Nonlinear State Estimation

Particle, Sigma-Points Filters

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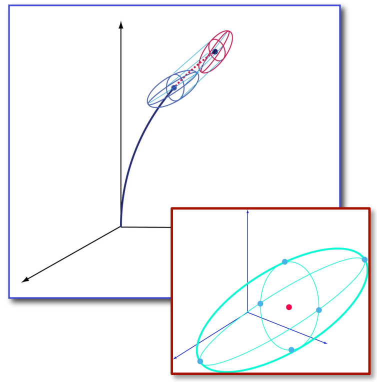
Optimal Control and Estimation, MAE 546
Princeton University, 2018

- Particle filter
- Sigma-Points ("Unscented Kalman") filter
  - Transformation of uncertainty
  - Propagation of mean and variance
- Helicopter, HIV state estimation examples
- Additional nonlinear filters



http://www.princeton.edu/~stengel/MAE546.html
http://www.princeton.edu/~stengel/OptConEst.html

Criticisms of the Basic Extended Kalman Filter*

**Julier and Uhlmann, 1997; Wan and van der Merwe, 2001*

- State estimate prediction is deterministic, i.e., not based on an expectation
  - (Not true; the state estimate is the expectation of the mean)
- State estimate update is linear (unless it is quasi-linear, iterated, or adapted)
- Jacobians must be evaluated to calculate covariance prediction and update

Transformation of Uncertainty

Nonlinear transformation of a random variable

$$\mathbf{y} = \mathbf{f}[\mathbf{x}]$$

where

$\mathbf{x}$ is a random variable with mean, $\bar{\mathbf{x}}$, and covariance, $\mathbf{P}_{\mathbf{xx}}$

Estimate of the <u>mean</u> and <u>covariance</u> of the transformation's output

$$\bar{\mathbf{y}}(\bar{\mathbf{x}}, \mathbf{P}_{\mathbf{xx}}) \text{ and } \mathbf{P}_{\mathbf{yy}}(\bar{\mathbf{x}}, \mathbf{P}_{\mathbf{xx}})$$

The transformation is said to be "unscented"*
if its probability distribution is
Consistent, Efficient, and Unbiased

Julier and Uhlmann, 1997

<u>Consistent</u> Estimate of a Dynamic State

$$\text{Let } \mathbf{x}_k \triangleq \mathbf{x}, \quad \mathbf{x}_{k+1} \triangleq \mathbf{y}$$

$$\bar{\mathbf{x}}_{k+1}(\bar{\mathbf{x}}_k, \mathbf{P}_{\mathbf{x}_k\mathbf{x}_k}) = \bar{\mathbf{y}}(\bar{\mathbf{x}}, \mathbf{P}_{\mathbf{xx}})$$

$$\mathbf{P}_{\mathbf{x}_{k+1}\mathbf{x}_{k+1}}(\bar{\mathbf{x}}_k, \mathbf{P}_{\mathbf{x}_k\mathbf{x}_k}) = \mathbf{P}_{\mathbf{yy}}(\bar{\mathbf{x}}, \mathbf{P}_{\mathbf{xx}})$$

Consistent state estimate converges in the limit

$$\left\{\mathbf{P}_{\mathbf{x}_{k+1}\mathbf{x}_{k+1}} - E\left[(\mathbf{x}_{k+1} - \bar{\mathbf{x}}_{k+1})(\mathbf{x}_{k+1} - \bar{\mathbf{x}}_{k+1})^T\right]\right\} \geq 0$$

$$\{\text{Estimated Covariance} - \text{Actual Covariance}\} \geq 0$$

<u>Lesson</u>: In filtering, add sufficient "process noise" to the filter gain computation to prevent filter divergence

Adding Process Noise Improves Consistency

- Satellite orbit determination
  - Aerodynamic drag produced unmodeled bias
  - Optimal filter did not estimate bias
- Process noise increased for filter design
  - Divergence is contained

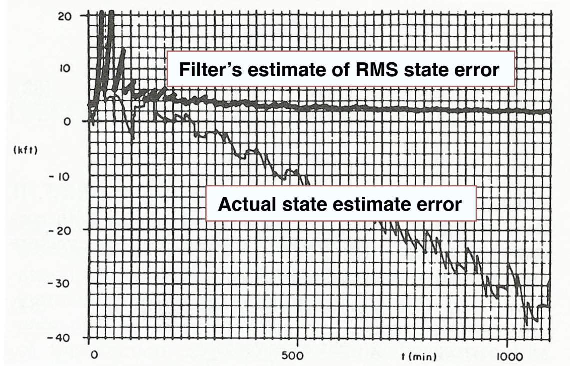


Fig. 6. Divergence due to ignoring drag in satellite navigation (forward component of position error).

