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LINEAR ALGEBRA AND ITS APPLICATIONS

Rigid relations in GL_2F

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Abstract

We describe all rigid relations in the group of invertible two by two matrices over an arbitrary field.

1. Introduction

The rigidity of relations in groups appears in connection with Hilbert's 16th problem on monodromy, the realization of finite groups as Galois groups over the rationals, and other problems in group theory.

Definition. We say that a relation $g_1 g_2 \dots g_n = 1$ in a group G is *rigid* if for any relation $g'_1 g'_2 \dots g'_n = 1$ with g_i similar to g'_i for $i = 1, 2, \dots, n$, there is $h \in G$ such that $g_i = h g'_i h^{-1}$ for all i (cf. [1–3]).

It is clear that every relation with $n \leqslant 2$ is rigid. For $n \geqslant 3$, if all but at most two g_i belong to the center of G, then the relation is rigid.

We showed in [3] that a relation $g_1 g_2 g_3 = 1_2$ in the group GL_2F, where F is a field, is rigid if at least one of the g_i has distinct eigenvalues and there is no common eigenvector over F for all g_i. In this paper, we extend our results by describing all rigid relations in GL_2F. These results are stated in the following three theorems. Recall that the center of GL_2F consists of scalar matrices.

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Theorem 1.1. *If* card$(F) = 2$, *then a relation* $g_1g_2\dots g_n = 1_2$ *in* GL_2F *is not rigid if and only if it has more than three nonscalar matrices.*

Theorem 1.2. *If* card$(F) = 3$, *then a relation* $g_1g_2\dots g_n = 1_2$ *in* GL_2F *is not rigid if and only if either*

(a) *it has more than three nonscalar matrices*
or
(b) *it has exactly three nonscalar matrices which have a common eigenvector over* F, *and one of them has distinct eigenvalues.*

Theorem 1.3. *If* card$(F) > 3$, *then a relation* $g_1g_2\dots g_n = 1_2$ *in* GL_2F *is not rigid if and only if either*
(a) *it has more than three nonscalar matrices*
or
(b) *it has exactly three nonscalar matrices with a common eigenvector over* F.

In the proofs of our main results we will use the following four lemmas which actually hold for an arbitrary group G. The proofs of Lemmas 1.4 and 1.6 are straightforward and are left to the reader.

Lemma 1.4. *A relation* $g_1g_2\dots g_n = 1$ *in a group* G *is rigid if and only if the relation obtained by dropping all trivial (identity) factors is rigid.*

Lemma 1.5. *Let* $g_1g_2\dots g_n = 1$ *and* $f_1f_2\dots f_n = 1$ *be two relations in a group* G *such that* g_i *is congruent to* f_i *modulo the center of* G *for every* i. *Then these relations are simultaneously rigid or not rigid.*

Proof. Suppose that $f_1f_2\dots f_n = 1$ is rigid. Let g_i' be similar to g_i and $g_1'g_2'\dots g_n' = 1$. We have $f_i = g_ic_i$ with central c_i. Then $f_i' = g_i'c_i$ is similar to f_i for all i and $f_1'f_2'\dots f_n' = 1$, so there is h in G such that $hf_ih^{-1} = f_i'$, hence $hg_ih^{-1} = g_i'$. Thus, $g_1g_2\dots g_n = 1$ is rigid. □

Lemma 1.6. *Let* $g_1g_2\dots g_n = 1$ *be a relation in a group* G. *Let* p *be a permutation of the set* $(1, 2, \dots, n)$. *Then there are* $f_i \in G$ *similar to* $g_{p(i)}$ *for all* i *such that* $f_1f_2\dots f_n = 1$ *and this relation is rigid if and only if* $g_1g_2\dots g_n = 1$ *is rigid.*

For example, $g_1g_2g_3 = 1$ is rigid if and only if the relation $g_2(g_2^{-1}g_1g_2)g_3 = 1$ is rigid. Here $n = 3$, p switches 1 and 2, $f_1 = g_2, f_2 = g_2^{-1}g_1g_2$, and $f_3 = g_3$.

