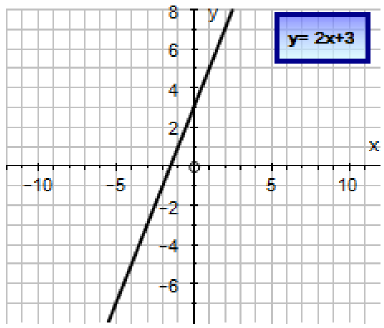
Line L_1: slope is 2 (y increases 2 units per each x-unit)

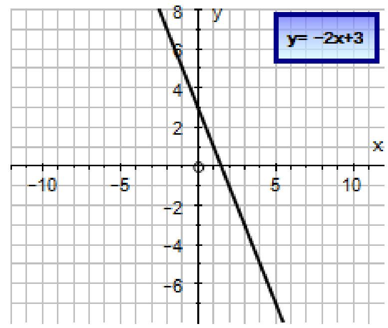
Line L_2: slope is -2 (y decreases 2 units per each x-unit)

In both cases c=0 (since the function passes through the origin)

EXAMPLE 3

Look at the graphs of two lines: L_1: $y=2x+3$ and L_2: $y=-2x+3$



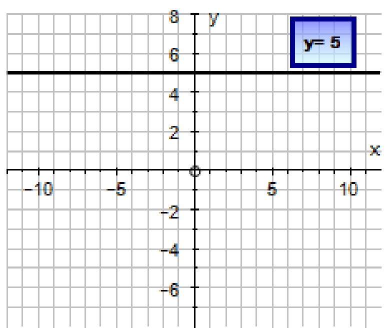


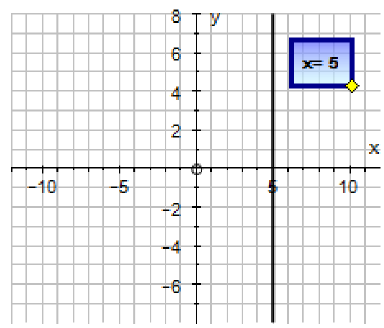
Line L_1: slope is 2 Line L_2: slope is −2

In both cases the y-intercept is 3

EXAMPLE 4

Look at the graphs of two lines: L_1: $y=5$ and L_2: $x=5$





♦ PARALLEL AND PERPENDICULAR LINES

Consider two lines: L_1: $y=m_1x+c_1$ and L_2: $y=m_2x+c_2$

Parallel lines:	$L_1 // L_2$	if	$m_1 = m_2$
Perpendicular lines:	$L_1 \perp L_2$	if	$m_2 = -1/m_1$

For example,

The lines $y=3x+5$ and $y=3x+8$ are parallel

The lines $y=3x+5$ and $y=-\frac{1}{3}x+8$ are perpendicular

♦ AN ALTERNATIVE FORMULA FOR A LINE

A more general formula for a line is

Equation of a line: **Ax+By=C**

If B≠0, we can solve for y and obtain the form $y=mx+c$
If B=0, we obtain a vertical line of the form $x=c$
If A=0, we obtain a horizontal line of the form $y=c$

EXAMPLE 5

- From Ax+By=C into the usual form
  The line $2x+3y=5$ may be expressed as $3y=-2x+5$ and finally
  $$y=-\frac{2}{3}x+\frac{5}{3}$$
- From the usual form into Ax+By=C
  a) The line $y=-3x+7$ may be expressed as
  $$3x+y=7$$
  b) The line $y=\frac{1}{2}x+\frac{2}{3}$ may be expressed as
  $$-\frac{1}{2}x+y=\frac{2}{3}$$
  We usually require the coefficients A,B,C to be integers.
  Multiplying by 6 we obtain
  $$-3x+6y=4$$
  c) The line $y=5$ may be expressed as $0x+y=5$
  d) The line $x=5$ may be expressed as $x+0y=5$

♦ GIVEN: A POINT AND A SLOPE

The line which

- passes through point $\mathbf{P(x_o,y_o)}$
- has slope $\mathbf{m}$

is given by

$$\boxed{y-y_o = m(x-x_o)}$$

EXAMPLE 6

The line which passes through point P(1,2), with slope m=3 is

$$y-2 = 3(x-1)$$

- Express in the form $y=mx+c$

$$y-2 = 3(x-1) \Leftrightarrow y=3x-3+2 \Leftrightarrow \underline{y=3x-1}$$

- Express in the form $ax+by=c$ or $ax+by+c=0$

$$y=3x-1 \Leftrightarrow \underline{3x-y=1} \quad \text{or} \quad \underline{3x-y-1=0}$$

♦ GIVEN: TWO POINTS

The line which passes through the points $P(x_1,y_1)$ and $Q(x_2,y_2)$ has slope

$$\boxed{m=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}}$$

and its equation is again given by the formula

$$\boxed{y-y_1 = m(x-x_1)}$$

EXAMPLE 7

Find the line which passes through the points P(1,2) and Q(4,7).

Express your answer in the form $ax+by=c$ where $a,b,c\in Z$ (integers).

Solution

The slope is $m=\frac{\Delta y}{\Delta x}=\frac{7-2}{4-1}=\frac{5}{3}$

The equation of the line is

$$y-2 = \frac{5}{3}(x-1)$$

$$\Leftrightarrow 3y-6 = 5(x-1)$$

$$\Leftrightarrow 3y-6 = 5x-5$$

and finally

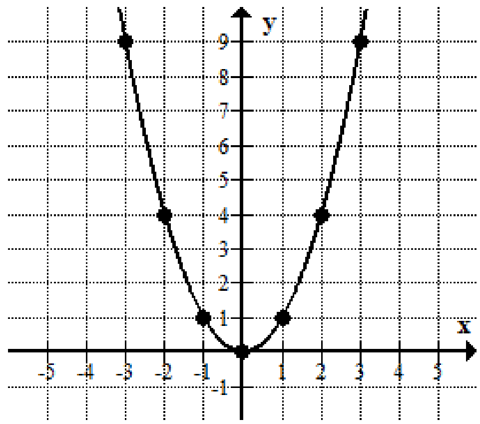
$$-5x+3y = 1$$

2.2 QUADRATICS (or QUADRATIC FUNCTIONS)

♦ THE SIMPLEST QUADRATIC: $y=x^2$

Consider the function $y=x^2$. Let us find some values

x	…	−3	−2	−1	0	1	2	3	…
$y=x^2$	…	9	4	1	0	1	4	9	…



Notice that x can take any value in R. We say that

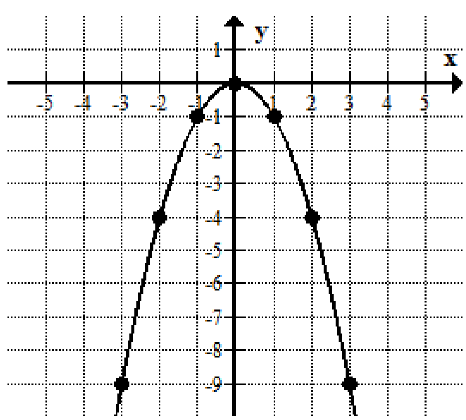
the **domain** of the function is $x \in R$

The result, i.e. the value of y, is always positive or 0. We say that

the **range** of the function is $[0,+\infty)$ (or simply $y \geq 0$).

The curve of this function is known as **parabola**.

We can easily see that the graph of the function $y=-x^2$ is

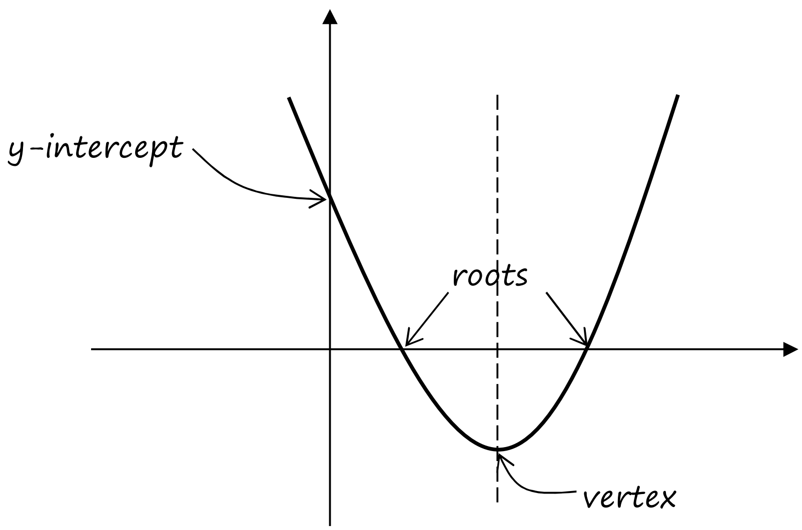


♦ THE QUADRATIC FUNCTION

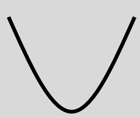
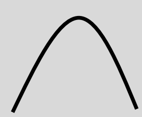
A **quadratic function** has the form

$$y=ax^2+bx+c$$

The graph of a quadratic is always a parabola. The basic characteristics of its graph as shown below:



1) **$a\neq 0$.** The sign of a shows the concavity of the function:

If **a>0** the graph looks like	If **a<0** the graph looks like
 (concave up)	 (concave down)

2) **Discriminant:** $\Delta=b^2-4ac$. It determines the number of roots

Δ>0: 2 roots	**Δ=0**: 1 root	**Δ<0**: No real roots

3) **x-intercepts** (or **roots**): $x_{1,2}=\dfrac{-b\pm\sqrt{\Delta}}{2a}$, (only if $\Delta\geq 0$)

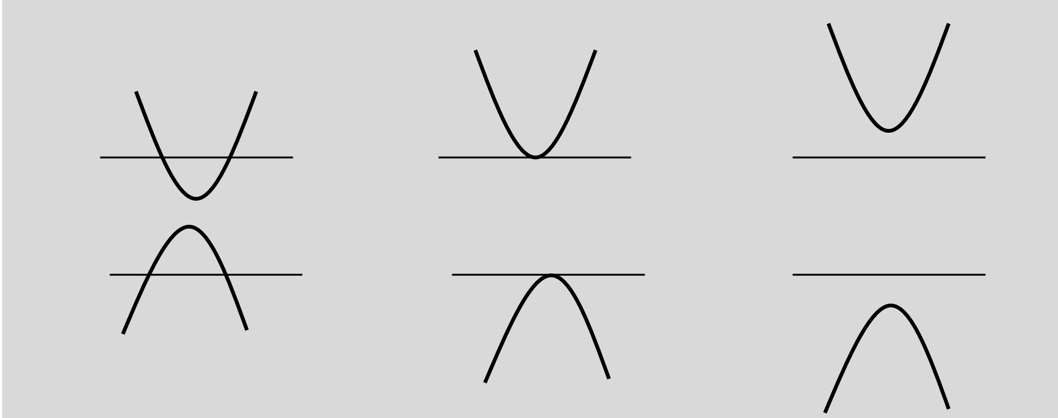
4) **y-intercept**: for $x=0$ we obtain $y=c$

5) **axis of symmetry:** $x=\dfrac{-b}{2a}$ (it's also the x-coordinate of the vertex)

If we know the two roots x_1,x_2 the vertex is at $x=\dfrac{x_1+x_2}{2}$

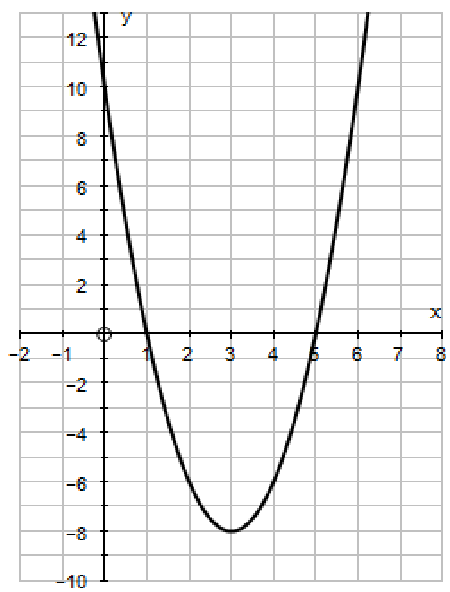
6) According to Δ, the graph looks like

Δ>0: 2 roots	**Δ=0**: 1 root	**Δ<0**: No real roots



EXAMPLE 1

Consider **$y=2x^2-12x+10$**



- $a=2$ (+tive), so the graph looks like **U** (concave up)
- $\Delta = b^2-4ac = 64>0$, thus two roots: $x_{1,2}=\dfrac{-b\pm\sqrt{\Delta}}{2a}$ = 1 and 5
- y-intercept: $y=10$
- Axis of symmetry: $x=\dfrac{-b}{2a}$ i.e. $x=3$. (Or otherwise $x=\dfrac{1+5}{2}=3$)

  For $x=3$, we obtain $y=-8$. Hence, the vertex is $V(3,-8)$

NOTICE FOR THE GDC (Casio)

We can find the roots 1 and 5 in

Equation – Polynomial (degree 2)

We can find more characteristics in **Graph mode: G-Solv (F5)**

Options	in our example
F1 (ROOT): for the roots	1 and 5
F2 (MAX) or F3 (MIN): for the vertex	(3,-8)
F4 (YCEPT): for y-intercept	10

♦ QUADRATIC INEQUALITIES

They have the form

$$ax^2+bx+c>0 \quad \text{or} \quad ax^2+bx+c\geq 0$$

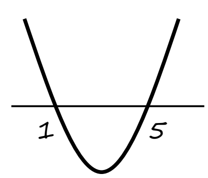
$$ax^2+bx+c<0 \quad \text{or} \quad ax^2+bx+c\leq 0$$

If we find the roots, the graph of the function gives a clear picture of the solutions.

For example, for

$$2x^2-12x+10 > 0$$

the roots are 1 and 5, the function is concave up, so it looks like



So it's positive for $x<1$ or $x>5$. We can also write $x\in]-\infty,1[\cup]5,+\infty[$

The inequality

$$2x^2-12x+10 \leq 0$$

has solutions $x\in[1,5]$.

NOTICE:

If we are given that

$$ax^2+bx+c>0 \quad \textbf{for any } x\in R$$

$$\text{or} \quad ax^2+bx+c<0 \quad \textbf{for any } x\in R$$

the graph does not intersect the x-axis, that is the quadratic has no real roots. Thus, $\Delta<0$

EXAMPLE 2

Let $\mathbf{f(x)=2x^2-4x+k}$. Find the values of k in each case below:

a) $f(x)=0$ has exactly one root (or two equal roots)
b) $f(x)=0$ has exactly two roots
c) $f(x)=0$ has no real roots
d) $f(x)=0$ has real roots
e) $f(x)>0$ for any $x\in R$
f) $f(x)\geq 0$ for any $x\in R$

Solution

All cases depend on the discriminant $\Delta=16-8k$

a) $\Delta=0$.

Hence, $16-8k=0 \Leftrightarrow 8k=16 \Leftrightarrow \mathbf{k=2}$

b) $\Delta>0$.

Hence, $16-8k>0 \Leftrightarrow 16>8k \Leftrightarrow \mathbf{k<2}$

c) $\Delta<0$.

Hence, $16-8k<0 \Leftrightarrow 16<8k \Leftrightarrow \mathbf{k>2}$

d) $\Delta\geq 0$. [in this case we have either one or two roots]

Hence, $16-8k\geq 0 \Leftrightarrow 16\geq 8k \Leftrightarrow \mathbf{k\leq 2}$

e) Since f(x) is always positive, it has no real roots. Thus, $\Delta<0$.

Hence, $16-8k<0 \Leftrightarrow 16<8k \Leftrightarrow \mathbf{k>2}$

f) Since f(x) is always positive or zero, it has either exactly one root or no real roots at all. Thus, $\Delta\leq 0$.

Hence, $16-8k\leq 0 \Leftrightarrow 8k\geq 16 \Leftrightarrow \mathbf{k\geq 2}$

♦ FORMS OF A QUADRATIC FUNCTION

1) Traditional form: $y=ax^2+bx+c$
2) Factorization form: $y=a(x-r_1)(x-r_2)$ [r_1,r_2 roots]
3) Vertex-form: $y=a(x-h)^2+k$ [(h,k) vertex]

NOTICE

- If we know the form $y=ax^2+bx+c$ the vertex is at

$$x=\frac{-b}{2a}$$

- If we know the form $y=a(x-r_1)(x-r_2)$, that is the roots r_1,r_2 the vertex is at their mid-point, that is

$$x=\frac{r_1+r_2}{2}$$

Since we know the x-coordinate of the vertex, that is **h**, we can also find the y-coordinate of the vertex, that is **k**. Thus we can derive the vertex form $y=a(x-h)^2+k$.

EXAMPLE 3

We consider again

$$y=2x^2-12x+10 \qquad (1)$$

We find the roots: 1 and 5. Therefore, the factorization is

$$y=2(x-1)(x-5) \qquad (2)$$

The vertex is at $x=\frac{-b}{2a}=\frac{12}{4}=3$ (or otherwise at $x=\frac{r_1+r_2}{2}=\frac{1+5}{2}=3$)

For x=3, it is y=-8, hence the vertex is (3,-8)

Therefore, the vertex-form of the quadratic is

$$y=2(x-3)^2-8 \qquad (3)$$

We may easily verify that forms (2) and (3) give (1).

Indeed,

$$y=2(x-1)(x-5) = 2(x^2-x-5x+5) = 2(x^2-6x+5) = 2x^2-12x+10$$

and

$$y=2(x-3)^2-8 = 2(x^2-6x+9)-8 = 2x^2-12x+18-8 = 2x^2-12x+10$$

♦ JUSTIFICATION OF THE VERTEX-FORM $\mathbf{y=a(x-h)^2+k}$

1) The point **(h,k)** is the vertex, i.e. a minimum or a maximum:

- If $a>0$, then

$$\begin{array}{ll} & a(x-h)^2 \geq 0 \qquad \text{(equality holds when } x=h) \\ \Rightarrow & a(x-h)^2+k \geq k \\ \Rightarrow & y \geq k \end{array}$$

  Therefore, at $x=h$ we obtain the minimum value $y=k$.

- If $a<0$, then

$$\begin{array}{ll} & a(x-h)^2 \leq 0 \qquad \text{(equality holds when } x=h) \\ \Rightarrow & a(x-h)^2+k \leq k \\ \Rightarrow & y \leq k \end{array}$$

  Therefore, at $x=h$ we obtain the maximum value $y=k$.

2) Any quadratic can be expressed in the vertex form, by the "completing the square" method.

For example, for the quadratic in EXAMPLE 3 above, we can work as follows

$$\begin{aligned} y = \underline{2x^2-12x}+10 &= 2(x^2-6x) +10 && \text{[only the first 2 terms]} \\ &= 2(x^2-6x\underline{+9-9})+10 && \text{[complete the square]} \\ &= 2(x-3)^2-18+10 \\ &= 2(x-3)^2-8 \end{aligned}$$

However, it is preferable to obtain the vertex-form as in example 3 above, that is by finding the vertex (h,k) and then expressing the quadratic as $y=a(x-h)^2+k$.

EXAMPLE 4

Let

$$y=-3x^2-15x+42 \quad (1)$$

By using the GDC,

we find the roots: -7 and 2. Thus the factorization is

$$y=-3(x+7)(x-2) \quad (2)$$

we find the vertex: V(-2.5, 60.75). Thus the vertex form is

$$y=-3(x+2.5)^2+60.75 \quad (3)$$

Notice: if you expand (2) or (3) you will obtain (1)

EXAMPLE 5

Consider $f(x)=3x^2+12x$. Find both **analytically** and **by GDC**

a) the roots and the factorization.

b) the equation of the axis of symmetry

c) the minimum value of y and the coordinates of the vertex.

d) the vertex form of f(x).

Solution

a) Analytically:

The factorization is $y = 3x^2+12x = 3x(x+4)$

So the roots are $x=0$, $x=-4$

By using GDC – Graph mode

The roots are $x=0$ and $x=-4$

So the factorization is $y=3(x-0)(x+4)$, that is $y=3x(x+4)$

b) $x=\frac{-b}{2a}=\frac{-12}{6}=-2$. That is $x=-2$.

c) Analytically:

For $x=-2$, it is $y=3(-2)^2+12(-2)=-12$. Thus $y_{min}=-12$

Thus the vertex is V(-2,-12)

By using GDC - mode: $y_{min}=-12$ and V(-2,-12).

d) $f(x) = 3(x+2)^2-12$

♦ VIETA FORMULAS

Consider the quadratic

$$y=ax^2+bx+c$$

Given that the real roots are r_1 and r_2, we define

S = the sum of the roots = r_1+r_2

P = the product of the roots = r_1r_2

Then, the Vieta formulas hold:

$$S = -\frac{b}{a}$$

$$P = \frac{c}{a}$$

Conversely, if we know the sum and the product of the roots, we can find a corresponding quadratic.

$$\boxed{y = x^2-Sx+P}$$

EXAMPLE 6

Consider again the quadratic function

$$y=2x^2-12x+10$$

The roots are 1 and 5 and indeed

their sum is $\quad S = -\frac{b}{a}=\frac{12}{2}=6$

their product is $\quad P = \frac{c}{a}=\frac{10}{2}=5$

Conversely, if we know that $S = 6$, and $P = 5$, the corresponding quadratic is

$$x^2-Sx+P$$

that is

$$x^2-6x+5$$

or any multiple of this, for example $2x^2-12x+10$.

2.3 FUNCTIONS, DOMAIN, RANGE, GRAPH

♦ DEFINITION

Let us formally introduce the notion of the **function:**

> $f: X \to Y$
>
> A **function** f from a set X to a set Y assigns
> to **each** element x of X
> a **unique** element y of Y

We write:

$f(x)=y$

$f: x \mapsto y$

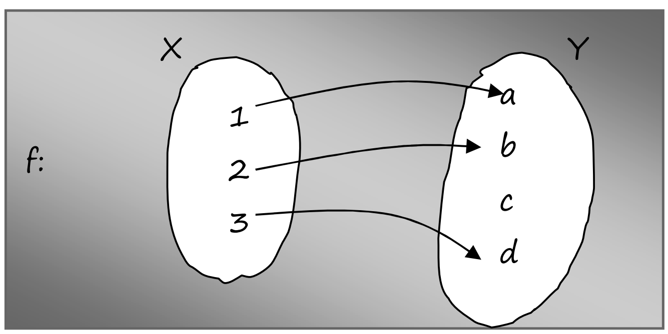
We say:

f maps x to y

y is the image of x

EXAMPLE 1

Let X={1,2,3} and Y={a,b,c,d}. The following is a function $f: X \to Y$



Indeed, **each** element of X has a **unique** image in Y.

We say

f	maps	1 to a	or	a is the image of 1
		2 to b		b is the image of 2
		3 to d		d is the image of 3

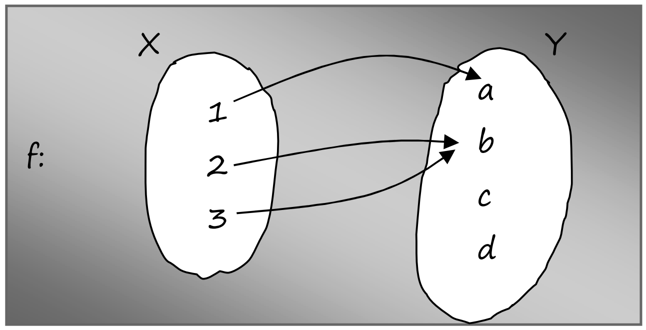
We write

	$f(1)=a$,	$f(2)=b$,	$f(3)=d$
or	$f: 1 \mapsto a$	$f: 2 \mapsto b$	$f: 3 \mapsto d$

EXAMPLE 2

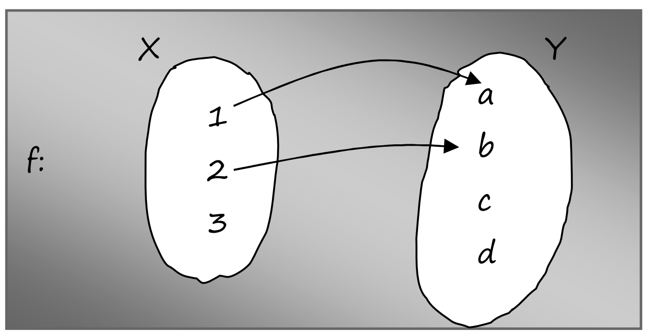
Let X={1,2,3} and Y={a,b,c,d}

- The following is a function



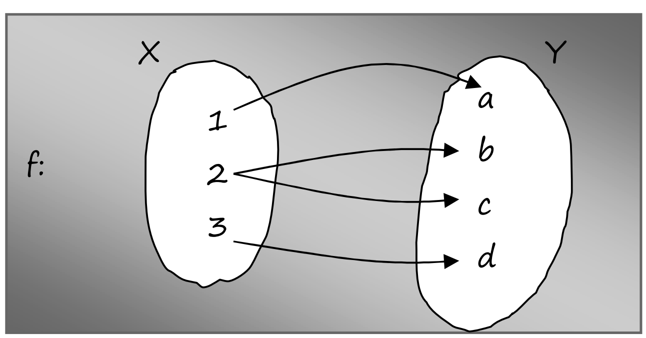
(we do not mind if two elements of X have the same image)

- Notice though that the following is not a function



(we said "**each** x of X", but here 3 has no image)

- Finally, the following is not a function



(we said "**unique** y of Y", but 2 has two images)

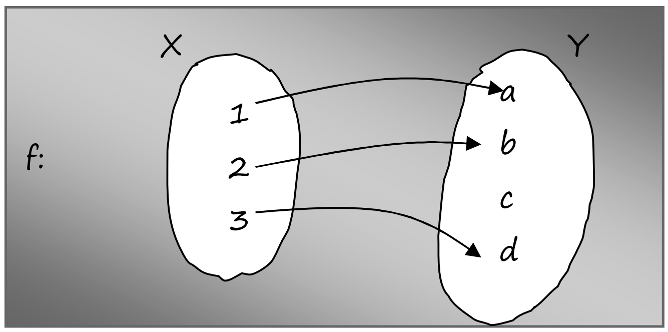
♦ DOMAIN AND RANGE

For a function f: X → Y,

The set of all x's involved is called **DOMAIN**

The set of all y's involved (only the images) is called **RANGE**

Consider again the function f: X → Y given by



Then DOMAIN : $x \in X = \{1,2,3\}$

RANGE : $y \in \{a,b,d\}$

We usually denote the domain by D_f and the range by R_f.

