Joint Reliability-Importance of Two Edges in an Undirected Network

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Key Words — **Joint reliability importance, Marginal reliability importance, Joint failure importance, Marginal failure importance, Undirected network**

Reader Aids —
Purpose: Widen state of the art
Special math needed for explanations: Probability, network theory
Special math needed to use results: Same
Results useful to: Reliability analysts

Summary & Conclusions — **This paper introduces: a) Joint Reliability Importance (JRI) of two edges in an undirected network, and b) concepts of Joint Failure Importance (JFI) and Marginal Failure Importance (MFI), duals of JRI & MRI, respectively,. The JRI of two edges in an undirected network is represented by the Marginal Reliability Importance (MRI) of each edge in sub-networks. Some relationships between JRI & MRI, JRI & JFI, and JFI & MFI are presented. From these relationships, we show that: a) JRI is an appropriate quantitative measure of interactions of two edges in a network w.r.t. Source-to-Terminal Reliability, and b) the sign of JRI of two edges can be determined without computing the JRI of these two edges in some special cases.**

1. INTRODUCTION

Acronyms

JFI	Joint Failure Importance
JRI	Joint Reliability Importance
MFI	Marginal Failure Importance
MRI	Marginal Reliability Importance
STR	Source-to-Terminal Reliability
w.r.t.	with respect to

Reliability importance of a component is a quantitative measure of the importance of the individual components in contributing to system reliability. It is defined as the rate at which system reliability improves as the component-reliability improves [3]. This paper extends this concept for 2 edges in an undirected network system and names it: JRI of 2 edges. In contrast, reliability importance of 1 edge is named MRI. In an undirected network, STR = Pr{source & terminal are connected by working edges} is our concern. In this system, therefore, an edge and STR correspond to a component and to system reliability respectively in conventional systems.

We also introduce the concepts of JFI & MFI, which are duals of JRI & MRI respectively. We establish some relationships between JRI & MRI, JRI & JFI, and JFI & MFI. From these relationships, we show that: a) JRI is an appropriate measure of interactions of two edges in a network w.r.t. STR, and b) the sign of JRI of two edges can be determined without computing the JRI of these two edges in some special cases.

Proofs of all lemmas & theorems are in the appendix.

Notation

V	set of vertices: $\{v_1, \ldots, v_m\}$
E	set of edges: $\{e_1, \ldots, e_n\}$
$G(V,E)$	an undirected graph G with vertex set V and edge set E
p_i, q_i	probability that $e_i \in E$ is [working, not working]; $p_i+q_i\equiv 1$
G^*i, $G-i$	G with e_i [contracted, deleted]
N	$[G(V,E), p_i, \{s,t\}]$: network N with edge reliability p_i and source & terminal are attached to G
$R(G)$, $F(G)$	STR: Pr{source & terminal [are, are not] connected by working edges}; $R(G)+F(G)\equiv 1$
X_i	indicator for e_i: $X_i=1$ if e_i is working, $X_i=0$ otherwise
$I_G(i)$	MRI of edge e_i: $\partial R(G)/\partial p_i$
$I_G(i,j)$	JRI of edge e_i & ej: $\partial^2 R(G)/\partial p_i \partial p_j$
$I'_G(i)$	MFI of edge e_i: $\partial F(G)/\partial q_i$
$I'_G(i,j)$	JFI of edge e_i & e_j: $\partial^2 F(G)/\partial q_i \partial q_j$

Other, standard notation is given in ''Information for Readers & Authors'' at the rear of each issue.

Nomenclature

s-t path: minimal set of edges whose working connects the source & terminal (simply termed as path)
s-t cut: minimal set of edges whose deletion disconnects the source & terminal (simply termed as cut)
irrelevant edge: edge which does not occur in any s-t path
series edges: edges which are incident to a common vertex of degree-2
parallel edges: edges with the same end-vertices

Assumptions

1. There are 2 states of each edge, *ie*, each edge is working or failed.
2. Failures of edges are mutually *s*-independent.

2. RELATIONSHIPS BETWEEN JRI & MRI

MRI of edge e_i in an undirected network is:

$$I_G(i) = \partial R(G)/\partial p_i. \tag{1}$$

JRI of 2 edges is defined similarly.

Definition 1. Given a stochastic network $N=[G(V,E), p_i, \{s,t\}]$, the JRI of 2 edges e_i & e_j therein is the partial derivatives of 2 edges w.r.t. the STR of the network:

$$I_G(i,j) = \partial^2 R(G)/\partial p_i \partial p_j. \quad (2)$$

Lemma 1 gives the explicit expression of JRI.

Lemma 1. JRI of 2 edges, $I_G(i,j)$, is:

$$I_G(i,j) = R(G^*i^*j) - R(G^*i-j) - R(G^*j-i) + R(G-i-j). \quad (3)$$

Lemma 2 shows that JRI can be represented by MRI of each edge in subgraphs. The following relationships between JRI & MRI result from lemma 1.