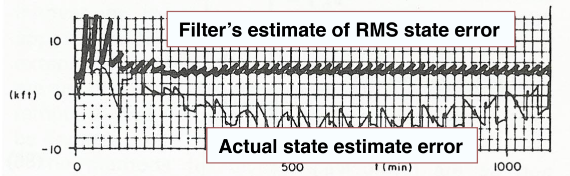


Fig. 7. Elimination of divergence by increasing assumed process noise.

Fitzgerald, 1971

Efficient and Unbiased Estimate of a Dynamic State

Efficient state estimator converges more quickly than an inefficient estimator

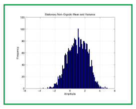
Add "just enough" process noise

$$\min_{\text{Added Process Noise}} \{\text{Estimated Covariance} - \text{Actual Covariance}\}$$

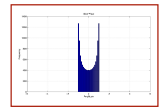
Unbiased estimate

$$\bar{\mathbf{x}}_{k+1} = E\left(\mathbf{x}_{k+1}\right)$$

Estimated Mean = Actual Mean



Empirical Determination of a Probability Distribution



- **Monte Carlo evaluation**
  - **Propagate random noise through nonlinearity for <u>given prior distribution</u> (e.g., Gaussian)**
  - **Generate histogram of output**
    - ***N* trials) of nonlinear system propagation**
    - **Histogram as numerical representation of distribution, *or***
    - **Functional minimization to identify associated theoretical distribution**
- **<u>Particle Filter</u>**
  - **Propagate many points to estimate <u>mean</u> and <u>standard deviation</u>**
  - **As *N* –> ∞, estimate error –> 0**

Empirical Determination of Mean and Variance

- **Sample mean for *N* data points, $x_1, x_2, \ldots, x_N$**

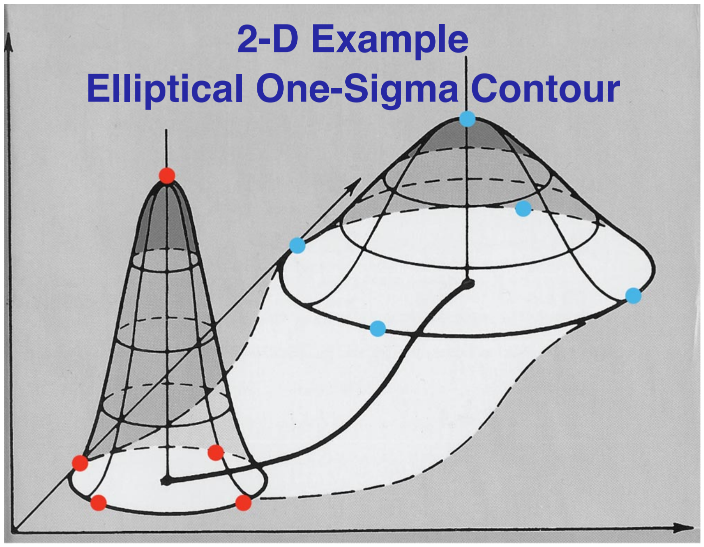
$$\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N}$$

- **Sample variance for same data set**

$$\sigma_x^2 = \frac{\sum_{i=1}^{N} \left(x_i - \bar{x}\right)^2}{\left(N-1\right)}$$

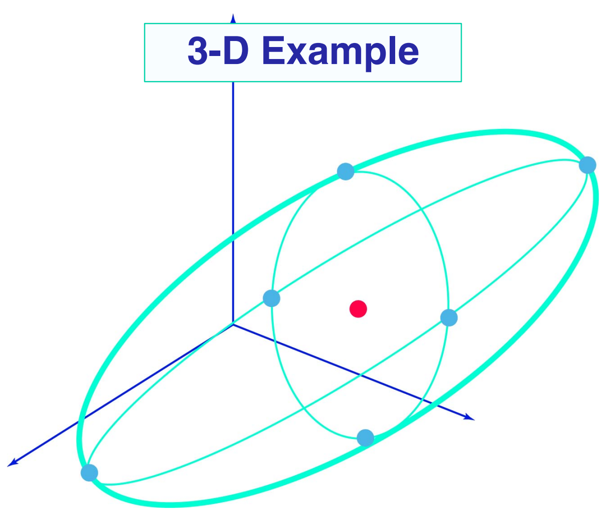
- **<u>Sigma Points Filter</u>:**
  - **Propagate limited number of points using nonlinear model to estimate mean and covariance**

Sigma Points of P_{xx} and Mean



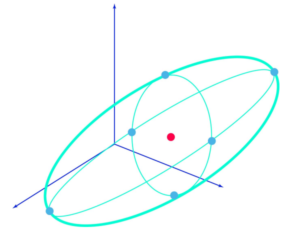
- Mean is center of ellipse
- Sigma points are ± major and minor axes of ellipse