The range is not necessarily the whole set Y, it may be part of Y.

Here, the sets X and Y are subsets of R, the set of real numbers.

Our functions usually have a specific pattern. For example, consider the function f which maps

$1 \mapsto 2 \qquad 2 \mapsto 4 \qquad 3 \mapsto 6 \qquad 4 \mapsto 8$ and so on

in other words f maps each value x to its double 2x.

We say that the function f: R→ R, is given by

$f: x \mapsto 2x$

or $f(x)=2x$

or $y=2x$

Thus the formula of the function gives any possible result, e.g.

$f(15)=30, \qquad f(2.4) = 4.8$ etc

If we restrict the function f from R to the interval X=[0,10], we still have the function f: X→ R, given by

$$f(x)=2x, \quad 0 \le x \le 10$$

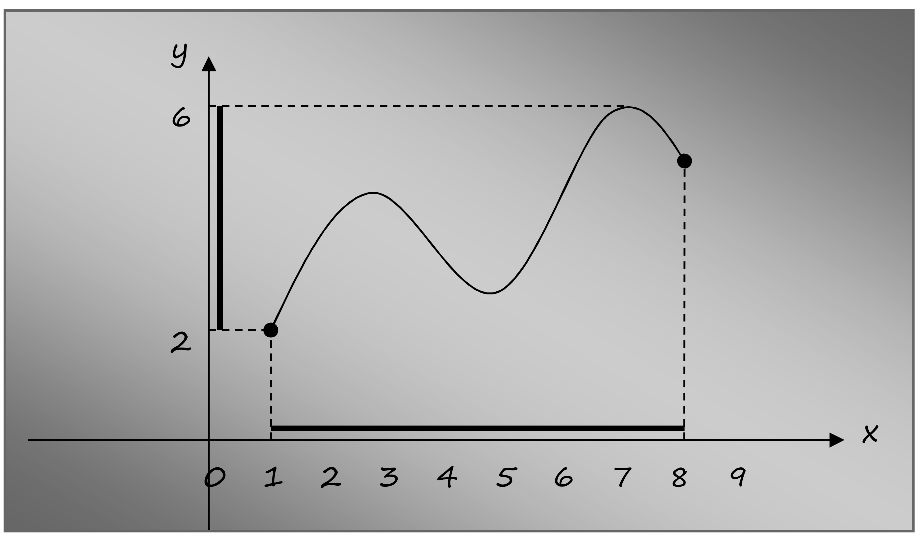
but now

DOMAIN : $x \in [0,10]$
RANGE : $y \in [0,20]$ (why?)

♦ GRAPH

We know that the pairs (x,y) that satisfy the equation of the function y=f(x) can be represented as points (x,y) on the Cartesian plane and form the **graph** of the function.

The graph clearly shows the DOMAIN and the RANGE of the function. For example,



DOMAIN: Projection on the x-axis, i.e. D_f: $x \in [1,8]$
RANGE: Projection on the y-axis, i.e. R_f: $y \in [2,6]$

We may observe, for example, that the points

(1,2), (5,3), (7,6), (8,5) lie on the curve.

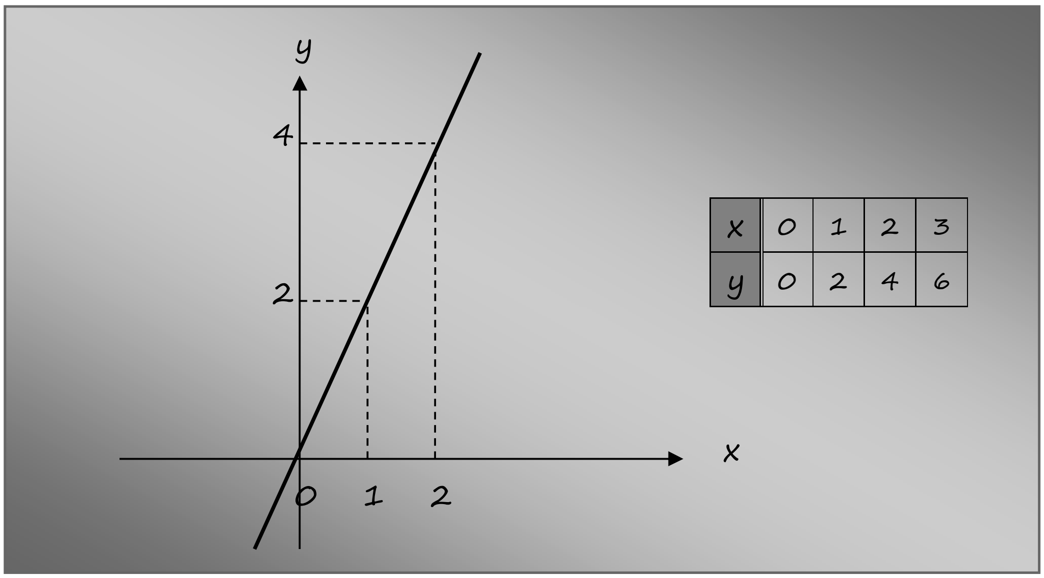
That implies

f(1)=2 f(5)=3 f(7)=6 f(8)=5

We have already studied the graphs of two families of functions; linear and quadratic functions. The graphs are straight lines and parabolas respectively.

EXAMPLE 3

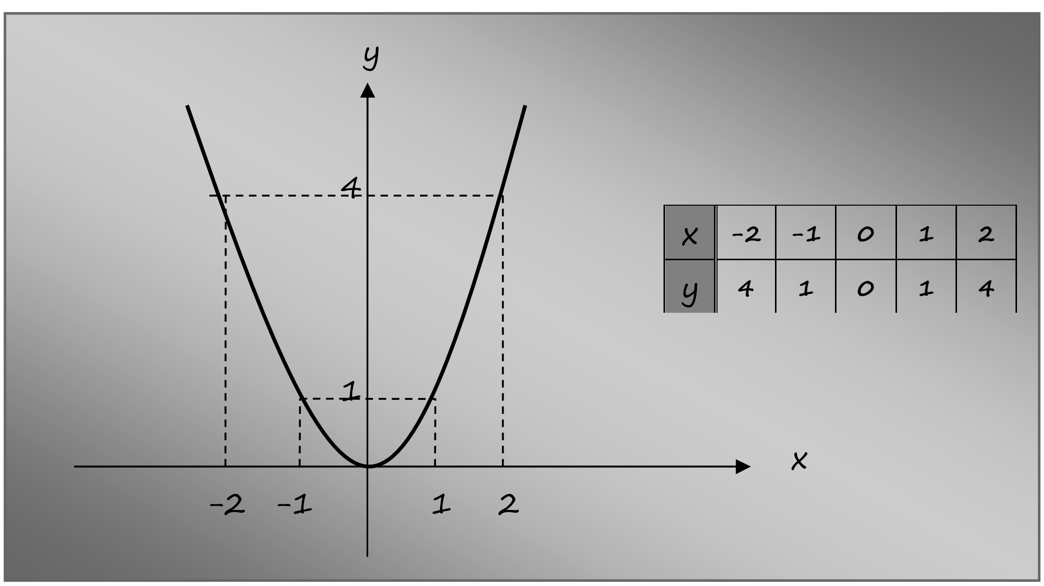
- $f(x)=2x$, or otherwise $y=2x$ is represented by the graph



x	0	1	2	3
y	0	2	4	6

Here D_f: $x\in R$ R_f: $y\in R$

- $f(x)=x^2$, or otherwise $y=x^2$ is represented by the graph



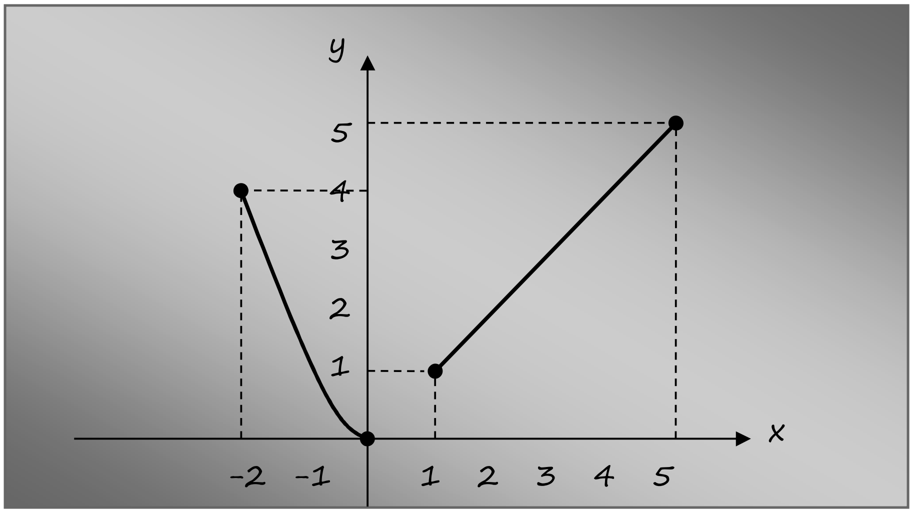
x	−2	−1	0	1	2
y	4	1	0	1	4

Here D_f: $x\in R$ R_f: $y\in[0,+\infty)$

EXAMPLE 4

Consider the function $f(x)=\begin{cases} x^2, & -2\le x\le 0 \\ x, & 1\le x\le 5 \end{cases}$

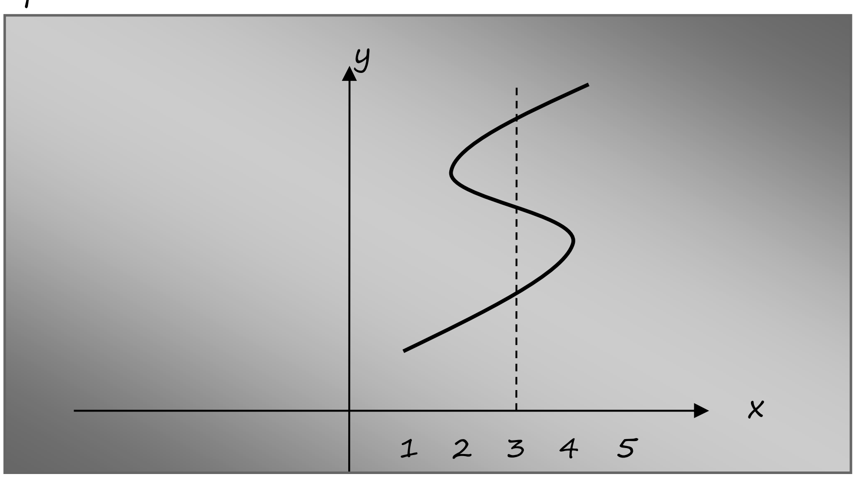
The graph is given below



Clearly, D_f: $x\in[-2,0]\cup[1,5]$ and R_f: $y\in[0,5]$

NOTICE:

The graph also shows if we have a function or not



This is not a function, since f(3) for example is not unique!

Vertical line test:

Any vertical line intersects the graph at most once.

♦ AN "AGGREEMENT" FOR THE DOMAIN

Usually, a function is simply given as a formula of the form $y=f(x)$, where x and y are real variables.

If the domain of the function is not given, we agree that

D_f **is R**

or D_f **is the largest possible subset of R**

For example,

- if f is given by $f(x)=2x$, we assume that $x \in R$
- if f is given by $f(x)=\frac{2}{x}$, we assume that $x \in R-\{0\} = R^*$

  (we may also write D_f: $x \neq 0$)

We mainly deal with the following cases

1. $f(x)$ is a function with no restrictions on x, for example a polynomial [say $f(x)=2x^3+3x^2+1$], then

   $\boxed{D_f = R}$

2. $f(x) = \frac{A}{B}$, then B cannot be 0, thus

   $\boxed{D_f = R - \{\text{roots of the equation } B=0\}}$

3. $f(x) = \sqrt{A}$, then $A \geq 0$.

   $\boxed{D_f = \text{the solution set of the inequality } A \geq 0}$

4. $f(x) = \log A$ or $f(x)=\ln A$, then $A>0$. [1]

   $\boxed{D_f = \text{the solution set of the inequality } A>0}$

5. $f(x)$ = is a combination of all the above. We find the subset of R where all our restrictions hold.

[1] The functions $f(x)=\log x$ and $f(x)=\ln x$ are not known yet. They will be introduced later on within this topic.

EXAMPLE 5

a) $f(x) = 3x-9$. Clearly, D_f: $x\in R$

b) $f(x) = \dfrac{5}{3x-9}$. Restriction: $3x-9 \neq 0$

Solve: $3x-9=0 \Leftrightarrow 3x=9 \Leftrightarrow x = 3$

Thus, D_f: $x\in R-\{3\}$. We may also write D_f: $x\neq 3$

c) $f(x) = \sqrt{3x-9}$. Restriction: $3x-9\geq 0$

Solve: $3x-9\geq 0 \Leftrightarrow 3x\geq 9 \Leftrightarrow x \geq 3$

Thus, D_f: $x\in[3,+\infty)$. We may also write D_f: $x\geq 3$

d) $f(x) = \ln(3x-9)$. Restriction: $3x-9>0$

Solve: $3x-9>0 \Leftrightarrow 3x>9 \Leftrightarrow x > 3$

Thus, D_f: $x\in(3,+\infty)$. We may also write D_f: $x>3$

e) $f(x) = \dfrac{x+2}{x^2-3x+2}$ Restriction: $x^2-3x+2\neq 0$

Solve: $x^2-3x+2=0 \Leftrightarrow x=1$ or $x=2$

Thus, D_f: $x\in R-\{1,2\}$

f) $f(x) = \sqrt{x-1}+\sqrt{2-x}$ Restrictions: $x-1 \geq 0$ and $2-x \geq 0$

Solve: $x-1 \geq 0 \Leftrightarrow x \geq 1$

$2-x \geq 0 \Leftrightarrow x \leq 2$

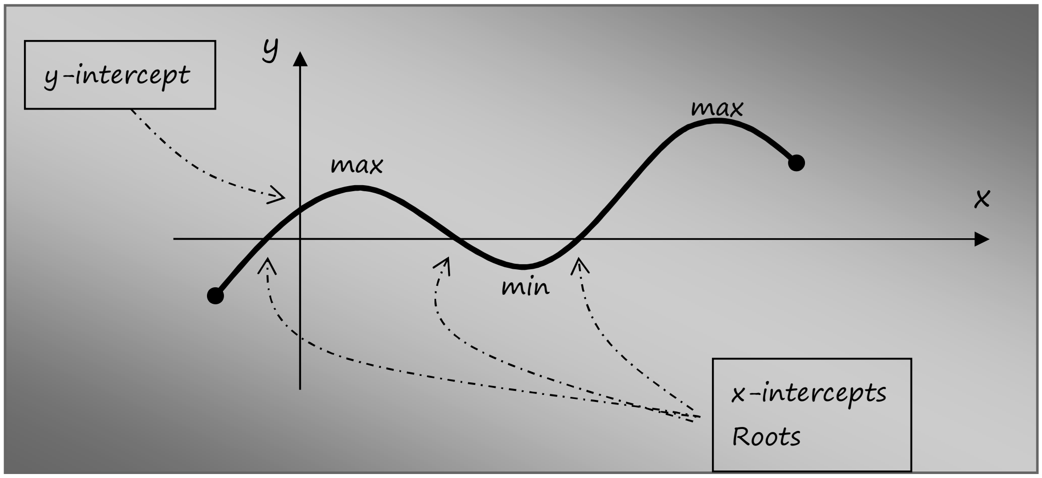
Thus, D_f: $x\in[1,2]$ We may also write D_f: $1\leq x\leq 2$

g) $f(x) = \dfrac{\sqrt{1-x^2}}{x}$ Restrictions: $1-x^2 \geq 0$ and $x\neq 0$

Solve: $1-x^2 \geq 0 \Leftrightarrow x^2 \leq 1 \Leftrightarrow -1\leq x\leq 1$

Thus, D_f: $x\in[-1,0)\cup(0,1]$

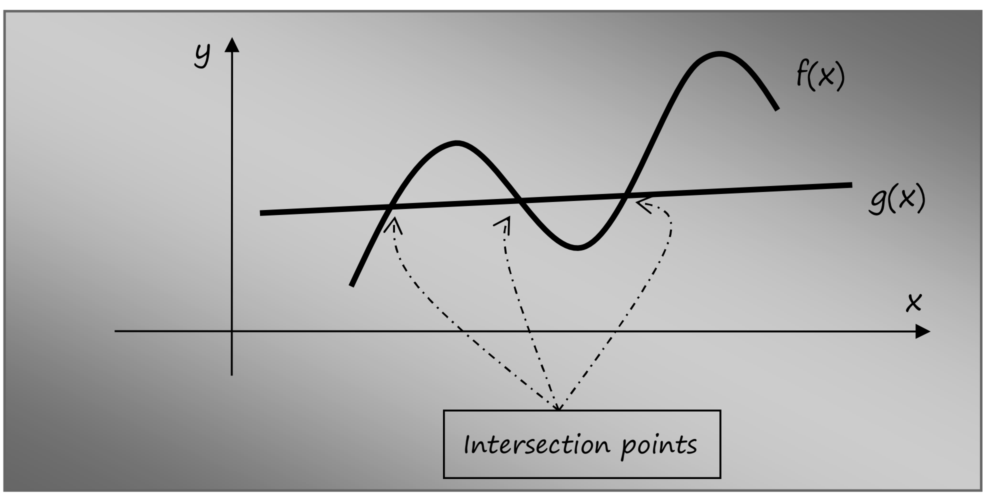
♦ SPECIFIC POINTS ON A GRAPH



For y=f(x)

- **y-intercept**: We set x=0 and find y
- **x-intercepts (roots)**: We solve the equation f(x)=0
- **local max-min:** (as shown above)

When we have two graphs y=f(x) and y=g(x), it also useful to know the **intersection points** of the two graphs



These points (x,y) can be found by solving the equation **f(x)=g(x)** to obtain x and then using either y=f(x) or y=g(x) to obtain y.

All notions above, namely ***y*-intercept**, ***x*-intercepts** (or **roots**), **max**, **min**, **intersection points** can be easily found in **GDC – Graph mode**.

EXAMPLE 6

Consider the functions $f(x)=(x-3)^2-4$ and $g(x)=x-5$.

For f :

y-intercept: for $x=0$, we obtain $\boxed{y=5}$

x-intercepts or roots: We solve $(x-3)^2-4=0$

$(x-3)^2-4=0 \Leftrightarrow (x-3)^2=4 \Leftrightarrow x-3=\pm 2 \Leftrightarrow x=2+3$ or $x=-2+3$

Hence $\boxed{x=5 \text{ or } x=1}$

max-min: for this particular function (quadratic), we know that there is only a minimum.

We have a min at the vertex, i.e. at point $\boxed{(3,-4)}$

We say: We have a <u>min at x=3.</u> The <u>min value is y=-4</u>

For intersection points of f and g :

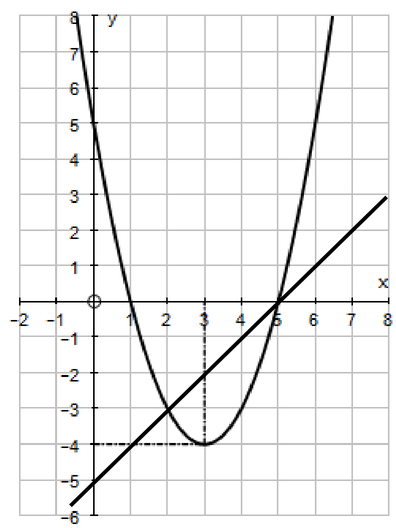
$f(x)=g(x) \Leftrightarrow (x-3)^2-4=x-5 \Leftrightarrow x^2-6x+9-4=x-5 \Leftrightarrow x^2-7x+10=0$

$\Leftrightarrow x=2$ or $x=5$

By using either $f(x)$ or $g(x)$ we find $y=-3$, $y=0$ respectively.

Hence, the curves intersect at points $\boxed{(2,-3) \text{ and } (5,0)}$

Indeed, the graphs of $f(x)$ and $g(x)$ are as follows



Remark: Confirm all the results by using **GDC – Graph mode.**

♦ SOVING EQUATIONS AND INEQUALITIES BY USING GRAPHS

We can solve

- equations of the form **f(x)=g(x)**
- inequalities of the form **f(x)>g(x)** or **f(x)≥g(x)**

by using **GDC – graph mode**

METHOD A: we find the intersection points of the graphs

$y_1 = f(x)$

$y_2 = g(x)$

Solutions of **f(x)=g(x)**: x-coordinates of intersection points

Solutions of **f(x)>g(x)**: intervals where $y_1=f(x)$ is above $y_2=g(x)$

METHOD B: we find the roots of the graph

$y_1 = f(x)-g(x)$

Solutions of **f(x)-g(x)=0**: the roots of the graph

Solutions of **f(x)-g(x)>0**: intervals where $y_1=f(x)-g(x)$ is positive

---

EXAMPLE 7

Consider again the functions of Example 6

$$f(x)=(x-3)^2-4 \quad \text{and} \quad g(x)=x-5.$$

a) Solve the equation $f(x)=g(x)$.

METHOD A: Look at the graphs of $y_1=f(x)$ and $y_2=g(x)$ (see Example 6). The intersection points occur at $\boxed{x=2,\ x=5}$

METHOD B: The equation can be written

$$f(x)-g(x) = (x-3)^2- 4 - (x-5) =0$$

Look at the graph of $y_1=f(x)-g(x)$ (see GDC). Roots: $\boxed{x=2,\ x=5}$

b) Solve the inequality $f(x)>g(x)$.

METHOD A: the graph of $y_1=f(x)$ is above $y_2=g(x)$ (see Example 6) when $\boxed{x<2 \text{ or } x>5}$

METHOD B: the graph of $y_1=f(x)-g(x)$ (see GDC) is positive outside the roots, that is when $\boxed{x<2 \text{ or } x>5}$

EXAMPLE 8

Solve the equation $2^x = 2x+3$.

(a) by using the function SolveN of your GDC

(b) by considering the graphs of

$$y_1=2^x$$
$$y_2=2x+3.$$

(c) by considering the graph

$$y=2^x - (2x+3)$$

Solution

(a) SolveN gives two roots:

$x = -1.29643 \cong -1.30$

$x = 3.24702 \cong 3.25$

For the following we need the diagrams

diagram 1 (for (b))

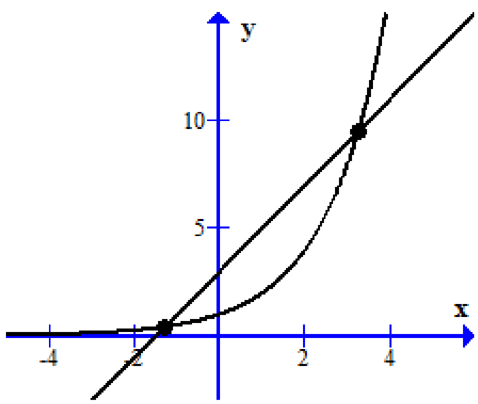
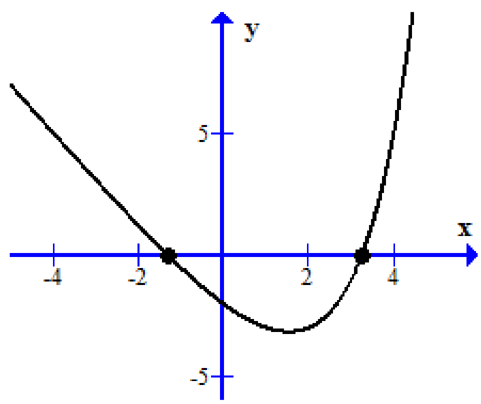


diagram 2 (for (c))



(b) Intersection points in diagram 1: $x \cong -1.30$ and $x \cong 3.25$

(c) Roots of the function in diagram 2: $x \cong -1.30$ and $x \cong 3.25$

Further question: (d) Solve the inequality $2^x < 2x+3$

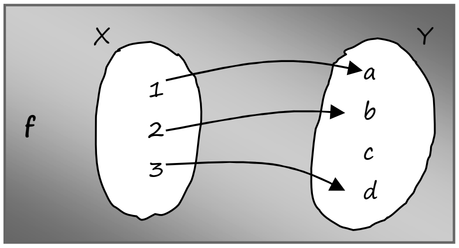
Solution

According to either diagram 1, or diagram 2

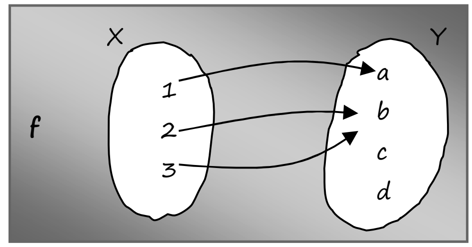
$$-1.30 < x < 3.25$$

♦ ONE-TO-ONE vs MANY-TO-ONE FUNCTIONS **(mainly for HL)**

Consider again the two functions below.



this function is **one-to-one**



this function is **many-to-one**

The formal definition for a one-to-one function says that different elements of X map to different elements of Y, that is

> A function f: $X \to Y$ is **one-to-one** if for any x_1, x_2 in X
>
> $$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$
>
> or equivalently (the contrapositive statement)
>
> $$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

(the contrapositive definition is more practical for exercises).

