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Robust inventory management with multiple supply sources

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ABSTRACT

We study a robust rolling horizon model for a periodically reviewed inventory system with multiple supply sources with general lead times. We restrict the demands in an uncertainty set without knowing the distributions. We prove that under some appropriate conditions, the robust optimal policy for multiple sourcing is a combination of the base-stock policy and a gap-of-base-stock policy with capping effect on supply sources except the fastest source. In particular, the structure of this policy is not a natural extension of the robust optimal policy for the system with two supply sources, for which the recent literature shows that the robust optimal policy is a dual-index, dual-base-stock policy that caps the slow order. We also derive closed-form expressions of the robust optimal policy for the central limit theorem (CLT) uncertainty set. We numerically testify to the effectiveness of the structure of the robust optimal policy for more than two supply sources using both simulation demand and real sales data. The computational results are promising.

1. Introduction

Global companies today increasingly endeavor to strike a balance between customers' responsiveness and cost-efficiency. Multiple sourcing is one of the key strategies companies employ to ensure profit; a large portion of enterprises practice strategies of multiple sourcing (Fig. 1). According to Abginehchi, Farahani, and Rezapour (2013), Sun Microsystems has several memory chip suppliers and splits demand among them based on their past performance. Japanese auto manufacturers also use a multi-supplier strategy to procure the majority of their components. From Wu, Wang, and Shang (2019), we know that Apple Inc. has sourced iPhone7′s key components from multiple suppliers, including Japan Display Inc.(JDI), South Korea's LG Display Co.(LG), Sharp Corp., etc. In vaccine supply, healthcare providers sourced influenza vaccine from Sanofi Pasteur, Seqirus, GlaxoSmithKline,etc. Also, Huawei P20 Pro[1] has multiple semiconductor suppliers around the world, including Samsung(61%), Hisilicon(27%), Skyworks(3%), Broadcom(2%), and others(7%). Different suppliers usually have different supply lead times and unit costs, and market demand is highly uncertain. Efficient inventory management for such a system requires managers to jointly balance the costs of ordering from different supply sources against inventory holding and shortage costs.

In most inventory models, it is generally assumed that the sales of products are relatively stable and the historical sales data is sufficient. Firms can first characterize the distribution of demand, then make the inventory planning based on it. However, in many practical problems, due to the lack of effective historical sales data (such as new products and short-lived products that have just been launched) or the non-stationary nature of the market (such as consumers' preference change), it is difficult for firms to completely characterize the demand distribution of a product (Dalrymple, 1987). Meanwhile, due to the fierce market competition, and in order to meet the personalized needs of consumers, companies nowadays especially e-commerce sell a wide variety of products, whereas a large part of them are slow-moving SKUs. Demand forecasting for slow-moving SKUs is usually difficult to do and the forecast accuracy could be very poor. In the meantime, common distribution functions may not fit their historical data well. In this case, it has the risk of model misspecification to assume that the demand distribution is known.

Thereby, how to make ordering decisions with only partial information about demand distribution is faced up by decision makers. When the mean and variance of the demand are available, Scarf (1958) and Gallego and Moon (1993) utilize min-max approach to minimize the worst-case total cost. Later, a less conservative approach called min-max regret is adopted by Yue, Chen, and Wang (2006) and Perakis and Roels (2008). They minimize a regret of newsvendor given mean with standard deviation and mean with support of the random demand respectively. Delage and Ye (2010) further consider the uncertainty of moments of

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[1] https://www.wsj.com/articles/how-huawei-took-over-the-world-11545735603.

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0377-2217/

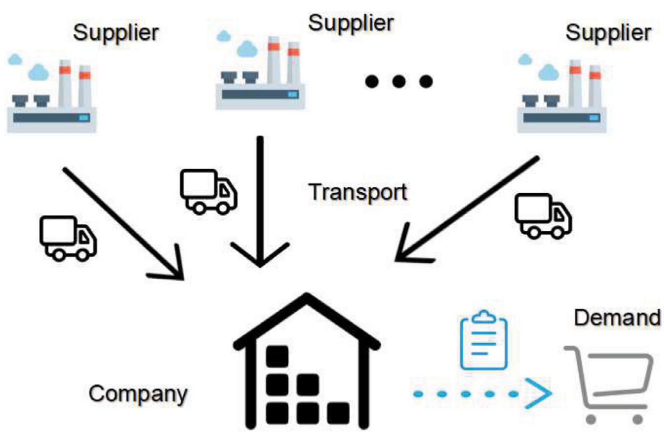

Fig. 1. Supply Chain with Multiple Sources.

the demand distribution and name the above optimization model as the "Distributionally Robust Optimization" model. In contrast, another method based on the robust optimization concentrates on finding a solution which is feasible for any realization taken by the uncertain parameters, such as making use of the most unfavorable realization of demand in determining the inventory policy. Not like the stochastic optimization, the robust optimization assumes that uncertain parameters vary over a deterministic set called "uncertainty set", which is centered around the nominal values of the uncertain parameters and the size could be adjusted to control the probabilistic protection. A general concern on the robust optimization method is over-conservatism. By introducing the concept of "budget of uncertainty" (Bertsimas & Sim, 2004), the level of conservatism can be relaxed. In addition to the low requirements on the demand information, another advantage of the robust optimization method over the stochastic optimization is its tractability. The decision maker can cope with the deterministic counterpart immune from curse of dimensionality.

In this paper, we study a periodic-review, infinite-horizon stochastic inventory system with multiple supply sources (delivery modes). The supply sources have different supply lead times and unit costs, and the shorter the lead time of a supplier, the higher its unit cost. Random demands can be correlated over time, and demands are constrained in an uncertainty set, whereas their underlying distributions are not known. We study a robust rolling horizon optimization model for the problem. Rolling horizon approach is a reactive scheduling method that iteratively solves the problem by moving forward the optimization horizon in every iteration/period. In each period, we solve a robust multi-period inventory problem over a prediction horizon (time window). After solving the problem, only the first period's decision is practically applied. Then, the system is updated and proceeds to the subsequent period. We employ the new information and solve an updated robust optimization model. Intuitively, when we implement robust optimization to model multi-period inventory problem, it is necessary to look a certain number of time periods ahead to aggregate knowledge before decision making. While this time window of the near future should not be too long, since information will be less reliable and computational cost is expensive.

For dual-sourcing problem, Sun and Van Mieghem (2019) show that the robust optimal policy is a dual-index, dual-base-stock policy that caps the slow order. We prove that the structure of the robust optimal policy for problems with more than two supply sources is *not* a natural extension of the optimal policy for the system with two supply sources. Rather, we show that, under some appropriate conditions, the robust optimal policy for sourcing from three or more suppliers is a base-stock policy for the fastest supply source; as for other supply sources, the optimal policy is a sort of "gap-of-base-stock" policy, which has capping effect for orders but different from the dual-sourcing case, the caps depend on the detailed pipeline information. This is the key finding of our paper. Based on the new structure of the robust optimal policy, we design a general policy called the MI-cap for managing inventory systems with multiple supply sources. We also study the central limit theorem (CLT) uncertainty set and derive closed-form expressions of the robust optimal policy called the RCLT.

In the numerical experiments, we investigate the performances of the proposed policies. We use both simulation demand and real sales data to compare the long-run average costs of our proposed policies, i.e., the MI-cap and the RCLT, with other two policies. One is the DI-cap proposed in Sun and Van Mieghem (2019) for the dual-sourcing problem. The other is the VSW, which extends the dual-index base-stock policy of Veeraraghavan and Scheller-Wolf (2008)) to the case of multiple-sourcing. In the first part of numerical experiments, we compare performances of the policies with demands generated from the normal and poisson distributions. The results show that the MI-cap has a prominent advantage over the other three policies, and the maximum improvement can exceed 10%. Regarding the average allocation of order quantity among suppliers, the numerical results show that for all cases the percentage of the cheapest source is more than 80%. The order percentages of other two sources vary from 0% to 13%, and the order percentage of the middle source is in general less than that of the fastest source. In the part with the real sales data, we find that the MI-cap has the lowest average cost. We study the relationship between performances of the policies and the statistical features of the sales data. The results show that as the number of days with 0 sales or the kurtosis of the data distribution increases, it is more likely that the MI-cap has a better performance than the RCLT. We also investigate the selection of budget of uncertainty parameter.

The remainder of this article is organized as follows: In Section 2, we review the related literature. In Section 3, we introduce the model setup. In Section 4, we obtain the robust optimal policy for static model and extend the policy to the dynamic model. In Section 5, we conduct the numerical study with both the simulation demand and the real data. Finally, we conclude in Section 6. Some additional numerical results and all of the proofs are contained in the appendix.

2. Literature review

We review the two streams of research: studies on multiple-sourcing inventory management and on robust optimization with inventory application.

2.1. Multiple-sourcing inventory management

Problems of inventory replenishment from more than one source have been studied for over half a century. The classic approach is the periodic-review dual-sourcing inventory model first taken up by Daniel (1963) and Fukuda (1964). They consider the system where the lead-time difference is only one period, and the optimal policy is a single-index dual-base-stock policy. Whittemore and Saunders (1977) find that when the dual-sourcing system has arbitrary lead times, the optimal policy is a function of a high-dimension vector of outstanding orders. Later, Li and Yu (2014) continue to characterize this dependence using the tool of multimodularity. They show the monotonicity of optimal order quantities over the outstanding orders. More details and information of related inventory control models are provided by Svoboda, Minner, and Yao (2017).

Due to the curse of dimensionality in dual-sourcing problems, scholars are increasingly blazing new trails by designing and applying high-efficiency heuristic policies. Veeraraghavan and Scheller-Wolf (2008) study a dual-index dual-base-stock policy (we abbreviate it as DI-VSW), which tracks two inventory positions and is verified to be near-optimal. The vector-base-stock (VBS) policy proposed by Sheopuri, Janakiraman, and Seshadri (2010) tracks the entire pipeline inventory vector and uses a vector of base-stock levels for the regular order. Hua, Yu, Zhang, and Xu (2015) generalize the VBS policy to a best weighted bounds (BWB) policy by using a weighted sum of lower and upper bounds on the regular order quantities. Sun and Van Mieghem (2019) study the dual-sourcing problem under a robust rolling horizon optimization framework. They show that the optimal robust policy is a modified dual-index policy, where there is a "cap" on the slow order quantity.