Sigma Points of P_{xx} and $E(x)$



- Mean is center of ellipsoid
- Sigma points are ± extremes of ellipsoid axes

Sigma Points of Estimate Uncertainty, $\mathbf{P_{xx}}$



State covariance matrix

$$\boxed{\mathbf{P_{xx}} : \text{ Symmetric, positive-definite covariance matrix}}$$

Eigenvalues are real and positive

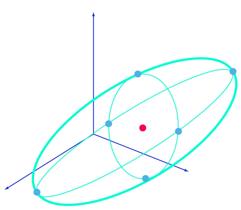
$$\left| s\mathbf{I}_n - \mathbf{P_{xx}} \right| = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$$

Eigenvectors

$$(\lambda_i \mathbf{I}_n - \mathbf{P_{xx}})\alpha \mathbf{e}_i = 0, \quad i = 1, n$$

Modal matrix *(not expected value symbol)*

$$\mathbf{E} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix}$$



Sigma Points of $\mathbf{P_{xx}}$

Diagonalized covariance matrix

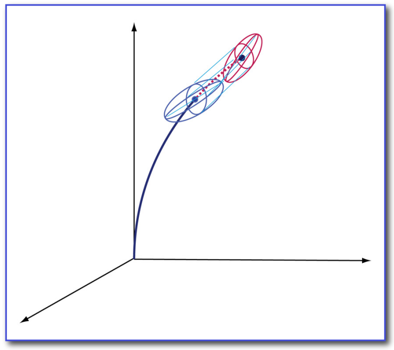
Eigenvalues are the Variances

$$\mathbf{\Lambda} = \mathbf{E}^{-1}\mathbf{P_{xx}}\mathbf{E} = \mathbf{E}^T\mathbf{P_{xx}}\mathbf{E} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

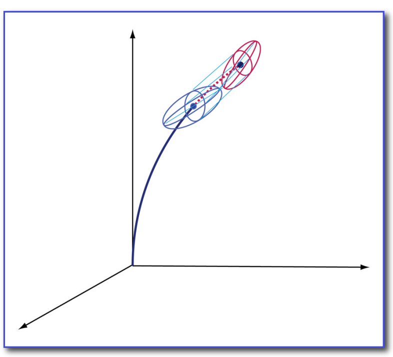
1) Principal axes of the covariance matrix are defined by modal matrix, **E**
2) Location of *2n* one-sigma points in state space given by

$$\begin{bmatrix} \pm\Delta\mathbf{x}(\sigma_1) & \pm\Delta\mathbf{x}(\sigma_2) & \cdots & \pm\Delta\mathbf{x}(\sigma_n) \end{bmatrix} = \mathbf{E}\begin{bmatrix} \pm\sigma_1 & 0 & \cdots & 0 \\ 0 & \pm\sigma_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \pm\sigma_n \end{bmatrix}$$

Propagation of the Mean Value and Covariance Matrix



Propagation of the Mean Value and the Sigma Points



- **Mean value at t_k**

$$\boxed{\bar{\mathbf{x}}(t_k) = \bar{\mathbf{x}}_k}$$

- **Sigma points (relative to mean value)**

$$\boldsymbol{\sigma}_{i_k} \triangleq \begin{cases} \bar{\mathbf{x}}_k - \Delta\mathbf{x}_k(\sigma_i), & i = 1,n \\ \bar{\mathbf{x}}_k + \Delta\mathbf{x}_k(\sigma_i), & i = (n+1),2n \end{cases}$$

Projection from the prior mean

$$\bar{\mathbf{x}}_{k+1} = \bar{\mathbf{x}}_k + \int_{t_k}^{t_{k+1}} \mathbf{f}\left[\bar{\mathbf{x}}(t),\mathbf{u}(t),\bar{\mathbf{w}}(t),t\right]dt$$

Nonlinear projection from each prior sigma point

$$\boldsymbol{\sigma}_{i_{k+1}} = \boldsymbol{\sigma}_{i_k} + \int_{t_k}^{t_{k+1}} \mathbf{f}\left[\boldsymbol{\sigma}_i(t),\mathbf{u}(t),\bar{\mathbf{w}}(t),t\right]dt, \quad i = 1,2n$$

Estimation of the Propagated Mean Value

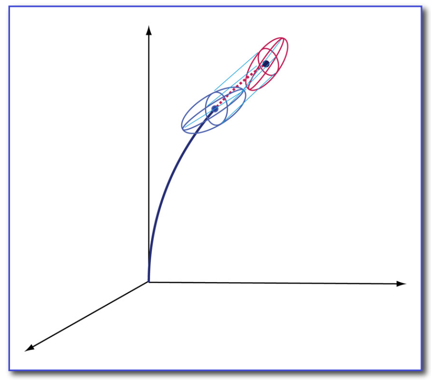
- **Assumptions:**
  - To 2nd order, the propagated probability distribution is symmetric about its mean
  - New mean is estimated as average or weighted average of projected points (arbitrary choice by user)

