Unscented Kalman Filter using Augmented State in the Presence of Additive Noise

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Abstract*—Unscented Kalman filter (UKF) using augmented state for the nonlinear dynamic systems in the presence of additive process and measurement noises is addressed. There are two alternative versions of UKF in which the system state may be augmented or not be augmented by the process and measurement noises. It was reported that the augmented UKF has better performance for dynamic systems than the nonaugmented UKF. However, the research results apply only to the dynamic system with weak noises, not to mention how the performance of these two UKFs compared in the presence of strong noises. Under the condition of $n+\kappa=const$, the basic difference in one filtering recursion between them is that the covariance of state vector in augmented UKF incorporates the process noises. This difference generally makes the gain in augmented UKF bigger than the counterpart in nonaugmented UKF. We cast augmented KF and nonaugmented UKF into a discrete-time dynamic system equation. Then we find that the nonaugmented UKF yields better performance than the augmented UKF in the case of strong noises, which contradicts with the results obtained in the presence of weak noises.

Keywords-unscented Kalman filter; moment matching method; augmented state

I. INTRODUCTION

Unscented Kalman filter (UKF) based on unscented transform (UT), is in light of the intuition that to approximate a probability distribution is easier than to approximate an arbitrary nonlinear transform, which is invented by Julier and Uhlmann [1]. It uses a finite set of points carefully designed by moment matching method, referred to as sigma points, to approach the nonlinear distribution in the Kalman filter framework. A specified nonlinear transformation can be applied to each sigma point, and the unscented estimate can be obtained by computing the statistics of the transformed set. The flaw in the extended Kalman filter (EKF) that results from propagating the mean and covariance through linear approximations of the nonlinear transformation is thus eliminated in the UKF, leading to theoretically better performance of the UKF. Furthermore, the UKF implementation does not need the calculation of any Jacobian or Hessian matrices, which results in considerable simply and suitable for real-time applications. The accuracy of the UKF can be compared to that of the second order EKF, but the computational order of the UKF is comparable to the EKF (see [2]-[6]). So, Julier et al. claimed that the UKF should be used instead of the EKF in virtually all nonlinear estimation problems [1].

The original UKF was first formulated in its augmented state form. It was reported in [7] that the augmented UKF yields better results than the nonaugmented UKF in the presence of additive weak noises. We point out that it is not sufficient to assert that augmented UKF is favored over the nonaugmented UKF in all situations as claimed in [7]. In other word, the choice of alternative UKFs is in general scenario-dependent. In this paper, we will show that the performance of UKF differs in the case of strong noises.

This paper is organized as follows. The filtering principle of UKF is formulated in Section II. Section III analyzes and compares the augmented and nonaugmented UKFs. The simulation results for one representative example are given in Section IV. Finally, a summary is given in Section V.

II. THE PRINCIPLE OF UKF

Consider a nonlinear dynamic system with additive noise

$$X(k+1)=f(k,X(k))+W(k) \tag{1}$$

$$Z(k+1)=h(k,X(k))+V(k) \tag{2}$$

where X is an $n\times 1$ random state vector, Z is the measurement vector, W is an $w\times 1$ zero-mean noise vector with covariance Q, and V is an $v\times 1$ zero-mean noise vector with covariance R. Both W and V are independent discrete-time process and measurement noises. The initial state $X_0\sim N(\bar{X}_0,P_0)$ is independent of W and V. Assume that both $f(\cdot)$ and $h(\cdot)$ are nonlinear function.

In some practical situations, some moments of the random variable X are known, but not the complete distribution. Given up to the qth moments of X, the number and locations of points χ_i with the associate probability mass p_i are designed such that up to the qth moments of X and χ are matched. For simplicity, it is easier to design probability mass function (pmf) of χ that matches a Gaussian-distributed random variable X with mean $\bar{X}$ and covariance P_X.

The weighted sigma points set $\{\chi_i,w_i,i=0,\cdots,N-1\}$ are captured in the following manner:

$$\sum_{i=0}^{N-1}w_i=1\ ,\ \sum_{i=0}^{N-1}w_i\chi_i=\bar{X}\ ,\ \sum_{i=0}^{N-1}\left(\chi_i-\bar{X}\right)\left(\chi_i-\bar{X}\right)^T w_i=P_X \tag{3}$$

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We only need consider how to design a particles set about a particles set about a Guassian random variable m with mean $0_{n\times1}$ and covariance of $I_{n\times n}$, where n is the rank of P_X. Then the design of sigma points $\{\chi_i, w_i, i=0,\cdots,N-1\}$ is simplified a standard problem of design $\{m_i, w_i, i=0,\cdots,N-1\}$ such that

$$\sum_{i=0}^{N-1} w_i = 1,\ \sum_{i=0}^{N-1} w_i m_i = 0_{n\times1},\ \sum_{i=0}^{N-1} m_i m_i^T w_i = I_{n\times n} \tag{4}$$

