International Journal of Production Research
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tprs20

A review of current inventory theory and its applications

SUMER C. AGGARWAL [a]

[a] The Pennsylvania State University, Pennsylvania, U.S.A

Version of record first published: 12 Mar 2007.

To cite this article: SUMER C. AGGARWAL (1974): A review of current inventory theory and its applications, International Journal of Production Research, 12:4, 443-482

To link to this article: http://dx.doi.org/10.1080/00207547408919568

INT. J. PROD. RES., 1974, VOL. 12, NO. 4, 443–482

A review of current inventory theory and its applications

SUMER C. AGGARWAL*

This paper looks at the inventory-related models published during the last ten years from the systems point of view. A chart is given to show the broad categories of inventory models. The paper first lists eleven quantifiable and four non-quantifiable variables (and their types) that go into inventory formulas, affect inventory decisions, and have already been identified by the researchers. Next, various research papers are grouped in six categories based on similarities of approaches used by these papers. These categories are : (1) models for determining optimum inventory policies, (2) lot size optimization, (3) optimization of various specific management objectives, (4) models for optimizing highly specialized inventory situations, (5) application of advanced mathematical theories to inventory problems, and (6) models bridging the gap between theory and practice. Next a synthesized review of recently published models belonging to each individual category is presented, and salient features of individual studies are highlighted. Further, this paper discusses the difficulties that most commonly arise in applying these models to routine practical inventory situations ; and it also lists the weaknesses (or limitations) of certain assumptions commonly made by the researchers for developing inventory models. In the end, the paper makes several suggestions about how the available results of inventory models can best be used in practice.

Introduction

With the advance of mathematical inventory theory and easy availability of cheaper computer time, more and more companies are giving up the use of classical EOQ formulae. These formulas are most commonly derived using static type of assumptions (i.e. constant rate of demand, constant lead time, etc.) and therefore are designated as the *static models*. There are a large number of other formulas, which are based on the assumption of one or more of the variables (rate of demand, lead time, or elements of cost, that affect the total inventory costs), being of constantly changing nature, are called *dynamic models*. The dynamic models require a considerable amount of computational effort, and nowadays fortunately most of the large sized companies are well equipped to handle such models.

To place all the published inventory models in one systematic format is more or less impossible ; but the main branches of inventory theory as regards *determination of inventory policies* can roughly be mapped in the form of one small tree as depicted in fig. 1. Each of the boxes of fig. 1 and its branches bursting downwards out of it are well-known in the literature. Most of the boxes at the lower-most ends of the branches can have down below them further a large number of branches depending upon the various combinations (or permutations) between : (i) types of lead times, (ii) types of cost functions for various cost-elements, (iii) types of constraints, etc. It must be noted that fig. 1 does not show separate branches for different types of lead times, because most researchers have either used constant lead times or zero lead times, and only in rare cases, variable lead times.

* The Pennsylvania State University, Pennsylvania, U.S.A.
Received by *Int. J. Prod. Res.*, 20 June 1972.

Published by Taylor & Francis Ltd, 10–14 Macklin Street, London WC2B 5NF.

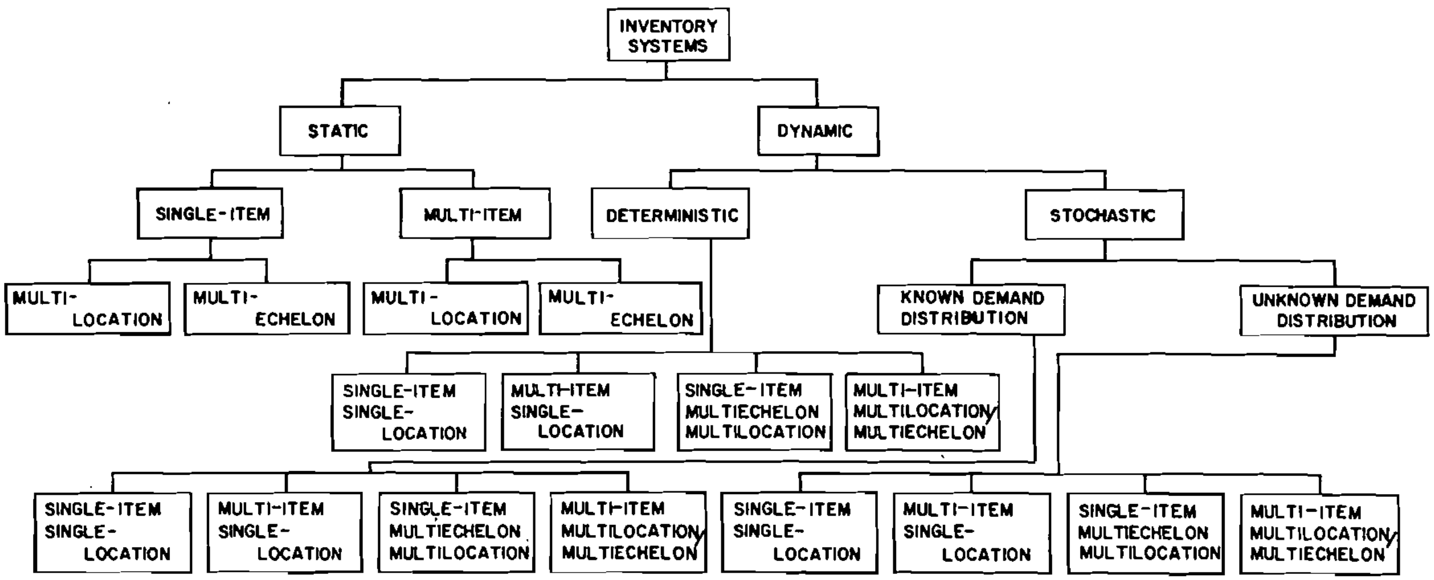

Fig. 1: A general schematic diagram for inventory models.

All-encompassing systems point of view

Possible number of inventory models

In the fast growing literature of inventory, different researchers have included different numbers and types of variables in their models, and as a result, the total number of inventory models has been increasing at a very fast rate. The main variables (or parameters) and the types of each of them that have till this time been distinctly identified and can as well be conveniently quantified are :

(1) *Types of demands.* (i) constant, (ii) known distribution, or (iii) unknown distribution.

(2) *Elements of inventory cost and their functions.* (i) production (or purchase) cost function may be : (*a*) linear, (*b*) convex, or (*c*) concave ; (ii) inventory holding cost function may be : (*a*) linear, (*b*) convex, or (*c*) concave ; and (iv) shortage (or backlogging) cost function may be : (*a*) linear, (*b*) convex, or (*c*) concave.

(3) *Kinds of leadtimes.* (i) zero, (ii) constant, (iii) known distribution, (iv) unknown distribution, or (v) dependent on the system state such as, the number of outstanding orders, etc.

(4) *Variations in the amounts actually received (or produced) as compared to the amounts ordered.* (i) 100% received, (ii) the received amount's distribution is known, or (iii) the amount received has unknown distribution.

(5) *Interdependence between demands of items.* (i) single item case, and (ii) multi-item case.

(6) *Interdependence between locations and/or echelons.* (i) single location (or single echelon), (ii) multilocation, (iii) multiechelon, or (iv) multi-echelon and multilocation.

(7) *Discounting of future costs.* (i) constant discount factor α, or (ii) variable α.

(8) *Various types of constraints/no-constraints.* (iA) no constraint on storage space, (iiA) limited storage space ; (iB) unlimited capital available for holding inventories, (iiB) limited amount of capital available ; (iC) no constraints on production capacity (or on supplies from vendors), (iiC) limited production capacity (or vendor's supplies) ; (iD) no limit on order-size, (iiD) minimum order size constraint, (iiiD) constraint of full batch order-size (full carloads for economy of transportation, etc.).

(9) *Types of backlogging policies.* (i) no backlogging, (ii) partial backlogging, or (iii) complete backlogging.

(10) *Conditions of obsolescence (or perishability) of products.* (i) no deterioration with time, or (ii) deterioration with time accompanied by proportional loss in value of the product.

(11) *Types of planning horizons.* (i) finite, or (ii) infinite.

The other well-known variables, which do affect inventory policies, but cannot be quantified are : (12) company's top management policy about continuance or discontinuance of a product or a group of products ; (13) changing government regulations about imports and exports of raw-materials or finished products ; (14) uncertainties of ever-changing economy and of the cost of capital ; and (15) changes in selling prices of the items, that certainly affect the demand rate of the products.

Grouping of research efforts based on similarity of approaches

There cannot be any rigid boundary lines between various groups of research directions (efforts), but still there are significant and quite evident reasons, based on definite similarities of approaches, that individual research papers may be placed under the following specific categories :

I. Models for determining optimum inventory policies.
II. Lot-size optimization.
III. Optimization of various specific management objectives.
IV. Models for optimizing highly specialized inventory situations.
V. Applications of advanced mathematical theories.
VI. Models bridging the gap between theory and practice.

In the following paragraphs, we summarize very briefly the major researches accomplished in the recent years. The emphasis throughout the summaries will be on the variables considered, special conditions or constraints included in the models and on the objective functions (or the objectives) being optimized.

I. *Models for determining optimum inventory policies*

(*a*) *Single-item multiperiod production and inventory models*: Zangwill (1966 b) analysed a deterministic multiperiod production and inventory model that had concave production costs and piecewise concave inventory costs. His model allowed backlogging. He utilized piecewise concave cost function for determining the minimum cost production schedule. First he determined a basic set B_k, then he determined the extreme point of B_k, $E(B_K)$. Next, stating X as the union of extreme points of B_k, $U_k B_k$, he minimized the total cost function, $F(Z)$ on set X at a point in D, $D = U_k E(B_k)$, where D was the dominant set. He also developed an algorithm to select the optimum vector from the dominant set, making use of the total cost function :

$$F(Z) = \sum_{i=1}^{n} [P_i(x_i) + H_i\{I_i\}. \tag{1}$$

Here $F(Z)$ is the total cost of producing the vector, $Z = (x_1, x_2, \ldots, x_n)$; $P_i(x_i)$ is the cost of completing production x_i in period i, and $H_i\{I_i\}$ is the inventory holding cost for holding I_i in period i.

Boylan (1967) extended the multiple (s, S) ordering case of single ordering decision (Boylan 1964) to the n-period case. He minimized $g_n(x)$, the expected cost over n-periods, when the starting stock for the first period was x, next he minimized the expected cost over n periods, where the initial stock was x and $g_n{}^*(x) = g_n(x) + c$. He showed that an ordering policy as a function of Y, $g_n{}^*(Y(x))$ was optimal for the first period of n-periods case, if for every x,

$$g_n{}^*(Y(x)) + k \,.\, \delta(Y(x) - x) - cx = f_n(x) \tag{2}$$

where c is a non-negative constant, k is a positive constant, and $\delta(x) = 1$ for $x > 0$, otherwise $\delta(x) = 0$. Using Bellman's principle, he proved that an ordering policy, which was optimal to use in the first of n periods, was also optimal to use, when there were n periods left in an m period case, $m > n$. He also found out the conditions which would ensure that for every n, some multiple (s, S) policy would be optimal for the first of n periods.

Porteus (1971) considered single product, periodic review, stochastic inventory model, with a concave increasing reordering cost function. He determined the optimum policy for a multiperiod problem, assuming that the number of units demanded in each period are independent, identically distributed random variables, described by a one-sided Polya-density ϕ. Assuming that the cost function, discount factors and probabilities of demand densities could be time-dependent, he developed $f_n(x)$, the infimum of the expected cost over periods $n, n-1, \ldots, 1$, as a function of the level of inventory x, before ordering in period n, which was,

$$f_n(x) = \inf_{y \geqslant x} \{c(y-x) + L(y) + \alpha f_{n-1} * \phi(y)\} \tag{3}$$

where c, y, $L(y)$ and α have their usual meanings and ϕ is a one-sided Polya frequency function. Using the eqn. (3), he showed that a generalized (s, S) policy was optimal. For proving the optimality of (s, S) policies, he used a generalization of the K-convex and quasi-concave functions to quasi-K-convex functions.

Next Porteus (1972) advanced his above results to a single product, single echelon, periodic review, stochastic, dynamic inventory model, where the ordering cost function was concave increasing (in place of being linear with a set-up cost). He again proved that a generalized (s, S) policy would be optimal in a finite horizon problem, where the probability density of demands were uniform or convolutions of a finite number of uniform and/or one-sided Polya densities. Such densities are not the one-sided Polya densities, for which a proof was given in the earlier paper (Porteus 1971).

Beesack (1967) considered a single-item periodic review model, while dividing the finite planning horizon into $(\lambda + N)$ periods of equal duration, during which an order could be placed. The λ denoted the fixed replenishment lead time, and the demand in period i, $(1 \leqslant i \leqslant \lambda + N)$, was an independent discrete random variable with redistribution ϕ_i. He assumed a fixed ordering (or set-up) cost; a linear unit purchase cost; a linear holding cost; and constant discount factor, α. All the unsatisfied demand was considered to be backlogged.

In general (s, S) policy does not remain optimal below a critical value of unit stockout cost (Arrow 1958). In place of unit stockout cost Beesack used a stockout constraint = (expected number of stockouts/expected demand during all the periods). He handled this constraint with Lagrangian multiplier technique, and minimized the total expected discounted cost:

$$E(x_1, Z_1) = \sum_{j=1}^{\lambda+N} \alpha^{j-1} Ex\{c(z_j) + h(y_j)\}, \tag{4}$$

here $Z_1 = (z_1, z_2, \ldots, z_n)$ is an ordering policy for periods 1 to N, and x_1 is the opening stock of period 1. Further, he developed a dynamic programming algorithm for determining the optimum ordering policy for a finite horizon case.

Falkner (1969) built a single item model to find the number of orders, the order quantities and the times at which the orders should be placed, for minimizing the total cost over a finite time horizon. He combined both the backlogging and no backlogging case, and found cost conditions for the optimal schedules having *regeneration point property* (Manne and Vienott 1967), that allowed the order quantities to be calculated from the requirement function. He applied his results to a purchase cost function (covering all units quantity discounts, and which was piecewise linear and right continuous).

