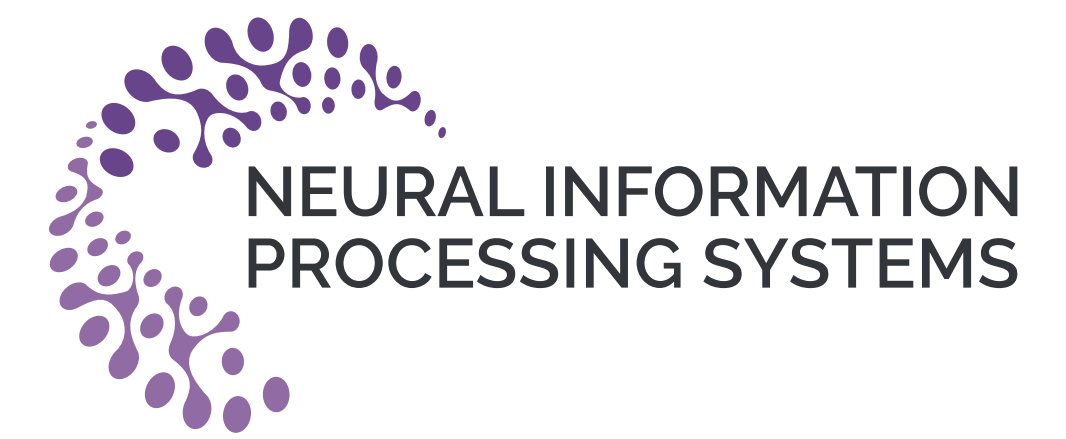


# Stochastic Gradients under Nuisances

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## Orthogonal Statistical Learning

**Motivation:** Many machine learning problems rely on a risk function that is only partially specified up to a class of risks under the unknown nuisance, wherein the risk of interest is the one under the true nuisance.