For stochastic inventory management with more than two suppliers and distinct lead times, Svoboda et al. (2017) summarize in Section 3.1: "Zhang (1996) and Feng, Gallego, Sethi, Yan, and Zhang (2005), who investigate the optimal policy for three supply modes with consecutive lead times, are among the very few authors to do so as the curse of dimensionality, already an issue for two sources with non-consecutive lead times, grows with a higher number of suppliers." Zhang (1996) investigate the optimal ordering policy for the three-supply-modes inventory system, where the lead times' difference is one period, then present a heuristic ordering policy with a simple structure and easily calculated order-up-to levels. Feng et al. (2005) come up with a periodic review inventory system endowed with three consecutive delivery modes and demand forecast updates. They find that in all cases they discussed about, there is a base-stock policy for fast and medium modes which is optimal. However, the optimal policy for the slow mode is not a base-stock policy in general. Feng, Yan, Zhang, and Sethi (2006b) analyze inventory systems with multiple (consecutive) delivery modes. Feng, Sethi, Yan, and Zhang (2006a) show by a counter-example that even when suppliers have consecutive lead times and unlimited capacity, base-stock policies are in general not optimal for the system. Apart from these papers, we consider the multiple supply sources with general lead times, which is one of our major contributions.

2.2. Robust optimization with inventory application

The original model about robust optimization can be traced back to Soyster (1973), who firstly focus on linear programming problems. Until entering 1990s, a leading breakthrough is started by Bental and Nemirovski (1998). Bental and Nemirovski (2000) then discuss the probability guarantees in robust optimization. Bertsimas and Sim (2004) introduce the concept of "budgets of uncertainty" to decrease the price of robustness. Gorissen and Den Hertog (2013) address the robust counterparts of optimization problems containing sums of maxima of linear functions and propose several less conservative reformulations. For the optimal and approximate policies, Chen and Zhang (2009) introduce the extended Affinely Adjustable Robust Counterpart described in Bental, Golany, and Shtern (2009). In terms of distributionally robust optimization, Bental, Bertsimas, and Brown (2010) propose a framework for robust optimization that relaxes the standard notion of robustness. Bertsimas, Gupta, and Kallus (2018) propose a schema for utilizing data to design uncertainty sets for robust optimization using statistical hypothesis tests. Bertsimas, Sim, and Zhang (2019) develop a modular and tractable framework for solving an adaptive distributionally robust linear optimization problem. Ben-Tal, El Housni, and Goyal (2020) introduce a new framework in which the uncertainty set is "approximated" by a "dominating" simplex and address the challenge in designing a practical piecewise affine policy. Interested readers are suggested to the book Ben-Tal, Ghaoui, and Nemirovski (2009) and papers Bertsimas, Brown, and Caramanis (2011) and Gabrel, Murat, and Thiele (2014) for more detailed descriptions on robust optimization.

For inventory management, Ben-Tal, Golany, Nemirovski, and Vial (2005) study a retailer-supplier flexible commitment problem in inventory management. Bertsimas and Thiele (2006) design a robust optimization model of inventory management that utilizes budgets of uncertainty, resulting in excellent performance, and find that the optimal robust policy is a base-stock policy. Bienstock and Ozbay (2008) present a family of algorithms based on decomposition and take on more general uncertainty sets. Bental et al. (2009) choose the method called Affinely Adjustable Robust Counterpart and extend this framework to control inventories in serial supply chains under demand uncertainly. See and Sim (2010) obtain the parameters of the replenishment policies by solving a tractable deterministic optimization problem in the form of a second-order cone optimization problem.

The CLT uncertainty sets are primarily studied by Bandi and Bertsimas (2012), who provide an in-depth analysis of the combination of robust optimization with the limit theorems of probability. Mamani, Nassiri, and Wagner (2017) derive closed-form ordering quantities for both symmetric and asymmetric uncertainty sets, under capacitated inventory constraints, in both static and dynamic settings. Sun and Van Mieghem (2019) extend their work to dual sourcing with non-zero lead times and provide closed-form robust dual sourcing policies. Our work keep up with Sun and Van Mieghem (2019) and further extend the model to multiple sourcing case. Using analogous methodology, we acquire the closed-form robust optimal policies for multiple sourcing under specific uncertainty sets.

3. Preliminaries

3.1. Stochastic inventory system with multiple supply sources

We consider a stochastic infinite-horizon inventory system with multiple supply sources (delivery modes), indexed by $1, 2, \cdots, m$. These m supply sources have different order costs and lead times. We assume, without loss of generality, that $c^1 > c^2 > \ldots > c^m = 0$, $0 = L^1 < L^2 < \ldots < L^m$, and $L^j,\ \ j = 2, \cdots, m$ are positive integers. That is, the supply source with the higher order cost has a shorter lead time, and we normalize the order cost of the slowest supply source and the lead time of the fastest supply source to zero. In each period t, there is a stochastic demand, $d_t \geq 0$. However, the distributions of d_t are unknown. Correlated demands and period-dependent moments are allowed. Demand is fulfilled by on-hand inventory, and unsatisfied demand is fully backlogged with a unit backlogging cost b per period. Each unit of leftover inventory incurs holding cost h per period. The objective is to make ordering decisions so as to minimize the long-run average cost of the inventory system.

Let x_t denote the net inventory level at the end of each period t. Then, for period t, the sequence of events is as follows.

1. At the beginning of each period t, the net inventory level x_{t-1} and all of the pipeline inventory $(q^1_{t-L^1}, \ldots, q^1_{t-1}, q^2_{t-L^2}, \ldots, q^2_{t-1}, \cdots, q^m_{t-L^m}, \ldots, q^m_{t-1})$ ordered in the previous periods are reviewed.
2. The order quantities from m supply sources are placed, denoted as $q^1_t, q^2_t, \ldots, q^m_t$, which incurs order costs $\sum_{j=1}^m c^j q^j_t$.
3. The in-transit orders due to arrive in period t, $q^1_{t-L^1}, q^2_{t-L^2}, \ldots, q^m_{t-L^m}$, are received and added to the net inventory.

4. The demand in this period is realized and fulfilled as much as possible by the on-hand inventory. The net inventory at the end of period t is then given by

$$x_t = x_{t-1} + q^1_{t-L^1} + q^2_{t-L^2} + \ldots + q^m_{t-L^m} - d_t. \tag{1}$$

If $x_t > 0$, then the excess inventory incurs a holding cost hx_t. On the other hand, if $x_t < 0$, then the system incurs a backlogging cost $-bx_t$.

5. The system proceeds to the subsequent period $t+1$ with initial net inventory level x_t and pipeline inventory $(q^1_{t-L^1+1}, \ldots, q^1_t, q^2_{t-L^2+1}, \ldots, q^2_t, \cdots, q^m_{t-L^m+1}, \ldots, q^m_t)$.

We define *the trimmed inventory position*, denoted by I^{t+k}_t, as the part of the *inventory position* comprised of the net inventory level at the beginning of period t and all of the in-transit orders that will arrive by period $t+k$. Specifically,

$$I^{t+k}_t = x_{t-1} + \sum_{i=t-L^1}^{(t-L^1+k)\wedge(t-1)} q^1_i + \sum_{i=t-L^2}^{(t-L^2+k)\wedge(t-1)} q^2_i + \cdots + \sum_{i=t-L^m}^{t-L^m+k} q^m_i,$$
$$k = 0, 1, \cdots, L^m - 1. \tag{2}$$

Note that the vector of the trimmed inventory positions $\mathbf{I}_t = (I^t_t, \ldots, I^{t+L^1}_t, \ldots, I^{t+L^2}_t, \ldots, I^{t+L^m-1}_t)$ is sufficient to characterize the system dynamics, and we will use $\mathbf{I}_t$ to define the inventory policy.

In what follows, to make the notations meticulous, let $I^n_t = I^{t+L^m-1}_t$ if $n \ge t + L^m$, and $\sum_{i=a}^{b} = 0$ when $a > b$. $\lfloor x \rfloor$ is the greatest integer not larger than x. $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$.

3.2. Robust rolling horizon model based on bounded partial-sum demand

We adopt the rolling horizon method to deduce the robust policy for the infinite horizon inventory system. We define the horizon length as H with $H \ge L^m + 1$. We denote the following index set:

$$\mathbb{N}^H_t = \{t, t+1, \ldots, t+H-1\}. \tag{3}$$

In our model, the distributions of the demands are not known. In each period, we consider the robust optimization objective and minimize the worst-case total cost in the planning horizon.

In period t, let q^j_n denote the order quantity from source j in period n as planned in the current period, where $n \in \mathbb{N}^H_t$. Here, to simplify the notations, we make the dependence of the variables on period t implicit. The decision variables of the optimization problem in period t is the following order quantities:

$$\mathbf{q}^1_t = (q^1_t, \cdots, q^1_{t+H-1}), \mathbf{q}^2_t = (q^2_t, \cdots, q^2_{t+H-1}), \ldots, \mathbf{q}^m_t$$
$$= (q^m_t, \cdots, q^m_{t+H-1}). \tag{4}$$

Given the vector of the trimmed inventory positions $\mathbf{I}_t$, and a suitable uncertainty set Ω_t of the demand in the planning horizon $\mathbf{d}_t = (d_t, \cdots, d_{t+H-1})$, the order quantities are determined by solving a static optimization problem that minimizes the following worst-case total cost (including ordering, inventory holding and backlogging costs from period t to $t+H-1$):

$$C_t(\mathbf{q}^1_t, \mathbf{q}^2_t, \ldots, \mathbf{q}^m_t; \mathbf{I}_t, \Omega_t) = \sum_{n=t}^{t+H-1}\left(\sum_{j=1}^{m} c^j q^j_n + \max_{\mathbf{d}_t\in\Omega_t}\{hx^+_n + bx^-_n\}\right)$$
$$= \sum_{n=t}^{t+H-1}\left(\sum_{j=1}^{m} c^j q^j_n + \max_{\mathbf{d}_t\in\Omega_t}\{hx_n, -bx_n\}\right), \tag{5}$$

where

$$x_n = I^n_t + \sum_{i=t}^{n-L^1} q^1_i + \sum_{i=t}^{n-L^2} q^2_i + \cdots + \sum_{i=t}^{n-L^m} q^m_i - \sum_{i=t}^{n} d_i \quad for \quad n \in \mathbb{N}^H_t, \tag{6}$$

and the second equality is because only one of x^+_n and x^-_n can be strictly positive.

