Pi à la Mode

Mathematicians tackle the seeming randomness of pi's digits

By IVARS PETERSON

Memorizing the digits of pi—the ratio of a circle's circumference to its diameter—presents a hefty challenge to anyone undertaking that quixotic exercise. Starting with 3.14159265, the decimal digits of pi run on forever, and there is no discernible pattern to ease the task.

The apparent randomness of pi's digits has long intrigued mathematician David H. Bailey of the Lawrence Berkeley (Calif.) National Laboratory. In the 1970s, when Bailey was a graduate student at Stanford University, he memorized the value of pi to more than 300 decimal places. It served "as a diversion during classroom lectures," Bailey confesses.

In 1986, after joining NASA's Ames Research Center in Mountain View, Calif., he tested a new supercomputer by having it compute pi to nearly 30 million digits (SN: 2/8/86, p. 91). Any errors would reflect problems in the computer. "The program actually did disclose hardware bugs," Bailey says.

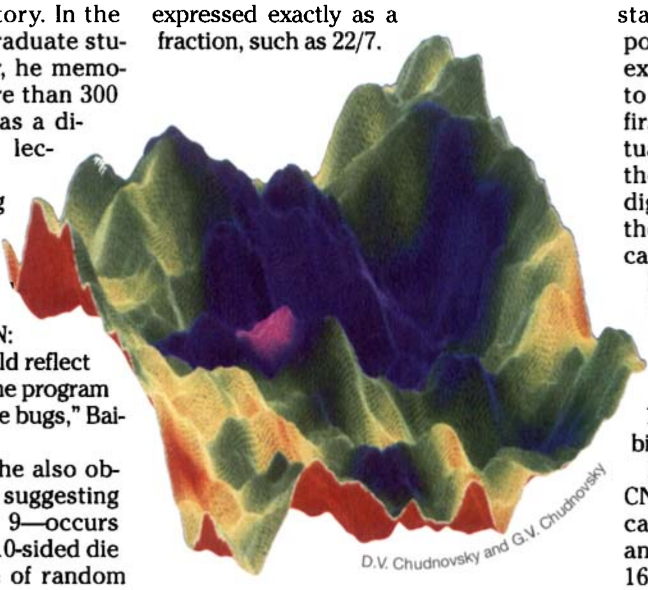
From the computer trial, he also obtained statistical evidence suggesting that every digit—from 0 to 9—occurs equally often. Tossing a fair 10-sided die would generate a sequence of random numbers having the same property.

However, like many mathematicians before him, Bailey could not prove that such a random distribution would hold beyond the first 30 million decimal digits of pi.

Now, there's a shard of hope that mathematicians may yet lay bare the apparent randomness of pi's infinite digits. Bailey and Richard E. Crandall of Reed College in Portland, Ore., have identified a potential link between two disparate mathematical fields—number theory and chaotic dynamics—that they suspect could lead to a proof that every digit occurs with the same frequency when pi is written out ad infinitum. Bailey and Crandall report their findings in the June EXPERIMENTAL MATHEMATICS. The characteristic would hold whether pi's in decimal form or in some other base.

If the Bailey-Crandall hypothesis pans out, "this would be one of the most spectacular results about pi ever," says Stan Wagon of Macalester College in St. Paul, Minn. Establishing the crucial mathematical link necessary for a proof, however, remains a difficult, unsolved problem.

Mathematicians have long known that pi is an irrational number. In other words, it can't be expressed exactly as a fraction, such as 22/7.



This irregular landscape represents the form that emerges when a computer plots the first 1 million decimal digits of pi as a random walk.

One consequence of pi's irrationality is that its endless string of digits never repeats in a cyclic fashion. Its digits show no apparent pattern and thus seem random although they can be derived from a formula.

Pi can be calculated to as many decimal places as desired by formulas that are sums of infinitely many terms, such as $\pi = 4/1 - 4/3 + 4/5 - 4/7 + 4/9 - \ldots$. The larger the number of terms, the more decimal places are revealed.

In 1909, mathematician Émile Borel introduced the concept of normality as one way to characterize the resemblance between the digits of pi and a sequence of random numbers. If a number is normal, digit sequences of the same length occur with the same frequency. Being normal is necessary but not sufficient for a number's digits to be random.

Pi would be considered normal to base 10 if any single digit appears one-tenth of the time, any two-digit combination one-hundredth of the time, any three-digit combination one-thousandth of the time, and so on.

Bailey and other researchers amassed statistical evidence in the 1980s supporting the notion that pi is normal. For example, one would expect the digit 7 to appear 1 million times among the first 10 million decimal digits of pi. It actually occurs 1,000,207 times—close to the expected value. Each of the other digits also turns up with approximately the same frequency, showing no significant departure from predictions.

In 1999, Yasumasa Kanada and his colleagues at the University of Tokyo computed pi to a record 206 billion decimal digits (SN: 10/16/99, p. 255). Their analysis shows that 7 appears 19,999,967,594 times among the first 200 billion decimal digits.

Last year, statistician Ted Jaditz of CNA Corp. in Alexandria, Va., systematically extended this type of frequency analysis of pi's digits to clusters up to 16 digits long. His statistical tests show no significant deviation from what would be expected for a string of random numbers.

A number is said to be "absolutely normal" if its digits are normal not only to base 10 but also to every integer base greater than or equal to 2. In base 2, for example, $\pi = 11.0010010000\ldots$. If pi is normal to base 2, the digits 1 and 0 would appear equally often.