Lemma 2. Given $N = [G(V,E), p_i, \{s,t\}]$, JRI of 2 edges in G is represented by MRI of each edge in subgraphs of G:

a. $I_G(i,j) = I_{G^*j}(i) - I_{G-j}(i)$ (4)

b. $I_G(i,j) = I_{G^*i}(j) - I_{G-i}(j)$. (5)

Alternatively, the following relationships hold between JRI & MRI.

Lemma 3. JRI of 2 edges in G is represented by MRI of each edge in subgraphs:

a. $I_G(i,j) = (I_G(i) - I_{G-j}(i))/p_j$ (6)

b. $I_G(i,j) = (I_G(j) - I_{G-i}(j))/p_i$. (7)

From lemma 2, the following interpretations are obtained.

a. If $I_G(i,j) \le 0$, then $I_{G^*j}(i) \le I_{G-j}(i)$ from lemma 2. This implies that e_i is more important w.r.t. STR when edge e_j is failed than when e_j is working.

b. If $I_G(i,j) \ge 0$, then $I_{G^*j}(i) \ge I_{G-j}(i)$ from lemma 2. This implies that e_i is more important w.r.t. STR when edge e_j is working than when e_j is failed.

These interpretations show that $I_G(i,j)$ is an appropriate quantitative measure for interactions of 2 edges w.r.t. STR. Ref [4] presents similar treatments independently.

Corollary 1 shows that the degree of interactions of 2 edges w.r.t. STR is bounded absolutely by 1.

Corollary 1. The value of JRI of 2 edges is between -1 & 1.

From lemma 1, the value of $I_G(i,j)$ is obtained by computing STR for the four subgraphs, G^*i^*j, G^*i-j, G^*j-i, $G-i-j$.

Many algorithms can compute $R(G)$. Since the factoring algorithm uses the pivoting of edges and is the most efficient [6,8], the R.K. Wood Algorithm [7,8] might be appropriate for the computation of $I_G(i,j)$. Let REL(G) be the factoring algorithm for computing STR of $N = [G(V,E), p_i, \{s,t\}]$; its basic steps in computing $I_G(i,j)$ are:

1. Generate G^*i^*j by contracting edges e_i & e_j in G. Generate G^*i-j (G^*j-i) by contracting e_i (e_j) and deleting e_j (e_i) in G. Generate $G-i-j$ by deleting both edges e_i & e_j in G.
2. Apply REL(G) to the 4 subgraphs generated in step 1.
3. Compute $I_G(i,j)$ using (3). □

Since the STR problem is NP-hard, the computation problem for $I_G(i,j)$ is also NP-hard. So, it is desirable to know the sign of $I_G(i,j)$. To determine the sign of $I_G(i,j)$ in an undirected network, the concepts of JFI & MFI must be introduced. Ref [4] contains a different way to determine the sign of $I_G(i,j)$ in a directed network.

3. RELATIONSHIPS BETWEEN JFI & MFI

JFI & MFI, duals of JRI & MRI, respectively are defined as follows.

Definition 2. For a given network $N = [G(V,E), p_i, \{s,t\}]$, JFI of 2 edges e_i & e_j is:

$$I'_G(i,j) \equiv \partial^2 F(G)/\partial q_i \partial q_j. \quad (8)$$

MFI of edge e_i is:

$$I'_G(i) \equiv \partial F(G)/\partial q_i. \quad (9)$$

Lemma 4 shows the relationship between JRI & JFI.

Lemma 4. The following relationship holds between JFI & JRI:

$$I'_G(i,j) = -I_G(i,j). \quad (10)$$

Lemmas 5 & 6 show the relationships between JFI & MFI.

Lemma 5. Given $N = [G(V,E), p_i, \{s,t\}]$, JFI of 2 edges in G is represented by MFI of each edge in subgraphs of G:

a. $I'_G(i,j) = I'_{G-j}(i) - I'_{G^*j}(i)$ (11)

b. $I'_G(i,j) = I'_{G-i}(j) - I'_{G^*i}(j)$. (12)

Lemma 6. The following alternate relationships hold between JFI & MFI:

a. $I'_G(i,j) = [I'_G(i) - I'_{G^*j}(i)]/q_j$ (13)

b. $I'_G(i,j) = [I'_G(j) - I'_{G^*i}(j)]/q_i$. (14)

4. DETERMINATION OF SIGN OF JRI OF 2 EDGES

For series & parallel edges, the sign of $I_G(i,j)$ is easily determined from lemma 2.

Corollary 2.

a. If edges (e_i,e_j) are in parallel, then $I_G(i,j) \leq 0$,
b. If edges (e_i,e_j) are in series, then $I_G(i,j) \geq 0$.