Graphically, it is easy to confirm that the function is one-to-one:

> **Horizontal line test:**
>
> Any horizontal line intersects the graph at most once.

EXAMPLE 9

Look at the functions of Example 3.

- the function $f(x)=2x$ is one-to-one, since

  $$f(x_1)=f(x_2) \Rightarrow 2x_1=2x_2 \Rightarrow x_1=x_2$$

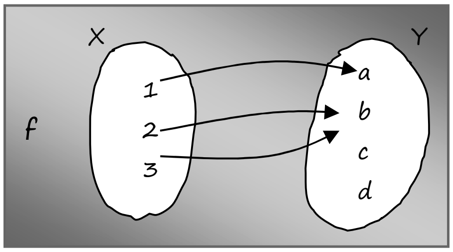
  OR, since any horizontal line intersects the graph at most once.

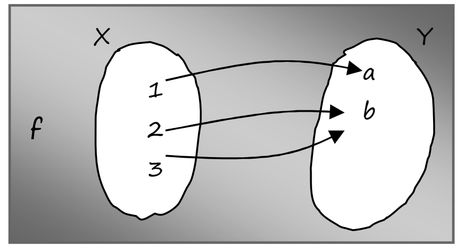
- the function $f(x)=x^2$ is many-to-one, since different elements may map to the same image, e.g. $f(2)=4$ but also $f(-2)=4$.

  OR, since a horizontal line may intersect the graph twice.

♦ ONTO FUNCTIONS **(only for HL – optional but good to know)**

Consider the following two functions





As you see, in the second example the range of f is coincides with Y. In other words, any element of Y is an image of some element of X.

We say that

f maps X **onto** Y or simply f is **onto**

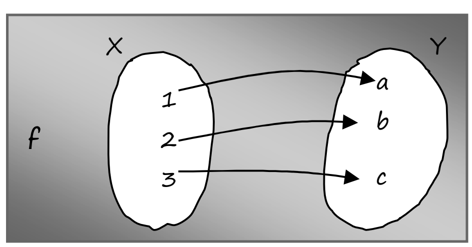
Notice though, that this property is "recoverable". Just ignore the elements of Y that are not images and the function becomes onto.

EXAMPLE 10

- the function f:R → R, given by f(x)=2x is **onto**, since the range of this function is R.
- the function f:R → R, given by $f(x)=x^2$ is **not onto**, since the range of this function is [0,+∞), which is a proper subset of R. However, if the function is given as f:R → [0,+∞), it is onto.

♦ 1-1 AND ONTO FUNCTIONS **(only for HL - optional)**

Consider the function



This is **one-to-one and onto.**

The function f:R→R, given by f(x)=2x, as well as any linear function, is one-to-one and onto.

2.4 COMPOSITION OF FUNCTIONS: f∘g

♦ DISCUSSION

Consider the function $f(x)=x^2$

Notice that

$$f(5) = 5^2$$
$$f(a) = a^2$$
$$f(3a+5) = (3a+5)^2$$
$$f(3x+5) = (3x+5)^2$$

In the last case the input value for f is another function of x.

In this way, we combine two functions,

$f(x)=x^2$ and $g(x)= 3x+5$

and create a new function $y=(3x+5)^2$.

This new function is denoted by **f∘g**.

♦ DEFINITION

For two functions f and g, the **composite function f∘g** is a new function defined by

$$(f\circ g)(x)=f(g(x))$$

The operation is called **composition**.
We say that f∘g is the **composite function** of f and g.

Therefore, for the functions $f(x)=x^2$ and $g(x)=3x+5$ given above, the procedure we follow in order to estimate $(f\circ g)(x)$ is

$$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= f(3x+5)\\ &= (3x+5)^2\end{aligned}$$

In the same way we can define the composite function $(g\circ f)(x)$. It is given by

$$(g\circ f)(x) = g(f(x))$$
$$= g(x^2)$$
$$= 3x^2+5$$

That is

$$(f\circ g)(x)= (3x+5)^2 \qquad \text{while} \qquad (g\circ f)(x)= 3x^2+5$$

NOTICE:

- In general

$$\mathbf{f\circ g \neq g\circ f}$$

- It is not necessary to write so analytically the answer. You can answer directly. Look at again

$$f(x)=x^2 \qquad \text{and} \qquad g(x)=3x+5$$

  For $f\circ g$ you just plug g into f.

$$(f\circ g)(x)= (3x+5)^2$$

  For $g\circ f$ you just plug f into g.

$$(g\circ f)(x)= 3x^2+5$$

- For three functions

$$f(x)=x^2, \quad g(x)=3x+5, \quad h(x)=\sqrt{x}$$

  we can define $(f\circ g\circ h)(x)$.

  We just plug h into g, to obtain

$$(g\circ h)(x)= 3\sqrt{x}+5$$

  and the result into f to obtain

$$(f\circ g\circ h)(x)= (3\sqrt{x}+5)^2$$

  We ca easily verify that

$$\mathbf{f\circ(g\circ h)=(f\circ g)\circ h}$$

EXAMPLE 1

Let $f(x)=2x^2-1$ and $g(x)=x+1$. Find

(a) $(f\circ g)(x)$ (b) $(g\circ f)(x)$ (c) $(f\circ g)(1)$ (d) $(g\circ f)(1)$

Solution

(a) $(f\circ g)(x) = 2(x+1)^2-1$

(b) $(g\circ f)(x) = (2x^2-1)+1 = 2x^2$

(c) From (a), we have

$$(f\circ g)(1)=7$$

(d) From (b), we have

$$(g\circ f)(1) = 2$$

Notice for questions (c) and (d)

For $(f\circ g)(1)$ and $(g\circ f)(1)$, it is not necessary to find $(f\circ g)(x)$ and $(g\circ f)(x)$ first. Alternatively, we can directly apply the definition as follows

(c) $(f\circ g)(1) = f(g(1)) = f(2) = 7$ [since $g(1)=2$]

(d) $(g\circ f)(1) = g(f(1)) = g(1) = 2$ [since $f(1)=1$]

Of course, if we are given a function f, we may also define the function $f\circ f$ in the obvious way:

$$(f\circ f)(x) = f(f(x))$$

That is, we plug f into itself.

For example, if $f(x)=2x-1$, then

$$(f\circ f)(x) = f(2x-1) = 2(2x-1)-1 = 4x-3$$

EXAMPLE 2

Let $f(x)=\dfrac{x+1}{2}$ and $g(x)=\sqrt{x}$

Find (a) $(f\circ g)(x)$ (b) $(g\circ f)(x)$

(c) $(f\circ f)(x)$ (d) $(g\circ g)(x)$

(e) $(f\circ f\circ f)(x)$ in two ways: as $f\circ(f\circ f)$ and as $(f\circ f)\circ f$

Solution

(a) $(f\circ g)(x)=\dfrac{\sqrt{x}+1}{2}$ (b) $(g\circ f)(x)=\sqrt{\dfrac{x+1}{2}}$

(c) $$(f\circ f)(x)=\frac{\frac{x+1}{2}+1}{2}=\frac{\frac{x+3}{2}}{2}=\frac{x+3}{4}$$

(d) $$(g\circ g)(x)=\sqrt{\sqrt{x}}=\sqrt[4]{x}$$

(e) $$(f\circ f\circ f)(x)=[f\circ(f\circ f)](x)=\frac{\frac{x+3}{4}+1}{2}=\frac{\frac{x+7}{4}}{2}=\frac{x+7}{8}$$

Or $$=[(f\circ f)\circ f](x)=\frac{\frac{x+1}{2}+3}{4}=\frac{\frac{x+7}{2}}{4}=\frac{x+7}{8}$$

- THE IDENTITY FUNCTION $i(x)$

It is the simple function that maps x to itself

$$i(x)=x \quad \text{or} \quad i: x\mapsto x$$

Notice that

$$(f\circ i)(x)=f(i(x))=f(x)$$

$$(i\circ f)(x)=i(f(x))=f(x)$$

That is

$$f\circ i=f \quad \text{and} \quad i\circ f=f$$

- PRESUPPOSITION FOR $f\circ g$ AND $g\circ f$ (**Mainly for HL**)

Let $f: A\to B$ and $g: B\to C$.

Then

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \\ x \arrow[r, mapsto] \arrow[rr, bend right, mapsto, "g\circ f"'] & f(x) \arrow[r, mapsto] & g(f(x)) \end{tikzcd}$$

That is in $g\circ f$, f is applied first and then g

Notice also that $g\circ f$ can be defined only if the Range of f is inside the Domain of g.

Similar observations may be done for $f\circ g$. Thus,

Function	**Observation**	**Presupposition**
$f\circ g$	g is applied first and then f	$R_g \subseteq D_f$
$g\circ f$	f is applied first and then g	$R_f \subseteq D_g$

2.5 THE INVERSE FUNCTION: f^{-1}

♦ DISCUSSION

Consider the function $f(x)=x+10$. It maps

$$0 \mapsto 10$$
$$1 \mapsto 11$$
$$2 \mapsto 12 \qquad \text{etc.}$$

The "inverse" procedure is also a function:

$$10 \mapsto 0$$
$$11 \mapsto 1$$
$$12 \mapsto 2 \qquad \text{etc.}$$

It is called the **inverse function** of f and it is denoted by f^{-1}.

Obviously

$$f^{-1}(x)=x-10$$

In fact, f and f^{-1} are **inverse** to each other.

♦ FORMAL DEFINITION

Let $f:R\to R$

The **inverse function** f^{-1} is a new function such that

$$f(x)=y \quad \Leftrightarrow \quad f^{-1}(y)=x.$$

♦ HOW DO WE FIND f^{-1} ?

Steps f is given	**Example** $f(x) = x+10$
1. Set $f(x)=y$	$x+10 = y$
2. Solve for x	$x = y-10$
3. Keep the solution but replace y by x	$f^{-1}(x)=x-10$

NOTICE:

1. The inverse function of f^{-1} is f itself. That is

$$\boxed{(f^{-1})^{-1} = f}$$

2. The domain of f becomes range of f^{-1} and vice-versa:

$$\boxed{D_{f^{-1}} = R_f} \qquad \boxed{R_{f^{-1}} = D_f}$$

3. It holds

$$\boxed{(f^{-1} \circ f)(x) = x = (f \circ f^{-1})(x) \qquad \text{(identity function)}}$$

For example, for $f(x) = x+10$ and $f^{-1}(x)=x-10$:

$(f \circ f^{-1})(x)=f(f^{-1}(x))=f(x-10)=(x-10)+10=x$

$(f^{-1} \circ f)(x)= f^{-1}(f(x))=f(x+10)=(x+10)-10=x$

EXAMPLE 1

Let $f(x)=3x+5$. Find (a) $f^{-1}(x)$ (b) $f^{-1}(11)$

Solution

(a) We follow the three steps:

- Set $3x+5=y$
- $3x+5=y \Leftrightarrow 3x = y-5 \Leftrightarrow x = \dfrac{y-5}{3}$
- $f^{-1}(x)= \dfrac{x-5}{3}$

(b) Since we know $f^{-1}(x)= \dfrac{x-5}{3}$, it is $f^{-1}(11) = 2$

Alternatively: It is not necessary to find $f^{-1}(x)$.

If $f^{-1}(11)=x$ then $f(x)=11$. Hence

$3x+5 = 11 \Leftrightarrow 3x = 6 \Leftrightarrow x=2.$

Thus, $f^{-1}(11) =2$

Remark:

Verify that

the inverse function of $f^{-1}(x)=\frac{x-5}{3}$ is $f(x)=3x+5$.

- Set $\frac{x-5}{3}=y$
- $\frac{x-5}{3}=y \Leftrightarrow x-5=3y \Leftrightarrow x=3y+5$
- The inverse function is $y=3x+5$

In other words f and f^{-1} are inverse to each other.

EXAMPLE 2

Let $f(x)=2x^2-1$ where $x\geq 0$. Find (a) $f^{-1}(x)$ (b) $f^{-1}(49)$

Solution

(a) We follow the three steps:

- Set $2x^2-1=y$
- $2x^2-1=y \Leftrightarrow 2x^2=y+1 \Leftrightarrow x^2=\frac{y+1}{2} \Leftrightarrow x=\sqrt{\frac{y+1}{2}}$
- $f^{-1}(x)=\sqrt{\frac{x+1}{2}}$

(b) Since we know $f^{-1}(x)=\sqrt{\frac{x+1}{2}}$, it is

$f^{-1}(49)=\sqrt{25}=5$

or again

$f^{-1}(49)=x$ implies $f(x)=49$

$\Leftrightarrow 2x^2-1=49 \Leftrightarrow x^2=25 \Leftrightarrow x=5$

So $f^{-1}(49)=5$

EXAMPLE 3

Let $f(x)=\dfrac{x+1}{x+2}$

(a) Show that $f^{-1}(x)=\dfrac{2x-1}{1-x}$

(b) Verify that $f \circ f^{-1}$ is the identity function [that is $(f \circ f^{-1})(x)=x$]

(c) Find the domain and the range of the functions f and f^{-1}

Solution

(a) $\dfrac{x+1}{x+2}=y \Leftrightarrow x+1=y(x+2)$

$\Leftrightarrow x+1=y(x+2)$

$\Leftrightarrow x+1=yx+2y$

$\Leftrightarrow x-yx=2y-1$

$\Leftrightarrow x(1-y)=2y-1$

$\Leftrightarrow x=\dfrac{2y-1}{1-y}$

Hence, $\boxed{f^{-1}(x)=\dfrac{2x-1}{1-x}}$

(b) $$(f \circ f^{-1})(x)=\frac{\frac{2x-1}{1-x}+1}{\frac{2x-1}{1-x}+2}=\frac{\frac{2x-1+1-x}{1-x}}{\frac{2x-1+2-2x}{1-x}}=\frac{\frac{x}{1-x}}{\frac{1}{1-x}}=x$$

That is $(f \circ f^{-1})(x)=x$ (identity function)

[In a similar way we can show that $(f^{-1} \circ f)(x)=x$]

(c) It is easier to find the domains of f and f^{-1}

$D_f = R-\{-2\}$. This is also R_{f-1}

$D_{f-1} = R-\{1\}$. This is also R_f

EXAMPLE 4

Let $f(x)=1-2x$ and $g(x)=\frac{1}{x}$. Find

(a) $(fog)(x)$ **(b)** $(gof)(x)$ **(c)** $(gof^{-1})(x)$

(d) $(fog^{-1})(x)$ **(e)** $(fog)^{-1}(x)$ **(f)** $(f^{-1}og^{-1})(x)$

Solution

(a) $(fog)(x) = f(g(x)) = f(\frac{1}{x}) = 1-2\frac{1}{x} = 1-\frac{2}{x}$

(b) $(gof)(x) = g(f(x)) = g(1-2x) = \frac{1}{1-2x}$

(c) We firstly need f^{-1}. Since $f(x)=1-2x$

$1-2x = y \Leftrightarrow 1-y = 2x \Leftrightarrow x = \frac{1-y}{2}$. Hence $f^{-1}(x)=\frac{1-x}{2}$

Now $(gof^{-1})(x) = \frac{2}{1-x}$

(d) We firstly need g^{-1}. Since $g(x)=\frac{1}{x}$

$\frac{1}{x} = y \Leftrightarrow x = \frac{1}{y}$. Hence $g^{-1}(x)=\frac{1}{x}$ [that is $g^{-1} = g$]

Then, $(fog^{-1})(x) = 1-\frac{2}{x}$

(e) We are looking for the inverse function of $(fog)(x) = 1-\frac{2}{x}$

$1-\frac{2}{x}=y \Leftrightarrow 1-y = \frac{2}{x} \Leftrightarrow x = \frac{2}{1-y}$. Thus , $(fog)^{-1}(x) = \frac{2}{1-x}$

(f) $(f^{-1}og^{-1})(x) = \frac{1-\frac{1}{x}}{2} = \frac{x-1}{2x}$

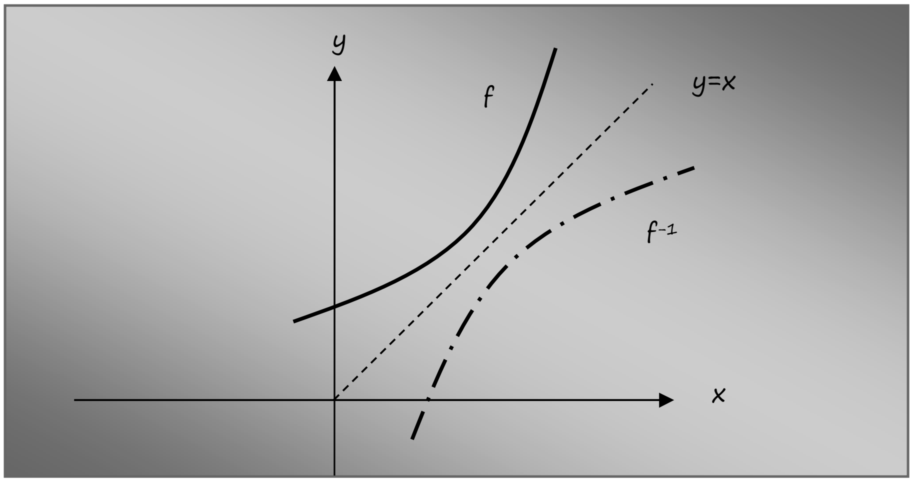
NOTICE:

Notice that $(fog)^{-1} \neq f^{-1}og^{-1}$. In fact it holds

$$\boxed{(fog)^{-1} = g^{-1}of^{-1}}$$

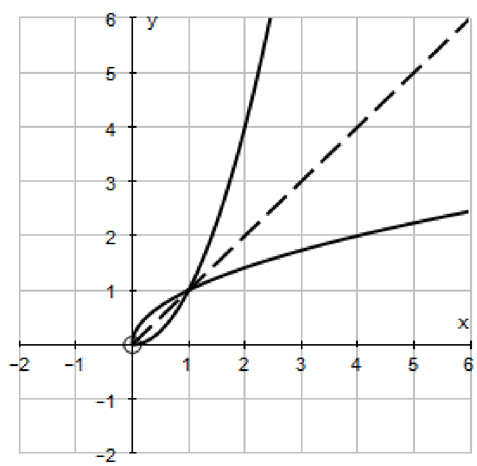
- GRAPH OF f^{-1}

The graph of f^{-1} is a reflection of f about the line y=x



EXAMPLE 5

If $f(x)=x^2$, for x≥0, then $f^{-1}(x)=\sqrt{x}$. Their graphs are



Notice: if f is increasing then f and f^{-1} may intersect **only on the line y=x.** Thus, in order to find the intersection points, instead of

$$f(x) = f^{-1}(x)$$

we can solve

$$f(x) = x$$

Here, $f(x)=x \Leftrightarrow x^2 = x \Leftrightarrow x^2-x = 0 \Leftrightarrow x(x-1)=0 \Leftrightarrow x=0$ or $x=1$

The intersection points are (0,0) and (1,1).

NOTICE:

We say that the function f is **self-inverse** if $f^{-1}=f$.

Then it also holds

$$(f\circ f)(x)=x$$

i.e. $f\circ f$ is the identity function I.

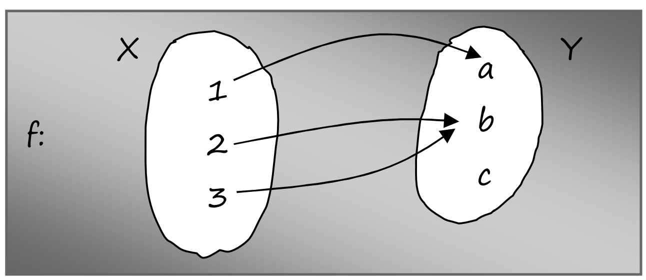
The graph of a self-inverse function is symmetric about y=x.

The simplest example is $f(x)=\frac{1}{x}$, since $f^{-1}(x)=\frac{1}{x}$.

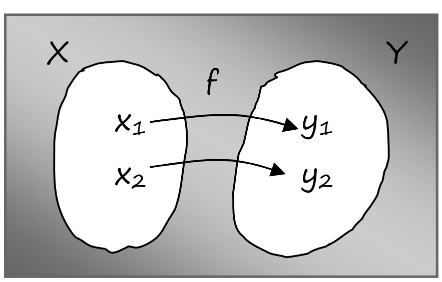
Another example is $f(x)=\frac{2x-6}{x-2}$ (please confirm!)

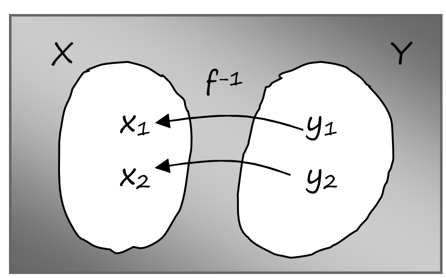
♦ PRESUPPOSITION FOR f^{-1} (**Mainly for HL**)

Consider the function



The inverse function f^{-1} doesn't exist, since $f^{-1}(b)$ is not uniquely determined (is it 2 or 3?). Hence, for f^{-1} to exist, different values of x should map to different values of y:





In other words, the function has to be **one-to-one**

(in fact, it has to be one-to-one and onto!)

NOTICE: Remember that

a function must satisfy the **vertical line test**.

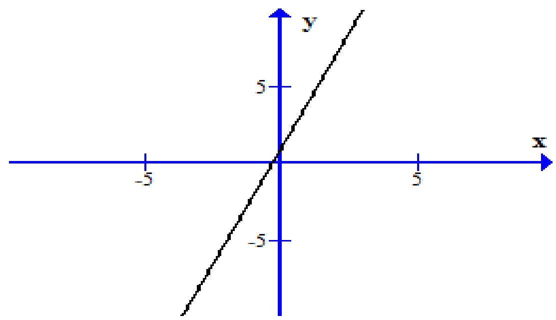
a "1-1" function must also satisfy the **horizontal line test**

Horizontal line test

Any horizontal line intersects the graph at most once

EXAMPLE 6

(a) The function f(x)=3x+1 is "1-1" since it is a straight line and satisfies the horizontal line test.



More mathematically:

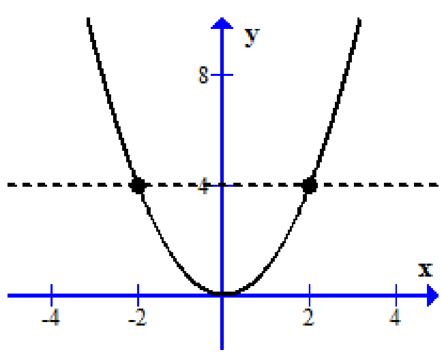
$$f(x_1)=f(x_2) \Rightarrow 3x_1+1=3x_2+1 \Rightarrow 3x_1=3x_2 \Rightarrow x_1=x_2$$

Hence f is "1-1" and f^{-1} exists.

We can easily find $f^{-1}(x)=\dfrac{x-1}{3}$

(b) The function $f(x)=x^2$ is not "1-1"

Indeed, f does not satisfies the horizontal line test, as two different values may map to the same image, for example f(-2)=4=f(2).



However,

- if we consider

$$f(x)=x^2, \quad x\geq 0$$

then f is "1-1" (horizontal line test) and f^{-1} exists.

$$f^{-1}(x)=\sqrt{x} \quad \text{(look at example 5)}$$

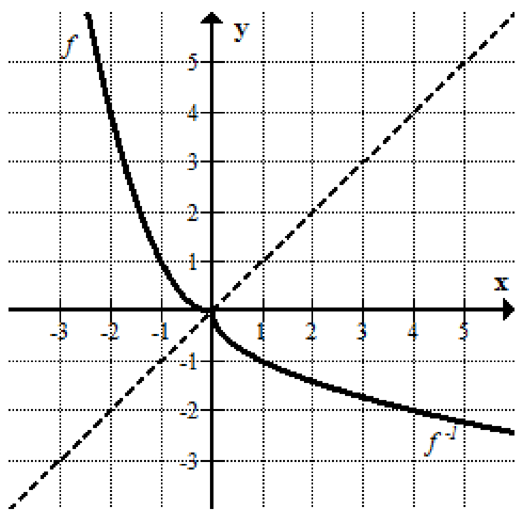
- Similarly, if we consider the restriction

$$f(x)=x^2, \quad x\leq 0$$

then f is "1-1" (horizontal line test) and f^{-1} exists. then

$$f^{-1}(x)=-\sqrt{x}$$

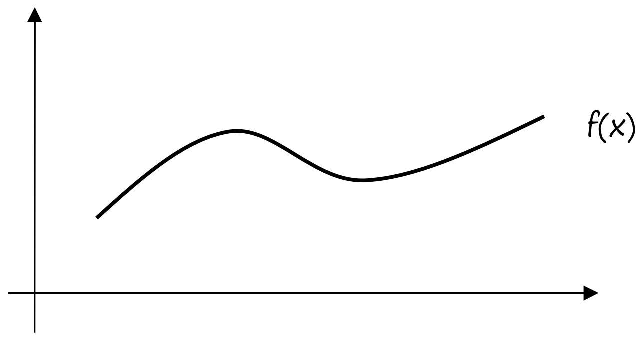
In this case the graphs of f and f inverse are as follows



2.6 TRANSFORMATIONS OF FUNCTIONS

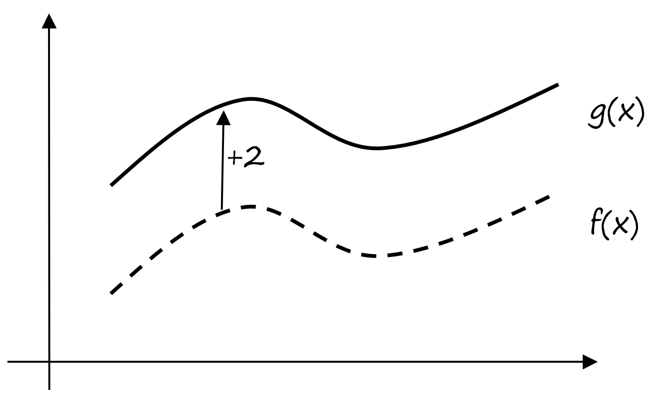
♦ DISCUSSION

Consider a function f(x).