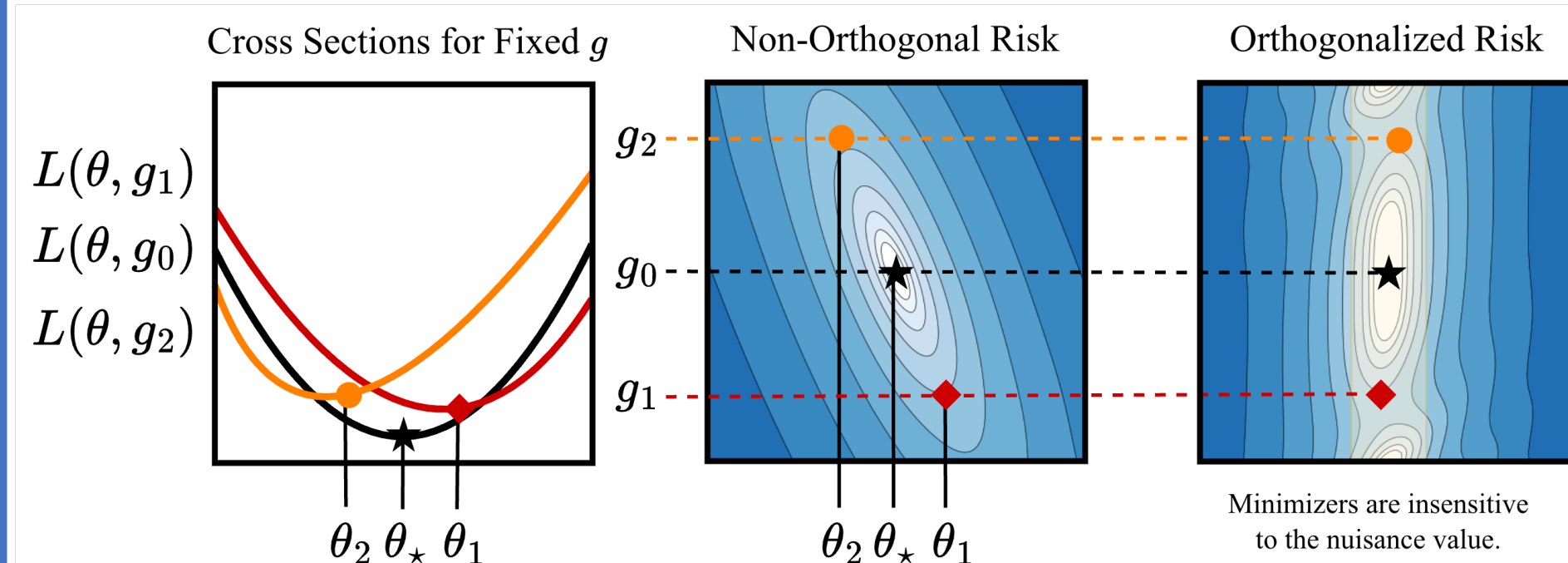
Risk function under unknown nuisance

$$\mathcal{L} := \{L(\cdot, g) : g \in \mathcal{G}\}$$

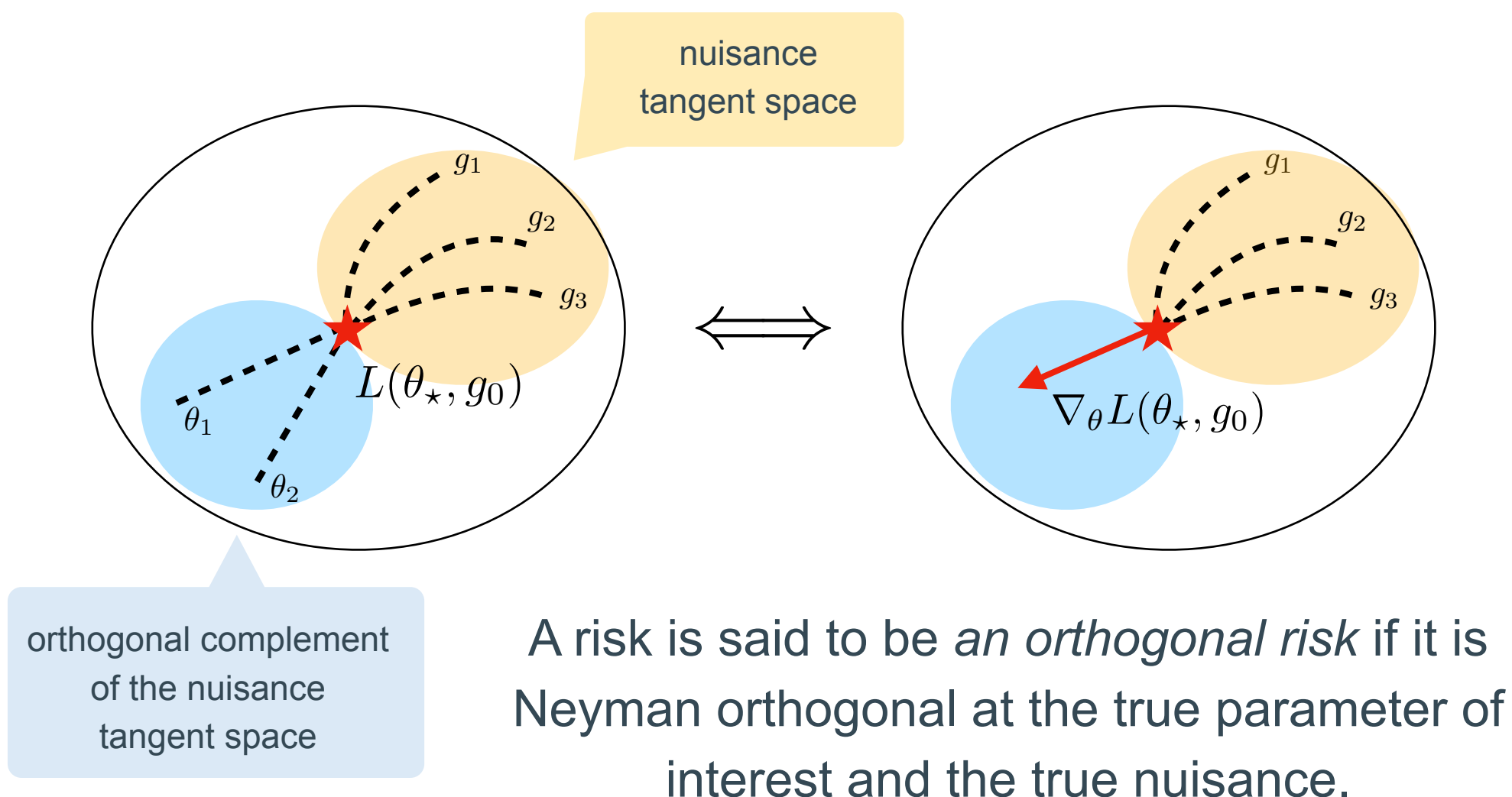
Risk minimization under **true nuisance**

$$\theta_\star = \arg \min_{\theta \in \Theta} [L_0(\theta) := \mathbb{E}_{Z \sim \mathbb{P}}[\ell(\theta, g_0; Z)]]$$

Learning the true nuisance would introduce additional statistical error. One way to make the risk insensitive to nuisance is to **orthogonalize** it.