(*b*) *Single-item, multiechelon, multiperiod models* : In another paper Zangwill (1966 a, b) considered the linking together of several single facilities models. He linked them in the form of an acyclic network of facilities (i.e. acyclic means each facility can receive input from any lower numbered facility, but it can produce only to supply to higher numbered facilities). His model considered concave production costs, piecewise concave inventory costs, and assumed that backlogging of unsatisfied demand was permitted. He minimized the total cost function :

$$F(Z)=P(Z)+\sum_{j=1}^{j=N}\sum_{i=1}^{i=N} H_i{}^j(Z), \tag{5}$$

where Z was the production schedule vector, n was the number of periods considered, N was the number of facilities and $H_i{}^j(Z)$ was the inventory cost function in terms of Z. He considered the set K having all factors $k_i{}^j$, where $k_i{}^j$ was $+1$ or 0 only. First he defined the basic set B_k, next he found the extreme points of the basic set, $E[B_k]$, and then minimized $F(Z)$, on the union of such extreme points, $U_{k\epsilon K}E[B_k]$. He also developed algorithms to calculate the minimum cost schedules for : (i) parallel facilities case and (ii) the facilities in series case.

Bessler and Vienott (1966) considered a single-item multiperiod multiechelon supply system, having n facilities. Each facility was assumed to order from an exogeneous source with delivery lag and proportional ordering costs. During the period, random demands got satisfied according to a given supply policy, that defined how excess stocks might be redistributed between facilities. Costs consisted of storage, shortage and transportation. An ordering policy for minimizing expected costs was determined. He showed that if the initial stock was small and certain other conditions were satisfied, then it would be optimal to order in period i, so that vector-stock at hand at n facilities became $\bar{Y}_i=(\bar{Y}_{i1}, \bar{Y}_{i2}, \bar{Y}_{i3}, \ldots, \bar{Y}_{in})$. The base-stock $\bar{Y}_i$ was obtained by minimizing a function of n variables. Special policy, where each facility (except facility 1) passed on its shortages to its direct supplier facility (each one has a direct supplier in series) was investigated. Next they developed shortage functions and the single period expected cost function. They also obtained upper and lower bounds for the base-stock level. Finally they gave the conditions, under which the optimal base-stock level and the stockout probability at facility j were lesser than those at facility k (having the same direct supplier). This main condition was identified as that the distribution of total demand at facility k in a period was spread less than at facility j.

Clark and Scarf (1960) stated that given the stock level at central warehouse 2 was x_2, and the stock level at satellite warehouse 1 was x_1 ; when satellite 1 received its stock from warehouse 2 only, and the demand was occurring at 1 only ; and assuming linear ordering cost, plus penalty cost for the satellite 1 (i.e. additional expected incremental cost of transporting from 2 to 1) they showed that,

$$C^n(x_1, x_2) = C_1{}^n(x_1) + C_2{}^n(x_2) \tag{6}$$

where $C^n(x_1, x_2)$ was the inventory cost function for the system with satellite stock-level x_1 and central warehouse stock level x_2, for the n-periods ; $C_1{}^n(x_1)$ was the cost of inventory in terms of x_1 ; but the separated penalty cost function $C_2{}^n(x_2)$ was in terms of x_2 alone. They also introduced upper and lower bounds for the penalty cost function. These bounds were again in terms of x_2 only. The simple cost approximations derived by them satisfied :

$$C_1{}^n(x_1) + C_2{}^n(x_2) \leqslant C^n(x_1, x_2) \leqslant C_1{}^n(x_1) + C_2{}^{\bar{n}}(x_2). \tag{6 a}$$

Hochstaedter (1970) extended the bounds concept of Clark and Scarf (1966) to one central warehouse and two-satellites case. He used fixed ordering costs and assumed that each satellite followed an (s^n, S^n) optimal policy, throughout n-periods, and then determined the upper and lower bounds for the cost function of the central warehouse alone. Finally he proved that the difference between these bounds for each of the periods was less than a fixed number. For the infinite horizon model, using discount factor $\alpha < 1$, this difference was shown to be less than a constant ; depending upon the means of demand, and holding and penalty costs for the periods. These results can easily be extended to more than 2-satellites cases as well.

Simon (1971) analysed a 2-echelon inventory model for consumable or repairable parts, where central warehouse (first echelon) used one-for-one replenishment policy, but the second echelon used continuous review $(s, S$ policy. Repairs were possible at all the echelons (i.e. facilities). Repairs and transportation timings were assumed to be deterministic, and failure process that generated demand for replacements was assumed to be Poisson. He derived exact expressions for stationary distributions of stock-on-hand, stock-in-repair and backlogged demand at each facility and also indicated how these expressions could be used for optimizing inventory operations.

Schwarz (1973) analysed one warehouse N-retailer deterministic inventory system. He determined the stocking policy, which minimized the average system cost per unit of time over the infinite time horizon. He also determined the necessary properties for the optimal policy, and the optimal solutions for the one-retailer and N-identical retailer problems. He further suggested heuristic methods for solving the general N-retailer problem, which were close to the optimal solution. His heuristic consisted of partitioning the problem into subset portions and using the single-cycling optimal solutions for one- or two-retailers problems ; and using simple single cycle solutions for more retailers' problems.

(c) *Multi-item multiperiod models* : Johnson (1967) considered a multi-item system (M = number of items), with periodic reviews. He assumed a single set-up cost K plus linear ordering cost c. Holding and shortage costs were represented as a function of x, by $1(x)$, x being the stock at the beginning of the period. Demand in a period was assumed to be ξ with the probability $\phi\ (x, \xi)$. The stock level after additions to stock have been made was represented by y and the usual assumption was $\phi(x, \xi) = \phi(y, \xi)$, for all x and y. He used

Markov programming and considered most general policy to order for x in σ (σ is the reorder region); where σ is a subset of unit vector W^M, up to a point $S_x \geqslant x$, $S_x \in \sigma^c = W^M - \sigma$; and not to order for $x \in \sigma^c$. He determined a reorder policy (order up to S for any point x in σ), which would minimize the infinite horizon expected discounted costs, and finally he gave computational procedures together with the derivation of bounds.

Curry, Skeith and Harper (1970) presented a multiproduct dependent inventory model. Their solution was a combination of mathematical optimization and simulation. They considered ordered triple policy (s, c, S), which meant that when any of the products dropped to its order point s, all products below their can-order point c must be brought up to their S levels. They considered N products with arbitrary demand distributions, and assumed lead times to be negligible. Their total cost function for the system was:

$$\text{Expected cost} = \sum_{i=1}^{n} \left\{ P_i(S, c) \left[\frac{D_i KI_i}{S_i} + \frac{S_i}{2} H_i \right] + (1 - P_i(S, c)) \left[\frac{D_i KJ_i}{S_i - R_i(S, c)} + \frac{S_i + R_i(S, c)}{2} H_i \right] \right\}, \tag{7}$$

where $P_i(S, c)$ was the probability of generating a joint replenishment for product i; D_i, the total demand for i; KI_i, the independent set-up cost for product i; KJ_i, the joint set-up cost for product i; H_i, the unit holding cost for product i; S_i, the maximum inventory level for i; c the can-order point for i; and $R_i(S, c)$ was the average joint replenishment point. $P_i(S, c)$ and $R_i(S, c)$ were determined with the help of a simulation model, using an iterative procedure. They found solution to (s, c, S) inventory levels for each of the products. Such policies may not provide optimum results but were found to be substantially better than those using individual product analyses.

Ignall and Veinott (1969) assumed proportional ordering costs and stochastic demands for all the products of a multiproduct system. They minimized expected cost during each finite time period. They also assumed that $\overline{Y}_t(H_t) \geqslant x_t$ for every t, where $\overline{Y}_t$ was an ordering policy out of a sequence of vector valued Borel functions, and H_t was the past history vector ($H_t = x_1, x_2, \ldots x_t$; $y_1, y_2, \ldots, y_{t-1}$; $D_1, D_2, \ldots, D_{t-1}$; representing the history of the process). Further they assumed that the holding and shortage cost function, $g(y_t, D_t)$ in period t was specified by Borel function of the starting inventory, y, and demand, D; and that the opening inventory level, $x_{t+1} = s(y_t, D_t)$ was a given Borel function from $X \times \mathscr{D}$. Assuming that each of the above three functions were measurable, they defined:

$$W(y, u) = cy + g(y, u) - \alpha c \,.\, s(y, u) \tag{8}$$

and

$$G(y) = \int_{\mathscr{D}} W(y, u)\, d\Phi(u), \tag{8 a}$$

where c was the linear cost of ordering a vector z, α the discount factor, and u stood for D_t, $G(y)$ was the total cost function, and $G(\,.\,)$ existed and was finite on Y, (Y, being a set of feasible stock levels).

Further assuming that any stock of product j left over after N periods could be discarded with a unit return c_j and that any blacklog could also be satisfied at unit cost, c_j; they derived the expected cost function for all the products over all the periods 1, 2, ..., N, starting with vector x_1 of initial inventories and using the policy $\bar{Y}$; which was:

$$f_N(x_1 | \bar{Y}) = \sum_{t=1}^{N} \alpha^{t-1} EG(y_t) \tag{9}$$

and the $\bar{Y}^*$ policy would be N-optimal for $x_1 \in X$, when

$$f_N(x_1 | \bar{Y}^*) = \min_{y} f_n(x_1 | \bar{Y}). \tag{9 a}$$

Further assuming that there was a Borrel function $\bar{y}(\,.\,)$ from X into Y, such that $\bar{y}(x) \geqslant x$ and $\bar{y}(x)$ uniquely minimized $G(\,.\,)$ over $[x, \infty] \cap Y$, $x \in X$; they showed that $\bar{y}(\,.\,)$ was the optimal one-period ordering rule. This means $\bar{Y} = (\bar{y}, \bar{y}, \ldots)$, or that the policy $\bar{y}(\,.\,)$ was used in each period, and hence was called the (stationary) *myopic policy*. Next they found the condition under which myopic policy was N-optimal for x_1, for every N, $N = 1, 2, \ldots, N$; and for every $x_1 \in X$. A policy $\bar{Y}^*$, being N-optimal was proved optimal under many additional criteria.

Further it was shown that $\bar{y}(\,.\,)$ had the *substitute property*, that means $\bar{y}(x) - x$ was non-increasing in x on X. That meant that increasing the initial inventory of product i reduced the starting inventory of product $j \neq i$; i.e. stocks of product i were substituted for stocks of product j. Application of these conditions to models with storage or investment limitations and to a multiechelon model were given. Under backlogging, the extension to fixed delivery-lag case was also given and finally, they analysed some non-stationary cases.

Morton (1971) analysed a case when order delivery is lagged and excess demand is lost. Assuming holding and penalty costs to be linear, he obtained good bounds for the optimal order policy and the associated minimum expected discounted cost function for the stationary problem. These bounds possessed a ' necessary ' interpretation in terms of costs to be incurred specifically for the review period (the period marked by the arrival of this order). Upper ordering bound was the myopic policy of Ignall and Veinott (1969). His experimental results for one-period and two-period lag problems supported the heuristic argument that myopic policies should be nearly optimum. An extension of the model to a partial backlogging situation indicated that for such a case also the optimal policies might be ' myopoid '.

(*d*) *Single-item, infinite horizon model*: Johnson developed a single-item infinite horizon model in his basic paper (1968), using the assumptions that: (i) ordering cost function was $M(i) + K(j)$, for changing the level of stock from i to j (i.e. $M(i) + K(j) = c(j-1) + K$); (ii) holding cost and shortage cost was $l(j)$, and (iii) the probability of demand $(j-k)$ was $\phi(j-k)$, when stock level was j. He found conditions to optimize the total discounted cost over infinite planning horizon and also the conditions for obtaining the optimum average cost. Bernoulli and geometric demand distributions were found to fit well with his method of finding conditions.

Further, he assumed $K(j) = K$, and $l(x) = -L(x)$ for $x > 0$; $l(x) = 0$, for $0 < x \leqslant I$ (where $I \geqslant 0$); and $l(x) = H(x - I)$, (where $x > I$); he found conditions for the optimum policy (s^*, S^*), so that $s^* = -1$ and $S^* = I$. Finally, using the same assumptions, but eliminating integerality restrictions on s and S, he developed formulae to calculate s^* and S^*, for the average cost criterion as well as for the total discounted cost criterion.

(*e*) *Single-item with* n-*classes of demand* : Evans (1968) considered a single product case, where two customers were of different types. Penalty cost for the lost sales were assumed to be different for the two classes of customers. He introduced two different shortage probabilities for the two classes. High penalty customers arrived independently, and the case when there was only one such customer was considered an extreme case. Single critical number policies both for restocking and sales purposes were found to be optimal in simple situations when the customers did not leave. If the priority customers started leaving when their demands were not satisfied, then simple convexity of Polya frequency functions of demand were not enough to guarantee that the optimal policy would be of a single critical number type.

Topkis (1968) considered a single item inventory system which had n-classes of stochastic demand of varying importance. When the demand for a given class arrived, the decision must be made whether to satisfy it or to hold the stocks for a more important class of demand. Additional stocks were available only at some fixed times in the future. The stocks needed to be rationed in order to balance the small cost of not supplying the present low class demand with the high penalty cost of possible non-fulfillment of a future higher class demand. His objective was to minimize the total expected value of all the costs, which included the penalty costs for unsatisfied demand, holding costs, ordering costs, and the end-salvage value (a negative cost). He considered cases, when in any given interval, the unsatisfied demand was either fully backlogged or entirely lost.

He determined conditions under which the optimal rationing policy between successive procurements of new stocks was determined by a set of critical rationing levels, such that at a given time, one satisfied the demand of a given class only, if no demand of a more important class remained unsatisfied and as long as the stock level did not fall below the critical rationing level, for that class at that time. He also gave conditions under which the optimal procurement policy at a given time could be determined by a single critical level in the usual manner. Finally, he gave conditions showing that the optimal rationing and procurement policies could be determined myopically, in other words these policies could be determined independently of costs, demand distributions, and optimal policies of other periods.

Hausmann and Thomas (1972) considered a special inventory rationing situation, where components were produced both for assembly and to fulfill the requirements of spare-parts. The assumption was that the spares had probabilistic demand, whereas the withdrawals for assembly were of deterministic nature. The relevant costs included set-up cost for replenishing the order, inventory carrying cost, and tbe back-order cost. They formulated an analytical model with the approximations of these characteristics, but could not solve its dynamic programme even with considerable amount of computing effort. Further they considered two types of operating policies—(1) a continuous review (Q, r) type policy, and (2) a single periodic review scheduling heuristic, which focused on scheduled assembly requirements ; and showed that the first policy was good when the spares demand was high compared to assembly requirements, and the second policy was good for the case with higher assembly requirements.

