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Safety stock planning under causal demand forecasting

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ABSTRACT

Mainstream inventory management approaches typically assume a given theoretical demand distribution and estimate the required parameters from historical data. A time series based framework uses a forecast (and a measure of forecast error) to parameterize the demand model. However, demand might depend on many other factors rather than just time and demand history. Inspired by a retail inventory management application where customer demand, among other factors, highly depends on sales prices, price changes, weather conditions, this paper presents two data-driven frameworks to set safety stock levels when demand depends on several exogenous variables. The first approach uses regression models to forecast demand and illustrates how estimation errors in this framework can be utilized to set required safety stocks. The second approach uses Linear Programming under different objectives and service level constraints to optimize a (linear) target inventory function of the exogenous variables. We illustrate the approaches using a case example and compare two methods with respect to their ability to achieve target service levels and the impact on inventory levels in a numerical study. We show that considerable improvements of the overly simplifying method of moments are possible and that the ordinary least squares approach yields a better performance than the LP-method, especially when the data sample for estimation is small and the objective is to satisfy a non-stockout probability constraint. However, if some of the standard assumptions of ordinary least squares regression are violated, the LP approach provides more robust inventory levels.

1. Introduction

Forecasting demand is undoubtedly one of the main challenges in supply chain management. Inaccuracy of forecasts leads to overstocks and respective markdowns or shortages and unsatisfied customers. To secure supply chain performance against forecast inaccuracy, an important countermeasure is safety stocks. The size of safety stocks required to obtain a certain customer service level depends on the degree of demand uncertainty and the corresponding forecast errors. Demand misspecification significantly affects the whole supply chain but its costs can be reduced by minimizing forecast errors (see Hosoda and Disney, 2009). An improvement of forecasts therefore directly results in inventory savings and service level improvements.

Mainstream inventory management models require specification of a demand distribution (e.g., normal, gamma; for an overview, see Silver et al., 1998) and are solved using stochastic calculus to derive required (safety) inventory levels. Alternatively, time series forecasting techniques are used and different measures for the resulting forecast error are then utilized to set safety stocks (see, e.g., Zinn and Marmorstein, 1990; Krupp, 1997). In practice, both the type of demand distribution and its parameters are unknown and need to be estimated. As a result, theory and practice often diverge, since theory does not sufficiently address the needs of practice and practice uses overly simplistic approaches to overcome the problem (Lee and Billington, 1992; Wagner, 2002; Tiwari and Gavirneni, 2007).

Accounting for estimation errors from having only a limited sample of historical demand observations is addressed by correcting the required estimation of the true demand standard deviation in Ritchken and Sankar (1984). The idea of estimating inventory control parameters treating historical demand as deterministic values and then optimizing performance in hindsight was proposed for the safety stock planning problem by Spicher (1975), further evaluated in Kässmann et al. (1986) and developed for an (s,S) inventory system in Iyer and Schrage (1992). Even though most forecasting and inventory models assume that demand is sampled from one distribution, it is more realistic that the kind of distribution may vary (Scarf, 1958). In order to circumvent the requirement to assume (and test) a theoretical demand distribution, a different approach is followed by robust, distribution free inventory models where the inventory is set to maximize worst case performance (for a review see

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Gallego and Moon, 1993). By giving less weight to outlier observations (e.g., using trimming), higher profits are sacrificed for obtaining a solution with lower variability. Lower variability thus reduces the risk associated with an inventory decision. More recently, Bertsimas and Thiele (2005, 2006) introduced a robust, data-driven approach that uses Linear Programming (LP) to obtain the required inventory level decisions based on historical data without making distributional assumptions. However, their approach works with a sample of observed demands only and does not take other factors such as price into account that might explain demand variations (Fildes et al., 2008).

Our research is motivated by an inventory planning problem in the retail sector where demand forecasting just based on historical demand did not provide satisfactory results due to strong demand dependency on other factors like prices, price changes, weather and others. For example, apart from price effects, retailers observe a strong increase in demand for salad on warm summer days in the barbecue season. Since time series models are unable to capture these effects, other forecasting methods were required. Econometrics provides a huge toolbox for estimation and statistical analysis, especially with respect to forecast errors. These methods allow decreasing safety stock by being able to explain a larger portion of the demand variability.

The contribution of this work is to discuss and promote integrated causal demand forecasting and inventory management in addition to the mainstream time series based approach. Our inventory planning approach accounts for the causal relationship between demand and external factors in order to explain larger portions of demand variability. We extend the data-driven approach to causal forecasting and compare it to existing methods such as regression analysis and the method of moments (MM). The data-driven approach directly estimates optimal inventory and safety stocks from historical demands and external factors whereas regression analysis and the method of moments is divided into demand prediction and inventory level determination. Demand variation is thereby separated into explained variation (with no need for protection by safety stock) and remaining (unexplained) variation which requires safety inventory.

In the following, Section 2 reviews the required basics from safety stock planning and the ordinary least squares framework and Section 3 presents the data-driven Linear Programming approach for integrated demand estimation and safety inventory planning. An illustrative example and a numerical comparison study including real data show the advantages and disadvantages of the application of the two proposed approaches in Section 4.

2. Safety stock basics and least squares estimation

We consider a perishable item, newsvendor model with zero lead time and periodic review. For demand estimation, $i=1,2,\ldots,n$ historical demand observations D_i are available. We assume that demand is fully observable. Each demand observation i can be partially explained by a set of m explanatory variables X_{ji}, $j=1,\ldots,m$.

