A Fourier representation of the diffusion MRI signal

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Introduction

The diffusion MRI community lacks a spectral perspective to investigate the intricate relationship between the diffusion MRI signals and cellular structures. Our study proposes a new simulation method that can provide a Fourier representation of the signal. Common spectral methods suffer from the singularity of the heat kernel, which requires infinite Fourier modes. We overcome the singularity by formulating the solution in a convolutional form and splitting the heat kernel into a singular part and a smooth part. Numerical experiments are performed to demonstrate the convergence of our method.

Bloch-Torrey equation

In diffusion MRI, a time-varying magnetic field gradient is applied to the tissue to encode the water diffusion. The magnetic field gradient can be fully characterized by a temporal function $f(t)$ which is the time profile, and a vector $\mathbf{g}$ which represents the direction and magnitude of the gradient. The complex transverse proton magnetization $M(\mathbf{x}, t)$ satisfies the Bloch-Torrey equation:

$$\frac{\partial}{\partial t}M(\mathbf{x},t) = -\imath\gamma f(t)\mathbf{g}\cdot\mathbf{x}M(\mathbf{x},t) + \nabla D_0 \nabla M(\mathbf{x},t), \mathbf{x}\in\Omega, \tag{1}$$

where $\gamma = 267.5rad\cdot ms^{-1}mT^{-1}$ is the gyromagnetic ratio of the proton, and D_0 is the intrinsic diffusion coefficient of proton in the domain Ω. The most commonly used time profile is the pulsed-gradient spin echo (PGSE) sequence.

Narrow pulse assumption

We restrict ourselves to the 2D Bloch-Torrey equation with impermeable interfaces. In addition, we derive our method under the narrow pulse assumption, where the pulse duration of the PGSE sequence δ is very short compared to the delay Δ ($\delta \ll \Delta$). The narrow pulse assumption allows us to transform the Bloch-Torrey equation to a diffusion equation:

$$\begin{aligned}
&\frac{\partial}{\partial t}w(\mathbf{x},t) = \nabla\cdot(D_0\nabla w(\mathbf{x},t)), \quad \mathbf{x}\in\Omega, t\in[0,\Delta-\delta],\\
&D_0\nabla w(\mathbf{x}_0,t)\cdot\mathbf{n} = D_0 N(\mathbf{x}_0,t,\mathbf{q}), \quad \mathbf{x}_0\in\Gamma, t\in[0,\Delta-\delta],\\
&w(\mathbf{x},0) = 0, \quad \mathbf{x}\in\Omega,
\end{aligned} \tag{2}$$

where w describes the transformed magnetization. The vector $\mathbf{q}$ is $\gamma\delta\mathbf{g}/2\pi$. The Neumann forcing term N is a known function that decays exponentially in time.

Singularity of the heat kernel

According to Haddar et al. 2018, the solution of Eq. 2 can be written as:

$$w(\mathbf{x},t) = \int_0^t\int_\Gamma D_0 G(\mathbf{x}-\mathbf{y}, t-\tau)u(\mathbf{y},\tau)ds_\mathbf{y}d\tau, \tag{3}$$

where u is an unknown density function. The heat kernel G is singular when τ is close to t (see Fig. 1). In theory, infinite Fourier modes are required due to the singularity.

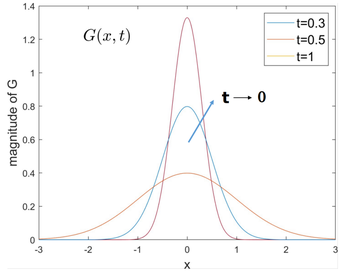

Figure 1: The heat kernel $G(\mathbf{x}, t)$ is singular when t is close to 0.

Fourier representation of dMRI signal

To overcome the singularity, we split Eq. 3 into a singular part and a smooth part ($w = w_{singular} + w_{smooth}$). We derived an asymptotic formula for the singular part on the domain boundary Γ, and the smooth part has a Fourier decomposition. In short, the two parts are:

$$w_{singular} = \int_{t-\eta}^t\int_\Gamma D_0 G(\mathbf{x}-\mathbf{y},t-\tau)u(\mathbf{y},\tau)ds_\mathbf{y}d\tau \simeq \sqrt{\frac{D_0\eta}{\pi}}u(\mathbf{x},t) + O(\eta^{3/2}), \tag{4}$$

$$w_{smooth} = \int_0^{t-\eta}\int_\Gamma D_0 G(\mathbf{x}-\mathbf{y},t-\tau)u(\mathbf{y},\tau)ds_\mathbf{y}d\tau = D_0\sum_\nu f(\nu,t)e^{2\pi\imath\nu\cdot\mathbf{x}}. \tag{5}$$

The parameter η is a controlling factor to separate the singular part from the heat kernel G. The Fourier-type decomposition of the smooth part of diffusion MRI signal appears in the Eq. 5.

Finally, the diffusion MRI signal s is

$$\begin{aligned}
s &= \int_\Omega M(\mathbf{x},t)d\mathbf{x}dt\\
&= D_0\rho^{-1}\int_\Gamma\int_0^{\Delta-\delta}\overline{N}(\mathbf{x},\Delta-\delta-\tau,\mathbf{q})w(\mathbf{x},\tau)d\tau ds_\mathbf{x} + |\Omega|\rho e^{-4\pi^2 D_0\|\mathbf{q}\|^2(\Delta-\delta)}.
\end{aligned}$$

Experiments

Convergence

We study the convergence with respect to the controlling factor η and the maximum frequency ν_{max}. The diffusion domains are circles of radius r. In Fig. 2, the left picture shows a clear convergence order of $3/2$ in η, and the error decreases exponentially due to the fast decay of Fourier coefficients.

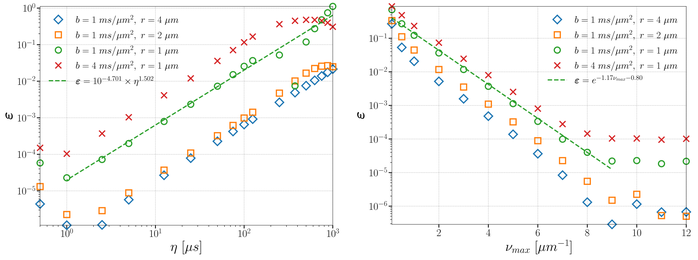

Figure 2: Convergence curves regarding η and ν_{max}.

Simulation on two axons Two axon shapes have been extracted from a microscopy image provided by AxonDeepSeg. We compare our method with finite element method. The two methods agree with each other.

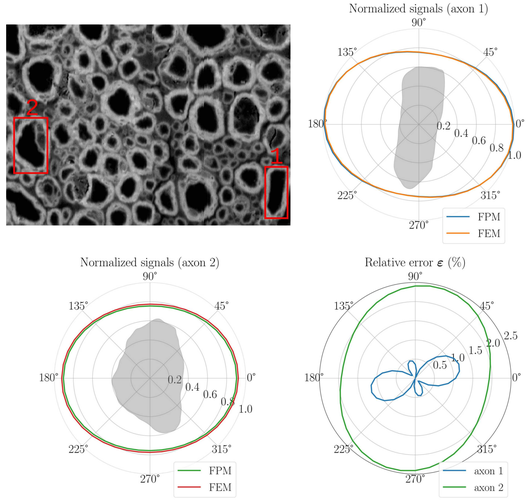

Figure 3: Comparison of our method with FEM.