(*f*) *Single-item expediting level policy*: Allen and D'Esopo (1968) defined and determined an expediting level ordering policy, and called it (X, E, Q) policy, where X was on-hand inventory less back orders, at which an order for Q units was placed to be received after a fixed lead time L; E was expediting level of inventory, at which an outstanding order was expedited and delivered after the expedited time $R < L$. They also developed the following expression for the total expected cost per unit time for an (X, E, Q) policy:

$$C = rcI + sS + AO + A'W + c'U \tag{10}$$

where C was the total expected cost; r the holding cost rate; c the unit purchase cost of item; I the expected on-hand inventory ignoring the back-orders; s the unit shortage cost; S the expected shortages per unit time, A the re-order cost; O the expected number of orders; A' the expedited order cost; W the expected number of orders expedited; c' the increment to item unit-cost for expediting; and U the expected number of units expedited. Treating the expedited time $(L - R)$ together with demand as a random variable, they developed individual expressions for each of the five terms on the right-hand side of eqn. (10).

(*g*) *Inventory policy base on expected total operating cost* (*ETOC*): Hayes (1969) introduced the concept of 'expected total operating cost' (ETOC) in relation to inventory policy estimation, assuming that the cost function was costwise linear. He showed that estimates based on classical procedures were often unsatisfactory, when analysed in terms of ETOC. He derived improved estimates, assuming that the probability demand function ϕ was known, and for specifying it completely, he used a set of known parameter values, $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)$; such that $\phi(\xi) = \phi(\xi|\boldsymbol{\theta})$. Next he determined the optimal inventory replenishment point y^*, assuming the loss function $l(y, \xi)$ as piecewise linear, that is:

$$l'(y) = (c_x + c_s) \, . \, \Phi(y) - c_s \tag{11}$$

which meant

$$y^* = \Phi^{-1}(\gamma|\boldsymbol{\theta}) \tag{11 a}$$

where c_x was the unit cost of excess stock; c_s the unit cost of shortage; $\Phi(y)$, the cumulative demand function; and γ, the cost ratio $c_s/(c_s + c_x)$. Finally, using Bayesian methodology, he derived prior distributions (within a class of functions), that were implied by his procedure of determining EOTC.

(*h*) *Dynamic programming approach to inventory analysis*: Beckman (1964) assumed demand distribution of a single item case to be Poisson or some modification thereof, and the delivery time to be fixed or Gamma distributed (while preserving the sequence of orders). All the excess demands were assumed to be backlogged. He formulated the problem in a dynamic programming version, and showed that the demand distribution for the delivery period must be negative binomial. He also derived formulas for reordering point and the order quantity, using geometric demand distribution.

II. *Lot-size optimization*

(*a*) *Multi-item single facility infinite planning horizon model*: Bomberger (1966) considered scheduling of several different items over the same facility on a repetitive basis because the facility could produce only one item at a time. He assumed a set-up cost and a set-up time in relation to production of each item. The demand rate for each item was considered known and constant over an infinite planning horizon, and all demands were to be fulfilled. He minimized the long-run average of the production costs plus inventory holding costs. First he defined C_i, the cost per unit time for the item i by the equation,

$$C_i = \frac{s_i}{T_i} + h_i(p_i - r_i)\frac{r_i T_i}{2p_i} \tag{12}$$

where s_i was the set-up cost for product i; T_i, the cycle length for item i; h_i, the unit inventory holding cost of i; p_i, the production rate of i; r_i, the demand rate of i, and t_i, the set-up time for i.

He found lower bound on the optimum repetitive schedule by choosing T_i, such that $\sum_{i=1}^{n} C_i$ got minimized. His minimum cost equation was:

$$C_i = \left(\frac{2A_i + \lambda t_i}{A_i + \lambda t_i}\right)((A_i + \lambda t_i)B_i)^{1/2} \tag{12 a}$$

where $A_i = s_i$; $B_i = h_i(p_i - r_i)r_i/2p_i$; and λ was the smallest non-negative number, for which

$$T_i = \sum_{1}^{n} \frac{t_i}{T_i} \leqslant 1 - \sum \frac{r_i}{p_i}.$$

Finally, from T_i's, he computed lot-sizes for individual items.

(*b*) *Single-item multistage production lot-size model*: Taha and Skeith (1970) developed a model for a single item multistage production-inventory system with static deterministic demand, where the product moved between stages in a serial sequence. They assumed that the unfilled demand was backlogged and allowed overproduction (with constraints) at different stages, such that each stage i might produce k_i batches, once every k_i cycles. Their decision variables were: k_i, the number of batches per run in stage i; Q_i the batch-size; and Q_s the shortage quantity of the finished product in the cycle. They assumed the movement from stage to stage in batches only, and that the stages were coupled, i.e. $Q_i = r_i Q_n$; where r_i was the number of units at stage i to produce one unit of Q_n, the finished product batch size, and n was the number of stages.

They optimized the decision variable by way of minimizing the sum of the inventory related costs per unit of time. Next they included in the model the constraints on the storage space between stages. Their solution provided an optimum detailed feasible production schedule plus the optimum lot-sizes, at each of the stages.

(*c*) *Economic release lot-size model*: Schussel (1968) developed new type of models for determining job-shop economic release lot-sizes (ELRS). For ELRS models, dollars invested in inventory value versus time-base were assumed to have a trapezoidal shape as compared to triangular shape of the classical EOQ models. He assumed that there was some holding time between the end of production and start of consumption, and it was the best representation of the realistic production environments. He included four costs in his models: (i) production cost, (ii) holding cost, (iii) cost of capital, and (iv) set-up cost; and developed three separate models: (1) general ELRS model, which took into account all the four costs, but ignored the concept of future cash flows; (2) model using equivalent median time-period, and treating the cash-flows as if they occurred at the median; and (3) the model that integrated discounted cash-flows continuously over time. Each of these models calculated overall least cost lot-sizes for individual items or sub-assemblies. Individual sub-assemblies' ELRS values could not be viewed independently because of mutual interdependence between them. Treating the computed ELRS values for individual sub-assemblies, as elements of final assembly, he developed simple reasoning to eliminate most of the individual sub-assembly ELRS policies and concluded: '(a) Never produce lower-level sub-assemblies in smaller lot sizes than those of the higher-level sub-assemblies. (b) Always produce low-level sub-assemblies in integer multiples of the calculated lot-release size of the final assembly.' He showed that his models could substantially reduce the production-inventory costs of a job-shop.

(*d*) *Economic lot-size for total planning horizon*: Diegel (1966) analysed the interaction of various cost elements, based on the assumption of their linear nature. His formulae were simple because of the assumptions: (i) demand could be approximated by a linear function; (ii) between any two set-ups, production continued at constant rate; (iii) demand and production overlapped in time, and inventories and backordering was allowed. He developed the optimum lot size formula for the total planning horizon:

$$q_i^* = \frac{Ta}{j} + \left(\frac{C - Ta}{j^2}\right)(2i - 1) \tag{13}$$

where q_i^* was the optimum size of ith lot; T, the total number of time units in planning horizon; a, the demand at the beginning; C, the total demand for the planning period; j, the total number of lots, and i was such that $1 \leqslant i \leqslant j$. He showed that the total cost of inventory operations got reduced to nearly 50% of what it would have been with the dynamic programming solution of the same problem. This became possible because Diegel relaxed the constraint of dynamic programming, where orders would be placed only at the beginning of each planning period. He assumed constant review policy and the orders could be placed at any time. This way his simple lot-size approach was found to be better than the complex dynamic version of economic lot size approach.

(*e*) *Dynamic economic lot-size*: Zangwill (1969) analysed two dynamic lot-size production systems: (1) single-item model with backlogging and (2) multi-echelon model with no backlogging. For each of the two models he determined production schedules that minimized the total production and inventory costs.

He represented the single product model with backlogging as a single source network under concave costs, and applied the theory of concave cost networks. He stated that to find the optimum production schedule x_i's, the conditions were:

$$\text{Minimize} \sum_{i=1}^{n} \{P_i(x_i) + H_i^+(I_i^+) + H_i^-(I_i^-)\}, \tag{14}$$

subject to :

$$\sum_{i=1}^{i=n} x_i = \sum_{i=1}^{n} r_i, \tag{14 a}$$

$$x_i + I_{i-1}^{+} - I_{i-1}^{-} - I_i^{+} + I_i^{-1} = r_i \; ; \quad i = 1, 2, \ldots, n, \tag{14 b}$$

$$(I_0^{+}) = I_0^{-1} + I_n^{+} = I_n^{-} = 0 \; ; \quad x_i \geqslant 0, \quad I_i^{+} \geqslant 0$$
$$\text{and} \quad I_i^{-} \geqslant 0 \; ; \quad i = 1, 2, \ldots, n \tag{14 c}$$

where x_i was the production in period i ; $P_i(x_i)$, the cost of producing x_i units, $H_i^{+}(I_i^{+})$, the cost of holding I_i^{+} units in inventory during period i ; $H_i^{-}(I_i^{-})$, the cost of shortage of I_i^{-} units in period i ; and r_i, the demand in period i. Defining that any period i for which $I_i = 0$, as *the regeneration point*, he showed that the optimal schedule has the property that between any two periods having production, there was always a regeneration point. He then developed an algorithm to calculate the optimum schedule.

Next he analysed a multiechelon system (i.e. where production of final product required several different processes/echelons in a serial sequence), permitting no backlogs of demand. Again he represented this system as a single source network that could be directly subjected to the analysis of the theory of concave cost networks. For determining the optimal production schedule in each facility, he stated that the following conditions must be satisfied :

$$\text{Minimize} \sum_{ij} \{P_{ij}(x_{ij}) + H_{ij}(I_{ij})\} \tag{15}$$

subject to :

$$\sum_{i=1}^{n} x_{i1} = \sum_{i=1}^{n} r_i, \tag{15 a}$$

$$I_{i-1,\,j} - I_{ij} + x_{ij} - x_{i,\,j+1} = 0 \; ; \quad j = 1, 2, \ldots, m-1 \; ; \quad i = 1, \ldots, n, \tag{15 b}$$

$$I_{i-1,\,m} - I_{i,\,m} + x_{i,\,m} = r_i, \quad i = 1, 2, \ldots, n, \tag{15 c}$$

$$I_{ij} \geqslant 0, \quad x_{ij} \geqslant 0, \quad \text{for all } i \text{ and } j, \tag{15 d}$$

$$I_{0j} = I_{nj} = 0, \quad \text{for all } j. \tag{15 e}$$

Where r_i was the demand for the finished product in period i ; x_{ij}, the units of production in period i on facility j ; I_{ij}, the inventory stored in facility j at the end of period i ; n the number of periods ; m the number of facilities in the sequence ; $P_{ij}(x_{ij})$, cost of producing x_{ij} units ; and $H_{ij}(I_{ij})$, cost of holding I_{ij} units in stock. Treating the system as a single source network, he determined the optimum production schedules for the system, as if they were the corresponding network optimal flows. Because optimal flow was always to be considered as an extreme flow, he searched for extreme flows, which required that any of the nodes can have at the most one positive input. Assuming that a total of z units, $z > 0$ were shipped into a node (i, j), (which is not the source), he showed that there were integers α and β ; $i \leqslant \alpha \leqslant \beta \leqslant n$, such that $z = \sum_{l=\alpha}^{l=\beta} r_l$. Finally he gave a dynamic programming algorithm for calculating those optimal flows as well.

III. *Optimization of various specific management objectives*

(*a*) *Minimize the customer-line items backordered* (*single item case*) : Hausman (1969) analysed a new measure of stock-out costs in the inventory control theory. That measure was the number or percentage of customer-line items (requests for a particular item by different customers), $Z(V, r)$ that were backordered. His expression for such a measure was :

$$Z(V, r) = b(V, r)/X(V, r) \tag{16}$$

where $b(V, r)$ was the number of units backordered ; $X(V, r)$ the average number of units backordered per customer-line-item backordered ; V, the total units demanded over lead time ; and r, the reorder point. Then the expectation of $Z(V, r)$ was placed equal to the expectation of the quotient given above, i.e. :

$$\bar{Z}(r) = \bar{b}(r)/\bar{X}(r) \tag{16 a}$$

where

$$\bar{b}(r) = \int_{r}^{\infty} (V-r) f_V(V)\, dV = \text{the expected number of backorders (in units) per lead time,} \tag{17}$$

$$\bar{X}(r) = \bar{X} = \text{the average number of units per customer line-item backordered.} \tag{17 a}$$

He combined the backorder cost per customer-line backordered with usual ordering and carrying cost and his expression for the total annual expected cost for the total inventory system became :

$$\text{Total annual cost} = \frac{\lambda}{Q} A + IC[Q/2 + r - \mu] + k_b \frac{\lambda}{Q\bar{X}} \bar{b}(r) \tag{18}$$

where λ was the expected annual demand ; Q, the order quantity ; A, the fixed ordering cost per order ; I, the annual carrying cost rate ; C, the unit variable cost (of purchase) ; r, the reorder point in units ; μ, the expected demand over the lead time ; k_b, per line-item backorder cost. Finally, differentiating total cost equation w.r.t. Q he determined the optimal order quantity. Applying his formulae to an actual inventory system, he showed that the cost of customer-line items backordered got reduced to 13% with no increase in total costs.

(*b*) *Minimize expected number of backorders* : Feeney and Sherbrooke (1966) have considered a special case, where the optimal policy is to reorder, whenever units were demanded. They assumed that the demand distribution could be any compound Poisson, and the resupply distribution might be arbitrary. Making use of queueing theorem by Palm (1938), they found out steady-state probabilities for the number of units in re-supply (or repair), and treated them as normalized values of the compound Poisson demand distribution based on the mean of the resupply distribution, but not the distribution itself. Knowing the steady-state probabilities, they computed the several measures of supply performance as a function of spare stock '*s*'. Using general inventory concepts, they minimized the total cost based on estimates of holding cost and supply performance cost.

Rose (1972) derived important measures of supply performance for a single-item $(S-1, S)$ inventory model. The performance measures were: the expected number of backorders, the expected supply and stockout times and the probability distribution for resupplying times. All demands were assumed to arise out of an arbitrary distribution and were fulfilled (not necessarily immediately). The delivery time of the suppliers were assumed constant. He also analysed the trade-offs between the number of spares, and the delivery time (expected item delay), to obtain a given level of supply performance at least cost.

