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LINEAR ALGEBRA AND ITS APPLICATIONS

On Nekrasov matrices [1]

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Abstract

In this paper we investigate nonsingularity of a generalized Nekrasov matrix and present some sufficient conditions for this matrix to be nonsingular on one hand. On the other hand, we also establish some sufficient and necessary conditions for a diagonally dominant matrix or a generalized Nekrasov matrix to be a generalized diagonally dominant matrix. All results in this presentation improve and generalize the corresponding results of Huang (T. Huang, Linear Algebra Appl. 225 (1995) 237) and Szulc (T. Szulc, Linear Algebra Appl. 225 (1995) 221). © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

By $\mathbb{C}^{nn}(\mathbb{R}^{nn})$ we denote the set of all complex (real) matrices of order n. Let $A = (a_{ij}) \in \mathbb{C}^{nn}$. Then we denote $|A| = (|a_{ij}|)$, and denote by $\tilde{A} = (\tilde{a}_{ij})$ the comparison matrix given by

$$\tilde{a}_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases}$$

We denote by $\langle n \rangle$ the set $\{1, 2, \ldots, n\}$. Let

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$$\Lambda_i(A) = \sum_{(i\neq)j=1}^{n} |a_{ij}|,$$

$$R_1(A) = \sum_{i=2}^{n} |a_{1i}|, \qquad R_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{R_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n} |a_{ij}|, \tag{1.1}$$

$$\alpha_A = \{i \in \langle n \rangle:\ |a_{ii}| = R_i(A)\},$$
$$\beta_A = \{i \in \langle n \rangle:\ |a_{ii}| = \Lambda_i(A)\}.$$

Let $A = (a_{ij}) \in \mathbb{C}^{nn}$. We consider the following conditions:

$$|a_{ii}| \geqslant \Lambda_i(A) \quad \text{for each } i \in \langle n \rangle, \tag{1.2}$$

$$|a_{ii}| > \Lambda_i(A) \quad \text{for each } i \in \langle n \rangle, \tag{1.3}$$

(1.2) holds and A is an irreducible matrix with $\beta_A \neq \langle n \rangle$, (1.4)

(1.2) holds and $\beta_A \neq \langle n \rangle$ and for any $i \in \beta_A$ there is a sequence of nonzero entries $a_{ii_1}, \ldots, a_{i_k j}$ with $j \notin \beta_A$, (1.5)

$$|a_{ii}| > R_i(A) \quad \text{for any } i \in \langle n \rangle, \tag{1.6}$$

It is well known that each of the above conditions is sufficient for A to be nonsingular (for example see [1], p. 222). The question posed by Bailey [2] and Szulc [1] is as follows: If conditions obtained by replacing $\Lambda_i(A)$ with $R_i(A)$ in Eqs. (1.4) and (1.5) and the conditions

$$|a_{ii}| \geqslant R_i(A) \quad \text{for any } i \in \langle n \rangle. \tag{1.7}$$

$$|a_{ii}| > \sum_{j=1}^{i-1} |a_{ij}| \frac{R_j(A)}{|a_{jj}|} \tag{1.8}$$

are still sufficient for A to be nonsingular?

Szulc provides a counter-example for the above question and shows that conditions (1.7) and (1.8) are sufficient conditions for A to be nonsingular (see Theorems 1,1′ and 1″ of [1]). But can we find a sufficient condition similar to (1.4) or (1.5) for A to be nonsingular under assumption of (1.7)? This is still a unsolved problem. To solve the above problem, we first investigate the (generalized) diagonally dominant matrix, and then present some sufficient conditions included in (1.8) as its special case and those similar to (1.4) or (1.5) for A to be nonsingular under assumption of (1.7). Now we introduce the content of the paper. In Section 2, we present notation and definitions.

In [3] Huang provides a sufficient and necessary condition such that a diagonally dominant matrix is a generalize diagonally dominant matrix (see Theorem 2 of [3]). But the inverse of a matrix is used in his conditions, which is difficult to verify in the practical application. In Section 3 we shall improve this result. This section is also a preparation for Section 4.

