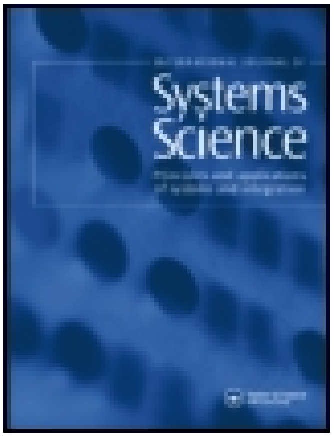

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Robust and optimal attitude control of spacecraft with disturbances

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Robust and optimal attitude control of spacecraft with disturbances

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In this paper, a robust and optimal attitude control design that uses the Euler angles and angular velocities feedback is presented for regulation of spacecraft with disturbances. In the control design, it is assumed that the disturbance signal has the information of the system state. In addition, it is assumed that the disturbance signal tries to maximise the same performance index that the control input tries to minimise. After proposing a robust attitude control law that can stabilise the complete attitude motion of spacecraft with disturbances, the optimal attitude control problem of spacecraft is formulated as the optimal game-theoretic problem. Then it is shown that the proposed robust attitude control law is the optimal solution of the optimal game-theoretic problem. The stability of the closed-loop system for the proposed robust and optimal control law is proven by the LaSalle invariance principle. The theoretical results presented in this paper are illustrated by a numerical example.

Keywords: spacecraft attitude control; robust control; optimal control; nonlinear systems

1. Introduction

1.1. Preliminary studies

Since spacecraft is in general subject to parameter uncertainties and disturbances in orbit, the robust attitude control problem of spacecraft has been studied by many researchers (e.g., Wie, Weiss, and Arapostathis 1989; Joshi, Kelkar, and Wen 1995; Karray and Modi 1995). Recently, Schaub, Akella, and Junkins (2000) presented an adaptive feedback control approach for nonlinear mechanical systems and applied this approach to spacecraft subject to actuator saturation constraints and parameter uncertainties. Park and Tahk (2001) presented a class of nonlinear robust attitude control laws for spacecraft with inertia uncertainties using the Cayley–Rodrigues parameters (Shuster 1993) and the modified Rodrigues parameters (Marandi and Modi 1987). Kim, Park, Bang, and Tahk (2003) proposed a proportional–integral–derivative control method for the attitude control of spacecraft with disturbances and gyroscope drifts. Also, they presented a covariance analysis method for evaluating the spacecraft attitude control system. Singh and Yim (2005) presented a nonlinear adaptive attitude control law for the control of the pitch angle of a satellite with parameter uncertainties using solar radiation torque. Zhou et al. (2010) developed a robust attitude control law for flexible spacecraft with parameter uncertainties. Liang, Wang, and Sun (2011) studied a robust decentralised coordinated attitude control of spacecraft formation in the presence of parameter uncertainties and external disturbances. Hu, Xiao, and Friswell (2011) investigated the robust fault-tolerant attitude control of an orbiting spacecraft with a combination of unknown actuator failure, input saturation, and external disturbances.

Subsequently, a considerable amount of efforts has been devoted to the optimal attitude control problem of spacecraft over the past several decades (e.g., Kumar 1965; Debs and Athans 1969; Dabbous and Ahmed 1982; Vadali, Kraige, and Junkins 1984; Tsiotras, Corless, and Rotea 1996). In these literatures, the authors studied the optimal attitude control problem of the angular velocity subsystem of spacecraft. As an extension of these early literatures, the optimal attitude control problem of the complete attitude motion of spacecraft, which includes dynamics as well as kinematics, has been studied: Carrington and Junkins (1986) used a polynomial expansion approach to approximate the solution of the Hamilton–Jacobi equation for the optimal attitude control problem of spacecraft. Tsiotras (1996) derived a class of globally asymptotically stabilising control laws for the complete attitude motion of non-symmetric spacecraft. Krstić and Tsiotras (1999) presented an inverse optimal control approach for constructing an optimal attitude control law for spacecraft. Park, Tahk, and Park (2001) proposed an optimal fuzzy attitude control law for spacecraft. Later, Park, Tahk, and Bang (2004) developed an optimal fuzzy attitude control law for spacecraft with control input constraint. Luo, Chu, and Ling (2005) studied the optimal attitude tracking control problem of rigid spacecraft with disturbances and an uncertain inertia matrix using the adaptive control method and the inverse optimality approach. Yang and Wu (2007)

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formulated the problem of finding an optimal rest-to-rest manoeuvring control law for spacecraft as a constrained nonlinear programming one and solved this optimal control problem by utilising an iterative procedure. Kuo, Kumar, Behdinan, and Fawaz (2008) presented an open-loop optimal attitude control law for stabilisation of miniature spacecraft using micro-electro-mechanical systems actuators. Pulecchi, Lovera, and Varga (2010) considered the problem of designing a discrete-time optimal attitude control law for stabilisation of magnetically actuated spacecraft. They proposed an approach to the tuning of various classes of projection-based control laws relying on periodic optimal output feedback control techniques. Horri, Palmer, and Roberts (2012) proposed an inverse optimal attitude control law for satellite to minimise the torque consumption and then modified their control law by gain scheduling to achieve a tradeoff enhancement compared with a linear state feedback controller.

1.2. *Motivation and contribution*

When we design an attitude control law for spacecraft, it is desirable that the attitude control law guarantees not only robustness with respect to disturbances but also optimality with respect to a performance index. With this motivation, in this paper, the robust and optimal attitude control problem is addressed for the complete attitude motion of spacecraft with disturbances. In order to describe the attitude motion of spacecraft with respect to the earth or inertia space, the orientation of one frame with respect to another frame should be specified. This work can be generally done through the use of the set of Euler angles or the unit quaternion. If we compare the set of Euler angles with the unit quaternion, the former has a singular configuration at some orientation angles and the latter also gives rise to a singular configuration at two equilibrium points. Such a mathematical singularity problem is the inherent property of the set of Euler angles and the unit quaternion (Wie 1995). However, the set of Euler angles allows us to interpret the orientation of spacecraft in an easy and physical way since the Euler angles physically represent the roll, pitch, and yaw angles of a body with respect to the local vertical and local horizontal frames. Therefore, the set of Euler angles is an useful kinematic parameter set to represent the orientation of spacecraft and will be used to describe the attitude motion of spacecraft in this paper.

In this paper, a robust and optimal attitude control law using the Euler angles and angular velocities feedback is proposed for regulation of the complete attitude motion of spacecraft with disturbances. After proposing a robust attitude control law that can stabilise the complete attitude motion of spacecraft with disturbances, the optimal attitude control problem of spacecraft is formulated as the optimal game-theoretic problem. Then it is shown that the proposed robust attitude control law is the solution of the optimal game-theoretic problem. In the optimal game-theoretic problem, it is assumed that the disturbance signal has the information of the system state and the control input and the disturbance signal are regarded as two players that compete each other for finding the optimal solution. The strategy of each player is based on a game point of view such that the control input tries to minimise the same performance index that the disturbance signal tries to maximise. The optimal game-theoretic problem considered in this paper is the worst-case problem since (i) the disturbance signal has the information of the system state and, thus, it has a direct impact on the system state; and (ii) the control input and the disturbance signal compete each other for finding the optimal solution and, thus, they conduct a keen competition on the edge of a knife. The optimal control scheme presented in this paper provides the analytic solution of the optimal attitude control problem coincided with the disturbance attenuation problem without the task of numerically solving the corresponding game-theoretic Hamilton–Jacobi equation. The stability of the closed-loop system for the proposed robust and optimal control law is proven by the LaSalle invariance principle in Khalil (1996). Compared to the previous studies, the robust and optimal attitude control design presented in this paper is, to the author's best knowledge, the first Euler angles and angular velocities feedback-based game-theoretic attitude control design that guarantees not only robustness with respect to disturbances but also optimality with respect to a performance index incorporating a penalty on both the angular position and velocity and the control input.