Remark 1. The above cost function follows Mamani et al. (2017) and Sun and Van Mieghem (2019), who put the robust part into the summation.

Define the partial sum demand, the lower bound and upper bound of the partial sum demand based on the uncertainty set as

$$D^n_t = \sum_{i=t}^{n} d_i, \ \ \underline{D}^n_t = \min_{\mathbf{d}_t\in\Omega_t}\sum_{i=t}^{n} d_i, \ \ \overline{D}^n_t = \max_{\mathbf{d}_t\in\Omega_t}\sum_{i=t}^{n} d_i. \tag{7}$$

If the uncertainty set is convex, the above two bounds could be solved efficiently. For example, if the uncertainty set is polyhedral, the problem is a linear programming. And for ellipsoidal uncertainty set, the problem is a QCLP (Quadratically Constrained Linear Programming). Both of them could be solved by general solvers. Particularly, Mamani et al. (2017) and Sun and Van Mieghem (2019) study an uncertainty set based on central limit theorem (CLT), under which $\underline{D}^n_t$ and $\overline{D}^n_t$ have closed-form expressions. Here we simply assume that these two bounds exist.

Assumption 1. $\underline{D}^n_t$ and $\overline{D}^n_t$ are solvable with admissible $\mathbf{d}_t$ in the uncertainty set Ω_t.

On the basis of (6) and (7), we obtain the following equations:

$$\max_{\mathbf{d}_t\in\Omega_t} hx_n = h\left(I^n_t + \sum_{j=1}^{m}\sum_{i=t}^{n-L^j} q^j_i\right) - h\min_{\mathbf{d}_t\in\Omega_t} D^n_t$$
$$= h\left(I^n_t + \sum_{j=1}^{m}\sum_{i=t}^{n-L^j} q^j_i\right) - h\underline{D}^n_t,$$
$$\max_{\mathbf{d}_t\in\Omega_t} -bx_n = -b\left(I^n_t + \sum_{j=1}^{m}\sum_{i=t}^{n-L^j} q^j_i\right) + b\max_{\mathbf{d}_t\in\Omega_t} D^n_t$$
$$= -b\left(I^n_t + \sum_{j=1}^{m}\sum_{i=t}^{n-L^j} q^j_i\right) + b\overline{D}^n_t.$$

Thus, by introducing auxiliary variables $\mathbf{z}_t = (z_t, \ldots, z_{t+H-1})$, the static robust multiple-sourcing optimization problem can be established as

$$\begin{aligned}
\min_{\mathbf{q}^1_t,\cdots,\mathbf{q}^m_t,\mathbf{z}_t\ge 0} \quad & \sum_{n\in\mathbb{N}^H_t}\left(\sum_{j=1}^{m} c^j q^j_n + z_n\right) \\
s.t. \quad & z_n \ge h\left(I^n_t + \sum_{j=1}^{m}\sum_{i=t}^{n-L^j} q^j_i\right) - h\underline{D}^n_t, \quad \forall n \in \mathbb{N}^H_t, \qquad \text{(RMS)} \\
& z_n \ge -b\left(I^n_t + \sum_{j=1}^{m}\sum_{i=t}^{n-L^j} q^j_i\right) + b\overline{D}^n_t, \quad \forall n \in \mathbb{N}^H_t.
\end{aligned}$$

Here z_n can be viewed as the maximum mismatch (either holding or penalty) cost in period n.

We name the above static model RMS (robust multiple sourcing), and denote it as $\mathbf{M}^H(\mathbf{I}_t, \Omega_t, \mathbb{N}^H_t)$, of which the state variables are $(\mathbf{I}_t, \Omega_t, \mathbb{N}^H_t)$. Then, we apply following schema to get the policy for the infinite periods problem.

1. In period t, we solve the model $\mathbf{M}^H(\mathbf{I}_t, \Omega_t, \mathbb{N}_t^H)$ and obtain optimal order quantities. We implement q_t^j to make orders at this period.
2. After demand d_t is realized, we move to period $t+1$, the state variables is updated to $(\mathbf{I}_{t+1}, \Omega_{t+1}, \mathbb{N}_{t+1}^H)$.
3. We solve the static model $\mathbf{M}^H(\mathbf{I}_{t+1}, \Omega_{t+1}, \mathbb{N}_{t+1}^H)$ and execute q_{t+1}^j. Repeat the process.

4. Robust policy for inventory model with multiple sources

In this section, we first analyze the optimality conditions for the static model and provide its closed-form solution. After that, we propose certain conditions under which the robust optimal policy maintains in the rolling horizon procedure. Additionally, we consider the CLT uncertainty set and present more compact expressions for the robust optimal order quantities. At last, we analyze the conditions required in the static model and dynamic model.

4.1. Closed-form solution to the static model

In this subsection, we study the solution to the static model $\mathbf{M}^H(\mathbf{I}_t, \Omega_t, \mathbb{N}_t^H)$. Before deriving the solution, we first study the substitution effect among supply sources.

Lemma 1. *For any $j = 1, \cdots, m-1$, if $c^j + L^j b > c^{j+1} + L^{j+1} b$, then the optimal solution to the static model $\mathbf{M}^H(\mathbf{I}_t, \Omega_t, \mathbb{N}_t^H)$ satisfies $\mathbf{q}_t^j = 0$, i.e., source j should never be used.*

Lemma 1 helps the decision maker to discern the worthless sources. In what follows, we assume that supply sources satisfy the following condition, which ensures that each source corresponds to an effective combination of cost and lead time and can not be replaced by any other source.

Condition 1. $c^j + L^j b < c^{j+1} + L^{j+1} b$ for any $j = 1, \cdots, m-1$.

Since problem RMS is a static model, for each period $n \in \mathbb{N}_t^H$, if there are orders due to arrive at this period as planned in period t, the quantity must be ordered from the cheapest source whose delivery can arrive at this period. Therefore, we have $q_{n-L^j}^j = 0$ for $n \ge t + L^{j+1}$. Otherwise, the quantity can be ordered from source $j+1$, which leads to a lower cost. Consequently, we can reduce the complexity of RMS by introducing cumulative orders Q_t^n, which represents the total order quantities planned in period t that will arrive before or at period n. Define $Q_t^{t-1} = 0$. Then Q_t^n can be computed by the following recursive equation: for $j = 1, \cdots, m$,

$$Q_t^n = Q_t^{n-1} + q_{n-L^j}^j, \quad t + L^j \le n \le t + L^{j+1} - 1, \tag{8}$$

where $L^{m+1} \doteq H$ and recall that $L^1 = 0$.

Lemma 2. *RMS is equivalent to the following problem:*

$$\begin{aligned} \min_{0 \le Q_t^t \le Q_t^{t+1} \le \cdots \le Q_t^{t+H-1}} \quad & \sum_{j=1}^{m-1} c^j \left(Q_t^{t+L^{j+1}-1} - Q_t^{t+L^j-1}\right) + \sum_{n=t}^{t+H-1} z_n \\ s.t. \quad & z_n \ge h(I_t^n + Q_t^n - \underline{D}_t^n), \quad \forall n \in \mathbb{N}_t^H, \\ & z_n \ge -b(I_t^n + Q_t^n - \overline{D}_t^n), \quad \forall n \in \mathbb{N}_t^H. \end{aligned} \tag{RMS-Q}$$

The above result shows that the planning horizon can be divided into m intervals. Each interval corresponds to one source—the cheapest source whose delivery can arrive in this interval. However, for each interval except the last one, i.e., $[t + L^m, t + H - 1]$, it could be optimal for some periods to receive nothing from the corresponding source and backlogging the demand to wait for the order from a cheaper source. In particular, when Condition 1 holds, if one postpones the order of certain source k, he should postpone the order to source $k+1$ (otherwise, source $k+1$ is a useless source). We define

$$\nu^j = \lfloor (c^j - c^{j+1})/b \rfloor, \quad j = 1, \cdots, m-1,$$

which is a threshold controlling the postponement of orders from source j to source $j+1$. To explain, source j corresponds to interval $[t + L^j, t + L^{j+1} - 1]$. However, periods in $[t + L^{j+1} - \nu^j, t + L^{j+1} - 1]$ should not receive orders from source j. Otherwise, if $q_n^j > 0$ for some $n \in [t + L^{j+1} - \nu^j, t + L^{j+1} - 1]$, we can order this quantity from source $j+1$ and bear the possible backlogging cost, which is at most $(t + L^{j+1} - n) b q_n^j$. But, this also brings reduction in the ordering cost by $(c^j - c^{j+1}) q_n^j$. By the definition of ν^j, it is ready to verify when $n \ge t + L^{j+1} - \nu^j$, we have $(c^j - c^{j+1}) > (t + L^{j+1} - n) b$. Thus, it is better to postpone the quantity to source $j+1$.

In general, solving problem RMS-Q is equivalent to minimize the summation of a series of piece-wise linear function. Next, we provide a condition that guarantees a closed-form optimal solution to problem RMS-Q.

Condition 2. $\left(\frac{h\underline{D}_t^n + b\overline{D}_t^n}{h+b} - I_t^n\right)^+$ is non-decreasing in $n = t, \cdots, t + L^m - 1$.

We note that Condition 2 here keeps in line with the premise required in Proposition 2 of Sun and Van Mieghem (2019); while we extend it to the problem with multiple sources. Next result characterizes the optimal solution to problem RMS-Q.

