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A Nonlinear Option Pricing Model Through the Adomian Decomposition Method

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Corresponding Author:	Oswaldo González Gaxiola, Ph.D. Universidad Autonoma Metropolitana-Cuajimalpa México DF, Deleg. Cuajimalpa MEXICO
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	Universidad Autonoma Metropolitana-Cuajimalpa
Corresponding Author's Secondary Institution:	
First Author:	Oswaldo González Gaxiola, Ph.D.
First Author Secondary Information:	
Order of Authors:	Oswaldo González Gaxiola, Ph.D.
	Juan Ruíz de Chávez, Ph. D.
	José Antonio Santiago, Ph. D.
Order of Authors Secondary Information:	
Author Comments:	manuscript and seven figures
Suggested Reviewers:	Jorge Velasco, Ph. D. Professor, Universidad Nacional Autonoma de Mexico jx.velasco@im.unam.mx He is expert in applied mathematics
	E. Babolian, Ph. D. Professor, Teacher Training University, Tehran, Iran babolian@saba.tmu.ac.ir He is expert in applied mathematics (Adomian decomposition method).
	R. K. Bera, Ph. D. Professor, National Institute of Technical Teacher's Training and Research, Kolkata, India rasjit@yahoo.com He is expert in applied mathematics
	M. A. Wazwaz, Ph. D. Professor, Saint Xavier University, Chicago, USA wazwaz@sxu.edu He is expert in applied mathematics (Adomian decomposition method)
	José Ramón Pintos, Ph. D. Professor, Universidad Politécnica de Valencia, Valencia, Spain jrpt60@gmail.com He is expert in applied mathematics (Black-Scholes equation)

Highlights for Review:

The most significant work is in Section 4, because there the nonlinear case is studied and ADM is implemented through two examples.

Oswaldo González-Gaxiola
José A. Santiago.

Significance and novelty of this paper:

Dear Editor:

In recent years the liquidity of financial markets and transaction costs have

become a topic of great interest in financial risk management. In this paper, a nonlinear model of option pricing that appears when the effects of market

illiquidity and transaction costs are taken into account and an approximation solution is obtained through the semi analytical Adomian decomposition method. Finally, two numerical examples are studied to demonstrate the efficiency of the method used. This equation (nonlinear Black-Scholes) has been extensively studied by others (e.g. numerical) methods but not by this method. Finally, we clarify that the examples studied are hypothetical.

Oswaldo González-Gaxiola

José A. Santiago

List of responses to the comments and changes

Dear Editor, thanks for your kind e-mail, we also thank the reviewers for their comments and, in this letter we answer the referee's comments.

The answers to the reviewer # 1 are:

(i) We have made a careful review of the english, adding each of the corrections made by the reviewer in the document: **Corrections 10 March 2015 - for IACM-D-15-00024.pdf**.

(ii) We added all the references recommended by the reviewer, that is, we have cited five new papers (putting here the number that appears in the revised manuscript):
[1] Abdelrazec, A., Pelinovsky, D.: Convergence of the Adomian decomposition method for initial-value problems. Numer. Methods Partial Differential Eq. **27**, 749-766 (2011)
[16] Duan, J. S.: Convenient analytic recurrence algorithms for the Adomian polynomials. Appl. Math. Comput. **217**, 6337-6348 (2011)
[17] Duan, J.S., Rach, R., Wazwaz, A. M.: A new modified Adomian decomposition method for higher-order nonlinear dynamical systems. CMES: Computer Modeling in Engineering & Sciences. **94**(1), 77-118 (2013)
[18] Duan, J. S., Rach, R., Wazwaz, A. M., Chaolu, T., Wang, Z.: A new modified Adomian decomposition method and its multistage form for solving nonlinear boundary value problems with Robin boundary conditions. Appl. Math. Model. **37**, 8687-8708 (2013)
[29] Rach, R.: A bibliography of the theory and applications of the Adomian decomposition method, 1961-2011. Kybernetes **41**(7-8), 1087-1148 (2012).

The answers to the reviewer # 2 comment by comment are:

(1) As stated in the Introduction, Section 2 goal is to present in a briefly and self-contained manner, the ADM method. In this sense, it would be desirable to deepen a bit more on the ADM method in the paper, in case the reader has not access to the two references provided, and add some more references, if needed.

Our response to this comment is:

The subject of this work, is not to make a mathematical analysis of ADM, the purpose is only to use the method for solving a nonlinear Black- Scholes model. Anyway, Section 2 has been revised and the following reference [29], was added in order to make it more friendly to the interested reader. This is a very complete compendium about ADM and its applications.
[29] Rach, R.: A bibliography of the theory and applications of the Adomian decomposition method, 1961-2011. Kybernetes **41**(7-8), 1087-1148 (2012).

(2) Given that the model with the selected payoff functions allow exact solutions, and presumably selected for comparative purposes with the AD method, it would also be interesting to find the solution (by means of the ADM method) of another nonlinear PDE arising from a financial application of interest that does not allow exact solution, to complement the paper.

Our response to this comment is:

In our examples, we have chosen the payoff functions such that the requirements of existence and uniqueness of the Cauchy problem posed in [27], were fulfilled, i.e.
"the payoff function f must satisfy: $f(e^x)$ is Lipschitz-continuous and $e^{-a\sqrt{x^2+1}}f(e^x)$ is bounded for some $a \geq 0$".
We have not set an example for which the exact solution is not known, because we would have no way to compare the solution given by the ADM.

(3) Develop more on obtaining the recursive formula in (10) or give references to the reader.

Our response to this comment is:

The recursive formula (10), is obtained from equation (9), by doing at each step $n = 0, 1, 2, \ldots$, that can be seen in the reference(included now in the manuscript):

[35] Wenhai, C., Zhengyi L.: An algorithm for Adomian decomposition method. Applied Mathematics and Computation **159**, 221-235 (2004).

Moreover, in order to make self-contained, we have added the next paragraph, in page 3 before formula (10):

"with u_0 identified as $f(x) + L_t^{-1} g(x,t)$, and therefore, we can write

$$\begin{aligned} u_0(x,t) &= f(x) + L_t^{-1} g(x,t), \\ u_1(x,t) &= -L_t^{-1} R u_0(x,t) - L_t^{-1} A_0(u_0), \\ &\vdots \\ u_{n+1}(x,t) &= -L_t^{-1} R u_n(x,t) - L_t^{-1} A_n(u_0, \ldots, u_n)." \end{aligned}$$

(4) Please say more on how to determine the number of terms (k) in (11), given a level of convergence and/or give references to deepen into the topic. When stated "The Adomain decomposition method need less work in comparison with traditional methods", elaborate more on that, in which order?, etc.