1.3. *Organisation*

The rest of this paper is organised as follows. Section 2 introduces the dynamic and kinematic equations of the rotational motion of spacecraft with disturbances. Section 3 presents a robust and optimal attitude control law using the Euler angles and angular velocities feedback for the attitude control of spacecraft with disturbances. Section 4 gives a numerical example to demonstrate the theoretical results presented in Section 3. This paper is concluded with some remarks in Section 5.

2. Attitude motion of spacecraft

2.1. *Dynamics*

The dynamic equation of the rotational motion of spacecraft with disturbances is given as follows (Junkins and Turnerm 1986):

$$J\dot{\omega}(t) = S(\omega(t))J\omega(t) + u(t) + G\xi(t), \qquad (1)$$

where $\omega(t) = [\,\omega_1(t)\ \omega_2(t)\ \omega_3(t)\,]^T \in \Re^3$ is the angular velocity vector of spacecraft in the spacecraft body-fixed frame, $u(t) = [\,u_1(t)\ u_2(t)\ u_3(t)\,]^T \in \Re^3$ is the control torque vector of spacecraft, $\xi(t) = [\,\xi_1(t)\ \xi_2(t)\ \xi_3(t)\,]^T \in \Re^3$ is the disturbance vector, $J \in \Re^{3\times 3}$ is the inertia matrix of spacecraft and satisfies $J = J^T > 0$, and $G \in \Re^{3\times 3}$ is the input matrix for $\xi(t)$. Moreover, $S(\omega(t)) \in \Re^{3\times 3}$ denotes a skew-symmetric matrix defined by

$$S(\omega(t)) \triangleq \begin{bmatrix} 0 & \omega_3(t) & -\omega_2(t) \\ -\omega_3(t) & 0 & \omega_1(t) \\ \omega_2(t) & -\omega_1(t) & 0 \end{bmatrix} \tag{2}$$

and has the following property:

$$\omega(t)^T S(\omega(t)) \equiv 0 \tag{3}$$

for all $\omega(t)$.

2.2. *Kinematics*

The kinematic equation of the rotational motion of spacecraft described in terms of the Euler angles is given as follows (Blakelock 1991):

$$\dot{r}(t) = F(r(t))\omega(t), \tag{4}$$

where $r(t) \triangleq [\,\phi(t)\ \theta(t)\ \psi(t)\,]^T \in \Re^3$ is the rotation vector consisting of the Euler angles $\phi(t)$, $\theta(t)$, and $\psi(t)$, which represent the roll, pitch, and yaw angles with respect to the local vertical and local horizontal frames, respectively, and $F(r(t)) \in \Re^{3\times 3}$ is a matrix defined by

$$F(r(t)) \triangleq \begin{bmatrix} 1 & \sin\phi(t)\tan\theta(t) & \cos\phi(t)\tan\theta(t) \\ 0 & \cos\phi(t) & -\sin\phi(t) \\ 0 & \sin\phi(t)\sec\theta(t) & \cos\phi(t)\sec\theta(t) \end{bmatrix}. \tag{5}$$

Remark 1: The matrix $F(r(t))$ of (5) holds for every orientation of the spacecraft body not only the pitch angles of $\theta(t) = (n + 1/2)\pi(t)$ rad, where n is an integer. Such a mathematical singularity problem for a certain orientation angle is the inherent property of all different sets of Euler angles. For example, the unit quaternion is a different set of Euler angles that has two equilibrium points at some rotation angles of a body about the Euler axis. The problem is that these two equilibrium points represent the same attitude of a body in the physical space and, thus, the sign ambiguity problem of the unit quaternion always happens at some rotation angles of a body about the Euler axis (Junkins and Turnerm 1986).

3. Main results

3.1. *Robust attitude control*

Before proceeding further, we have the following by the Rayleigh–Ritz's theorem (Horn 1985):

$$\lambda_{\min}(A)\|x(t)\|^2 \le x(t)^T A x(t), \tag{6}$$

where $A \in \Re^{n\times n}$ is a Hermitian matrix, $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of a given matrix, $x(t) \in \Re^n$, and $\|\cdot\|$ denotes the Euclidean norm of a vector (i.e., $\|x(t)\| \triangleq \sqrt{x(t)^T x(t)} = \sqrt{\sum_{i=1}^n x_i(t)^2}$ for $x(t) \in \Re^n$). Also, the spectral norm of a matrix $B \in \Re^{m\times n}$ is defined by

$$\|B\|_M \triangleq \sqrt{\lambda_{\max}(B^T B)}, \tag{7}$$

where $\lambda_{\max}(\,\cdot\,)$ denotes the largest eigenvalue of a given matrix (Horn 1985).

In the following, a robust attitude control law is proposed for asymptotically stabilising the complete attitude motion of spacecraft described by (1) and (4).

Theorem 1: *For the complete attitude motion of spacecraft given by (1) and (4), let the control law be*

$$u(t) = -K_\omega\omega(t) - F(r(t))^T K_r r(t), \tag{8}$$

where $K_\omega = K_\omega^T > 0$ and $K_r = K_r^T > 0$ are 3×3 positive definite Hermitian matrices and suppose that the following conditions hold:

$$\|\xi(t)\| \le \alpha\|\omega(t)\| \ (\triangleq \delta) \tag{9}$$

for all $\xi(t)$ and $\omega(t)$ and

$$\lambda_{\min}(K_\omega) > \alpha\|G\|_M, \tag{10}$$

where $\alpha > 0$ and $\delta \ge 0$ are constants. Then the control law $u(t)$ of (8) is asymptotically stabilising for the complete attitude motion of spacecraft.

Proof: With the choice of the following Lyapunov function candidate:

$$V(\omega(t), r(t)) = \frac{1}{2}\omega(t)^T J\omega(t) + \frac{1}{2}r(t)^T K_r r(t), \tag{11}$$

taking the time derivative of $V(\omega(t), r(t))$ of (11) along the trajectories of the closed-loop system with the control law

$u(t)$ of (8) and using the properties of (3) and (6), we obtain

$$\begin{aligned}\dot{V}(\omega(t), r(t)) &= \omega(t)^T J\dot{\omega}(t) + r(t)^T K_r \dot{r}(t)\\ &= \omega(t)^T \left(S(\omega(t))J\omega(t) - K_\omega \omega(t)\right.\\ &\quad - F(r(t))^T K_r r(t)\\ &\quad \left.+ G\xi(t)\right) + r(t)^T K_r F(r(t))\omega(t)\\ &= -\omega(t)^T K_\omega \omega(t) + \omega(t)^T G\xi(t)\\ &\le -\lambda_{\min}(K_\omega)\|\omega(t)\|^2 + \|\omega(t)\|\|G\|_M\|\xi(t)\|.\end{aligned} \tag{12}$$