The previous sigma points set $\{\chi_i, w_i\}$ can be obtained from the points set $\{m_i, w_i\}$ of (4) by transformation

$$\chi_i = A\left[m_i^T, 0_{n\times1}\right]^T + \bar{X} \tag{5}$$

$$P_X = A\,diag(I_{n\times n}, 0_{n\times1})A^T \tag{6}$$

III. THE STEPS OF UKF

Now outline the steps of UKFs based on nonaugmented UT or augmented state in one filtering recursion for the nonlinear dynamic system defined in (1) and (2).

A. The steps of nonaugmented UKF

1) The random vector $\hat{X}(k)$ is approximated by $2n+1$ symmetric sigma points

$$\begin{cases} \chi_0(k) = \hat{X}(k) & i=0 \\ \chi_i(k) = \hat{X}(k) + (\sqrt{(n+\kappa)P_{X(k)}})_i & i=1,\cdots,n \\ \chi_{i+n}(k) = \hat{X}(k) - (\sqrt{(n+\kappa)P_{X(k)}})_i & i=1,\cdots,n \end{cases} \tag{7}$$

$$\begin{cases} w_0 = \kappa/(n+\kappa) & i=0 \\ w_i = 1/[2(n+\kappa)] & i=1,\cdots,n \\ w_{i+n} = 1/[2(n+\kappa)] & i=1,\cdots,n \end{cases} \tag{8}$$

where $\left(\sqrt{P}\right)_i$ is the ith column of the matrix square root of P, and w_i is the weight associated with the ith sigma point. The scalar κ is a scaling parameter. When state variable X is assumed Gaussian random variable, a useful heuristic suggestion is to select $n+\kappa=3$. If a different distribution for variable X is assumed, then an appropriate choice of κ might be decided by simulation.

2) Initiate each point in points set $\{\chi_i(k), w_i, i=0,\cdots,2n\}$ through the function

$$\chi_i(k+1\,|\,k) = f\left(k, \chi_i(k)\right) \quad i=0,\cdots,2n \tag{9}$$

3) The mean of $X(k+1\,|\,k)$ is given by

$$\hat{X}(k+1\,|\,k) = \sum_{i=0}^{2n} w_i \chi_i(k+1\,|\,k) \tag{10}$$

4) The covariance of $\chi(k+1\,|\,k)$

$$\Delta\chi_i(k+1\,|\,k) = \chi_i(k+1\,|\,k) - \hat{X}(k+1\,|\,k) \tag{11}$$

$$P_{\chi(k+1|k)} = \sum_{i=0}^{2n} w_i \Delta\chi_i(k+1\,|\,k)\Delta\chi_i(k+1\,|\,k)^T \tag{12}$$

5) Incorporating the process noise in predicted covariance of state vector

$$P_{X(k+1|k)} = P_{\chi(k+1|k)} + Q \tag{13}$$

6) Measurement prediction is given by $\chi_i(k+1\,|\,k)$

$$\xi_i(k+1) = h\left(k+1, \chi_i(k+1\,|\,k)\right) \tag{14}$$

7) The mean of $Z(k+1\,|\,k)$ is given by

$$\hat{Z}(k+1\,|\,k) = \sum_{i=0}^{2n} w_i \xi_i(k+1) \tag{15}$$

8) The covariance of $\xi(k+1)$

$$\Delta\xi_i(k+1) = \xi_i(k+1) - \hat{Z}(k+1\,|\,k) \tag{16}$$

$$P_{\xi(k+1)} = \sum_{i=0}^{2n} w_i \Delta\xi_i(k+1)\Delta\xi_i(k+1)^T \tag{17}$$

$$P_Z = P_{\xi(k+1)} + R \tag{18}$$

9) The cross covariance of $X(k+1\,|\,k)$ and $Z(k+1\,|\,k)$

$$P_{XZ} = \sum_{i=0}^{2n} w_i \Delta\chi_i(k+1\,|\,k)\Delta\xi_i(k+1)^T \tag{19}$$

10) The gain $K(k+1)$

$$K(k+1) = P_{XZ}P_Z^{-1} \tag{20}$$

11) State update equation

$$\Delta Z(k+1) = Z(k+1) - \hat{Z}(k+1\,|\,k) \tag{21}$$

$$\hat{X}(k+1) = \hat{X}(k+1\,|\,k) + K(k+1)\Delta Z(k+1) \tag{22}$$

12) State's covariance update equation

$$P_{X(k+1)} = P_{X(k+1|k)} - K(k+1)P_Z K^T(k+1) \tag{23}$$

B. The steps of augmented UKF

The step of augmented UKF is similar to, not the same as that of nonaugmented UKF. The sigma points χ_i^a and the augmented state X^a are the counterparts of χ_i and X, $P_{X^a(k+1|k)}$ and P_{Z^a} are the counterparts of $P_{X(k+1|k)}$ and P_Z. We outline the steps of augmented UKF as follows.