**Intuition of Neyman Orthogonality:** The parameter tangent space is **orthogonal** to the nuisance tangent space.



## Classical Stochastic Gradient Algorithm

$$Z_1, \dots, Z_n \stackrel{\text{i.i.d.}}{\sim} \mathbb{P}$$

learning rate

$$(\text{SGD}) \quad \theta^{(n)} = \theta^{(n-1)} - \eta \nabla_{\theta} \ell(\theta^{(n-1)}, \hat{g}; Z_n)$$

plug-in nuisance estimator  
(independent of samples)

**Theorem 1.** The standard SGD iterate expected errors satisfy

$$O((1 - \mu\eta/2)^n + \eta + \|\hat{g} - g_0\|_{\mathcal{G}}^2) \quad (\text{Nuisance sensitive})$$

For an orthogonal risk, the iterates expected errors satisfy

$$O((1 - \mu\eta/2)^n + \eta + \|\hat{g} - g_0\|_{\mathcal{G}}^4) \quad (\text{Nuisance Insensitive})$$

SGD  
Optimization Error

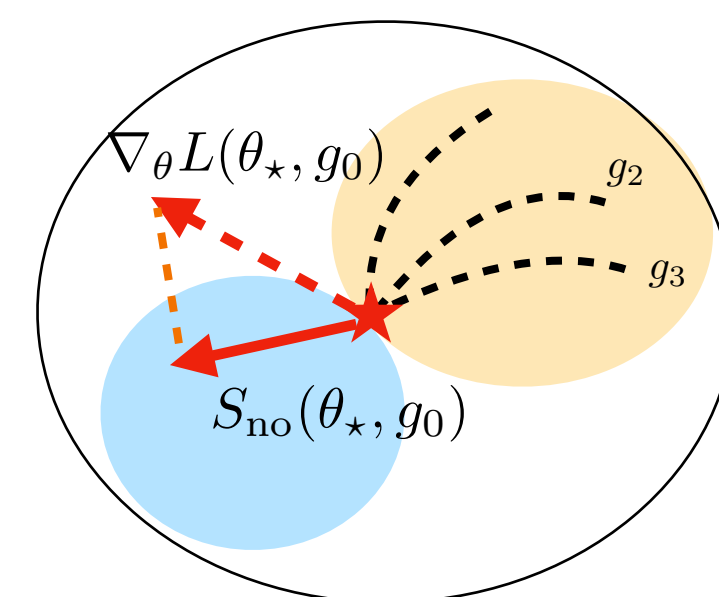
Nuisance  
Estimation Error

Using an orthogonal loss can further remove the nuisance estimation error. However, it is not always possible to handcraft orthogonalized objectives. Sequentially *orthogonalizing first-order information* is more **flexible**.

## Orthogonalized Stochastic Gradients

$$S_{\text{no}}(\theta, g; z) = \nabla_{\theta} \ell(\theta, g; z) - \Gamma_0 \nabla_g \ell(\theta, g; z)$$

orthogonalizing operator



**Intuition:**

Project the standard gradient oracle onto the *orthogonal complement* of the nuisance tangent space.