Subsequently, if two conditions in (9) and (10) hold, then Equation (12) becomes

$$\begin{aligned}\dot{V}(\omega(t), r(t)) &\le -\lambda_{\min}(K_\omega)\|\omega(t)\|^2 + \|\omega(t)\|\|G\|_M\|\xi(t)\|\\ &\le -\left(\lambda_{\min}(K_\omega) - \alpha\|G\|_M\right)\|\omega(t)\|^2\\ &= -\left(\lambda_{\min}(K_\omega) - \alpha\|G\|_M\right)\left(\tfrac{\delta}{\alpha}\right)^2 \le 0\end{aligned} \tag{13}$$

for all $\omega(t)$. Since $V(\omega(t), r(t))$ is radially unbounded, every trajectory is included in a bounded set $\Omega_a = \{(\omega(t), r(t)) \in \Re^3 \times \Re^3 \mid V(\omega(t), r(t)) \le a\}$ for all $a > 0$. Also, since $V(\omega(t), r(t))$ is differentiable and positive definite and $\dot{V}(\omega(t), r(t)) \le 0$ for all $\omega(t)$, the LaSalle invariance principle (Khalil 1996) implies that every trajectory approaches the largest invariant set $\mathcal{A}_1$ in the set $\mathcal{B}_1 = \{(\omega(t), r(t)) \in \Re^3 \times \Re^3 \mid \dot{V}(\omega(t), r(t)) = 0\} = \{(\omega(t), r(t)) \in \Re^3 \times \Re^3 \mid \omega(t) = 0\}$ as $t \to \infty$. In the set $\mathcal{A}_1$, we obtain $\omega(t) = 0$ and $\dot{\omega}(t) = 0$ and, thus, $u(t) = -G\xi(t)$ from Equation (1) and $\|u(t)\| = \|G\xi(t)\| \le \|G\|_M\|\xi(t)\| \le \alpha\|G\|_M\|\omega(t)\| = 0$ from the condition in (9) and $\omega(t) = 0$. Therefore, in the set $\mathcal{A}_1$, the control torque vector becomes $u(t) = 0$, and this implies that the rotation vector becomes $r(t) = 0$ since $u(t) = -K_\omega\omega(t) - F(r(t))^T K_r r(t) = 0$ from Equation (8), $u(t) = 0$, and $\omega(t) = 0$. Hence, we conclude that $\mathcal{A}_1 = \{(\omega(t), r(t)) \in \Re^3 \times \Re^3 \mid \omega(t) = 0, r(t) = 0\}$. This completes the proof. □

Remark 2: We observe that the robust attitude control law in (8) does not require any information of the inertia of spacecraft. When we develop spacecraft, it is generally difficult to exactly know the inertia values of the spacecraft body since some components of spacecraft such as actuators, sensors, payloads, electronics, and electric wires are integrated with the spacecraft body and the inertia values of the spacecraft body are calculated or simulated with these components. Therefore, the robust attitude control law in (8) provides convenience to us when we design an attitude control law in the case that we do not exactly know the inertia values of the spacecraft body.

Remark 3: From the condition in (9), we see that the magnitude of disturbances is bounded by $\alpha\|\omega(t)\|$ ($\triangleq\delta$). Since we know the angular velocity vector $\omega(t)$ and can adjust the constant α, we can control a permissible level of disturbances when we design the robust attitude control law in (8). Since $G \in \Re^{3\times3}$ is the input matrix for the disturbance vector $\xi(t)$, the input matrix G plays the role of transmitting the disturbance signal to spacecraft. Therefore, in order to obtain the input matrix G, we may apply the known disturbance signal to spacecraft and detect the torque that is generated by the known disturbance signal. In the condition in (10), assume that the input matrix G is given. Then, as the constant α increases, $\lambda_{\min}(K_\omega)$ shall also increase in order to satisfy the condition in (10). From the result, if we want to design the robust attitude control law in (8) having a large robustness margin with respect to disturbances, it should be designed with large values of α and $\lambda_{\min}(K_\omega)$ satisfying the conditions in (9) and (10).

Specially, the following lemma is presented for the case of knowing the values of the inertia elements of spacecraft and $G = 0_{3\times3}$, where $0_{3\times3}$ denotes the 3×3 zero matrix.

Lemma 1: *For the complete attitude motion of spacecraft given by* (1) *and* (4)*, let the input matrix for the disturbance vector be $G = 0_{3\times3}$ and the feedback gain matrix K_ω for the control law u(t) of* (8) *be*

$$K_\omega = \beta J, \tag{14}$$

where $\beta > 0$ is a constant. Then the maximal decay rate of the closed-loop system with the control law u(t) of (8) becomes β.

Proof: Under the assumption of $G = 0_{3\times3}$, if we take the time derivative of the Lyapunov function candidate $V(\omega(t), r(t))$ of (11) along the trajectories of the closed-loop system with the control law $u(t)$ of (8) and use the property of (3), we obtain

$$\begin{aligned}\dot{V}(\omega(t), r(t)) &= \omega(t)^T J\dot{\omega}(t) + r(t)^T K_r \dot{r}(t)\\ &= \omega(t)^T (S(\omega(t))J\omega(t) - K_\omega\omega(t)\\ &\quad - F(r(t))^T K_r r(t))\\ &\quad + r(t)^T K_r F(r(t))\omega(t)\\ &= -\omega(t)^T K_\omega \omega(t) \le 0\end{aligned} \tag{15}$$

for all $\omega(t)$. Since $V(\omega(t), r(t))$ is radially unbounded, every trajectory is included in a bounded set $\Omega_b = \{(\omega(t), r(t)) \in \Re^3 \times \Re^3 \mid V(\omega(t), r(t)) \le b\}$ for all $b > 0$. Also, since $V(\omega(t), r(t))$ is differentiable and positive definite and $\dot{V}(\omega(t), r(t)) \le 0$ for all $\omega(t)$, the LaSalle invariance principle (Khalil 1996) implies that every trajectory approaches the largest invariant set $\mathcal{A}_2$ in the set $\mathcal{B}_2 = \{(\omega(t), r(t)) \in \Re^3 \times \Re^3 \mid \dot{V}(\omega(t), r(t)) = 0\} = \{(\omega(t), r(t)) \in \Re^3 \times \Re^3 \mid \omega(t) = 0\}$ as $t \to \infty$. In the set $\mathcal{A}_2$, we obtain $\omega(t) = 0$ and $\dot{\omega}(t) = 0$ and, thus, $u(t) = -G\xi(t) = 0$ from Equation (1) and the assumption of $G = 0_{3\times3}$. Therefore, in the set $\mathcal{A}_2$, the control torque vector becomes $u(t) = 0$, and this implies that the rotation vector becomes $r(t) = 0$ since $u(t) = -K_\omega\omega(t) - F(r(t))^T K_r r(t) = 0$ from Equation (8), $u(t) = 0$, and $\omega(t) = 0$. Hence, we conclude that $\mathcal{A}_2 = \{(\omega(t), r(t)) \in \Re^3 \times \Re^3 \mid \omega(t) = 0, r(t) = 0\}$.