1) The augmented random state vector X^a is defined as follows

$$X^a = \begin{bmatrix} X^T & W^T & V^T \end{bmatrix}^T \tag{24}$$

Note that X^a is approximated by $2L+1$ ($L=n+w+v$) symmetric sigma points $\{\chi_i^a, w_i^a, i=0,\cdots,2L\}$ defined as follows

$$\begin{cases} \chi_0^a = \hat{X}^a & i=0 \\ \chi_i^a = \hat{X}^a + (\sqrt{(L+\kappa^a)P_{X^a}})_i & i=1,\cdots,L \\ \chi_{i+L}^a = \hat{X}^a - (\sqrt{(L+\kappa^a)P_{X^a}})_i & i=1,\cdots,L \end{cases} \tag{25}$$

$$\begin{cases} w_0^a = \kappa^a/(L+\kappa^a) & i=0 \\ w_i^a = 1/[2(L+\kappa^a)] & i=1,\cdots,L \\ w_{i+L}^a = 1/[2(L+\kappa^a)] & i=1,\cdots,L \end{cases} \tag{26}$$

Assume that χ_i^x is a column vector which is composed of χ_i^a from 1-dimension to n-dimension, χ_i^w is a column vector which is composed of χ_i^a from $n+1$-dimension to $n+w$-dimension, and χ_i^v is a column vector which is composed of χ_i^a from $n+w+1$-dimension to $n+w+v$-dimension.

2) Prediction equations

$$\chi_i^x(k+1|k) = f^a\left(k, \chi_i^x(k), \chi_i^w(k)\right),\ i = 0,\cdots,2L \quad (27)$$

$$\hat{X}^a(k+1|k) = \sum_{i=0}^{2L} w_i^a \chi_i^x(k+1|k) \quad (28)$$

$$\Delta\chi_i^a(k+1|k) = \chi_i^x(k+1|k) - \hat{X}^a(k+1|k) \quad (29)$$

$$P_{X^a(k+1|k)} = \sum_{i=0}^{2L} w_i^a \Delta\chi_i^a(k+1|k)\Delta\chi_i^a(k+1|k)^T \quad (30)$$

$$\xi_i^a(k+1) = h^a\left(k+1, \chi_i^x(k+1|k), \chi_i^v(k)\right) \quad (31)$$

$$\hat{Z}^a(k+1|k) = \sum_{i=0}^{2L} w_i \xi_i^a(k+1) \quad (32)$$

$$\Delta\xi_i^a(k+1) = \xi_i^a(k+1) - \hat{Z}^a(k+1|k) \quad (33)$$

$$P_{Z^a} = \sum_{i=0}^{2L} w_i \Delta\xi_i^a(k+1)\Delta\xi_i^a(k+1)^T \quad (34)$$

$$P_{X^aZ^a} = \sum_{i=0}^{2L} w_i \Delta\chi_i^a(k+1|k)\Delta\xi_i^a(k+1)^T \quad (35)$$

3) Update equations

$$K^a(k+1) = P_{X^aZ^a} P_{Z^a}^{-1} \quad (36)$$

$$\Delta Z^a(k+1) = Z(k+1) - \hat{Z}^a(k+1|k) \quad (37)$$

$$\hat{X}^a(k+1) = \hat{X}^a(k+1|k) + K^a(k+1)\Delta Z^a(k+1) \quad (38)$$

$$P_{X^a(k+1)} = P_{X^a(k+1|k)} - K^a(k+1)P_{Z^a}K^a(k+1)^T \quad (39)$$

C. Comparison of UKFs

In [7], it has been proved that under the condition of $n+\kappa = n+m+\kappa^a = const$, following equation in one recursion is satisfied

$$\hat{X}^a(k+1|k) = \hat{X}(k+1|k) \quad (40)$$

$$P_{X^a(k+1|k)} = P_{X(k+1|k)} \quad (41)$$

Note that the nonlinear measurement function $h(\bullet)$ makes a difference between the augmented UKF and the nonaugmented UKF in one recursion of UKF. If the transformed sigma points $\chi_i(k+1|k)$ for (14) can not be redrawn to match the same mean $\hat{X}(k+1|k)$ and the covariance $P_{X^a(k+1|k)}$ as $\chi_i^x(k+1|k)$, the nonaugmented UKF is different from the augmented UKF under the condition above described. Since $\chi_i^x(k+1|k)$ with mean $\hat{X}^a(k+1|k)$ and covariance $P_{X^a(k+1|k)}$ incorporates the process noise, the following inequations can be obtained