Let's think of the new function $g(x)=f(x)+2$

In fact, we add 2 units to any value of $y=f(x)$, thus the whole graph of f(x) moves 2 units up.



We say that this is a **vertical translation** of the graph.

In a similar way we can describe other transformations of f(x), not only in a vertical direction (applied on y) but also in a horizontal direction (applied on x).

Let us present the most important transformations in a concise way!

♦ THE BASIC TRANSFORMATIONS

Consider the original function $y=f(x)$.
(In the following tables we assume $a>0$ and $b>1$)

VERTICAL TRANSFORMATIONS				
Function	**Transformation**		**Example: $f(x)=x^2$**	
$f(x)+a$	vertical **translation** a units up	$\uparrow a$	$g(x)=x^2+2$	$\uparrow 2$
$f(x)-a$	vertical **translation** a units down	$\downarrow a$	$g(x)=x^2-2$	$\downarrow 2$
$bf(x)$	vertical **stretch** with **scale factor b**	$\updownarrow \times b$	$g(x)=2x^2$	$\updownarrow \times 2$
$f(x)/b$	vertical **stretch** with **scale factor 1/b** (shrink)	$\div b$	$g(x)=x^2/2$	$\div 2$
$-f(x)$	**reflection** in the x-axis		$g(x)=-x^2$	

Now, as far as the horizontal transformations below are concerned, we obtain, perhaps, the opposite of what we expect!

HORIZONTAL TRANSFORMATIONS				
Function	**Transformation**		**Example: $f(x)=x^2$**	
$f(x+a)$	horizontal **translation** a units to the left	$\xleftarrow{a}$	$g(x)=(x+2)^2$	$\xleftarrow{2}$
$f(x-a)$	horizontal **translation** a units to the right	$\xrightarrow{a}$	$g(x)=(x-2)^2$	$\xrightarrow{2}$
$f(bx)$	horizontal **stretch** with **scale factor 1/b** (shrink)	$\div b$	$g(x)=(2x)^2$	$\div 2$
$f(x/b)$	horizontal **stretch** with **scale factor b**	$\leftrightarrow \times b$	$g(x)=(x/2)^2$	$\leftrightarrow \times 2$
$f(-x)$	**reflection** in the y-axis		$g(x)=(-x)^2$	

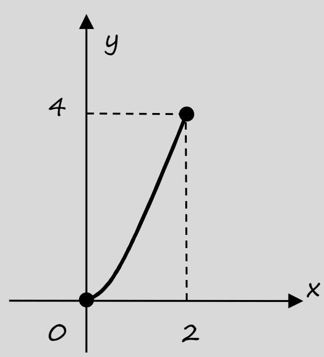
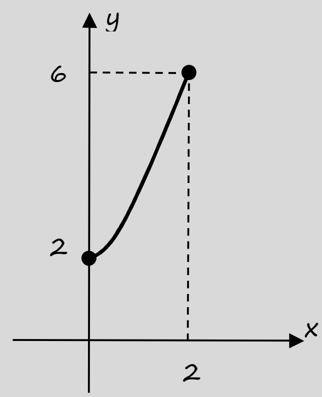
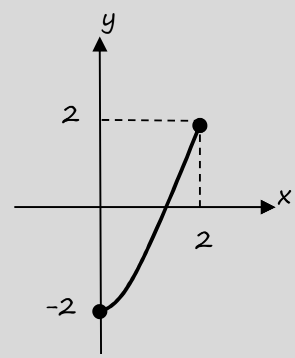
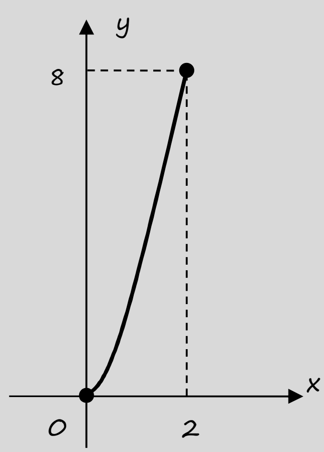
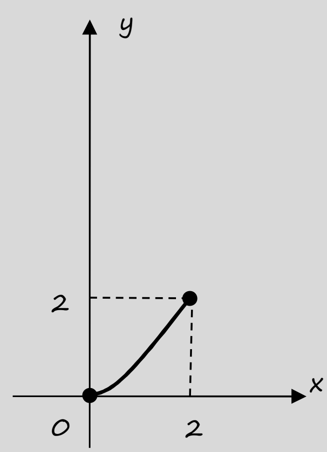
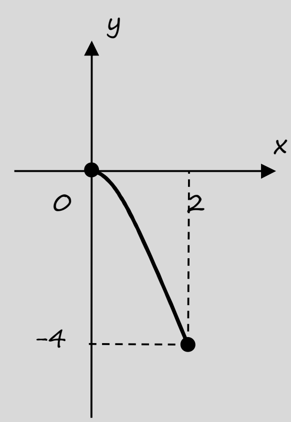
EXAMPLE 1

Let us observe the basic transformations of the function

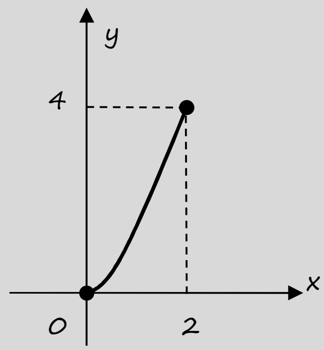
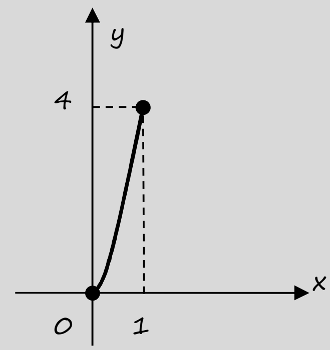
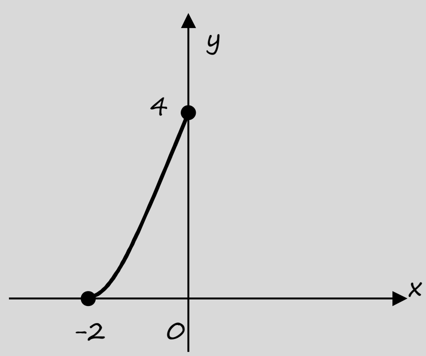
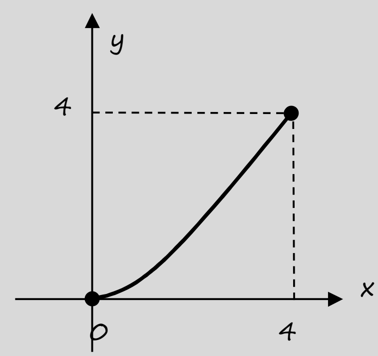
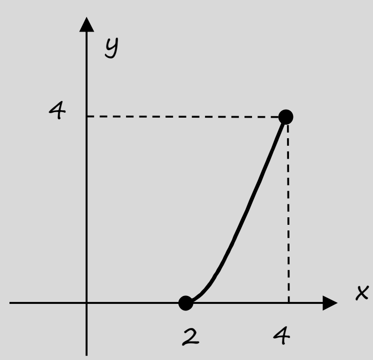
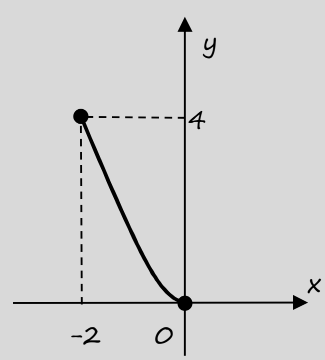
$$f(x) = x^2, \quad 0 \le x \le 2$$

in connection with the two tables above.

Let us see the vertical transformations first

VERTICAL TRANSFORMATIONS		
$f(x)=x^2$ [original function] 	$f(x)=x^2+2$ [2 units up] 	$f(x)=x^2-2$ [2 units down] 
$f(x)=2x^2$ [vertical stretch, s.f. 2] 	$f(x)=x^2/2$ [vertical stretch s.f. ½ That is shrink (÷2)] 	$f(x)=-x^2$ [reflection in x-axis] 

Next, we observe the horizontal transformations

HORIZONTAL TRANSFORMATIONS	
$f(x)=x^2$ [original function]  	$f(x)=(2x)^2$ [horizontal stretch, s.f. ½ That is shrink (÷2)] 
$f(x)=(x+2)^2$ [2 units to the left]  	$f(x)=(x/2)^2$ [horizontal stretch, s.f. 2] 
$f(x)=(x-2)^2$ [2 units to the right]  	$f(x)=(-x)^2$ [reflection in y-axis] 

EXAMPLE 2

Let **A(6,10)** be a point on the curve of $y=f(x)$.

Let us present some basic transformations as well as the corresponding images of the point A.

VERTICAL TRANSFORMATIONS			
Function	**Transformation**		**Image of A**
$f(x)+2$	*vertical* **translation** *2 units up*	↑ 2	A′(6,12)
$f(x)-2$	*vertical* **translation** *2 units down*	↓ 2	A′(6,8)
$2f(x)$	*vertical* **stretch** *with* **scale factor 2**	↕ ×2	A′(6,20)
$f(x)/2$	*vertical* **stretch** *with* **scale factor 1/2** *(shrink)*	÷2	A′(6,5)
$-f(x)$	**reflection** *in the x-axis*		A′(6,-10)

HORIZONTAL TRANSFORMATIONS			
Function	**Transformation**		**Example: $f(x)=x^2$**
$f(x+2)$	*horizontal* **translation** *2 units to the left*	← 2	A′(4,10)
$f(x-2)$	*horizontal* **translation** *2 units to the right*	→ 2	A′(8,10)
$f(2x)$	*horizontal* **stretch** *with* **scale factor 1/2** *(shrink)*	÷2	A′(3,10)
$f(x/2)$	*horizontal* **stretch** *with* **scale factor 2**	↔ ×2	A′(12,10)
$f(-x)$	**reflection** *in the y-axis*		A′(-6,10)

NOTICE:

The horizontal translation by a units (to the right or to the left)

is also denoted by the translation vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$

A vertical translation by b units (up or down)

is also denoted by the translation vector $\begin{pmatrix} 0 \\ b \end{pmatrix}$

The combination of those two translations is denoted by $\begin{pmatrix} a \\ b \end{pmatrix}$

Of course we may have a combination of several simple transformations.

For example, $2f(x-3)+5$ implies

a vertical stretch with scale factor 2, followed by

a horizontal translation 3 units to the right, followed by

a vertical translation 5 units up

NOTICE:

Remember the vertex form of a quadratic function

$$y=a(x-h)^2+k$$

This is a combination of transformations of the simple quadratic function $y=x^2$

Indeed, If $a>0$

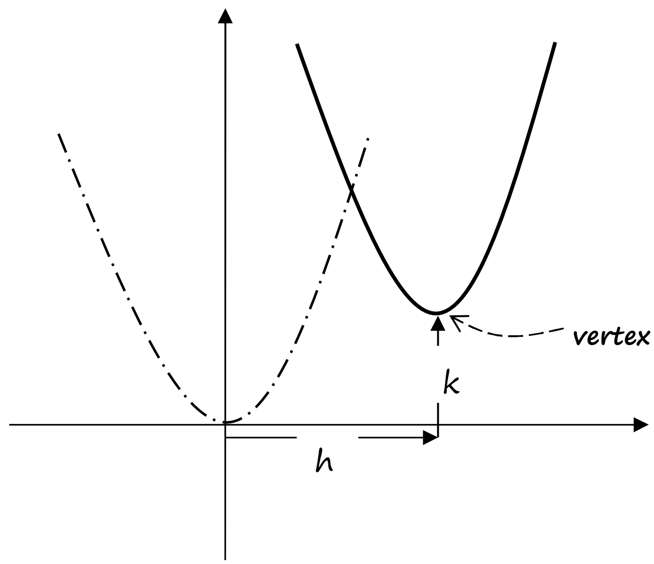
x^2	original function
ax^2	<u>vertical stretch</u> by scale factor a
$a(x-h)^2$	<u>horizontal translation</u> by **h units**
$a(x-h)^2+k$	<u>vertical translation</u> by **k units**

(if $a<0$, we also have a reflection about x-axis)

The two translations by $\begin{pmatrix} h \\ k \end{pmatrix}$ imply that the initial vertex (0,0) of the function x^2 moves

h units horizontally, and
k units vertically,

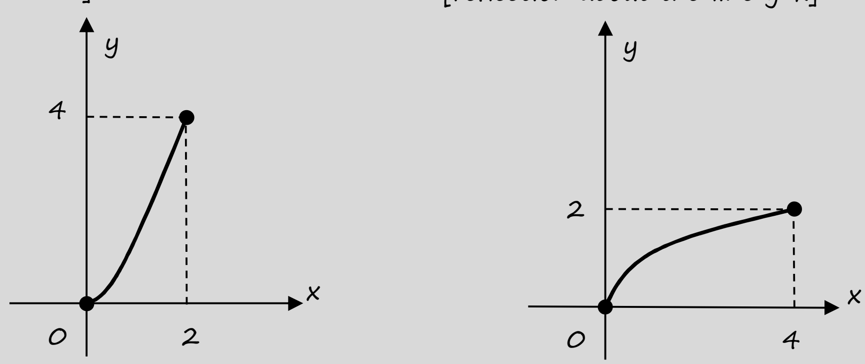
thus its new position is (h,k)



♦ THE INVERSE FUNCTION TRANSFORMATION

We have already seen that $f^{-1}(x)$ causes a reflection in the line y=x.

$f(x)=x^2$ [initial function]	$f^{-1}(x)=\sqrt{x}$ [reflection about the line y=x]



The image of the point A(2,4) is A′(4,2)

NOTICE:

Mind the order when applying composite transformations.

For example, the transformation $y=2f(x)+3$ consists of the following two single transformations:

- $f(x)$
- $2f(x)$
- $2f(x)+3$

Be careful! The reverse order will result to

- $f(x)$
- $f(x)+3$
- $2[f(x)+3] = 2f(x)+6$

Indeed, in a vertical stretch by s.f. 2 we multiply not only $f(x)$ but the whole expression by 2.

Similarly, the transformation $y=f(2x+6)$ consists of:

- $f(x)$
- $f(x+6)$
- $f(2x+6)$

Be even more careful now! In horizontal transformations, only x changes from one form to another. The reverse order will result to

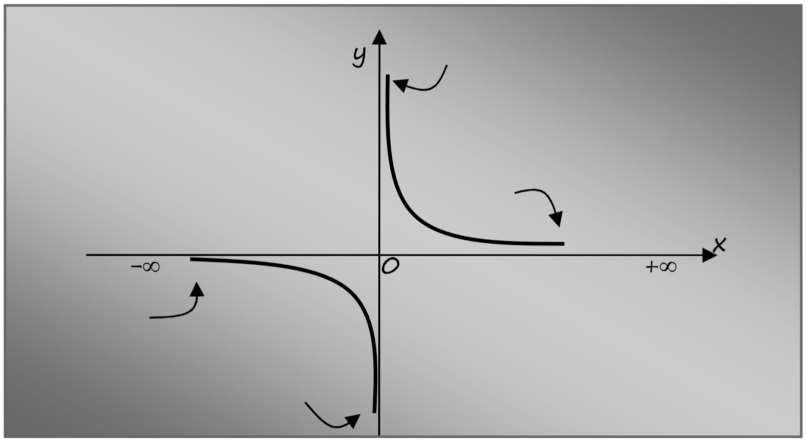
- $f(x)$
- $f(2x)$
- $f(2(x+6)) = f(2x+12)$!!!

Otherwise, if we express $f(2x+6)$ as $f(2(x+3))$, the correct order is

- $f(x)$
- $f(2x)$
- $f(2(x+3))$

2.7 ASYMPTOTES

Look at the graph of the function $f(x)=\frac{1}{x}$



Notice: as x tends to +∞ the value of y tends to 0 **(the x-axis)**

Also as x tends to −∞ the value of y approaches 0 **(the x-axis)**

We say that

the x-axis (that is the line **y=0**) is a **horizontal asymptote**

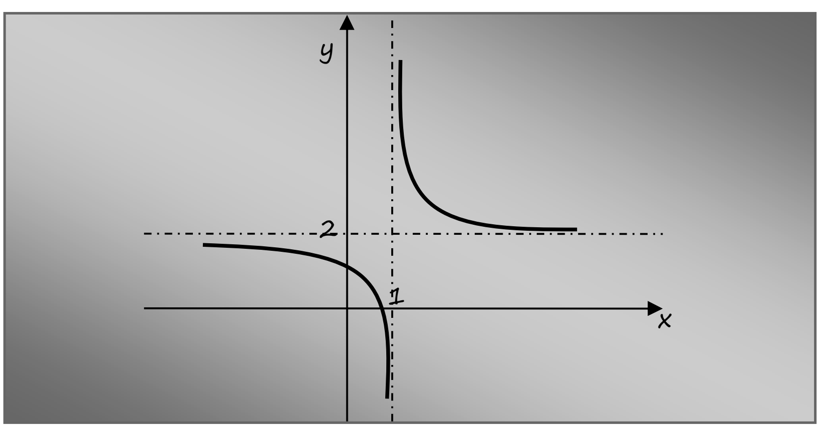
Moreover,

for values of x near 0 **(y-axis),** the value of y tends to +∞ or +∞

We say that

the y-axis (that is the line **x=0**) is a **vertical asymptote**

Similarly, for $g(x)=\frac{1}{x-1}+2$ (f moved 1 unit right and 2 units up).



Now the line y=2 is a horizontal asymptote

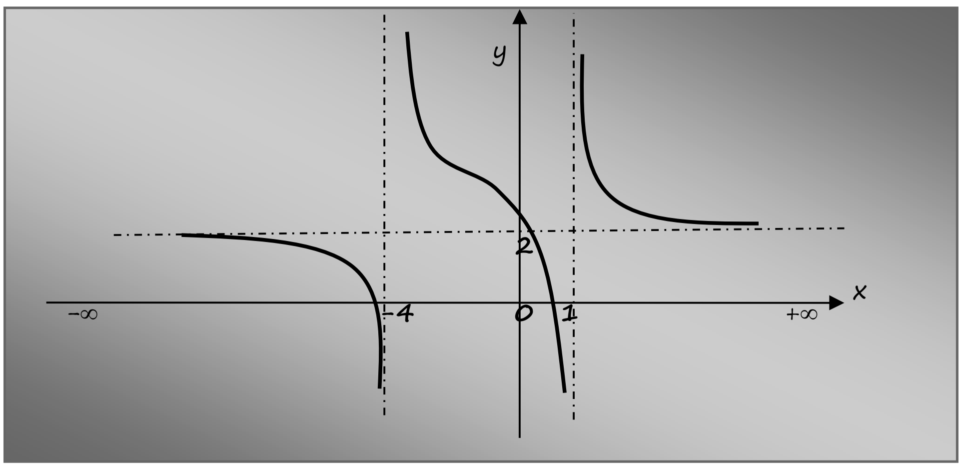
the line x=1 is a vertical asymptote

In general,

For Vertical Asymptotes: we are looking at points x=a where the function is not defined

For Horizontal Asymptotes: we observe what happens if x tends to $+\infty$ or $-\infty$. If the function approaches the line y=b we say that y=b is a horizontal asymptote!

In the following graph:



The function is not defined at x=-4 and x=1, so

the lines x=-4 and x=1 are vertical asymptotes

As x tends to $+\infty$ or $-\infty$ the graph approaches the line y=2, so

the line y=2 is a horizontal asymptote

In this section we concentrate on rational functions of the form

$$f(x)=\frac{Ax+B}{Cx+D}$$

and their asymptotes. It can be shown that such a function can be derived from original function

$$f(x)=\frac{1}{x}$$

by a sequence of transformations.

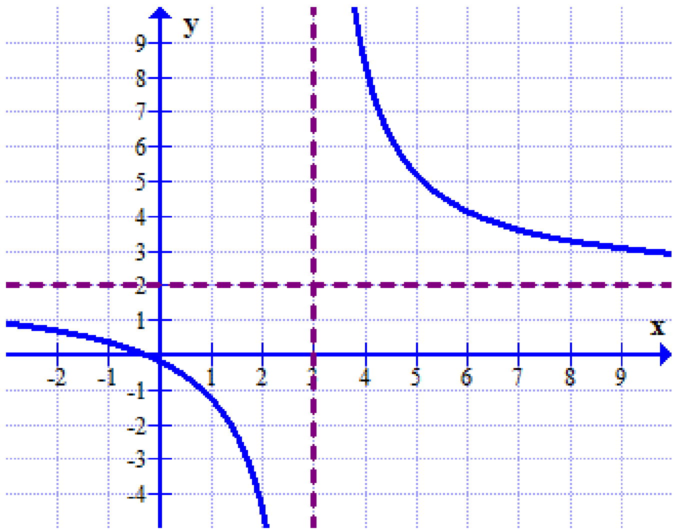
♦ RATIONAL FUNCTIONS OF THE FORM $f(x)=\dfrac{Ax+B}{Cx+D}$,

These functions possess one vertical and one horizontal asymptote.

For example, the function

$$f(x)=\frac{4x+1}{2x-6}$$

looks like



1) Vertical Asymptotes: ***x=a***

At points where the function is not defined.

We solve

$$2x-6=0 \Leftrightarrow x=3$$

Hence

$$\boxed{\text{The line } x=3 \text{ is a vertical asymptote}}$$

2) Horizontal Asymptotes: *y=b*

The line

$$y=\frac{A}{C} \quad \text{is a horizontal asymptote}$$

(we consider only the leading coefficients!)

For our example,

$$y=\frac{4}{2}=2,$$

Hence

$$\boxed{\text{The line } y=2 \text{ is a horizontal asymptote}}$$

Notice

The domain is $x \neq 3$ while the vertical asymptote is $x=3$.

The range is $y \neq 2$ while the vertical asymptote is $y=2$.

Two short explanations for the horizontal asymptote:

- The function can be written as follows:

$$f(x)=\frac{4x+1}{2x-6}=\frac{2(2x-6)+13}{2x-6}=\frac{2(2x-6)}{2x-6}+\frac{13}{2x-6}=2+\frac{13}{2x-6}$$

As x tends to $+\infty$ or $-\infty$ the fraction $\frac{13}{2x-6}$ approaches 0.

- If we divide everything by x we obtain:

$$f(x)=\frac{4x+1}{2x-6}=\frac{4+\frac{1}{x}}{2-\frac{6}{x}}$$

As x tends to $+\infty$ or $-\infty$ the fractions $\frac{1}{x}$ and $\frac{6}{x}$ approach 0.

In both cases f(x), that is the value of y, approaches 2.

EXAMPLE 1

Look at some rational functions and their asymptotes:

Function	Vertical Asymptotes (denominator = 0)	Horizontal Asymptote (divide leading coefficients)
$f(x)=\frac{3x-7}{x-5}$	$x=5$	$y=3$
$f(x)=\frac{3x-7}{2x-5}$	$x=\frac{5}{2}$	$y=\frac{3}{2}$
$f(x)=\frac{8x-7}{2x+4}$	$x=-2$	$y=4$
$f(x)=\frac{7}{x-5}$	$x=5$	$y=0$
$f(x)=\frac{7}{x-5}+3$	$x=5$	$y=3$

EXAMPLE 2

Let $f(x)=\dfrac{3x+2}{x-4}$

We can easily find that the inverse function is $f^{-1}(x)=\dfrac{4x+2}{x-3}$

Notice what happens with the asymptotes:

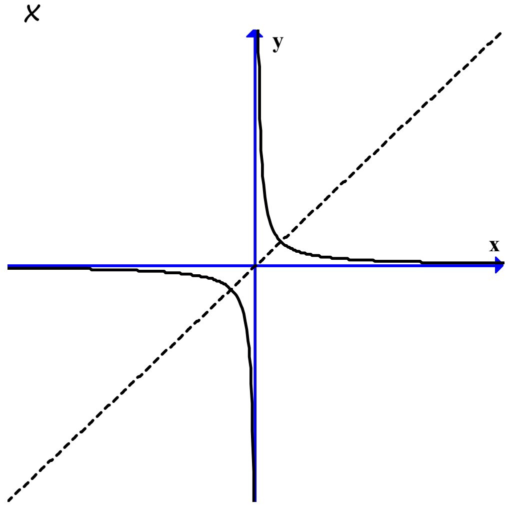
	Domain	Range	V.A.	H.A
f(x)	$x\neq 4$	$y\neq 3$	$x=4$	$y=3$
$f^{-1}(x)$	$x\neq 3$	$y\neq 4$	$x=3$	$y=4$

♦ SELF-INVERSE FUNCTIONS

A function is said to be **self-inverse** if $f^{-1}(x)=f(x)$

Such a function is **symmetric in the line y=x**.

For example $f(x)=\dfrac{1}{x}$ is a self-inverse function.



Indeed, $y=\dfrac{1}{x}\Leftrightarrow x=\dfrac{1}{y}$ hence, $f^{-1}(x)=\dfrac{1}{x}$

Several rational functions are self-inverse. For example

$$f(x)=\frac{2x+3}{x-2}=f^{-1}(x)$$

The asymptotes for those two functions are x=2 and y=2.