In Section 4 we present a sufficient and necessary condition such that the (irreducible) matrix with condition (1.7) is a generalized diagonally dominant matrix. In Section 5 we provide some sufficient conditions similar to (1.4) or (1.5) or (1.8) for the matrix with condition (1.7) to be nonsingular. The results given in the section generalize the recent result. Also some interesting corollaries are given in this section.

2. Notation and definitions

Here we give some notation and definitions used in this article.

By $A \geqslant (>)0$ we denote the *nonnegative* (positive) *matrix*, i.e., each element of the matrix A is nonnegative (positive). By α' we denote the set $\langle n \rangle \setminus \alpha$. By $A[\alpha, \beta]$ we denote the submatrix of A whose rows are indexed by α and columns by β. For simplicity, we denote by $A[\alpha]$ the principal submatrix of A whose rows and columns are indexed by α. Let x be a vector. By x_i we denote the ith element of x.

Matrices satisfying conditions (1.3), (1.2), (1.4) and (1.5) are called *strictly diagonally dominant* (s.d.d.) *matrix*, *diagonally dominant* (d.d.) *matrix*, *irreducible diagonally dominant* (i.d.d.) *matrix and chain diagonally dominant* (c.d.d.) *matrix*, respectively; A is said to be a *generalized diagonally dominant* (g.d.d.) *matrix* if there is a positive diagonal matrix D such that AD is an s.d.d. matrix; a *Nekrasov matrix* [1] if condition (1.6) is satisfied.

Let $A \in \mathbb{R}^{nn}$ be a *Z-matrix*, i.e., A can be expressed as $A = sI - B$ where $s > 0$ and $B \geqslant 0$. A is said to be a nonsingular *M-matrix* if $s > \rho(A)$, the spectral radius of A. A *directed graph* Γ is a pair of sets (V, E) where $E \subseteq V \times V$. The sets V and E are called the *vertex set* and *arc set*, respectively. Unless otherwise specified, we take $V = \langle n \rangle$ and identify Γ with its arc set. A *path from i to j in* Γ is a sequence of vertices $\sigma = (i_0, i_1, \ldots, i_k)$ where $i_0 = i$ and $i_k = j$ such that (i_0, i_1), $(i_1, i_2), \ldots, (i_{k-1}, i_k)$ are arcs of Γ. If $\sigma_1 = (i_0, i_1, \ldots, i_k)$ and $\sigma_2 = (i_k, i_{k+1}, \ldots, i_t)$ are paths in Γ, then the *concatenation path* [4] of σ_1 and σ_2 is a path $(i_0, \ldots, i_k, \ldots, i_t)$, and denote by (σ_1, σ_2).

If Γ_1 and Γ_2 are graphs we define the *product graph* $\Gamma_1\Gamma_2$ by $(i, j) \in \Gamma_1\Gamma_2$ if and only if there exists $k \in \langle n \rangle$ such that $(i, k) \in \Gamma_1$ and $(k, j) \in \Gamma_2$. Since the product is associative, we define powers of Γ inductively by $\Gamma^{k+1} = \Gamma^k\Gamma$ for $k \geqslant 1$. We denote the *diagonal graph*, $\{(i, i)\colon i \in \langle n \rangle\}$ by Δ. The *reflexive transitive closure* [4] $\overline{\Gamma}$ of a graph Γ is defined to be $\overline{\Gamma} = \Delta \cup \Gamma \cup \Gamma^2 \cup \cdots$

For $A \in \mathbb{C}^{nn}$, *the* (directed) *graph* of A is defined by

$$\Gamma(A) = \{(i, j) \mid a_{ij} \neq 0\}.$$

Definition 2.1. $A \in \mathbb{C}^{nn}$ is said to be a *generalized Nekrasov matrix* if all diagonal elements of A are nonzero and A satisfies condition (1.7).

Notice that a Nekrasov matrix is a generalized Nekrasov matrix.