Our response to this comment is:

For a more detailed about convergence, the following two references have been added, (also suggested by the first reviewer)

[1] Abdelrazec, A., Pelinovsky, D.: Convergence of the Adomian decomposition method for initial-value problems. Numer. Methods Partial Differential Eq. **27**, 749-766 (2011)

[36] Yong-Chang, J., Chuangyin, D., Yoshitsugu, Y.: An extension of the decomposition method for solving nonlinear equations and its convergence. Computers and Mathematics with Applications **55**, 760-775 (2008),

Morover, two new graphs have been included (fig. 4 and fig. 7), that show the goodness of the used method with respect to the convergence speed.

(5) In regard of Example 1, explain why the selection of payoff function f(S), or clarify if the choice was made only because with this payoff function one gets an exact solution by means of [15], (now ref. [20] in the revised manuscript). Although the paper intention is to show the power of the Adomain method, the authors selected a financial application. It would be desirable to have a financial interpretations in example 1, i.e., explain the payoff function (and the solution) in financial terms. Why do not try to solve the problem of an European call or put option?

Our response to this comment is:

As the aim of this work is to show the power of the method, the examples in this paper are hypothetical, and we have great care with the requirements to posed a well defined Cauchy problem such that the solution exists through ADM. On the other hand, in section 4: **A nonlinear case**, we mentioned that the nonlinear Black-Scholes model, works well in the context of european and american options, into a market where the nonlinearity is a result of illiquidity. Perhaps in a future work, we will be interested in financial problems.

(6) In Figures 1 to 5, it would have been interesting to graph the exact solution versus the approximation solution obtained by adding terms in the series expansion. In this way, one can see the relationship between the goodness of fit to an exact solution and adding terms in the series expansion solution. Also, by self-content purposes, include the expression of the exact solution obtained in [15] , (ref. [20] in the revised manuscript).

Our response to this comment is:

We added the equation (18), i.e. the exact solution founded in [20]:

$$u(S,t) = S - \frac{\sqrt{S_0}}{\rho}\Big(\sqrt{S}\exp(\frac{r+\frac{\sigma^2}{4}}{2})t + \frac{\sqrt{S_0}}{4}\exp(r+\frac{\sigma^2}{4})t\Big). \tag{18}$$

We also have included two graphs (fig. 4 and 7) that show the successive approximation to the solution.

Noname manuscript No.
(will be inserted by the editor)

A Nonlinear Option Pricing Model Through the Adomian Decomposition Method

O. González-Gaxiola · J. Ruíz de Chávez · J. A. Santiago

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Abstract Recently the liquidity of financial markets and transaction costs have become a topic of great interest in financial risk management. In this paper, a nonlinear model of option pricing that occurs when the effects of market illiquidity and transaction costs are taken into account and an approximate solution is obtained through the Adomian decomposition method. Finally, two numerical examples are investigated to demonstrate the efficiency of our approach.

Keywords Option pricing · Nonlinear Black-Scholes equation · Illiquid markets · Adomian decomposition method · Adomian polynomials

1 Introduction

Most of the phenomena that arise in the real world are described by nonlinear differential equations (ordinary and partial) and in some case by nonlinear integral equations. However, most of the methods developed so far in mathematics are valid only for solving linear differential equations. The decomposition method as developed by the mathematician George Adomian (1922-1996) has been very useful in applied mathematics, the applied sciences and engineering [32,33]. The Adomian decomposition method (ADM) has the advantage that its sequence of approximate solutions converges to the exact solution in the vast majority of important cases for very few

O. González-Gaxiola
Departamento de Matemáticas Aplicadas y Sistemas, Universidad Autónoma Metropolitana-Cuajimalpa. Vasco de Quiroga 4871, Santa Fe, Cuajimalpa, 05300, México D.F.
E-mail: ogonzalez@correo.cua.uam.mx

J. Ruíz de Chávez
Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa. San Rafael Atlixco 186, A.P. 55534, Col. Vicentina, Iztapalapa, 09340, México D.F.

J. A. Santiago
Departamento de Matemáticas Aplicadas y Sistemas, Universidad Autónoma Metropolitana-Cuajimalpa. Vasco de Quiroga 4871, Santa Fe, Cuajimalpa, 05300, México D.F.

prerequisites which are naturally satisfied when modeling physical phenomena, and applications can be easily handled for a wide class of ordinary and partial differential equations encompassing both linear and nonlinear cases [18]. In this paper, we will use the ADM specifically to solve the nonlinear Black-Scholes equation that arises when considering the market model in which volatility is not constant as a result of considering liquidity and transaction costs as set out in the work cited in [7,9,11, 22]. We will see that the nonlinear term can be easily handled with the help of the Adomian polynomials.
In Section 2, we present, in a brief and self-contained way, the ADM and several references are presented to delve deeper into the subject and to study its mathematical foundation that is beyond the scope of this present work. In Section 3, the Black-Scholes model is addressed in its standard linear version; by contrast, in Section 4, we show how the ADM can be implemented in order to solve a nonlinear model involving a modified volatility due to the market illiquidity. We also show, through two numerical examples, the efficacy and accuracy of the method by contrasting the results with the exact solution found indirectly in [20]. Finally, in Section 5, we present our conclusions.

2 Analysis of the Method

The ADM is a technique to solve ordinary and nonlinear differential equations. Using this method, it is possible to express analytic solutions in terms of a rapdily converging series [4]. In a nutshell, the method identifies and separates the linear and nonlinear parts of a differential equation. By inverting and applying the highest order differential operator that is contained in the linear part of the equation, it is possible to express the solution in terms of the rest of the equation affected by the inverse operator. At this point, the solution is proposed by means of a decomposition series with terms that will be determined by recursion and that gives rise to the solution components [34]. The nonlinear part is expressed in terms of the Adomian polynomials. The initial or the boundary conditions and the terms that contain the independent variables will be considered as the initial approximation. In this way and by means of a recurrence relation, it is possible to calculate the terms of the series by recursion that give the approximate solution of the differential equation. A reference to the topic of the general progress of the Adomian decomposition method can be found in [29].
Given a partial (or ordinary) differential equation

$$Fu(x,t) = g(x,t) \tag{1}$$

with the initial condition

$$u(x,0) = f(x), \tag{2}$$

where F is a differential operator that could itself, in general, be nonlinear and therefore includes linear and non-linear terms.
In general, equation (1) is be written as

$$L_t u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t) \tag{3}$$

where $L_t = \frac{\partial}{\partial t}$, R is the linear remainder operator that could include partial derivatives with respect to x, N is a nonlinear operator which is presumed to be analytic and g is a non-homogeneous term that is independent of the solution u.
Solving for $L_t u(x,t)$, we have

$$L_t u(x,t) = g(x,t) - Ru(x,t) - Nu(x,t). \tag{4}$$

As L is presumed to be invertible, we can apply $L_t^{-1}(\cdot) = \int_0^t (\cdot)dr$ to both sides of equation (4), obtaining

$$L_t^{-1} L_t u(x,t) = L_t^{-1} g(x,t) - L_t^{-1} Ru(x,t) - L_t^{-1} Nu(x,t). \tag{5}$$