**Example.** [Partially Linear Model]

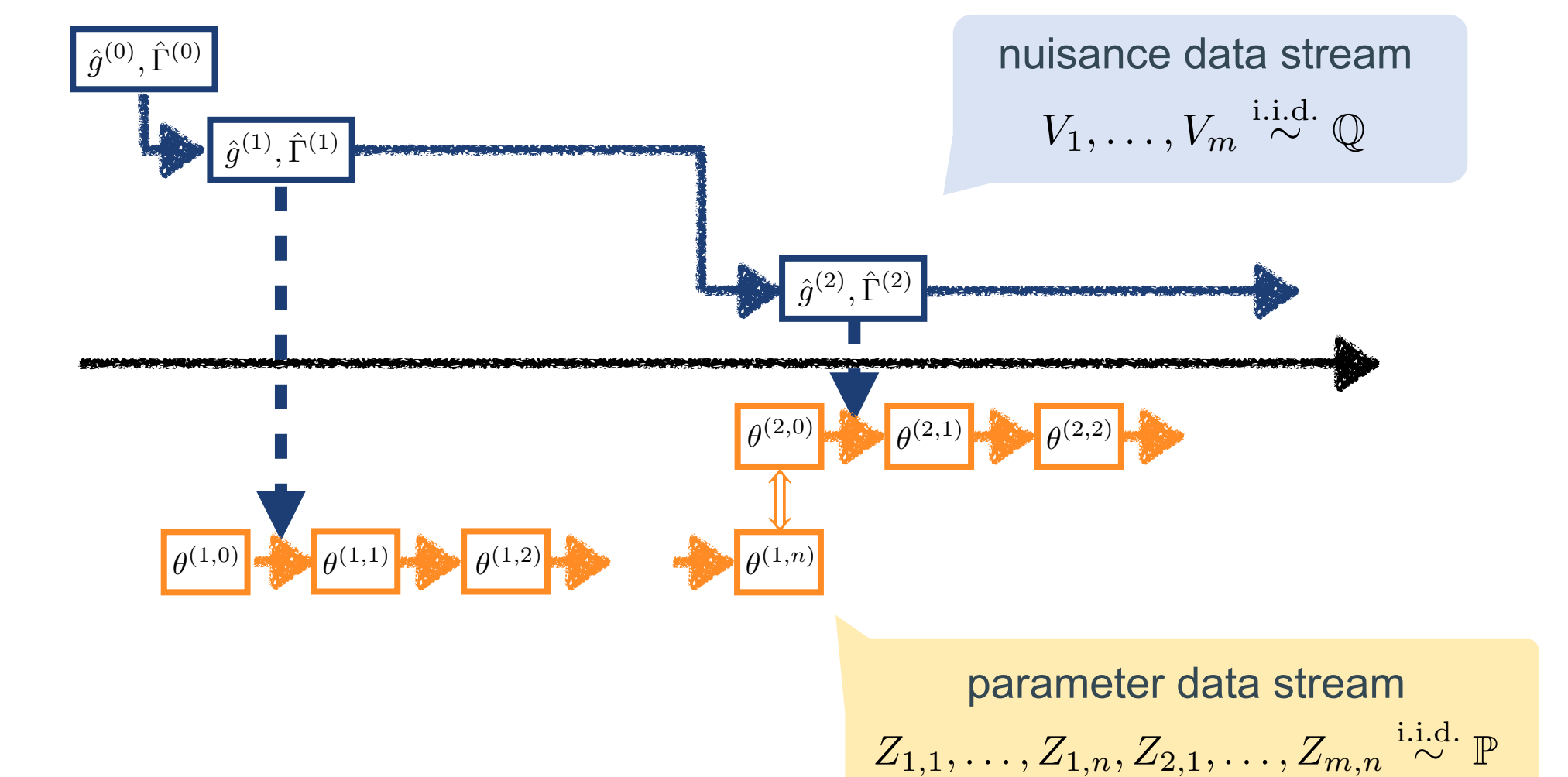
$$\begin{aligned} Y &= \langle \theta_\star, X \rangle + g_0(W) + \epsilon & Z &= (X, Y, W) \sim \mathbb{P} \\ U &= g_0(W) + \xi & V &= (X, U, W) \sim \mathbb{Q} \end{aligned}$$

Consider the following non-orthogonal loss

$$\ell(\theta, g; z) = \frac{1}{2}(y - g(w) - \langle \theta, x \rangle)^2$$

The true nuisance and the orthogonalizing operator are

$$g_0(W) = \mathbb{E}[U | W] \quad \Gamma_0 : g \mapsto \mathbb{E}[\mathbb{E}[X | W]g(W)]$$



(OSGD)

$$\theta^{(m,n)} = \theta^{(n-1)} - \eta \hat{S}_{\text{no}}^{(m)}(\theta^{(m,n-1)}, \hat{g}^{(m)}; Z_{m,n})$$

approximated stochastic gradient oracle  
 $\hat{S}_{\text{no}}^{(m)}(\theta, g; z) = \nabla_{\theta} \ell(\theta, g; z) - \hat{\Gamma}^{(m)} \nabla_g \ell(\theta, g; z)$

**Theorem 2.** OSGD iterate expected errors satisfy

$$O((1 - \mu\eta/2)^{mn} + n^{-1} + \eta + m^{-(2\alpha-1)/\alpha})$$

nuisance insensitive rate

**Assumptions**

- $\|\hat{g}^{(m)} - g_0\|_{\mathcal{G}}^2 = O_p(m^{-(2\alpha-1)/2\alpha})$
- $\|\hat{\Gamma}^{(m)} - \Gamma_0\|^2 = O_p(m^{-(2\alpha-1)/2\alpha})$
- $(\eta n)^{-1} = O(1)$