Definition 2.2. Let $A \in \mathbb{C}^{nn}$, D, $-L$ and $-U$ are the diagonal, lower and upper triangular parts of A, respectively. A path $(i_0, i_1, \ldots, i_k)$ from i_0 to i_k in $\Gamma(A)$ for $k \geqslant 2$ is called a *path with property p* if $(i_0, i_1, \ldots, i_{k-1})$ is a path in $\Gamma(D-L)$ and $(i_{k-1}, i_k) \in \Gamma(U)$, we denote by $i \rightarrow^p j$ a path in $\Gamma(A)$ from i to j with property p.

Notice that if $(i_0, i_1, \ldots, i_k)$ is a path with property p from i_0 to i_k in $\Gamma(A)$, then $(i_0, i_1, \ldots, i_k)$ is a path with $i_0 \geqslant i_1 \geqslant \cdots \geqslant i_{k-1}$ and $j = i_k > i_{k-1}$, and vice versa.

Definition 2.3. A generalized Nekrasov matri A is said to satisfy the *chain condition* if for any $i \in \alpha_A$ there exists $j \in \alpha'_A$ such that there is a concatenation path $(i_0, \ldots, i_1, \ldots, i_k)$ where $i_0 = i$ and $i_k = j$ from i to j in $\Gamma(A)$ with $i_0,\ i_1, \ldots, i_k$ pairwise distinct and $i_t \rightarrow^p i_{t+1}$, $t = 0, 1, \ldots, k-1$.

3. On the diagonally dominant matrix

What are the equivalent conditions of a g.d.d. matrix under the assumption that the matrix is a d.d. matrix? In [3] the author provides an answer to this question. But the condition given in [3] is so complicated that it is difficult to verify. Here we improve the result of [3] and present some equivalent conditions of a g.d.d. matrix, which answers the above question.

Recalling the definition of β_A in Section 1 we give the following lemma.

Lemma 3.1. *Let $A \in \mathbb{C}^{nn}$ be a d.d matrix. If $\tilde{A}[\beta_A]$ is a nonsingular M-matrix, then $\tilde{A}$ is a nonsingular M-matrix.*

Proof. If $\beta_A = \emptyset$, then A is an s.d.d. matrix. This implies that conclusion holds. Now let $\beta = \beta_A \neq \emptyset$. By the hypothesis, we have $\beta \neq \langle n \rangle$. Obviously, there is a permutation matrix P such that

$$P\tilde{A}P^{\mathrm{T}} = \begin{bmatrix} \tilde{A}_{11} & -|A_{12}| \\ -|A_{21}| & \tilde{A}_{22} \end{bmatrix},$$

where $\tilde{A}_{22} = \tilde{A}[\beta]$. Noting the definition of β, we obtain

$$-|A_{21}|e_1 + \tilde{A}_{22}e_2 = 0,$$
$$\tilde{A}_{11}e_1 - |A_{21}|e_2 > 0,$$

where $e_i = (1,\dots,1)^{\mathrm{T}}$ is conformable with A_{ii}, $i = 1,2$. From the hypothesis that $\tilde{A}_{22}$ is nonsingular it follows that $e_2 = \tilde{A}_{22}^{-1}|A_{21}|e_1$. By the above strict inequality we have $(\tilde{A}_{11} - |A_{12}|\tilde{A}_{22}^{-1}|A_{21}|)e_2 > 0$, i.e., the Shur complement S_β of A_{22} in $P\tilde{A}P^{\mathrm{T}}$ satisfies that $S_\beta e_2 > 0$. From Lemma 2.3 of [5] it follows that S_β is a Z-matrix. By Theorem 6.2.3(I_{27}) of [6], S_β is a nonsingular M-matrix. It follows from [7], p. 128, that $P\tilde{A}P^{\mathrm{T}}$ (and hence of $\tilde{A}$) is a nonsingular M-matrix. □

Lemma 3.2. *Let $A \in \mathbb{C}^{nn}$. Then A is a g.d.d. matrix if and only if $\tilde{A}$ is a nonsingular M-matrix*