Miller (1971) built one-period multi-item inventory model and tried to minimize functions of item backorders subject to a budget constraint. He considered two criterion functions: (1) the sum of expected item backorders and (2) the expected joint backorder. His main result is that the joint backorder criterion is convex for arbitrary, twice differentiable demand distributions. He also compared the solutions obtained with each of the two criterion functions and indicated that criterion (1) recommended buying of too few of the high cost items; but criterion (2), under suitable conditions, recommended buying of more of the high-cost items.

(*c*) *Optimize service levels*: Herron (1967) discussed (Q, r) policies, where Q is the quantity ordered, when a reorder point r is reached. The Q based on Wilson formula might generally be accompanied by excess costs. He developed graphical and algebraic methods suitable for computer applications for determining accurately the minimum cost values of Q and r. His formula for total cost K was:

$$K = A\lambda/Q + IC(0{\cdot}5Q + t\sigma)\nu\Phi\lambda/Q \tag{19}$$

where A was the reordering cost per re-order; λ, annual demand; I, the annual holding cost rate; C, the unit cost of the item; s, the safety stock or expected stock on hand at the time of arrival of order, $s = r - \mu$, μ being the expected demand during lead time; $t = s/\sigma$, σ being the standard deviation between the actual and expected demand during lead time; ν, the dollar stock-out penalty per stock-out; and Φ, the probability of stock-out at each replenishment occasion. He determined minimum cost values of Q^* and r^* using the assumptions: (1) stock-out penalty was proportional to number of stock-outs; (ii) stock-out penalty was proportional to annual number of units out of stock; (iii) service level was based on number of stock-out occasions; (iv) service level was based on fraction of units, out-of-stock. He plotted curves for minimum cost ratios K^*/ICQ_w (Q_w being Wilson's formula Q), order quantity ratios Q^*/Q_w, minimum cost service levels, and safety stock ratios s^*/σ, each of these against the abscissa of Q_w/σ. His formulas could be used with cost computers and resulted in substantial savings.

(*d*) *Minimize long-run expected cost per unit of time*: Agin (1966) analysed a model where every N-periods an order was placed for an amount that brought the level of inventory (on hand plus on order) up to a level S. The demand was assumed to be random, backordering was allowed, lead time was random, and receipts of orders could cross in time. The demand process was considered independent of lead time. He formulated a cost function in which costs reflected a measure of stock deficit. The probability that the stock deficit remained less than or equal to x was designated by $\phi(x : N)$. The expression for expected costs per unit time, $C(N, S)$ was :

$$C(N, S) = \frac{C_k}{N} + C_w \,.\, S + C_h \int_0^S (S-x)\, d_x \,.\, \phi(x:N) + C_p \int_S^\infty (x-S)\, d_x \,.\, \phi(x:N) \qquad (20)$$

where C_k was the fixed cost to order and review ; C_w, the warehouse cost per unit of stock per unit time ; C_h, the holding cost per unit of stock per unit time ; C_p, the penalty cost per unit of stock per unit time.

Knowing that with random lead time, determination of equilibrium distribution of stock deficit was not possible, he considered a substitute problem, whose minimum solution could yield values, N_O, S_O, which were quite close to N^*, S^*.

Using a given mean and variance of demand, and a given mean, variance and distribution of lead time, he computed the mean variance and equilibrium distribution of stock deficit. Next, he found another distribution (with mean and variance) for stock deficit, which maximized the expected cost per unit of time. Finally he minimized the maximized cost function w.r.t. N and S. This method could be used, even when the functional form of the distribution of demand was not known. He evaluated this minimax method for various values of input parameters and found ranges for which it gave values of N_O and S_O very close to N^* and S^*.

(*e*) *Optimize* (*or reduce*) *lead times* : Gross and Soriano (1969) studied the effects of reducing lead times of inventory systems through simulation. They assumed that both the demand and lead time were stochastic. The simulation was based on the user specified periodic review policy with flexible (s, S) values. Output in the form of performance measures was (i) the reductions in allowable safety stock, and (ii) the accompanying reduction in average on-shelf inventory level for the prescribed system performance requirements. Both the outputs were treated as a function of reduction in mean lead time. Reductions in lead time were made possible by airlift instead of surface transportation. Total savings as a result of reductions in pipeline and on-shelf inventories were compared against increase in cost for faster replenishment. Some generalizations about the effect of reducing lead times on the allowable reductions in on-shelf inventory and number of shortages were given.

(*f*) *Optimize* (*or reduce*) *multi-item inventory through design commonality* : Rutenberg (1971) treated the design commonality problem as a cost-balance problem. On one side was the cost of disutility of refusing to provide each segment of customers with an item fitting its exact requirements, and on the other side was the cost of producing and inventorying each item. He considered N demand classes (different sizes of the item), their per-period demands were $d_1, d_2, \ldots, d_N$. The M items (or classes) were considered for production and stocking, $M \leqslant N$. He assumed $c_i(p_i)$ to be the total cost function of producing and carrying quantity p_i of item i during each period, and that there were economies of scale in purchasing, manufacturing and holding so that each $c_i(p_i)$ was concave. He defined x_{ij} as the quantity of item i used for satisfying demand of class j and $c_{ij}(x_{ij})$ as the concave disutility cost associated with using item i in demand class j. The problem was depicted as a single-source multiple-destination graph with the objective of finding minimum cost flows that would satisfy the demands. The flows represented the depth of the product-line (the optimum number of items to be produced). Most multi-item problems had a special structure that the N-classes of demand could be placed in an ordering so that for a given item i, c_{ij} were monotonically non-decreasing about the minimum c_{ij}, and the same ordering was valid for all i; then in the optimum solution, there would be no gaps in the block of adjacent demands filled by a particular item.

Sometimes sub-assemblies could interact, then commonality choice of one sub-assembly would affect optimal commonality decision of the other sub-assemblies. This way the choices cascading into one another needed to be solved simultaneously. This problem was solved taking two stages at a time, starting from the final demands at the final (finish product) stage. He developed computational procedures and illustrated the solution methods with numerical examples.

IV. *Models for optimizing highly specialized inventory situations*

(*a*) *Stochastic lead times*: Kaplan (1970) developed a dynamic inventory model with stochastic lead times. He assumed that the orders did not cross in time, and that the arrival probabilities of the outstanding orders were independent of the number and size of the outstanding orders. Using these assumptions, he showed that the sequential multidimensional (multivector) problem, normally prevailing with the random lead time model, could be reduced to a sequence of one-dimensional minimizations over one variable. His model included ordering costs $c(z)$; holding cost, $h(w)$; and a shortage cost $p(-w)$; where z was the amount purchased and w the end-of-period inventory on hand. He minimized the expected present value of all the future costs using discount factor $\alpha < 1$. His expected discounted holding and shortage costs of j periods in future (with no possibility of restocking inventory during future j periods), $L_j(y)$ was:

$$L_j(y) = \alpha^j \int_0^\infty l(y-\xi)\phi_{j+1}(\xi)\, d\xi \tag{21}$$

where y was the beginning level of inventory after delivery, and ϕ_j, the j-fold convolution of ϕ, and ξ, the demand variable.

He determined optimum ordering policies with the assumption of convex expected holding and shortage costs, linear ordering costs, and a fixed set-up cost (paid when the order was placed). These policies were found to be somewhat like those of deterministic lead times, but they had slightly different single period critical numbers (i.e. re-order points). He showed that the optimal ordering policies for each period was a function of the variable representing the stock on-hand plus all the outstanding orders.

(*b*) *Imperfect demand information*: Morey (1970) considered a single-product, periodic review system, assuming the true demand in the field to be a random variable ξ with a known distribution function. The demand transmitted to stocking point was a different random variable, η. The difference between these two random variables was due to human or mechanical transmitting errors. Stocking point policy was to supply to the field the amount $g(\eta)$, which was the function of the observed demand, η; or the amount y, which was on hand, (one of these two, whichever was smaller). His cost function consisted of the purchase cost, $c(z)$; the inventory holding cost, $h(x)$ (because of error in demand information transmittal); the shortage cost, $p(x)$, and the salvage value of the excess amount received in the field, $r(x)$. It was assumed by him that purchasing was less costly than to incur shortages in the field. He developed loss functions (the equations for the expected holding, shortage plus the negative salvage costs), and determined qualitative orderings of the critical recorder points as functions of demands and losses involved. He further analysed how inventory systems' costs grew both as a function of ρ, the correlation coefficient between the true and the transmitted demand and also as a function of σ_n, the standard deviation of the transmitted demand. His second analysis was about the relative efficiency of a few issuing strategies used by the stocking point. Each time he determined optimal stocking policy by putting differential of the loss function equal to zero. He found that the use of the strategy, $g(\eta)=F_{\xi|\eta}{}^{-1}\left(\frac{p-c}{p-r}\right)$ recovered 50–58% of the costs incurred due to errors in the transmitted demands.

Iglehart and Morey (1972) analysed a situation when the stock records were not in agreement with the actual physical stock. They derived expressions to calculate the proper frequency and depth of inventory counts and suggested modifications (additional buffer stock) to the given stocking policy so as to minimize total cost per unit time for the given probability of a warehouse denying to fulfill the orders between inventory counts. Their formulas were simple and of classical EOQ type. The calculated periodic physical count intervals were based on one of the two criteria: (1) the cumulative number of demands since last count, and (2) the elapsed time since last count.

Thompson (1966) built an idealized model of the production-inventory decision system. He defined the ratio of the standard deviation of inventory change (or production) to standard deviation of demand to be the *amplification of fluctuations in inventory change* (*or production*) over the fluctuations in demand. In relation to these ratios, he studied the overall effects of: (*a*) the average forecast error, (*b*) the variance of the forecast error, (*c*) the serial correla tion of demands from period to period, and (*d*) the correlation between the errors of a two-period forecast.

When the variance of the forecast error was zero, the amplification could be calculated easily, but when the errors were having non-zero variances, the calculations became very difficult, and hence he used simulation. He found that when demands were not serially correlated, the perfect forecasts yielded the smallest amplifications. Errors of over-estimates or missed directions yielded greater amplifications than those given by underestimations. As the variance of the forecast errors increased, amplifications increased; and amplifications were accentuated in case of correlated errors in a two-period forecast.

(*c*) *Gamma distributed demand*: Sivazlian (1971) assumed stationary periodic review (s, S) policy with stationary demands per period that had a gamma distribution of order r and parameter μ. He assumed complete back-logging and defined h as the unit holding cost and p as the unit penalty cost. He performed the dimensional analysis of the basic equations:

$$M(s^*, Q^*)=K \quad \text{and} \quad \left.\frac{\partial M(s^*, x)}{\partial x}\right|_{x=Q^*}=0 \tag{22}$$

and

$$\mathscr{L}\{M(s^*, x)\} = \bar{M}(x^*, z) = \mathscr{L}\frac{L(x^*+x)}{1-\bar{\phi}(z)} \tag{22 a}$$

where x was the end-of-period stock level, and the Laplace transform of a function $f(x)$, $(0 < x < \infty)$ was $\bar{f}(z) = \mathscr{L}\{f(x)\} = \int_0^\infty \exp(-zx) f(x)\, dx$, and $Q^* = S^* - s^*$.

He showed that it was possible to express the quantities μs^* and μQ^* as a function of three factors : p/h, $2K\mu/h$ and r ; that reduced the number of input parameters from five to three. Next he computed the values of μs^* and μQ^* for a range of values of these three input factors, and plotted them as groups of curves on the logarithmic scales.

(*d*) *Exponential smoothing of demand* : Landi and Johnson (1967) considered a single-item, fixed interval stock replenishment policy that would minimize inventory level variance, while fulfilling a sequence of random pairwise uncorrelated demands. They analysed how order level variances could be reduced at the expense of slightly larger average fluctuations in the inventory levels. They proved that such a re-order rule should be based on exponentially weighted moving average to smoothen the random components of the demand sequence with mean zero and variance that either grew or decayed geometrically or remained constant in time. The smoothing constant was equal to $C_I/(C_I + rC_\theta)$; where C_I was the cost proportional to inventory variance ; C_θ, the cost proportional to order-level variance ; r, the constant of the above said geometric series. The rule for determining the reorder level was in the form of the equation

$$\theta_k = f_k + \alpha\left[\sum_{j=1}^{j=T}(f_{k-j} - \theta_{k-j}) - I_k\right] \tag{23}$$

where θ_k was the quantity ordered at the end of the period k ; f_k. the forecast of demand during the period $k+T+1$; and I_k, the inventory level at the end of the period k.

(*e*) *Stockout penalties considered as modified future demand pattern* : Schwartz (1966) stated that instead of imposing direct penalties for stockouts, we could modify the future demand pattern and he termed it as ' perturbed demand or PD '. He showed that under a set of special conditions, PD system could be characterized by :

$$\lambda = \frac{\lambda_0}{1+\alpha I} = f(\alpha) \tag{24}$$

where λ_0 was the expected demand rate with no stockouts ; λ, the demand rate when stockouts did occur ; α, the ratio of demand when stocks exhausted to the total demand ; and I, the loss of sales in each disappointment (i.e. shortage).

Next he showed (Schwartz 1970) how optimal policies could be determined with PD assumptions. He defined M_i, the amount to be ordered in each cycle ; and L_i, the re-order level when there was some unsatisfied demand (i.e. L_i always represented negative inventory). Unsatisfied demand could be backlogged or lost. He termed unit sales revenue as C ; unit holding cost as H ; and disappointment factor as α (i.e. limit of the ratio of demand backlogged or lost to the total demand). Then $(1-\alpha)$ became the service level of the inventory policy. If in each cycle, decisions were made for $M_i = M$; and $L_i = L$; then α became L/M.

Assuming external demand rate to be λ, he introduced the constraint relating to L_i and M_i ; that is $\Omega_i(M_i, L_i) = 0$. He also assumed that the demand rate followed the basic perturbed demand function given in eqn. (24). Then the determination of the optimal policy involved selection of M and L, subject to the constraint $\Omega(M, L) = 0$. His equation for the net profit per unit of time was :

$$P = \frac{\lambda_0 C(M-L)}{M+LI} - \frac{H}{2}\frac{(M-L)^2}{M}. \tag{25}$$

He determined optimal policies under various conditions by placing dP/dL every time equal to zero and then found optimum values of L. He also demonstrated that the perturbed demand concept was operationally practical because the optimum policies could be easily calculated. Further he derived optimal policies for the constand demand rate and proved that the results of constant demand situation were equally valid for distributed demands as well.

