A Fourier representation of the diffusion MRI signal

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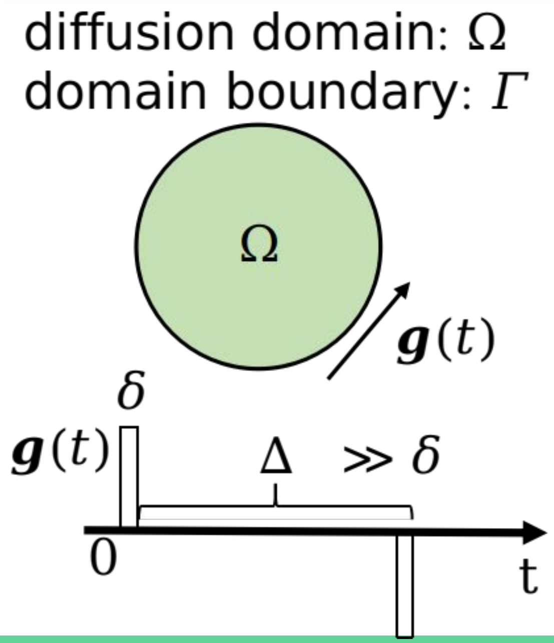
Context & Objective

In diffusion MRI, a time-varying magnetic field gradient $\mathbf{g}(t)$ is applied to the tissue to encode the water diffusion. The transverse water proton magnetization $M(\mathbf{x}, t)$ satisfies the Bloch-Torrey equation:

$$\frac{\partial}{\partial t} M(\mathbf{x}, t) = \mathcal{D}_0 \Delta M(\mathbf{x}, t) - i\gamma \mathbf{x} \cdot \mathbf{g}(t) M(\mathbf{x}, t), \mathbf{x} \in \Omega.$$

$$\text{dMRI signal } s = \int_\Omega M(\mathbf{x}, TE) d\mathbf{x}$$



Objective:

Design a simulation method that solves the Bloch-Torrey equation and provides a Fourier representation of the magnetization.

Challenges

Bloch-Torrey equation

$$\overbrace{\frac{\partial}{\partial t}M(\mathbf{x},t) = \mathcal{D}_0 \Delta M(\mathbf{x},t)}^{\textit{diffusion equation}} - i\gamma \mathbf{x}\cdot \mathbf{g}(t) M(\mathbf{x},t), \;\; x \in \Omega.$$

$$\textit{Initial condition}: \;\; M(\mathbf{x}, t=0) = \rho(\mathbf{x}), \;\; x \in \Omega$$

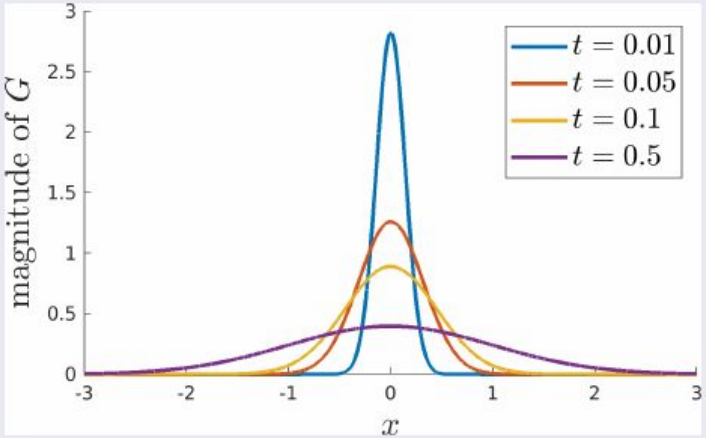
$$\textit{Boundary condition}\ (\textit{impermeable}): \;\; \mathcal{D}_0 \nabla M(\mathbf{x},t)\cdot \mathbf{n} = 0, \;\; x \in \Gamma$$

Singularity of the heat kernel

The heat kernel $G(\mathbf{x}, t)$ is singular.

$$G(\mathbf{x},t) = \frac{1}{(4\pi\mathcal{D}_0 t)^{d/2}} e^{-\frac{\|\mathbf{x}\|^2}{4\mathcal{D}_0 t}}$$

$$\lim_{t\to 0} G(\mathbf{x},t) = \delta_{\mathbf{x}}$$



Transform Bloch-Torrey to diffusion equation

Narrow-pulse assumption to transform the Bloch-Torrey equation

- $\delta \ll \Delta \longrightarrow M(\mathbf{x}, \delta) = M(\mathbf{x}, 0) \cdot e^{-i\delta\gamma \mathbf{g}\cdot\mathbf{x}} = \rho e^{-i\delta\gamma \mathbf{g}\cdot\mathbf{x}}$
- change of variable: $\omega(\mathbf{x}, t) \equiv M(\mathbf{x}, t+\delta) - \rho e^{-\mathcal{D}_0\|\mathbf{g}\|^2 t - i\mathbf{x}\cdot\mathbf{g}}$
- diffusion equation about ω:

$$\frac{\partial}{\partial t}\omega(\mathbf{x}, t) = \mathcal{D}_0 \Delta\omega(\mathbf{x}, t)$$

$$\omega(\mathbf{x}, t = 0) = 0$$

$$\nabla\omega(\mathbf{x}, t) \cdot \mathbf{n} = -i\mathbf{n} \cdot \mathbf{g} e^{-\mathcal{D}_0\|\mathbf{g}\|^2 t - i\mathbf{x}\cdot\mathbf{g}}$$