Proof. Well-known (e.g. see [6]). □

Theorem 3.3. *Let $A \in \mathbb{C}^{nn}$ be a d.d. matrix. Then the following statements are equivalent*:

(1) *A is a g.d.d. matrix.*

(2) *A is a c.d.d. matrix.*

(3) *$\tilde{A}[\beta_A]$ is a nonsingular M-matrix.*

(4) *There exists an integer $k, k \in \langle n\rangle$ such that the set series $\{\beta_1,\dots,\beta_k\}$ has the following property*:

$$\emptyset = \beta_k \subseteq \cdots \subseteq \beta_1 \subseteq \langle n\rangle \text{ and } \beta_k \neq \cdots \neq \beta_1 \neq \langle n\rangle,$$

where $\beta_1 = \beta_A$, $\beta_i = \beta_{A_{i-1}}$, and $A_{i-1} = A[\beta_{i-1}]$, $i = 2,\dots,k$.

Proof. (1) $\iff$ (2): Follows immediately from Theorem 6.2.3 (L_{32}) of [6] and Lemma 3.2.

(1) $\Rightarrow$ (3): Let (1) hold. Then $\tilde{A}$ is a nonsingular M-matrix from Lemma 3.2. In view of Theorem 6.2.3(A_1) of [6], $\tilde{A}[\beta_A]$ is a nonsingular M-matrix, which is (3).

(3) $\Rightarrow$ (4): Let $\beta_1 = \beta_A$. Then $\beta_1 \subseteq \langle n\rangle$. If $\beta_1 = \langle n\rangle$, then $\tilde{A}$ is a nonsingular M-matrix from (3), but from the definition of β_A we obtain $\tilde{A}e = 0$, which is a contradiction. Hence $\beta_1 \neq \langle n\rangle$. If $\beta_1 \neq \emptyset$, then let $\beta_2 = \beta_{A[\beta_1]}$, and hence $\beta_2 \subseteq \beta_1$. Assume that $\beta_2 = \beta_1$, then $\tilde{A}[\beta_1]e = 0$. This implies that $\tilde{A}[\beta_1]$ is singular, which contradicts condition (3). Hence $\beta_2 \neq \beta_1$. It follows that $\tilde{A}[\beta_2]$ is a nonsingular M-matrix. If β_2 is nonempty, then going on in this way, by finite steps we can find an integer k, $k \in \langle n\rangle$ such that A_{k-1} is an s.d.d. matrix. In this case, $\beta_k = \emptyset$. (4) is proved.

(4) $\Rightarrow$ (1): Let (4) hold and $\beta_k = \emptyset$. Considering the principal submatrix $\tilde{A}_{k-1} = \tilde{A}[\beta_{k-1}]$, clearly, $\tilde{A}_{k-1}$ is an s.d.d. matrix, and hence $\tilde{A}_{k-1}$ is a nonsingular M-matrix. Now we consider $\tilde{A}_{k-2} = \tilde{A}[\beta_{k-2}]$. Obviously, $\tilde{A}_{k-2}$ is a d.d. matrix. From $\beta_{k-2} = \beta_{\tilde{A}_{k-1}}$ and Lemma 3.1 it follows that $\tilde{A}_{k-2}$ is a nonsingular M-matrix. Repeating the above program one can conclude that $\tilde{A}[\beta_1]$ is a nonsingular M-matrix. From Lemma 3.1 it follows that the statement (1) holds. □

Corollary 3.4. *Let $A \in \mathbb{C}^{nn}$. Then A is a c.d.d matrix if and only if each principal submatrix of A is a c.d.d. matrix.*

Proof. Obviously, it need only show that necessity holds. Let A_1 be any principal submatrix of A. Then A_1 is a d.d. matrix. Because a c.d.d. matrix is a g.d.d. matrix, $\tilde{A}$ is a nonsingular M-matrix by Lemma 3.2, and so is $\tilde{A}_1$, i.e., A_1 is also a g.d.d. matrix. It follows from Theorem 3.3 that A_1 is a c.d.d. matrix. □