Then, with the choice of K_ω given in (14), Equation (15) becomes

$$\begin{aligned}\dot{V}(\omega(t), r(t)) &= -\omega(t)^T K_\omega \omega(t) \\ &= -\beta\omega(t)^T J\omega(t) - \beta r(t)^T K_r r(t) \\ &\quad + \beta r(t)^T K_r r(t) \\ &= -2\beta\left(\tfrac{1}{2}\omega(t)^T J\omega(t) + \tfrac{1}{2} r(t)^T K_r r(t)\right) \\ &\quad + \beta r(t)^T K_r r(t) \\ &= -2\beta V(\omega(t), r(t)) + \beta r(t)^T K_r r(t).\end{aligned} \tag{16}$$

From Equation (16), we obtain

$$\dot{V}(\omega(t), r(t)) + 2\beta V(\omega(t), r(t)) = \beta r(t)^T K_r r(t) \geq 0 \tag{17}$$

for all $\omega(t)$ and $r(t)$ and, thus,

$$V(\omega(t), r(t)) \geq V(\omega(t_0), r(t_0))\, e^{-2\beta(t-t_0)} \tag{18}$$

for all $\omega(t)$ and $r(t)$, where t_0 is the initial time. Equation (18) implies that the maximal decay rate of the closed-loop system is β. This completes the proof. □

Remark 4: From the result of Lemma 1, we see that the feedback gain matrix K_ω has a close relation with the inertia of spacecraft and the maximal decay rate of the closed-loop system. Therefore, the result of Lemma 1 can be used as a guideline to select K_ω. In a practical point of view, the values of the inertia elements of spacecraft may have an impact on the attitude control performance and, thus, K_ω should be chosen carefully.

3.2. Optimal attitude control

For the optimal attitude control of the complete attitude motion of spacecraft given by (1) and (4), let us consider the worst-case problem that the disturbance signal has the information of the system state and the disturbance signal tries to maximise the same performance index that the control input tries to minimise, which is called the optimal game-theoretic problem (Dorato, Abdallah, and Cerone 1995). Note that Dorato et al. (1995) presented the optimal game-theoretic problem for linear systems only. Since the complete attitude motion of spacecraft given by (1) and (4) describes a nonlinear system, it may be a formidable and intractable task to solve the corresponding optimal game-theoretic problem. In the following, this optimal game-theoretic problem will be analytically solved without the task of numerically solving the corresponding game-theoretic Hamilton–Jacobi equation, and the robust attitude control law in (8) will be shown to be the optimal solution of this optimal game-theoretic problem.

Theorem 2: *For the complete attitude motion of spacecraft given by* (1) *and* (4)*, the control law in* (8) *and the disturbance signal given by*

$$\xi(t) = W^{-1} G^T \omega(t), \tag{19}$$

where $W > 0$ is a 3×3 positive definite Hermitian matrix, are optimal with respect to the performance index

$$\mathcal{P} = \frac{1}{2}\int_0^\infty \left[x(t)^T Q x(t) + 2u(t)^T N x(t) + u(t)^T R u(t) - \xi(t)^T W \xi(t)\right] dt, \tag{20}$$

where $x(t) \triangleq [\omega(t)^T \;\; r(t)^T]^T \in \Re^{6\times1}$ is the state vector and $Q \in \Re^{6\times6}$, $N \in \Re^{3\times6}$, and $R \in \Re^{3\times3}$ are defined by

$$Q \triangleq \begin{bmatrix} K_\omega - G W^{-1} G^T & 0_{3\times3} \\ 0_{3\times3} & K_r F(r(t)) K_\omega^{-1} F(r(t))^T K_r \end{bmatrix}, \tag{21}$$

$$N \triangleq \begin{bmatrix} 0_{3\times3} & K_\omega^{-1} F(r(t))^T K_r \end{bmatrix}, \tag{22}$$

and

$$R \triangleq K_\omega^{-1}, \tag{23}$$

respectively, and K_ω satisfies

$$K_\omega > G W^{-1} G^T. \tag{24}$$

Proof: The game-theoretic Hamilton–Jacobi equation associated with the performance index $\mathcal{P}$ of (20) and the complete attitude motion of spacecraft described by (1) and (4) are given by

$$\begin{aligned}-\frac{\partial\mathcal{P}}{\partial t} = \min_{u(t)} \max_{\xi(t)} \Bigg\{ &\frac{1}{2} x(t)^T Q x(t) + u(t)^T N x(t) \\ &+ \frac{1}{2} u(t)^T R u(t) - \frac{1}{2}\xi(t)^T W \xi(t) \\ &+ \frac{\partial\mathcal{P}}{\partial\omega(t)}[J^{-1} S(\omega(t)) J\omega(t) + J^{-1} u(t) \\ &+ J^{-1} G\xi(t)] + \frac{\partial\mathcal{P}}{\partial r(t)}[F(r(t))\omega(t)]\Bigg\},\end{aligned} \tag{25}$$

where $\partial\mathcal{P}/\partial\omega(t)$ and $\partial\mathcal{P}/\partial r(t)$ denote the gradients of $\mathcal{P}$ with respect to $\omega(t)$ and $r(t)$, respectively, and are given by the row vectors. The optimal solutions of $u(t)$ and $\xi(t)$ to Equation (25), which are denoted with $u^*(t)$ and $\xi^*(t)$, respectively, can be found by setting the gradients of the right-hand side term in (25) with respect to $u(t)$ and $\xi(t)$ to the zero vectors. By this procedure, one can obtain $u^*(t)$ and $\xi^*(t)$ as follows:

$$u^*(t) = -F(r(t))^T K_r r(t) - K_\omega J^{-1} \left[\frac{\partial\mathcal{P}}{\partial\omega(t)}\right]^T \tag{26}$$

and

$$\xi^*(t) = W^{-1} G^T J^{-1} \left[\frac{\partial \mathcal{P}}{\partial \omega(t)} \right]^T. \tag{27}$$

If we substitute $u^*(t)$ of (26) and $\xi^*(t)$ of (27) into Equation (25), we obtain

$$\begin{aligned}
-\frac{\partial \mathcal{P}}{\partial t} = \min_{u(t)} \max_{\xi(t)} \Bigg\{ & \frac{1}{2}\omega(t)^T K_\omega \omega(t) - \frac{1}{2}\omega(t)^T G W^{-1} G^T \omega(t) \\
& - \left[\frac{\partial \mathcal{P}}{\partial \omega(t)} \right] J^{-1} F(r(t))^T K_r r(t) \\
& + \left[\frac{\partial \mathcal{P}}{\partial \omega(t)} \right] J^{-1} S(\omega(t)) J \omega(t) \\
& + \left[\frac{\partial \mathcal{P}}{\partial r(t)} \right] F(r(t))\omega(t) - \frac{1}{2} \left[\frac{\partial \mathcal{P}}{\partial \omega(t)} \right] J^{-1} K_\omega J^{-1} \left[\frac{\partial \mathcal{P}}{\partial \omega(t)} \right]^T \\
& + \tfrac{1}{2} \left[\frac{\partial \mathcal{P}}{\partial \omega(t)} \right] J^{-1} G W^{-1} G^T J^{-1} \left[\frac{\partial \mathcal{P}}{\partial \omega(t)} \right]^T \Bigg\}.
\end{aligned} \tag{28}$$

Now, consider the Lyapunov function candidate $V(\omega(t), r(t))$ of (11). Then $V(\omega(t), r(t))$ of (11) solves Equation (28). Indeed, note that

$$\frac{\partial V}{\partial t} = 0, \ \frac{\partial V}{\partial \omega(t)} = \omega(t)^T J, \ \frac{\partial V}{\partial r(t)} = r(t)^T K_r. \tag{29}$$

Moreover, if we replace $\mathcal{P}$ of (28) by $V(\omega(t), r(t))$ of (11) and use the property of (3), then we see that $V(\omega(t), r(t))$ of (11) becomes the solution of Equation (28). Therefore, with $V(\omega(t), r(t))$ of (11), $u^*(t)$ of (26) and $\xi^*(t)$ of (27) become $u(t)$ of (8) and $\xi(t)$ of (19), respectively.