In a newsvendor context under profit maximization, cost minimization, or a service level constraint, the target inventory level (single period order quantity) B and the implied safety stock level SI are set as follows. Assume that period demand follows a (continuous) theoretical demand distribution with probability density function f and cumulative distribution function F. Let μ and σ denote the respective mean and standard deviation. In a cost-based framework, let ν denote the unit penalty cost for not satisfying a demand (underage cost) and h be the unit inventory holding cost (overage cost). In a service level (SL) framework, let P_1 denote a required non-stockout probability (α-service level) and P_2 denote a required fill-rate (β-service level).

$$B=\mu+SI \tag{1}$$

$$F(B)=P_1=\frac{\nu}{\nu+h} \tag{2}$$

$$1-\frac{1}{\mu}\int_B^{\infty}(x-B)f(x)dx=P_2 \tag{3}$$

The target inventory level B thus consists of two components: mean demand and safety inventory SI (1) and depends on the target P_1 or P_2 service level or, equivalently, on penalty and holding costs. Eqs. (2) and (3) represent the standard formulas to set target inventory levels under service level constraints (see, e.g., Silver et al., 1998). For the widely used case of normally distributed demands (forecast errors), letting $f_{0,1}$ and $F_{0,1}$ denote the respective standardized normal density and distribution, the required safety factors under specified service level constraints become

$$B=\mu+k\sigma,\quad k_1=F_{0,1}^{-1}(P_1)=F_{0,1}^{-1}\left(\frac{\nu}{\nu+h}\right),$$
$$k_2=G^{-1}\left(\frac{(1-P_2)\mu}{\sigma}\right),\quad G(k_2)=f_{0,1}(k_2)-k_2(1-F_{0,1}(k_2)) \tag{4}$$

If only a sample of historical demand observations is available, besides making distributional assumptions about f, we need estimators for the parameters μ and σ in order to evaluate (1) and (2). These estimators can either directly be obtained from calculating mean and standard deviation of the sample as in the method of moments or as a function of other variables.

The classic approach in Econometrics to estimate the functional relationship between demand and any hypothesized factors is ordinary least squares (OLS) regression. Demand D_i is expressed as a (linear) function of explaining variables X_{ji} with the objective to minimize the sum of the squared errors. In the following the main results required for the purpose of safety stock setting are illustrated for a single explanatory variable first and then summarized for the multi-variable case using matrix notation.

2.1. The single-variable case

With $m=1$, demand can be expressed as the sum of an absolute term, a dependent term, and an error term u_i.

$$D_i=\beta_0+\beta_1X_i+u_i \tag{5}$$

Given the observations (X_i, D_i), the OLS estimators for β_0 and β_1 are (see, e.g., Gujarati and Porter, 2009):

$$\hat{\beta}_1=\frac{\sum_{i=1}^n(X_i-\overline{X})(D_i-\overline{D})}{\sum_{i=1}^n(X_i-\overline{X})^2} \tag{6}$$

$$\hat{\beta}_0=\overline{D}-\hat{\beta}_1\overline{X} \tag{7}$$

where $\overline{D}$ denotes the average observed demand and $\overline{X}$ the average value observed for the explanatory variable.

For the purpose of inventory management and safety stock planning, the inventory for a given situation with known value of the explanatory variable X_0 (e.g., the next day's sales price) is required. For this purpose, the estimated standard deviation of $D(X_0)$ is of most interest.

$$\hat{\sigma}=\sqrt{\frac{\sum_{i=1}^n(D_i-\hat{\beta}_0-\hat{\beta}_1X_i)^2}{n-2}} \tag{8}$$

$$\hat{\mathrm{V}}\mathrm{ar}(D(X_0))=\hat{\sigma}^2\left(1+\frac{1}{n}+\frac{(X_0-\overline{X})^2}{\sum_{i=1}^n(X_i-\overline{X})^2}\right) \tag{9}$$

Expression (9) consists of two parts, an estimator for the standard deviation of the error term u, as commonly used in inventory models with theoretical (normal) demand distribution, and the standard deviation of the estimator which accounts for the additional risk of the sample. Obviously, for $n \to \infty$ the latter term converges to zero. Compared to using the method of moments for estimating the standard deviation, the percentage reduction in safety stock using regression can be stated by R^2 as the fraction of explained variation of total variation in observations. In an idealized setting where the regression function perfectly explains demand fluctuations, R^2 would be equal to 1 and no safety stock would be required. Any demand variation could be explained by changes in the exogenous variables that are known in advance. In the opposite case where the portion of explained variation approaches 0, results obtained by regression analysis are identical to those generated by the method of moments.

According to the classical OLS assumptions, we assume that the error term u_i is normally distributed with mean zero and constant variance σ^2 (homoscedasticity assumption). Furthermore, error terms of different observations are independent. Given that these assumptions hold, the least squares estimator $\hat{\beta}$ is the best linear unbiased estimator (BLUE) and linear in D_i. Its expected value equals the true value of β, i.e. it is unbiased. Furthermore, the estimator for the coefficient β has the smallest variance compared to other linear unbiased estimators, i.e. it is efficient or "best". For a detailed overview on assumptions and methods, see Gujarati and Porter (2009).

2.2. The multi-variable case

In the general (linear) case with multiple regression variables, we present the results illustrated in the previous section in matrix notation. The demand model becomes

$$D_i = \beta_0 + \sum_{j=1}^{m} \beta_j X_{ji} + u_i \tag{10}$$

$$D = X\beta + u \tag{11}$$

where D is the vector of demands, β is the vector of m coefficients, X is the $m \times n$ matrix of observations of explanatory variables and u is the vector of errors. The estimator for the coefficients β_j is

$$\hat{\beta} = (X'X)^{-1}X'D \tag{12}$$

with X′ denoting the transpose of X.