Proposition 1. *If Conditions 1 and 2 hold, then the optimal solution to problem RMS-Q is given by*

$$Q_t^{n*} = \begin{cases} \left(\dfrac{h\underline{D}_t^n + b\overline{D}_t^n}{h+b} - I_t^n\right)^+, & t + L^j \le n \le t + L^{j+1} - \nu^j - 1, \\ & \quad j = 1, \cdots, m-1, \\ Q_t^{t+L^{j+1}-\nu^j-1*}, & t + L^{j+1} - \nu^j \le n \le t + L^{j+1} - 1, \\ & \quad j = 1, \cdots, m-1, \\ \left(\dfrac{h\underline{D}_t^n + b\overline{D}_t^n}{h+b} - I_t^{t+L^m-1}\right)^+, & t + L^m \le n \le t + H - 1. \end{cases} \tag{9}$$

Proposition 1 posits that under certain initial condition, model RMS-Q admits a closed-form solution. However, when Condition 2 is violated, the solution properties will be much more complicated even for the dual-sourcing case as shown by Sun and Van Mieghem (2019), not to mention for the multiple-sourcing case.

The above static robust optimal policy is a multi-index base-stock policy, in terms of the cumulative order quantities. Accordingly, we can get the optimal order quantities q_n^{j*} from Q_t^{n*}.

Corollary 1. *If Conditions 1 and 2 hold, then the optimal order quantities can be computed as follows. For $j = 1, \cdots, m$,*

$$q_n^{j*} = \begin{cases} Q_t^{t+L^j*} - Q_t^{t+L^j-\nu^{j-1}-1*}, & n = t \\ Q_t^{n+L^j*} - Q_t^{n+L^j-1*}, & t+1 \le n \le t + L^{j+1} - L^j - 1 \\ 0, & o.w. \end{cases} \tag{10}$$

where $\nu^0 \doteq 0$.

4.2. Robust optimal policy for dynamic model

In the preceding section, we show that if Conditions 1 and 2 hold, the static problem has a closed-form solution. With regard to the rolling horizon procedure, the state variables keep changing. Hence, we need a condition to guarantee that Condition 2 preserves over time, which we remark as Condition 3. It extends condition in Proposition 3 of Sun and Van Mieghem (2019).

Condition 3. For any n, $h(\underline{D}_t^n - \underline{D}_t^{n-1}) + b(\overline{D}_t^n - \overline{D}_t^{n-1})$ is non-decreasing in $t \in [n - L^m,\ n-1]$.

In addition, for the dynamic model, Condition 1 is not sufficient enough to sustain the optimal structure during the rolling procedure. In particular, when $\nu^j > 0$, the optimal solution to the static model is to postpone some orders from source j to source $j+1$, resulting in an increasing inventory position for period $t + L^{j+1}$. But it may conflict with Condition 2, which requires the trimmed inventory position possessing a decreasing trend if robust cumulative demands do not increase across the horizon. In this case, we can not guarantee that the dynamic model has closed-form solutions for all periods. Therefore, we propose Condition 4 as following.

Condition 4. $\nu^j = \lfloor (c^j - c^{j+1})/b \rfloor = 0$ for $j = 1, \cdots, m-1$.

Condition 4 is equivalent to $\max_{j=1,\ldots,m-1}\{c^j - c^{j+1}\} < b$. That is, the backlog cost is greater than the difference between ordering costs of any two consecutive sources. This is not strict because nowadays firms especially e-commerce companies usually target a very high service level agreement due to fierce market competition, and thus the shortage cost is usually set to be quite high.

Furthermore, for the initial static model, we need a little stronger condition stated as below.

Condition 5. For the initial static model, i.e., $t = 1$, $(\frac{h\underline{D}_t^n + b\overline{D}_t^n}{h+b} - I_t^n)$ is non-decreasing in $n = t, \cdots, t + L^m - 1$.

Note that Condition 5 can be easily satisfied. For example, the initial pipeline inventory is zero, that is, there is no in-transit inventory at the beginning of the system.

With Conditions 3, 4, and 5, we have the following proposition describing the dynamic robust optimal policy.

Proposition 2. *If Conditions 3, 4, and 5 hold, then Proposition 1 holds for each period's static model in the rolling procedure. And we have the following optimal order quantities:*

$$q_t^{1*} = (S_t^{1*} - I_t^t)^+, \tag{11}$$

$$q_t^{j*} = [(S_t^{j*} - I_t^{t+L^j})^+ - (S_t^{j-*} - I_t^{t+L^j-1})^+]^+, \quad j = 2, \cdots, m, \tag{12}$$

where $S_t^{1*}, S_t^{j*}, S_t^{j-*}$ are given by

$$S_t^{1*} = \frac{h\underline{D}_t^t + b\overline{D}_t^t}{h+b}, \quad S_t^{j*} = \frac{h\underline{D}_t^{t+L^j} + b\overline{D}_t^{t+L^j}}{h+b},$$
$$S_t^{j-*} = \frac{h\underline{D}_t^{t+L^j-1} + b\overline{D}_t^{t+L^j-1}}{h+b}. \tag{13}$$

Proposition 2 implies that for the fastest supply source, the robust optimal policy is a base-stock policy. However, for the other sources, the robust optimal order quantities are essentially determined by a series of gap-of-base-stock. Intuitively, $(S_t^{j*} - I_t^{t+L^j})^+$ computes the needed order quantity when only pipelines ordered before period t are taken into account. It does not capture the impact of order quantities placed in period t from other sources which have shorter lead times than source j, and $-(S_t^{j-*} - I_t^{t+L^j-1})^+$ accounts this impact. This structure of our policy is a distinguishing point comparing to Sun and Van Mieghem (2019)'s DI-cap. In fact, there occurs a transformed cap imposed on each source's order quantity (except the fastest one). To some extent, it is consistent with the cap for the slow source in Proposition 3 of Sun and Van Mieghem (2019), whereas the caps for the intermediate sources depend on the detailed pipeline information.

Corollary 2. *For any $j \geq 2$, the robust optimal order quantity is capped by $\bar{q}_t^{j*}$, which has the following form*

$$\bar{q}_t^{j*} = [S_t^{j*} - S_t^{j-*} - I_t^{t+L^j} + I_t^{t+L^j-1}]^+. \tag{14}$$

The optimal order quantity can be expressed as

$$q_t^{j*} = \min\{(S_t^{j*} - I_t^{t+L^j})^+, \bar{q}_t^{j*}\}. \tag{15}$$

Particularly, when $j = m$, we have $I_t^{t+L^m} = I_t^{t+L^m-1}$, thus

$$\bar{q}_t^{m*} = S_t^{m*} - S_t^{m-*} = \frac{h(\underline{D}_t^{t+L^m} - \underline{D}_t^{t+L^m-1}) + b(\overline{D}_t^{t+L^m} - \overline{D}_t^{t+L^m-1})}{h+b}. \tag{16}$$

Corollary 2 implies that for those intermediate sources, the transformed cap is equal to the difference of two order-up-to levels minus the pipeline part counting by $(I_t^{t+L^j} - I_t^{t+L^j-1})$. The difference of two order-up-to levels shows the effect of incoming demand at $t + L^j$, and the larger the demands arrive, the looser the cap becomes. On the other hand, the difference of the two trimmed inventory positions equals to the in-transit order that is due to arrive at $t + L^j$, and more in-transit orders leads to a tighter cap. The decision maker should choose appropriate order quantities by estimating the hybrid effect of these two factors. Worth mention here, the cap of the slowest source is a special upper bound constraint, because it is independent of the inventory positions. When $m = 2$, Corollary 2 degenerates to the case in Sun and Van Mieghem (2019).

In addition to giving the closed-form expression for the robust optimal policy, Proposition 2 also provides a new policy structure for managing inventory systems with multiple supply sources. The policy structure is significantly different from the order-up-to type of policies and is not a natural extension of the structure of the DI-cap in Sun and Van Mieghem (2019) either. Hence, we also propose a general heuristic policy called MI-cap for the stochastic inventory systems with multiple supply sources. Here MI represents multi-index (the policy is defined by a vector of trimmed inventory positions). In particular, the MI-cap takes the base-stock levels $(S^{1*},\ S^{j*},\ S^{j-*},\ j = 2, \ldots, m)$ as policy parameters, and the order quantities are determined based on Proposition 2. If the demand data is available (either generated from the known demand distribution or the historical demand data), the parameters can then be optimized by the simulation method. In the numerical study, we will examine the performance of the MI-cap policy.

At last, Condition 3 is not strict, it can be easily satisfied under proper uncertainty sets. Like Sun and Van Mieghem (2019), we consider the CLT uncertainty set. For $n \in \mathbb{N}_t^H$, d_n has finite mean μ_n and standard deviation σ_n. Then the form of CLT uncertainty set is as follows:

$$\Omega_t^{CLT} = \Bigg\{ (d_t, \ldots, d_{t+H-1}) : -\Gamma_n \leq \sum_{i=t}^{n} \frac{d_i - \mu_i}{\sigma_t^n} \leq \Gamma_n, \mu_n - \hat{\Gamma}_n\sigma_n \leq d_n \leq \mu_n + \hat{\Gamma}_n\sigma_n, \forall n \in \mathbb{N}_t^H \Bigg\}, \tag{17}$$

where σ_t^n is the standard deviation of partial sum demands $\sum_{i=t}^n d_i, n \in \mathbb{N}_t^H$. Meanwhile, $\Gamma_n \geq 0$ and $\hat{\Gamma}_n \geq 0$ are tunable parameters called "budget of uncertainty" introduced by Bertsimas and Sim (2004). In particular, the larger their values are, the more conservative the model's solution will be. Hence, they can be adjusted depending on the trade-off between robustness and performance that the decision maker attempts to achieve. From Lemma 2 of Sun and Van Mieghem (2019),