Now, let the condition in (24) hold. Taking the time derivative of $V(\omega(t), r(t))$ of (11) along the trajectories of the closed-loop system with $u(t)$ of (8) and $\xi(t)$ of (19) and using the property of (3) yield

$$\begin{aligned}
\dot{V}(\omega(t), r(t)) &= \omega(t)^T J \dot{\omega}(t) + r(t)^T K_r \dot{r}(t) \\
&= \omega(t)^T \big[S(\omega(t)) J \omega(t) - F(r(t))^T K_r r(t) \\
&\quad - K_\omega \omega(t) + G W^{-1} G^T \omega(t) \big] \\
&\quad + r(t)^T K_r F(r(t)) \omega(t) \\
&= -\omega(t)^T \left(K_\omega - G W^{-1} G^T \right) \omega(t) \le 0
\end{aligned} \tag{30}$$

for all $\omega(t)$. Since $V(\omega(t), r(t))$ is radially unbounded, every trajectory is included in a bounded set $\Omega_c = \{(\omega(t), r(t)) \in \Re^3 \times \Re^3 \mid V(\omega(t), r(t)) \le c\}$ for all $c > 0$. Also, since $V(\omega(t), r(t))$ is differentiable and positive definite and $\dot{V}(\omega(t), r(t)) \le 0$ for all $\omega(t)$, the LaSalle invariance principle (Khalil 1996) implies that every trajectory approaches the largest invariant set $\mathcal{A}_3$ in the set $\mathcal{B}_3 = \{(\omega(t), r(t)) \in \Re^3 \times \Re^3 \mid \dot{V}(\omega(t), r(t)) = 0\} = \{(\omega(t), r(t)) \in \Re^3 \times \Re^3 \mid \omega(t) = 0\}$ as $t \to \infty$. In the set $\mathcal{A}_3$, we obtain $\omega(t) = 0$ and $\dot{\omega}(t) = 0$ and, thus, $u(t) = -G\xi(t) = -GW^{-1}G^T\omega(t) = 0$ from Equations (1) and (19), and $\omega(t) = 0$. Therefore, in the set $\mathcal{A}_3$, the control torque vector becomes $u(t) = 0$, and this implies that the rotation vector becomes $r(t) = 0$ since $u(t) = -K_\omega\omega(t) - F(r(t))^T K_r r(t) = 0$ from Equation (8), $u(t) = 0$, and $\omega(t) = 0$. Hence, we conclude that $\mathcal{A}_3 = \{(\omega(t), r(t)) \in \Re^3 \times \Re^3 \mid \omega(t) = 0, r(t) = 0\}$. This completes the proof.□

Remark 5: If we set $G = 0_{3\times3}$ in (1) and $W = 0_{3\times3}$ in (20) and apply Theorem 2, we obtain the optimal attitude control problem of spacecraft without disturbances. Hence, the optimal game-theoretic problem considered in this section can be regarded as the generalisation of the optimal attitude control problem of spacecraft.

Remark 6: The optimal attitude control law $u(t)$ of (8) with respect to the performance index in (20) can be expressed as

$$u(t) = -R^{-1} \{ [I_{3\times3} \ \ 0_{3\times3}] + N \} \begin{bmatrix} \omega(t) \\ r(t) \end{bmatrix}, \tag{31}$$

where R and N are defined in (23) and (22), respectively, and $I_{3\times3}$ denotes the 3×3 identity matrix. Therefore, from Equations (20) and (31), we see that the functions of the weight matrix N are to impose a cross-penalty on the control input and state vectors and to adjust a feedback gain for the Euler angles.

Remark 7: As shown in Equations (20)–(23), the feedback gain matrix K_ω has an impact on the state and control input. Indeed, the penalty on the control input decreases as the penalty on the state increases. In this case, the control input becomes aggressive in order to make the closed-loop system stable within a short period of time. On the other hand, the penalty on the control input increases as the penalty on the state decreases. This makes the control input less aggressive and results in slow response of the closed-loop system state. In these situations, as shown in Equations (20) and (21), the penalty on the Euler angles can be adjusted by the feedback gain matrix K_r besides the feedback gain matrix K_ω. Therefore, the result of Theorem 2 provides a desirable feature of the optimal control scheme.

Remark 8: In the point of view of spacecraft, any disturbing torque appearing in spacecraft may be regarded as disturbances whatever they come from. External disturbing torques applied to spacecraft and internal disturbing torques induced by spacecraft are examples of disturbances. Indeed, the solar radiation applied to spacecraft makes external disturbing torques. Also, the momentum wheel or thruster gives rise to internal disturbing torques. These unwanted disturbing torques make the spacecraft body rotate and/or translate at some speed. Thus, we see that these unwanted disturbing torques are closely related with the angular velocity of spacecraft. Therefore, without loss of generality, the disturbance signal $\xi(t)$ of (19) makes sense. For the matrices W and G for the disturbance signal $\xi(t)$ of (19), if the constant α in (9) is chosen such that

$$\alpha \ge \| W^{-1} G^T \|_M, \tag{32}$$

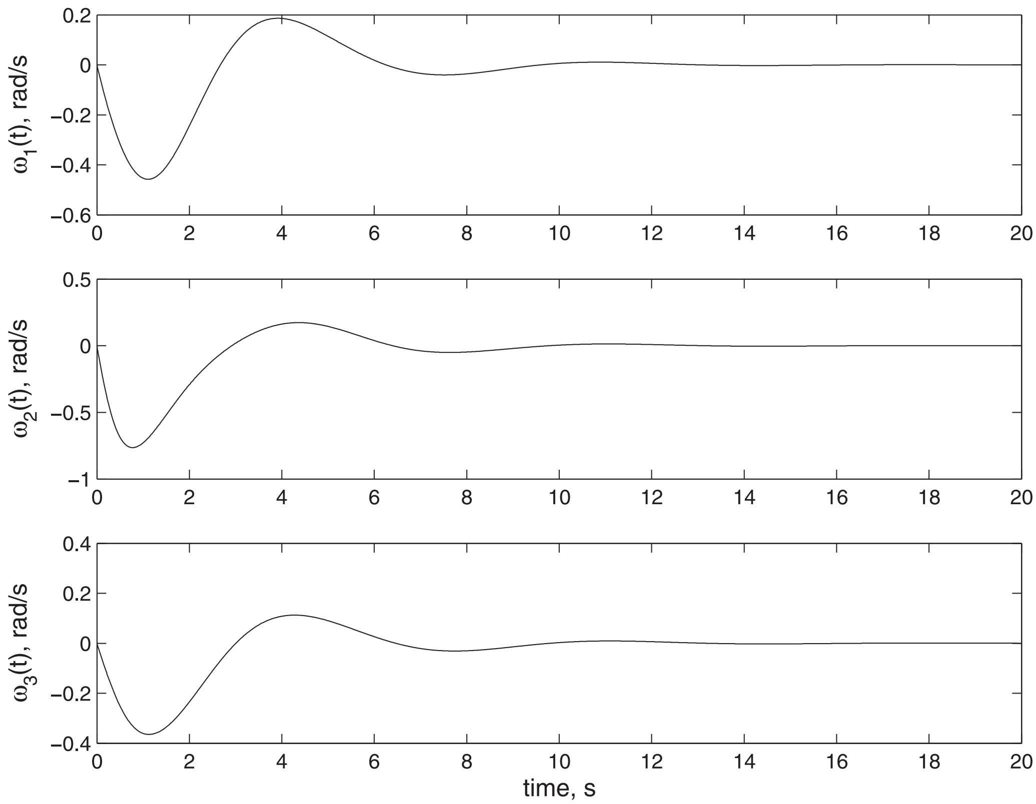

Figure 1. Time histories of the angular velocities.