Corollary 3.5. *Let $A \in \mathbb{C}^{nn}$ be a d.d. matrix. If $|\beta_A| = 1$, then A is a g.d.d. matrix if and only if all diagonal elements of A are nonzero.*

Proof. If A is a g.d.d. matrix, then $\tilde{A}$ is a nonsingular M-matrix. From Theorem 6.2.3 (A_1) of [6] it follows that all diagonal elements of A are nonzero. Conversely, let $i \in \beta_A$, then $\tilde{A}[\beta_A] = |a_{ii}| > 0$, hence $\tilde{A}[\beta_A]$ is a nonsingular M-matrix. From Theorem 3.3 it follows that A is a g.d.d. matrix. □

4. On the generalized Nekrasov matrix

Throughout the rest of this paper we always assume that $A = D - L - U$, where D, $-L$ and $-U$ are the diagonal and lower and upper triangular parts of A, respectively.

Lemma 4.1. *Let all diagonal elements of A be nonzero. Then* $R_i(A) = |a_{ii}|\,((|D| - |L|)^{-1}|U|e)_i$

Proof. This can be found in [8], p. 239. □

Lemma 4.2. *Let all diagonal elements of A be nonzero. Then* $\Gamma((|D| - |L|)^{-1}|U|) = \overline{\Gamma(L)}\Gamma(U)$.

Proof. It is readily to see $(|D| - |L|)^{-1}|U| = (I - |D|^{-1}|L|)^{-1}|D|^{-1}|U|$. Noting that $I - |D|^{-1}|L|$ is a lower triangular Z-matrix with all positive diagonal elements, thus $I - |D|^{-1}|L|$ is a nonsingular M-matrix. Hence from Lemma 2.2 of [4] we obtain

$$\begin{aligned}\Gamma((|D| - |L|)^{-1}|U|) &= \Gamma((I - |D|^{-1}|L|)^{-1}|D|^{-1}|U|) \\ &= \Gamma((I - |D|^{-1}|L|)^{-1})\Gamma(|D|^{-1}|U|) \\ &= \overline{\Gamma(|D|^{-1}|L|)}\Gamma(|D|^{-1}|U|).\end{aligned}$$

Since $|D| = \mathrm{diag}(|A|)$ is a positive diagonal matrix, $\Gamma(|D|^{-1}|U|) = \Gamma(|U|) = \Gamma(U)$ and $\Gamma(|D|^{-1}|L|) = \Gamma(|L|) = \Gamma(L)$, which leads to the desired result. □

Lemma 4.3. *Let all diagonal elements of A be nonzero. Then A is a g.d.d. matrix if and only if* $I - (|D| - |L|)^{-1}|U|$ *is a nonsingular M-matrix.*

Proof. Follows immediately from Corollary 7.5.22(1) of [6]. □

Lemma 4.4. *Let all diagonal elements of A be nonzero. Then* $(i,j) \in \overline{\Gamma(L)}\Gamma(U)$ *if and only if there is a path from i to j in* $\Gamma(A)$ *with property p, i.e.,* $i \rightarrow^p j$.

Proof. Follows immediately from Definition 2.2. □

Lemma 4.5. *Let* $A \in \mathbb{C}^{nn}$ *be a generalized Nekrasov matrix. Then A satisfies the chain condition if and only if* $I - (|D| - |L|)^{-1}|U|$ *is a c.d.d matrix.*