An equivalent expression to (5) is

$$u(x,t) = f(x) + L_t^{-1} g(x,t) - L_t^{-1} Ru(x,t) - L_t^{-1} Nu(x,t). \tag{6}$$

where $f(x)$ is the constant of integration with respect to t that satisfies $L_t f = 0$. In equations where the initial value $t = t_0$, we can conveniently define L^{-1}.
The ADM proposes a decomposition series solution $u(x,t)$ given as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{7}$$

The nonlinear term $Nu(x,t)$ is given as

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n) \tag{8}$$

where $\{A_n\}_{n=0}^{\infty}$ is the Adomian polynomials sequence established in [6,34].
The Adomian polynomials have been studied in a formal manner in [16].
Substituting (7) and (8) into equation (6), we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + L_t^{-1} g(x,t) - L_t^{-1} R \sum_{n=0}^{\infty} u_n(x,t) - L_t^{-1} \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n), \tag{9}$$

with u_0 identified as $f(x) + L_t^{-1} g(x,t)$, and therefore, we can write

$$\begin{aligned}
u_0(x,t) &= f(x) + L_t^{-1} g(x,t), \\
u_1(x,t) &= -L_t^{-1} Ru_0(x,t) - L_t^{-1} A_0(u_0), \\
&\vdots \\
u_{n+1}(x,t) &= -L_t^{-1} Ru_n(x,t) - L_t^{-1} A_n(u_0, \ldots, u_n).
\end{aligned}$$

From which we can establish the following recurrence relation, that is obtained in a explicit way for instance in reference [35],

$$\begin{cases} u_0(x,t) = f(x) + L_t^{-1} g(x,t), \\ u_{n+1}(x,t) = -L_t^{-1} Ru_n(x,t) - L_t^{-1} A_n(u_0, u_1, \ldots, u_n), \ n = 0, 1, 2, \ldots. \end{cases} \tag{10}$$

Using (10), we can obtain an approximate solution of (1), (2) as

$$u(x,t) \approx \sum_{n=0}^{k} u_n(x,t), \text{ where } \lim_{k\to\infty} \sum_{n=0}^{k} u_n(x,t) = u(x,t). \tag{11}$$

This method has been successfully applied to a large class of both linear and nonlinear problems [17]. The Adomian decomposition method requires far less work in comparison with traditional methods [1]. This method considerably decreases the volume of calculations. The decomposition procedure of Adomian easily obtains the solution without linearizing the problem by implementing the decomposition method rather than the standard methods. In this approach, the solution is found in the form of a convergent series with easily computed components; in many cases, the convergence of this series is extremely fast and consequently only a few terms are needed in order to have an idea of how the solutions behave. Convergence conditions of this series have been investigated by several authors, e.g., [2,3,12,13].

3 The Black-Scholes equation (the standard case)

This model is a partial differential equation whose solution describes the value of an European Option, see [8,28]. Nowadays, it is widely used to estimate the pricing of options other than the European ones. For an European call or put on an underlying stock paying no dividends, the equation is:

$$V_\tau(S,\tau) + \frac{1}{2}\sigma^2 S^2 V_{SS}(S,\tau) + rSV_S(S,\tau) - rV(S,\tau) = 0, \tag{12}$$

where V is the price of the option as a function of underlying price S and time τ; with $0 \le \tau < T$, and $S \ge 0$; here the risk-free interest rate r and the volatility σ are assumed to be constant.

There are many varieties of options. European options may only be exercised on the maturity date. American options may be exercised any time up to and including the maturity date. The Asian option is an option whose payoff depends on the average price of the underlying asset during the period since the issue of the option until its expiration date, e.g., see [26] for details.

In the case of a European option, the value of the like-call option can be obtained from (12) with the boundary conditions:

$$\begin{cases} V_c(S,T) = \max(S-K,0) & : \tau = T \\ V_c(0,\tau) = 0 & : S = 0 \\ V_c(S,\tau) \to S - Ke^{-r(T-\tau)} & : S \to \infty. \end{cases}$$

We have that

$$V_c(S,\tau) = SN\left(\frac{1}{\sigma\sqrt{T-\tau}}\left[\ln(\frac{S}{K}) + (r+\frac{1}{2}\sigma^2)(T-\tau)\right]\right)$$
$$-e^{-r(T-\tau)}KN\left(\frac{1}{\sigma\sqrt{T-\tau}}\left[\ln(\frac{S}{K}) + (r-\frac{1}{2}\sigma^2)(T-\tau)\right]\right),$$

where $N(\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\eta} e^{-\frac{s^2}{2}} dx$ and $K > 0$ is the strike price.
Here $S = S(\tau)$ for $\tau \in [0,T]$; the solution of (12) provides both an option pricing formula for a European option and a hedging portfolio that replicates the contingent claim assuming that:

1. The asset price S or the value of the underlying asset follows a geometric Brownian motion.
2. The trend or drift μ which measures the average rate of growth of the asset price, the volatility σ which measures the standard deviation of the returns and the riskless interest rate r are all constant for $0 \leq \tau \leq T$, and no dividends are paid in that time period.
3. The market is assumed to be frictionless, thus there are no transaction costs such as fees or taxes, the interest rates for borrowing and lending money are equal, all parties have immediate access to any information, and all securities and credits are available at any time and any size. That is, all variables are perfectly divisible and may assume any real number. Moreover, individual trading is assumed to not influence the price.
4. There are no arbitrage opportunities, meaning that there are no opportunities for instantly extracting a risk-free profit.

Under these assumptions the market is complete, which means that any derivative and any asset can be replicated or hedged with a portfolio of other assets in the market.
The linear equation (12) has been studied using various mathematical techniques, for example, from the point of view of the operator theory in [25]; from the point of view of mathematical physics in [21,30], and through the ADM as a particular case study in [19]. In the next section, we study the Black-Scholes model for a nonlinear case.

4 A nonlinear case

It is easy to conceive that the restrictive assumptions mentioned at the end of the previous section are never actually fulfilled in real life as stated in [5]. Due to transaction costs, see for example, [7,11], large investor preferences, see [23,24], and for incomplete markets [31], such assumptions turn unrealistic and the classical model becomes a strongly nonlinear, possibly degenerate, parabolic convection-diffusion equation, where both the volatility σ and the drift μ can depend on the time τ, the stock price S or the derivatives of the option price V itself.
There are several transaction cost models from the most relevant class of nonlinear Black-Scholes equations for European and American options with a constant drift μ and a nonconstant modified volatility function

$$\hat{\sigma}^2 = \hat{\sigma}^2(\tau, S, V_S, V_{SS}). \tag{13}$$

There have been many approaches to improve the aforementioned model by treating the volatility in different ways, e.g., using a modified volatility function $\hat{\sigma}$ to model the effects of transaction costs, illiquid markets and large traders, which is the reason for the nonlinearity of (12). Illiquid markets and large trader effects have been modeled by several authors. In [24], Frey and Stremme and later Frey and Patie [23] consider these effects on the price and deduced the result

$$\hat{\sigma}(\tau,S,V_S,V_{SS}) = \frac{\sigma}{1-\rho S\lambda(S)u_{SS}}, \tag{14}$$

where σ the traditional volatility, ρ constant and λ (price of the risk) is a continuous positive function describing the liquidity profile of the market. Bordag and Chmakova in [10] assume that λ is a constant in order to solve a nonlinear Black-Scholes equation with the modified volatility (14).