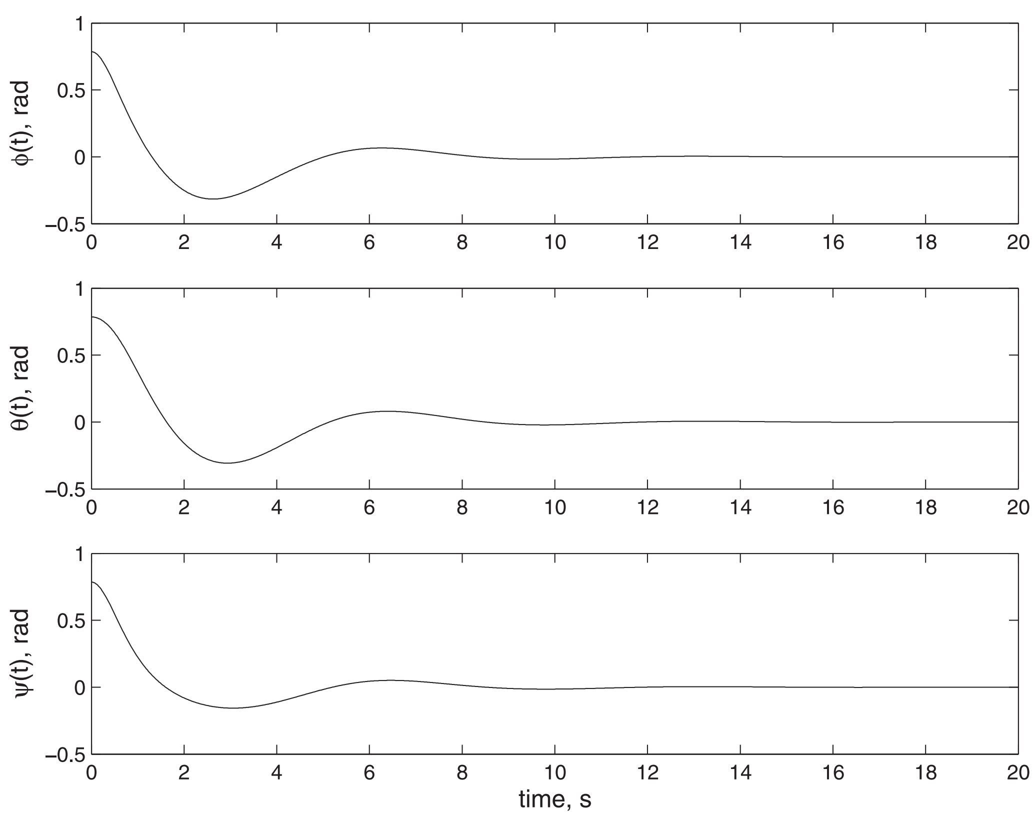

Figure 2. Time histories of the Euler angles.

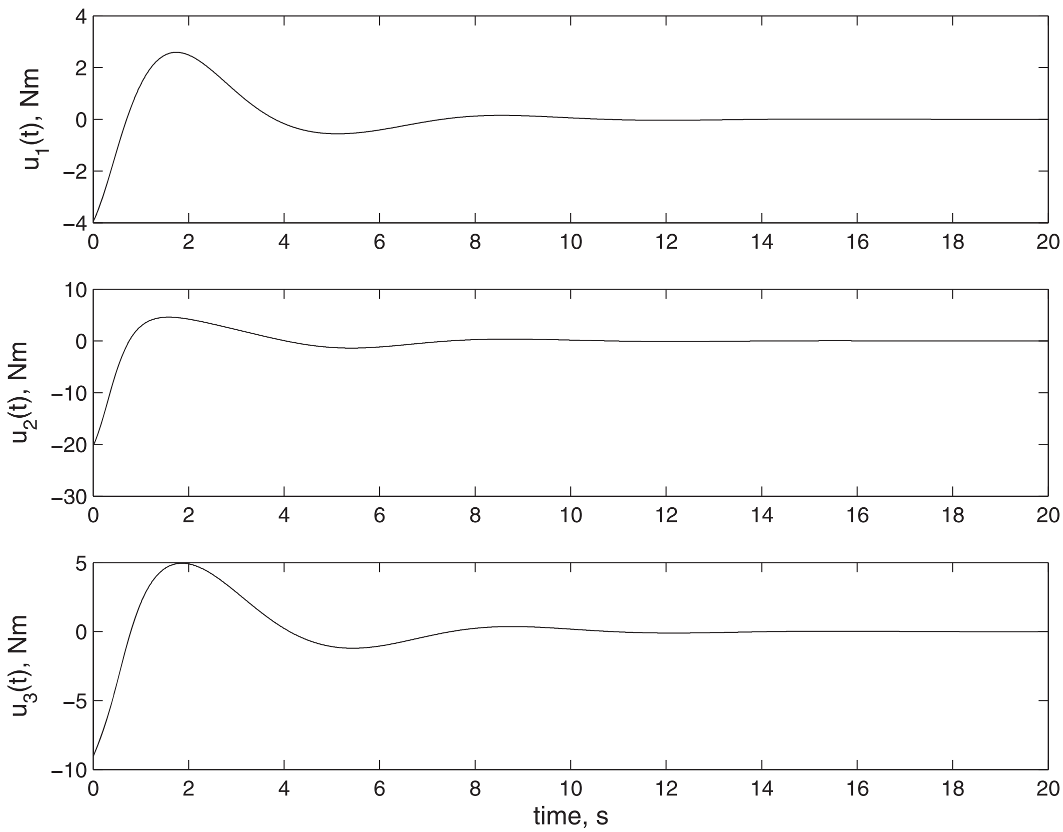

Figure 3. Time histories of the control inputs.

then $\xi(t)$ of (19) satisfies the condition in (9) as follows:

$$\begin{aligned}\|\xi(t)\| &= \|W^{-1}G^T\omega(t)\| \\ &\le \|W^{-1}G^T\|_M\,\|\omega(t)\| \\ &\le \alpha\|\omega(t)\|\ (\triangleq \delta)\end{aligned} \tag{33}$$

for all $\omega(t)$. Therefore, with the relation of Theorems 1 and 2, if three conditions in (10), (24), and (32) hold for the control law $u(t)$ of (8) and the disturbance signal $\xi(t)$ of (19), then $u(t)$ of (8) becomes the robust and optimal attitude control law for the complete attitude motion of spacecraft given by (1) and (4) with respect to the performance index in (20). Note that the disturbance signal does not have to specifically follow Equation (19), and Equation (19) is only used to prove the robustness property of the control law $u(t)$ of (8). In other words, the disturbance signal $\xi(t)$ of (19) is only used to prove that the inequality in Equation (33) is valid.

4. A numerical example

This section presents a numerical example to illustrate the theoretical results presented in Section 3.

In the numerical example, the equations of motion given by (1) and (4) are used as the numerical simulation model of spacecraft. In Equation (1), it is assumed that the input matrix for the disturbance signal is $G = \mathrm{diag}[1, 1.5, 2]$ and the inertia matrix of spacecraft is $J = \mathrm{diag}[5, 10, 15]\,\mathrm{kgm}^2$, where 'diag' implies a diagonal matrix. Moreover, $\xi(t)$ of (19) and $u(t)$ of (8) are used as the disturbance signal and the control law, respectively.