The required variance estimators become (see, e.g., Gujarati and Porter, 2009)

$$E(u'u) = \sigma^2 (X'X)^{-1} \tag{13}$$

$$\hat{\sigma}^2 = \frac{E(u'u)}{n-m} \tag{14}$$

$$\hat{\mathrm{V}}\mathrm{ar}(D(X_0)) = \hat{\sigma}^2 (1 + X_0'(X'X)^{-1}X_0) \tag{15}$$

2.3. Violations of ordinary least squares assumptions

In real-world applications, several of the above stated OLS assumptions might be violated. In the following, we discuss heteroscedasticity, non-normal residuals, and errors with non-zero mean. Violations can be corrected either by adjusting the estimation procedure itself or by altering inventory planning.

If the zero mean assumption for the error terms does not hold, there are two possibilities. Either all error terms have a common mean other than zero. This simply adds to the constant, resulting in a biased estimate for the intercept $\hat{\beta}_0$ but with no effect on the estimation procedure. On the other hand, if the mean varies with each observation, there are more parameters than observations and the regression coefficients are biased which results in holding higher safety stocks to account for the larger portion of unexplained variability.

It is a common phenomenon in retail that one observes larger variations in demand when prices are especially low under promotions or extreme weather conditions or vice versa. As a consequence, the error term is heteroscedastic and the least squares estimator is no longer BLUE. It is still unbiased and linear, but not efficient. This means that standard OLS does not yield the minimum variance estimator (Gujarati and Porter, 2009). A large variance then results in building up additional safety stock. To compensate this effect, we can adjust the estimation procedure according to Park (1966). This method assumes the following functional form for the error term:

$$\sigma_i^2 = \sigma^2 X_i^{\theta} e^{r_i} \tag{16}$$

with r_i as the error in estimating the variance of the original error term u. Then, using $\hat{u}_i^2$ as an approximation for the unknown σ_i^2,

$$\ln \hat{u}_i^2 = \ln \sigma^2 + \theta \ln X_i + r_i \tag{17}$$

serves to estimate parameter θ with OLS-regression. The demand function is subsequently divided by $X_i^{\theta/2}$ to establish homoscedasticity and regression analysis is performed on the transformed equation.

Finally, we assess the impact of non-normal demand on our model estimation approaches. Apart from normal demand, another widely used distribution in inventory theory is the gamma distribution in which case the OLS estimators are still BLUE. We can thus adjust the inventory planning approach for non-normal residuals in order to better achieve the required service level.

3. Data-driven linear programming

Rather than following the indirect (parametric) approach to first estimate a demand model and then determine the inventory level B, we propose an integrated approach and assume that the required inventory level B is a linear function of the explanatory variables $X_0 = (X_{j0})_{j=1,\cdots,m}$given by

$$B = \beta_0 + \sum_{j=1}^{m} \beta_j X_{j0}. \tag{18}$$

The decision variables are the parameters β_j and indirectly the resulting inventory levels y_i and satisfied demands s_i for each demand observation i. The problem can be formulated as a newsvendor model with a cost minimization objective or with a service level constraint.

3.1. The cost model

Assume a single period newsvendor framework with unit holding cost h and unit penalty cost v. The goal is to find the target inventory function parameters such that total costs comprising of inventory holding cost and shortage penalties are minimized. Other objectives like profit maximization can be incorporated in a straightforward manner. Further, logical constraints relate inventories, satisfied demands, and demand observations.

$$\min C = \sum_{i=1}^{n} (hy_i + v(D_i - s_i)) \tag{19}$$

$$\text{s.t. } y_i \geq \sum_{j=0}^{m} \beta_j X_{ji} - D_i \quad i=1,\cdots,n \tag{20}$$

$$s_i \leq D_i \quad i=1,\cdots,n \tag{21}$$

$$s_i \leq \sum_{j=0}^{m} \beta_j X_{ji} \quad i=1,\cdots,n \tag{22}$$

$$s_i, y_i \geq 0; \ \ \beta_j \in \Re \quad i=1,\cdots,n; \ \ j=0,\cdots,m \tag{23}$$

The objective function (19) is the sum of holding costs and penalty costs overall individual demand observation and their positioning above (inventory y_i) or below (shortage $D_i - s_i$) the linear target inventory level function. Eq. (20) together with the minimization objective determines excess inventory y_i corresponding to each demand D_i. (21) and (22) enforce that the sales quantity s_i under a given demand is equal to the minimum of demand D_i and supply.