The fact that 2 edges, (e_i,e_j), are in series means that the paths containing e_i contain e_j at the same time. Similarly, the fact that 2 edges, (e_i,e_j), are in parallel means that the cuts containing e_i contain e_j at the same time. In general, therefore, there are 3 situations for the relations of 2 edges, (e_i,e_j):

a. There exists no path containing both e_i & e_j, *ie*, there exist some cuts containing both e_i & e_j.
b. There exists no cut containing both e_i & e_j, *ie*, there exist some paths containing both e_i & e_j.
c. There exist some paths containing both e_i & e_j AND there exist some cuts containing both e_i & e_j. □

It is impossible to have no cut containing both e_i & e_j AND no path containing both e_i & e_j. A special case of situation #a is parallel edges whereas a special case of situation #b is series edges. We show that the sign of JRI is nonpositive for situation #a.

Theorem 1. If there exists no path containing both e_i & e_j, then JRI of e_i & e_j is nonpositive.

Theorem 2. If there exists no cut containing both e_i & e_j, then JRI of e_i & e_j is nonnegative.

5. NUMERICAL EXAMPLES

5.1 *Bridge Network*

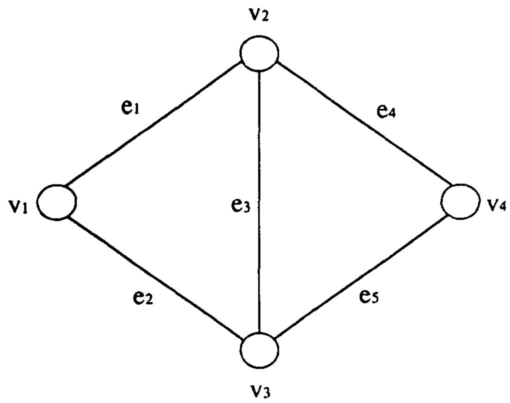
Consider the bridge network in figure 1. There are 10 pairs of edges. Edge-pairs are classified into 3 situations:

a. Pairs of edges such that there exists no path containing both edges: $C_1 = \{(e_1,e_2), (e_4,e_5)\}$.
b. Pairs of edges such that there exists no cut containing both edges: $C_2 = \{(e_1,e_4), (e_2,e_5)\}$.
c. Pairs of edges such that there exist some paths and some cuts containing both edges: $C_3 = \{(e_1,e_3), (e_2,e_3), (e_3,e_4), (e_3,e_5), (e_1,e_5), (e_2,e_4)\}$.

For simplicity, let $p_i = p$ for all edges. Then in this simple network, JRI of edges (e_1,e_2) is:

$$I_G(1,2) = R(G^*1^*2) - R(G^*1-2) - R(G^*2-1) + R(G-1-2)$$

$$= (p+p-p^2) - (p+p^2-p^3) - (p+p^2-p^3) + 0$$



[$N=G(V,E)$; $p_i = p$ for all $i=1-5$; $(s,t) = (V_1, V_4)$]
Figure 1. Bridge Network

$$= 2p^3 - 3p^2.$$

Similarly,

$$I_G(1,3) = I_G(1,5) = 2p^3 - 3p^2 + p.$$

$$I_G(1,4) = 2p^3 - 3p^2 + 1.$$

From the symmetric topology of the pairs of edges, we know that:

$$I_G(1,2) = I_G(4,5),$$

$$I_G(1,3) = I_G(1,5) = I_G(2,3) = I_G(3,5) = I_G(2,4) = I_G(3,4),$$

$$I_G(1,4) = I_G(2,5).$$

The values of $I_G(i,j)$ for $0 \leq p \leq 1$ are illustrated in figure 2. This example illustrates that JRI of 2 edges in situation #a is nonpositive, whereas JRI of 2 edges in situation #b is nonnegative for all values of p. However, JRI of 2 edges in situation #c has either positive or negative values depending on the values of p.

5.2 *Power Supply*

Consider the power supply [5]. The reliability (logic) block diagram (RBD) and network representation of this system are in figures 3 & 4. There are 66 pairs of edges. Pairs of edges are classified as follows:

a. $C_1 =$

$\{(e_1,e_2),(e_1,e_3),(e_1,e_4),(e_2,e_3),(e_2,e_4),(e_3,e_4),(e_{11},e_{12})\}$,