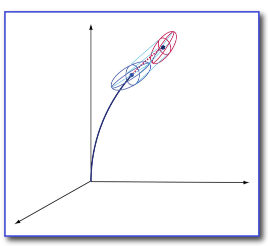
Ensemble Average for the Mean Value

$$\hat{\mathbf{x}}_{k+1} = \frac{\overline{\mathbf{x}}_{k+1} + \sum_{i=1}^{2n} \boldsymbol{\sigma}_{i_{k+1}}}{2n+1}$$

Weighted Ensemble Average for the Mean Value

$$\hat{\mathbf{x}}_{k+1} = \frac{\overline{\mathbf{x}}_{k+1} + \xi \sum_{i=1}^{2n} \boldsymbol{\sigma}_{i_{k+1}}}{2\xi n+1}$$





Projected Covariance Matrix

Unbiased ensemble estimate of the covariance matrix

$$\mathbf{P}_{\mathbf{x}_{k+1}\mathbf{x}_{k+1}} = \frac{1}{(2n+1)-1}\left\{\left(\overline{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{k+1}\right)\left(\overline{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{k+1}\right)^T + \sum_{i=1}^{2n}\left(\boldsymbol{\sigma}_{i_{k+1}} - \hat{\mathbf{x}}_{k+1}\right)\left(\boldsymbol{\sigma}_{i_{k+1}} - \hat{\mathbf{x}}_{k+1}\right)^T\right\}$$

This estimate neglects effects of disturbance uncertainty during the state propagation from t_k to t_{k+1}

Does not require calculation of Jacobian matrices
Does require calculation of nonlinear integrals

Sigma Points of Disturbance Uncertainty, Q

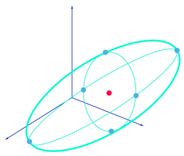
$\mathbf{Q}: \quad (s \times s)$ Symmetric, positive-definite covariance matrix

$$\left|s\mathbf{I}_s - \mathbf{Q}\right| = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_s) \qquad [s = \text{Laplace operator}]$$

$$(\lambda_i \mathbf{I}_s - \mathbf{Q})\alpha \mathbf{e}_i = 0, \quad i = 1, s$$

$$\mathbf{E}_\mathbf{Q} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_s \end{bmatrix}_\mathbf{Q}$$

- Eigenvalues
- Modal Matrix
- Transformation



$$\boldsymbol{\Lambda}_\mathbf{Q} = \mathbf{E}_\mathbf{Q}^T \mathbf{Q} \mathbf{E}_\mathbf{Q} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_s \end{bmatrix}_\mathbf{Q} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sigma_s^2 \end{bmatrix}_\mathbf{Q}$$

$$\begin{bmatrix} \pm\Delta\mathbf{w}(\sigma_1) & \pm\Delta\mathbf{w}(\sigma_2) & \cdots & \pm\Delta\mathbf{w}(\sigma_s) \end{bmatrix} = \mathbf{E}_\mathbf{Q} \begin{bmatrix} \pm\sigma_1 & 0 & \cdots & 0 \\ 0 & \pm\sigma_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \pm\sigma_s \end{bmatrix}_\mathbf{Q}$$

Propagation of the Disturbed Mean Value

Sigma points of disturbance (relative to mean value)

$$\boldsymbol{\omega}_{i_k} \triangleq \begin{cases} \overline{\mathbf{w}}_k + \Delta\mathbf{w}_k(\sigma_i), & i = 1, s \\ \overline{\mathbf{w}}_k - \Delta\mathbf{w}_k(\sigma_i), & i = (s+1), 2s \end{cases}$$

Incorporation of effects of disturbance uncertainty on state propagation

$$\left(\overline{\mathbf{x}}_{\boldsymbol{\omega}_i}\right)_{k+1} = \overline{\mathbf{x}}_k + \int_{t_k}^{t_{k+1}} \mathbf{f}\left[\overline{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\omega}_i(t), t\right] dt, \quad i = 1, 2s$$

Calculation of nonlinear integrals

Estimation of the Propagated Mean Value with Disturbance Uncertainty

Estimate now includes effect of disturbance uncertainty

Estimate of the mean is the average or weighted average of projected points

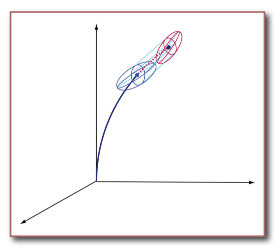
Ensemble Average for the Mean Value

$$\hat{\mathbf{x}}_{k+1} = \frac{\bar{\mathbf{x}}_{k+1} + \sum_{i=1}^{2n} \boldsymbol{\sigma}_{i_{k+1}} + \sum_{i=1}^{2s} \left(\bar{\mathbf{x}}_{\omega_i}\right)_{k+1}}{2(n+s)+1}$$