3.2. The service level model

In addition to the variables defined in Section 3.1, let γ_i define a binary demand satisfaction indicator for observation i. In the following we assume either a required non-stockout probability P_1, or a fill rate P_2. The goal is to find the target inventory function parameters such that inventory holding costs are minimized subject to a service level constraint.

$$\min C = \sum_{i=1}^{n} h y_i \tag{24}$$

$$\text{s.t. } y_i \geq \sum_{j=0}^{m} \beta_j X_{ji} - D_i \quad i=1,\cdots,n \tag{25}$$

$$s_i \leq D_i \quad i=1,\cdots,n \tag{26}$$

$$s_i \leq \sum_{j=0}^{m} \beta_j X_{ji} \quad i=1,\cdots,n \tag{27}$$

$$D_i - \gamma_i M \leq \sum_{j=0}^{m} \beta_j X_{ji} \quad i=1,\cdots,n \tag{28}$$

$$\sum_{i=1}^{n} \gamma_i \leq n(1-P_1) \tag{29}$$

$$\sum_{i=1}^{n} s_i \geq P_2 \sum_{i=1}^{n} D_i \tag{30}$$

$$s_i, y_i \geq 0, \ \ \gamma_i \in \{0,1\}, \ \ \beta_j \in \Re \quad i=1,\cdots,n \quad j=1,\cdots,m \tag{31}$$

In (28) M is a large number (e.g., the maximum demand observation). If a non-stockout probability is required, then (28) ensures that the satisfaction indicator γ_i for demand D_i becomes one if demand exceeds supply. Summing overall observations n, (29) states that a maximum of $(1-P_1)n$ observations are allowed to result in a stockout. Note that because of integer n, certain values of the required service level P_1 will result in an overachievement of service. Under a fill-rate service constraint expressed in (30), total sales have to exceed the fraction P_2 of total demand.

4. Numerical examples

In a controlled simulation experiment, we compare the proposed methods. First, we generate data where all OLS assumptions are met, vary the sample size and then subsequently relax the OLS assumptions. Finally, we apply the proposed methods to real data.

In the following, we assume that true demand is a (linear) function of the sales price p plus some normally distributed error term u with zero mean and variance σ^2, $D = a - bp + u$. For a general review of price-dependent demand, see Petruzzi and Dada (1999). In practice, the parameters a, b, and σ^2 are unknown and need to be estimated. For each randomly generated problem instance, we normalize the price range to $p \in [0,1]$ and draw the known parameters a, b and σ as follows:

- market size $a \sim U(1000, 2000)$,
- slope $b \sim U(500, 1000)$,
- demand volatility is generated such that the coefficient of variation (cv) at mean price p equals 0.3 or 0.5, i.e. $cv = \sigma/\mu = 0.3$ or 0.5, respectively.

For a given instance, we sample $n \in \{50, 200\}$ normally distributed demand observations where the price is uniformly chosen from the interval [0,1]. Fig. 1 shows an illustrative example with 200 demand observations drawn for an instance with $a = 1471.8$, $b = 702.72$, and $\sigma = 360.7$.

From all n observations, the models are estimated and inventory levels are set such that a desired service level (either non-stockout probability or fill rate) of either 90% or 95% is met. For the cost model, we use a unit penalty cost of $v = 9$ instead of a non-stockout probability of 90% and $v = 19$ for the 95% case such that these two parameter settings yield the same critical fractile. Further, $h = 1$.

The safety stocks SI (except for the LP method) are set according to (1)–(4). For known parameters, given the price, mean demand is determined and the required safety stocks are obtained from (4). Ignoring demand dependency from prices and estimating demand parameters from the sample only (method of moments), μ and σ in (1)–(4) are replaced by the respective estimators $\hat{\mu}$and $\hat{\sigma}$. Using the OLS method, $\hat{\mu} = \hat{\beta}_0 + \hat{\beta}_1 p$ and σ is estimated by (9).

For the data set shown in Fig. 1 and $P_1 = 90\%$ this yields the following results: (i) if all parameters are known, $Q = 1471.81 - 702.72p + 462.2$, (ii) for the method of moments we estimate $\hat{\mu} = 1123.2$ and $\hat{\sigma} = 418.79$ and the inventory level resulting from (1)–(4) is $Q_{MM} = 1659.93$, (iii) OLS estimation yields $\hat{\beta}_0 = 1491.53$, $\hat{\beta}_1 = 748.46$, $\hat{\sigma} = 359.29$ with inventory level function

$$Q_{OLS} = 1491.53 - 748.46p + 459.89\sqrt{1 + \frac{1}{200} + \frac{(p-0.5)^2}{16.45}}$$
$$\approx 1491.53 - 748.46p + 461.61,$$

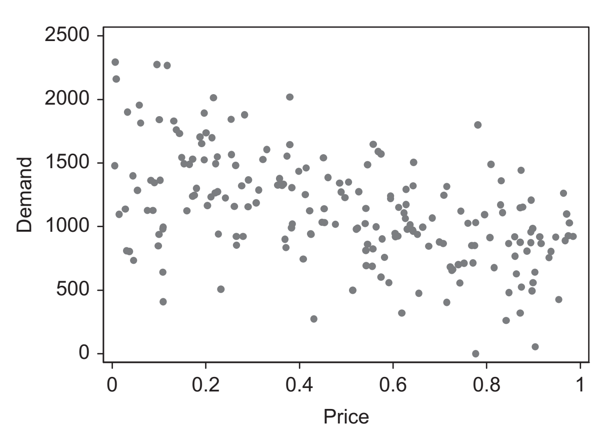

Fig. 1. Price-demand sampling example.

(iv) the Linear Programming solution is $Q_{LP}=2010.29-940.29p$. Fig. 2 illustrates the resulting inventory levels as a function of price for the different methods of estimation.

After estimating the above inventory levels for n observations, further 100,000 demands are generated by first sampling a price uniformly from the interval [0,1] and then sampling the demand under the true price-response function to test the estimates on out-of-sample observations.

The experiment was conducted for 500 randomly generated instances. Tables 1 and 2 summarize the average results with respect to achieved service level and average inventory level. Values given in Table 1 (2) are estimated from a sample of $n=50$ (2 0 0). Note that the LP cost approach was not applied to the cases with a fill rate constraint, since the cost-service equivalence in this case depends on mean demand and standard deviation.