In [22,23], the authors derived a generalized Black-Scholes pricing PDE that, for the case where the interest rate $r \geq 0$ and the reference volatility $\sigma > 0$ are constant, assumes the form

$$\begin{cases} V_\tau(S,\tau) + \frac{1}{2}\sigma^2 S^2 \cdot \frac{V_{SS}(S,\tau)}{(1-\rho S\lambda(S)V_{SS}(S,\tau))^2} + rSV_S(S,\tau) - rV(S,\tau) = 0, \\ V(S,T) = f(S), \;\; 0 < S < \infty. \end{cases} \tag{15}$$

In [27], the authors establish the existence and uniqueness of a classical solution to this nonlinear PDE, for the case where the payoff function f satisfies: $f(e^x)$ is Lipschitz-continuous and $e^{-a\sqrt{x^2+1}}f(e^x)$ is bounded for some $a \geq 0$. Recently two numerical studies of (15) have been completed in [14] and [15].

If we consider the special case where the price of risk is one unit, $\lambda(S) = 1$, in this case, and if we further assume that $||\rho SV_{SS}|| < \varepsilon$ for some $\varepsilon > 0$ that is small enough (low impact of hedging, assumption A4 of [23]) and using the functional approximation $\frac{1}{(1-F)^2} \approx 1 + 2F + \mathscr{O}(F)^3$, we find that eq. (15) becomes

$$\begin{cases} V_\tau(S,\tau) + \frac{1}{2}\sigma^2 S^2 V_{SS}(S,\tau)(1+2\rho SV_{SS}(S,\tau)) + rSV_S(S,\tau) - rV(S,\tau) = 0, \\ V(S,T) = f(S), \;\; S \in [0,\infty). \end{cases} \tag{16}$$

Note by considering the translation $t = T - \tau$, and denoting $V(S,\tau) = u(S,t)$, problem (16) assumes the form

$$\begin{cases} u_t(S,t) + \frac{1}{2}\sigma^2 S^2 u_{SS}(S,t)(1+2\rho Su_{SS}(S,t)) + rSu_S(S,t) - ru(S,t) = 0, \\ u(S,0) = f(S), \;\; S \in [0,\infty). \end{cases} \tag{17}$$

An exact solution of equation (17) has been obtained in [20], as

$$u(S,t) = S - \frac{\sqrt{S_0}}{\rho}\Big(\sqrt{S}\exp(\frac{r+\frac{\sigma^2}{4}}{2})t + \frac{\sqrt{S_0}}{4}\exp(r+\frac{\sigma^2}{4})t\Big), \tag{18}$$

where S_0 is the initial stock price.

4.1 The nonlinear case solved by the ADM

Comparing (17) with equation (3), we have that $g(S,t)=0$, and

$$L_t = \frac{\partial(\cdot)}{\partial t},\ R = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2(\cdot)}{\partial S^2} + rS\frac{\partial(\cdot)}{\partial S} - r,\ N = \rho\sigma^2 S^3 \Big(\frac{\partial^2(\cdot)}{\partial S^2}\Big)^2. \tag{19}$$

Referring to equation (10), establish the nonlinear recursion relation

$$\begin{cases} u_0(S,t) = f(S), \\ u_{n+1}(S,t) = -L_t^{-1}Ru_n(S,t) - \rho\sigma^2 S^3 L_t^{-1} A_n(u_0,u_1,\dots,u_n),\ n=0,1,2,\dots, \end{cases} \tag{20}$$

where $\left(\frac{\partial^2 u}{\partial S^2}\right)^2$, the nonlinear term,

$$F(u) = u_{ss}^2 = \sum_{n=0}^{\infty} A_n(u_0,u_1,\dots,u_n). \tag{21}$$

Referring to equation (7), the solution comprises its solution components

$$u = \sum_{n=0}^{\infty} u_n,$$

and the nonlinearity comprises the Adomian polynomials

$$F(u) = u_{ss}^2 = (u_{0,ss} + u_{1,ss} + u_{2,ss} + u_{3,ss} + u_{4,ss} + u_{5,ss} + \cdots)^2 \tag{22}$$

$$= u_{0,ss}^2 + 2u_{0,ss}u_{1,ss} + 2u_{0,ss}u_{2,ss} + 2u_{0,ss}u_{3,ss} + 2u_{0,ss}u_{4,ss} + 2u_{0,ss}u_{5,ss} + u_{1,ss}^2$$

$$+2u_{1,ss}u_{2,ss} + 2u_{1,ss}u_{3,ss} + 2u_{1,ss}u_{4,ss} + 2u_{1,ss}u_{5,ss} + u_{2,ss}^2 + 2u_{2,ss}u_{3,ss} + 2u_{2,ss}u_{4,ss}$$

$$+2u_{2,ss}u_{5,ss} + u_{3,ss}^2 + 2u_{3,ss}u_{4,ss} + 2u_{3,ss}u_{5,ss} + u_{4,ss}^2 + 2u_{4,ss}u_{5,ss} + u_{5,ss}^2 + \cdots.$$

The above expression can be rearranged by grouping all terms such that the sum of subscripts of u_n are equal. This procedure gives the Adomian polynomials [34] for the quadratic differential nonlinearity u_{ss}^2:

$$A_0 = u_{0,ss}^2$$

$$A_1 = 2u_{0,ss}u_{1,ss}$$

$$A_2 = 2u_{0,ss}u_{2,ss} + u_{1,ss}^2$$

$$A_3 = 2u_{0,ss}u_{3,ss} + 2u_{1,ss}u_{2,ss}$$

$$A_4 = 2u_{0,ss}u_{4,ss} + 2u_{1,ss}u_{3,ss} + u_{2,ss}^2$$

$$A_5 = 2u_{0,ss}u_{5,ss} + 2u_{1,ss}u_{4,ss} + 2u_{2,ss}u_{3,ss}$$

$$\vdots$$

Further study on the convergence of the ADM, which is beyond the scope of application of this work, can be found in references [1,36].