First, in the disturbance signal $\xi(t)$ of (19), it is assumed that $W = I_{3\times 3}$. Next, the feedback gain matrices for the control law $u(t)$ of (8) are chosen to satisfy three conditions in (10), (24), and (32). Specifically, with the given matrices $W = I_{3\times 3}$ and $G = \mathrm{diag}[1, 1.5, 2]$, one can obtain $\|W^{-1}G^T\|_M = 2$, $\|G\|_M = 2$, and $GW^{-1}G^T = \mathrm{diag}[1, 2.25, 4]$. Thus, the constant α in (9) is chosen as $\alpha = 2$ to satisfy the condition in (32) such that $\alpha\,(= 2) \ge \|W^{-1}G^T\|_M\,(= 2)$. The feedback gain matrices for the control law $u(t)$ of (8) are selected as $K_\omega = \mathrm{diag}[5, 10, 15]$ and $K_r = \mathrm{diag}[5, 10, 15]$ to satisfy two conditions in (10) and (24) such that $\lambda_{\min}(K_\omega)\,(= 5) > \alpha\|G\|_M\,(= 4)$ and $K_\omega\,(= \mathrm{diag}[5, 10, 15]) > GW^{-1}G^T$ $(= \mathrm{diag}[1, 2.25, 4])$. Finally, it is assumed that the initial conditions for the Euler angles at the initial time $t_0 = 0\,\mathrm{s}$ are $\phi(t_0) = \theta(t_0) = \psi(t_0) = \pi/4\,\mathrm{rad}$. Also, it is assumed that the initial conditions for the angular velocities at the initial time $t_0 = 0\,\mathrm{s}$ are $\omega_1(t_0) = \omega_2(t_0) = \omega_3(t_0) = 0\,\mathrm{rad/s}$.

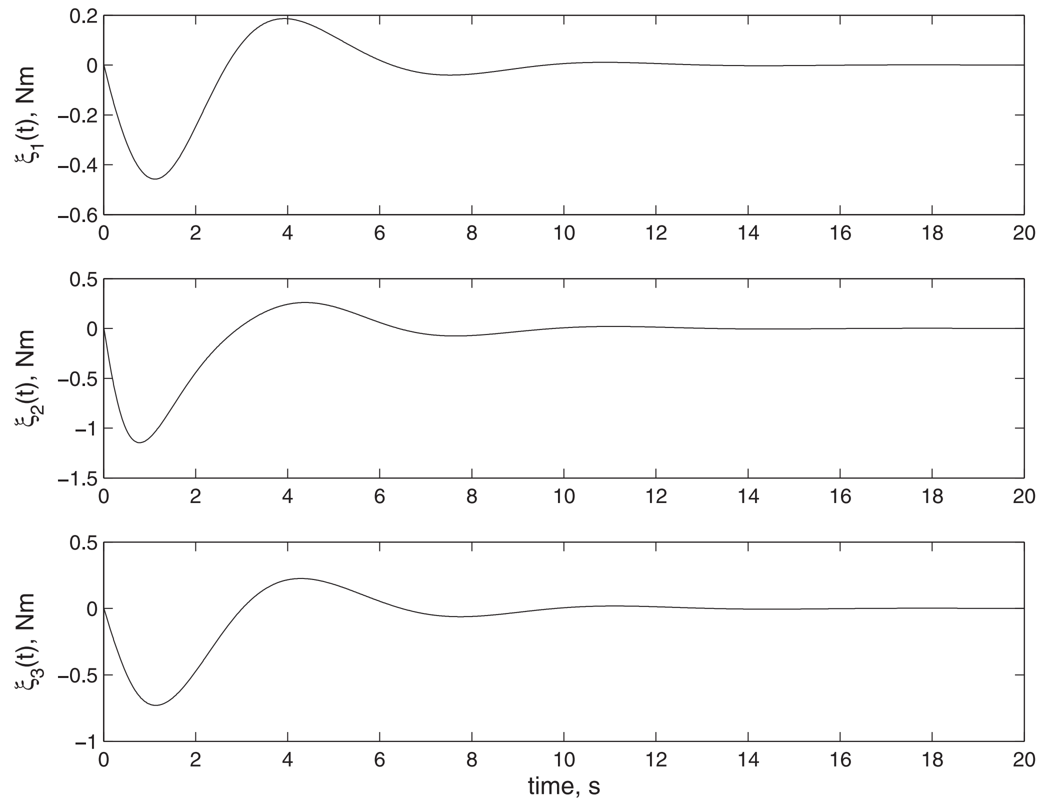

Figure 4. Time histories of the disturbances.

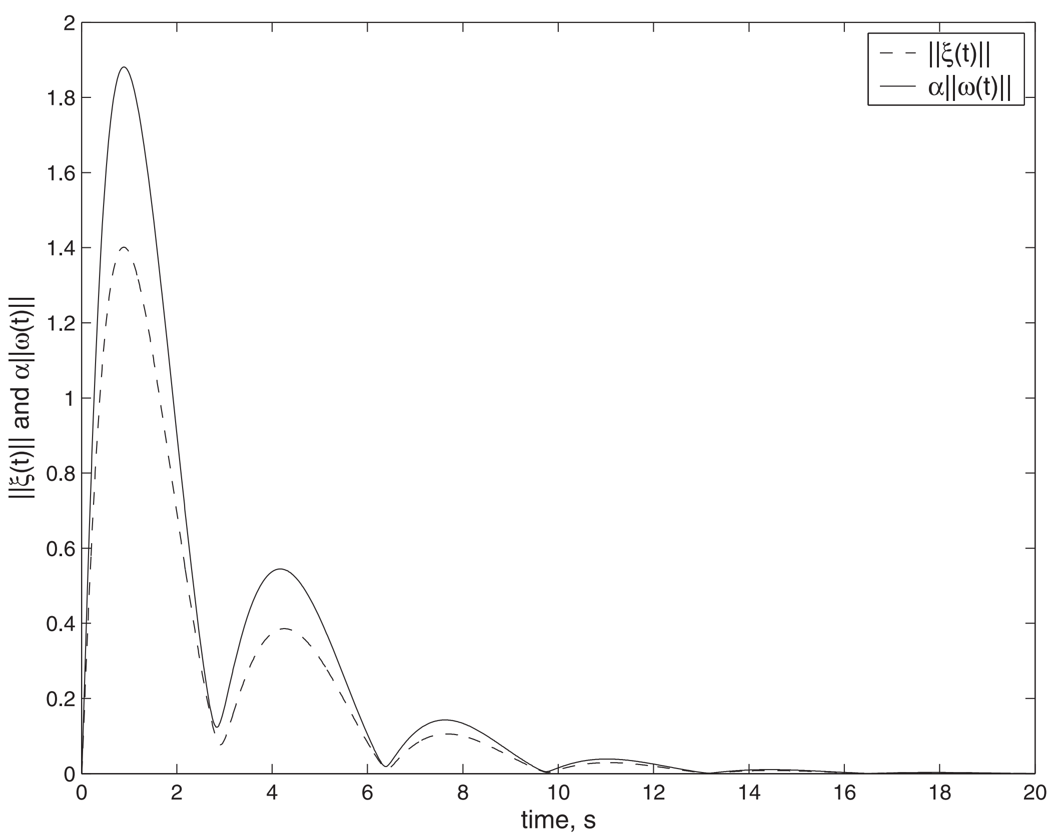

Figure 5. Time histories of the Euclidean norms of $\xi(t)$ and $\alpha\omega(t)$ with the given $\alpha = 2$.