The method of moments not incorporating the price dependency performs worst with an average inventory increase of 20% (6%) for a required non-stockout probability and 33% (9%) for a fill-rate constraint if the coefficient of variation is 0.3 (0.5). The latter is caused by the fact that under a fill-rate constraint the safety factor in (4) depends on the demand forecast, too. Inventory level determination based on OLS slightly underachieves the service goals but reduces excess inventory. The LP service level approach significantly underachieves the required service targets, however, this gap closes with a larger sample size and when a fill-rate rather than a non-stockout constraint is used. In contrast, the LP cost approach performs significantly better with average service levels slightly below target. Note that the underachievement of required service levels comes along with lower inventory levels.

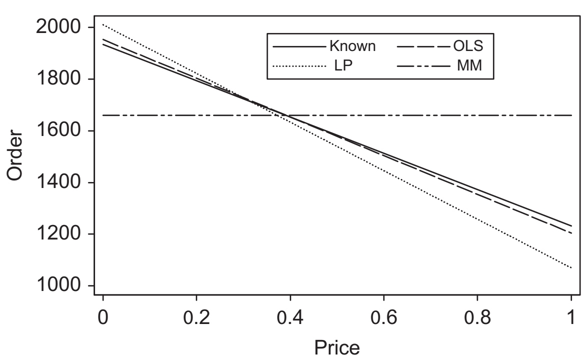

Fig. 2. Inventory level functions.

4.1. Sample size effects

In the previous section, we have observed that the LP service approach results in a P_1 level more than 5% below target for small sample sizes. We estimate the parameters from samples of 20, 50, 100, 200 and 500 observations for a coefficient of variation of 0.5 and $P_1=95\%$. The results from testing the estimates on 100,000 out-of-sample observations are shown in Table 3.

For a sample of 20 observations, the P_1 level achieved with the LP service approach is almost 10% below target, whereas for a sample size of 500 the service level nearly matches its target. In contrast, the LP cost approach and OLS estimation attain higher service levels even with small sample sizes. The MM constantly achieves a service level above OLS and LP, but at the cost of a very high inventory level in all cases.

Kässmann et al. (1986) have come to similar results when comparing a service level to a cost approach. The OLS-based model follows a feedforward mechanism which determines optimal inventory levels based on sales forecasts. As opposed to this method the service level approach works as feedback model that corrects for positive and negative target deviations resulting from the parameter estimates. Similar to our results, the service-oriented feedback approach also achieves lower service levels than the feedforward approach. We can thus conclude that the sample size plays an important role concerning the accuracy of the estimates. Since the MM hedges against uncertainty with high inventory levels, its average service and inventory level are less affected.

4.2. Violations of OLS assumptions

To assess the impact of violations of OLS assumptions, data is generated similarly to the above procedure with the following modifications. Table 4 contains a summary of the results for $n=200$.

In order to test the effect of error terms with non-zero mean, we draw normally distributed disturbances with uniformly distributed mean e:

$$u \sim N(e,\sigma^2) \quad \text{with } e \sim U(150,300).$$

Note that the mean varies with each observation. The results show that both OLS and the LP cost model cope well with the zero

Table 1
Numerical results ($n=50$).

$n=50$		Average service levels (standard deviation)					Average inventory levels (standard deviation)				
cv	SL (%)	Known	MM	OLS	LP service	LP cost	Known	MM	OLS	LP Service	LP Cost
0.3	$P_1=90$	0.9	0.8922	0.8904	0.8441	0.8817	449.2	536.6	444.7	392	443.1
		(0.0009)	(0.0357)	(0.0353)	(0.0481)	(0.0458)	(122.6)	(117.8)	(132.4)	(123.2)	(139)
0.3	$P_1=95$	0.95	0.945	0.9419	0.9129	0.945	563.2	672.1	557	507.8	548
		(0.0007)	(0.0247)	(0.0252)	(0.0375)	(0.0375)	(153.6)	(145.6)	(164.5)	(156.3)	(171.2)
0.5	$P_1=90$	0.9	0.8892	0.8868	0.8465	0.8821	741.7	783.2	724.3	649	729.8
		(0.0009)	(0.0355)	(0.036)	(0.0479)	(0.0453)	(203.9)	(203.2)	(217.9)	(207.2)	(230.7)
0.5	$P_1=95$	0.95	0.9409	0.9386	0.9155	0.9302	931.7	979.6	905.7	844.1	905.3
		(0.0007)	(0.0251)	(0.0261)	(0.0375)	(0.037)	(255.5)	(251.6)	(270.2)	(263.9)	(284.2)
0.3	$P_2=90$	0.9001	0.897	0.8969	0.8932	–	163.2	225.8	163.1	157.3	–
		(0.0004)	(0.0208)	(0.0188)	(0.0204)		(42.1)	(47.3)	(49.1)	(51.1)	
0.3	$P_2=95$	0.95	0.9477	0.9469	0.9433	–	265	348.6	263.7	255.3	–
		(0.0003)	(0.0152)	(0.0141)	(0.0155)		(70)	(73.3)	(79.7)	(81)	
0.5	$P_2=90$	0.9008	0.8935	0.893	0.8895	–	389.1	424	379.6	373.2	–
		(0.0006)	(0.0242)	(0.0238)	(0.0259)		(103.6)	(105.1)	(114.6)	(118.8)	
0.5	$P_2=95$	0.9504	0.9443	0.9435	0.9397	–	563.1	604.8	546.5	537.4	–
		(0.0004)	(0.0176)	(0.0176)	(0.0204)		(151.3)	(152.6)	(165.3)	(169.5)	

Table 2
Numerical results ($n=200$).