2.8 EXPONENTS – THE EXPONENTIAL FUNCTION a^x

♦ THE EXPONENTIAL 2^x

Let us define the power 2^x, as x moves along the sets

$N = \{0,1,2,3,...\}$	**Natural numbers**
$Z = \{...,-3,-2,-1,0,1,2,3,...\}$	**Integers**
$Q = \{\text{fractions } \frac{m}{n} \mid m,n\in Z,\ n\neq 0\}$	**Rational numbers**
$R = Q$ + irrational numbers[†]	**Real numbers**

1) If $x=n\in N$, then

$$2^0 = 1$$

$$2^n = 2\cdot 2\cdot 2\cdots 2 \text{ (n times)}$$

For example $2^3=8$

2) If $x=-n$, where $n\in N$, then

$$2^{-n} = \frac{1}{2^n}$$

Thus we know 2^x for any $x\in Z$.

For example $2^{-3}=\frac{1}{2^3}=\frac{1}{8}$

3) If $x=\frac{m}{n}$, where $m,n\in Z$, $n\neq 0$, then

$$2^{\frac{m}{n}} = \sqrt[n]{2^m}$$

Thus we know 2^x for any $x\in Q$

For example, $2^{\frac{2}{3}}=\sqrt[3]{2^2}=\sqrt[3]{4}$, $\quad 2^{\frac{3}{2}}=\sqrt{2^3}=\sqrt{8}$, $\quad 2^{\frac{1}{2}}=\sqrt{2}$

[†] That is numbers that cannot be expressed as fractions, eg π, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$

4) If x=irrational, then

2^x = given by a calculator!

The definition is beyond our scope, thus we trust technology!

Thus we know 2^x for any $x \in R$

For example, $2^\pi = 8.8249779$

In general, if $a>0$ we define

$a^0 = 1$

$a^n = a \cdot a \cdot a \cdots a$ (n times)

$a^{-n} = \dfrac{1}{a^n}$

$a^{\frac{m}{n}} = \sqrt[n]{a^m}$

a^x = given by a calculator!

NOTICE

- If $a<0$, a^x is defined only for $x=n \in Z$
- $0^x=0$ only if $x \neq 0$
- 0^0 is not defined

♦ PROPERTIES

All known properties of powers are still valid for exponents $x \in R$

(1) $a^x a^y = a^{x+y}$ (3) $(ab)^x = a^x b^x$ (5) $(a^x)^y = a^{xy}$

(2) $\dfrac{a^x}{a^y} = a^{x-y}$ (4) $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Here $a,b>0$ and $x,y \in R$

EXAMPLE 1

- $5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$
- $\left(\dfrac{1}{5}\right)^{-2} = \dfrac{1}{5^{-2}} = 5^2 = 25$
- $\left(\dfrac{3}{5}\right)^{-2} = \left(\dfrac{5}{3}\right)^{2} = \dfrac{25}{9}$
- $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$ or $8^{2/3} = (2^3)^{2/3} = 2^{3\cdot(2/3)} = 2^2 = 4$
- $27^{-4/3} = \sqrt[3]{27^{-4}} = \sqrt[3]{\dfrac{1}{27^4}} = \sqrt[3]{\left(\dfrac{1}{27}\right)^4} = \sqrt[3]{\left(\dfrac{1}{27}\right)}^{4} = \left(\dfrac{1}{3}\right)^4 = \dfrac{1}{81}$

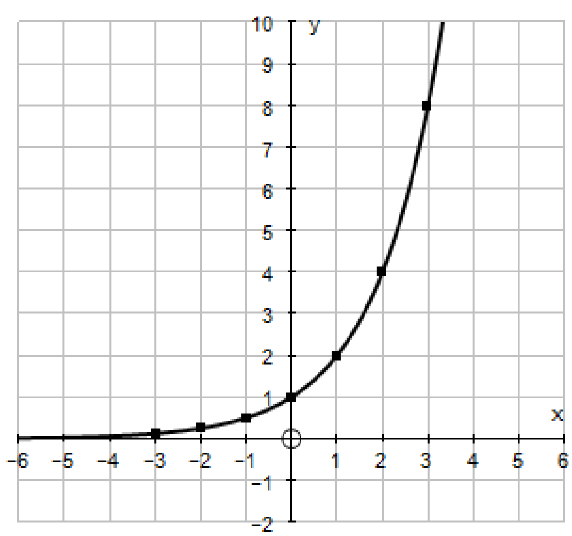
♦ THE EXPONENTIAL FUNCTION $f(x)=a^x$ (where $a>0$)

Consider $f: R \to R$

$f(x)=2^x$

Let us estimate some values

x	...	-3	-2	-1	0	1	2	3	...
$y=2^x$	...	1/8	1/4	1/2	0	1	4	8	...



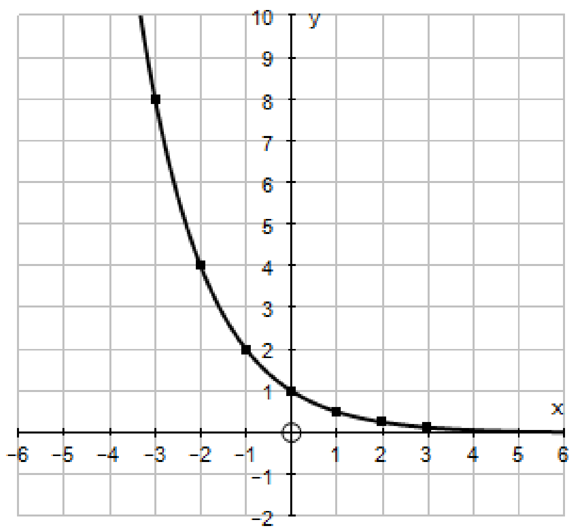
Domain: $x \in R$
Range: $y>0$

Consider now $g: R \to R$

$g(x)=0.5^x$ [that is $g(x)=\left(\frac{1}{2}\right)^x=\frac{1}{2^x}$]

Let us estimate some values

x	…	−3	−2	−1	0	1	2	3	…
$y=2^x$	…	8	4	2	0	1/2	1/4	1/8	…



Domain: $x \in R$

Range: $y>0$

NOTICE

1) $f(x)=a^x$ is <u>always</u> positive (even if $x<0$)

2) $g(x)=\left(\frac{1}{a}\right)^x=\frac{1}{a^x}=a^{-x}$. Thus, $g(x)$ is a reflection of $f(x)=a^x$ about the y-axis [look at the graphs of f(x) and g(x) above]

3) if $a>1$, then $f(x)=a^x$ **increases** (the graph looks like that of 2^x)
if $a<1$, then $f(x)=a^x$ **decreases** (the graph looks like that of 0.5^x)
if $a=1$, then $f(x)=1^x=1$ is constant

4) if $a\neq1$, function $f(x)= a^x$ is "one-one", i.e.

$$a^x=a^y \Rightarrow x=y$$

This property helps us to solve exponential equations!

EXAMPLE 2

Solve the following equations

(a) $2^{3x-1} = 2^{x+2}$ (b) $2^{3x-1} = 4^{x+2}$ (c) $4^{3x-1} = 8^{x+2}$

(d) $\dfrac{1}{2^{3x-1}} = 4^{x+2}$ (e) $\sqrt{2}^{\,3x-1} = 4^{x+2}$

Solution

Our attempt will be to induce a common base in both sides

(a) We have already a common base. Thus

$$2^{3x-1} = 2^{x+2} \Leftrightarrow 3x-1 = x+2 \Leftrightarrow 2x = 3$$

$$\Leftrightarrow \boxed{x=3/2}$$

(b) We can write $4=2^2$. Thus

$$2^{3x-1} = 4^{x+2} \Leftrightarrow 2^{3x-1} = 2^{2x+4} \Leftrightarrow 3x-1 = 2x+4$$

$$\Leftrightarrow \boxed{x = 5}$$

(c) We can write $4=2^2$ and $8=2^3$. Thus

$$4^{3x-1} = 8^{x+2} \Leftrightarrow 2^{6x-2} = 2^{3x+6} \Leftrightarrow 6x-2 = 3x+6$$

$$\Leftrightarrow 3x=8 \Leftrightarrow \boxed{x = 8/3}$$

(d) We apply the property $\dfrac{1}{2^n}=2^{-n}$. Thus

$$\frac{1}{2^{3x-1}} = 4^{x+2} \Leftrightarrow 2^{-3x+1} = 2^{2x+4} \Leftrightarrow -3x+1 = 2x+4$$

$$\Leftrightarrow 5x = -3 \Leftrightarrow \boxed{x=-3/5}$$

(e) We apply the property $\sqrt{2} = 2^{1/2}$. Thus

$$\sqrt{2}^{\,3x-1} = 4^{x+2} \Leftrightarrow 2^{\frac{3x-1}{2}} = 2^{2x+4} \Leftrightarrow \frac{3x-1}{2} = 2x+4$$

$$\Leftrightarrow 3x-1= 4x+8 \Leftrightarrow \boxed{x=-9}$$

♦ THE NUMBER e

There is a specific irrational number

$$e=2.7182818...$$

which plays an important role in mathematics. The number e is almost as popular as the irrational number $\pi=3.14...$

An approximation of e is given below. Consider the expression

$$\left(1+\frac{1}{n}\right)^n$$

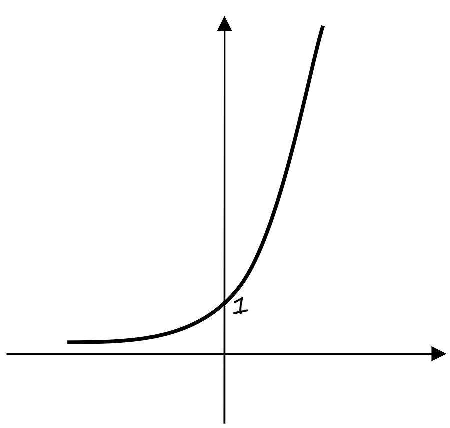
For $n=1$	the result is	2
For $n=2$	the result is	2.25
For $n=10$	the result is	2.5937424...
For $n=100$	the result is	2.7048138...
For $n=1000$	the result is	2.7169239...
For $n=10^6$	the result is	2.7182804...

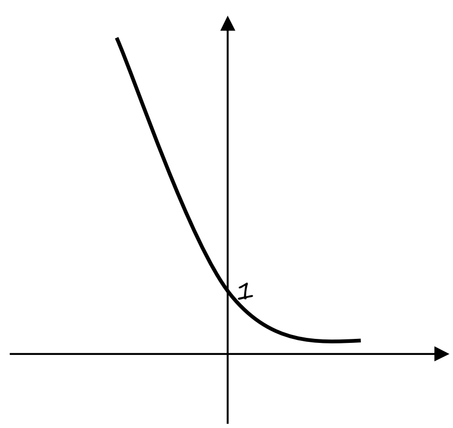
As n tends to $+\infty$ this expression tends to $e=2.7182818...$

♦ THE EXPONENTIAL e^x

The exponential function $f(x)=e^x$ appears in many applications. The graph looks like any function of the form $f(x)=a^x$. We present the graphs of

$f(x) = e^x$ $f(x)=e^{-x}$

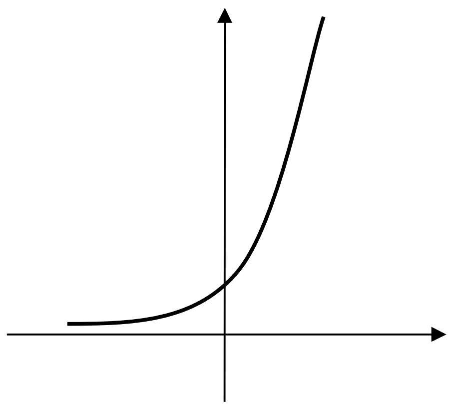




- ASYMPTOTES OF EXPONENTIAL FUNCTIONS

Observe the exponential functions ($a>0$, $a\neq 1$)

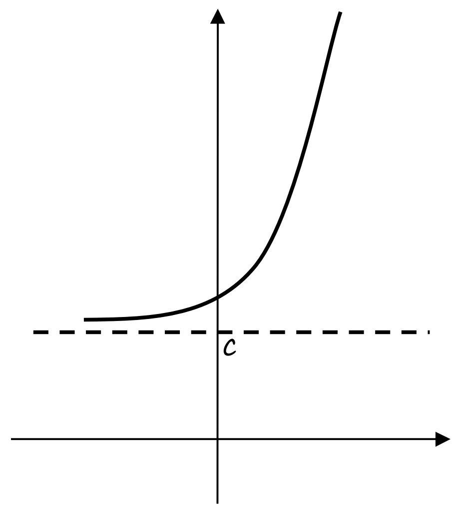
$f(x)=a^x$



horizontal asymptote: $y=0$

y-intercept: $y=1$

$f(x)=a^x+c$



horizontal asymptote: $y=c$

y-intercept: $y=c+1$

EXAMPLE 3

Function	Horizontal Asymptote	y-intercept
$f(x)=2^x$	line $y=0$	$y=1$
$f(x)=2^{-x}$	line $y=0$	$y=1$
$f(x)=e^x$	line $y=0$	$y=1$
$f(x)=e^{3x}$	line $y=0$	$y=1$
$f(x)=3e^x$	line $y=0$	$y=3$
$f(x)=-3e^x$	line $y=0$	$y=-3$
$f(x)= e^x+5$	line $y=5$	$y=6$
$f(x)=3e^x+5$	line $y=5$	$y=8$
$f(x)=e^{x-2}$	line $y=0$	$y=e^{-2}$

2.9 LOGARITHMS – THE LOGARITHMIC FUNCTION $y=\log_a x$

♦ THE LOGARITHM $\log_2 x$

This number is called **logarithm of x to the base 2**. It is connected to the exponential 2^x. The definition is given by

$$\boxed{\log_2 x = y \Leftrightarrow 2^y = x}$$

For example,

$\log_2 8 = 3$, since $2^3=8$

$\log_2 16 = 4$, since $2^4=16$

$\log_2 1024 = 10$, since $2^{10}=1024$

etc.

For example, for $\log_2 8=?$, we think in the following way:

$2^{\text{what exponent}}$ gives 8?

The answer is 3

Hence $\log_2 8=3$

Working in the same way let us find $\log_2 64=?$

It is $\log_2 64=6$

However, for $\log_2 10=?$, we should think:

$2^{\text{what exponent}}$ gives 10 ?

OK, this is difficult to answer!!!

Our calculator gives $\log_2 10=3.321928\ldots$

This implies that

$$2^{3.321928\ldots}=10$$

EXAMPLE 1

Find $\log_2 32$, $\log_2 2^5$, $\log_2 2^{100}$, $\log_2 2^{1453}$, $\log_2 2$, $\log_2 1$

- $\log_2 32=5$
- $\log_2 2^5=5$
- $\log_2 2^{100}=100$ Notice, in general $\boldsymbol{\log_2 2^x=x}$
- $\log_2 2^{1453}=1453$
- $\log_2 2=1$
- $\log_2 1=0$

♦ THE LOGARITHM $\log_a x$

In exactly the same way, for any base $a>0$, $a\neq 1$ we define

$$\boxed{\log_a x = y \Leftrightarrow a^y=x}$$

For example, $\log_3 9 = 2$ (since $3^2=9$)

NOTICE

Once upon a time $\log_{10} x$ has been the most popular logarithm!!!

Due to its popularity, for this particular logarithm the base 10 is usually omitted

We write $\boldsymbol{\log x}$ instead of $\boldsymbol{\log_{10} x}$

For example,

$\log 100 = 2$, since $10^2=100$

$\log 1000 = 3$, since $10^3=1000$

$\log 10000 = 4$, since $10^4=10000$

Notice: use your GDC to confirm these results

Clearly,

$$\log 10=1 \quad \text{and} \quad \log 1=0$$

EXAMPLE 2

- $\log_{10}1000000 = 6$,
- $\log 10^7=7$ Notice, in general $\log 10^x=x$
- $\log 10^{1453}=1453$

But also for very small numbers

- $\log_{10}0.001 = -3$,
- $\log_{10}0.000001 = -6$,

NOTICE (Just for information)

In some way, the logarithm to the base 10 indicates the size of the number!

Indeed, since $\log 100=2$ and $\log 999=2.99957$

any 3-digit integer has a logarithm within the interval $[2,3)$

Similarly, any 10-digit number has a logarithm within $[9,10)$

Any n-digit number has a logarithm between n-1 and n.

Question: how many digits does the number 2^{100} have?

The GDC gives $\log 2^{100} = 30.1$

Therefore, the number 2^{100} has 31 digits!

♦ THE NATURAL LOGARITHM **lnx**

The most frequently used logarithm is the logarithm to the base

$$e=2.7182818\ldots$$

Instead of $\log_e x$, we denote it by

$$\ln x$$

Hence,

$$\boxed{\ln x=y \Leftrightarrow e^y=x}$$

♦ THE LOGARITHMIC FUNCTION **$y=\log_a x$**

A new function is defined

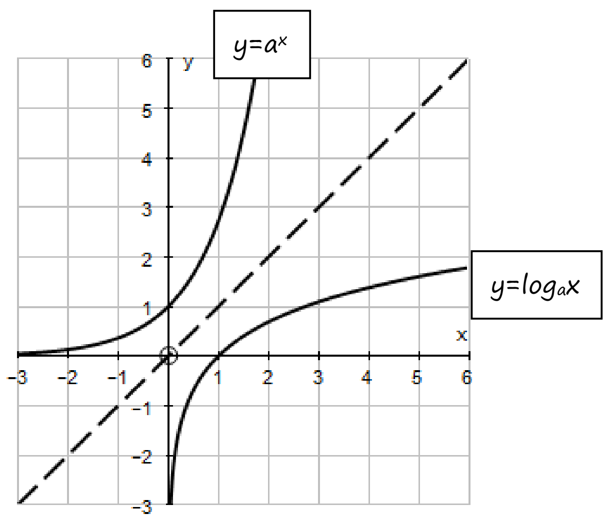
$$y = \log_a x$$

In fact, this is the inverse function of the exponential function $y=a^x$

$$\boxed{\text{If } f(x)=a^x \text{ then } f^{-1}(x)=\log_a x}$$

Indeed, $a^x=y \Leftrightarrow x= \log_a y$, hence $f^{-1}(x)=\log_a x$

If $a>1$ (for example if $a=2$), the graphs of these two functions look like



Observations:

- For $y=a^x$: **Domain**: $x \in R$ **Range**: $y \in R_+$ (i.e. $y>0$)
- For $y=\log_a x$: **Domain**: $x \in R_+$ (i.e. $x>0$) **Range**: $y \in R$
- The x-axis is a horizontal asymptote of $y=a^x$
- The y-axis is a vertical asymptote of $y=\log_a x$
- $y=a^x$ always passes through $(0,1)$
- $y=\log_a x$ always passes through $(1,0)$

♦ BASIC PROPERTIES OF LOGARITHMS

For any base a ($a>0$, $a\neq 1$)

- $\log_a 1 = 0$
- $\log_a a = 1$
- $\log_a a^x = x$
- $a^{\log_a x} = x$

The first three results can be directly confirmed by the definition of logarithm. For the last one, set $y=\log_a x$. The definition implies $a^y=x$. Replace back $y=\log_a x$ and the result is immediate!

♦ FOUR ALGEBRAIC LAWS

For simplicity reasons, we use log instead of $\log_a$.

1) $\log xy = \log x + \log y$
2) $\log \frac{x}{y} = \log x - \log y$
3) $\log x^n = n\log x$
4) $\log \frac{1}{x} = -\log x$

or

- $\log x + \log y = \log xy$
- $\log x - \log y = \log \frac{x}{y}$
- $n\log x = \log x^n$
- $-\log x = \log \frac{1}{x}$

Proofs (consider all logarithms to be of base a)

For all of them we follow the same method! We check if $a^{LHS} = a^{RHS}$

1) $a^{LHS} = xy$ and $a^{RHS} = a^{\log_a x+\log_a y} = a^{\log_a x}a^{\log_a y} = xy$

2) $a^{LHS} = x/y$ and $a^{RHS} = a^{\log_a x-\log_a y} = a^{\log_a x}/a^{\log_a y} = x/y$

3) $a^{LHS} = x^n$ and $a^{RHS} = a^{n\log_a x} = (a^{\log_a x})^n = x^n$

4) this is a special case of 2) if we set $x=1$, as well as of 3) if $n=-1$

NOTICE

The first two laws can be combined in the following way:

$$\boxed{\log A+\log B-\log C+\log D = \log\frac{ABD}{C}}$$

If we also have coefficients we can work as in the following example

$$\begin{aligned}\mathbf{2}\log A+\mathbf{3}\log B-\mathbf{4}\log C+\mathbf{5}\log D &= \log A^2+\log B^3-\log C^4+\log D^5\\ &= \log\frac{A^2B^3D^5}{C^4}\end{aligned}$$

Thus

$$\boxed{\mathbf{2}\log A + \mathbf{3}\log B - \mathbf{4}\log C + \mathbf{5}\log D = \log\frac{A^2B^3D^5}{C^4}}$$

This is the way **<u>we collect many logs into one log</u>**.

For example

$$\mathbf{2}\log 3 + \mathbf{3}\log 4 - \mathbf{4}\log 2 = \log\frac{3^2 4^3}{2^4} = \log 36$$

or

$$\mathbf{2}\ln 3 + \mathbf{3}\ln 4 - \mathbf{4}\ln 2 = \ln\frac{3^2 4^3}{2^4} = \ln 36$$

Look at also the opposite direction

$$\boxed{\log\frac{A^2B^3D^5}{C^4} = \mathbf{2}\log A + \mathbf{3}\log B - \mathbf{4}\log C + \mathbf{5}\log D}$$

This is the way **<u>we split one log into many logs</u>**.

For example

$$\log 72 = \log(8\times 9) = \log 2^3 3^2 = \mathbf{3}\log 2 + \mathbf{2}\log 3$$

or

$$\ln 72 = \ln(8\times 9) = \ln 2^3 3^2 = \mathbf{3}\ln 2 + \mathbf{2}\ln 3$$

EXAMPLE 3

Suppose lnx=**a**, lny=**b**, lnz=**c**. Express the following in terms of **a,b,c.**

$\ln xy$, $\ln x^2$, $\ln\frac{y}{z}$, $\ln\frac{x^3y}{z^2}$, $\ln\frac{1}{x}$, $\ln\sqrt{x}$,

Solution

- $\ln xy = \ln x + \ln y = \mathbf{a+b}$
- $\ln x^2 = 2\ln x = \mathbf{2a}$
- $\ln\frac{y}{z} = \ln y - \ln z = \mathbf{b-c}$
- $\ln\frac{x^3y}{z^2} = 3\ln x + \ln y - 2\ln z = \mathbf{3a+b-2c}$
- $\ln\frac{1}{x} = \ln 1 - \ln x = 0-a = \mathbf{-a}$ [or $\ln\frac{1}{x} = \ln x^{-1} = -\ln x = \mathbf{-a}$]
- $\ln\sqrt{x} = \ln x^{1/2} = \frac{1}{2}\ln x = \mathbf{\frac{a}{2}}$

EXAMPLE 4

Suppose ln2=**m**, ln5=**n**. Express the following in terms of **m, n**.

$\ln 10$, $\ln 50$, $\ln 2.5$

Solution

- $\ln 10 = \ln(2\times 5) = \ln 2 + \ln 5 = \mathbf{m+n}$
- $\ln 50 = \ln(2\times 5^2) = \ln 2 + 2\ln 5 = \mathbf{m+2n}$
- $\ln 2.5 = \ln\frac{5}{2} = \ln 5 - \ln 2 = \mathbf{n-m}$

♦ SIMPLE LOGARITHMIC EQUATIONS

They have the form

$$\log_a x = b$$

We use the definition to solve them:

$$x = a^b$$

EXAMPLE 5

Solve the logarithmic equations

(a) $log_2(x+2)=3$ (a) $log(x+2)=3$ (c) $ln(x+2)=3$

Solution

(a) $x+2=2^3 \Leftrightarrow x+2=8 \Leftrightarrow x=6$

(b) $x+2=10^3 \Leftrightarrow x+2=1000 \Leftrightarrow x=998$

(c) $x+2=e^3 \Leftrightarrow x=e^3-2$

Notice

Of course the solutions may be obtained by a GDC.

For (a) and (b), **SolveN** gives the exact solutions x=6 and x=998
For (c) it gives an approximation x≅18.1
(this is not the **exact** solution, it is the approximate value of e^3-2).

Furthermore, if the equation contains a parameter, for example

$$log_2(x+a)=3$$

we cannot use GDC. The solution must be expressed in terms of a:

$$x+a=2^3 \Leftrightarrow x=8-a$$

In paper 2 (GDC allowed) we can use our calculator to solve more complicated logarithmic equations'

In paper 1 (GDC not allowed) we have to present the analytical solution for equations which involve more than one logarithms.

Our target will be to bring them in one of the forms

- **logA = logB** so that **A = B**
- **log$_b$A=c** so that **A=b^c** by definition

The resulting equations will be easier to deal with.

The following examples will clarify what we mean.

EXAMPLE 6

Solve the equations

(a) $\log_2 x + \log_2(x+2) = \log_2 3$

(b) $\log_2 x + \log_2(x+2) = 3$

(c) $\log_2 x + \log_2(x-2) - \log_2(x-\frac{3}{4}) = \log_2 3$

Solutions

(a) We obtain $\log_2 x(x+2) = \log_2 3$

Hence

$$x(x+2)=3 \quad \Leftrightarrow \quad x^2+2x-3=0$$

The solutions are $x=1$ and $x=-3$

The second solution is rejected since $x>0$ by the original equation. Therefore $\boxed{x=1}$.

(b) We obtain $\log_2 x(x+2) = 3$

Hence

$$x(x+2)=2^3 \quad \Leftrightarrow \quad x^2+2x-8=0$$

The solutions are $x=2$ and $x=-4$

The second solution is rejected since $x>0$ by the original equation. Therefore $\boxed{x=2}$.