Example 1

In this first example, we consider the particular case of (17) such that $r = 0$; this case was proposed in [23] in which the authors studied a model of an illiquid market where the implementation of a dynamic hedging strategy has a feedback effect on the price process of the underlying asset. Here we propose $f(S) = S + 10(\sqrt{S} + \frac{1}{4})$, $\sigma = 0.2$, $\lambda(S) = 1$ and $|\rho| = 0.01$, noting that the prerequisite in [27] is satisfied. That is, for $a = 2$, it holds that

$$|e^{-4\sqrt{S^2+1}} f(e^S)| \leq 2 \text{ for every } S \in [0, \infty).$$

In this example, we denote the operators as

$$L_t = \frac{\partial(\cdot)}{\partial t}, \; R = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2(\cdot)}{\partial S^2}, \; N = \rho\sigma^2 S^3 \Big(\frac{\partial^2(\cdot)}{\partial S^2}\Big)^2.$$

In the ADM framework, we choose $u_0(S,t) = S + 10(\sqrt{S} + \frac{1}{4})$ and therefore, the first several Adomian polynomials are given as

$$A_0 = u_{0,ss}^2 = \frac{25}{4S^3},$$

$$A_1 = 2u_{0,ss}u_{1,ss} = -6.25 \times 10^{-2} t S^{-3},$$

$$A_2 = 2u_{0,ss}u_{2,ss} + u_{1,ss}^2 = 1.7188 \times 10^{-4} t^2 S^{-3}.$$

With these results and using (19) and (20) and considering the change of variable $t = T - \tau$, implying $L_t^{-1} = -\int_0^t$, and if $t \in [0, T]$, we calculate

$$\begin{aligned} u_1(S,t) &= -L_t^{-1} R u_0(S,t) - \rho\sigma^2 L_t^{-1} S^3 A_0(u_0) \\ &= -5 \times 10^{-2} t\sqrt{S} - 2.5 \times 10^{-3} t, \end{aligned}$$

$$\begin{aligned} u_2(S,t) &= -L_t^{-1} R u_1(S,t) - \rho\sigma^2 L_t^{-1} S^3 A_1(u_0, u_1) \\ &= 1.25 \times 10^{-5} t^2\sqrt{S} + 1.25 \times 10^{-5} t^2, \end{aligned}$$

$$\begin{aligned} u_3(S,t) &= -L_t^{-1} R u_2(S,t) - \rho\sigma^2 L_t^{-1} S^3 A_2(u_0, u_1, u_2) \\ &= -2.0833 \times 10^{-8} t^3\sqrt{S} - 2.2917 \times 10^{-8} t^3. \end{aligned}$$

Thus the solution of (17) is

$$u_{ADM}(S,t) = u_0(S,t) + u_1(S,t) + u_2(S,t) + u_3(S,t) + \cdots \tag{23}$$

In Table 1, we compare the solution of the nonlinear Black-Scholes equation obtained from (23) with the exact solution obtained in [20], in which a solution of (17) is obtained indirectly using traveling waves in porous media, equation (18). In figures 1, 2 and 3, we display the values of the option against the underlying asset prices obtained from ADM and the exact values for $t = 0.5$, $t = 1.0$ and $t = 1.5$ years. In addition, in figure 4, we show that the first terms values of the partial sum are not distinguish of the exact solution. All the numerical work was accomplished with the Mathematica software package.

	t=0.5			t=1.0			t=1.5		
S	$u_{ex}(S,t)$	$u_{ADM}(S,t)$	$\lvert\frac{u_{ADM}(S,t)-u_{ex}(S,t)}{u_{ex}(S,t)}\rvert$	$u_{ex}(S,t)$	$u_{ADM}(S,t)$	$\lvert\frac{u_{ADM}(S,t)-u_{ex}(S,t)}{u_{ex}(S,t)}\rvert$	$u_{ex}(S,t)$	$u_{ADM}(S,t)$	$\lvert\frac{u_{ADM}(S,t)-u_{ex}(S,t)}{u_{ex}(S,t)}\rvert$
0.0	2.51253	2.49875	0.00548378	2.52513	2.49751	0.01093530	2.53778	2.49628	0.0163547
0.5	10.1013	10.0521	0.00486605	10.1316	10.0332	0.00971248	10.1621	10.0143	0.0145394
1.0	13.5376	13.4738	0.00471328	13.5753	13.4475	0.00940872	13.6131	13.4213	0.0140864
1.5	16.2906	16.2156	0.00460693	16.3340	16.1837	0.00919712	16.3774	16.1519	0.0137706
2.0	18.6901	18.6055	0.00452265	18.7381	18.5690	0.00902937	18.7864	18.5324	0.0135202
2.5	20.8635	20.7706	0.00445178	20.9158	20.7299	0.00888827	20.9682	20.6891	0.0133095
3.0	22.8764	22.7760	0.00439009	22.9325	22.7314	0.00876545	22.9887	22.6869	0.0131261
3.5	24.7676	24.6603	0.00433518	24.8272	24.6123	0.00865610	24.8869	24.5643	0.0129628
4.0	26.5626	26.4488	0.00428552	26.6254	26.3975	0.00855719	26.6883	26.3463	0.0128150
4.5	28.2788	28.1589	0.00424006	28.3447	28.1047	0.00846664	28.4107	28.0504	0.0126798
5.0	29.9292	29.8035	0.00419806	29.9979	29.7464	0.00838298	30.0668	29.6893	0.0125548

Table 1: The values of u_{ADM} and u_{ex} and the relative difference between them for $t = 0.5$, 1.0 and 1.5 years