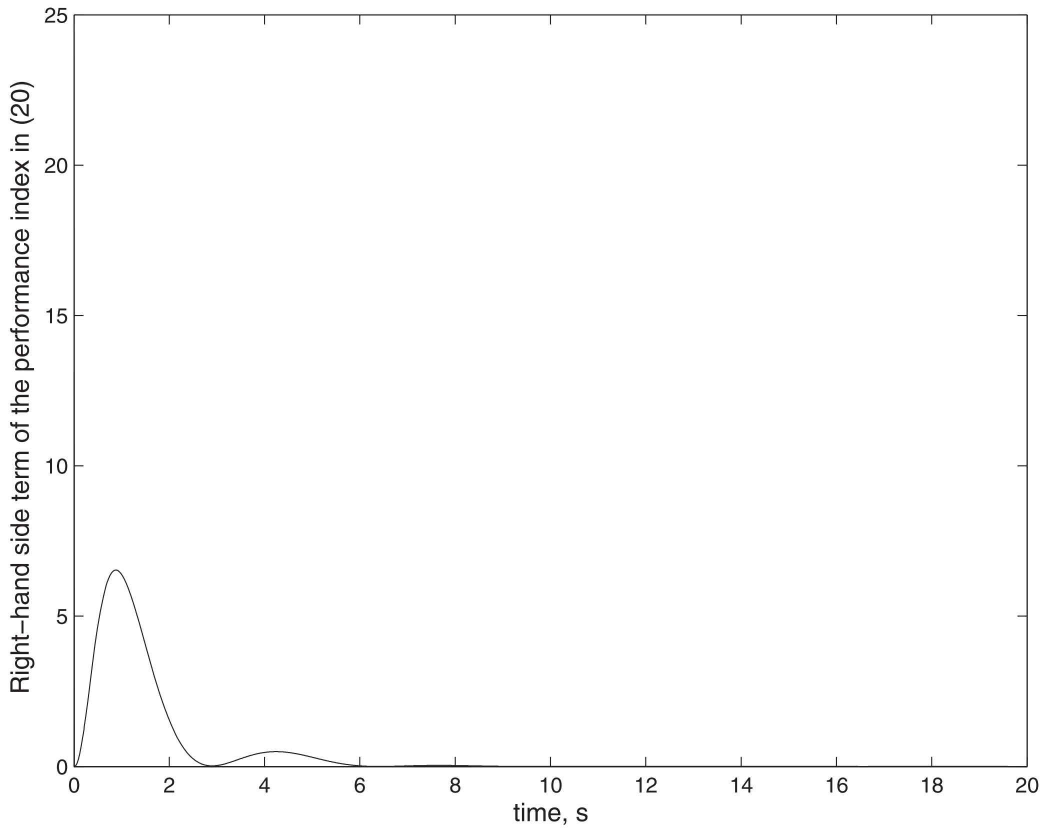

Figure 6. Time histories of the right-hand side term of the performance index in (20).

Figures 1 and 2 show the numerical simulation results for the system given by (1) and (4) with the disturbance signal $\xi(t)$ of (19) assumed in this section and the control law $u(t)$ of (8) designed in this section. The numerical simulation results shown in Figures 1 and 2 demonstrate asymptotic stability of the closed-loop system of spacecraft. Figures 3 and 4 show the control inputs and the disturbances, respectively. As illustrated in Figures 3 and 4, the control inputs try to overcome the disturbances with the purpose of stabilising the closed-loop system of spacecraft. Figure 5 shows the trajectories of $\|\xi(t)\|$ and $\alpha\|\omega(t)\|$ with the given $\alpha = 2$. In Figure 5, the solid and dashed lines represent the trajectories of $\alpha\|\omega(t)\|$ and $\|\xi(t)\|$, respectively. From Figure 5, we see that the condition in (9) is satisfied with the given $\alpha = 2$. Figure 6 shows the right-hand side term of the performance index in (20) (i.e., $1/2[x(t)^T Qx(t) + 2u(t)^T Nx(t) + u(t)^T Ru(t) - \xi(t)^T W\xi(t)]$). As shown in Figure 6, the value of the right-hand side term of the performance index in (20) becomes large in the initial time domain and small as time goes by. The numerical simulation result shown in Figure 6 demonstrates a desirable feature of the optimal control scheme because it implies more aggressive control action far away from the equilibrium point in order to make the closed-loop system stable within a short period of time and, eventually, less aggressive control action as the closed-loop system state approaches the equilibrium point. Indeed, as shown in Figures 1–3, the control law allows for increasingly corrective control action when the closed-loop system state starts deviating from the intended operating point. On the other hand, the control law makes the control action less aggressive as the closed-loop system state approaches the equilibrium point.

5. Conclusion

In this paper, the robust and optimal attitude control problem is considered for the complete attitude motion of spacecraft with disturbances. The robust attitude control law using the Euler angles and angular velocities feedback is proposed, and it does not require any information of the inertia of spacecraft. After presenting a robust attitude control law having a large robustness margin with respect to disturbances, the optimal attitude control problem of spacecraft is formulated as the optimal game-theoretic problem. Then it is shown that the proposed robust attitude control law is the optimal solution of the optimal game-theoretic problem. In the optimal game-theoretic problem, it is assumed that the disturbance signal has the information of the system state. Moreover, it is assumed that the control input and the disturbance signal are regarded as two players that compete each other for finding the optimal solution. The optimal control scheme presented in this paper provides the analytic solution of the optimal attitude control problem coincided with the disturbance attenuation problem without the task of numerically solving the corresponding game-theoretic Hamilton–Jacobi equation. The theoretical results presented in this paper are illustrated by numerical simulations. The numerical simulation results demonstrate that the control inputs provided by the robust and optimal attitude control law try to overcome the disturbances, and, eventually, the robust and optimal attitude control law stabilises the attitude angles and angular velocities of spacecraft.

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Yonmook Park received his BS and MS degrees in control and instrumentation engineering from Korea University, Republic of Korea, in 1995 and 1999, respectively, and his PhD degree in aerospace engineering from Korea Advanced Institute of Science and Technology, Republic of Korea, in 2004. From November 2003 to September 2004, he was a senior researcher and the satellite attitude control team leader at the Satellite Technology Research Centre, Korea Advanced Institute of Science and Technology, Republic of Korea. From October 2004 to April 2012, he was a senior engineer at the Mechatronics and Manufacturing Technology Centre, Samsung Electronics Co., Ltd., Republic of Korea. From May 2008 to December 2009, he was a visiting scientist at the Laboratory for Manufacturing and Productivity, Department of Mechanical Engineering, Massachusetts Institute of Technology, USA. Since June 2012, he has been the chief technology officer at the Synapse Imaging Co., Ltd., Republic of Korea. He was awarded the Silver Prize in the ninth Annual Humantech Thesis Prize by Samsung Electronics Co., Ltd. in February 2003, the Best Conference Paper Award in the IEEE International Conference on Mechatronics and Automation in August 2007, and the Meritorious Service Award by Samsung Electronics Co., Ltd. in January 2008. He served as a chair of the fourth Asian Control Conference in September 2002 and as a session chair of the IEEE International Conference on Mechatronics and Automation in August 2007. His current research interests include spacecraft attitude control, robust control, optimal control, precision control, fuzzy control, neural network, pattern recognition, scanning probe microscope, and vision camera-based wafer and flat panel display inspection.

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