(c) We obtain $\log_2 \frac{x(x-2)}{(x-\frac{3}{4})} = \log_2 3$

Hence

$$\frac{x(x-2)}{(x-\frac{3}{4})}=3 \quad \Leftrightarrow \quad x^2-2x=3x-\frac{9}{4} \quad \Leftrightarrow \quad x^2-5x+\frac{9}{4}=0$$

The solutions are $x=4.5$ or $x=0.5$

The second solution is rejected since $x>2$. Therefore, $\boxed{x=4.5}$

Notice

Use your **GDC – SolveN** to confirm the results

♦ CHANGE OF BASE

Consider the equation

$$a^x=b$$

If you apply log_a on both sides you obtain

$$x= log_a b$$

However, we can apply any logarithm:

$$a^x=b \Rightarrow log\ a^x = logb \Rightarrow xloga = logb \Rightarrow x = \frac{logb}{loga}$$

$$a^x=b \Rightarrow ln\ a^x = lnb \Rightarrow xlna = lnb \Rightarrow x = \frac{lnb}{lna}$$

$$a^x=b \Rightarrow log_c\ a^x = log_c b \Rightarrow xlog_c a = log_c b \Rightarrow x = \frac{log_c b}{log_c a}$$

Thus

$$log_a b = \frac{logb}{loga} = \frac{lnb}{lna} = \frac{log_c b}{log_c a}$$

This tells us that we can change $log_a b$ into $\frac{log_c b}{log_c a}$, in any base we like.

The formula

$$\boxed{log_a b = \frac{log_c b}{log_c a}}$$

is known as the "**change of base formula**".

For example

$$log_2 5$$

can be changed to

$$\frac{log5}{log2} \quad or \quad \frac{ln5}{ln2} \quad or \quad \frac{log_3 5}{log_3 2} \quad etc$$

Use your GDC to confirm that all these are equal to

2.322…

EXAMPLE 7

Suppose lnx=**a**, lny=**b**, lnz=**c**. Express the following in terms of **a,b,c**.

$\ln xy$, $\ln x^2$, $\ln\frac{y}{z}$, $\ln\frac{x^3y}{z^2}$, $\ln\frac{1}{x}$, $\ln\sqrt{x}$,

Solution

- $\ln xy = \ln x + \ln y = \mathbf{a+b}$
- $\ln x^2 = 2\ln x = \mathbf{2a}$
- $\ln\frac{y}{z} = \ln y - \ln z = \mathbf{b-c}$
- $\ln\frac{x^3y}{z^2} = 3\ln x + \ln y - 2\ln z = \mathbf{3a+b-2c}$
- $\ln\frac{1}{x} = \ln 1 - \ln x = 0-a = \mathbf{-a}$ [or $\ln\frac{1}{x} = \ln x^{-1} = -\ln x = \mathbf{-a}$]
- $\ln\sqrt{x} = \ln x^{1/2} = \frac{1}{2}\ln x = \mathbf{\frac{a}{2}}$

EXAMPLE 8

Suppose ln2=**m**, ln5=**n**. Express the following in terms of **m, n**.

$\ln 10$, $\ln 50$, $\ln 2.5$, $\ln 0.4$, $\log_5 e$, $\log_4 5^3$

Solution

- $\ln 10 = \ln(2\times5) = \ln 2 + \ln 5 = \mathbf{m+n}$
- $\ln 50 = \ln(2\times5^2) = \ln 2 + 2\ln 5 = \mathbf{m+2n}$
- $\ln 2.5 = \ln\frac{5}{2} = \ln 5 - \ln 2 = \mathbf{n-m}$
- $\ln 0.4 = \ln\frac{2}{5} = \ln 2 - \ln 5 = \mathbf{m-n}$
- $\log_5 e = \frac{\ln e}{\ln 5} = \mathbf{\frac{1}{n}}$
- $\log_4 5^3 = 3\log_4 5 = 3\frac{\ln 5}{\ln 4} = 3\frac{\ln 5}{\ln 2^2} = \mathbf{\frac{3n}{2m}}$

EXAMPLE 9

Solve the equation

$$\log_4(x+12) = 1 + \frac{1}{2}\log_2 x$$

Solution

We have to change base 4 to 2.

$$\frac{\log_2(x+12)}{\log_2 4} = 1 + \frac{1}{2}\log_2 x$$

$$\Leftrightarrow \frac{\log_2(x+12)}{2} = 1 + \frac{1}{2}\log_2 x$$

$$\Leftrightarrow \log_2(x+12) = 2 + \log_2 x$$

$$\Leftrightarrow \log_2(x+12) - \log_2 x = 2$$

$$\Leftrightarrow \log_2 \frac{x+12}{x} = 2$$

$$\Leftrightarrow \frac{x+12}{x} = 4$$

$$\Leftrightarrow x+12 = 4x$$

$$\Leftrightarrow 3x = 12$$

$$\Leftrightarrow x = 4$$

2.10 EXPONENTIAL EQUATIONS

In these equations the unknown x is in the exponent. The simplest exponential equation has the form

$$a^x=b$$

If we apply $\log_a$ the solution is $x=\log_a b$

If we apply log or ln the solution is $x=\dfrac{\log b}{\log a}$ or $x=\dfrac{\ln b}{\ln a}$

EXAMPLE 1

Solve the equation $2(5^x) = 9$. Express the result in the form $\dfrac{\log a}{\log b}$.

Solution

We first divide by 2 and then apply log

$$5^x = 4.5 \Leftrightarrow \log 5^x = \log 4.5 \Leftrightarrow x\log 5 = \log 4.5 \Leftrightarrow \boxed{x = \frac{\log 4.5}{\log 5}}$$

Notice

If we use ln( ), the answer will be $\boxed{x = \dfrac{\ln 4.5}{\ln 5}}$

If we use $\log_5$( ), the answer will be $\boxed{x = \log_5 4.5}$

Whenever we see exponentials of base e, it is preferable to use ln( ).

EXAMPLE 2

Solve the equation $10e^{2x} = 85$

Solution

We first divide by 10:

$$10e^{2x} = 85 \Leftrightarrow e^{2x} = 8.5 \Leftrightarrow \ln e^{2x} = \ln 8.5 \Leftrightarrow 2x = \ln 8.5 \Leftrightarrow \boxed{x = \frac{\ln 8.5}{2}}$$

EXAMPLE 3

Solve the equation $5^x = 2^{x+1}$. Express the result in the form $\frac{\ln a}{\ln b}$.

Solution

Method A: Let us apply ln on both sides

$$5^x = 2^{x+1} \Leftrightarrow \ln 5^x = \ln 2^{x+1}$$

$$\Leftrightarrow x\ln 5 = (x+1)\ln 2$$

$$\Leftrightarrow x\ln 5 = x\ln 2 + \ln 2$$

$$\Leftrightarrow x\ln 5 - x\ln 2 = \ln 2$$

$$\Leftrightarrow x(\ln 5 - \ln 2) = \ln 2$$

$$\Leftrightarrow x = \frac{\ln 2}{\ln 5 - \ln 2} \Leftrightarrow \boxed{x = \frac{\ln 2}{\ln \frac{5}{2}}}$$

Method B: Simplify the equation to the form $a^x=b$; then apply ln

$$5^x = 2^{x+1} \Leftrightarrow 5^x = 2^x 2$$

$$\Leftrightarrow \frac{5^x}{2^x} = 2$$

$$\Leftrightarrow \left(\frac{5}{2}\right)^x = 2$$

$$\Leftrightarrow x\ln\left(\frac{5}{2}\right) = \ln 2$$

$$\Leftrightarrow \boxed{x = \frac{\ln 2}{\ln \frac{5}{2}}}$$

Remarks

- This is the exact answer. If we are looking for an answer to 3sf, the calculator gives x=0.756.
- We can use any logarithm instead of ln( ), for example log( ).
- If an expression in the form $\log_a b$ is required, the answer is

$$\left(\frac{5}{2}\right)^x = 2 \Leftrightarrow x = \log_{\frac{5}{2}} 2$$

♦ EXPONENTIAL MODELLING (GROWTH OR DECAY)

In many applications a quantity increases or decreases exponentially according to time.

Suppose that a population **P** at time **t** (years after a certain time) is given by the formula

$$\boxed{P=P_o e^{kt}}$$

- If k>0, the population increases (e.g. $P=1000e^{0.2t}$)
- If k<0, the population decreases (e.g. $P=1000e^{-0..2t}$)

Question 1: What is the **initial** population (at starting time)?

Initial means t=0. Since $e^0=1$

$$P=P_o$$

Thus, the coefficient P_o is always the initial value of P.

Let us consider the case where the initial population is 1000 and P is given by

$$P=1000e^{0.2t}$$

Question 2: What is the population after 3 years?

For t=3

$$P=1000e^{0.2\times3}=1822$$

Question 3: The population after t years is 2500. Find t.

$$\begin{aligned} 1000e^{0.2t}=2500 &\Leftrightarrow e^{0.2t}=2.5 \\ &\Leftrightarrow \ln e^{0.2t}=\ln 2.5 \\ &\Leftrightarrow 0.2t=\ln 2.5 \\ &\Leftrightarrow t=\frac{\ln 2.5}{0.2}=4.58 \text{ years} \end{aligned}$$

(OR directly by GDC, t = 4.58 years)

Question 4: The population doubles after t years. Find t.

It's the same as in Question 3. We set P=2000, or in general $P=2P_o$

Sometimes the constant k is not known. We are given some information to find its value. Suppose

$$P=P_oe^{kt}$$

Question 5: Given that the population doubles every 4 years, find k.

For t=4, $P=2P_o$,

$$P_Oe^{k4} = 2P_o \Leftrightarrow e^{4k} = 2$$
$$\Leftrightarrow 4k = \ln 2$$
$$\Leftrightarrow k = \frac{\ln 2}{4} = 0.173$$

Let us see an example of decay.

EXAMPLE 4

The mass m of a radio-active substance at time t hours is given by

$$\mathbf{m = 4e^{-kt}}$$

(a) The mass is 1 kg after 5 hours. Find k.

(b) What is the mass after 3 hours?

(c) The mass reduces to a half after t hours. Find t.

Solution

(a) For t=5, m=1, thus

$$4e^{-k5} = 1$$

$$\Leftrightarrow e^{-5k} = \frac{1}{4} \Leftrightarrow e^{5k} = 4 \Leftrightarrow 5k = \ln 4 \Leftrightarrow \mathbf{k} = \frac{\ln 4}{5} \ (\cong \mathbf{0.277})$$

Therefore,

$$\mathbf{m = 4e^{-0.277t}}$$

(b) For t=3,

$$m = 4e^{-0.277 \times 3} = \mathbf{1.74}$$

(c) For m=2,

$$4e^{-0.277t} = 2$$

$$\Leftrightarrow e^{-0.277t} = 0.5 \Leftrightarrow -0.277t = \ln 0.5 \Leftrightarrow t = \frac{\ln 0.5}{-0.277} = \mathbf{2.50\ hours}$$

This time (the quantity reduces to a half) is known as **half-life time.**

♦ MORE EXPONENTIAL EQUATIONS **(mainly for HL)**

Let us look at some additional examples

EXAMPLE 5

Solve the equation

$$6^x 7^{x-1} = 3^{x-2}$$

Express the result in the form $\frac{\ln a}{\ln b}$

Solution

Although we can apply ln( ) on both sides and obtain

$$x\ln 6 + (x-1)\ln 7 = (x-2)\ln 3$$

which is a linear equation and can be solved as usual, I will recommend the quicker method: to simplify first the equation to the form $a^x=b$;

$$6^x 7^{x-1} = 3^{x-2} \Leftrightarrow \frac{6^x 7^x}{7} = \frac{3^x}{3^2}$$

$$\Leftrightarrow \frac{6^x 7^x}{3^x} = \frac{7}{3^2}$$

$$\Leftrightarrow 14^x = \frac{7}{9}$$

$$\text{(now apply ln)} \quad \Leftrightarrow x\ln 14 = \ln\frac{7}{9}$$

$$\Leftrightarrow \boxed{x = \frac{\ln(7/9)}{\ln 14}}$$

Notice:

Mind the following (common mistake)

$$A \pm B = C \quad \textbf{does not imply} \quad \log A \pm \log B = \log C$$

$$\textbf{it implies} \quad \log(A \pm B) = \log C$$

If an equation contains a sum of exponentials, it doesn't help to apply a logarithm, as $\log(a^x \pm b^x)$ cannot be simplified.

In such an equation we usually substitute an exponential by a new variable y.

EXAMPLE 6

Solve the equations:

(a) $6e^x+\dfrac{12}{e^x}=17$ (b) $6(10^{2x})+12=17(10^x)$

Solution

(a) Let $y=e^x$. Then

$$6y+\frac{12}{y}=17 \Leftrightarrow 6y^2-17y+12=0$$

There are two solutions: $y=\dfrac{3}{2}$, $y=\dfrac{4}{3}$

- For $y=\dfrac{3}{2}$, $e^x=\dfrac{3}{2} \Leftrightarrow \boxed{x=\ln\dfrac{3}{2}}$
- For $y=\dfrac{4}{3}$, $e^x=\dfrac{4}{3} \Leftrightarrow \boxed{x=\ln\dfrac{4}{3}}$

(b) Let $y=10^x$. Then

$$6y^2-17y+12=0$$

There are two solutions: $y=\dfrac{3}{2}$, $y=\dfrac{4}{3}$

- For $y=\dfrac{3}{2}$, $10^x=\dfrac{3}{2} \Leftrightarrow \boxed{x=\log\dfrac{3}{2}}$
- For $y=\dfrac{4}{3}$, $10^x=\dfrac{4}{3} \Leftrightarrow \boxed{x=\log\dfrac{4}{3}}$

EXAMPLE 7

Solve the system of equations

$$2(3^x)-3(2^y)=-22 \quad \text{and} \quad 5(3^x)+\frac{1}{2}(2^y)=9$$

Solution

Let $A=3^x$ and $B=2^y$. Then

$$2A-3B=-22 \quad \text{and} \quad 5A+\frac{1}{2}B=9$$

The solution is A=1, B=8. Hence,

$$3^x=1 \Leftrightarrow x=\log_3 1 \Leftrightarrow \boxed{x=0} \qquad \text{and} \qquad 2^y=8 \Leftrightarrow y=\log_2 8 \Leftrightarrow \boxed{y=3}$$

ONLY FOR

HL

2.11 POLYNOMIAL FUNCTIONS (for HL)

♦ DEFINITION

A **polynomial function**, or simply a **polynomial** is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, all $a_i \in R$ and $n \in N$.

The highest power of x is called **degree** of the polynomial. We write

$$\deg f(x) = n$$

For example

$f(x) = 5x^4+3x^2-7x+2$	$\deg f(x)=4$
$g(x) = x^5-2x^3+5x-7$	$\deg g(x)=5$

We also use the following terminology for polynomials of a particular degree:

$\deg f(x) = 0$	$f(x)=a$	(constant function)
$\deg f(x) = 1$	$f(x)=ax+b$	(linear function)
$\deg f(x) = 2$	$f(x)=ax^2+bx+c$	(quadratic function)
$\deg f(x) = 3$	$f(x)=ax^3+bx^2+cx+d$	(cubic function)
$\deg f(x) = 4$	$f(x)=ax^4+bx^3+cx^2+dx+e$	(quartic function)

Notice though that the degree of the zero polynomial $f(x)=0$ is undefined[‡]

[‡] In some books the degree of the zero polynomial is defined to be -1 or $-\infty$.

♦ ADDITION AND MULTIPLICATION OF POLYNOMIALS

When we add or multiply polynomials the result is also a polynomial. We perform these operations in the obvious way!.

EXAMPLE 1

Let $f(x) = 3x^2-2x+5$ and $g(x)= 2x^3-7x+1$

Then

$$f(x)+g(x) = (3x^2-2x+5)+(2x^3-7x+1) = 2x^3+3x^2-9x+6$$

$$f(x)g(x) = (3x^2-2x+5)(2x^3-7x+1)$$

$$= 6x^5-21x^3+3x^2-4x^4+14x^2-2x+10x^3-35x+5$$

$$= 6x^5-4x^4-11x^3+17x^2-37x+5$$

Here, deg $f(x)=2$, deg $g(x)=3$ while

$deg[f(x)+g(x)]=3$ $deg[f(x)g(x)]=5$

In general

If deg $f(x)=n$, def $g(x)=m$ with $n>m$ (i.e max degree = n)

$deg[f(x)+g(x)] = n$ $deg[f(x)g(x)] = n+m$

If deg $f(x)=n$, def $g(x)=n$ (equal degrees)

$deg[f(x)+g(x)] \le n$ $deg[f(x)g(x)] = 2n$

NOTICE

Look at the last line: it is not $deg[f(x)+g(x)] = n$ since $f(x)$ and $g(x)$ may have opposite leading coefficients; for example

$f(x)=3x^2+7x$, $g(x)= -3x^2+2$ $(n=2)$

Then

$f(x)+g(x)= (3x^2+7x)+(-3x^2+2)=-7x+2$ deg=1<2

$f(x)g(x)= (3x^2+7x)(-3x^2+2)=-9x^4-21x^3+6x^2+14x$ deg=4

♦ DIVISION OF POLYNOMIALS

Since $(2x)(3x+1)=6x^2+2x$, we can derive that

$$\frac{6x^2+2x}{2x}=3x+1$$

But how can we divide polynomials in general?

REMEMBER When we divide two integers, say $a:b$ or $\frac{a}{b}$, we obtain

$$a=bq+r$$

where q=quotient and r=remainder ($0\le r<b$)

For example 23:5 gives quotient=4 and remainder=2, so

$$23=5\cdot 4+2$$

The same applies for polynomials

> If we divide two polynomials, $f(x)$ by $g(x)$, we obtain two polynomials
>
> the **quotient** $q(x)$
>
> the **remainder** $r(x)$
>
> such that
>
> $$f(x)=g(x)q(x)+r(x)$$
>
> where
>
> $r(x)=0$ or $\deg r(x) < \deg g(x)$

Let us describe the process of **long division** by using an example.

EXAMPLE 2

We will divide $f(x)=2x^3-4x^2+5x-1$ by $g(x)= x^2+3x+1$

[As the way of dividing varies in different countries we present two methods: the left to the right and the right to the left division]

left to the right method	instructions	right to the left method
$2x^3-4x^2+5x-1 \mid x^2+3x+1$	**step 1**	$x^2+3x+1\overline{)2x^3-4x^2+5x-1}$
$2x^3-4x^2+5x-1 \mid x^2+3x+1$ $\mid 2x$	**step 2** divide $2x^3:x^2=2x$	$2x$ $x^2+3x+1\overline{)2x^3-4x^2+5x-1}$
$2x^3-4x^2+5x-1 \mid x^2+3x+1$ $2x^3+6x^2+2x \mid 2x$	**step 3** multiply $2x$ by $g(x)$	$2x$ $x^2+3x+1\overline{)2x^3-4x^2+5x-1}$ $2x^3+6x^2+2x$
$2x^3-4x^2+5x-1 \mid x^2+3x+1$ $-$ $\underline{2x^3+6x^2+2x} \mid 2x$ $-10x^2+3x-1$	**step 4** subtract	$2x$ $x^2+3x+1\overline{)2x^3-4x^2+5x-1}$ $-$ $\underline{2x^3+6x^2+2x}$ $-10x^2+3x-1$

repeat with $-10x^2+3x-1$ and x^2+3x+1

left to the right method	instructions	right to the left method
$2x^3-4x^2+5x-1 \mid x^2+3x+1$ $-$ $\underline{2x^3+6x^2+2x} \mid 2x-10$ $-10x^2+3x-1$	**step 5** divide $-10x^2:x^2=-10$	$2x-10$ $x^2+3x+1\overline{)2x^3-4x^2+5x-1}$ $-$ $\underline{2x^3+6x^2+2x}$ $-10x^2+3x-1$
$2x^3-4x^2+5x-1 \mid x^2+3x+1$ $-$ $\underline{2x^3+6x^2+2x} \mid 2x-10$ $-10x^2+3x-1$ $-10x^2-30x-10$	**step 6** multiply -10 by $g(x)$	$2x-10$ $x^2+3x+1\overline{)2x^3-4x^2+5x-1}$ $-$ $\underline{2x^3+6x^2+2x}$ $-10x^2+3x-1$ $-10x^2-30x-10$
$2x^3-4x^2+5x-1 \mid x^2+3x+1$ $-$ $\underline{2x^3+6x^2+2x} \mid 2x-10$ $-10x^2+3x-1$ $-$ $\underline{-10x^2-30x-10}$ $33x+9$	**step 7** subtract	$2x-10$ $x^2+3x+1\overline{)2x^3-4x^2+5x-1}$ $-$ $\underline{2x^3+6x^2+2x}$ $-10x^2+3x-1$ $-$ $\underline{-10x^2-30x-10}$ $33x+9$

Hence, $q(x)=2x-10$, $r(x)=33x+9$ and

$$2x^3-4x^2+5x-1=(x^2+3x+1)(2x-10)+(33x+9)$$

NOTICE

In number theory, the division a=bq+r also gives $\frac{a}{b}=q+\frac{r}{b}$

For example $\frac{23}{5}=4+\frac{3}{5}$.

Similarly, the division of polynomials gives

$$\frac{f(x)}{g(x)}=q(x)+\frac{r(x)}{g(x)}$$

In our example

$$\frac{2x^3-4x^2+5x-1}{x^2+3x+1} = 2x-10 + \frac{33x+9}{x^2+3x+1}$$

If r(x)=0, then f(x)=g(x)q(x). Then we say that

f(x) is divisible by g(x)
or g(x) divides exactly f(x)
or g(x) is a factor of f(x)

EXAMPLE 3

Let us divide $f(x)=2x^3+2x^2-x-1$ by $g(x)=2x^2-1$

We present the long division in one step:

left to the right method		right to the left method
$\begin{array}{r\|l} 2x^3+2x^2-x-1 & 2x^2-1 \\ -\;\; 2x^3 \qquad\quad -x & x+1 \\ \hline 2x^2 \quad -1 & \\ -\;\; 2x^2 \quad -1 & \\ \hline 0 & \end{array}$	notice that the remainder r(x) is 0	$\begin{array}{rl} & x+1 \\ \hline 2x^2-1 \,) & 2x^3+2x^2-x-1 \\ - & 2x^3 \qquad\quad -x \\ \hline & 2x^2 \quad -1 \\ - & 2x^2 \quad -1 \\ \hline & 0 \end{array}$

Therefore,

$$2x^3+2x^2-x-1 = (2x^2-1)(x+1)$$

or otherwise

$$\frac{2x^3+2x^2-x-1}{2x^2-1} = x+1$$

♦ THE FACTOR THEOREM

$$f(x) \text{ is divisible by } (x-\mathbf{a}) \quad \Leftrightarrow \quad f(\mathbf{a}) = 0$$

or otherwise

$$(x-\mathbf{a}) \text{ is a factor of } f(x) \quad \Leftrightarrow \quad \mathbf{a} \text{ is a root of } f(x)$$

Proof

($\Rightarrow$) If $f(x)$ is divisible by $(x-\mathbf{a})$ then $f(x)=(x-\mathbf{a})q(x)$ for some $q(x)$

then $f(\mathbf{a})=0$

($\Leftarrow$) Let $f(\mathbf{a})=0$. We divide $f(x)$ by $(x-\mathbf{a})$ and obtain

$$f(x)=(x-\mathbf{a})q(x)+r \qquad (r \text{ must be constant})$$

But then, $f(\mathbf{a})=0 \Rightarrow r=0$. That is

$$f(x)=(x-\mathbf{a})q(x)$$

ie $f(x)$ is divisible by $(x-\mathbf{a})$

♦ THE REMAINDER THEOREM

When $f(x)$ is divided be $(x-\mathbf{a})$ the remainder is $f(\mathbf{a})$

Proof

We divide $f(x)$ by $(x-\mathbf{a})$. Suppose $f(x)=(x-\mathbf{a})q(x)+r$. Then $f(\mathbf{a}) = r$

EXAMPLE 4

Let $f(x)=x^3+x^2-x+2$. Find the remainder when $f(x)$ is divided by

$$(x-1),\ (x+1),\ (x-2),\ (x+2)$$

$f(1)=3$, hence the remainder when $f(x)$ is divided by $(x-1)$ is 3

$f(-1)=3$, hence the remainder when $f(x)$ is divided by $(x+1)$ is 3

$f(2)=12$, hence the remainder when $f(x)$ is divided by $(x-2)$ is 12

$f(-2)=0$, hence $f(x)$ is divisible by $(x+2)$, ie $(x+2)$ is a factor of $f(x)$

EXAMPLE 5 (the factor theorem for quadratics)

Let $f(x)=ax^2+bx+c$ be a quadratic with two roots p and q, that is

$$f(p)=0 \text{ and } f(q)=0$$

Then $f(x)$ is divisible by $(x-p)$ and $(x-q)$. Indeed, we know that

$$f(x) = a(x-p)(x-q)$$

EXAMPLE 6

Solve the equation $x^3+x^2-x+2=0$.

If we know one root then we may use division to find the remaining roots.

In Example 4, we saw that -2 is a root. Hence $(x+2)$ is a factor.

We divide x^3+x^2-x+2 by $(x+2)$ and get $q(x)= x^2-x+1$ (left as exercise)

The equation takes the form

$$(x+2)(x^2-x+1)=0$$

However, the quadratic x^2-x+1 has no real roots, so our equation has only one root, ie $x=-2$.

EXAMPLE 7

Let $f(x) = x^3-6x^2+11x-6$. Solve the equation $f(x) = 0$.

Solution

We can easily observe that $x=1$ is a solution since $f(1)=0$.

We divide $f(x)$ by the factor $(x-1)$ and find the quotient (x^2-5x+6).

(it is left as exercise!)

The equation takes the form

$$(x-1)(x^2-5x+6)=0$$

But the quadratic (x^2-5x+6) has two roots, $x=2$ and $x=3$. Thus the equation has three solutions, namely 1,2 and 3.