Weighted Ensemble Average for the Mean Value

$$\hat{\mathbf{x}}_{k+1} = \frac{\bar{\mathbf{x}}_{k+1} + \xi \left[\sum_{i=1}^{2n} \boldsymbol{\sigma}_{i_{k+1}} + \sum_{i=1}^{2s} \left(\bar{\mathbf{x}}_{\omega_i}\right)_{k+1} \right]}{2\xi(n+s)+1}$$

Covariance Propagation with Disturbance Uncertainty



Unbiased sampled estimate of the covariance matrix

$$\mathbf{P}_{\mathbf{x}_{k+1}\mathbf{x}_{k+1}} = \frac{1}{\left[2(n+s)+1\right]-1}\left(\mathbf{P}_{mean} + \mathbf{P}_{sigma} + \mathbf{P}_{disturbance}\right)$$

$$\mathbf{P}_{mean} = \left(\bar{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{k+1}\right)\left(\bar{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{k+1}\right)^T$$

$$\mathbf{P}_{sigma} = \sum_{i=1}^{2n} \left(\boldsymbol{\sigma}_{i_{k+1}} - \hat{\mathbf{x}}_{k+1}\right)\left(\boldsymbol{\sigma}_{i_{k+1}} - \hat{\mathbf{x}}_{k+1}\right)^T$$

$$\mathbf{P}_{disturbance} = \sum_{i=1}^{2s} \left[\left(\bar{\mathbf{x}}_{\omega_i}\right)_{k+1} - \hat{\mathbf{x}}_{k+1}\right]\left[\left(\bar{\mathbf{x}}_{\omega_i}\right)_{k+1} - \hat{\mathbf{x}}_{k+1}\right]^T$$

Sigma-Points ("Unscented Kalman") Filter

System Vector Notation

System vector

$$\mathbf{v} \triangleq \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \\ \mathbf{n} \end{bmatrix}$$

$$\dim(\mathbf{v}) = (n + r + s) \times 1$$

Expected value of system vector

$$\hat{\mathbf{v}}_0 = \begin{bmatrix} \hat{\mathbf{x}}_0 \\ \hat{\mathbf{w}}_0 \\ \hat{\mathbf{n}}_0 \end{bmatrix} = E\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{w}_0 \\ \mathbf{n}_0 \end{bmatrix} \triangleq \boldsymbol{\chi}_0 = \begin{bmatrix} \boldsymbol{\chi}_0^{\mathbf{x}} \\ \boldsymbol{\chi}_0^{\mathbf{w}} \\ \boldsymbol{\chi}_0^{\mathbf{n}} \end{bmatrix}$$

Propagation of the mean

$$\boldsymbol{\chi}_{k+1}^{\mathbf{x}} = \boldsymbol{\chi}_k^{\mathbf{x}} + \int_{t_k}^{t_{k+1}} \mathbf{f}\left[\boldsymbol{\chi}^{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\chi}^{\mathbf{w}}(t), t\right] dt$$

Measurement vector, corrupted by noise

$$\boldsymbol{\psi} = \mathbf{h}\left(\boldsymbol{\chi}^{\mathbf{x}}, \boldsymbol{\chi}^{\mathbf{n}}\right)$$

Analogous to $\mathbf{z}(t) = \mathbf{H}\mathbf{x}(t) + \mathbf{n}(t)$

Matrix Array of System and Sigma-Point Vectors

Expected value of system vector

$$\boldsymbol{\chi}_0 = \hat{\mathbf{v}}_0 = \begin{bmatrix} \hat{\mathbf{x}}_0 \\ \hat{\mathbf{w}}_0 \\ \hat{\mathbf{n}}_0 \end{bmatrix}; \quad \dim(\boldsymbol{\chi}_0) = (n+r+s)\times 1 \triangleq L\times 1$$

Weighted sigma points for system vector

$$\boldsymbol{\chi}_i = \begin{Bmatrix} \hat{\mathbf{v}}_i + \xi(\mathbf{S})_i, & i = 1, L \\ \hat{\mathbf{v}}_i - \xi(\mathbf{S})_i, & i = L+1, 2L \end{Bmatrix}; \quad \dim(\boldsymbol{\chi}_i) = 2L\times 1$$

$\mathbf{S}$: Square root of $\mathbf{P}$; $\quad (\mathbf{S})_i \triangleq i^{th}$ column of $\mathbf{S}$

Matrix of mean and sigma-point vectors

$$\mathbf{X} \triangleq \begin{bmatrix} \boldsymbol{\chi}_0 & \boldsymbol{\chi}_1 & \cdots & \boldsymbol{\chi}_{2L} \end{bmatrix}; \quad \dim(\mathbf{X}) = L\times(2L+1)$$