Lemma 1.7. *Suppose that for some even* $m > 3$ *every relation in a group* G *with* m *noncentral factors is not rigid. Then every relation with* $n > m$ *noncentral factors is not rigid.*

Proof. It suffices to prove the conclusion with $n = m + 1$ and $n = m + 2$. Let $n = m + 1$ and $g_1 g_2 \dots g_n = 1$ in G with noncentral g_i. Suppose that the relation is rigid. Since g_n is not central and n is odd, $g_i g_{i+1}$ cannot be central for some $i = 1, 3, \dots, n-1$. Let $g_i g_{i+1}$ be noncentral. Replacing g_i and g_{i+1} by their product, $g_i g_{i+1}$, we obtain a relation with m noncentral factors. Since it is not rigid, the original relation is not rigid.

Let now $n = m + 2$. If $g_{n-1} g_n$ is not central, we are done as before. Otherwise, $g_m g_{n-1} g_n$ is not central. Replacing g_m, g_{n-1}, and g_n by their product, $g_m g_{n-1} g_n$, we obtain a relation with m noncentral factors. Since it is not rigid, the original relation is not rigid. □

By Lemmas 1.4 and 1.5, the question of rigidity for a relation can be reduced to the same question for a (possibly shorter) relation with noncentral factors. By Lemma 1.7, to prove that every relation with more than three noncentral factors is not rigid, it suffices to prove that:

(4) every relation $g_1 g_2 g_3 g_4 = 1_2$ with nonscalar g_i in $GL_2 F$ is not rigid.

By the above mentioned result of [3], the rest of Theorems 1.1, 1.2, and 1.3 is reduced to the following statement.

(5) A relation $g_1 g_2 g_3 = 1_2$ with nonscalar $g_i \in GL_2 F$ is rigid if and only if one of the following three cases occurs:

1. $\mathrm{card}(F) = 2$;
2. $\mathrm{card}(F) = 3$, all g_i have a common eigenvector over F and each of them has equal eigenvalues;
3. there is no common eigenvector over F for all g_i.

2. Proof of (4)

Now we will look at the relation, $g_1 g_2 g_3 g_4 = 1_2$, where the g_i are nonscalar matrices in $GL_2 F$ and prove that it is not rigid. Let U_i be the conjugacy class of g_i.

Lemma 2.1. *Let g in $GL_2 F$ be a matrix which is not similar to $g_1 g_2$ but belongs to both $U_1 U_2$ and $U_4^{-1} U_3^{-1}$. Then the relation $g_1 g_2 g_3 g_4 = 1_2$ is not rigid.*

Proof. We write $g = g'_1 g'_2 = g'^{-1}_4 g'^{-1}_3$ with $g'_i \in U_i$ and g not similar to $g_1 g_2$. We have $g'_1 g'_2 g'_3 g'_4 = 1_2$. If there were an h such that $g'_1 = h g_1 h^{-1}$ and $g'_2 = h g_2 h^{-1}$ then $g = g'_1 g'_2 = h g_1 g_2 h^{-1}$, but this is not true by our choice of g. □

We split our proof of (4) into three cases:

1. $\mathrm{card}(F) = 2$,
2. $\mathrm{card}(F) = 3$,
3. $\mathrm{card}(F) \geqslant 4$.

1. card$(F) = 2$. In this case, $GL_2F = SL_2F = PGL_2F = PSL_2F \cong S_3$, the symmetric group on three letters. We have three conjugacy classes, T_i with $i = 1, 2, 3 =$ the orders of the elements in T_i.