(*f*) *Optimizing stock depletion under monopolistic situations* : Mossin (1966) formulated a discounted value function for a monopolistic situation. The monopolist was assumed to start at time $(t=0)$, with a fixed stock X_0. The reduction in inventory was supposed to be $X' = -(s+w)$; where s was sales ; w, the deterioration and X', the time derivative of X. The rate of deterioration, v, was assumed to be at a constant proportion of X, viz. $w = vX$. The relation between sales per unit time and the prevailing price, p, was set at :

$$p = a - bse^{-gt} \tag{26}$$

where a, b were constants ; g, the rate of demand growth at the given price (g could be negative as well). The gross cash inflow would then be ps. The inventory holding cost was assumed to be continuously payable at dollars h per unit in stock, i.e. hX was paid out from the revenue. The objective was to maximize the discounted value V of the future net cash inflows (using interest rate r per unit of time). Next he defined a time function $X(T) = 0$, such that after T, cash inflow became zero. As a result, the objective function V became :

$$V = \int \exp(-rt)\,[as - bs^2 \exp(-gt) - hX]\,dt. \tag{27}$$

This meant that the problem was to find the time-function of s, which would maximize V ; where $p > 0$, $s > 0$, and $0 \leqslant t \leqslant T$.

Using calculus of variations, he developed the equation for the maximum value of V :

$$V = \frac{2b(v+r)k^2}{(g-r)(2v+r+g)} \exp[(g-r)T] + \frac{bk^2}{2v+r+g} \exp[-2(v+r)T] - \left(\frac{bk^2}{g-r} + \frac{hX_0}{v+r}\right) \tag{27 a}$$

where, $k = [(r+v)a+h]/2b(r+v)$ and the initial condition $X(0) = X_0$.

He also discussed the economic meaning of this value function, and stated that the reduction in gross revenue from selling at lower prices was balanced by costs associated with sales over elongated period of time (i.e. interest, deterioration and inventory holding costs).

(*g*) *Joint set-up costs* : Ignall (1969) analysed minimum average cost ordering policies for continuously reviewed two-product inventory systems. Times of demand for two products A and B were assumed to be determined by two independent Poisson processes with mean arrival rates λ_A, λ_B respectively. The unit holding cost for A was h_A and for B was h_B. He assumed that when a product was ordered, a set-up cost occurred and was equal to K when only one product was ordered, but mK, $(1 \leqslant m \leqslant 2)$ when both the products were ordered. The replenishment was assumed to be instantaneous. The optimal policy was defined as the one that would minimize the long-run average costs. Since all demands could be met, unit purchase price became irrelevant. The random joint policy (s, c, S) for the two-product problem was considered to gain the benefit of joint set-up cost savings. Assuming some maximum allowable total inventory, M, he found that (s, c, S) were not always optimal even for the simple two-product joint set-up cost case. To find the regions of the given optimal stationary policies in parameter space, he had to make use of Markov renewal programming.

Lippman (1969) analysed a deterministic single product, discrete review, finite time horizon inventory problem. The holding cost in each period was assumed to be a non-decreasing concave function ; ordering cost function was neither convex nor concave, but a multiple set-up cost function. He assumed that the ordering cost in period i was the cost of the cumulative set-ups (each set-up having a production capacity of M_i), and that the cost of each set-up was a non-decreasing concave function of the amount produced. He denoted the N-period cost of the production schedule x by :

$$\mathscr{C}(x) \equiv \sum_{i=1}^{N} [c_i(x_i) + h_i(y_i)] \tag{28}$$

where $\mathscr{C}(x)$ was the N-period production cost ; $c_i(\ .\ )$, the multiple set-up cost function ; and $h_i(y_i)$, the non-decreasing holding cost function. He determined the optimal schedules, which were such that for each period : (*a*) there were no partially used set-ups in period i if the inventory at the beginning of i was positive and (*b*) the inventory at the end of period i was less than M_i. Using these constraints, he developed a computational algorithm for finding the optimum production schedule.

Next he analysed the stationary, infinite horizon case of the multiple set-up cost problem. He defined (*i. j*)-*periodic schedule* as the one that had a cycle length of i periods and during each cycle, it ordered fractional capacity (of a single set-up) only in jth period (if at all) ; it had ending inventory of zero, and the ending inventory in each of its periods (except the last period) in the cycle was strictly between zero and M. Whereas *schedule*-∞ was stated to be that schedule, during which orders were placed in each period for the minimum number of production set-ups (considering initial inventory) for satisfying the requirements of that period only.

Next he investigated whether these types of schedules existed and were they unique. He used two criteria of optimality : (1) minimum cost per unit of time, and (2) the minimal discounted cost and proved that there existed a countable set S of schedules which possessed the (i, j)-*periodic schedule* property in addition to the two properties (a) and (b) of finite horizon case above. Further, it was shown that if the ratio of the demand per period, r to M_i (i.e. r/M_i) was rational, then S contained a schedule with minimal discounted cost.

Further Lippman (1971) considered the problem of scheduling the production of a single product at each instant during a time horizon of length $T(\leqslant \infty)$, so as to minimize average cost per unit time. Backlogging of demand and disposal of stocks were not allowed. Costs incurred included linear inventory holding cost and the multiple set-up cost function. He studied a special case with $K=0$ (i.e. the standard EOQ case) and derived some results about the EOQ model. He also found conditions for the optimal production schedule for $T=\infty$ and $T<\infty$. It was shown that for a finite period an optimal production schedule may have several orders of two different sizes only, in contrast to EOQ case, where every order must be of the same size. Finally, simple algorithms to find optimal policies for both $T<\infty$ and $T=\infty$ were given.

(h) *Items with stochastic field life functions*: Pierskalla (1967 b) performed depletion analysis of the stock of an item, where the field life of the item was a random variable $X(S)$, which was in turn dependent on the age S of the item at the time of its issue. He assumed that at the beginning, the system had n identical items in stock with varying initial ages $S_1 < S_2, \ldots, < S_n$, where $S_1 > 0$. New items were not added to stocks, and an item was issued only when the entire useful life of the previously issued item (in field) was ended.

Assuming that each item obeyed one of the family of decreasing field life value functions, he proved that (FIFO) was the optimal issuing policy (i.e. it maximized total field life), provided $E[X(S)] = aS + b$, where $0 \geqslant a \geqslant -1$, $b>0$, i.e. where item had a linearly decreasing mean value function. Further it was shown that if no stockouts occurred, then (FIFO) policy was optimal for the dynamic model also, where N new items are added to inventory, one each at the N different future times, $t_1, t_2, \ldots, t_N$ (each of the new items had its age $S=0$).

In another paper (Pierskalla 1967 a), he considered the cases of several demand sources as well, assuming that there was a penalty cost p (for installation, etc.), whenever an item was issued. He considered the most general form of function $L(S)$ to be a concave non-increasing function for $S \in [0, t]$, and $L(S) = L(t) = c > 0$ for $S \geqslant t$; where c was a real number, $t = j/\nu$; j, the number of items issued from inventory and ν, the number of demand sources. This type of $L(S)$ was found to be an approximation to the general decreasing S-shape curve, and he also presented the optimal policies or bounds on the optimal policies for all these cases.

Later Pierskalla and Roach (1972) derived optimal issuing policies for some particular classes of perishable inventory problems under 3 objective functions: (1) maximize utility of inventories, (2) minimize total average amount backlogged or lost, and (3) minimize total number of items reaching the oldest category. The stock of an item was assumed to be grouped into categories according to shelf-age. Demand occurred for each of the categories and could be satisfied by inventory units from that particular category or from any 'younger' category as well. They proved that for most of the abovementioned objectives the optimal policy was of FIFO type.

(*i*) *Inventory of rented equipment*: Whisler (1967) studied the inventory levels for the equipment to be rented by a company. He assumed that the company had a pool of equipment whose number could be changed only at the beginning of a period. The number in the pool at the beginning of nth period was denoted by x_n, and the number after placing the order for renting additional equipment by y_n. At the beginning of period n, z_n equipments were assumed to be with user-customers and therefore $(x_n - z_n)$ remained in the pool. The customers could return the equipment after use, at the beginning of a period. He assumed that $z(t) \leqslant y_n$ for all t in the nth period, where $z(t)$ was the number of equipments in use with customers at time t during the period. The value of $z(t)$ changed with time as new demands occurred, for $x_n \geqslant 0$, $y_n \geqslant 0$, $z_n \geqslant 0$.

He drew analogies: (1) between pool equipment and the parallel service channels of queueing systems, and (2) between service times for customers and service times of the pool equipment with the user-customers and hence treated the models of rented equipment exactly like queueing theory models.

He developed a dynamic programming model for the cost function, which was:

$$C_n(x, z) = \min_{y \geqslant z} \{a(y-x) + cy + L_z(y) + \alpha \sum_{j=0}^{y} C_{n-1}(y, j) p_{z,j}(T, y)\} \tag{29}$$

where $C_n(x, z)$ was the nth period minimum cost for the beginning values of x and z; a, the unit ordering cost (or $a = (-d)$, d being the unit return cost); c, the unit rental cost per period, $L_z(y)$, one period expected holding and shortage cost; α, the discount factor; T, the length of period under consideration; $p_{z,j}(T, y)$, the probability of j units being in use at the end of the period, given z were in use at the beginning of the period and y were rented for the pool during the period; and $n = 1, 2, \ldots, N$. He proved that at the beginning of each period, two real numbers (called the upper and lower critical numbers) could be computed. If x_n was larger than the upper critical number, then for obtaining optimum cost, y_n was reduced to the upper critical number $u_n(z)$; and if x_n was smaller than the lower critical number, then for optimum cost, the quantity was brought up to the lower critical number $t_n(z)$. If x_n was in-between the two critical numbers, then no change in the number of rented equipment was to be made. Finally, he showed that for the infinite-horizon case $C^*(x, z) = \lim . C_n(x, z)$ as $n \to \infty$.

(*j*) *Style goods inventory problem*: Murray and Silver (1966) treated the single item style goods problem. They assumed finite selling period and variable selling rate. Vendor could purchase or manufacture these items only a few times during the season, and cost was dependent on the time at which the item was obtained. Unit revenue from sales also varied. Both the cost and revenue functions were assumed to be deterministic and known in advance. They stated that the sales potential was greatly uncertain, but better forecasts could be developed as actual sales became known. They assumed that the initial state of knowledge (at T_1, the time of first acquisition) could be expressed by a density function of two parameter beta family:

$$\begin{aligned}(d/dp_0 pr\{p \leqslant p_0\}) &= f_\beta(p_0 | r_0, n_0) \\ &= (B(r_0, n_0 - r_0))^{-1} P_0^{r_0 - 1}(1-p_0)^{n_0 - r_0 - 1}; \\ &\qquad 0 \leqslant p_0 \leqslant 1; \quad r_0 > 0; \quad n_0 > r\end{aligned} \tag{30}$$

where,

P_0 = vendor's initial estimate about the customer selecting the item under consideration, $0 \leqslant p_0 \leqslant 1$

r_0, n_0 = vendor's initial state of information on the parameters r and n

$$\left.\begin{aligned} r &= r_0 + \sum_{j=1}^{i-1} r_j \\ n &= n_0 + \sum_{j=1}^{i-1} N_i \end{aligned}\right\} \text{State of information at } T_i$$

T_i = Time of ith buying opportunity

$i = 1, 2, \ldots, m$ = the number of opportunities available

$j = 0, 1, 2, \ldots, N_i$; ($n_0, N_1, N_2, \ldots$ were known in advance)

$= N_i$ customers produced exactly j units of demand

N_i = The number of customers that purchase the item during the period T_i to T_{i+1}

r_i = The units of item demanded by N_i customers during T_i to T_{i+1}

They treated sales potential as a subjective random variable, whose distribution changed with time (as per Bayes' rule). They set up an equation for the largest expected profit (i.e. for the optimum policy), which specified for each T_i, the quantity to be purchased x_i, for all possible values of r and y (y was stock on hand before purchasing), at that time. A dynamic programming state-cum-cost equation represented both the unsold inventory of the item and the current knowledge about the sales potential. Iterating that equation, they found the expected profit in the remaining time of the selling period, i.e. $g_i(y, r)$ values for $i = 1, 2, 3, \ldots, m$ and then selected the optimum policy by inspection. Numerical examples were discussed to show how this type of formulation of the problem with the current state of knowledge improved adaptively as the actual demand became known.

Recently Arunachalam (1972) developed an inventory model in which season length was variable for the seasonal style goods. He assumed that the demand was of a simple contagion type (i.e. that the past demands influenced the future demand occurrences). He first showed that an (s, S) optimal order policy was the function of the length of selling season and then determined the optimal duration and timing of the selling season using a computational algorithm.

(*k*) *Dynamic inventory of liquids* : Self and Lewis (1969) analysed a multi-product liquid inventory system. Production and demand rates for the products were assumed to be constant. Only one item could be produced or replenished at a time because of single production facility, but the demand for all products continued simultaneously. There were a given number of storage tanks, each with a fixed capacity, and one tank could contain only one product at a time. If for some tank product changeover became necessary, it involved a changeover cost. Optimum sequence for the changeover was known from the prior technical analysis, and it was assumed that all the demand would be met. The total cost equation for n replenishments through all products was defined by :

$$K = \sum_{i=1}^{k} A_i + \tfrac{1}{2}n \sum_{i=1}^{k} \left(1 - \frac{d_i}{p_i}\right) d_i b_i I_i \tag{31}$$

where K was total cost ; A_i, changeover cost for ith product ; p_i, production rate for ith product ; d_i, demand rate for the ith product ; b_i, unit cost of ith product ; I_i, unit holding cost of ith product as a percentage of b_i. Differentiating the above equation, he calculated n^* the optimum number of cycles through all the products, Q_i^* optimum order size for each of the ith product, and $Q_{\max i}^*$, the optimal maximum inventory level for each of the ith products.