Initialize Filter

State and covariance estimates

$$\hat{\mathbf{x}}_o = E[\mathbf{x}(0)] = \boldsymbol{\chi}^{\mathbf{x}}(0)$$

$$\mathbf{P}^{\mathbf{x}}(0) = E\left\{[\mathbf{x}(0) - \hat{\mathbf{x}}(0)][\mathbf{x}(0) - \hat{\mathbf{x}}(0)]^T\right\}$$

Covariance matrix of system vector

$$\mathbf{P}^{\mathbf{v}}(0) = E\left\{[\mathbf{v}(0) - \hat{\mathbf{v}}(0)][\mathbf{v}(0) - \hat{\mathbf{v}}(0)]^T\right\}$$

$$= \begin{bmatrix} \mathbf{P}^{\mathbf{x}}(0) & 0 & 0 \\ 0 & \mathbf{Q}^{\mathbf{w}}(0) & 0 \\ 0 & 0 & \mathbf{R}^{\mathbf{n}}(0) \end{bmatrix}$$

Propagate State Mean and Covariance

Incorporate disturbance sigma points

$$\left(\boldsymbol{\chi}_i^{\mathbf{x}}\right)_{k+1} = \left(\boldsymbol{\chi}_i^{\mathbf{x}}\right)_k + \int_{t_k}^{t_{k+1}} \mathbf{f}\left[\left(\boldsymbol{\chi}_i^{\mathbf{x}}\right)(t), \mathbf{u}(t), \left(\boldsymbol{\chi}_i^{\mathbf{w}}\right)(t), t\right] dt$$

Ensemble average estimates of mean and covariance

$$\hat{\mathbf{x}}_{k+1}(-) = \sum_{i=0}^{2L} \eta_i \, \boldsymbol{\chi}_i^{\mathbf{x}}\Big|_{k+1}$$

η_i : **Weighting factor**

Typically

$$\eta_i = \begin{cases} 1/(L+1), & i = 0 \\ 1/2(L+1), & i = 1, 2L \end{cases}$$

$$\mathbf{P}_{k+1}^{\mathbf{x}}(-) = \sum_{i=0}^{2L} \eta_i \left\{\left[\boldsymbol{\chi}_i^{\mathbf{x}}(-) - \hat{\mathbf{x}}(-)\right]_{k+1}\left[\boldsymbol{\chi}_i^{\mathbf{x}}(-) - \hat{\mathbf{x}}(-)\right]_{k+1}^T\right\}$$

after *Wan and van der Merwe, 2001*

Incorporate Measurement Error in Output

Mean/sigma-point projections of measurement

$$\left(\boldsymbol{\psi}_i\right)_{k+1} = \mathbf{h}\left[\left(\boldsymbol{\chi}_i^{\mathbf{x}}\right)_{k+1}, \left(\boldsymbol{\chi}_i^{\mathbf{n}}\right)_{k+1}\right], \quad i = 0, 2L$$

Weighted estimate of measurement projection

$$\hat{\mathbf{y}}_{k+1}(-) = \sum_{i=0}^{2L} \eta_i \, \boldsymbol{\psi}_i\Big|_{k+1}$$

Incorporate Measurement Error in Covariance

Prior estimate of measurement covariance

$$\boxed{\mathbf{P}_{k+1}^{\mathbf{y}}(-) = \sum_{i=0}^{2L} \eta_i \left\{ \left[\boldsymbol{\psi}_i(-) - \hat{\mathbf{y}}(-) \right]_{k+1} \left[\boldsymbol{\psi}_i(-) - \hat{\mathbf{y}}(-) \right]_{k+1}^T \right\}}$$

Prior estimate of state/measurement cross-covariance

$$\boxed{\mathbf{P}_{k+1}^{\mathbf{xy}}(-) = \sum_{i=0}^{2L} \eta_i \left\{ \left[\boldsymbol{\chi}_i^{\mathbf{x}}(-) - \hat{\mathbf{x}}(-) \right]_{k+1} \left[\boldsymbol{\psi}_i(-) - \hat{\mathbf{y}}(-) \right]_{k+1}^T \right\}}$$

after *Wan and van der Merwe, 2001*

Compute Kalman Filter Gain

Original formula (eq. 3, Lecture 18)

$$\boxed{\mathbf{K}_k = \mathbf{P}_k(-)\mathbf{H}_k^T \left[\mathbf{H}_k \mathbf{P}_k(-) \mathbf{H}_k^T + \mathbf{R}_k \right]^{-1}}$$

Sigma points version w/index change

$$\boxed{\mathbf{K}_{k+1} = \mathbf{P}_{k+1}^{\mathbf{xy}}(-) \left[\mathbf{P}_{k+1}^{\mathbf{y}}(-) \right]^{-1}}$$