$$Z_i^a(k+1|k) \neq Z_i(k+1|k) \quad (42)$$

$$P_{X^aZ^a} \neq P_{XZ} \quad (43)$$

$$P_{Z^a} \neq P_Z \quad (44)$$

$$K^a(k+1) \neq K(k+1) \quad (45)$$

$$P_{X^a(k+1)} \neq P_{X(k+1)} \quad (46)$$

Remarks: For the UKF is based on the framework of Kalman filter, its performance is tuned by the gain K of the filter. The absolute value of the gain in the augmented UKF is generally bigger than the counterpart in nonaugmented UKF in the presence of noise. So, the performance of UKF is scenario-dependent.

IV. SIMULATION

The performance of two UKFs is compared by using the mean squared error (MSE) defined by

$$MSE(\kappa) = \frac{1}{M}\sum_{i=1}^{M}(x_n - \hat{x}_n)^2 \quad (47)$$

A representative discrete-time dynamic system equation can be written as

$$x_k = \frac{1}{2}x_{k-1} + 25\frac{x_{k-1}}{1+x_{k-1}^2} + 8\cos\left(1.2(k-1)\right) + e_k \quad (48)$$

$$z_k = \frac{x_k^2}{20} + v_k,\ \ k = 1,\cdots,L \quad (49)$$

The reference data were generated by using $x_0 = 0.1$ and $L = 5000$. The bimodality makes this problem more difficult to address by using conventional methods. The initial conditions were $\hat{x}_0 = 0$ and $P_0 = 10$.

A larger number of simulation runs were carried out. The performance of two UKFs, namely, augmented UKF with augmented state variable and nonaugmented UKF with nonaugmented state variable, was compared in Fig.1 and Fig.2 for 30 runs. The former UKF takes $\kappa = 1$, the latter one takes $\kappa = 3$. Fig.1 plots their MSEs in the presence of $e_k \sim N(0,1)$ and $v_k \sim N(0,1)$. Fig.2 plots their MSEs in the presence of $e_k \sim N(0,10)$ and $v_k \sim N(0,1)$.

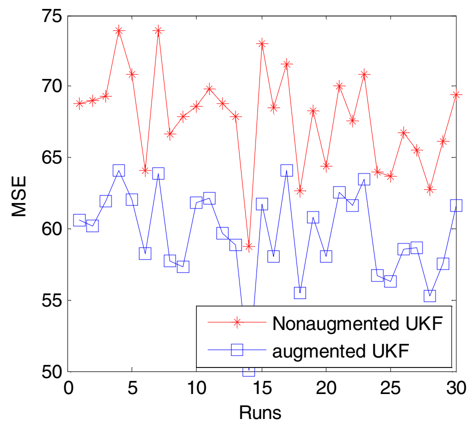

Fig. 1. Performance comparison ($e_k \sim N(0,1)$)

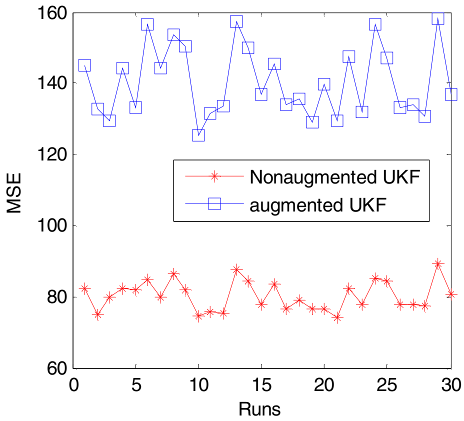

Fig. 2. Performance comparison ($e_k \sim N(0,10)$)

From Fig.1, it can be seen that the augmented UKF outperforms the nonaugmented UKF in the presence of the process noise $e_k \sim N(0,1)$. However, from Fig.2, it also can be seen that the nonaugmented UKF has better performance than the augmented UKF in the presence of the process noise $e_k \sim N(0,10)$. Obviously, they contradict each other. The contrary results obtained in different level of noise showed that the UKF is scenario-dependent.

V. Conclusion

The focus of this paper is on the analysis of the augmented UKF. The filtering performance of the augmented UKF is affected by the process and measurement noises. By comparison of the performance of the augmented UKF and the nonaugmented UKFs, we find that the UKF is scenario-dependent. Simulation results agree well with the analyses.

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