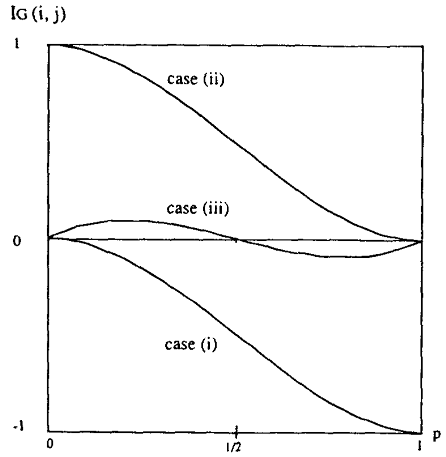


Figure 2. Values of $I_G(i,j)$ for $0 \leq p \leq 1$

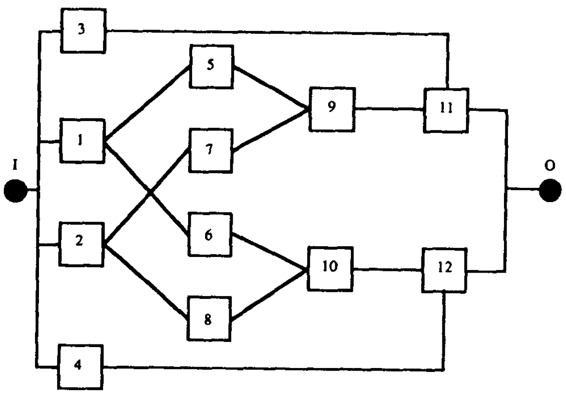
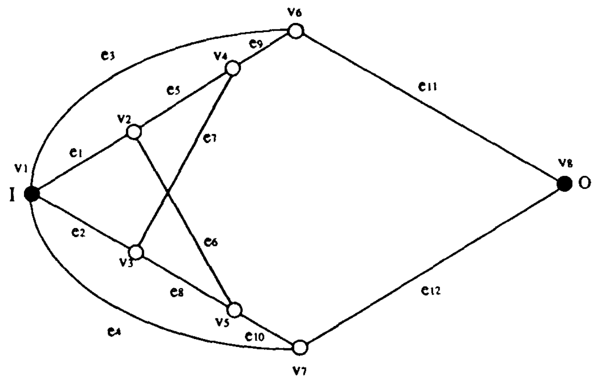


Figure 3. Reliability (Logic) Block Diagram of Power Supply



$[N=G(V,E)$; $p_i = p$ for all $i=1-5$; $(s,t) = (V_1, V_8)]$
Figure 4. Network Representation of Power Supply

b. $C_2 = \{(e_3,e_{11}),(e_4,e_{12})\}$,
c. $C_3 = \{$remaining pairs of edges$\}$.

MRI of each edge and JRI of pairs containing edge e_3 are illustrated in tables 1 & 2 for various values of $p_i=p$. Tables 1 & 2 illustrate the results of theorems 1 & 2.

TABLE 1
MRI of Each Edge for Various Values of p

	Edge					
p	e_1 (e_2)	e_3 (e_4)	e_5 (e_8)	e_6 (e_7)	e_9 (e_{10})	e_{11} (e_{12})
.1	1.774E-3	9.735E-1	8.503E-4	6.690E-3	1.027E-3	1.138E-1
.2	1.188E-2	1.808E-1	5.900E-3	1.835E-2	8.254E-3	2.354E-1
.3	3.084E-2	2.392E-1	1.635E-2	2.953E-2	2.569E-2	3.431E-1
.4	5.044E-2	2.625E-1	2.861E-2	3.773E-2	5.017E-2	4.188E-1
.5	5.908E-2	2.446E-1	3.564E-2	3.955E-2	7.080E-2	4.468E-1
.6	5.075E-2	1.904E-1	3.235E-2	3.272E-2	7.571E-2	4.185E-1
.7	3.049E-2	1.180E-1	2.048E-2	1.972E-2	6.091E-2	3.393E-1
.8	1.090E-2	5.230E-2	7.771E-3	7.321E-3	3.404E-2	2.280E-1
.9	1.348E-3	1.186E-2	1.060E-3	1.002E-3	9.623E-3	1.088E-1
.91	9.683E-4	9.472E-3	7.725E-4	7.324E-4	7.852E-3	9.726E-2
.92	6.682E-4	7.373E-3	5.417E-4	5.152E-4	6.245E-3	8.582E-2
.93	4.387E-4	5.558E-3	3.616E-4	3.452E-4	4.809E-3	7.452E-2
.94	2.698E-4	4.018E-3	2.270E-4	2.174E-4	3.551E-3	6.337E-2
.95	1.518E-4	2.743E-3	1.305E-4	1.258E-4	2.476E-3	5.237E-2
.96	7.534E-5	1.725E-3	6.646E-5	6.437E-5	1.590E-3	4.153E-2
.97	3.088E-5	9.531E-4	2.772E-5	2.700E-5	8.969E-4	3.087E-2
.98	8.702E-6	4.157E-4	8.285E-6	7.927E-6	3.994E-4	2.039E-2
.99	1.132E-6	1.019E-4	1.013E-6	1.073E-6	9.996E-5	1.010E-2

Table 2 shows that JRI is an appropriate measure of the importance of edge-pairs in contributing to STR. In other words, when a pair is the one which contains edge e_3, the most appropriate edge in this pair can be identified by examining JRI as illustrated in table 2. Table 2 indicates that either e_4 or e_{12} requires more reliability in order to compensate for the deterioration of e_3. When e_3 improves, however, it indicates that improvement of e_{11} causes improvement of e_3 to become more valuable.