$n=200$		Average service levels (Standard deviation)					Average inventory levels (Standard deviation)				
cv	SL (%)	Known	MM	OLS	LP Service	LP Cost	Known	MM	OLS	LP Service	LP Cost
0.3	$P_1=90$	0.9 (0.0009)	0.8967 (0.0173)	0.8982 (0.0175)	0.8805 (0.0225)	0.896 (0.0217)	452.7 (119)	541.7 (104.9)	453.3 (124.5)	431.1 (119.2)	452.6 (125.6)
0.3	$P_1=95$	0.95 (0.0007)	0.949 (0.0113)	0.9483 (0.0118)	0.9333 (0.0177)	0.9455 (0.0164)	567.5 (149.1)	678.4 (130.5)	567.5 (155.2)	536 (149)	566.1 (158.8)
0.5	$P_1=90$	0.9 (0.0009)	0.8945 (0.0177)	0.895 (0.018)	0.881 (0.022)	0.8964 (0.0213)	745.3 (198.1)	793.5 (188.9)	737.4 (205.8)	710.4 (198.2)	746.3 (209.2)
0.5	$P_1=95$	0.95 (0.0007)	0.9455 (0.0121)	0.9454 (0.0125)	0.9339 (0.0174)	0.9456 (0.0163)	936.1 (248.3)	992.5 (235.2)	922.1 (256.3)	885.5 (249)	934.8 (264.8)
0.3	$P_2=90$	0.9001 (0.0004)	0.8986 (0.0102)	0.8996 (0.0097)	0.899 (0.0103)	–	164.4 (40.9)	226.2 (36.4)	165.2 (44.2)	161.3 (45.8)	–
0.3	$P_2=95$	0.95 (0.0003)	0.9492 (0.0073)	0.9494 (0.007)	0.9487 (0.0077)	–	274 (87.7)	350 (58.7)	272.1 (82.1)	267.8 (83.6)	–
0.5	$P_2=90$	0.9007 (0.0006)	0.8963 (0.0122)	0.8967 (0.0121)	0.8982 (0.0128)	–	390.8 (100.7)	427.9 (93.8)	384.4 (106)	383.8 (110)	–
0.5	$P_2=95$	0.9504 (0.0004)	0.9468 (0.0087)	0.9469 (0.0087)	0.9478 (0.0095)	–	565.7 (147.1)	611.2 (137.5)	554.4 (153.6)	556.5 (158.5)	–

Table 3
Sample size effects.

$cv=0.5$, $P_1=95\%$	Average service levels (standard deviation)					Average inventory levels (standard deviation)				
n	Known	MM	OLS	LP Service	LP Cost	Known	MM	OLS	LP Service	LP Cost
20	0.95 (0.0007)	0.9304 (0.0479)	0.9255 (0.0501)	0.8541 (0.0766)	0.9107 (0.0639)	931.3 (242.9)	970.4 (300.9)	900.8 (311.5)	716 (282.6)	922.2 (387.3)
50	0.95 (0.0007)	0.9409 (0.0251)	0.9386 (0.0261)	0.9155 (0.0375)	0.9302 (0.037)	931.7 (255.5)	979.6 (251.6)	905.7 (270.2)	844.1 (263.9)	905.3 (284.2)
100	0.95 (0.0007)	0.9437 (0.0186)	0.9437 (0.0185)	0.9231 (0.0265)	0.9421 (0.0240)	922.2 (248.4)	977.2 (238.2)	907.2 (256.9)	841.1 (236.9)	916.9 (263.5)
200	0.95 (0.0007)	0.9455 (0.0121)	0.9454 (0.0125)	0.9339 (0.0174)	0.9456 (0.0163)	936.1 (248.3)	992.5 (235.2)	922.1 (256.3)	885.5 (249)	934.8 (264.8)
500	0.95 (0.0007)	0.9461 (0.0075)	0.946 (0.0072)	0.9419 (0.0097)	0.9484 (0.0094)	918.2 (245)	976.2 (227)	902.6 (248.2)	891.9 (245.8)	918.9 (253.3)

mean assumption violation. They achieve average service levels very close to the ones obtained by the known function. The service level model performs significantly better for the target fill rate than for in-stock probabilities, where it achieves only 98% of the target P_1 service level.

Heteroscedasticity can be represented by standard deviations σ_i which depend on the level of the explanatory variable price. We assume the functional form (see Park, 1966):

$$\sigma_i^2 = \sigma^2 X_i^\theta e^{r_i}$$

with $\theta=3$ so that variations in demand increase with higher prices as displayed in Fig. 3.

The LP service and the cost model underachieve the P_1 service level but at lower inventory levels. In contrast, OLS overachieves a P_1 target of 90%, but underachieves the 95% target. In both cases, controlling for heteroscedasticity according to Park (1966) leads to average P_1 service levels closer to the target at lower inventory levels (marked with an asterisk).

For P_2 service level constraints, the LP service model achieves a slightly lower service level, but at significantly less inventory. The OLS and LP model achieve higher fill rates with lower inventory than the known case. This at first sight counterintuitive result can be explained by the determination of service levels for different prices. While service level optimization with known parameters and the correction for heteroscedasticity lead to a good fit for all observations due to the price-dependent structure of the error term, OLS and LP achieve high service levels for low prices and thus small σ, but only small service levels for high prices which in the end averages to the required target fill rate.