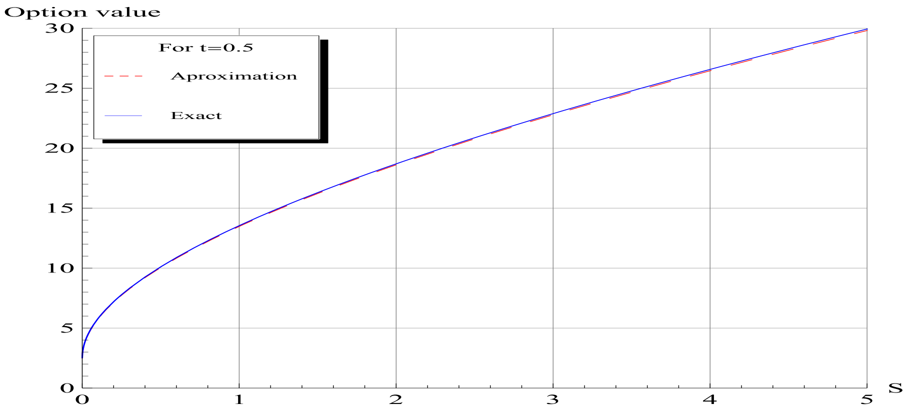

Fig. 1: Graph of the values of u_{ADM} and u_{ex} for $t = 0.5$ years

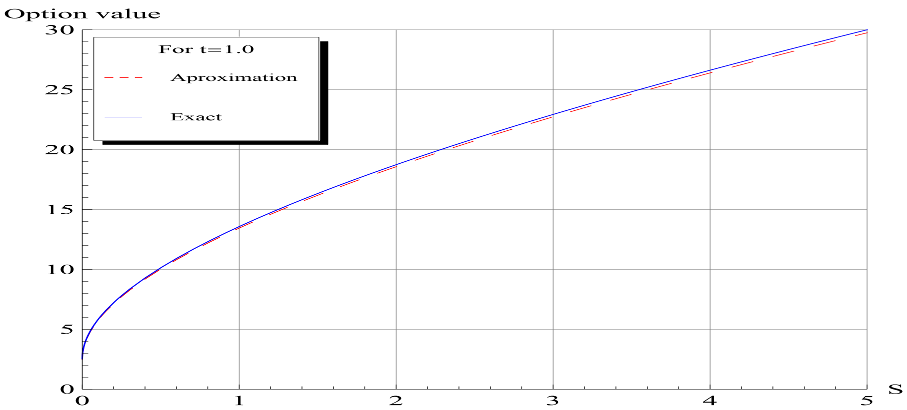

Fig. 2: Graph of the values of u_{ADM} and u_{ex} for $t = 1.0$ years

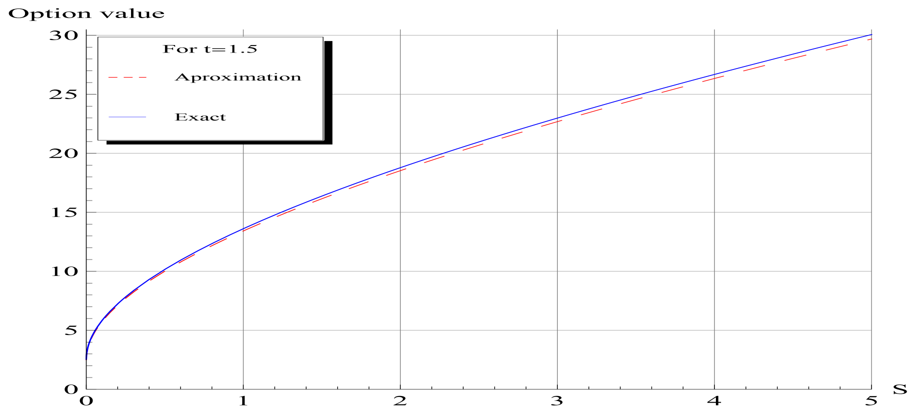

Fig. 3: Graph of the values of u_{ADM} and u_{ex} for $t = 1.5$ years

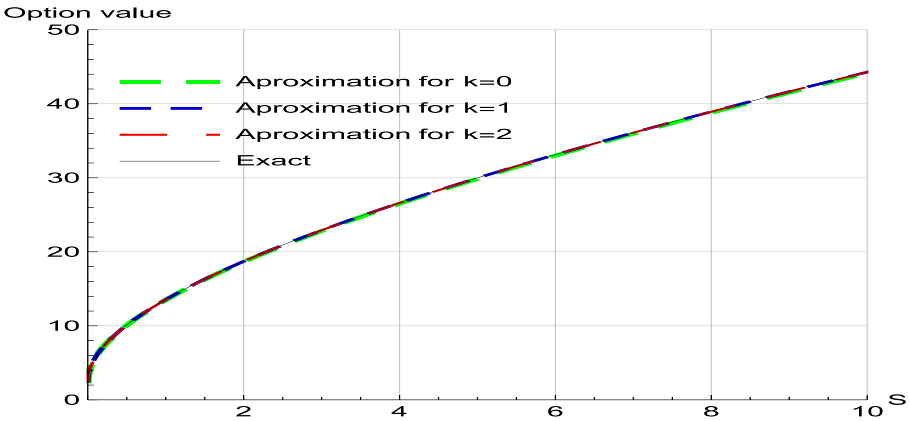

Fig. 4: Graph of the values of u_{ADM} for $k = 0,1,2$ and u_{ex} for $t = 1.0$ years

Example 2

In the second example, assume $f(S) = S + 200\sqrt{S} + 100$, $r = 0.06$, $\sigma = 0.4$ and $\rho = -0.01$. Notice as well that the prerequisite in [27] is fulfilled, that is, for $a = 5$ we have that

$$|e^{-5\sqrt{S^2+1}} f(e^S)| \leq 3 \text{ for every } S \in [0,\infty).$$

In the ADM framework, we have $u_0(S,t) = S + 200\sqrt{S} + 100$ thus

$$A_0 = u_{0,ss}^2 = \frac{2500}{S^3},$$

$$A_1 = 2u_{0,ss}u_{1,ss} = -\frac{250t}{S^3},$$

$$A_2 = 2u_{0,ss}u_{2,ss} + u_{1,ss}^2 = \frac{12.5t^2}{S^3},$$

$$A_3 = 2u_{0,ss}u_{3,ss} + 2u_{1,ss}u_{2,ss} = \frac{5t}{\sqrt{S^3}} - \frac{0.3125t^3}{S^3},$$

$$A_4 = 2u_{0,ss}u_{4,ss} + 2u_{1,ss}u_{3,ss} + u_{2,ss}^2 = \frac{1.2136\times 10^{-2}t^4 - 0.3St^2}{S^3}.$$

As before, using (19) and (20) and considering the translation $t = T - \tau$, implying $L_t^{-1} = -\int_0^t$, and if $t \in [0,T]$, we calculate

$$\begin{aligned} u_1(S,t) &= -L_t^{-1}Ru_0(S,t) - \rho\sigma^2 L_t^{-1}S^3A_0(u_0) \\ &= 2t + 10t\sqrt{S} + 0.06St, \end{aligned}$$

$$\begin{aligned} u_2(S,t) &= -L_t^{-1}Ru_1(S,t) - \rho\sigma^2 L_t^{-1}S^3A_1(u_0,u_1) \\ &= -0.25t^2\sqrt{S} + 0.14t^2, \end{aligned}$$

$$\begin{aligned} u_3(S,t) &= -L_t^{-1}Ru_2(S,t) - \rho\sigma^2 L_t^{-1}S^3A_2(u_0,u_1,u_2) \\ &= 4.1667\times 10^{-3}t^3\sqrt{S} - 9.4667\times 10^{-3}t^3, \end{aligned}$$

$$\begin{aligned} u_4(S,t) &= -L_t^{-1}Ru_3(S,t) - \rho\sigma^2 L_t^{-1}S^3A_3(u_0,u_1,u_2,u_3) \\ &= -5.2084\times 10^{-5}\sqrt{S}t^4 + 2.67\times 10^{-4}t^4 - 0.004t^2S^{\frac{3}{2}}, \end{aligned}$$

$$\begin{aligned} u_5(S,t) &= -L_t^{-1}Ru_4(S,t) - \rho\sigma^2 L_t^{-1}S^3A_4(u_0,\ldots,u_4) \\ &= -5.2085\times 10^{-5}\sqrt{S}t^5 - 6.5374\times 10^{-6}t^5 - 1.2\times 10^{-4}t^3S^{\frac{3}{2}} + 1.6\times 10^{-4}St^3, \end{aligned}$$

Thus the solution of (17) is

$$u_{ADM}(S,t) = u_0(S,t) + u_1(S,t) + u_2(S,t) + u_3(S,t) + u_4(S,t) + u_5(S,t) + \cdots. \quad (24)$$

As in the first example, we display the corresponding results in Table 2 and figures 5 and 6 below. In addition, in figure 7, we show that the first terms values of the partial sum are not distinguish of the exact solution.