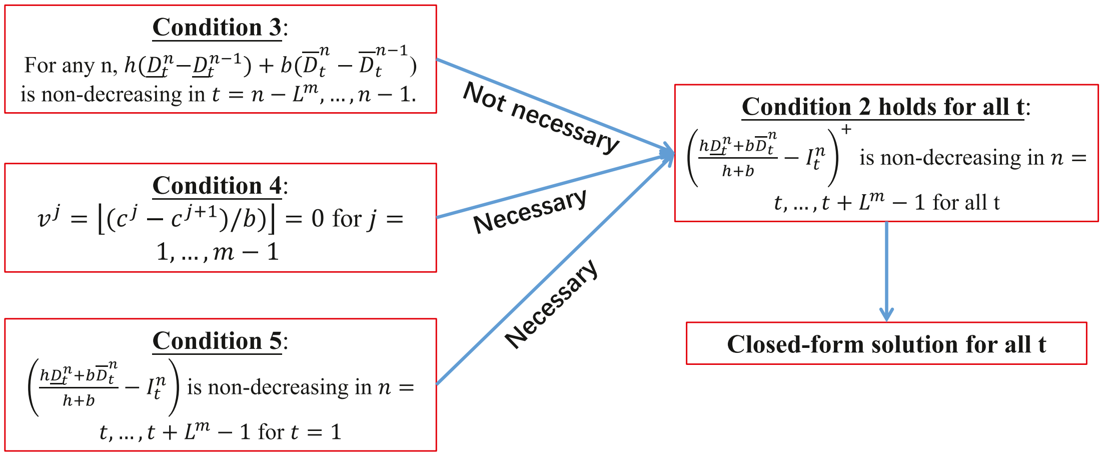

Fig. 2. Relevance of Conditions to Propositions.

$$\Delta d_t^n = \min\left\{\sum_{i=t}^{n} \hat{\Gamma}_i \sigma_i, \min_{j=t,\ldots,n-1}\left\{\Gamma_j \sigma_t^j + \sum_{i=j+1}^{n} \hat{\Gamma}_i \sigma_i\right\},\right.$$
$$\left. \Gamma_n \sigma_t^n, \min_{j=n+1,\ldots,t+H-1}\left\{\Gamma_j \sigma_t^j + \sum_{i=n+1}^{j} \hat{\Gamma}_i \sigma_i\right\}\right\}.$$

We next provide the RCLT policy with its i.i.d. version.

Corollary 3. *Suppose demands are in a CLT uncertainty set Ω_t^{CLT} defined as Eq. (17). If $\Delta d_t^n - \Delta d_t^{n-1}$ is non-decreasing in t where $n - L^m \le t \le n-1$ for any n, the robust optimal multiple-sourcing policy works as:*

$$S_t^{1*} = \mu_t + \left(\frac{b-h}{b+h}\right)\Delta d_t^t, \tag{18}$$

$$S_t^{j*} = \sum_{i=t}^{t+L^j} \mu_i + \left(\frac{b-h}{b+h}\right)\Delta d_t^{t+L^j},$$
$$S_t^{j-*} = \sum_{i=t}^{t+L^j-1} \mu_i + \left(\frac{b-h}{b+h}\right)\Delta d_t^{t+L^j-1}, \quad j = 2, \cdots m. \tag{19}$$

Corollary 4. *Suppose demands are in an i.i.d. CLT uncertainty set, then $\mu_n = \mu, \sigma_t^n = \sigma\sqrt{n-t+1}$ and $\sigma_n = \sigma$ for all n. In view of uniformity, we set $\hat{\Gamma}_n = \Gamma_n = \Gamma$ for all n and obtain more compact parameters:*

$$S^{1*} = \mu + \left(\frac{b-h}{b+h}\right)\Gamma\sigma, \tag{20}$$

$$S^{j*} = (1 + L^j)\mu + \left(\frac{b-h}{b+h}\right)\sqrt{L^j+1}\Gamma\sigma,$$
$$S^{j-*} = L^j\mu + \left(\frac{b-h}{b+h}\right)\sqrt{L^j}\Gamma\sigma, \quad j = 2, \cdots m. \tag{21}$$

The proofs of these two corollaries are not included in the appendix since their processes are apparent. And similar information can be found in the proofs of Propositions 4 and 5 in Sun and Van Mieghem (2019). The Corollary 4 shows that under the i.i.d. CLT uncertainty set, the robust optimal base-stock levels have simple and intuitive expressions. Specifically, each base-stock level contains two parts: the mean value of the lead time demand; and the safety stock for the lead time demand that depends on the service level ratio, the variance of the lead time demand and the budget of the uncertainty of the model. In the above result, if we set $\Gamma = 0$, i.e., no budget for the uncertainty, then the base-stock level is simply the mean value; while as the value of Γ increases, the base-stock level increases, the inventory policy becomes more conservative regarding to the uncertainty.

4.3. Discussion of conditions

So far, we have proposed 5 conditions. Condition 1 is assumed without loss of generality, when it fails, orders from some sources (violating the condition) are zero. The other conditions are all important to our propositions. Their relevance is reflected in Fig. 2. Condition 2 is necessary to Proposition 1. From the structure of model RMS-Q, we know that the objective function is the summation of a series of piecewise linear functions. Moreover, the problem could be divided into a series of subproblems with respect to Q_t^n, and each subproblem is convex with a constraint $Q_t^{n-1} \le Q_t^n \le Q_t^{n+1}$. When the variables are free, we can sequentially solve the unconstrained piecewise linear functions, and the optimal solution is exactly $(\frac{h\underline{D}_t^n + b\overline{D}_t^n}{h+b} - I_t^n)^+$. The positive part means that the partial sum of order quantities should be non-negative. As a result, Condition 2 ensures that the optimal solutions to these separate unconstrained subproblems also satisfy the inequalities $Q_t^t \le Q_t^{t+1} \le \ldots \le Q_t^{t+H-1}$. In other words, Condition 2 guarantees the closed-form solution to the static model. When it is violated, some of inequalities $Q_t^t \le Q_t^{t+1} \le \ldots \le Q_t^{t+H-1}$ in problem RMS-Q will be binding. For this case, we have to numerically solve the problem. For Conditions 3, 4, and 5, we next provide some discussions on their "necessity" or "sufficiency" using some examples. For the sake of brevity, all examples are constructed under the dual-sourcing case. We firstly show that Condition 4 is necessary for Proposition 2.

Discussion of Condition 4. Assuming the demand is constant d at each period, so the uncertainty set is $\Omega_t = \{\mathbf{d}_t = (d, \cdots, d) | \mathbf{d}_t \in \mathbb{R}_+^H\}$ for any t. We then have $\overline{D}_t^n = \underline{D}_t^n = D_t^n = (n-t+1)d$, which makes Condition 3 hold for the system. Let the initial on-hand inventory be zero and there is no pipeline inventory, which means $I_1^n = 0, n = 1, \cdots, L^2$. Let

$$R_t^n = S_t^n - I_t^n = \left(\frac{h\underline{D}_t^n + b\overline{D}_t^n}{h+b} - I_t^n\right). \tag{22}$$

For period 1, we have

$$R_1^n = \left(\frac{h\underline{D}_1^n + b\overline{D}_1^n}{h+b} - I_1^n\right) = (D_1^n - I_1^n) = nd.$$

Hence, R_1^n is nondecreasing in $n = 1, \cdots, L^2$, which implies that Condition 2 holds for period1's static model $\mathbf{M}^H(\mathbf{I}_1, \Omega_1, \mathbb{N}_1^H)$. By applying Corollary 1 to the static model for $t = 1$, we obtain the following optimal order quantities for period 1:

$$q_1^{1*} = Q_1^{1*} - 0 = (S_1^1 - I_1^1)^+ = d,$$

$$q_1^{2*} = Q_1^{1+L^2*} - Q_1^{1+L^2-\nu^1-1*} = (\nu^1 + 1)d.$$

For period $t = 2$, we have the updated trimmed inventory position as

$$I_2^n = \begin{cases} 0, & n = 2, \cdots, L^2, \\ (\nu^1 + 1)d, & n = L^2 + 1. \end{cases}$$

In the meantime,

$$R_2^n = \begin{cases} (n-1)d, & n = 2, \cdots, L^2, \\ (n-1)d - (1+\nu^1)d, & n = L^2 + 1. \end{cases}$$

Since $L^2 - 1 > L^2 - 1 - \nu^1$, it turns out that $R_2^{L^2+1} < R_2^{L^2}$ as long as $\nu^1 \geq 1$. But Condition 2 requires $(R_2^n)^+$ is non-decreasing in $n = 2, \cdots, L^2 + 1$. Therefore, Condition 2 does not hold for $\mathbf{M}^H(\mathbf{I}_2, \Omega_2, \mathbb{N}_2^H)$ if $\nu^1 \geq 1$. This indicates that without Condition 4, Condition 3 is not able to ensure that Condition 2 holds during the rolling procedure. In Section 5.1.5, the simulation results manifest the consequence when Condition 4 is violated.

Discussion of Condition 3. We next construct an example to show that Condition 2 may still hold even if Condition 3 is violated. Let $\Omega_1 = \{(d, \ldots, d)\}$ and $\Omega_2 = \{(d-1, \ldots, d-1)\}$. We notice that Condition 3 is violated for this case. We set $L^2 = 5$, the initial on-hand inventory is $2d$ and pipeline inventory $(0, 2d, 0, d)$. It is easy to get $\mathbf{I}_1 = (2d, 2d, 4d, 4d, 5d)$, and with Eq. (22), $(R_1^1, R_1^2, R_1^3, R_1^4, R_1^5) = (-d, 0, -d, 0, 0)$, which means Condition 2 is satisfied for period 1. Also, we can get $q_1^{1*} = 0, q_1^{2*} = d$ after simple calculation, which makes $\mathbf{I}_2 = (d, 3d, 3d, 4d, 5d)$. We have $(R_2^2, R_2^3, R_2^4, R_2^5, R_2^6) = (-1, -2-d, -3, -4, -5)$. We can also get $h(\underline{D}_1^n - \underline{D}_1^{n-1}) + b(\overline{D}_1^n - \overline{D}_1^{n-1}) = (h+b)d$ and $h(\underline{D}_2^n - \underline{D}_2^{n-1}) + b(\overline{D}_2^n - \overline{D}_2^{n-1}) = (h+b)(d-1)$. Although Condition 3 fails, Condition 2 still holds for $\mathbf{M}^H(\mathbf{I}_2, \Omega_2, \mathbb{N}_2^H)$. Hence, Condition 3 is not a necessary condition for Proposition 2. However, the advantage of this condition is that it does not depend on the inventory states and can be easily checked.