Next, we replace our normally distributed observations by demand following the gamma distribution (see, e.g., Burgin, 1975). Demand is distributed according to a $\Gamma(\lambda,\rho)$ distribution with $1/\lambda$ being the scale and ρ as shape parameter. Standard deviation and mean can be computed according to

$$\sigma = \sqrt{\frac{\rho}{\lambda}} = \mathrm{cv}(a-0.5b)$$

$$\mu = \frac{\rho}{\lambda} = a-bp$$

and thus λ and ρ can be derived. For a coefficient of variation (cv) of 0.5, we compute required safety stock for normally and gamma distributed errors with safety factors based on Strijbosch and Moors (1999). A comparison of the resulting inventory and service level shows that OLS also works well with gamma demand if the resulting inventory function is adjusted by a gamma safety factor (marked with an asterisk). Otherwise, if no attention is paid to the type of demand distribution, OLS underachieves the required service level. In contrast, the LP model does not require any prior knowledge on the type of demand distribution. Again, the LP cost model proves to be superior to the service level model for P_1 concerning achievement of the target service level. For P_2, the LP service approach performs well and produces results close to the target service level without accumulating large excess inventories.

Table 4
Violations of OLS assumptions ($n=200$).

$n=200$		Average service levels (Standard deviation)				Average inventory levels (Standard deviation)			
Viol. Ass.	SL (%)	Known	OLS	LP Service	LP Cost	Known	OLS	LP Service	LP Cost
Non-zero mean	$P_1=90$	0.8977	0.8971	0.8801	0.896	452.3	452.4	431.4	453.8
		(0.0178)	(0.0179)	(0.0229)	(0.0219)	(123.8)	(123.8)	(120.2)	(126.7)
	$P_1=95$	0.9481	0.9476	0.9323	0.945	566.8	566.7	535	565.8
		(0.0121)	(0.0122)	(0.0181)	(0.0166)	(154.6)	(154.5)	(147.4)	(156.4)
	$P_2=90$	0.8993	0.8991	0.8984	–	140	140	137	–
		(0.009)	(0.009)	(0.0095)		(42.9)	(43)	(43.4)	
	$P_2=95$	0.9493	0.9491	0.9484	–	239.8	239.9	236.7	–
		(0.0067)	(0.0067)	(0.0071)		(71.1)	(71.1)	(71.5)	
Heteroscedasticity	$P_1=90$	0.9	0.9165	0.8857	0.8951	177.4	222.7	150.4	176.6
		(0.0009)	(0.017)	(0.0231)	(0.0226)	(48.6)	(66.1)	(44.2)	(52.3)
			0.906*				183.7*		
			(0.0242)*				(57)*		
	$P_1=95$	0.95	0.9449	0.9383	0.9458	222.5	278.6	196.4	223.8
		(0.0006)	(0.0132)	(0.018)	(0.0169)	(60.9)	(82.8)	(58.8)	(65.5)
			0.9524*				230.7*		
			(0.0169)*				(72.2)*		
	$P_2=90$	0.9	0.9008	0.8994	–	47.2	35.4	20	–
		(0.0002)	(0.0076)	(0.0101)		(9.2)	(9.3)	(6.8)	
			0.901*				49.1*		
			(0.0071)*				(13.2)*		
	$P_2=95$	0.95	0.9603	0.9493	–	81.9	67.8	51	–
		(0.0002)	(0.0061)	(0.0071)		(18.3)	(20.6)	(16.2)	
			0.9509*				85.6*		
			(0.0061)*				(25.3)*		
Gamma	$P_1=90$	0.8931	0.8905	0.8816	0.8961	740.2	737.2	719.9	767.3
		(0.0013)	(0.0174)	(0.0208)	(0.0205)	(206.1)	(216)	(218.2)	(231.5)
		0.9005*	0.8979*			768.7*	765.4*		
		(0.0012)*	(0.0168)*			(214.1)*	(224.2)*		
	$P_1=95$	0.9322	0.9299	0.9342	0.9456	920.5	916.1	965.6	1038.4
		(0.0009)	(0.0136)	(0.0163)	(0.0149)	(256.4)	(268)	(292.1)	(316.1)
		0.9498*	0.9478*			1039.2*	1033.9*		
		(0.0007)*	(0.0113)*			(289.5)*	(302.2)*		
	$P_2=90$	0.8891	0.8875	0.8965	–	394.4	393	415.5	–
		(0.001)	(0.0139)	(0.0147)		(105.1)	(113.5)	(127.3)	
		0.8999*	0.8986*			419.5*	418.9*		
		(0.0007)*	(0.0113)*			(117)*	(123.4)*		
	$P_2=95$	0.9334	0.9318	0.9464	–	568.1	565.6	654.8	–
		(0.0008)	(0.0108)	(0.0114)		(154.1)	(164.6)	(198)	
		0.9495*	0.9482*			661.5*	659*		
		(0.0007)*	(0.0081)*			(184.2)*	(193.3)*		

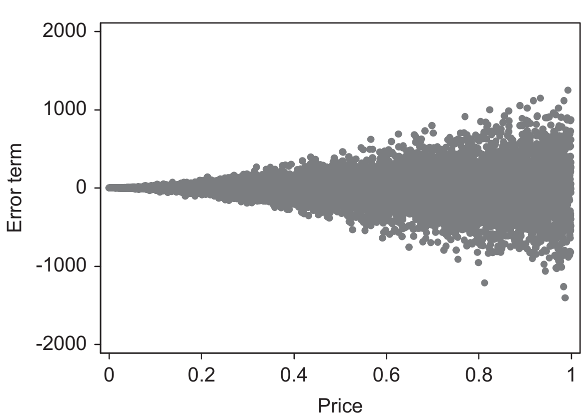

Fig. 3. Heteroscedasticity.