APPENDIX (PROOFS)

Proof of Lemma 1

The number of all possible states of 2 edges is 4. By pivoting on edges e_i & e_j, $R(G)$ is:

$$R(G) = p_i p_j R(G|X_i=X_j=1) + p_i q_j R(G|X_i=1,X_j=0)$$
$$+ q_i p_j R(G|X_i=0,X_j=1) + q_i q_j R(G|X_i=X_j=0)$$
$$= p_i p_j R(G^*i^*j) + p_i q_j R(G^*i-j) + q_i p_j R(G^*j-i)$$
$$+ q_i q_j R(G-i-j). \quad \text{(A-1)}$$

Differentiating both sides of (A-1) w.r.t. p_i & p_j yields:

TABLE 2
JRI of (e_3, e_j) for various values of p, $J = 1 \sim 12$, $j \neq 3$

	Pair										
p	(e_3,e_1)	(e_3,e_2)	(e_3,e_4)	(e_3,e_5)	(e_3,e_6)	(e_3,e_7)	(e_3,e_8)	(e_3,e_9)	(e_3,e_{10})	(e_3,e_{11})	(e_3,e_{12})
.1	−1.084E-3	−1.084E-3	−8.388E-3	−8.522E-4	−6.457E-3	−7.340E-3	3.070E-5	−9.825E-4	8.230E-4	9.734E-1	−2.445E-2
.2	−8.882E-3	−8.882E-3	−2.966E-2	−6.162E-3	−1.579E-2	−2.173E-2	−2.228E-4	−8.450E-3	4.422E-3	9.021E-1	−7.952E-2
.3	−2.832E-2	−2.832E-2	−6.-72E-2	−1.882E-2	−2.165E-2	−3.765E-2	−2.814E-3	−3.023E-2	7.534E-3	7.910E-1	−1.458E-1
.4	−5.741E-2	−5.741E-2	−9.610E-2	−3.817E-2	−2.481E-2	−5.338E-2	−9.599E-3	−7.078E-2	4.298E-1	6.434E-1	−2.101E-1
.5	−8.496E-2	−8.496E-2	−1.240E-1	−5.762E-2	−2.637E-2	−6.543E-2	−1.855E-2	−1.240E-1	−6.836E-1	4.717E-1	−2.588E-1
.6	−9.547E-2	−9.547E-2	−1.300E-1	−6.661E-2	−2.468E-2	−6.753E-2	−2.376E-2	−1.717E-1	−2.005E-2	2.998E-1	−2.771E-1
.7	−7.986E-2	−7.986E-2	−1.064E-1	−5.769E-2	−1.782E-2	−5.518E-2	−2.034E-2	−1.905E-1	−2.607E-2	1.561E-1	−2.540E-1
.8	−4.489E-2	−4.489E-2	−6.154E-2	−3.401E-2	−7.997E-3	−3.175E-2	−1.025E-2	−1.647E-1	−2.018E-2	5.987E-2	−1.901E-1
.9	−1.185E-2	−1.185E-2	−1.809E-2	−9.764E-3	−1.243E-3	−9.189E-3	−1.818E-3	−9.533E-2	−7.401E-7	1.228E-2	−9.941E-2
.91	−9.542E-3	−9.542E-3	−1.483E-2	−7.960E-3	−9.110E-4	−7.515E-3	−1.357E-3	−8.658E-2	−6.190E-3	9.741E-7	−8.962E-2
.92	−7.478E-3	−7.478E-3	−1.185E-2	−6.323E-3	−6.408E-4	−5.991E-3	−9.728E-4	−7.758E-2	−5.046E-3	7.537E-3	−7.977E-2
.93	−5.665E-3	−5.665E-3	−9.165E-3	−4.861E-3	−4.280E-4	−4.626E-3	−6.635E-4	−6.837E-2	−3.984E-3	5.652E-3	−6.987E-2
.94	−4.108E-3	−4.108E-3	−6.801E-1	−3.583E-3	−2.671E-4	−3.426E-3	−4.246E-4	−5.897E-2	−3.017E-3	4.067E-3	−5.993E-2
.95	−2.809E-3	−2.809E-3	−4.769E-3	−2.494E-1	−1.525E-4	−2.398E-3	−2.484E-4	−4.940E-2	−2.158E-3	2.766E-3	−4.997E-2
.96	−1.766E-3	−1.766E-3	−3.081E-3	−1.598E-3	−7.647E-5	−1.546E-3	−1.285E-4	−3.970E-2	−1.422E-3	1.735E-3	−3.999E-2
.97	−9.729E-4	−9.729E-4	−1.749E-3	−8.999E-4	−3.129E-5	−8.768E-4	−5.466E-5	−2.987E-2	−8.237E-4	9.5591-4	−3.000E-2
.98	−4.224E-4	−4.224E-4	−7.848E-4	−3.999E-4	−9.000E-6	−3.929E-4	−1.609E-5	−1.996E-2	−3.769E-4	4.165E-4	−2.000E-2
.99	−1.029E-4	−1.029E-4	−1.981E-4	−1.000E-4	−1.132E-6	−9.900E-5	−1.967E-6	−9.995E-3	−9.704E-5	1.020E-4	−1.000E-2