Using $Q_{\max i}^*$ values, they represented replenishments and demands of products by geometric figures (the triangles), and found the specific solution of a two-product, three-tank dynamic liquid storage problem. They came across two basic constraints : (i) product-dominance, when one product bounded by a fixed volume $Q_{\max i}$ restricted the cycle time T to a maximum value, and (ii) tank-release dominance, when the demand portion of one product did not release the shared tank for use by the other product during the replenishment portion of this second product. Algebraically the solution procedure involved maximizing the cycle time T. They started with a reasonably large T and it was reduced systematically in a manner directed by the product-tank interactions to yield an optimal value of T. Inferences concerning larger quantities of products and larger number of tanks were also given.

(*l*) *Facilities in series inventory model* : Love (1972) analysed an n-period single product inventory model with known demands and separable concave production and storage costs. The model had N-facilities (numbering 1, 2, ..., N) of production in series. He showed that if the storage and production costs were non-decreasing in time, then an optimal schedule had the property that if in a given period, facility j produced, then facility $j+1$ did also produce. Such a nested structure was exploited in an algorithm to find the optimum schedule. For the stationary infinite horizon case, the algorithm yielded a periodic optimal schedule.

V. *Applications of advanced mathematical theories*

(*a*) *Linear programming applied to a ' variable* S ' *inventory model* : Sargent and Bradley (1969) stated that inventory models with periodic review policy (when backordering was allowed) were basically linear in nature. Generally the simplest policy with the linear programming was of (S, T) type. They generalized (S, T) in such a way that S varied from period to period as a general linear function of demand and net inventory, and called it generalized policy, the ' variable S ' model. Demand forecaster was considered to be linear. Stationary as well as non-stationary demand processes were analysed. Lead time was assumed to be an integer multiple of the review period and an independent and identically distributed random variable. They developed the individual linear equations for : (1) end-of-period inventory, (2) demand process, (3) Box and Jenkin's demand forecaster for any future period, (4) the amounts of outstanding orders, and (5) the error between the desired inventory level at the end of each period and its actual value.

They used z-transforms of all these linear equations to solve for I_n, the net inventory at the end of nth period, and $Q_{n+\tau}$, the order size at the end of nth period ; where τ was the replenishment lead time. Further they analysed this linear model for stability, the transient response, steady state response, and for the effects of most common non-linearities. For each of these characteristics, it was shown that the assumed linearity of the above equations essentially held good all the time for their ' variable S ' policy.

(*b*) *Hanssmann's inventory control model* : Lampkin (1966) first discussed the Hanssmann's inventory control model (Hanssmann 1959) of partially finished goods. Hansmann's model has assumed that individual shortages costed nothing, complete backordering was possible and the average rate of demand was a function of the customer's average waiting time. The stock was reviewed simultaneously at all sub-stores at equal spaced weekly intervals. Demands on stores were continuous and independently normally distributed. Central store also ordered simultaneously along with sub-stores, but it had the knowledge of sub-store orders. Hanssmann introduced two functions : (1) the normalized overage, $A_n(I_n)$ and (2) the normalized shortage, $B_n(I_n)$, where I_n was the inventory at the end of the nth week, and A_n, the monotonic function of I_n and so $Y = A_n[B_n^{-1}(x)]$, was the well-defined function. Hanssmann tabulated the values of A_n and B_n.

For two stores in series problem Hanssmann used the order rule :

$$E(V_{l+k'+1}) = I_0 \tag{32}$$

$$x = I_0 + (l + k' + 1)s\alpha(t) - V_0 \tag{32 a}$$

where I_0 was the expected end-of-zero-period inventory at store 2 : V_i, the stock level at store 2 at the end of week i ; l, the number of orders in the pipeline between store 1 and store 2 ; k', an integer which was the average time to fill an order x from supplier to store 1 ; s, the mean sales rate ; $\alpha(t)$, the delay function of t, the average delivery time maintained by store 2. Next Hanssmann derived the expected profit equations for : (*a*) single-store, (*b*) two-stores in series, and (*c*) central store/substore problem.

Lampkin modified Hanssmann's ordering rules slightly for the case, when store 1 was empty at the end of week zero and used the rule :

$$x = I_1 - V_0 - \text{orders by store 2 outstanding} \tag{33}$$

where I_1 was the inventory at store 2 at the end of period 1. He showed that the distribution of V_T and its variance were different from that obtained by Hanssmann and then he calculated the improved expected profit per week using his modified ordering rule. Next, Lampkin analysed the case of central store/sub-store problem for the conduction when the stock at store 1 was insufficient to meet all demands. His rule for sharing shortages between the sub-stores was that ' for each sub-store, the difference between the outstanding orders on the sub-store and the total in the pipeline to the sub-store should be kept proportional to the average demand from that particular sub-store '. He again formulated the expression for the improved expected weekly profit with his modified ordering rules.

(*c*) *Theory of regenerative stochastic processes applied to inventory control* : Hurter and Kaminsky (1967) applied the theory of regenerative stochastic processes for finding the limiting distribution of the number of units in storage for a single item inventory storage system. Stocked items were issued from storage, one at a time, in a stochastic manner ; with times between individual issues N_t, being the random variable. Two modes of replenishment were assumed to be available : (1) regular replenishment—when the inventory was depleted, then immediately the replenishment was made to a level S_1, and (2) opportunity replenishment—opportunities occurred at random points in time with times between opportunities T_t being independent random variables. The ordering policy in the opportunity situation was similar to (s, S) policy. When an opportunity (or the discounted sales price) arose, the stock was replenished up to its maximum capacity S_2, provided n items were in storage, where $0 < n \leqslant L_2$.

They analysed only the case when $L_2 \geqslant S_1$; so under opportunity ordering conditions, the system was replenished with $S_2 - n$ units, if it contained at the most L_2 units. They studied the limiting behaviour of the stochastic processes $\{Z_t, t \geqslant 0\}$, where Z_t represented the number of units in storage at time t. The system was stated to be in state n at time t, if there were n units in storage. They also assumed ϵ to be the event ' when the system was in state n, where $0 < n \leqslant L_2$, and the opportunity arose '. Whenever ϵ occurred at some time t, the system was replenished to its capacity and hence $Z_t = S_2$. Whenever ϵ occurred, the system repeated itself and began anew, this way ϵ was a regenerative or recurrent unit. They showed that $\{Z_t, t \geqslant 0\}$ was a *regenerative stochastic process*.

Assuming V to be the random variable, that represented the time between occurrences of ϵ, and $v(t)$ the probability density function of V, they developed the expression for EV, the expected time between occurrences of ϵ, that was

$$EV = \int_0^\infty tv(t)\, dt.$$

Next they considered the distribution of the stochastic process $\{Z_t, t \geqslant 0\}$; and using the key renewal theory (Prabhu 1965), they developed the expression for lim . $t \to \infty$ $P\{Z_t = n\}$, the limiting state probabilities for the three cases : (i) $L_2 < n \leqslant S_2$, (ii) $S_1 < n \leqslant L_2$ and (iii) $0 < n \leqslant S_1$.

The significance of their method of analysis was that it could be easily extended to the general case where the intervals between the demands were independent and identically distributed random variables.

(d) *Minimizing the concave cost-function over the solution set of a Leontief substitution system* : Vienott (1969) first defined *Leontief* as a matrix A that had exactly one positive element in each column and a non-negative (column) vector x for which Ax was positive (i.e. had all positive components). He mentioned Dantzig's linear programme for finding a (column) vector $x = (x_j)$ called optimal, that minimized cx, subject to

$$Ax = b,\ x \geqslant 0 \tag{34}$$

where $A = (a_{ij})$ was a given Leontief matrix ; $b = (b_i)$ was a given non-negative column vector, and $c = (c_j)$ was a given row vector. He further stated that if there was an optimal solution, then the simplex method would produce an optimal basic solution and an associated optimal basis matrix, which according to Dantzig was Leontief and had a non-negative inverse, and that basis matrix was optimal for all non-negative b.

Next he used the partitioned matrix :

$$A = \begin{vmatrix} A_{11}, A_{12}, \dots A_{1N} \\ A_{21}, A_{22}, \dots A_{2N} \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ A_{N1}, A_{N2}, \dots A_{NN} \end{vmatrix}$$

for defining the *block triangular* concept, and called it upper (lower) block triangular, if $A_{ij}=0$, for all $i>j$ $(i<j)$; and block triangular, if it was either upper or lower block triangular. It was also mentioned that a block triangular matrix with non-positive off-diagonal matrices was Leontief if and only if each diagonal matrix was Leontief. He used Dantzig's method for minimizing cx, where $b \geqslant 0$ and A was an upper (lower) block triangular Leontief matrix. That method involved solving sequentially N linear programmes with their constraint matrices equal to the diagonal matrices, $A_{11}, A_{22}, \dots A_{NN}$, $(A_{NN}, A_{N-1}, \dots, A_{22}, A_{11})$. He used S to denote the set of programmes x, for which $x_i x_j = 0$, for all pairs (i, j) in the specified set.

If $X(b)$ was the set of solutions to eqn. (34), A was Leontief and $b \geqslant 0$; then (34) with or without S was designated as a *Leontief substitution system* and $x(b)\cap S$ as its solution set. Irrespective of (34) being a Leontief substitution system or not, a theorem stated that x was an extreme point of $X(b)\cap S$ if and only if $x \in S$ and x was an extreme point of $X(b)$. Another lemma indicated that if a concave function defined on the solution set $X(b)\cap S$ of a Leontief substitution system assumed its minimum on the set then it did so at an extreme point of the set. The minimum was achieved when $X(b)\cap S$ was bounded.

He formulated a large number of problems in the form of models, minimizing concave cost functions over the solution sets of the Leontief substitution systems. The formulated problems included cases of : (1) single and multi-facility economic lot size, (2) lot-size smoothing, (3) warehousing, (4) product assortment, (5) investment consumption, (6) batch queueing, (7) capacity expansion, and (8) reservoir control. The cost functions of each one of these problems were concave. He also gave dynamic programming recursions for finding the optimal solutions. With his D.P. algorithms, the computational effort increased only algebraically with the size of the problem and not exponentially (as it happens with most other D.P. algorithms).

(*e*) *Multicommodity network flows* : Creameans, Smith and Tyndall (1970) discussed a solution procedure for multicommodity network flows with resource constraints and delivery requirements, for the minimum cost case and developed an extension to permit substitution of resources. With linear programming, the number of potential vectors in most applications was found to be so large that the standard arc chain formulation became impractical. They used an extension of the column generation technique of the multicommodity network flow problem that considered network chain selection and resource allocation simultaneously. The flows were constrained by resource availability and network capacity. A minimum cost procedure was formulated and its computational algorithm was also given.

(*f*) *Least* d-*majorized network flows*: Vienott (1971) stated that for any feasible network flow model, there was a flow which simultaneously minimized every d-Schur convex function of the flows emanating from a single node called the source. The vector of flows coming out from the source, while giving minimum flow was unique and was called the *least* d-*majorized flow*. This flow was found to be such that it was determinable by way of solving the problem for the special case, where the d-Schur function was separable and quadratic. After the flow had been determined, the solution of the dual problem was equivalent to finding the conjugate of a function appearing in the dual objective function for the above flow. When the function was separable, this computation was found to be extremely simple. Vienott extended his results to the situations where the variables must be integers. He also showed that an important special case of the problem could be solved geometrically by choosing from among paths joining two points in the plane and lying between two given non-intersecting paths, the path with the minimum Euclidian length (i.e. the string solution). He applied his results to : (i) production-distribution models, (ii) stochastic inventory redistribution models, (iii) deterministic price speculation and storage model, (iv) series multiechelon inventory model with zero lead time, and (v) several maximum likelihood problems.

(*g*) *Results of queueing theory applied to inventory problems*: Gross and Harris (1971) discussed $(S-1, S)$ inventory models in which the time required for order replenishment was dependent on the number of orders outstanding. They assumed demand to be Poisson random variable with mean λ, and lead time was considered state-dependent in either of the two ways : (1) the portion of the lead time corresponding to actual filling of orders (i.e. service time) was an exponentially distributed random variable with distribution function $B_{T_m}(t) = 1 - \exp[-\mu(m)t]$, where m was the number of outstanding orders just after filling the previous order and $B_{T_m}(t)$ was the distribution function of time, and (2) the instantaneous probability at an arbitrary point in time of an order being filled (i.e. departure point state dependence) in an infinitesimal interval of time, Δt was $\mu(n)\Delta t + 0(\Delta t)$, where n was the number of outstanding orders at any arbitrary point of time. They investigated several models for each state. Orders were assumed to follow single-server queue principles, and using queueing theory results, an expected inventory cost function in terms of S was developed for obtaining the optimum value of S. The cost function included carrying costs and out-of-stock costs. They also considered models which included additional costs dependent on a service rate parameter (order fulfillment rate), μ, and optimized the cost function for S and μ.

Gross and Harris (1973) extended their above work on $[S-1, S]$ ordering policies to a general continuous review (s, S) type policy with complete backlogging and state dependent stochastic lead times. The demand on the inventory system was assumed to be a Poisson process, and the order filling process was dependent on the number of outstanding orders (i.e. state dependent). They obtained inventory state probabilities (needed for optimizing the cost function) via queueing analysis of the order filling process. Orders were assumed to arrive at a single channel facility where service was state dependent in either of the two ways mentioned in the above paragraph. They showed that the queueing model for the ordering process had Poisson input, and a batch service of batch size $Q = (S - s)$. The model assuming instantaneous state dependence was analysed using Chapman–Kolmogorov approach, and the model using departure-point state dependence utilized an imbedded Markov–Chain approach to calculate steady state system probabilities needed for the inventory cost function.

(*h*) *Decomposition algorithm for arborescence inventory systems* : Kalymon (1972) developed an algorithm for solving arborescence-structured production and inventory systems. Arborescence structures represent multiechelon production systems in which each facility gets input from a unique immediate processor. He assumed deterministic demands and no backlogging, and tried to schedule production over a finite planning horizon to minimize production and holding costs. His algorithm is suitable for systems whose all facilities with followers incur set-up change costs, linear production costs, and linear holding costs. General costs are incurred at the lowest echelon facilities (those without followers) ; if these costs are assumed concave, then this algorithm becomes highly efficient. The algorithm makes use of known results on the structure of optimal policy in arborescence systems and decomposes the problem into single stage problems at each lowest echelon facility. The decomposition is obtained by enumerating implicitly the feasible production set-up patterns at facilities with followers. The computation effort required by this algorithm increases linearly with the number of lowest echelon facilities, but it increases exponentially with the number of facilities with followers.