In summary, the impact of violations of the standard OLS assumptions varies not only with the type of assumption, but also with the kind of service level used. While poor performance of OLS can be compensated by applying problem-specific remedies, the LP approach still outperforms these countermeasures in several cases in terms of inventory level.

4.3. Real data

Since this research was originally inspired by an inventory management application at a large European retail chain, we assess the different models based on real data from 64 stores for a newsvendor-type product. The data contains daily sales for salad, customer demand, prices and weather information. Since true demand cannot be observed when demand exceeds supply, we exclude days when the product stocks out. After obtaining historical data for the first 50 days, the model parameters are specified in order to establish a function that is able to predict future demand. There are no structural breaks, seasonality or other factors during the time horizon considered. Therefore, it is reasonable to assume that customer demand is alike during the course of the experiment. Consequently, the functions established are then used to generate forecasts for the following days. For each one of these approximately 220 days (depending on product availability), predicted order quantities are compared to the observed demand realizations.

The explanatory variables are price, weather and weekdays. Prices recorded range from 0.29 € to 1.49€ and daily temperatures (w) lie between -6 and 32 °C. Furthermore, due to variations in demand during the week, weekdays are also included into the model. We segment weekdays into three categories: Tuesdays and Wednesdays generally exhibit the lowest demand levels

Table 5
Example of the resulting order functions for one store.

	Order function Q	P_1 service level	Inventory level
MM	84.01	0.87	34.42
OLS	$83.55–37.07p-0.25w$-$6.91d_1+13.98d_2+16.63$	0.89	28.75
LP Service	$82.61-25.57p-0.15w$-$10.31d_1+31.72d_2$	0.83	21.69
LP Cost	$95.03-22.5p-19.25d_1+13.25d_2$	0.87	28.46

Table 6
Results for real data.

$n=50$	Average service levels (Standard deviation)				Average inventory levels (Standard deviation)			
SL (%)	MM	OLS	LP Service	LP Cost	MM	OLS	LP Service	LP Cost
$P_1=90$	0.8997 (0.0436)	0.8712 (0.0938)	0.8051 (0.0945)	0.8722 (0.064)	50.2 (17.7)	36.5 (16.4)	28.5 (11.3)	38.5 (14.7)
$P_2=90$	0.8653 (0.0430)	0.8579 (0.0602)	0.8561 (0.0709)	–	26.6 (10.5)	17.6 (8.8)	17.2 (8.6)	–

(denoted by d_1), Fridays and Saturdays when maximum demand is observed (denoted by d_2) and Mondays and Thursdays. d_1 and d_2 are binary variables. Note that Mondays and Thursdays do not have an indicator variable to avoid collinearity.

Table 5 shows an example of a single store with the respective order functions for MM, OLS, LP Service and LP Cost and $P_1=90\%$. The last two columns contain the P_1 service level achieved and the resulting inventory levels.

Estimation of the multi-variable regression model yields goodness of fit measures R^2 between 0.25 and 0.74 with an average of 0.44. All estimated regression models are statistically significant. Results of our analysis are shown in Table 6. We observe that the method of moments best achieves the target P_1 service level of 90%, but incurs inventory levels more than 30% above those attained with the other approaches. The LP cost approach exhibits the lowest variability compared to OLS and LP Service. Its constant service level comes at the cost of a slightly higher inventory level. In terms of underachievement, the LP service model only achieves a P_1 10% below target. This can be explained by sample size effects. Since the sample contains 50 observations, the LP service model can be expected to perform worse than the other approaches due to the feedback mechanism. Using a larger sample size would solve this problem, but add new issues such as seasonality and trend.

The target fill rate is achieved equally well by all three approaches. The MM approaches hedge against uncertainty by accumulating larger inventory levels since it does not take any factors into account that allow to explain demand fluctuations. For example, instead of stocking less on the low-demand weekdays and thereby reducing average inventory levels, the MM approach holds the same amount of inventory as on Fridays and Saturdays when demand is high. The LP Service and OLS approach take these effects into account which leads to overall lower inventory levels.

5. Conclusions

This work presents an integrated framework for demand estimation and safety stock planning in environments where demand depends on several external factors such as price and weather where a time series model would thus be inadequate. Using basic results from econometrics for causal demand forecasting and error estimation as well as a Linear Programming, data-driven method, target inventory levels to minimize cost or achieve desired service levels are calculated.

We illustrate and compare the proposed methods in a numerical experiment and a retail case application. As expected, the method of moments can achieve required service levels but at the cost of significantly higher inventory levels. The data-driven approach provides a robust method for inventory level determination. However, especially under non-stockout probability constraints, the data-driven approach exhibits an underachievement of required service levels if the size of the available data sample is too small. This problem can be avoided by applying the cost-service equivalence and using a cost minimization data-driven approach instead. The OLS-approach shows the best performance as long as the assumptions are valid. In case of misspecifications (heteroscedasticity, gamma distributed residuals), the robust approach dominates. Misspecification, however, can be overcome and appropriately addressed by adjusting the demand estimation or inventory level determination.

For further research, one interesting area is to extend the framework to non-perishable products and to the joint estimation of multiple demand series incorporating dependencies between store demands.

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