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An upper bound for the permanent of a nonnegative matrix

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Abstract

Let A be a fully indecomposable, nonnegative matrix of order n with row sums $r_1, \ldots, r_n$, and let s_i equal the smallest positive element in row i of A. We prove the permanental inequality

$$\mathrm{per}(A) \leqslant \prod_{i=1}^{n} s_i + \prod_{i=1}^{n} (r_i - s_i)$$

and characterize the case of equality. In 1984 Donald, Elwin, Hager, and Salamon gave a graph-theoretic proof of the special case in which A is a nonnegative integer matrix.

1. Introduction

The literature of combinatorial matrix theory includes inequalities for the permanent of a nonnegative matrix under different hypotheses and in terms of various parameters. (See Minc [1], ch. IV.) In 1984 Donald et al. [2] proved

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that if A is a fully indecomposable, nonnegative integer matrix with row sums $r_1, \ldots, r_n$, then

$$\operatorname{per}(A) \leqslant 1 + \prod_{i=1}^{n}(r_i - 1). \tag{1}$$

In a follow-up paper [3] they characterized the case of equality in Eq. (1). The terminology and techniques used in Refs. [2,3] were graph-theoretic. In this paper we give a short, matrix-theoretic proof of a generalization of Eq. (1) to nonnegative real matrices.

Let $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ be real numbers with $r_i - s_i \geqslant s_i > 0$ for $i = 1, \ldots, n$. A matrix of the form

$$C = \begin{bmatrix} r_1 - s_1 & 0 & \cdots & 0 & s_1 \\ s_2 & r_2 - s_2 & \cdots & 0 & 0 \\ 0 & s_3 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & r_{n-1} - s_{n-1} & 0 \\ 0 & 0 & \cdots & s_n & r_n - s_n \end{bmatrix} \tag{2}$$

is a *diagonally dominant cycle matrix*. (The bipartite graph whose edges are defined by the positive elements of C is a cycle on $2n$ vertices.) Note that C is fully indecomposable, has row sums $r_1, \ldots, r_n$, and

$$\operatorname{per}(C) = \prod_{i=1}^{n} s_i + \prod_{i=1}^{n}(r_i - s_i).$$

Up to permutations and a contraction operation defined below, diagonally dominant cycle matrices are the only ones for which equality holds in our main result.

Consider the matrices

$$A = \left[\begin{array}{c|c|c} X & x_{n-1} & x_n \\ \hline 0 \cdots 0 & a_{n,n-1} & a_{n,n} \end{array}\right], \qquad A' = [X \mid a_{n,n}x_{n-1} + a_{n,n-1}x_n].$$

Here A is a matrix of order n with 0's everywhere in row n except for two nonzero elements, $a_{n,n-1}$ and $a_{n,n}$. The matrix A' is of order $n-1$ and is obtained from A by adding $a_{n,n-1}$ times column n to $a_{n,n}$ times column $n-1$ and then deleting row n and column n. We say that A *row-contracts* to A'. Observe that $\operatorname{per}(A) = \operatorname{per}(A')$ and that if A is fully indecomposable, then so is A'. In case $a_{n,n-1} = a_{n,n}$ we refer to an *equi-contraction*. An equi-contraction with $a_{n,n-1} = a_{n,n} = 1$ is a unit *equi-contraction*. Row-contractions may be carried on any row that contains two nonzero elements and all other elements equal to 0. Suppose that a sequence of equi-contractions and row and column permutations transforms A to a matrix B. Then we say that A *equi-contracts* to B.

We now state our main theorem. The proof is in Section 2.

Theorem. *Let A be a fully indecomposable, real nonnegative matrix of order n with row sums $r_1, \ldots, r_n$ $(n \geqslant 2)$. Let s_i equal the smallest positive element in row i $(i = 1, \ldots, n)$. Then the permanent of A satisfies*

$$\operatorname{per}(A) \leqslant \prod_{i=1}^{n} s_i + \prod_{i=1}^{n} (r_i - s_i). \tag{3}$$

Equality holds if and only if A equi-contracts to a diagonally dominant cycle matrix.

Let us draw an immediate corollary, which summarizes the main results in [2] and [3].

Corollary (Donald et al. [2,3]). *If A is a fully indecomposable, nonnegative integer matrix of order n $(n \geqslant 2)$ with row sums $r_1, \ldots, r_n$, then*

$$\operatorname{per}(A) \leqslant 1 + \prod_{i=1}^{n} (r_i - 1). \tag{4}$$

Equality holds if and only if A equi-contracts to a diagonally dominant cycle matrix of the form Eq. (2) with $s_1 = \cdots = s_n = 1$ using only unit equi-contractions and row and column permutations.

Proof. Apply the theorem and note that for integer matrices we have $s_i \geqslant 1$ and that the upper bound in Eq. (3) is largest when $s_1 = \cdots = s_n = 1$; in this case each equi-contraction must be a unit equi-contraction. □

2. Proof of the theorem

We shall assume that

$$s_1 = \cdots = s_n = 1. \tag{5}$$

This is sufficient because we may replace A by the matrix obtained by dividing row i by s_i for $i = 1, \ldots, n$ to bring about this situation, and the general inequality Eq. (3) follows from the special case we treat by the multilinearity of the permanent.