	$t=0.0$			$t=0.5$			$t=1.0$		
S	$u_{ex}(S,t)$	$u_{ADM}(S,t)$	$\lvert\frac{u_{ADM}(S,t)-u_{ex}(S,t)}{u_{ex}(S,t)}\rvert$	$u_{ex}(S,t)$	$u_{ADM}(S,t)$	$\lvert\frac{u_{ADM}(S,t)-u_{ex}(S,t)}{u_{ex}(S,t)}\rvert$	$u_{ex}(S,t)$	$u_{ADM}(S,t)$	$\lvert\frac{u_{ADM}(S,t)-u_{ex}(S,t)}{u_{ex}(S,t)}\rvert$
0.0	100.000	100.000	0.0000000	105.127	101.034	0.0389338	110.517	102.131	0.0758797
0.5	241.921	248.992	0.0292285	250.629	249.997	0.0025216	259.689	250.978	0.0335439
1.0	301.000	311.000	0.0332226	311.190	312.001	0.0026061	321.771	312.941	0.0274418
1.5	346.449	358.696	0.0353514	357.777	359.698	0.0053692	369.525	360.609	0.0241282
2.0	384.843	398.985	0.0367478	397.130	399.988	0.0071966	409.861	400.877	0.0219196
2.5	418.728	434.539	0.0377605	431.860	435.546	0.0085351	445.458	436.415	0.0203004
3.0	449.410	466.731	0.0385405	463.307	467.742	0.0095724	477.688	468.595	0.0190354
3.5	477.666	496.374	0.0391661	492.265	497.390	0.0104119	507.367	498.228	0.0180126
4.0	504.000	524.000	0.0396825	519.253	525.022	0.0111101	535.026	525.847	0.0171561
4.5	528.764	549.977	0.0401185	544.631	551.005	0.0117033	561.034	551.818	0.0164268
5.0	552.214	574.574	0.0404915	568.662	575.608	0.0122146	585.660	576.410	0.0157941

Table 2: The values of u_{ADM} and u_{ex} and the relative difference between them for $t = 0.0$, 0.5 and 1.0 years

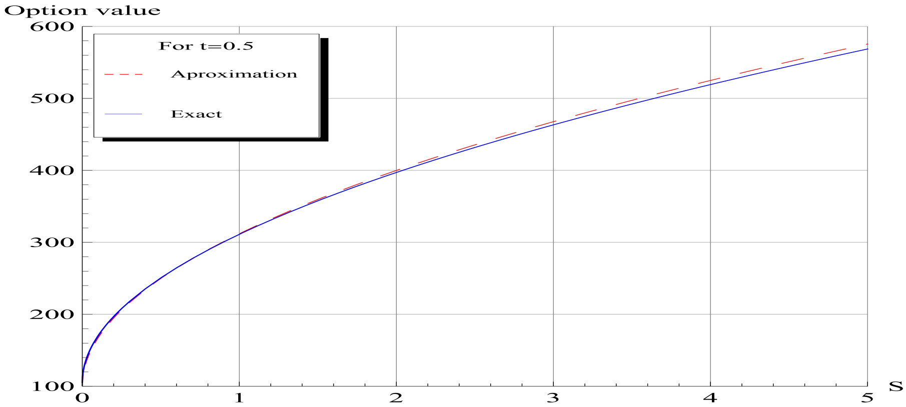

Fig. 5: Graph of u_{ADM} and u_{ex} for $t = 0.5$ years

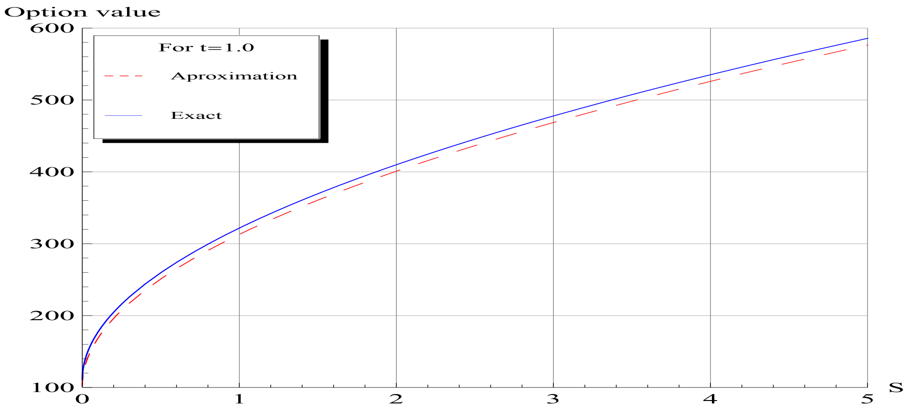

Fig. 6: Graph of u_{ADM} and u_{ex} for $t = 1.0$ years

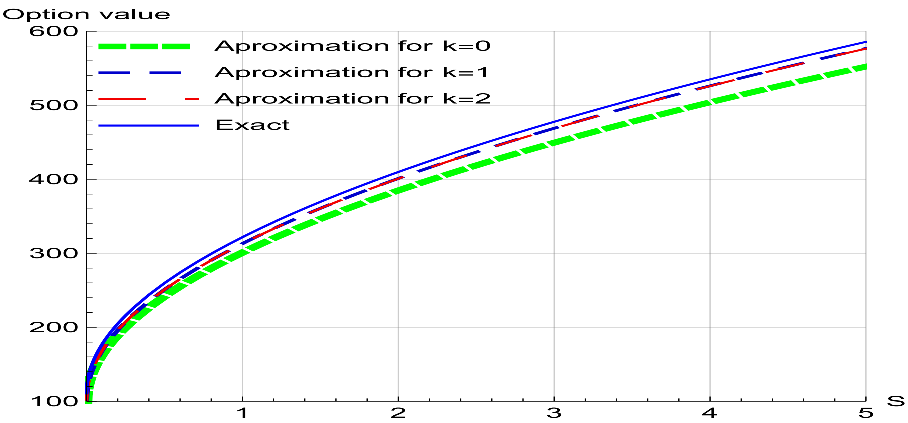

Fig. 7: Graph of the values of u_{ADM} for $k = 0, 1, 2$ and u_{ex} for $t = 1.0$ years

We remark that, as these examples demonstrate, the Adomian decomposition method avoids several difficulties in the calculation including massive computational work are required, e.g., by discretization techniques, in determining the approximate analytic solution.