Notice also that the full factorization of the cubic equation gives

$$x^3-6x^2+11x-6=0$$

$$\Leftrightarrow (x-1)(x-2)(x-3)=0$$

$$\Leftrightarrow x=1 \text{ or } x=2 \text{ or } x=3$$

REMARK (useful for guessing roots)

Consider the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad \text{where } a_i \in Z$$

We may look for roots among the following

Potential integer roots: $\pm$ factors of a_0

Potential rational roots: $\pm \dfrac{\text{factor of } a_0}{\text{factor of } a_n}$

EXAMPLE 8

Let $f(x) = 2x^3-7x^2-17x+10$.

Potential integer roots: $\pm1, \pm2, \pm5, \pm10$

Potential rational roots: $\pm\dfrac{1}{2}, \pm\dfrac{5}{2}$ ($\pm\dfrac{2}{2}$ and $\pm\dfrac{10}{2}$ are integers)

Among those, we can verify that

$$f(-2)=0, \quad f(5)=0, \quad f(1/2)=0.$$

We could also find the first root, say $x=-2$, and then divide $f(x)$ by the factor $(x+2)$ to obtain the remaining quadratic factor.

Indeed, the long division will give

$$2x^3-7x^2-17x+10 = (x+2)(2x^2-11x+5)$$

and the quadratic factor has two roots, $x=5$ and $x=1/2$.

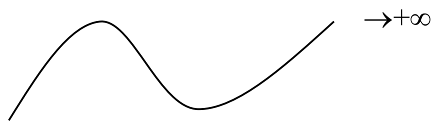
♦ THE GRAPH OF A CUBIC FUNCTION

Consider a cubic function

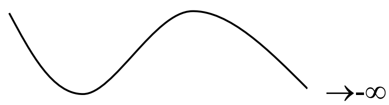
$$f(x) = ax^3+bx^2+cx+d$$

The leading coefficient a determines the behavior of the graph towards the right end:

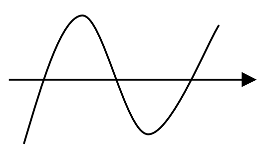
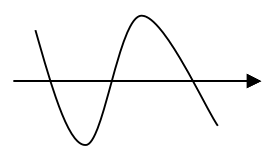
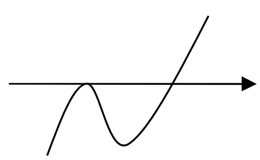
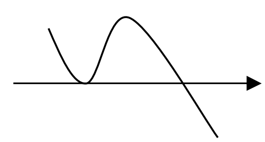
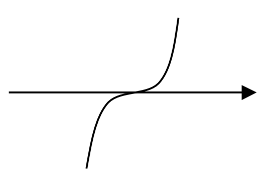
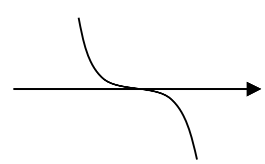
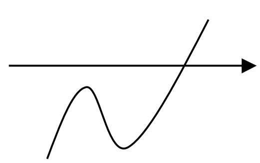
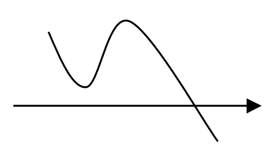
- If $a>0$, for large values of x, $f(x)\to+\infty$ and the graph looks like



- If $a<0$, for large values of x, $f(x)\to-\infty$ and the graph looks like



The factorization of the cubic function determines the position of the graph in relation to the x-axis:

f(x)	a>0	a<0
$a(x-r_1)(x-r_2)(x-r_3)$		
$a(x-r_1)^2(x-r_2)$		
$a(x-r_1)^3$		
$a(x-r_1)(x^2-px+qx)$ *irreducible*		

2.12 SUM AND PRODUCT OF ROOTS (for HL)

The fundamental theorem of algebra said that a polynomial of degree n has n complex roots. Here, we denote by

$S = r_1+r_2+\cdots+r_n$ the sum of the roots

$P = r_1r_2\cdots r_n$ the product of the roots

♦ QUADRATIC FUNCTIONS

We have seen that for a quadratic function

$$f(x) = ax^2+bx+c \qquad (1)$$

there are always two complex roots r_1 and r_2.

We may have

- r_1,r_2 real, $r_1\neq r_2$
- r_1,r_2 real, $r_1=r_2$
- r_1,r_2 conjugate complex roots

In any case, the factorization over C is

$$f(x) = a(x-r_1)(x-r_2)$$

Thus

$$f(x) = a(x^2-r_1x-r_2x+r_1r_2)$$

$$= ax^2-a(r_1+r_2)x+ar_1r_2x \qquad (2)$$

By comparing (1) and (2) we obtain

$b=-a(r_1+r_2)$ and $c= ar_1r_2$

and finally

$$\boxed{S= r_1+r_2=-\frac{b}{a}} \qquad \boxed{P = r_1r_2=\frac{c}{a}}$$

These relations are known as **Vieta formulae.**

- CUBIC FUNCTIONS

Consider now the cubic function

$$f(x) = ax^3+bx^2+cx+d \qquad (1)$$

According to the fundamental theorem of algebra the factorization of f(x) over C is

$$f(x) = a(x-r_1)(x-r_2)(x-r_3)$$

The constant term is

$$-ar_1r_2r_3$$

Thus, by (1)

$$d= -ar_1r_2r_3 \Rightarrow r_1r_2r_3 = -\frac{d}{a}$$

The coefficient of x^2 is

$$-ar_3-ar_2-ar_1 = -a(r_1+r_2+r_3)$$

Thus, by (1)

$$b= -a(r_1+r_2+r_3) \Rightarrow r_1+r_2+r_3 = -\frac{b}{a}$$

Hence,

$$\boxed{S= r_1+r_2+r_3=-\frac{b}{a}} \qquad \boxed{P = r_1r_2r_3=-\frac{d}{a}}$$

Notice

Usually a cubic function is expressed in the form

$$f(x) = a_3x^3+a_2x^2+a_1x+a_0$$

The Vieta formulae take the form

$$\boxed{S= r_1+r_2+r_3=-\frac{a_2}{a_3}} \qquad \boxed{P = r_1r_2r_3=-\frac{a_0}{a_3}}$$

♦ THE GENERAL CASE

Consider the general form of a polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \qquad (1)$$

According to the fundamental theorem of algebra the factorization of f(x) over C is

$$f(x) = a_n(x-r_1)(x-r_2)\ldots(x-r_n)$$

The constant term is

$$(-1)^n a_n r_1 r_2 \cdots r_n$$

Thus, by (1)

$$a_0 = (-1)^n a_n r_1 r_2 \cdots r_n \Rightarrow r_1 r_2 \cdots r_n = (-1)^n \frac{a_0}{a_n}$$

The coefficient of x^{n-1} is

$$-a_n r_1 - a_n r_2 - \cdots - a_n r_n = -a_n(r_1 + r_2 + \cdots + r_n)$$

Thus, by (1)

$$a_{n-1} = -a_n(r_1 + r_2 + \cdots + r_n) \Rightarrow r_1 + r_2 + \cdots + r_n = -\frac{a_{n-1}}{a_n}$$

Hence,

$$\boxed{S = r_1 + r_2 + \cdots + r_n = -\frac{a_{n-1}}{a_n}} \qquad \boxed{P = r_1 r_2 \cdots r_n = (-1)^n \frac{a_0}{a_n}}$$

NOTICE (just for information!)

By considering the coefficients of x^{n-2}, x^{n-3} etc we similarly obtain

The sum S_2 of all possible pairs $r_i r_j$ is $\dfrac{a_{n-2}}{a_n}$

The sum S_3 of all possible triples $r_i r_j r_k$ is $-\dfrac{a_{n-3}}{a_n}$

and so on.

EXAMPLE 1

Let $f(x)=2x^3+ax^2+bx+c$

The sum of the roots is 3.5, the product of the roots is -5 and the polynomial is divided by (x+2). Find the values of a,b and c.

Solution

$$S=-\frac{a_2}{a_3} \Rightarrow -\frac{a}{2}=3.5 \Rightarrow a = -7$$

$$P=(-1)^3\frac{a_0}{a_3} \Rightarrow -\frac{c}{2}=-5 \Rightarrow c = 10$$

By the factor theorem

$$\begin{aligned} f(-2)=0 \quad &\Rightarrow -16+4a-2b+c = 0 \\ &\Rightarrow -16-28-2b+10 = 0 \\ &\Rightarrow b=-17 \end{aligned}$$

EXAMPLE 2

Let $f(x)=ax^4-10x^3+bx+c$

The sum of the roots is 2, the product of the roots is -5. and the polynomial is divided by (x-1). Find the values of a,b and c.

Solution

$$S=-\frac{a_3}{a_4} \Rightarrow \frac{10}{a}=2 \Rightarrow a = 5$$

$$P=(-1)^4\frac{a_0}{a_4} \Rightarrow \frac{c}{a}=-5 \Rightarrow c = -25$$

By the factor theorem

$$\begin{aligned} f(1)=0 \quad &\Rightarrow a-10+b+c = 0 \\ &\Rightarrow 5-10+b-25 = 0 \\ &\Rightarrow b=30 \end{aligned}$$

2.13 RATIONAL FUNCTIONS – PARTIAL FRACTIONS (for HL)

♦ RATIONAL FUNCTIONS

A **rational** function has the form

$$\boxed{f(x)=\frac{p(x)}{q(x)}}$$

where p(x) and q(x) are polynomials.

For example

$$f(x)=\frac{2x-5}{x^2-4x+3}$$

We have already seen rational functions of the form

$$f(x)=\frac{Ax+B}{Cx+D}$$

and their asymptotes.

Again, for the asymptotes of a rational function in general, we work as follows:

1) Vertical Asymptotes: **x=a**

At points where the function is not defined. Thus, we solve the equation q(x)=0.
For example,

$$f(x)=\frac{2x-5}{x^2-4x+3}$$

we solve

$$x^2-4x+3=0 \Leftrightarrow x=1 \text{ or } x=3$$

Hence

$$\boxed{\text{The lines } x=1 \text{ and } x=3 \text{ are vertical asymptotes}}$$

2) Horizontal Asymptotes: y=b

We only consider the leading coefficients of p(x) and q(x).

We distinguish three cases:

- $\deg p(x) = \deg q(x)$, $y = \dfrac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}$
- $\deg p(x) < \deg q(x)$, $y=0$
- $\deg p(x) > \deg q(x)$, there is no horizontal asymptote

For example,

$f(x) = \dfrac{4x^2-3x+1}{2x^2+7x-6}$ The line $y=2$ is a horizontal asymptote

$f(x) = \dfrac{3x+1}{2x^2+7x-6}$ The line $y=0$ is a horizontal asymptote

$f(x) = \dfrac{4x^2-3x+1}{2x-6}$ There is no horizontal asymptote

Notice also that,

if $f(x) = \dfrac{p(x)}{q(x)}$ has a horizontal asymptote $y=b$,

then $g(x) = \dfrac{p(x)}{q(x)} + c$ has a horizontal asymptote $y=b+c$

as $g(x)$ is the function $f(x)$ shifted c units up.

EXAMPLE 1

Function	Vertical Asymptotes (denominator = 0)	Horizontal Asymptote (divide leading coefficients)
$f(x) = \dfrac{7x^2-5x+1}{x^2-3x+2}$	$x=1$, $x=2$	$y=7$
$f(x) = \dfrac{7x^2-5x+1}{2x^2-6x+4}$	$x=1$, $x=2$	$y=7/2$
$f(x) = \dfrac{-5x+1}{x^2-3x+2}$	$x=1$, $x=2$	$y=0$
$f(x) = \dfrac{-5x+1}{x^2-3x+2}+8$	$x=1$, $x=2$	$y=8$
$f(x) = \dfrac{7x^2-5x+1}{-3x+6}$	$x=2$	none

EXAMPLE 2

Find the intercepts the domain and the asymptotes of the function

$$f(x)=\frac{x^2-6x+8}{x^2-4x+3}$$

Use your GDC to sketch the graph of f(x) and hence find its range.

Solution

It is

$$f(x)=\frac{x^2-6x+8}{x^2-4x+3}=\frac{(x-2)(x-4)}{(x-1)(x-3)}$$

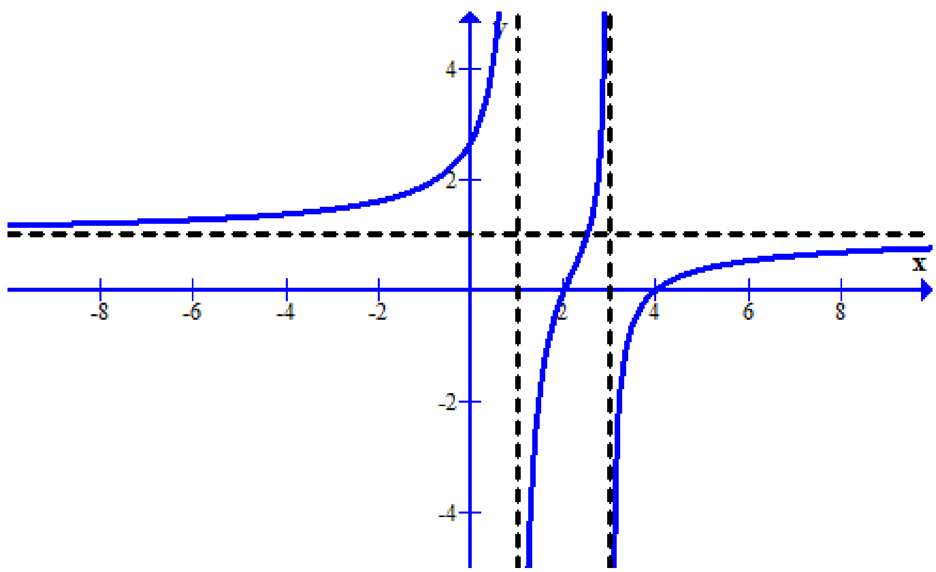
x-intercepts (or roots): x=2, x=4

y-intercept: For x=0, y=8/3

Domain: $x\neq 1$, $x\neq 3$

VA: x=1, x=3

HA: y=1



According to the graph the **range** is $y\in R$

Notice: that the value of the asymptote y=1 is not excluded from the range.

EXAMPLE 3

Find the intercepts the domain and the asymptotes of the function

$$f(x)=\frac{x^2-3x-4}{x^2-4x+3}$$

Use your GDC to sketch the graph of f(x) and hence find its range.

Solution

It is

$$f(x)=\frac{x^2-3x-4}{x^2-4x+3}=\frac{(x+1)(x-4)}{(x-1)(x-3)}$$

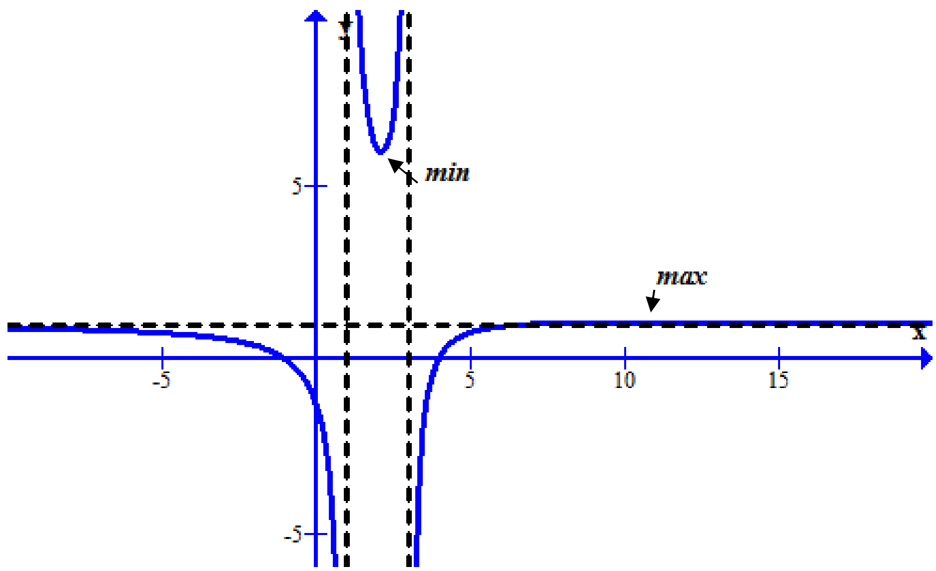
x-intercepts (or roots): $x=-1$, $x=4$

y-intercept: For $x=0$, $y=-4/3$

Domain: $x\neq1$, $x\neq3$

VA: $x=1$, $x=3$

HA: $y=1$



By using the GDC, we find that:

there is a **local min** at (2.1,5.95) and a **local max** at (11.9,1.05)

According to the graph the **range** is $y\in\,]-\infty,1.05]\cup[5.95,+\infty[$

Later on we will be able to find the local min and the local max without a GDC, by using derivatives!

♦ OBLIQUE ASYMPTOTES

We have seen that for a rational function of the form

$$f(x)=\frac{ax^2+bx+c}{dx+e}$$

there is no horizontal asymptote. But there is an **oblique asymptote.** If the division of the two polynomials gives the quotient $q(x)=Ax+B$ and the remainder r, then

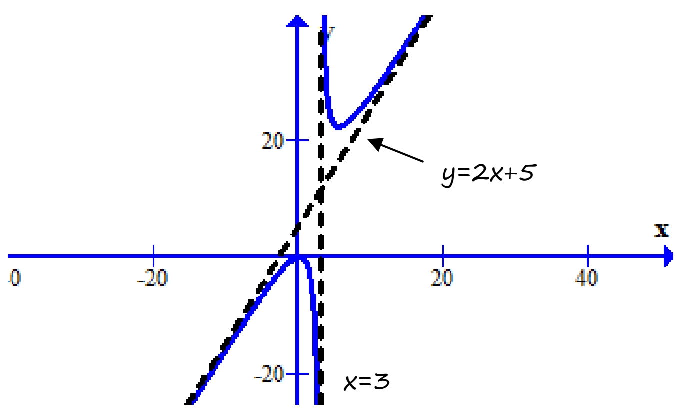
$$f(x)=\frac{ax^2+bx+c}{dx+e}=(Ax+B)+\frac{r}{dx+e}$$

As x→±∞, the last fraction tends to 0 and thus f(x) → Ax+B.
That is the graph of y=f(x) approaches the oblique line **y=Ax+B**.

EXAMPLE 4

$$f(x)=\frac{4x^2-2x+1}{2x-6}$$

- The vertical asymptote is **x=3**.
- There is no horizontal asymptote.
- As $4x^2-2x+1$ divided by $2x-6$ gives q(x)=2x+5 (and r=31) the oblique asymptote is **y=2x+5**.



Justification: $f(x)=\frac{4x^2-2x+1}{2x-6}=2x+5+\frac{31}{2x-6}\to 2x+5$ as $x\to\pm\infty$

__Notice__. The same situation occurs for a rational function $f(x)=\frac{p(x)}{q(x)}$ where deg q(x) is **one less** than deg p(x).

♦ PARTIAL FRACTIONS (only the easiest case)

We only consider rational functions of the form

$$f(x)=\frac{a'}{ax^2+bx+c} \qquad \text{and} \qquad f(x)=\frac{a'x+b'}{ax^2+bx+c}$$

If the denominator ax^2+bx+c has two roots, say $x=r_1$ and $x=r_2$, we can express the functions in the form of **partial fractions**:

$$f(x)=\frac{A}{x-r_1}+\frac{B}{x-r_2}$$

We will demonstrate the way by using an example.

EXAMPLE 5

$$f(x)=\frac{3x-5}{x^2-4x+3}.$$

The denominator has two roots: x=1, x=3. Thus

$$f(x)=\frac{A}{x-1}+\frac{B}{x-3}$$

Method 1

$$\frac{A}{x-1}+\frac{B}{x-3}=\frac{A(x-3)+B(x-1)}{(x-1)(x-3)}=\frac{(A+B)-(3A+B)}{(x-1)(x-3)}$$

Comparing with the numerator of the original function

$$A+B = 3$$

$$3A+B = 5$$

The solution of the system gives **A=1** and **B=2**.

Method 2

$$\frac{3x-5}{x^2-4x+3}=\frac{A}{x-1}+\frac{B}{x-3}$$

Multiply by (x-1)(x-3):

$$A(x-3)+B(x-1) = 3x-5$$

For x=3 we obtain: $2B=4 \Rightarrow \mathbf{B=2}$

For x=1 we obtain: $-2A=-2 \Rightarrow \mathbf{A=1}$

Therefore,

$$f(x)=\frac{1}{x-1}+\frac{2}{x-3}$$

2.14 POLYNOMIAL AND RATIONAL INEQUALITIES (for HL)

Let f(x) be a polynomial. By factorizing f(x) we can easily sketch a graph and thus solve the **polynomial inequalities**

$f(x)>0$ $f(x)<0$ $f(x)\geq 0$ $f(x)\leq 0$

When we factorize f(x) we may find

- <u>linear factors</u> of the form $(x-a)$
- <u>irreducible quadratic factors</u> of the form (x^2+bx+c) [with $\Delta<0$]

Only the roots of the linear factors affect the inequality. We can sketch a graph of the polynomial, having in mind that

in a **single** root the graph **crosses** the x-axis

in a **double** root the graph just **touches** the x-axis

In general, for a root which is repeated **n times**

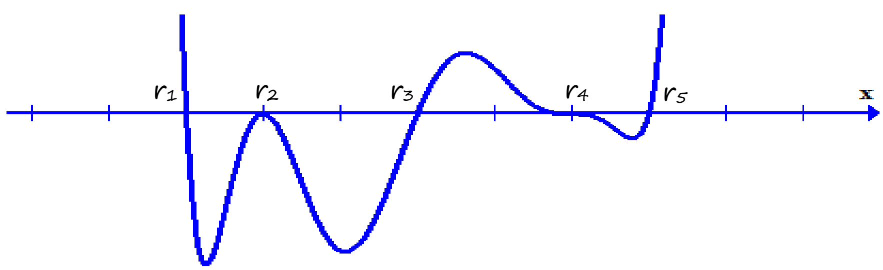
if n is **odd** it behaves as a single root (change of sign)

if n is **even** it behaves as a double root (no change of sign)

For example, if

$$f(x) = a(x-r_1)(x-r_2)^2(x-r_3)(x-r_4)^3(x-r_5)$$

and $a>0$ the graph looks like



The sign of a shows the behavior of the curve towards $+\infty$.

Now the signs of the function are shown in the table below

x	$-\infty$	r_1		r_2		r_3		r_4		r_5		$+\infty$
f(x)	+	0	−	0	−	0	+	0	−	0	+	

EXAMPLE 1

Solve the inequality

$$2x^3-7x^2-17x+10>0$$

Solution

We have seen earlier that this cubic function has three single roots, -2, 0.5 and 5. Thus the inequality becomes

$$2(x+2)(x-0.5)(x-5)>0$$

We obtain

x	$-\infty$		-2		0.5		5		$+\infty$
$f(x)$		$-$	0	$+$	0	$-$	0	$+$	

Hence, the solution is $x\in\ ]-2,0.5[\ \cup\ ]5,+\infty[$

EXAMPLE 2

Solve the inequalities

(a) $3(x-1)^2(x-5)>0$

(b) $3(x-1)^2(x-5)\geq 0$

(c) $3(x-1)^2(x-5)(x^2+1)\geq 0$

Solution

The quadratic factor x^2+1 in (c) has no real roots (irreducible). It is always positive so it doesn't affect the sign of the polynomial.

We obtain

x	$-\infty$		1		5		$+\infty$
$f(x)$		$-$	0	$-$	0	$+$	

Hence, the solutions are

(a) $x>5$

(b) $x=1$ or $x\geq 5$

(c) $x=1$ or $x\geq 5$

For a rational function of the form $\frac{f(x)}{g(x)}$ remember that

$$\frac{f(x)}{g(x)} > 0 \Leftrightarrow f(x)g(x) > 0$$

Therefore, by factorizing f(x) and g(x) we can think of the polynomial f(x)g(x) and thus solve the **rational inequalities**

$$\frac{f(x)}{g(x)}>0 \qquad \frac{f(x)}{g(x)}<0 \qquad \frac{f(x)}{g(x)}\geq 0 \qquad \frac{f(x)}{g(x)}\leq 0$$

In case the inequality is either ≥ or ≤, remember to include the roots of the numerator f(x) and exclude the roots of the denominator g(x).

EXAMPLE 3

Solve the inequalities

$$(a)\ \frac{(x-1)(x-3)^2}{(x-2)(x^2+x+1)} \leq 0, \qquad (b)\ \frac{(x-1)(x^2+x+1)}{(x-3)^2(x-2)} \geq 0$$

(factorization is already given).

Solution

Notice that the same factors appear in both inequalities. If we multiply all factors we obtain the polynomial

$$(x-1)(x-2)(x-3)^2(x^2+x+1)$$

We obtain

x	$-\infty$		1		2		3		$+\infty$
$f(x)$		$+$	0	$-$	0	$+$	0	$+$	

Hence, the solutions are

(a) $x \in [1,2[\cup \{3\}$

[we exclude the root x=2 of the denominator]

(b) $x \in]-\infty,1] \cup]2,3[\cup]3,+\infty[$.

[we exclude the roots x=2 and x=3 of the denominator]

Mind the difference between equations and inequalities.

EXAMPLE 4

Solve (a) $\frac{x+1}{x-2}=x-3$ (b) $\frac{x+1}{x-2}\geq x-3$

(a) $\frac{x+1}{x-2}=x-3 \Leftrightarrow x+1=(x-2)(x-3)$

$\Leftrightarrow x+1=x^2-5x+6$

$\Leftrightarrow x^2-6x+5=0$

$\Leftrightarrow x=1 \text{ or } x=5$

(b) we present two solutions, one without GDC, one by GDC.