$$I_G(i,j) = \partial^2 R(G)/\partial p_i \partial p_j = R(G^*i^*j) - R(G^*i-j)$$

$$- R(G^*j-i) + R(G-i-j). \qquad Q.E.D.$$

Proof of Lemma 2

Since $G^*j-i = (G-i)^*j$, (3) can be rewritten as $I_G(i,j) = R(G^*i^*j) - R(G^*i-j) - \{R[(G-i)^*j] - R(G-i-j)\}$. Consider MRI of e_j in G^*i:

$$I_{G^*i}(j) = \partial R(G^*i)/\partial p_j$$

$$= \partial[p_j(R(G^*i^*j) + q_j R(G^*i-j)]/\partial p_j$$

$$= R(G^*i^*j) - R(G^*i-j). \qquad \text{(A-2)}$$

Similarly,

$$R(G-i^*j) - R(G-i-j) = I_{G-i}(j). \qquad \text{(A-3)}$$

Therefore,

$$I_G(i,j) = I_{G^*i}(j) - I_{G-i}(j).$$

Interchanging the roles of i & j yields:

$$I_G(i,j) = I_{G^*j}(i) - I_{G-j}(i). \qquad Q.E.D.$$

Proof of Lemma 3

We begin with $I_G(i)$, then —

$$I_G(i) = \partial R(G)/\partial p_i = \partial\{p_j R(G^*j) + q_j R(G-j)\}/\partial p_i$$

$$= p_j \partial R(G^*j)/\partial p_i + q_j \partial R(G-j)/\partial p_i$$

$$= p_j I_{G^*j}(i) + q_j I_{G-j}(i)$$

$$= p_j \{I_{G^*j}(i) - I_{G-j}(i)\} + I_{G-j}(i). \qquad \text{(A-4)}$$

Substituting (4) into (A-4) yields:

$$I_G(i) = p_j I_G(i,j) + I_{G-j}(i). \qquad \text{(A-5)}$$

Arranging the terms in (A-5) yields:

$$I_G(i,j) = [I_G(i) - I_{G-j}(i)]/p_j.$$

Interchanging the roles of i & j yields:

$$I_G(i,j) = [I_G(j) - I_{G-i}(j)]/p_i. \qquad Q.E.D.$$

Proof of Corollary 1

The value of MRI in any graph is between 0 & 1 [2]. From lemma 2, $I_G(i,j) = I_{G^*j}(i) - I_{G-j}(i)$. Since $0 \leq I_{G^*j}(i) \leq 1$ and $0 \leq I_{G-j}(i) \leq 1$, the value of $I_G(i,j)$ is between -1 & 1. Q.E.D

Proof of Lemma 4

Proof is similar to that of lemma 1, so it is easy to see that:

$$I'_G(i,j) = F(G-i-j) - F(G^*i-j) - F(G^*j-i)$$

$$+ F(G^*i^*j)$$

Since $F(G) = 1 - R(G)$,

$$I'_G(i,j) = 1 - R(G-i-j) - (1 - R(G^*i-j))$$
$$- (1 - R(G^*j-i)) + (1 - R(G^*i^*j))$$
$$= -[R(G^*i^*j) - R(G^*i-j) - R(G^*j-i)$$
$$+ R(G-i-j)]$$
$$= -I_G(i,j) \qquad Q.E.D.$$

Proof of Lemma 5

Proof is similar to that of lemma 2, so JFI is easily expressed by:

$$I'_G(i,j) = F(G-j-i) - F((G-j)^*i) - [F(G^*j-i)$$
$$- F(G^*j^*i)] \tag{A-6}$$

Since $F(G) = q_jF(G-j) + p_jF(G^*j)$,

$$\partial F(G)/\partial q_j = I'_G(j) = F(G-j) - F(G^*j).$$

Thus,

$$I'_{G-j}(i) = \partial F(G-j)/\partial q_i = F(G-i-j) - F(G-j^*i) \tag{A-7}$$

$$I'_{G^*j}(i) = \partial F(G^*j)/\partial q_i = F(G^*j-i) - F(G^*i^*j). \tag{A-8}$$