Post-Measurement State and Covariance Estimate

State estimate update

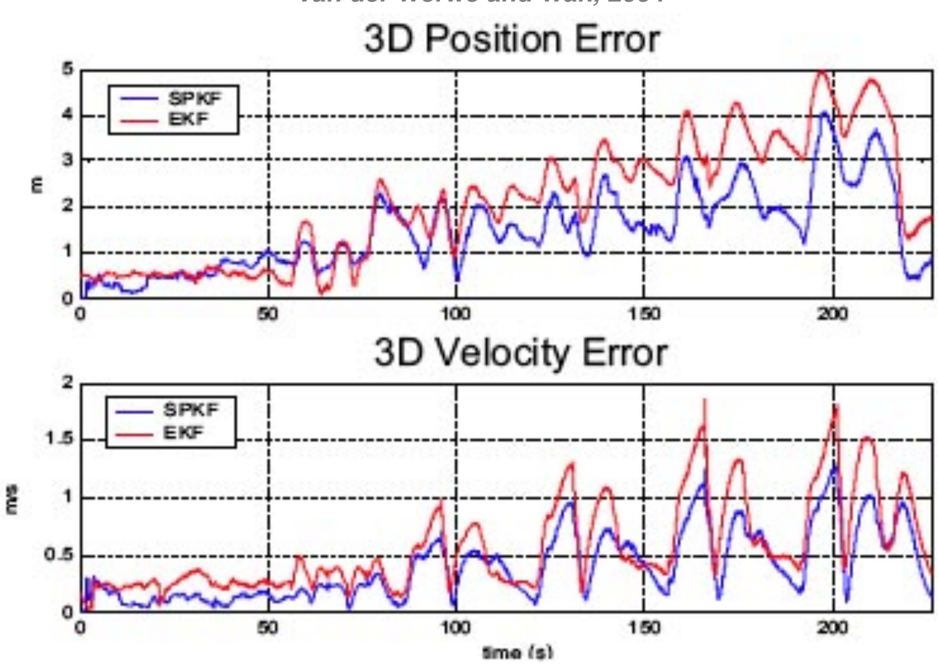
$$\boxed{\hat{\mathbf{x}}_{k+1}(+) = \hat{\mathbf{x}}_{k+1}(-) + \mathbf{K}_{k+1}\left[\mathbf{z}_{k+1} - \hat{\mathbf{y}}_{k+1}(-)\right]}$$

Covariance estimate "update"

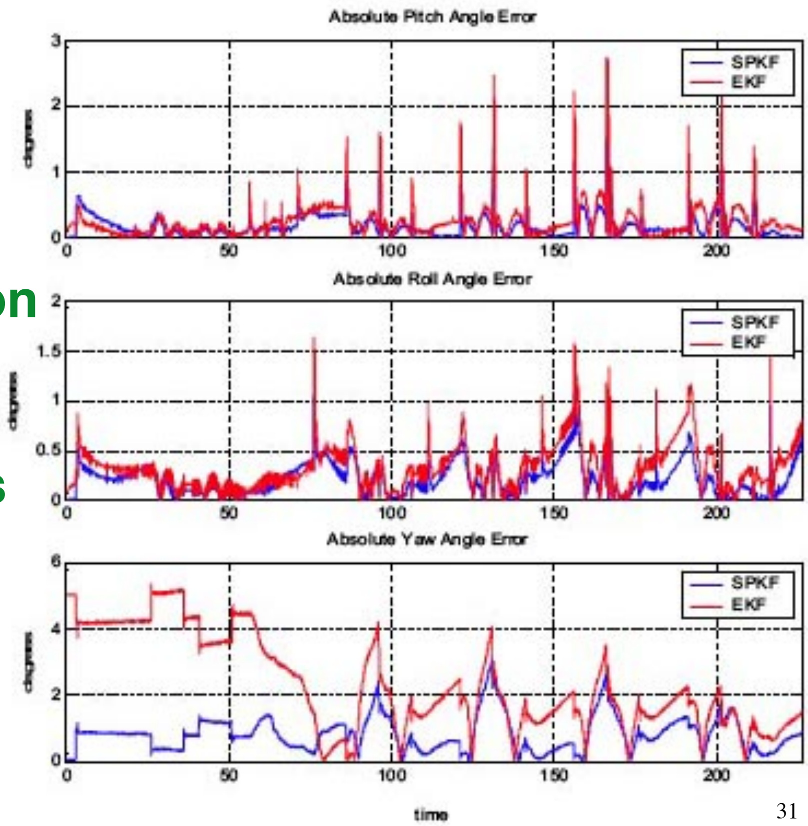
$$\boxed{\mathbf{P}^{\mathbf{x}}_{k+1}(+) = \mathbf{P}^{\mathbf{x}}_{k+1}(-) - \mathbf{K}_{k+1}\mathbf{P}^{\mathbf{y}}_{k+1}(-)\mathbf{K}^{T}_{k+1}}$$