The theorem holds for matrices of order 2. We suppose that $n \geqslant 3$ and proceed by induction on the sum of the elements $\sigma(A)$ in $A = [a_{ij}]$. This is a real parameter of induction, and we take as our base case $6 \leqslant \sigma(A) < 8$. In this case, Eq. (5) implies that A contains at least two 0's, and the full indecomposability of A implies these cannot be in the same row or column. With this information and Eq. (5), the theorem is readily verified. We henceforth suppose that $\sigma(A) \geqslant 8$.

Suppose that some row sum of A is 2, say, $r_n = 2$. Then a unit equi-contraction on row n produces a fully indecomposable matrix A' of order $n-1$ with row sums $r_1, \ldots, r_{n-1}$ and $\sigma(A') = \sigma(A) - 2$. By induction

$$\operatorname{per}(A) = \operatorname{per}(A') \leqslant 1 + \prod_{i=1}^{n-1}(r_i - 1) = 1 + \prod_{i=1}^{n}(r_i - 1)$$

with equality if and only if A' (and hence A) equi-contracts to a diagonally dominant cycle matrix. We henceforth suppose that

$$r_i > 2 \quad (i = 1, \ldots, n). \tag{6}$$

Let E_{ij} denote the matrix of order n with a 1 in position (i,j) and 0's elsewhere. Recall that a fully indecomposable matrix $A = [a_{ij}]$ is *nearly decomposable* provided $A - a_{ij}E_{ij}$ is partly decomposable for all (i,j) for which $a_{ij} \neq 0$. A result of Minc [4] asserts that *the number of nonzero elements in a nearly decomposable matrix of order n* $(n \geqslant 3)$ *is at most* $3(n-1)$. (See also [1], pp. 90–91.)

Our matrix A cannot be a nearly decomposable $(0,1)$-matrix for then Minc's result would imply that $r_i = 2$ for some i, contrary to Eq. (6). We now suppose that either A is not a $(0,1)$-matrix or that A is a $(0,1)$-matrix that is not nearly decomposable. In either case we may assume without loss of generality that $a_{nn} \geqslant 1$ and that for a suitable positive number β the matrix $A - \beta E_{nn}$ is fully indecomposable with no positive element less than 1. Observe that

$$\operatorname{per}(A) = \beta \operatorname{per}(A(n|n)) + \operatorname{per}(A - \beta E_{nn}). \tag{7}$$

The matrix $A(n|n)$ need not be fully indecomposable, but after row and column permutations

$$A(n|n) = \begin{bmatrix} A_1 & O & O & \cdots & O \\ * & A_2 & O & \cdots & O \\ \vdots & \vdots & \ddots & & \vdots \\ * & * & \cdots & A_{m-1} & O \\ * & * & \cdots & * & A_m \end{bmatrix},$$

where A_j is a fully indecomposable matrix of order n_j, say $(j = 1, \ldots, m)$. Let the row sums of A_1 be $q_1, \ldots, q_{n_1}$. Then $q_i \leqslant r_i$ for $i = 1, \ldots, n_1$. By the full indecomposability of A, strict inequality holds for at least one i, say, $q_1 \leqslant r_1 - 1$. We claim that

$$\operatorname{per}(A_1) \leqslant \prod_{i=1}^{n_1}(r_i - 1) \tag{8}$$

with strict inequality in case $n_1 \geqslant 2$. This inequality is clear if $n_1 = 1$, while for $n_1 \geqslant 2$ the induction hypothesis and Eq. (6) imply that

$$\operatorname{per}(A_1) \leqslant 1 + \prod_{i=1}^{n_1}(q_i - 1) \leqslant 1 + (r_1 - 2)\prod_{i=2}^{n_1}(r_i - 1) < \prod_{i=1}^{n_1}(r_i - 1).$$

An inequality similar to Eq. (8) holds for each A_j ($j = 1, \dots, m$), and thus

$$\operatorname{per}(A(n|n)) = \prod_{j=1}^{m}\operatorname{per}(A_j) \leqslant \prod_{i=1}^{n-1}(r_i - 1). \tag{9}$$

The row sums of $A - \beta E_{nn}$ are $r_1, \dots, r_{n-1}, r_n - \beta$. By induction

$$\operatorname{per}(A - \beta E_{nn}) \leqslant 1 + (r_n - \beta - 1)\prod_{i=1}^{n-1}(r_i - 1) \tag{10}$$

with equality if and only if $A - \beta E_{nn}$ equi-contracts to a diagonally dominant cycle matrix. Now Eqs. (7), (9) and (10) imply the desired inequality (3) under the assumption (5).

Finally, suppose equality holds in (3) under the hypotheses (5) and (6). Then equality holds in Eq. (9) and thus each diagonal block A_j of $A(n|n)$ is of order 1, that is, $A(n|n)$ is lower triangular with diagonal elements $r_1 - 1, \dots, r_{n-1} - 1$. Moreover, equality holds in Eq. (10), and thus $A - \beta E_{nn}$ equi-contracts to a diagonally dominant cycle matrix. It now follows that A is a diagonally dominant cycle matrix. □

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