Discussion of Condition 5. Firstly, if Condition 5 does hold, the closed-form solution to the static model in period 1 is guaranteed. Next, we construct an example to show that if Condition 5 does not hold, then Condition 2 may not hold for the static model in period 2 even if Condition 3 holds. We keep the same setting as the former part except for $\Omega_2 = \{(d+1, \ldots, d+1)\}$. With such an uncertainty set for period 2, it is obvious that Condition 3 is now satisfied since $h(\underline{D}_1^n - \underline{D}_1^{n-1}) + b(\overline{D}_1^n - \overline{D}_1^{n-1}) = (h+b)d$ and $h(\underline{D}_2^n - \underline{D}_2^{n-1}) + b(\overline{D}_2^n - \overline{D}_2^{n-1}) = (h+b)(d+1)$. Moreover, the discussions on Condition 3 indicates that $(R_1^1, R_1^2, R_1^3, R_1^4, R_1^5) = (-d, 0, -d, 0, 0)$, which satisfies condition 2 but violates condition 5 in period 1. By similar deduction to former part, we have $(R_2^2, R_2^3, R_2^4, R_2^5, R_2^6) = (1, 2-d, 3, 4, 5)$. When $d > 1$, Condition 2 fails for $\mathbf{M}^H(\mathbf{I}_2, \Omega_2, \mathbb{N}_2^H)$. Therefore, Condition 5 is a necessary condition for Proposition 2.

5. Numerical study

In this section, we conduct numerical experiments to assess the performance of the robust optimal policy with three supply sources under different scenarios. In Section 5.1, we assume the demand distribution is known, our main purpose is to investigate the effectiveness of the MI-cap policy structure and the extra benefits brought by intermediate supply sources. In Section 5.2, we assume the demand distribution is unknown and only some historical data of the demand is available. We use a dataset provided by Alibaba.com to conduct the numerical experiments in this part.

Table 1
Adjustable Parameters of Policies.

Policy	Parameters
MI-cap	$S^1, S^2, S^{2-}, S^3, S^{3-}$
DI-cap	S^1, S^2, cap
RCLT	Γ
VSW	V_1, V_2, V_3

Table 2
Parameter Values for Numerical Experiments.

Parameter	Values
c^1	120
c^2	102,103,104,105
c^3	100
L^1	0
L^2	3,4,5,6
L^3	20
h	3,6,9
b	$\sim$
$b/(h+b)$	0.9,0.95,0.99
μ	100
σ/μ (for normal demand)	0.1,0.2,0.3

We consider four policies: the MI-cap and the RCLT defined in Section 4, the DI-cap with the fastest and slowest sources studied in Sun and Van Mieghem (2019), and the triple-index base-stock policy, denoted as VSW, which is an extension of the dual-index base-stock policy of Veeraraghavan and Scheller-Wolf (2008). In particular, comparison with the DI-cap shows the value of providing more supply sources; while comparison with the VSW shows the value of parameters S^{j-*}, which serves as a part of "cap" for the intermediate source. The control parameters of the four policies are presented in Table 1. For each policy, the parameters are optimized via a simulation-based procedure. Specifically, we first generate N periods of demand samples from the demand distribution as the training dataset. Given a policy and one set of its parameters' values, we simulate the average cost of the system following the policy and find the optimal policy parameters. Then, we simulate the average cost of the system under the policy with optimized parameters over another T periods of demand samples, i.e., the testing dataset.

For the MI-cap, DI-cap, and VSW, we employ the optimization toolbox (interior-point algorithm) of MATLAB to search for the optimal parameters. For the RCLT, if the demand distribution is unknown, we first calculate the mean and variance of the demand samples of the training dataset; and given parameter Γ, the policy is determined according to Corollary 4. We then apply a grid search algorithm to find the best Γ based on simulation. Hence, the selection of parameter Γ is performance-driven. The searching interval is set as [0,3]. Note that we select Γ in such a way out of a fair comparison purpose. The same selection strategy has been employed by Sun and Van Mieghem (2019). We refer the readers to Bertsimas and Thiele (2006) and Mamani et al. (2017) for more discussions on the selection of the budget of uncertainty when considering the robustness of the policy. We will numerically study how the performance and robustness of RCLT are affected by the value of Γ in Section 5.2.2.

5.1. Numerical experiment with known demand distribution

In this subsection, we assume the demand information is known. In the experiments, we generate 10^4 periods of demand from normal (or Poisson) distribution as training dataset, i.e., $N = 10^4$. The mean and variance of the normal distribution are in Table 2. We generate another 10^5 periods of demand from the same distribution as the testing dataset, i.e., $T = 10^5$. We first

Table 3
Results of Simulation on Normal Distribution.

Policy	Counts	Ratio	Average improvement	Max improvement
MI-cap	420	97.222%	–	–
DI-cap	12	2.7778%	2.25%	14.52%
RCLT	0	0%	17.81%	31.64%
VSW	0	0%	8.48%	12.87%

Table 4
Results of Simulation on Poisson Distribution.

Policy	Counts	Ratio	Average improvement	Max improvement
MI-cap	137	95.139%	–	–
DI-cap	7	4.8611%	1.92%	12.54%
RCLT	0	0%	17.03%	30.48%
VSW	0	0%	8.45%	11.83%

compare the performances of the four policies under different parameters' combinations. Then we study the allocation of order quantity among suppliers and examine the value of the intermediate source by changing its ordering cost. In the appendix, we also discuss the value of parameter S^{j-*}, which serves as a part of "cap" for the intermediate source, and explore performances of the policies when Condition 4 is violated. When we calculate the long-run average cost, we normalize the cheapest unit ordering cost to zero to highlight the cost that is affected by the inventory policy.

5.1.1. Performance of policies

By combining different values of system parameters as shown in Table 2, we investigate the performance of the MI-cap against other three polices under 432 scenarios for the normal demand and 144 scenarios for the Poisson demand. We gather the outperforming cases of each policy and the results for the normal (Poisson) demand is provided in Columns 2 and 3 in Table 3 (4). As expected, the MI-cap dominates other three policies in almost all scenarios. Theoretically, when the demand distribution is known, the performance of the MI-cap should be better than the DI-cap because the latter is a special case of the former. However, here because the optimization of policy parameters and the evaluation of the expected long-run average cost are based on the simulation data, which incurs simulation error. Thus there are a few cases where the DI-cap is better than the MI-cap policy, particularly for cases where the intermediate source is rarely used. In these cases, the gap between two polices is very small, and their parameters optimized by training dataset may not always be optimal on the testing dataset. In fact, we find that the MI-cap performs always better than the DI-cap on the training dataset.

In order to show the gap of performances between the MI-cap and the other policies, we define the following measure:

$$\text{average percentage cost improvement} = \frac{1}{\tau}\sum_{trial=1}^{\tau}\frac{C^{o.p.} - C^{MI}}{C^{o.p.}},$$

where $C^{o.p.}$ denotes the cost of the DI-cap or RCLT, VSW. The results are presented in Columns 4 and 5 in Tables 3 and 4. We can see that the MI-cap shows a prominent advantage over other three policies, and the maximum improvement can exceed 10%. The comparison with the DI-cap shows the value of the intermediate source, while the comparison with the VSW shows the value of the "cap".

5.1.2. Order quantity allocation

In practice, when implementing a sourcing strategy, companies might need to sign a capacity reservation contract with suppliers, thus it is crucial to determine the average allocation of order quantity among suppliers. To shed some light on this issue, we count the percentages of orders from three sources under the MI-cap policy for 432 scenarios.

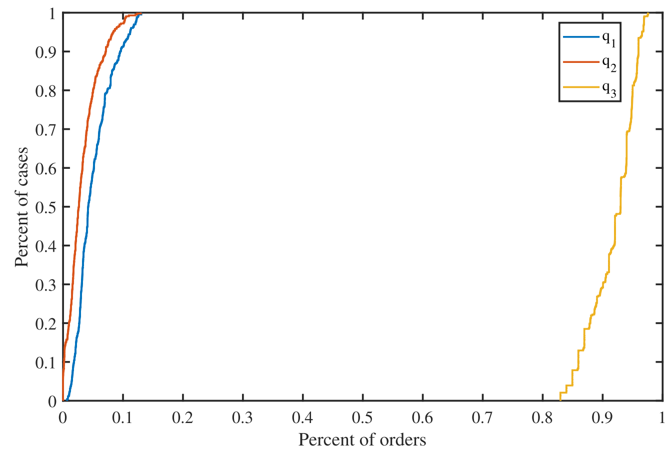

Fig. 3. The Cumulative Distributions of Percents of Orders under 432 Scenarios.

The empirical distributions of order percentages from three sources are shown in Fig. 3. We can see that for all cases the percentage of the cheapest source is more than 80%, and for more than 70% cases, the percentage is more than 90% and for about 20% cases, the percentage is more than 95%. Meanwhile, the order percentages of the other two sources vary from 0 to 13%, and the curve of the middle source is always at the left-hand-side of the curve of the fastest source, which implies that the order percentage of the middle source is in general less than that of the fastest source. And for about 15% cases, the middle source will never be used, in which case, the performance of the MI-cap should be very close to that of the DI-cap (theoretically they should be equal).

Next, we further explore the value of the intermediate source. Fig. 4(a) depicts the impact of the ordering cost of the intermediate source on the cost of the MI-cap. Since the DI-cap does not use the intermediate source, its performance is not affected by the ordering cost of the intermediate source. As the cost of the intermediate source c^2 increases, the cost of the MI-cap increases while the increment decreases. When c^2 is high enough, the cost of the MI-cap coincides with the cost of the DI-cap. Fig. 4(b) shows the allocation of the order quantity among three suppliers. We can see that most of the inventory is ordered through the cheapest source 3. Meanwhile, as c^2 increases, the percentage of source 2 decreases, whereas percentages of the other two sources increase. When c^2 is high enough, the percentage of source 2 becomes 0, and the percentages of the other two sources become stable. We remark that if we fix the cost of source 2 and vary its lead time, similar results will be obtained. In the appendix, we provide an example on how orders from three suppliers and the net inventory level change over time for a randomly selected case under the normal demand.