Solution to the diffusion equation

$$\omega(\mathbf{x}, t) = \mathcal{D}_0 G * \mu(\mathbf{x}, t) = \mathcal{D}_0 \int_0^t \int_\Gamma G(\mathbf{x} - \mathbf{y}, t - \tau)\mu(\mathbf{y}, \tau) ds_{\mathbf{y}} d\tau$$

Convolution theorem guarantees a Fourier representation of $\omega(\mathbf{x}, t)$.

Overcome the singularity of G

Divide the integration in time

$$\omega(\mathbf{x}, t) = \int_{t-\eta}^{t} \dots \, d\tau + \int_{0}^{t-\eta} \dots \, d\tau$$
$$= \omega_{singular}(\mathbf{x}, t) + \omega_{smooth}(\mathbf{x}, t)$$

$$\omega_{singular} \simeq \sqrt{\frac{\mathcal{D}_0 \eta}{\pi}} \mu(\mathbf{x}, t) + O(\eta^{3/2})$$
$$\omega_{smooth} = \mathcal{D}_0 \sum_{\nu} f(\nu, t) e^{2\pi i \nu \cdot \mathbf{x}}$$

The unknown function μ and the Fourier coefficients f can be efficiently solved by using a recursive method[1].

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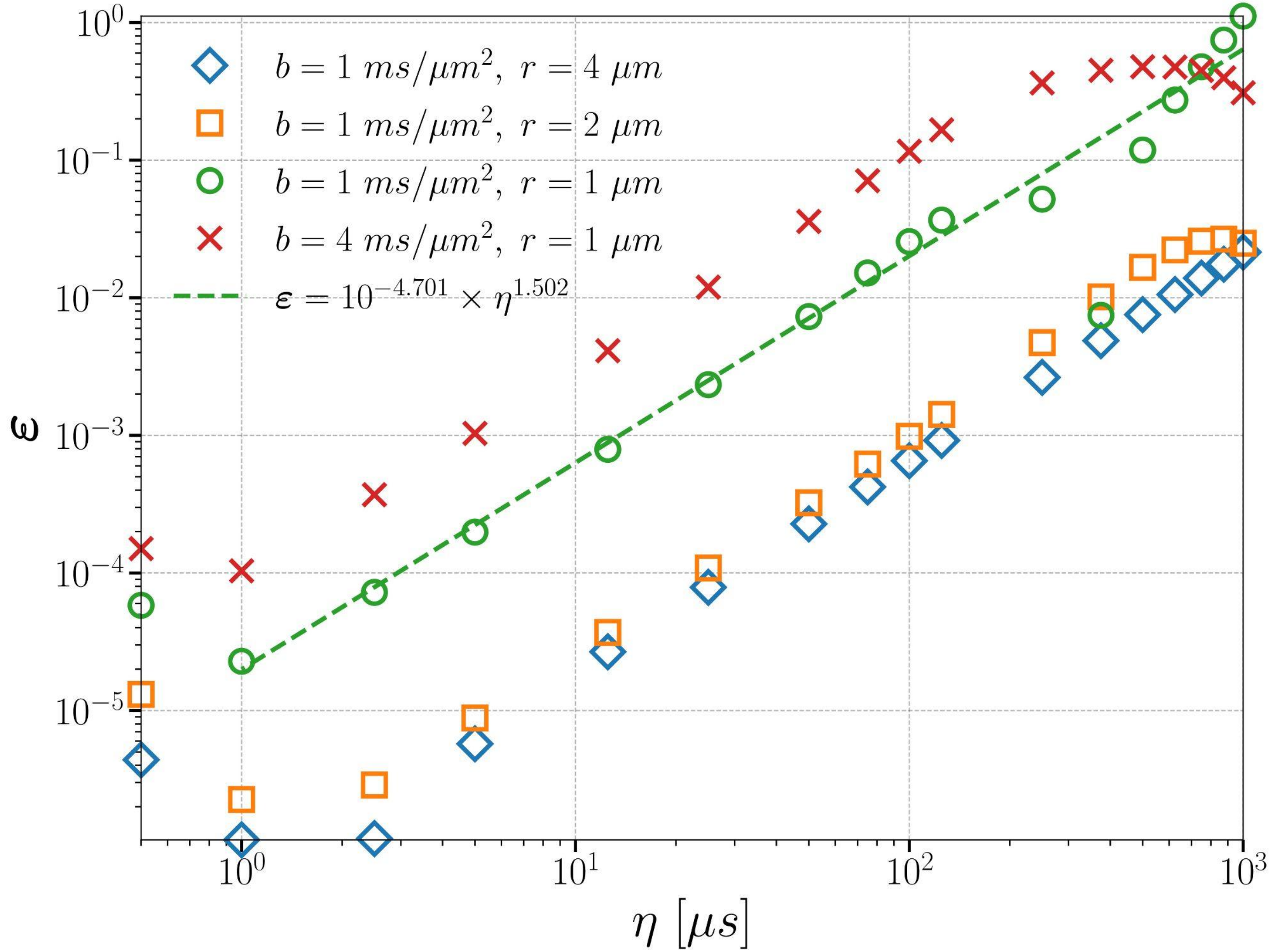
1. Greengard, Leslie, and John Strain. "A fast algorithm for the evaluation of heat potentials." *Communications on Pure and Applied Mathematics* 43.8 (1990): 949-963.