Therefore, from (A-6) - (A-8),

$$I'_G(i,j) = I'_{G-j}(i) - I'_{G^*j}(i).$$

Interchanging the roles of i & j yields:

$$I'_G(i,j) = I'_{G-i}(j) - I'_{G^*i}(j). \qquad Q.E.D.$$

Proof of Lemma 6

Proof is similar to that of lemma 3, so:

$$I'_G(i) = \partial F(G)/\partial q_i = \partial[q_j(F(G-j) + p_jF(G^*j))]/\partial q_i$$
$$= p_jI'_{G^*j}(i) + q_jI'_{G-j}(i)$$
$$= I'_{G^*j}(i) + q_j[I'_{G-j}(i) - I'_{G^*j}(i)]. \tag{A-9}$$

From lemma 5,

$$I'_G(i,j) = [I'_G(i) - I'_{G^*j}(i)]/q_j.$$

Interchanging the roles of i & j yields:

$$I'_G(i,j) = [I'_G(j) - I'_{G^*i}(j)]/q_i. \qquad Q.E.D.$$

Proof of Corollary 2

a. When edges (e_i,e_j) are in parallel, then e_i is irrelevant in G^*j, so $I_{G^*j}(i) = 0$. Then, from lemma 2,

$$I_G(i,j) = I_{G^*j}(i) - I_{G-j}(i) = 0 - I_{G-j}(i) \le 0.$$

b. When edges (e_i,e_j) are in series, then e_i is irrelevant in $G-j$, so $I_{G-j}(i) = 0$. Then, from lemma 2,

$$I_G(i,j) = I_{G^*j}(i) - I_{G-j}(i) = I_{G^*j}(i) \ge 0. \qquad Q.E.D.$$

Proof of Theorem 1

It is sufficient to show that $I_G(i) - I_{G-j}(i) \le 0$ in (6). Let there be paths:

A_{k_1} — contains e_i, $k_1 = 1, \ldots, n_1$,

A_{k_2} — contains e_j, $k_2 = n_1+1, \ldots, n_2$,

A_{k_3} — contains neither e_i nor e_j, $k_3 = n_2+1, \ldots, n_3$.

Since there exists no path containing both e_i & e_j, STR of G is:

$$R(G) = \Pr\left\{\left(\bigcup_{k_1=1}^{n_1} A_{k_1}\right) \cup \left(\bigcup_{k_2=n_1+1}^{n_2} A_{k_2}\right)\right.$$
$$\left.\cup \left(\bigcup_{k_3=n_2+1}^{n_3} A_{k_3}\right)\right\}$$
$$= \Pr\{\cup A_{k_1})\} + \Pr\{(\cup A_{k_2})\} + \Pr\{(\cup A_{k_3})\}$$
$$- \Pr\{(\cup A_{k_1}) \cap (\cup A_{k_2})\} - \Pr\{(\cup A_{k_2}) \cap (\cup A_{k_3})\}$$
$$- \Pr\{(\cup A_{k_1}) \cap (\cup A_{k_3})\}$$
$$+ \Pr\{(\cup A_{k_1}) \cap (\cup A_{k_2}) \cap (\cup A_{k_3})\}. \tag{A-10}$$

Since $I_G(i) = \partial R(G)/\partial p_i$ and the term containing p_i is the probability of the set containing A_{k_1}, we have,

$$I_G(i) = \partial \Pr\{(\cup A_{k_1})\}/\partial p_i - \partial \Pr\{(\cup A_{k_1}) \cap (\cup A_{k_2})\}/\partial p_i$$
$$- \partial \Pr\{(\cup A_{k_1}) \cap (\cup A_{k_3})\}/\partial p_i$$
$$+ \partial \Pr\{(\cup A_{k_1}) \cap (\cup A_{k_2}) \cap (\cup A_{k_3})\}/\partial p_i. \tag{A-11}$$

The reliability of $G-j$ is:

$$R(G-j) = \Pr\{(\cup A_{k_1}) \cap (\cup A_{k_3})\}$$
$$= \Pr\{(\cup A_{k_1})\} + \Pr\{(\cup A_{k_3})\} - \Pr\{(\cup A_{k_1}) \cap (\cup A_{k_3})\}. \tag{A-12}$$

Differentiating (A-12) w.r.t. p_i yields,

$$\partial R(G-j)/\partial p_i = \partial \Pr\{(\cup A_{k_1})\}/\partial p_i$$
$$- \partial \Pr\{(\cup A_{k_1}) \cap (\cup A_{k_3})\}/\partial p_i. \tag{A-13}$$