Example: Simulated Helicopter UAV Flight

van der Werwe and Wan, 2004



Comparison of Pitch, Roll, and Yaw Errors



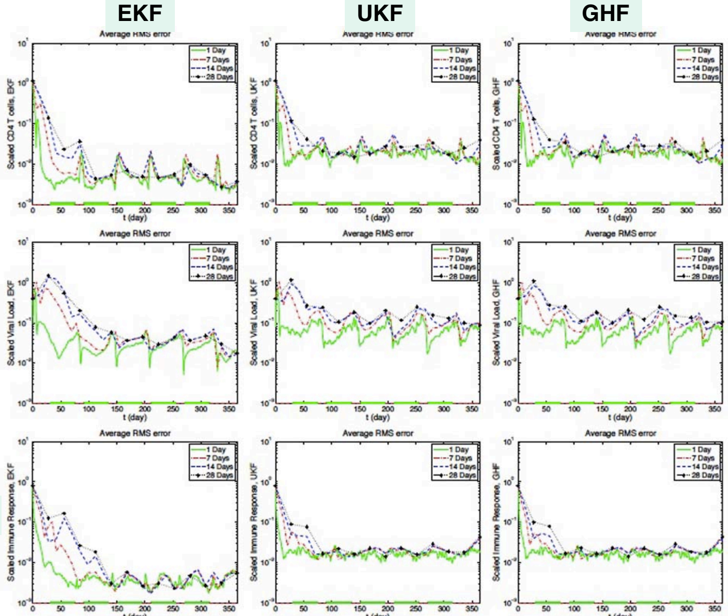
Comparison of RMS Errors for EKF, UKF, and Gauss-Hermite Filter (GHF)

EKF UKF GHF

CD4+ Cells

HIV Virions

Effector Cells



Banks et al, A COMPARISON OF NONLINEAR FILTERING APPROACHES IN THE CONTEXT OF AN HIV MODEL, 2010

from Banks Abstract

"Numerical experiments reveal that the GHF is the most computationally expensive algorithm, while the EKF is the least expensive one. In addition, computational experiments suggest that there is little difference in the estimation accuracy between the UKF and GHF. When measurements are taken as frequently as every week to two weeks, the EKF is the superior filter. When measurements are further apart, the UKF is the best choice in the problem under investigation."

Banks et al, A COMPARISON OF NONLINEAR FILTERING APPROACHES IN THE CONTEXT OF AN HIV MODEL, 2010

Comparison of Various Filters for "Blind Tricyclist Problem" *(Psiaki, 2013)*

TABLE 1 Performance metrics for the filters for the blind tricyclist problem.

Filter	Terminal RMS Position Error (m)	Terminal % of Cases Above 99.99% χ^2 Bound	Mean and Maximum Computation Time (s)	Filter Tuning Parameters
EKF	6.67	97	0.08, 0.24	N/A
UKF A	21.61	47	1.18, 1.27	$\alpha = 0.1, \beta = 2, \kappa = -9$
UKF B	487.71	37	1.18, 1.23	$\alpha = 0.01, \beta = 2, \kappa = 0$
PF	77.08	65	249.7, 314.6	3000 particles, 400 min diversity
PF B	636.06	23	680.9, 725.4	10,000 particles, 5000 min diversity
BLSF	4.35	N/A	197.94	N/A
BSEKF A	2.52	26	60.84, 69.23	$n = 30$ samples of smoothing
BSEKF B	2.50	26	110.6, 162.0	$n = 40$ samples of smoothing

Comparison of Various Filters for "Blind Tricyclist Problem" *(Psiaki, 2013)*

TABLE 2 Performance metrics for the filters for the bimodal one-dimensional problem.

Filter	RMS State Error	Total % of Cases Above 99.99% χ^2 Bound	Filter Tuning Parameters
EKF	21.53	47.3	N/A
UKF A	26.97	28.6	$\alpha = 0.1, \beta = 2, \kappa = 1$
UKF B	475.90	33.5	$\alpha = 0.01, \beta = 2, \kappa = 0$
PF A	4.75	0.9	400 particles, 50 min diversity
PF B	4.68	0.4	1600 particles, 200 min diversity
BSEKF A	12.24	30.9	$n = 30$ samples of smoothing
BSEKF B	12.26	29.8	$n = 40$ samples of smoothing

Definitions

- **EKF: Extended Kalman filter**
- **UKF: Unscented Kalman (Sigma-Points) filter**
- **GHF: Gauss-Hermite filter***
- **PF: Particle filter**
- **BLSF: Batch least-squares filter**
- **BSEKF: Backward-smoothing EKF**

* See https://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=6838186
https://en.wikipedia.org/wiki/Gauss–Hermite_quadrature

Observations

- Jacobians vs. no Jacobians
- Number of nonlinear propagation steps
- Gaussian vs. approximate non-Gaussian distributions
- Best choice of averaging weights is problem-dependent
- Comparison of filters is problem-dependent

- Are these filters better than a quasi-linear filter? *(TBD)*

More on Nonlinear Estimators

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Next Time:
Adaptive State Estimation