per element varies little between the strata, and where the S_h^2 refers to a multitude of different items, many of them percentages. If the sampling fractions f_h are small, we can also disregard the $(1 - f/k_h)$ and have

$$nV^2 = \sum W_h k_h \left(\sum \frac{W_h}{k_h}\right). \tag{11.7.6}$$

This compares the actual variance to the variance of proportionate sampling, when this is optimum (S_h^2 and J_h constant).

The same result is obtained if different sampling rates are applied to parts of an unrestricted random sample. The initial size is $m = \Sigma\, n_h k_h = Nf$, then $n_h = mW_h/k_h$ are selected in the hth part, altogether $n = \Sigma\, n_h = m\Sigma\, W_h/k_h$. If the mean is computed as $\Sigma\, k_h y_h / \Sigma\, k_h$, the variance can be denoted as $(\Sigma\, n_h k_h^2 S^2)/(\Sigma\, n_h k_h)^2$; the ratio of this to S^2/n is

$$\frac{(\sum n_h)(\sum n_h k_h^2)}{(\sum n_h k_h)^2} = \left(\sum \frac{W_h}{k_h}\right) \sum W_h k_h. \tag{11.7.6$'$}$$

Expanding the above, we obtain

$$\begin{aligned} nV^2 &= \sum_h^H W_h^2 + \sum_{h<i}^H \frac{W_h W_i}{k_h k_i}(k_h^2 + k_i^2) \\ &= \left(\sum_h^H W_h^2 + 2\sum_{h<i}^H W_h W_i\right) + \sum_{h<i}^H \frac{W_h W_i}{k_h k_i}(k_h^2 + k_i^2 - 2k_h k_i) \\ &= 1 + \sum_{h<i}^H W_h W_i \frac{(k_h - k_i)^2}{k_h k_i}. \end{aligned} \tag{11.7.7}$$

The second term expresses the relative increase in variance due to departures from proportionate sampling. For any fixed spread from the lowest to highest weights (from $k_{\min} = 1$ to $k_{\max} = K$), and for any fixed weight $W_{\min}$ for the lowest group, the maximum loss is attained when all of the remaining weight is subject to K. *Thus, for any fixed spread among the weights, the greatest loss is incurred when there are only two strata.* Hence, the case of two strata ($H = 2$) is interesting because it is an extreme as well as a common case. The increase in variance when $H = 2$, $k_1 = 1$, and $k_2 = K$ is

$$nV^2 = 1 + W(1 - W)\frac{(K-1)^2}{K}. \tag{11.7.8}$$

The increase is the same whether the larger or smaller stratum has the greater weight, and reaches its maximum when both strata are equal. For large K it approaches and is a little greater than $W(1 - W)(K - 2)$.

For the general case of many strata we must confine ourselves to a few helpful illustrations. For comparability we suppose that the range of weights is from 1 to H, the same as the number of strata. We use (11.7.6)