We have a normal subgroup $H = T_1 \cup T_3$ of order 3 in GL_2F. It is of index 2. Considering $g_1g_2g_3g_4 = 1_2$ modulo H, we conclude that the number of g_i outside of H, i.e., in T_2, is even. Using Lemma 1.6, we can assume that in the case when the number is not zero we have $g_1, g_2 \in T_2$. Now we have $U_1U_2 = H = T_2T_2 = T_3T_3 = (U_3U_4)^{-1}$. We pick $g \in H = T_1 \cup T_3$ which is not similar to g_1g_2. We choose $g'_i \in U_i$ such that $g'_1g'_2 = g = (g'_3g'_4)^{-1}$.

2. card$(F) = 3$. In this case the group $G = GL_2F$ is of order 48. Following the notation of [3], G has the following eight similarity classes: $\pm C_1$ (scalar matrices), $C_2, \pm C_3$, all of determinant 1; $T_0, \pm T_1$, all of determinant -1. We have $\operatorname{tr}(C_1) = \operatorname{tr}(C_3) = 2$, $\operatorname{tr}(C_2) = 0 = \operatorname{tr}(T_0)$, $\operatorname{tr}(T_1) = 1$.

By Lemma 1.6, we can assume that $\det(g_1g_2) = 1$. Then g_1g_2 belongs to one of the five classes $\pm C_1, C_2, \pm C_3$. By Lemma 1.5, we can assume that g_1g_2 belongs to one of the three classes C_1, C_2, C_3.

In the case $g_1g_2 \in C_1 = \{1_2\}$, using the multiplication table for C_j in [3], we see that both U_1U_2 and $(U_3U_4)^{-1}$ contain C_2. So we are done by Lemma 2.1 with $g \in C_2$.

In the case $g_1g_2 \in C_2$, using the multiplication table for C_j in [3], we see that both U_1U_2 and $(U_3U_4)^{-1}$ contain C_3, except in the case when U_1U_2 or $(U_3U_4)^{-1}$ is $T_1(-T_1)$ or C_2C_2. By Lemmas 1.5 and 1.6, we can assume that $U_1 = U_2 = C_2$ or $U_1 = -U_2 = T_1$. When $U_1 = U_2 = C_2$, we have $U_1U_2 = C_1 \cup -C_1 \cup C_2$ and the table shows that $(U_3U_4)^{-1} \cap U_1U_2 = C_2$ only in the case when $(U_3U_4)^{-1} = T_0T_1 = C_2 \cup C_3 \cup -C_3$. By Lemmas 1.5, 1.6, and 2.1, we can assume that $U_1 = C_2, U_2 = T_0^{-1} = T_0, U_3 = C_2, U_4 = T_1^{-1} = -T_1$. In this case $U_1U_2 = T_0 \cup T_1 \cup -T_1 = (U_3U_4)^{-1}$, so we are done by Lemma 2.1. When $U_1 = -U_2 = T_1$, we have $U_1U_2 = C_1 \cup -C_3 \cup C_2$, and the table shows that $(U_3U_4)^{-1}$ contains C_1 or $-C_3$, so we are done by Lemma 2.1.

Finally, in the case when $g_1g_2 \in C_3$, using the multiplication table for C_j in [3], we see that both U_1U_2 and $(U_3U_4)^{-1}$ contain $-C_3$, except in the case when U_1U_2 or $(U_3U_4)^{-1}$ is $T_1T_1 = (-T_1)(-T_1) = -C_1 \cup C_3 \cup C_2$. By Lemmas 1.5 and 1.6, we can assume that $U_1 = U_2 = T_1$. Now $(U_3U_4)^{-1}$ contains C_2 unless $(U_3U_4)^{-1} = C_2C_3 = C_3 \cup -C_3$. Using Lemmas 1.5 and 1.6, we are reduced to the case when $U_1 = T_1, U_2 = C_2, U_3 = C_3^{-1} = C_3, U_4 = T_1^{-1} = -T_1$. In this case, $T_0 \cup T_1 = (U_3U_4)^{-1} \subset U_1U_2$, so we are done by Lemma 2.1.