Solution without GDC (analytical)

Now we **cannot** cross multiply! We move everything to the LHS:

$$\frac{x+1}{x-2}-(x-3)\geq 0 \Leftrightarrow \frac{x+1-(x-3)(x-2)}{x-2}\geq 0$$

$$\Leftrightarrow \frac{x+1-x^2+5x-6}{x-2}\geq 0$$

$$\Leftrightarrow \frac{-x^2+6x-5}{x-2}\geq 0$$

$$\Leftrightarrow \frac{-(x-1)(x-5)}{x-2}\geq 0$$

We obtain

x	$-\infty$		1		2		5		$+\infty$
f(x)		+	0	−	‖	+	0	−	

Hence, the solution is $x\in\ ]-\infty,1] \cup\ ]2,5]$

Solution by GDC

We sketch the graph of $f(x)=\frac{x+1}{x-2}-(x-3)$

We construct a table as above with **all the critical values:**

- the roots of the function: $x=1$, $x=5$
- the values where the function is not defined: $x=2$

Based on the graph we complete the signs on the table as above

---

2.15 MODULUS EQUATIONS AND INEQUALITIES (for HL)

Remember that, if a is a positive constant,

$$|x| = a \Leftrightarrow x=a \text{ or } x=-a$$

$$|x| < a \Leftrightarrow -a<x<a$$

$$|x| > a \Leftrightarrow x<-a \text{ or } x>a$$

In this way we can solve easy equations or inequalities involving only one absolute value.

EXAMPLE 1

(a) $|2x-3|=5 \Leftrightarrow 2x-3=5 \text{ or } 2x-3=-5$

$\Leftrightarrow 2x=8 \text{ or } 2x=-2$

$\Leftrightarrow x=4 \text{ or } x=-1$

(b) $|2x-3|<5 \Leftrightarrow -5<2x-3<5$

$\Leftrightarrow -2<2x<8$

$\Leftrightarrow -1<x<4$

(c) $|2x-3|>5 \Leftrightarrow 2x-3<-5 \text{ or } 2x-3>5$

$\Leftrightarrow 2x<-2 \text{ or } 2x>8$

$\Leftrightarrow x<-1 \text{ or } x>4$

EXAMPLE 2

(a) $\left|\frac{x-1}{x-2}\right|=5 \Leftrightarrow \frac{x-1}{x-2}=5 \text{ or } \frac{x-1}{x-2}=-5$

$\Leftrightarrow x-1=5x-10 \text{ or } x-1=-5x+10$

$\Leftrightarrow 4x=9 \text{ or } 6x=11$

$\Leftrightarrow x=9/4 \text{ or } x=11/6$

The inequality here is more complicated.

(b) $\left|\frac{x-1}{x-2}\right|<5 \quad \Leftrightarrow \quad -5<\frac{x-1}{x-2}<5$

We solve separately,

$$\frac{x-1}{x-2}<5 \Leftrightarrow \frac{x-1}{x-2}-5<0 \Leftrightarrow \frac{x-1-5x+10}{x-2}<0 \Leftrightarrow \frac{-4x+9}{x-2}<0$$

We obtain

x	$-\infty$	2		9/4		$+\infty$
f(x)	$-$	0	$+$	0	$-$	

Thus x<2 or x> 9/4 (1)

Similarly

$$\frac{x-1}{x-2}>-5 \Leftrightarrow \frac{x-1}{x-2}+5>0 \Leftrightarrow \frac{x-1+5x-10}{x-2}>0 \Leftrightarrow \frac{6x-11}{x-2}>0$$

We obtain

x	$-\infty$	11/6		2		$+\infty$
f(x)	$+$	0	$-$	0	$+$	

Thus x<11/6 or x> 2 (2)

(1) and (2) together give: x<11/6 or x>9/4

Alternative solution:

Since both sides of the inequality are positive

$$\left|\frac{x-1}{x-2}\right|<5 \Leftrightarrow \left|\frac{x-1}{x-2}\right|^2<5^2 \Leftrightarrow (x-1)^2<25(x-2)^2$$

$$\Leftrightarrow \quad x^2-2x+1<25(x^2-4x+4)$$

$$\Leftrightarrow \quad 24x^2-98x+99>0$$

$$\Leftrightarrow \quad 24(x-9/4)(x-11/6)>0$$

$$\Leftrightarrow \quad x<11/6 \text{ or } x>9/4$$

Things become even more complicated when more that one absolute values are involved or there is a variable outside the absolute value. We have to find first the zeros of the absolute values and investigate the different cases.

EXAMPLE 3

(a) $|x-1|=3x+2$

We find the zeros: $x-1=0 \Leftrightarrow x=1$

CASE 1: $x<1$

$|x-1|=3x+2 \Leftrightarrow -x+1 = 3x+2 \Leftrightarrow 4x=-1 \Leftrightarrow x=-1/4$ (accepted)

CASE 2: $x>1$

$|x-1|=3x+2 \Leftrightarrow x-1 = 3x+2 \Leftrightarrow 2x=-3 \Leftrightarrow x=-3/2$ (rejected)

Final answer (the union of the two cases): $x=-1/4$

(b) $|x-1|<3x+2$

We find the zeros: $x-1=0 \Leftrightarrow x=1$

CASE 1: $x<1$

$|x-1|<3x+2 \Leftrightarrow -x+1 < 3x+2 \Leftrightarrow 4x>-1 \Leftrightarrow x>-1/4$

Thus $x>-1/4$

CASE 2: $x>1$

$|x-1|<3x+2 \Leftrightarrow x-1 < 3x+2 \Leftrightarrow 2x>-3 \Leftrightarrow x>-3/2$ (rejected)

Thus $x>1$

Final answer (the union of the two cases): $x>-1/4$

Alternative graphical solution for $|x-1|-3x-2<0$

We can easily sketch the graph of $f(x)=|x-1|-3x-2$

We know that the graph consists of linear parts.

For $x=1$, $f(1)=-5$
For $x=0$ (before 1): $f(0)=-1$
For $x=2$ (after 1): $f(2)=-7$

We sketch the graph and observe which part is negative.

EXAMPLE 4

$$|x-1|+|x-2|=x$$

We find the zeros of the absolute values: $x=1$ and $x=2$

CASE 1: $x<1$

$-x+1-x+2=x \Leftrightarrow 3x = 3 \Leftrightarrow x=1$ (rejected)

CASE 2: $1\le x<2$

$x-1-x+2=x \Leftrightarrow x=1$ (accepted)

CASE 3: $x\ge 2$

$x-1+x-2=x \Leftrightarrow x=3$ (accepted)

Final answer (the union of the three cases): $x=1$ or $x=3$

EXAMPLE 5

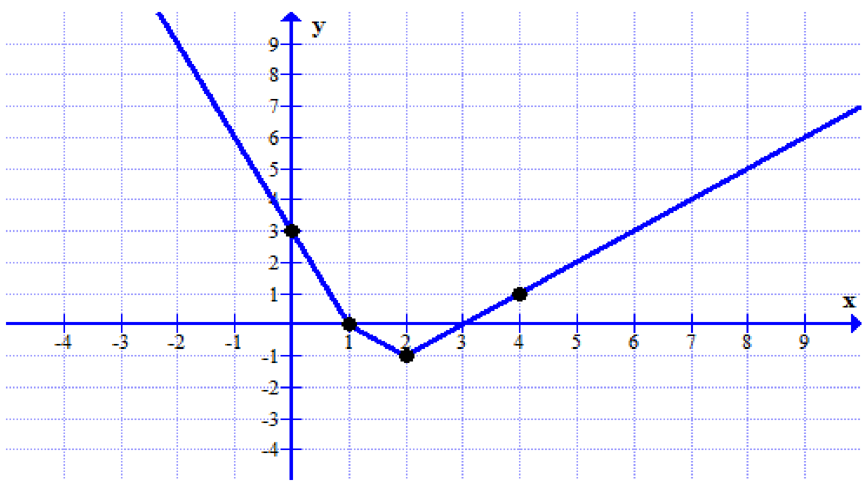
Sketch the graph of $f(x)=|x-1|+|x-2|-x$

We find the zeros of the absolute values: $x=1$ and $x=2$

We know that the graph consists of linear parts. Thus we need 4 points on the graph, the two values above, one before, one after:

$$f(1)=0, \quad f(2)=-1, \quad f(0)=3, \quad f(4)=1$$

We just connect them:



2.16 SYMMETRIES OF f(x) – MORE TRANSFORMATIONS (for HL)

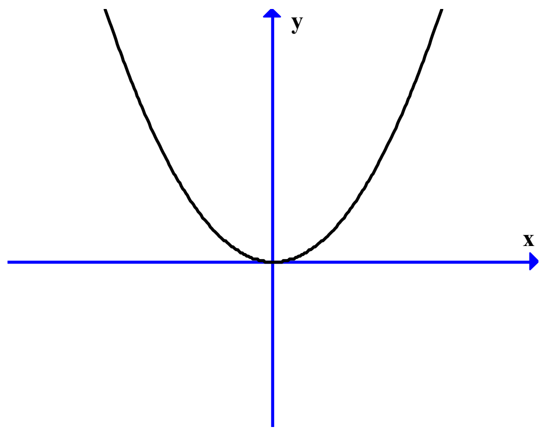
♦ EVEN AND ODD FUNCTIONS

A function is said to be **even** if

$$f(-x)=f(x)$$

Such a function is **symmetric in y-axis**.

For example $f(x)=x^2$ is an even function.

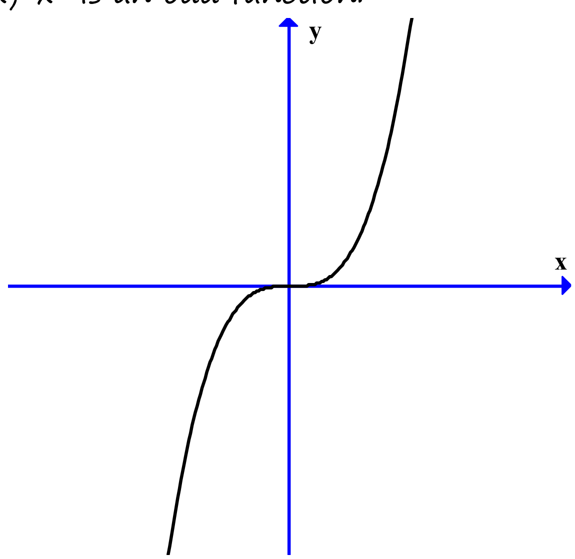


A function is said to be **odd** if

$$f(-x)=-f(x)$$

Thus a function is **symmetric about the origin**.

For example $f(x)=x^3$ is an odd function.

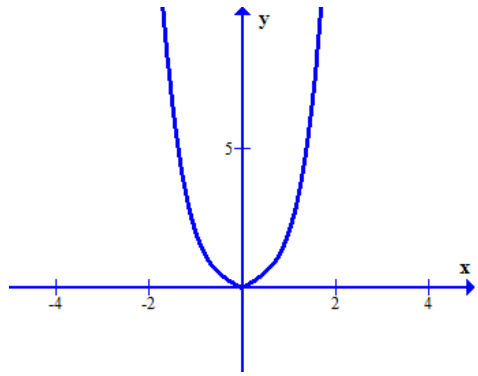
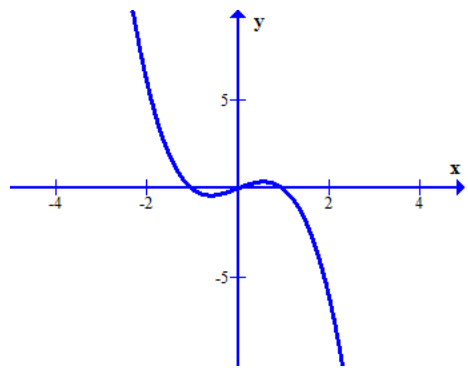
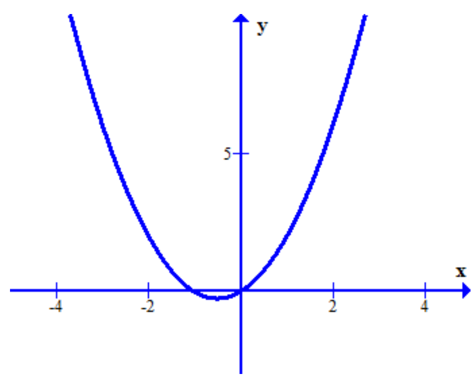


EXAMPLE 1

Investigate whether the following functions are even or odd.

a) $f(x)=x^4+|x|$ b) $g(x)=x-x^3$ c) $h(x)=x+x^2$

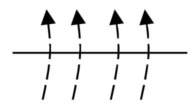
Solution

(a)	$f(-x) = (-x)^4+\|-x\|$ $= x^4+\|x\|$ $= f(x)$ hence the function is **even**.	
(b)	$g(-x) = (-x)-(-x)^3$ $=-x+x^3$ $=-(x-x^3)$ $=-g(x)$ hence the function is **odd**.	
(c)	$h(-x) = (-x)+(-x)^2$ $= -x+x^2$ hence the function is **neither even nor odd**.	

♦ THE ABSOLUTE VALUE TRANSFORMATIONS

Consider the initial function f(x).

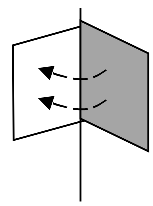
(a) The new function $|f(x)|$



- preserves any positive part of f(x)
- reflects any negative part of f(x) in x-axis

    [this is because $f(x)<0$ implies that $|f(x)|=-f(x)$]

(b) The new function $f(|x|)$

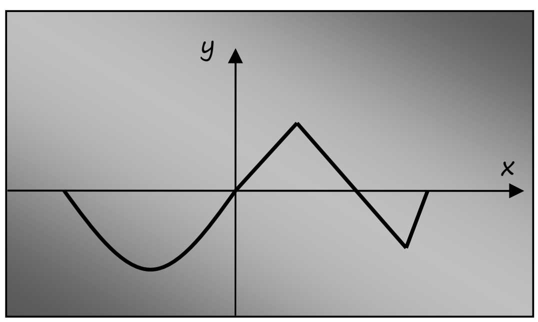


- ignores f(x) for $x<0$
- reflects f(x), $x\geq 0$ in y-axis

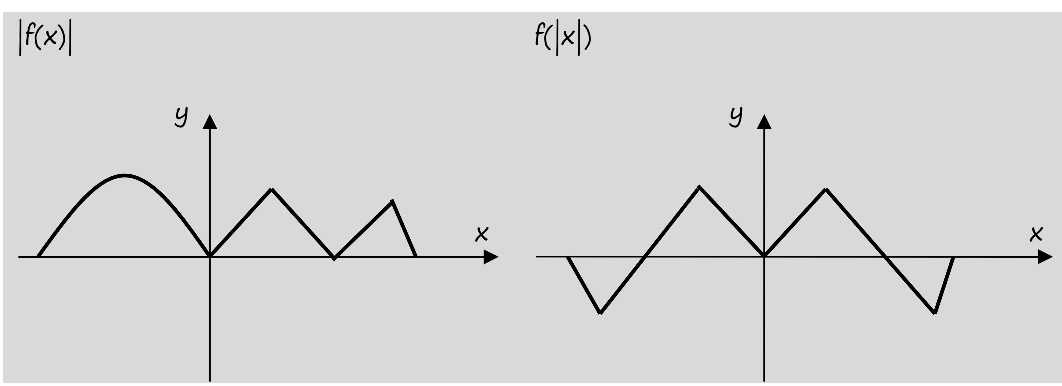
    [this is because $x<0$ implies that $f(|x|)=f(-x)$]

EXAMPLE 2

Let f(x) have the graph

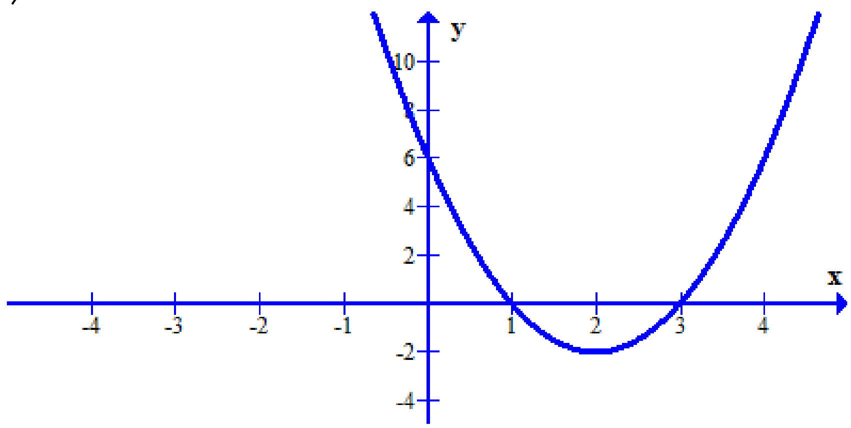


Then

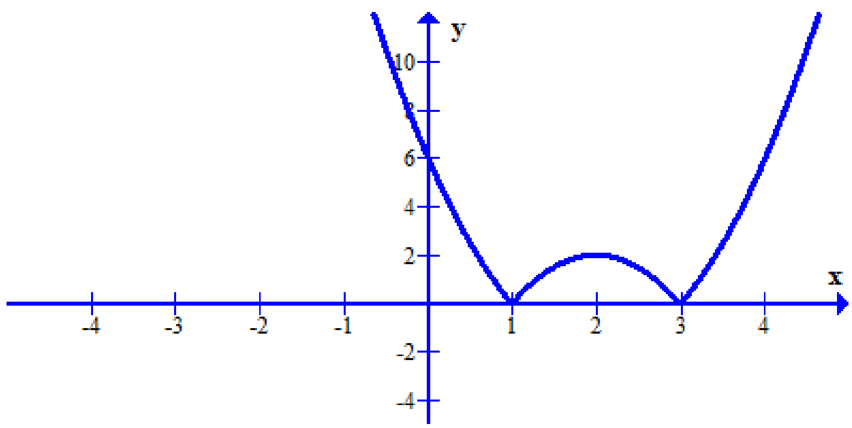


EXAMPLE 3

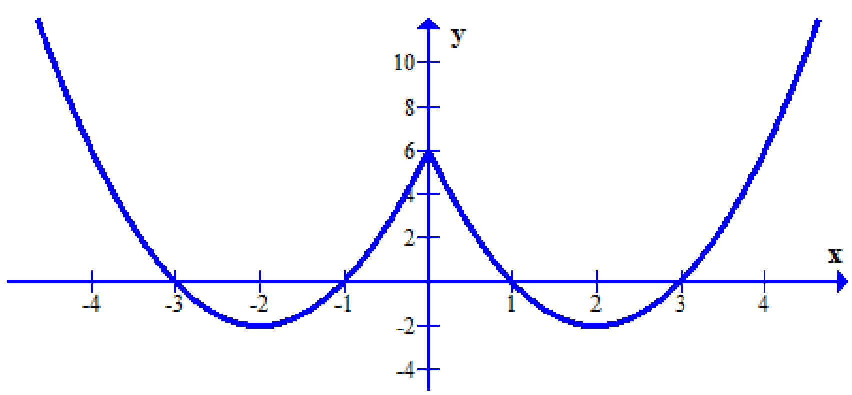
Let $f(x) = 2x^2 - 8x + 6$



Then $|f(x)| = |2x^2 - 8x + 6|$ has the graph



while $f(|x|) = 2|x|^2 - 8|x| + 6$ has the graph



♦ THE RECIPROCAL FUNCTION $\frac{1}{f(x)}$

Another transformation of the function f(x) is $g(x)=\frac{1}{f(x)}$

We notice the following:

- If x=a is a root of f(x) then g(x) is not defined at x=a (V.A.)
- If x=a is vertical asymptote of f(x) then x=a is a root of g(x)
- Any y=a concept (H.A., y-intercept etc) becomes $y=\frac{1}{a}$

Thus, in order to sketch the graph of the reciprocal function $\frac{1}{f(x)}$ we follow the rules:

1) V.A. become roots and roots become V.A.

2) H.A. y=a becomes H.A. $y=\frac{1}{a}$

3) Any characteristic point (x, y) becomes $(x, \frac{1}{y})$

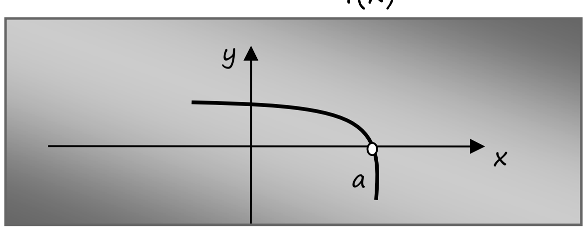
max at (x, y) becomes **min** at (x, 1/y) (and vice versa)
y-intercept (0, y) becomes ***y-intercept*** (0, 1/y), etc.

4) If f(x) is positive/negative, g(x) is also positive/negative

5) If f(x) is increasing, g(x) is decreasing (and vice versa)

NOTICE

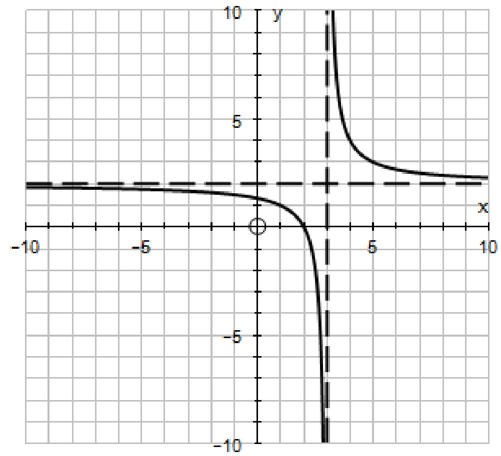
- In fact, the V.A. x=a becomes not exactly a root but a point of discontinuity on x axis, since $g(x)=\frac{1}{f(x)}\neq 0$. The graph looks like



- If y=0 is a HA, in the reciprocal y tends to +∞ or −∞ accordingly.

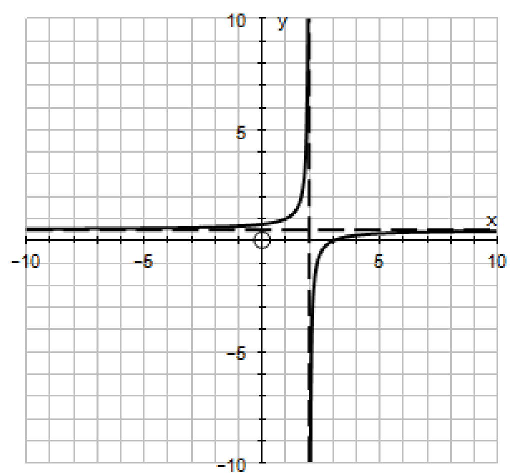
EXAMPLE 4

Consider the function $f(x)=\dfrac{2x-4}{x-3}$



Observations on f(x)	**Conclusions for $\dfrac{1}{f(x)}$**
Root: x=2	V.A: x=2
V,A,: x=3	Root: x=3
H.A.: y=2	H.A.: y=1/2
y-intercept y=4/3	y-intercept y=3/4

For $\dfrac{1}{f(x)}$ (i.e. $\dfrac{x-3}{2x-4}$) we indicate roots, asymptotes and carry on



- THE TRANSFORMATION $y=[f(x)]^2$

Consider the function y=f(x)

What about the function $y=[f(x)]^2$?

(1) Whatever lies on the line y=1 remains the same

(2) Whatever lies on the line y=-1 goes to y=1.

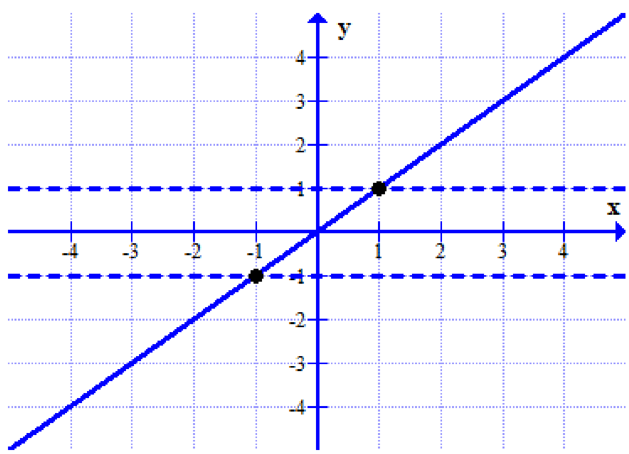
(3) For the positive part of the function:

We stretch everything above y=1: 2 becomes 4 etc

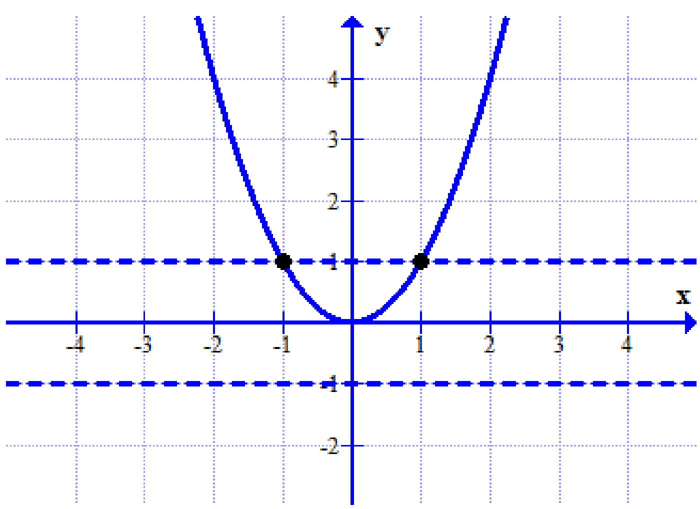
We shrink everything below y=1: 1/2 becomes 1/4 etc

(4) The negative part becomes positive and behaves as in 3

The easy example of f(x)=x is indicative.



We obtain



which is the known curve of $y=x^2$ ☺