Results: convergence of our method

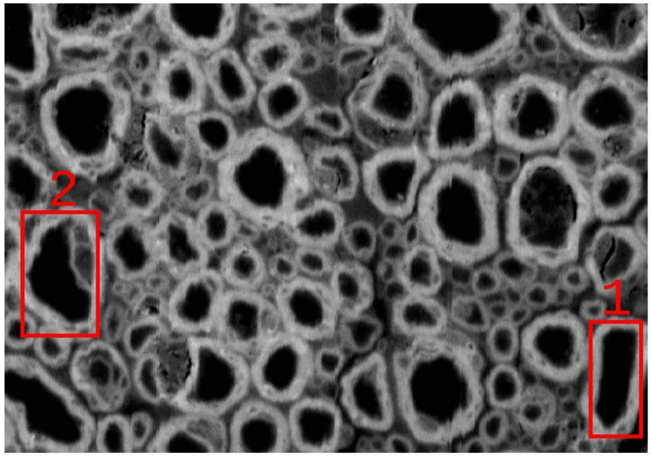
Simulation on circles with radius r.

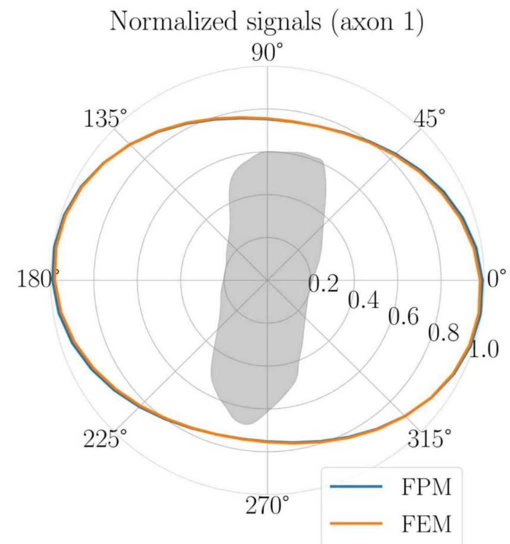


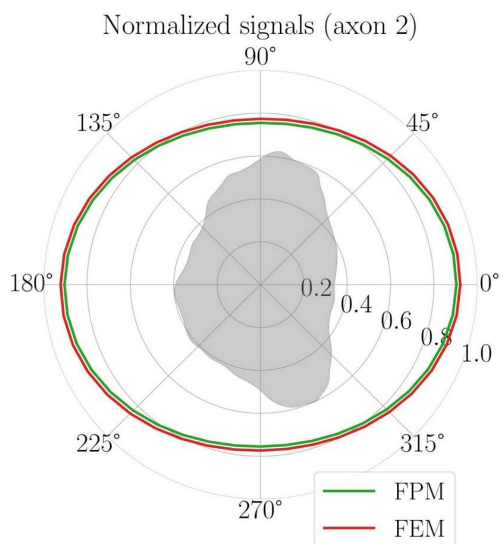
Results: temporal evolution of the spectrum

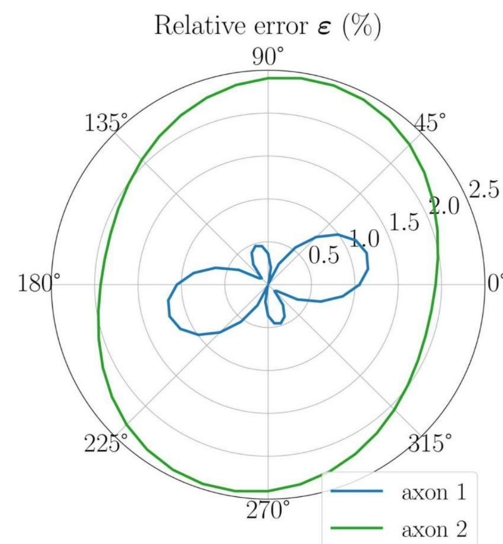
Simulation on a disk with radius r=5 micrometer.
(Please click the black box below to watch the video.)

Results: simulations on two axons









Axon image belongs to *AxonDeepSeg* (https://axondeepseg.rtfd.io/)