STAT 499: Undergraduate Research

Week 4: Estimation in noisy settings

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n this lecture note, we consider such a linear model $y = \mathbf{X}\theta^* + w$, and mainly study the consistency of the Lagrangian Lasso solution:

$$\min_{\theta \in \mathbb{R}^d} \left\{ \frac{1}{2n} \|y - \mathbf{X}\theta\|_2^2 + \lambda_n \|\theta\|_1 \right\}. \tag{4.1}$$

Extension of restricted nullspace. Define the set

$$\mathbb{C}_\alpha(S) := \{\Delta \in \mathbb{R} : \|\Delta_{S^c}\|_1 \leq \alpha \|\Delta_S\|_1\}.$$

This definition generalizes the set $\mathbb{C}(S)$ used in our definition of the restricted nullspace property, which corresponds to the special case $\alpha = 1$.

RE condition. The matrix $\mathbf{X}$ satisfies the restricted eigenvalue (RE) condition over S with parameters (k, α) if

$$\frac{1}{n} \|\mathbf{X}\Delta\|_2^2 \geq k \|\Delta\|_2^2 \text{ for all } \Delta \in \mathbb{C}_\alpha(S). \tag{4.2}$$

Assumption 4.1 *In the case of hard sparse, we assume that*

($\mathbf{A_1}$) *The vector θ^* is supported on a subset $S \subset \{1, \ldots, d\}$ with $|S| = s$.*

($\mathbf{A_2}$) *The design matrix $\mathbf{X}$ satisfies the RE condition with parameter $(k, 3)$:*

$$\frac{1}{n} \|\mathbf{X}\Delta\|_2^2 \geq k \|\Delta\|_2^2 \textit{ for all } \Delta \in \mathbb{C}_3(S).$$

Theorem 4.2 (Theorem 7.13 in Wainwright's book) *Under above assumption, for any solution $\widehat{\theta}$ of the Lagrangian Lasso (4.1) with $\lambda_n \geq 2\|\frac{\mathbf{X}^T w}{n}\|_\infty$, we have*

$$\|\widehat{\theta} - \theta^*\|_2^2 \leq \frac{3}{k} \sqrt{s} \lambda_n.$$

Proof: Our first step is to show that, under the condition $\lambda_n \geq 2 \left\| \frac{\mathbf{X}^T w}{n} \right\|_\infty$, the error vector $\widehat{\Delta} = \widehat{\theta} - \theta^*$ belongs to $\mathbb{C}_3(S)$.

Define the Lagrangian $L(\theta; \lambda_n) = \frac{1}{2n} \|y - \mathbf{X}\theta\|_2^2 + \lambda_n \|\theta\|_1$. Since $\widehat{\theta}$ is optimal, we have

$$L\left(\widehat{\theta}; \lambda_n\right) \leq L(\theta^*; \lambda_n).$$

That is,

$$\begin{aligned}
&\frac{1}{2n} \|y - \mathbf{X}\widehat{\theta}\|_2^2 + \lambda_n \|\widehat{\theta}\|_1 \leq \frac{1}{2n} \|y - \mathbf{X}\theta^*\|_2^2 + \lambda_n \|\theta^*\|_1 \\
\iff &\frac{1}{2n} \|\mathbf{X}\theta^* + w - \mathbf{X}\widehat{\theta}\|_2^2 + \lambda_n \|\widehat{\theta}\|_1 \leq \frac{1}{2n} \|w\|_2^2 + \lambda_n \|\theta^*\|_1 \\
\iff &\frac{1}{2n} (\mathbf{X}\widehat{\Delta} - w)^T (\mathbf{X}\widehat{\Delta} - w) + \lambda_n \|\widehat{\theta}\|_1 \leq \frac{1}{2n} w^T w + \lambda_n \|\theta^*\|_1.
\end{aligned}$$

Rearranging it, we have

$$0 \le \frac{1}{2n}\|\mathbf{X}\widehat{\Delta}\|_2^2 \le \frac{w^{\mathrm{T}}\mathbf{X}\widehat{\Delta}}{n} + \lambda_n \left\{\|\theta^*\|_1 - \|\widehat{\theta}\|_1\right\}.$$