Subtracting (A-13) from (A-11) yields,

$$\partial R(G)/\partial p_i - \partial R(G-j)/\partial p_i = \partial \Pr\{(\cup A_{k_1}) \cap (\cup A_{k_2}) \cap (\cup A_{k_3})\}/\partial p_i - \partial \Pr\{(\cup A_{k_1}) \cap (\cup A_{k_3})\}/\partial p_i. \quad \text{(A-14)}$$

Rewrite $\Pr\{(\cup A_{k_1}) \cap (\cup A_{k_2}) \cap (\cup A_{k_3})\}$ in r.h.s. of (A-14):

$$\Pr\{(\cup A_{k_1}) \cap (\cup A_{k_2}) \cap (\cup A_{k_3})\} = \Pr\{(\cup A_{k_1}) \cap (\cup A_{k_2})\} \cdot \Pr\{(\cup A_{k_3}) | (\cup A_{k_1}) \cap (\cup A_{k_2})\}. \quad \text{(A-15)}$$

Since $0 \le \Pr\{(\cup A_{k_3}) | (\cup A_{k_1}) \cap (\cup A_{k_2})\} \le 1$, for all p_i,

$$\Pr\{(\cup A_{k_1}) \cap (\cup A_{k_2}) \cap (\cup A_{k_3})\} \le \Pr\{(\cup A_{k_1}) \cap (\cup A_{k_2})\},$$

for all p_i.

Thus,

$$\partial \Pr\{(\cup A_{k_1}) \cap (\cup A_{k_2}) \cap (\cup A_{k_3})\}/\partial p_i \le \partial \Pr\{(\cup A_{k_1}) \cap (\cup A_{k_2})\}/\partial p_i. \quad \text{(A-16)}$$

Therefore,

$$\partial R(G)/\partial p_i - \partial R(G-j)/\partial p_i \le 0. \quad \text{(A-17) } Q.E.D.$$

Proof of Theorem 2

Since $I'_G(i,j) = -I_G(i,j)$ from lemma 4, we need to show that $I'_G(i,j) \le 0$. Then, from lemma 6, it is sufficient to show that,

$$I'_G(i) - I'_{G^*j}(i) \le 0.$$

Let there be cuts:

C_{k_1} — contains e_i, $k_1 = 1, \ldots, m_1$,

C_{k_2} — contains e_j, $k_2 = m_1+1, \ldots, m_2$,

C_{k_3} — contains neither e_i nor e_j, $k_3 = m_2+1, \ldots, m_3$.

Then, $F(G)$ is:

$$F(G) = \Pr\left\{\left(\bigcup_{k_1=1}^{m_1} C_{k_1}\right) \cup \left(\bigcup_{k_2=m_1+1}^{m_2} C_{k_2}\right) \cup \left(\bigcup_{k_3=m_2+1}^{m_3} C_{k_3}\right)\right\}$$

$$= \Pr\{\cup C_{k_1})\} + \Pr\{(\cup C_{k_2})\} + \Pr\{(\cup C_{k_3})\} - \Pr\{(\cup C_{k_1}) \cap (\cup C_{k_2})\} - \Pr\{(\cup C_{k_2}) \cap (\cup C_{k_3})\} - \Pr\{(\cup C_{k_1}) \cap (\cup C_{k_3})\} + \Pr\{(\cup C_{k_1}) \cap (\cup C_{k_2}) \cap (\cup C_{k_3})\}. \quad \text{(A-18)}$$

The rest of the proof is similar to that of theorem 1, so:

$$I'_G(i) = \partial F(G)/\partial q_i = \partial \Pr\{(\cup C_{k_1})\}/\partial q_i - \partial \Pr\{(\cup C_{k_1}) \cap (\cup C_{k_2})\}/\partial q_i - \partial \Pr\{(\cup C_{k_1}) \cap (\cup C_{k_3})\}/\partial q_i + \partial \Pr\{(\cup C_{k_1}) \cap (\cup C_{k_2}) \cap (\cup C_{k_3})\}/\partial q_i. \quad \text{(A-19)}$$

$$F(G^*j) = \Pr\{(\cup C_{k_1}) \cap (\cup C_{k_3})\} = \Pr\{(\cup C_{k_1})\} + \Pr\{(\cup C_{k_3})\} - \Pr\{(\cup C_{k_1}) \cap (\cup C_{k_3})\}. \quad \text{(A-20)}$$

$$I'_{G^*j}(i) = \partial F(G^*j)/\partial q_i = \partial \Pr\{(\cup C_{k_1})\}/\partial q_i - \partial \Pr\{(\cup C_{k_1}) \cap (\cup C_{k_3})\}/\partial q_i. \quad \text{(A-21)}$$

From (A-19) & (A-21),

$$I'_G(i) - I'_{G^*j}(i) \le 0. \quad \text{(A-22) } Q.E.D.$$

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Joint Reliability-Importance of Two Edges in an Undirected Network

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