5.2. Numerical study with real sales data of an E-commerce company

In this section, we use a real dataset to evaluate the performances of the policies. The dataset is provided by Alibaba.com, it contains transactional data of 272,328 SKUs sold on TMall.com from Jan 2017 to Jul 2017.[2] For data preprocessing, we first remove those SKUs that has less than 10,000 orders. We then use these orders' information to estimate the daily demand of each SKU. Consider that some items have short sales seasons, we only retain SKUs that have sales seasons across the whole 212 days, i.e., those

[2] https://tianchi.aliyun.com/competition/entrance/231623/information.

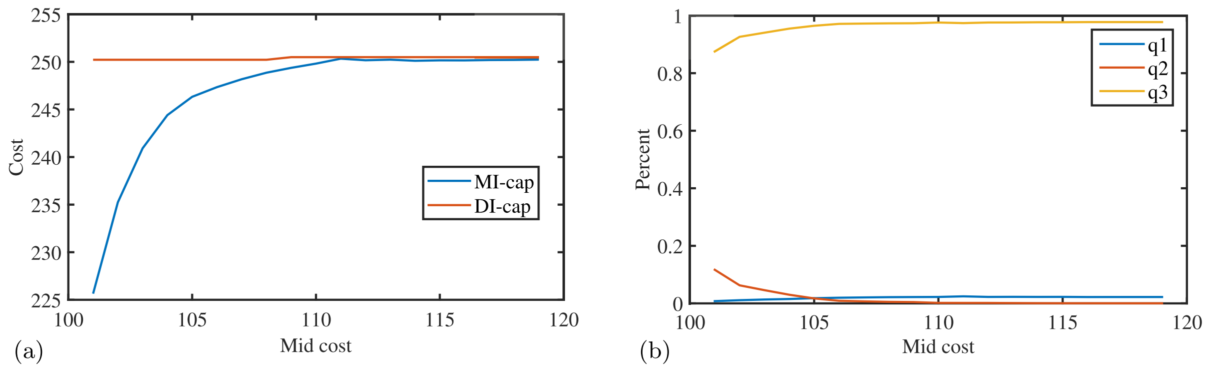

Fig. 4. (a) Impact of Mid-Cost. (b) Impact of Mid-Cost on Percentage of Order Quantities. The other parameters are fixed at: $c^1 = 120,\ c^3 = 100,\ L^1 = 0,\ L^2 = 2,\ L^3 = 5,\ h = 3,\ b = 57,\ \mu = 100,\ \sigma = 20$.

Table 5
Results of Real Data.

Policy	Counts	Ratio	Average cost
MI-cap	295	26.32%	3013.6
DI-cap	177	15.78%	3146.3
RCLT	541	48.26%	3143.8
VSW	108	9.63%	3135.7

Table 6
Results of the Logistic Regression Model.

	Constant	NOZ	Mean	Variance	Skewness	Kurtosis
Coefficients	0.3943	−0.1901	0.0471	−0.4189	0.0587	−0.4761
P-Value	0	0.0052	0.664	0.214	0.775	0.024

Table 7
Comparison results on data group by NOZ.

NOZ	Policy	Counts	Ratio	Average cost
> 20	MI-cap	25	36.23%	5502
	DI-cap	6	8.70%	5757
	RCLT	27	39.13%	6175
	VSW	11	15.94%	5856.5
≤ 20	MI-cap	270	25.67%	2850.4
	DI-cap	171	16.25%	2975.1
	RCLT	514	48.86%	2944.9
	VSW	97	9.22%	2957.3

have sales data at the first day and the last day. We finally obtain 1121 SKUs, each SKU has a training dataset with 212 periods of demand. For the testing dataset, we apply bootstrapping to the historical data and extend them to 10^5 periods. The system parameters are fixed at: $c^1 = 120,\ \ c^2 = 104,\ \ c^3 = 100,\ \ L^1 = 0,\ L^2 = 2,\ L^3 = 5,\ h = 3,\ b = 57$.

5.2.1. Performances of policies

Table 5 shows the average cost and the percentage of outperform cases of each policy among 1121 SKUs. We can see that the MI-cap again has the lowest average cost in the real dataset; though in this case, each of the other policies has the best performance for a proportion of the SKUs.

To gain a better understanding of the performances of the MI-cap and the RCLT in a real dataset, we next investigate the relationship between the features of the sales data of a SKU and the relative performance of the two policies by a logistic regression model. Specifically, we label SKUs that the RCLT has a better performance than the MI-cap as class 1, and the others as class 0. We then extract statistic features from each SKU's sales data, including NOZ (number of days with 0 sale), mean, variance, skewness and kurtosis. After computing these features, we apply a Z-Score standardization to normalize the scale of the data of different features. We then build a logistic regression model for the classification of the SKUs. The coefficients of the regression model and the corresponding p-values are shown in Table 6. We can see that the impacts of NOZ and kurtosis are statistically significant. Specifically, as the NOZ or the kurtosis of the data distribution increases, it is more likely that the MI-cap has a better performance than the RCLT.

We next further investigate the impact of zero values in the demand data. We divide the SKUs into different groups according to the NOZ. Then we compare performances of the four policies under different groups of SKUs. The results are shown in Table 7 (We can also use the average intermittent days of the demand data to divide SKUs. And similar results will be obtained).

Table 7 shows that for the two groups with NOZ $\leq$ 20, the average cost of the RCLT is lower than the DI-cap and the VSW. Whereas for the group with NOZ>20 (which includes only a small portion of the SKUs), the average cost of the RCLT is the highest. Therefore, practical firms could classify the SKUs into different groups according to the value of NOZ, and the RCLT is more appropriate for those with small NOZ. Intuitively, the RCLT is derived based on the CLT uncertainty set, which is constructed by the mean and standard deviation of the demand (assuming the set of possible demand value is Mean -/+ k*Std).

When the NOZ or the kurtosis of the demand data is large, it would be beneficial to construct a more sophisticated uncertainty set that includes the information of NOZ and kurtosis. For example, one can include zero into the set of possible values and assign certain probability (or a lower bound of this probability) to the zero value. But in this case, the distributionally robust framework (e.g., Delage & Ye, 2010), i.e., the uncertainty set characterizes the possible demand distributions instead of the possible values of the demand, would be more appropriate to model the optimization problem. Because in our current robust optimization framework, only the support of the demand is used to define the optimization model.

5.2.2. Effect of Γ on RCLT.

The numerical experiments reported so far focus mainly on the value of the structures of the robust optimal policies, i.e., the performance side of the MI-cap and RCLT policies. We next explore the impact of the budget of uncertainty parameter Γ on the performance of the RCLT to shed some lights on the robustness side of the policy.

We randomly select two SKUs and adopt the RCLT policy. Fig. 5 presents the performances of the RCLT policy with different values of Γ for the two SKUs. We can observe that the average cost of the RCLT policy is first decreasing in Γ and then increasing in Γ.

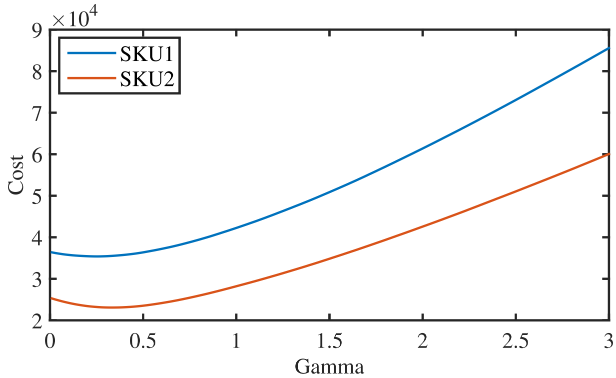

Fig. 5. Impact of Selection of Γ.

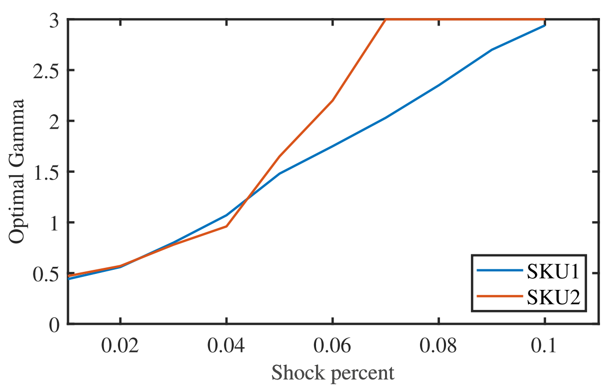

Fig. 6. Relationship between the Optimal Γ and Shock Percent.

Meanwhile, the optimal value of Γ is relatively small (around 0.4). As we discussed earlier, Γ is the parameter that controls the trade-off between performance and robustness. Therefore, it is expected that as the value of Γ increases, the RCLT would be more robust. To illustrate that point, we introduce some random shock into the sales data of the experiment to make the demands more difficult to predict, in which case the robustness of the policy is crucial. Specifically, when bootstrapping the testing dataset, we increase the daily demand to λ_s times with certain probability p_s. Here λ_s is the shock scale and p_s is the shock percent. We fix $\lambda_s = 5$ to investigate the relationship between the shock percent and the optimal Γ of the RCLT. Fig. 6 demonstrates that as the shock percent increases, the optimal Γ increases, which implies that with a large Γ, the RCLT is more robust to the demand surges.