VI. *Models bridging the gap between theory and practice*

(*a*) *Adaptive forecasting applied to inventory* : Packer (1967) discussed a case study of selected inventory items stocked by a large manufacturer. He augmented the existing methods of inventory decision-making of the company with : (i) the exponential smoothing of the demand forecasts of each of the items, and (ii) the statistical determination of optimum safety stock level of each.

Computerized simulation of individual items was used to determine the most representative values of parameter α for use in the exponential smoothing formula. First he built a simulation model to correspond with the existing practices and tested it for its correspondence with the real world. He found that the near-optimal value of α was 0·4 for assets (i.e. manufactured for sale) inventory items ; and the near-optimal value of α was 0·1 for the maintenance items. Next, through simulation experiments, he developed curves showing change in the average investments and in the number of stockouts (as compared to those with the ongoing inventory procedures) plotted against the safety factor for various samples (inventory classes) of items. (Actually the simulation first computed these values for individual items and then aggregated them to obtain the average values for each of the given inventory classes.) He also computed the annual savings in the inventory related costs for each class.

(*b*) *Adaptive control limits* : Eilon and Elmaleh (1970) suggested a method for computing periodwise control limits (i.e. s, S values) based on Winter's demand-forecasting procedures (Winters 1960) which took into account the seasonal fluctuations and trends.

They simulated an inventory system which was assumed to have three inventory stages in series : (i) raw materials stage, (ii) work-in-progress stage, and (iii) finished goods stage. Each time the simulation was run for 500 periods. They analysed about 200 different cases which included different combinations of demand distributions, seasonal factors, cycle times and safety factors. Values of the parameters, such as demand distributions, trends and cycle times for one single item were taken from a food company's records.

They made comparisons between the results of two policies : (1) *fixed limits* when the values of (s, S) were determined at the beginning of each simulation and kept constant throughout and (2) *adaptive limits*, when the method of determining (s, S) was just like that of the fixed limits policy, but the s, S limits for each period were based on its own demand forecast. They found that in most cases, the total costs were low and the percentage of satisfied customers were higher with adaptive limits, as compared to those obtained with the 'fixed limits' method. Particularly in the case of normal demand distribution, for a given percentage of service level, the costs using adaptive control limits were found to be about 30% less than those with fixed limits policy. In the inventory literature, prior to this study, there had been little mention of adaptive limits policies, and hence this paper is of classical nature.

(c) *From mathematical theory to a workable model* : Gross, Harris and Robers (1971) investigated a continuous review inventory control system. They utilized the stochastic extension of the basic economic lot size model, that included a fixed reorder lead time and probabilistic demand. Mathematically, for a continuous system (when the demand distribution is constant from period to period), an s, S (or equivalently an s, Q) type policy is optimum. The items studied by them did not have constant demand patterns, rather the actual demands indicated the presence of trend effects and seasonal fluctuations. They recomputed the values of s and Q at the beginning of each new period (or every month) to take into account the changes in demand pattern.

They had assumed that the demand during each month was normally distributed random variable and therefore they had to forecast month-to-month mean demand by utilizing exponential smoothing with trend and seasonal terms added on. Wagner's model and algorithms (Wagner 1969) were used to determine the month-to-month values of the reorder point, s and the optimum order quantity, Q. Their model assumed that each item of inventory was independent of all the other items, ordering and holding costs were time-independent, and complete backlogging was allowed at a penalty cost of ϕ per unit. Wagner model basically minimizes the expected average cost per unit of time (or per month). Instead of first estimating the unit backordering cost ϕ, they started with the management estimates of the permissible average number of backorders over a lead time, and from there working backwards they calculated the optimal values of s, ϕ and Q, by making use of Wagner's algorithms.

For multi-item situations, they suggested the computation of s and Q for each individual item independently and recommended that as and when one item's inventory in the group of items (purchased from the same supplier) went below its level 's', then all the other items in that group having inventory levels within $+10\%$ of their respective current 's' values should also be ordered along with.

Using the above type month-to-month recomputed s, Q values, simulation runs were made for five typical items of a company, covering a period of one year, which showed that the average inventory level of these items got reduced by about 44% with their (s, Q) policy as compared to that resulting from the existing company procedures. They also found that their models were insensitive to perturbations in all the parameters except the lead-time and the fixed ordering costs.

(*d*) *Synthesis of priority scheduling and inventory control*: Berry (1971) examined the interdependence between the problems of priority scheduling and inventory control. He conducted simulation experiments to evaluate the gains in performance, when the inventory data was used in the scheduling rules. He studied only one inventory control policy (R, Q), where R was the reorder point and Q, the fixed order size. The values of R and Q were calculated with the help of the single-item deterministic formulae :

$$Q=\sqrt{\left(\frac{2AS}{C}\right)} \tag{35}$$

$$R=\bar{D}+k\sigma_D \tag{36}$$

where A was the demand rate ; S, the ordering (or set-up) cost ; C, the unit holding cost ; $\bar{D}$, average demand during the lead time ; σ_D, standard deviation of the demand distribution for lead time ; k, the level of protection expressed in terms of the number of standard deviations above $\bar{D}$; and $k\sigma_D$, the buffer stock level.

Using inventory data, he develops three scheduling rules : (i) Minimum slack-time per remaining operation rule—means selecting that order next, which has the smallest ratio of slack time to the number of remaining operations, (ii) Critical ratio rule—where critical ratio = rate of inventory depletion ÷ rate of production = (on-hand inventory ÷ manufacturing lead time)—dictates that the order with the smallest critical ratio should be processed next ; and (iii) Two-class SPT/Q rule—stresses that the orders waiting to be processed at a machine are first divided into two categories : (1) for items having inventory shortages and (2) for items that have some inventories left ; then the shortage category items are given higher dispatching priority and within this category, the order with the shortest processing time is dispatched first ; and only after all the first-category orders have been processed, the orders from the second category using SPT-rule are dispatched.

Simulation runs were made for the above three rules and also for two other most commonly used rules (i) SPT-rule and (ii) FCFS-rule. The last two rules served as reference points for comparing the gains in performance resulting from the use of the above three rules. His results indicated that the scheduling rules incorporating inventory information improved the performance as regards the total inventory related costs and percentage machine utilization. However, the results obtained from the Critical-Ratio Rule indicated that the increase and availability in the amount of information for scheduling purposes would not always lead to improved performance.

(*e*) *The distribution system simulator*: Connors *et al.* (1972) constructed a large-scale physical distribution simulation model. This model was designated as *distribution system simulator* (DSS). The user was expected to respond to a questionnaire which contained options from which he could choose those options that represented his own distribution system. The options allowed the user to include each of the major factors involved in the operations of his distribution system such as the demand characteristics of each of the products, buying patterns of the customers, order filling policies, redistribution policies, transportation policies, distribution channels, factory locations, production capacities, and other elements. User subroutines might also be used to incorporate functions such as vehicle scheduling, forecasting, production scheduling, and pricing.

DSS has basically been designed on the lines of ' total system approach '. The DSS first generates a computer programme for running the model on the computer after the user's answers to the above mentioned questionnaire plus the DSS are submitted to the computer. The resulting programme will have all the options ; and the user, in addition to the programme, gets the formats of input data and output reports. The user of DSS need not be familiar with computer programming at all, for running simulation experiments on his distribution system.

Limitations of theoretical inventory models

The large number of research studies and the models discussed above have covered a large number of the practical situations ; but they do not in any way exhaust the possibilities of formulating the possible millions of additional models. However they do indicate that virtually for each group of similar items (or a family of homogeneous items), there must be a specific inventory policy suiting each individual item, stocked by a company. From company to company, even for the same item, the demand patterns, lead time fluctuations, and cost structures are likely to be different. Therefore, for optimizing inventory operations, each company needs to compute its specific optimum inventory policy for each of its thousands of items, for the then-existing values of the respective parameters of that item.

As the time passes, the company may find that even for a particular item, the conditions in relation to the demand, the cost elements, the supply, the lead time, etc., have completely changed, and in consequence may conclude that the previously computed policy (whether s, S ; R, Q ; T, S or any other) is not any longer for the current values of the item parameters. Further, it may as well happen that over a period of time, the management objectives have totally changed—either from optimization of inventory-related costs to the optimization of the number of shortages or backorders per period ; or from optimization of inventory costs to working within the limited available capital or keeping the total number of orders per period below a particular limit, etc.

The above discussions clearly indicate that for keeping the inventory operations of a company optimal ; the management will have to compute period after period (daily, weekly or monthly) the optimum values of the limits of inventory levels (either in terms of (s, S) or (R, Q) or (R, T), etc.) for each of the thousands of the stocked items, and they may even have to use different models for the same item or the family (or group) of items from period to period.

Even with the present high speed of computers, so wide a use of all the different models by a company seems impractical because it is highly unlikely that the company's analysts will be easily able to programme the computers for deciding for each item when to switch from one type of an inventory model to another type of inventory model. Further, to incorporate all the optimizing models in a single inventory control system can be an extremely costly proposition. If it is assumed that the decision to switch from one policy to another for the same item will be made by the company analysts, even then the computer programmes for individual items will have to be updated quite frequently and such a set-up will constantly require the efforts of a large number of analysts and programmers. The costs of such additional personnel will most likely offset the savings resulting from the extra efforts spent on matching the most suitable and optimizing inventory policy with each of the items of the system at tbe beginning of each period.

If the interdependence between the items of a group and/or number of storage locations is also to be taken into account, then the management must develop some guidelines for grouping the items and the locations. For example, the items purchased from each supplier may be grouped together ; or the items going into a particular sub-assembly may make one group ; or the items which possess similar physical characteristics like dollar-value, size, weight, shelf-life, etc. may be placed in one group ; or the items are grouped customer-wise, or distribution centre-wise ; or the grouping may be accomplished on a combined basis of one or more of the just-stated different ways of grouping. Now if the optimization is to be sought for the individual groups, then the company must use multi-item and/or multi-echelon models, which are quite complicated and require much more computing time, as compared to the computing time requirements of the single-item models. Till this time, there has hardly been any published report about the use of multi-item and/or multi-echelon models by the companies that stock thousands of items. So, even the large number of available optimizing-type models do not help the practising managers in optimizing their operations ideally, because they cannot find resources at a reasonable cost to keep each of the inventory items geared to an ideally suiting inventory policy model, all the time.

Apart from the above considerations, different authors have made different assumptions about the situations they have modeled, and most of them point out the following limitations as regards those assumptions.

(1) It is extremely difficult to estimate the cost of shortages or back-ordering.

(2) The common assumption of the selling price remaining constant over a period of time, does not always hold good. It is a common phenomenon that companies quite often reduce their selling prices just to clear their existing inventories or spend heavily on promotion for increasing their sales rate (the cost of promotion again comes out of the sales revenue).

(3) The often-used assumption of zero lead time or constant lead time is generally not representative of the actually fluctuating lead times of the real-life situations.

(4) For the unbounded horizon cases, the assumptions of stable periodic demand distribution and stationary costs are far off the field conditions ; so how the policies computed for the unbounded horizon models can be optimal.

(5) In several models, the authors make the assumption of convex cost functions. Such an assumption may make the mathematical manipulation of equations easier and may help them determine mathematically the optimum policies ; but from the practical point of view, the assumption of convex cost function is highly questionable, because a convex cost function can occur only under abnormal conditions of limited capacity, limited storage space, limited availability of capital, or limited capacity of the suppliers, etc.

(6) The common assumption that the demand distribution of each individual item is totally independent of each other or their distributions are independent of the selected replenishment policy or of any other managerial action is not always correct.

Suggestions for practicability

Realizing fully well that if the company is dealing in only one or a couple of items, commodities, or specialized products, then even the most sophisticated inventory model can at the most provide only certain estimates which do help the management in making day-to-day decisions ; but simultaneously the management always takes into account the factors like the changing economy, changes in international markets, and the impending government regulations. The inventory models can be really helpful only when they can relieve the management of the routine decision-making functions related to the inventory operations of thousands of their stocked items.

Computations resulting from the optimizing models may suggest different review periods for different items. Under such conditions, if a particular group of items is to be ordered from a particular supplier, it may happen that all the items in the group do not have the same optimum review period, then some of the items may have to be reviewed (at the time of ordering), earlier than their respective optimum review-periods, and a few others may have to be reviewed late after their optimum review-time. Such adjustments in review periods will certainly lead to a loss of optimization in relation to their ideal inventory policies.

Considering the types of difficulties and limitations discussed above, it seems desirable that tables (similar to the sampling tables of quality control) for the most powerful policies like (s, S), (s, Q), (T, S), (R, T), etc. for various combinations of demand distributions, cost structures, replenishment lead time distributions, etc. may be developed. Each such table may cover a wide range of the numerical values of the parameters associated with it. Such tables can prove extremely handy to the management in computerizing the inventory control of thousands of their stocked items as well as in reviewing the inventory limits of each of the items, from period to period. These tables may even be very helpful in grouping of items for the given lengths of review periods, or on individual supplier basis, etc.

Forrestor's simulation experiments clearly suggest (Forrestor 1961) that if the demand information from all the distribution points of a company can be simultaneously fed back to all the in-between stocking levels instead of through the chain of successive levels of inventory points, then the inventory operations of the company will be more stable and as a result less costly. Such a type of feedback system can be made all the more effective in terms of choosing the optimum values of inventory limits and the review periods, if we had these values readily available from the suggested type of tables. As and when prepared, these tables are likely to prove extremely useful because they can directly fulfill the operational needs of the inventory control managers and hopefully will help them in achieving the near-optimum inventory control for the company as a whole.

L'auteur y

Cet article considère du point de vue des systèmes, les modèles ayant rapport aux inventaires, qui ont été publiés au cours des dix dernières années.
donne un graphique montrant les vastes catégories de modèles d'inventaires.

L'article dresse d'abord une liste de onze variables dont la quantité peut être déterminée et de quatre variables qu'on ne peut pas quantifier (et les types de ces variables) ; ces variables font partie des formules d'inventaires, affectent les décisions prises à propos des inventaires et ont déjà été identifiées par les chercheurs.