Bolstered by the statistical evidence now available, mathematicians generally consider pi and several other fundamental constants, such as the square root of 2 and the natural logarithm of 2 (log 2), to be absolutely normal. However, they haven't yet mathematically proved that even one of these fundamental constants is normal to a particular base, much less to all bases.

"Current techniques don't even begin to dent the problem," says Peter B.

Borwein of Simon Fraser University in Burnaby, B.C.

Indeed, mathematicians have very little definitive information about the digits of pi and other irrational constants. "It is not even known that all digits appear infinitely often," Wagon remarks. In the case of pi, for example, no one can yet rule out the possibility that at some point beyond the range of current computations of pi's value, its decimal digits revert to a string constrained to, say, only the digits 1 and 0. If this were so, Wagon points out, it would alter the relative frequency of the digits.

A flavor of pi's normality

Proof of the normality of pi may come from a link between number theory and chaotic dynamics. While this link is too complicated to explain briefly for pi, the example of log 2 (the logarithm of 2 to base e, where e is the fundamental constant 2.718281828 . . . .) illustrates the mathematicians' new approach.

Log 2 can be obtained to any desired number of decimal places from the expression given below:

$\log 2 = 1/2 + 1/8 + 1/24 + 1/64 + \ldots$,
where each term has the form $1/k2^k$,
starting at $k = 1$

This value works out to 0.6931471805599453 . . . .

Bailey and Crandall have proposed that the normality of log 2 to base 2 is linked to a particular iterative process, or dynamical map, that generates a sequence of numbers between 0 and 1. Here's the mathematical form for this dynamical map:

$x_n = (2x_{n-1} + 1/n) \bmod 1$

Starting with $x_0 = 0$, each iteration, n, uses the previous result, x_{n-1}, as the input for calculating the next number, x_n. The term "mod 1" is an instruction to use only the fractional remainder of each iteration's result as input for the next iteration. In other words, no input is ever larger than 1.

The process generates the following sequence: $x_0 = 0, x_1 = 0, x_2 = 1/2, x_3 = 1/3, x_4 = 11/12, x_5 = 1/30, x_6 = 7/30, x_7 = 64/105, x_8 = 289/840 \ldots$. If it could be proved that the erratically fluctuating numbers x_n are evenly distributed between 0 and 1, log 2 would be deemed normal to base 2.

Establishing the same equidistribution property for a different, more complicated dynamical map would lead to a proof that pi is normal to base 16 (or, equivalently, to base 2). That would be a significant step toward the long-sought goal of proving pi's absolute normality. *—I.P.*

An amazing 1995 discovery in number theory provided the first hint of a new way to tackle the question of pi's normality.

Bailey, Borwein, and Simon Plouffe of the University of Quebec at Montreal unexpectedly found a simple formula that enables one to calculate isolated digits of pi—say, the trillionth digit—without computing and keeping track of all the preceding digits (SN: 10/28/95, p. 279). How such a formula possibly could arise constitutes a mystery in itself, mathematicians say.

The only catch is that the formula works for base 2 and 16 but not base 10.

So, it's possible to use the formula for determining that, say, the five-trillionth binary digit of pi is 0 (SN: 10/17/98, p. 255). But there's no way to convert the result into its decimal equivalent without knowing all the binary digits that come before the one of interest.

Similar formulas are now known for computing arbitrary, isolated digits of other mathematical constants, including log 2.

Bailey suspected early on that the existence of such formulas might have something to do with the normality of pi, log 2, and other mathematical constants. The formulas also reminded him of those used in certain computer algorithms to generate so-called pseudorandom numbers (SN: 11/9/91, p. 300). Computers follow such recipes to create strings of digits that pass for random numbers, which are often required for games, simulations, and other applications.

Crandall, who heads the Center for Advanced Computation at Reed, then identified a link between normality and so-called chaotic sequences of numbers that fall between 0 and 1. These sequences are extremely sensitive to tiny changes in the formula or starting point, and they hop from one value to another in an erratic manner. They've been used to model a variety of natural phenomena (SN: 10/31/98, p. 285).

Together, Bailey and Crandall established that if they could prove that the numbers of particular chaotic sequences (see box) are evenly distributed between 0 and 1, the normality to base 2 of pi and log 2 would follow automatically.

"What we have done is to translate a heretofore unapproachable problem to a more tractable question in the field of chaotic processes," Bailey says. "At the very least, we have shown why the [binary] digits of pi and log 2 appear to be random."

"This is interesting work," Borwein comments. "The link between [chaotic dynamics] and normality was a surprise to me."

Bailey and Crandall are now taking a closer look at the link between their hypothesis and algorithms for generating pseudorandom numbers. "This is likely the best route to making further progress on the problem" of proving that pi's digits are normal no matter how far out they go, Bailey says.

Some mathematicians are pessimistic about whether the Bailey-Crandall approach will eventually lead to a normality proof for pi and other fundamental constants. The pair's hypothesis about chaotic sequences may itself be too hard to prove, says Jeffrey C. Lagarias of AT&T Labs–Research at Florham Park, N.J.

Crandall responds by quoting German mathematician Carl Ludwig Siegel: "One cannot guess the real difficulties of a problem before having solved it."

As it stands now, pi and its mathematical cousins continue to present tantalizing mysteries that push mathematical research in new, unexpected directions. □

I. Peterson

Using the digits of pi to guide her selection of colors, artist Arlene Stamp of Calgary, Alberta, added an element of unpredictability to this tiling design in a Toronto subway station.