3. card$(F) > 3$. If two (or more) of our matrices are diagonalizable over F, then we can assume that they are g_1 and g_3 by Lemma 1.6. By Lemma 5.1 of [3], U_1U_2 contains all matrices of determinant $\det(g_1g_2)$, and the same is true for $(U_3U_4)^{-1}$. So we are done by Lemma 2.1. Suppose now that at most one matrix g_i is diagonalizable. By Lemma 1.6, we can assume that g_2 and g_3 are not diagonalizable. By Lemma 5.1 of [3], both U_1U_2 and $(U_3U_4)^{-1}$ contain all diagonal nonscalar matrices of determinant $\det(g_1g_2)$. When card$(F) > 5$ or g_1g_2

is not diagonalizable, we are done by Lemma 2.1. Assume now that $g_5 = g_1g_2$ is diagonalizable. Without loss of generality we can assume that g_5 is diagonal. Now we take a diagonal matrix $h_1 = \mathrm{diag}(w, 1)$ and set $g'_i = h_1g_ih_1^{-1}$ for $i = 1,2$ and $g'_i = g_i$ for $i = 3,4$. Then $g'_1g'_2g'_3g'_4 = 1_2$ and $\mathrm{tr}(g'_2g'_3) = c_2w + c_0/w + c_1 = \alpha(w)$ with fixed $c_j \in F$ and $c_2 \neq 0$ because g_2 and g_3 are not diagonalizable. Since $\mathrm{card}(F) \geqslant 4$, we can find $w \neq 0$ such that $\alpha(w) \neq \alpha(1) = \mathrm{tr}(g_2g_3)$. Then $g'_2g'_3$ is not similar to g_2g_3, hence there is no h such that $g_i = hg'_ih^{-1}$ for all i.

3. Proof of (5)

Let $g_1g_2g_3 = 1_2$ in GL_2F with nonscalar g_i. By [3], the relation is rigid if at least one of the g_i has distinct eigenvalues and there is no common eigenvector over F for all g_i. So it suffices to prove (5) in the following two cases:

Case (3.1): All g_i have a common eigenvector over F. We can simultaneously (i.e., by the same matrix for all 3 factors) conjugate our matrices to be equal to upper triangular matrices. So

$$g_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}.$$

If one of the matrices, say g_3 (see Lemma 1.6), has distinct eigenvalues (which is impossible when $\mathrm{card}(F) = 2$), we can assume that $b_3 = 0$. If one of the matrices g_1, g_2 is not diagonal, then we can take g'_i to be the transpose of g_i for $i = 1,2$ and $g'_3 = g_3$. We have $g'_1g'_2g'_3 = 1_2$. Since $g'_3 = g_3$, any matrix h which conjugates g_3 with g'_3 must commute with g_3 and thus must be diagonal, but then it does not conjugate either g_1 or g_2 to its transpose. (This was observed in p. 182 of [3].) Thus, the relation is not rigid in this case.

Consider now the case when all three matrices g_i are diagonal (and g_3 still has distinct eigenvalues). Then we take $g'_3 = g_3$,

$$g'_1 = \begin{pmatrix} a_1 & 1 \\ 0 & d_1 \end{pmatrix}$$

and $g'_2 = g_1'^{-1}g_3^{-1}$. Then g'_i is similar to g_i for $i = 1,2$ (since the g_i are not scalar), $g'_1g'_2g'_3 = 1_2$, and no matrix h conjugates g_i to g'_i for all i (to work for $i = 3$, the matrix h must be diagonal). Thus, the relation is not rigid in this case.

Next we consider the case when each matrix g_i has equal eigenvalues, i.e., $d_i = a_i$. We choose

$$g'_1 = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g_1 \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a_1 & \alpha b_1 \\ 0 & d_1 \end{pmatrix}$$

and

$$g_2' = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} g_2 \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a_2 & \beta b_2 \\ 0 & d_2 \end{pmatrix}.$$

We want $g_1'g_2' = g_3^{-1} = g_3'^{-1}$, which gives a linear equation for α, β with a nonzero constant term b_3. We do not want α to be 0 or 1 and we do not want β to be 0. This excludes at most three values for α.