Now since θ^* is S-sparse, we can write

$$\|\theta^*\|_1 - \|\widehat{\theta}\|_1 = \|\theta^*_S\|_1 - \left\|\theta^*_S + \widehat{\Delta}_S\right\|_1 - \left\|\widehat{\Delta}_S\right\|_1.$$

Thus,

$$\begin{aligned} 0 \le \frac{1}{n}\|\mathbf{X}\widehat{\Delta}\|_2^2 &\le 2\frac{w^{\mathrm{T}}\mathbf{X}\widehat{\Delta}}{n} + 2\lambda_n \left\{\|\theta^*_S\|_1 - \left\|\theta^*_S + \widehat{\Delta}_S\right\|_1 - \left\|\widehat{\Delta}_{S^c}\right\|_1\right\} \\ &\overset{\text{(i)}}{\le} 2\left\|\mathbf{X}^{\mathrm{T}}w/n\right\|_\infty \|\widehat{\Delta}\|_1 + 2\lambda_n \left\{\left\|\widehat{\Delta}_S\right\|_1 - \left\|\widehat{\Delta}_{S^c}\right\|_1\right\} \\ &\overset{\text{(ii)}}{\le} \lambda_n \left\{3\left\|\widehat{\Delta}_S\right\|_1 - \left\|\widehat{\Delta}_{S^c}\right\|_1\right\}, \end{aligned} \tag{4.3}$$

where step (i) follows from a combination of Hölder's inequality and the triangle inequality, whereas step (ii) follows from the choice of λ_n.

Inequality (4.3) shows that $\widehat{\Delta} \in \mathbb{C}_3(S)$, so that the RE condition may be applied. Doing so, we obtain

$$\kappa\|\widehat{\Delta}\|_2^2 \le 3\lambda_n\sqrt{s}\|\widehat{\Delta}\|_2,$$

which implies the consistency. ■

Example 7.14 in Wainwright's book. Consider the classical linear Gaussian model, where $w \in \mathbb{R}^n$ has i.i.d. $\mathcal{N}(0, \sigma^2)$ entries.

Consider $\mathbf{X} \in \mathbb{R}^{n\times d}$ is fixed. Suppose that $\mathbf{X}$ satisfies the RE condition, and that it is C-column normalized, i.e.,

$$\max_{j=1,\dots,d} \frac{\|\mathbf{X}(\cdot, j)\|_2}{\sqrt{n}} \le C.$$

Thus, the random variable $\|\frac{\mathbf{X}^T w}{n}\|_\infty$ corresponds to the absolute maximum of d zero-mean Gaussian variables, each with variance at most $\frac{C^2\sigma^2}{n}$, since

$$var(\frac{\mathbf{X}(\cdot, j)^T w}{n}) = \frac{\sigma^2}{n^2}\sum_{i=1}^n \mathbf{X}(i, j)^2 = \sigma^2\frac{\|\mathbf{X}(\cdot, j)\|_2^2}{n^2} \le \frac{C^2\sigma^2}{n}.$$

The standard Gaussian tail bounds states that, for any $i \in \{1, \dots, d\}$,

$$P\left(\left|\frac{\mathbf{X}(\cdot, j)^T w}{n}\right| \ge t\right) \le 2\exp\{-\frac{nt^2}{2C^2\sigma^2}\} \text{ for all } t > 0.$$

Thus, for all $\delta > 0$,

$$P\left(\left\|\frac{\mathbf{X}^T w}{n}\right\|_\infty \geq C\sigma\left(\sqrt{\frac{2\log d}{n}} + \delta\right)\right) \leq \sum_{j=1}^{d} P\left(\frac{\mathbf{X}(\cdot, j)^T w}{n} \geq C\sigma\left(\sqrt{\frac{2\log d}{n}} + \delta\right)\right)$$
$$\leq 2d\exp\left\{-\frac{nC^2\sigma^2(\sqrt{\frac{2\log d}{n}} + \delta)^2}{2C^2\sigma^2}\right\}$$
$$\leq 2e^{-n\delta^2/2}.$$

If we set $\lambda_n = 2C\sigma(\frac{2\log d}{n} + \delta)$, then the theorem implies that

$$\|\widehat{\theta} - \theta^*\|_2 \leq \frac{6C\sigma}{k}\sqrt{s}\left\{\frac{2\log d}{n} + \delta\right\}.$$