5 Conclusion and summary

In this paper, we briefly discussed the Adomian decomposition method for approximate analytic solutions of both nonlinear ordinary and partial differential equations, and applied this method for the solution of the nonlinear Black- Scholes equation. We have also studied two hypothetical examples through the ADM. In the first example, we consider the case for which $r = 0$. As shown in table 1, the percentage error is not greater than 1.63 % despite that we have only considered an ADM approximation of third degree. In the second example, we considered a case in which $r \neq 0$ and as shown in table 2, the error percentage is not exceeding 7.58 % by considering an ADM approximation of fifth degree; in both examples, we have graphically compared our approximate analytic solutions by the ADM with the exact result found indirectly in [20] in the framework of a fluid traveling through porous media. It is also observed from the graphs 1, 2, 3, 5 and 6 that the option always exceeds the price of the underlying asset as a result of variable volatility. The results presented here provide further evidence of the practicality and efficiency of the Adomain decomposition method to find solutions of nonlinear partial differential equations such as the nonlinear Black-Scholes equation. In this paper, the MATHEMATICA software package was used to calculate the decomposition series.

References

1. Abdelrazec, A., Pelinovsky, D.: Convergence of the Adomian decomposition method for initial-value problems. Numer. Methods Partial Differential Eq. **27**, 749-766 (2011)
2. Abbaoui, K., Cherruault, Y.: Convergence of Adomian's method applied to differential equations. Comput. Math. Appl. **28**(5), 103-109 (1994)
3. Abbaoui, K., Cherruault, Y.: New ideas for proving convergence of decomposition methods. Comput. Math. Appl. **29**(7), 103-108 (1995)
4. Adomian, G.: Solving Frontier Problems of Physics: The Decomposition Method, Boston, MA: Kluwer Academic Publishers (1994)
5. Ankudinova, J., Ehrhardt, M.: On the numerical solution of nonlinear Black-Scholes equations. Computers and Mathematics with Applications **56**, 799-812 (2008)
6. Babolian, E., Javadi, Sh.: New method for calculating Adomian polynomials. Applied Math. and Computation **153**, 253-259 (2004)
7. Barles, G., Soner, H. M.: Option pricing with transaction costs and a nonlinear Black-Scholes equation. Finance Stoch. **2**, 369-397 (1998)
8. Black, F., Scholes, M.: The Pricing Options and Corporate Liabilities. Journal of Political Economy **81**(3), 637-654 (1973)
9. Bordag, L. A.: On option-valuation in illiquid markets: invariant solutions to a nonlinear model. Math. Control Theory and Finance, 71-94 (2008)
10. Bordag, L.A., Chmakova, A.: Explicit solutions for a nonlinear model of financial derivatives. Int. J. Theor. Appl. Finance **10**, 1-21 (2007)
11. Boyle, P., Vorst, T.: Option replication in discrete time with transaction costs. J. Finance **47**, 271-293 (1992)

12. Cherruault, Y.: Convergence of Adomian's method. Kybernetes **18**(2), 31-38 (1989)
13. Cherruault, Y., Adomian, G.: Decomposition methods: a new proof of convergence. Math. Comput. Modelling **18**(12), 103-106 (1993)
14. Company R., Jódar, L., Ponsoda, E., Ballester, C.: Numerical analysis and simulation of option pricing problems modeling illiquid markets. Comp. and Math. with Applications **59**, 2964-2975 (2010)
15. Company, R., Jódar L., Pintos J. R.: Numerical analysis and computing for option pricing models in illiquid markets. Math. and Computer Modelling **52**, 1066-1073 (2010)
16. Duan, J. S.: Convenient analytic recurrence algorithms for the Adomian polynomials. Appl. Math. Comput. **217**, 6337-6348 (2011)
17. Duan, J.S., Rach, R., Wazwaz, A. M.: A new modified Adomian decomposition method for higher-order nonlinear dynamical systems. CMES: Computer Modeling in Engineering & Sciences. **94**(1), 77-118 (2013)
18. Duan, J. S., Rach, R., Wazwaz, A. M., Chaolu, T., Wang, Z.: A new modified Adomian decomposition method and its multistage form for solving nonlinear boundary value problems with Robin boundary conditions. Appl. Math. Model. **37**, 8687-8708 (2013)
19. El-Wakil, S. A., Abdou, M. A., Elhanbaly, A.: Adomian decomposition method for solving the diffusion-convection-reaction equations. Applied Math. and Comp. **177**, 729-736 (2006)
20. Esekon, J. E.: Analytic solution of a nonlinear Black-Scholes equation. Int. J. of Pure and Applied Math. **82**(4) 547-555 (2013)
21. Fassari, S., Rinaldi, F.: On Some Potential Applications of the Heat Equation with a Repulsive Point Interaction to Derivative Pricing. Rendiconti di Matematica **31**, 35-52 (2011)
22. Frey, R.: Market illiquidity as a source of model risk in dynamic hedging, ed. by R. Gibson, Risk Publications, Risk Publications, London (2000)
23. Frey, R., Patie, P.: Risk Management for Derivatives in Illiquid Markets: A Simulation Study. Advances in Finance and Stochastics. 137-159 (2002)
24. Frey, R., Stremme, A.: Market volatility and feedback effects from dynamic hedging. Math. Finance **4**, 351-374 (1997)
25. González-Gaxiola, O., Santiago, J. A.: The Black-Scholes Operator as the Generator of a C_0-Semigroup and Applications. Int. J. of Pure and Applied Math. **76**(2), 191-200 (2012)
26. Hull, J.C.: Options, Futures and Other Derivatives, 5th ed., Prentice Hall, New Jersey (2002)
27. Liu, H., Yong J.: Option pricing with an illiquid underlying asset market. Journal of Economic Dynamics and Control **29**, 2125-2156 (2005)
28. Merton, R.C.: Theory of Rational Options Pricing. Bell Journal of Economic and Management Science **4**(1), 141-183 (1973)
29. Rach, R.: A bibliography of the theory and applications of the Adomian decomposition method, 1961-2011. Kybernetes **41**(7-8), 1087-1148 (2012)
30. Romero, J. M., González-Gaxiola, O., Ruiz de Chávez, J., Bernal-Jaquez, R.: The Black-Scholes Equation and Certain Quantum Hamiltonians. Int. Journal of Pure and Applied Math. **67**(2), 165-173 (2011)
31. Soner, H. M., Touzi, N.: Superreplication under gamma constraints. SIAM J. Contr. Optim. **39**, 73-96 (2001)
32. Tatari, M., Dehghan, M.: The use of the Adomian decomposition method for solving multipoint boundary value problems. The Royal Swedish Academy of Sciences **73**, 672-676 (2006)
33. Wazwaz, A. M.: Partial Differential Equations: Methods and Applications. Lisse, Netherlands: Balkema Publishers (2002)
34. Wazwaz, A. M.: A new algorithm for calculating Adomian polynomials for nonlinear operators. Appl. Math. and Computation **111**(1), 53-69 (2000)
35. Wenhai, C., Zhengyi L.: An algorithm for Adomian decomposition method. Applied Mathematics and Computation **159**, 221-235 (2004)
36. Yong-Chang, J., Chuangyin, D., Yoshitsugu, Y.: An extension of the decomposition method for solving nonlinear equations and its convergence. Computers and Mathematics with Applications **55**, 760-775 (2008).