If $\mathrm{card}(F) > 3$ we can find an α which works. Now if $hg_3h_1 = g_3' = g_3$, i.e., h commutes with g_3, it also commutes with g_1 so it cannot conjugate g_1 to $g_1' \neq g_1$. Thus, the relation is not rigid.

Finally, when $\mathrm{card}(F) = 3$, our relation modulo the center is

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^3 = 1_2$$

with $b = 1$ or 2. We have to prove that this relation is rigid. Let $g_1'g_2'g_3' = 1_2$ with g_i' similar to

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

Without loss of generality we can assume that $g_3' = g_3$. Since $\mathrm{tr}(gg_3^{-1}) \neq \mathrm{tr}(g)$ when g is not upper triangular, the equality $\mathrm{tr}(g_2') = \mathrm{tr}(g_1'^{-1}g_3'^{-1}) = \mathrm{tr}(g_1'^{-1})$ implies that the matrix $g_2'^{-1}$ is upper triangular, hence $g_2' = g_2$, hence $g_1' = g_1$. Thus the relation is rigid.

Case (3.2): Every g_i has equal eigenvalues but there is no common eigenvector. Assume now that the g_i have no common eigenvector over F and that each of them has equal eigenvalues. We will prove that the relation is rigid.

If the eigenvalue of g_3 belongs to F, by simultaneous conjugation and scaling (see Lemma 1.5), we can arrange

$$g_3 = \begin{pmatrix} 1 & b_3 \\ 0 & 1 \end{pmatrix}$$

with $b_3 \neq 0$. Now we consider

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}.$$

Since there is no common eigenvector over $F, c_1 \neq 0$. Conjugating all three matrices by an upper triangular matrix, we make g_1 a companion matrix, i.e.,

$$g_1 = \begin{pmatrix} 0 & b_1 \\ 1 & d_1 \end{pmatrix}.$$

The form of the matrix g_3 is preserved (with possibly different b_3). We have $d_1 = \mathrm{tr}(g_1), b_1 = -\det(g_1)$, and $d_1^2 + 4b_1 = 0$ because g_1 has equal eigenvalues.

Now we have

$$g_2 = g_1^{-1}g_3^{-1} = \begin{pmatrix} -d_1/b_1 & 1 \\ 1/b_1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -b_3 \\ 0 & 1 \end{pmatrix};$$

hence $\mathrm{tr}(g_2) = -d_1/b_1 - b_3/b_1$ and $\det(g_2) = 1/b_1$. The equality $\mathrm{tr}(g_2)^2 - 4\det(g_2) = 0$ gives $(d_1 + b_3)^2 + 4b_1 = 0$. Together with the above equality $d_1^2 + 4b_1 = 0$, this gives $b_3 = -2d_1$. (This is impossible when $2F = 0$.)

So we see that the relation $g_1g_2g_3 = 1_2$ can be conjugated by a matrix to a standard form, where g_1, g_3, and hence g_2 are determined by the traces and determinants of g_i. Thus, the relation is rigid.

Now we assume that the eigenvalue of each g_i is not in F. Then $2F = 0$. Passing to a quadratic field extension F' of F, we conclude that all g_i have a common eigenvector over F', because the previous case was impossible in characteristic 2. This implies that the g_i commute with each other. So $g_1 \in F + Fg_3$. We write $g_1 = a1_2 + bg_3$ with $a, b \in F$. Then $\det(g_1) = a^2 + \det(g_3)b^2$. Since $\det(g_3)$ is not a square in F, we conclude that a, b are determined by $\det(g_i)$. Therefore g_1 is determined by g_3, hence g_2 is determined by g_3. Thus, the relation $g_1g_2g_3 = 1_2$ is rigid.

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