The results of this section show that each of the MI-cap and the RCLT has its own merit. Recall that these two policies have the same structure but the policy parameters are determined by different methods. In particular, the parameters of the MI-cap are optimized by a simulation procedure based on the historical data. While for the RCLT, we first use the historical data to construct a CLT uncertainty set of demand and then determine the parameters (except the budget of uncertainty) by the closed form-solution. In practice, firms can build a classification model for SKUs based on the statistical features of the demand data and employ different policies for different groups of SKUs. For instance, the MI-cap are more suitable to SKUs that have common feature of larger kurtosis and more zero sales days. On the contrary, managers are inclined to adopt the RCLT for SKUs with smaller kurtosis and less zero sales days and for situations where the unpredictable demand surge may occur.

Lastly, we remark that in the study of Section 5.2, we essentially considered an i.i.d. demand process with an unknown distribution and we eliminated the autocorrelation structure of the demand process. In fact, with the CLT uncertainty set, we are capable of incorporating the correlation of demand into the model. We also apply data generated by multivariate normal distribution and AR(1) process to take demands' correlation into account. We compare the performances of policies obtained by Corollary 4 and Corollary 3. The results indicate that if the correlation information of the demand is available, it is rather beneficial to adopt the RCLT uncertainty set incorporating the information. Due to the paper length limit, this part is provided in the appendix.

6. Conclusion

In this article, we allow multiple supply sources with different lead times in an inventory system. We build up a static robust model for multiple sourcing and extend it to the rolling horizon framework under certain conditions, placing very few requirements on the demands. We show a high-quality policy for a general case with a simplification for the CLT uncertainty set. This robust optimal policy is a base-stock policy for the fastest source and sort of gap-of-base-stock policy for the other ones. Under the CLT uncertainty set, we get closed-form expressions for all of its control parameters. Based on the structure of the policy, we propose a new class of policies called MI-cap for the multiple sourcing inventory models. One of its most appealing features is that the policy is amenable to a wide variety of parameters, which should greatly facilitates real-world adoption.

In the first part of numerical experiments, we compare performances of the MI-cap, the RCLT, the DI-cap and the VSW with demands generated from the normal and poisson distributions. The results show that the MI-cap has a prominent advantage over the other three policies, and the maximum improvement can exceed 10%. We also investigate the average allocation of order quantity among suppliers. The results show that for all cases the percentage of the cheapest source is more than 80%. Meanwhile, the order percentages of the other two sources vary from 0 to 13%, and the order percentage of the middle source is in general less than that of the fastest source. We also explore the value of the intermediate source along with the capping effect. In the second part, we use real sales data from an E-commerce company in China. We find that the MI-cap has the lowest average cost. We study the relationship between performances of the policies and the statistical features of the sales data. In particular, the impacts of the number of days with 0 sales and kurtosis are statistically significant. As the number of days with 0 sales or the kurtosis of the data distribution increases, it is more likely that the MI-cap has a better performance than the RCLT. Firms may classify SKUs into different groups accordingly and employ the corresponding most suitable policy. We also investigate the selection of budget of uncertainty parameter.

There are several future research directions worth exploring. The first direction is to consider more sophisticated uncertainty sets of demand including more demand distribution information, i.e., the kurtosis of the distribution and the probability of zero demand. Anther research direction worth pursuing is to extend the current model to the case with supply disruption and supply capacity. Moreover, studying other inventory systems such as the multi-echelon inventory system with multiple delivery modes and the perishable inventory system with dual-sourcing are also interesting.

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Supplementary material

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References

Abginehchi, S., Farahani, R. Z., & Rezapour, S. (2013). A mathematical model for order splitting in a multiple supplier single-item inventory system. *Journal of Manufacturing Systems, 32*(1), 55–67.

Bandi, C., & Bertsimas, D. (2012). Tractable stochastic analysis in high dimensions via robust optimization. *Mathematical Programming, 134*(1), 23–70.

Ben-Tal, A., El Housni, O., & Goyal, V. (2020). A tractable approach for designing piecewise affine policies in two-stage adjustable robust optimization. *Mathematical Programming, 182*(1), 57–102.

Ben-Tal, A., Ghaoui, L. E., & Nemirovski, A. (2009). *Robust optimization*. Princeton, NJ: Princeton University Press.

Ben-Tal, A., Golany, B., Nemirovski, A., & Vial, J. P. (2005). Retailer-supplier flexible commitments contracts: A robust optimization approach. *Manufacturing & Service Operations Management, 7*(3), 248–271.

Bental, A., Bertsimas, D., & Brown, D. B. (2010). A soft robust model for optimization under ambiguity. *Operations Research, 58*(4), 1220–1234.

Bental, A., Golany, B., & Shtern, S. (2009). Robust multi-echelon multi-period inventory control. *European Journal of Operational Research, 199*(3), 922–935.

Bental, A., & Nemirovski, A. (1998). Robust convex optimization. *Mathematics of Operations Research, 23*(4), 769–805.

Bental, A., & Nemirovski, A. (2000). Robust solutions of linear programming problems contaminated with uncertain data. *Mathematical Programming, 88*(3), 411–424.

Bertsimas, D., Brown, D. B., & Caramanis, C. (2011). Theory and applications of robust optimization. *SIAM Review, 53*(3), 464–501.

Bertsimas, D., Gupta, V., & Kallus, N. (2018). Data-driven robust optimization. *Mathematical Programming, 167*(2), 235–292.

Bertsimas, D., & Sim, M. (2004). The price of robustness. *Operations Research, 52*(1), 35–53.

Bertsimas, D., Sim, M., & Zhang, M. (2019). Adaptive distributionally robust optimization. *Management Science, 65*(2), 604–618.

Bertsimas, D., & Thiele, A. (2006). A robust optimization approach to inventory theory. *Operations Research, 54*(1), 150–168.

Bienstock, D., & Ozbay, N. (2008). Computing robust basestock levels. *Discrete Optimization, 5*(2), 389–414.

Chen, X., & Zhang, Y. (2009). Uncertain linear programs: Extended affinely adjustable robust counterparts. *Operations Research, 57*(6), 1469–1482.

Dalrymple, D. J. (1987). Sales forecasting practices: Results from a united states survey. *International journal of Forecasting, 3*(3–4), 379–391.

Daniel, K. H. (1963). A delivery-lag inventory model with emergency. *Multistage Inventory Models and Techniques*. H. E. Scarf, D. M. Gilford, M. W. Shelly, eds. Stanford University Press, Stanford, CA. 32–46

Delage, E., & Ye, Y. (2010). Distributionally robust optimization under moment uncertainty with application to data-driven problems. *Operations Research, 58*(3), 595–612.

Feng, Q., Gallego, G., Sethi, S. P., Yan, H., & Zhang, H. (2005). Periodic-review inventory model with three consecutive delivery modes and forecast updates. *Journal of Optimization Theory & Applications, 124*(1), 137–155.

Feng, Q., Sethi, S., Yan, H., & Zhang, H. (2006a). Are base-stock policies optimal in inventory problems with multiple delivery modes? *Operations Research, 54*, 801–807.

Feng, Q., Yan, H., Zhang, H., & Sethi, S. P. (2006b). Optimality and nonoptimality of the base-stock policy in inventory problems with multiple delivery modes. *Journal of Industrial and Management Optimization, 2*(1), 19–42.

Fukuda, Y. (1964). Optimal policies for the inventory problem with negotiable leadtime. *Management Science, 10*(4), 690–708.

Gabrel, V., Murat, C., & Thiele, A. (2014). Recent advances in robust optimization: An overview. *European Journal of Operational Research, 235*(3), 471–483.

Gallego, G., & Moon, I. (1993). The distribution free newsboy problem: Review and extensions. *Journal of the Operational Research Society, 44*(8), 825–834.

Gorissen, B. L., & Den Hertog, D. (2013). Robust counterparts of inequalities containing sums of maxima of linear functions. *European Journal of Operational Research, 227*(1), 30–43.

Hua, Z., Yu, Y., Zhang, W., & Xu, X. (2015). Structural properties of the optimal policy for dual-sourcing systems with general lead times. *IIE Transactions, 47*(8), 841–850.

Li, Q., & Yu, P. (2014). Multimodularity and its applications in three stochastic dynamic inventory problems. *Manufacturing & Service Operations Management, 16*(3), 455–463.

Mamani, H., Nassiri, S., & Wagner, M. R. (2017). Closed-form solutions for robust inventory management. *Management Science, 63*(5), 1625–1643.

Perakis, G., & Roels, G. (2008). Regret in the newsvendor model with partial information. *Operations Research, 56*(1), 188–203.

Scarf, H. E. (1958). A min-max solution of an inventory problem. *Studies in the Mathematical Theory of Inventory & Production*. Stanford University Press, California, 201–209

See, C., & Sim, M. (2010). Robust approximation to multiperiod inventory management. *Operations Research, 58*(3), 583–594.

Sheopuri, A., Janakiraman, G., & Seshadri, S. (2010). New policies for the stochastic inventory control problem with two supply sources. *Operations Research, 58*(3), 734–745.

Soyster, A. L. (1973). Convex programming with set-inclusive constraints and applications to inexact linear programming. *Operations research, 21*(5), 1154–1157.

Sun, J., & Van Mieghem, J. A. (2019). Robust dual sourcing inventory management: Optimality of capped dual index policies and smoothing. *Manufacturing & Service Operations Management, 21*(4), 912–931.

Svoboda, J., Minner, S., & Yao, M. (2017). Typology and literature review on multiple supplier inventory control models. *Social Science Research Network*.

Veeraraghavan, S., & Scheller-Wolf, A. (2008). Now or later: A simple policy for effective dual sourcing in capacitated systems. *Operations Research, 56*(4), 850–864.

Whittemore, A. S., & Saunders, S. C. (1977). Optimal inventory under stochastic demand with two supply options. *SIAM Journal on Applied Mathematics, 32*(2), 293–305.

Wu, J., Wang, H., & Shang, J. (2019). Multi-sourcing and information sharing under competition and supply uncertainty. *European Journal of Operational Research, 278*(2), 658–671.

Yue, J., Chen, B., & Wang, M. (2006). Expected value of distribution information for the newsvendor problem. *Operations Research, 54*(6), 1128–1136.

Zhang, V. L. (1996). Ordering policies for an inventory system with three supply modes. *Naval Research Logistics, 43*(5), 691–708.