L'auteur groupe ensuite en six catégories divers articles de la recherche faite sur ce sujet, en basant ses catégories sur les caractères communs des différentes méthodes d'approche utilisées dans ces articles. Ces catégories sont les suivantes : (1) Les modèles qui permettent de déterminer des plans d'inventaires optimisés. (2) L'optimisation de la dimension d'un lot. (3) L'optimization de divers objectifs de gestion spécifiques. (4) Les modèles qui permettent d'optimiser des situations dans lesquelles les inventaires sont très spécialisés. (5) L'application de théories mathématiques avancées à des problèmes d'inventaires. (6) Les modèles qui permettent de combler la brèche qui existe entre la théorie et la pratique.

Puis on présente une synthèse des modèles publiés récemment qui appartiennent à chacune des catégories et on souligne les caractéristiques des études particulières.

De plus cet article discute les difficultés qui se présentent le plus couramment lorsqu'on applique ces modèles à des situations d'inventaires pratiques courants ;
et cet article expose également les faiblesses (ou limites) de certaines hypothèses que
font couramment les chercheurs pour développer les modèles d'inventaires.

Enfin, l'article suggère plusieurs façons de mettre en pratique le mieux possible les résultats des modèles d'inventaires dont on dispose.

Eine

Diese Arbeit betrachtet die während der vergangenen zehn Jahre veröffentlichten
ge o c ns ct . Ei e
Tabelle zeigt die groben Kategorien der Lageraufnahmeschemata. Die Arbeit führt
lageraufnahmebezogenen Schemata vom Verfahrensstandpunkt aus gesehen.
zuerst elf meßbare und vier nicht meßbare Veränderliche (und deren Typen) auf, die Lageraufnahmeformeln beeinflussen, auf Inventurentscheidungen einwirken und schon von den Forschern identifiziert worden sind. Dann werden verschiedene Forschungsstudien auf der Grundlage von Ähnlichkeiten in der Handhabung des Themas durch diese Studien in sechs Kategorien eingeteilt. Die Kategorien lauten wie folgt : (1) Schemata zur Festlegung optimaler Lageraufnahmemethoden, (2) höchstmögliche Verbesserung der Postengrößen, (3) größtmögliche Verbesserung verschiedener spezifischer Managementziele, (4) Schemata zur größtmöglichen Verbesserung hochspezialisierter Lageraufnahmesituationen, (5) Anwendung fortgeschrittener mathematischer Theorien auf Probleme der Lageraufnahme und (6) Schemata, die die Kluft zwischen Theorie und Praxis überbrücken. Dann wird ein synthetischer Überblick über kürzlich veröffentlichte Schemata in jeder einzelnen der dargestellten Kategorien gegeben, und hervorragende Punkte einzelner Studien werden hervorgehoben. Außerdam bespricht diese Arbeit die Schwierigkeiten, die sich gewöhnlich bei der Anwendung dieser Schemata auf Routinelageraufnahmesituationen in der Praxis ergeben. Auch werden die Schwächen (oder Unzulänglichkeiten) gewisser Annahmen aufgeführt, die häufig von den Forschern für die Entwicklung von Lageraufnahmemethoden zugrundegelegt werden. Zum Schluß macht die Arbeit mehrere Vorschläge für die beste Nutzung der verfügbaren Ergebnisse von Lageraufnahmeschemata in der Praxis.

References

AGIN, N., 1966, A minimax inventory model, *Mgmt Sci.*, **12**, 517.

ALLEN, S. G., and D'ESOPO, D. A., 1968, An ordering policy for stock items when delivery can be expedited, *Ops Res.*, **16**, 880.

ARROW, K. J., KARLIN, S., and SCARF, H., 1958, *Studies in Mathematical Theory of Inventory and Production* (Stanford University Press).

ARUNACHALAM, R., 1972, Management of seasonal style goods inventories, *Ops Res.*, **20**, 265.

BECKMAN, M. J., 1964, Dynamic programming and inventory control, *Opl Res. Q.*, **15**, 389.

BEESACK, P. R., 1967, A finite horizon dynamic inventory model with a stockout constraint, *Mgmt Sci.*, **13**, 618.

Berry, W. L., 1971, Priority scheduling and inventory control in job-lot manufacturing systems, paper presented at the A.I.I.E. 22nd Annual Conference, Boston, May.

Bessler, S. A., and Veinott, A. F., Jr., 1966, Optimal policy for a dynamic multi-echelon inventory model, *Nav. Res. Logist. Q.*, **13,** 355.

Bomberger, E. E., 1966, A dynamic programming approach to a lot-size scheduling problem, *Mgmt Sci.*, **12,** 778.

Boylan, E. S., 1964, Multiple (s, S) policies, *Econometrica*, **32,** 399 ; 1967, Multiple (s, S) policies and the n-period inventory problem, *Mgmt Sci.*, **14,** 196.

Clark, A. J., and Scarf, H., 1960, Optimal policies for a multi-echelon inventory problem, *Mgmt Sci.*, **6,** 475.

Connors, M. M., Coray, C., Cuccaro, C. J., Green, W. K., Low, D. W., and Markowitz, H. M., 1972, The distribution system simulator, *Mgmt Sci.*, **18,** B-425.

Cremeans, J. E., Smith, R. A., and Tyndall, G. R., 1970, Optimal multi-commodity network flows with resource allocation, *Nav. Res. Logist. Q.*, **17,** 269.

Currey, G. L., Skeith, R. W., and Harper, R. G., 1970, A multi-product dependant inventory model, *A.I.I.E. Trans.*, **2,** 263.

Diegel, A., 1966, A linear approach to the dynamic inventory problem, *Mgmt Sci.*, **12,** 530.

Eilon, S., and Elmaleh, J., 1970, Adaptive limits in inventory control, *Mgmt Sci.*, **16,** B-533.

Evans, R. V., 1968, Sales and restocking policies in a single-item inventory system, *Mgmt Sci.*, **14,** 463.

Falkner, C. H., 1969, Optimal ordering policies for a continuous-time deterministic inventory model, *Mgmt Sci.*, **15,** 672.

Feeney, G. R., and Sherbrooke, C. C., 1966, The $(s-1, s)$ inventory policy under compound Poisson demand, *Mgmt Sci.*, **12,** 391.

Forrester, J. W., 1961, *Industrial Dynamics* (Cambridge, Massachusetts : MIT Press).

Gross, D., and Harris, C. M., 1971, On one-for-one ordering inventory policies with state-dependent lead times, *Ops Res.*, **19,** 735; 1973, Continuous review (s, S) inventory models with state-dependent lead times, *Mgmt Sci.*, **19,** 567.

Gross, D., Harris, C. M., and Robers, P. D., 1971, Bridging the gap between mathematical inventory theory and the construction of a workable model—a case study, presented to the XVIII International meeting of TIMS in Washington, D.C.

Gross, D., and Soriano, A., 1969, The effect of reducing lead-time on inventory levels—simulation analysis, *Mgmt Sci.*, **16,** B-610.

Hausman, W. H., 1969, Minimizing customer-line items backordered in inventory control, *Mgmt Sci.*, **15,** B-628.

Hausman, W. H., and Thomas, L. J., 1972, Inventory control with probabilistic demand and periodic withdrawals, *Mgmt Sci.*, **18,** 265.

Hanssmann, F., 1959, Optimal inventory location in control and production networks, *Ops Res.*, **7,** 483.

Hayes, R. H., 1969, Statistical estimation problems in inventory control, *Mgmt Sci.*, **15,** 686.

Herron, D. P., 1967, Inventory management for minimum cost, *Mgmt Sci.*, **14,** B-219.

Hochstaedtler, D., 1970, An approximation of the cost-function for multi-echelon inventory model, *Mgmt Sci.*, **16,** 716.

Hurter, P. A., and Kaminsky, F. C., 1967, An application of regenerative stochastic processes to a problem in inventory control, *Ops Res.*, **15,** 467.

Iglehart, D. L., and Morey, R. C., 1972, Inventory systems with imperfect asset information, *Mgmt Sci.*, **18,** B-388.

Ignall, E., 1969, Optimal continuous review policies for two-product inventory systems with joint set-up costs, *Mgmt Sci.*, **15,** 278.

Ignall, E., and Veinott, A. F., Jr., 1969, Optimality of myopic inventory policies for several substitute products, *Mgmt Sci.*, **15,** 284.

JOHNSON, E. L., 1967, Optimality and computation of (σ, S) policies in the multi-item infinite horizon inventory problem, *Mgmt Sci.*, **13**, 475; 1968, On (s, S) policies, *Mgmt Sci.*, **15**, 80.

KALYMON, B. A., 1972, A decomposition algorithm for arborescence inventory systems, *Ops Res.*, **20**, 860.

KAPLAN, R. S., 1970, A dynamic inventory model with stochastic lead-times, *Mgmt Sci.*, **16**, 491.

LAMPKIN, W., 1966, A second look at Hanssmann's inventory control model with special reference to the central store/substore problem, *Ops. Res.*, **14**, 59.

LANDI, D. M., and JOHNSON, B. M., 1967, Optimal linear inventory control and first order exponential smoothing, *Ops Res.*, **15**, 342.

LIPPMAN, S. A., 1969, Optimal inventory policy with multiple set-up costs, *Mgmt Sci.*, **16**, 118; 1971, Economic order quantities and multiple set-up costs, *Mgmt Sci.*, **18**, 39.

LOVE, S. F., 1972, A facilities in series inventory model with nested schedules, *Mgmt Sci.*, **18**, 327.

MANNE, A. S., and VEINOTT, A. F., JR., 1967, Optimal plant size with arbitrary increasing time-paths of demand, *Investments for Capacity Expansion-Size, Location and Time-phasing*, edited by A. S. Manne (Cambridge, Massachusetts : MIT Press), pp. 178–190.

MILLER, B. L., 1971, A multi-item inventory model with joint backorder criterion. *Ops Res.*, **19**, 1467.

MOREY, R. C., 1970, Inventory systems with imperfect demand information, *Nav. Res. Logist. Q.*, **17**, 287.

MORTON, T. E., 1971, The near myopic nature of the lagged-proportional inventory-cost problem with lost sales, *Ops Res.*, **19**, 1708.

MOSSIN, J., 1966, On a class of optimal stock depletion policies, *Mgmt Sci.*, **13**, 120.

MURRAY, G. R., and SILVER, E. A., 1966, A Bayesian analysis of the style goods inventory problem, *Mgmt Sci.*, **12**, 785.

PACKER, A. H., 1967, Simulation and adaptive forecasting as applied to inventory control, *Ops Res.*, **15**, 660.

PALM, C., 1938, Analysis of the Erlang traffic formula for busy-signal arrangements, *Ericsson Tech.*, **5**.

PIERSKALLA, W. P., 1967 a, Optimal issuing policies in inventory management, *Mgmt Sci.*, **13**, 395; 1967 b, Inventory depletion management with stochastic field-life functions, *Mgmt Sci.*, **13**, 877; 1972, Optimal issuing policies for perishable inventories, *Mgmt Sci.*, **18**, 603.

PORTEUS, E. L., 1971, On the optimality of generalized (s, S) policies, *Mgmt Sci.*, **17**, 411; 1972, The optimality of generalized (s, S) policies under uniform demand densities, *Mgmt Sci.*, **18**, 644.

PRABHU, N. U., 1965, *Stochastic Processes : Basic Theory and Its Applications* (New York : Macmillan).

ROSE, M., 1972, The ($S-1$, S) inventory model with arbitrary backordered demand and constant delivery times, *Ops Res.*, **20**, 1022.

RUTENBERG, D. P., 1971, Design commonality to reduce multi-item inventory : optimal depth of a product-line, *Ops Res.*, **19**, 491.

SARGENT, R. G., and BRADLEY, H. E., 1969, A ' variant S ' inventory model, *Mgmt Sci.*, **15**, 716.

SCHUSSEL, G., 1968, Job-shop lot release sizes, *Mgmt Sci.*, **14**, B–449.

SCHWARTZ, B. L., 1966, A new approach to stockout penalties, *Mgmt Sci.*, **12**, B–538; 1970, Optimal inventory policies in perturbed demand models, *Mgmt Sci.*, **16**, B–509; 1973, A simple continuous review deterministic one-warehouse N-retailer inventory problem, *Mgmt Sci.*, **19**, 555.

SELF, G. D., and LEWIS, H. S., 1969, Geometric solution of a dynamic liquid inventory system, *A.I.I.E. Trans.*, **1**, 199.

SIMON, R. M., 1971, Stationary properties of a two-echelon inventory model for low demand items, *Ops Res.*, **19**, 761.

SIVAZLIAN, B. D., 1971, Dimensional and computational analysis in stationary (s, S) inventory problems with gamma distributed demand, *Mgmt Sci.*, **17**, B–307.

TAHA, H. A., and SKEITH, R. W., 1970, Economic lot-sizes in multiple production systems, *A.I.I.E. Trans.*, **2**, 157.

THOMPSON, H. E., 1966, Sales forecasting errors and inventory fluctuations : Random errors and random sales, *Mgmt Sci.*, **12**, 448.

TOPKIS, D. M., 1968, Optimal ordering and rationing policies in a non-stationary dynamic inventory model with n demand classes, *Mgmt Sci.*, **15**, 160.

VEINOTT, A. F., JR., 1966, The status of mathematical inventory theory, *Mgmt Sci.*, **12**, 745 ; 1969, Minimum concave-cost solution of Leontief substitution models of multifacility inventory systems, *Ops Res.*, **17**, 262 ; 1971, Least d-majorized network flows with inventory and statistical applications, *Mgmt Sci.*, **17**, 547.

WAGNER, H. M., 1969, *Principles of Operations Research* (Englewood Cliffs, New Jersey : Prentice-Hall), pp. 811–821.

WHISLER, W. D., 1967, A stochastic inventory model for rented equipments, *Mgmt Sci.*, **13**, 640.

WINTERS, P. R., 1960, Forecasting sales by exponentially weighted moving averages, *Mgmt Sci.*, **6**, 324.

ZANGWILL, W. I., 1966 a, A deterministic multi-product facility production and inventory model, *Ops Res.*, **14**, 486 ; 1966 b, A deterministic multi-period production scheduling model with backlogging, *Mgmt Sci.*, **13**, 105 ; 1967, The piecewise concave function, *Mgmt Sci.*, **13**, 900 ; 1969, A backlogging model and a multi-echelon model of a dynamic lot-size production system—A network approach, *Mgmt Sci.*, **15**, 506.