Proof. Let $B = I - (|D| - |L|)^{-1}|U|$. Then we have $\alpha_A = \beta_B$ from Lemma 4.1, and hence $\alpha'_A = \beta'_B$. Let A satisfy the chain condition. Then for any $i \in \beta_B = \alpha_A$, there exists $j \in \beta'_B$ and pairwise distinct integers $i = i_1, i_2, \ldots, i_k = j$ such that $i_1 \rightarrow^p i_2 \rightarrow^p \cdots \rightarrow^p i_k$. From Lemma 4.4 it follows that $(i_1, i_2, \ldots, i_k)$ is a path from i to j in $\overline{\Gamma(L)}\Gamma(U)$, and hence $(i_1, i_2, \ldots, i_k)$ is a path from i to j in $\Gamma((|D| - |L|)^{-1}|U|)$ from Lemma 4.2. Since $i_1, i_2, \ldots, i_k$ are pairwise distinct, $(i_1, i_2, \ldots, i_k)$ is a path from i to j in $\Gamma(B)$. Hence B is a c.d.d. matrix from Lemma 4.1. Conversely, let B be a c.d.d. matrix. Then for any $i \in \alpha_A = \beta_B$, there exists $j \in \alpha'_A$ and pairwise distinct integers $i = i_1, i_2, \ldots, i_k = j$ such that $(i_1, i_2, \ldots, i_k)$ is a path from i to j in $\Gamma(B)$. Hence $(i_1, i_2, \ldots, i_k)$ is a path from i to j in $\Gamma((|D| - |L|)^{-1}|U|)$. The result follows immediately from Lemmas 4.2 and 4.4. □

The following theorem is a characterization of a g.d.d. matrix under the assumption that the matrix is a generalized Nekrasov matrix.

Theorem 4.6. *Let* $A \in \mathbb{C}^{nn}$ *be a generalized Nekrasov matrix. Then A is a g.d.d matrix if and only if A satisfies the chain condition.*

Proof. From Lemma 4.1 it follows that A is a generalized Nekrasov matrix if and only if $(|D| - |L|)^{-1}|U|e \leqslant e$, i.e., $I - (|D| - |L|)^{-1}|U|$ is a d.d. matrix. Hence $I - (|D| - |L|)^{-1}|U|$ is a g.d.d. matrix (and hence of a nonsingular M-matrix) if and only if $I - (|D| - |L|)^{-1}|U|$ is a c.d.d. matrix by Theorem 3.3. From Lemmas 4.3 and 4.5 it follows that A is a g.d.d matrix if and only if A satisfies the chain condition. □

Corollary 4.7. *Let* $A \in \mathbb{C}^{nn}$ *be a generalized Nekrasov matrix. If for each* $i \in \alpha_A$

$$|a_{ii}| > \sum_{j=1}^{i-1} |a_{ij}| \frac{R_j(A)}{|a_{jj}|},$$

then A is a g.d.d. matrix.

Proof. Let k be a maximal integer in α_A. Then $k \neq n$ and $\sum_{j=k+1}^{n} |a_{kj}| \neq 0$ from hypothesis of this theorem. This implies that there is an integer j, $k < j \leqslant n$ such that $a_{kj} \neq 0$. By the assumption of k we have $j \in \alpha'_A$. Noting $(k,k) \in \Gamma(D-L)$ and $(k,j) \in \Gamma(U)$ with $k < j$, hence there exists $j \in \alpha'_A$ such that $k \to^p j$.

Now let q be a maximal integer in $\alpha_A \setminus k$. Then there exists an integer t, $q < t \leqslant n$ such that $a_{qt} \neq 0$. Hence $q \to^p t$. Clearly, $t = k$ or $t \in \alpha'_A$. If the first case occurs, then, combining the first paragraph of this proof we have $q \to^p k \to^p j \in \alpha'_A$.

Repeating the previous proof, we know that for any $i \in \alpha_A$ there exists $j \in \alpha'_A$ such that $i \to^p \cdots \to^p j$. Hence A satisfies the chain condition, and thus A is a g.d.d. matrix from Theorem 4.6. □

Let $\gamma = \{j\colon\ U_{*j} = 0\}$, where by U_{*j} we mean the jth column of U. Clearly $1 \in \gamma$.

If the matrix is assumed to be irreducible, then we obtain the following theorem.

Theorem 4.8. *Let $A \in \mathbb{C}^{nn}$ be an irreducible generalized Nekrasov matrix. Then A is a g.d.d. matrix if and only if there exists an integer $k \in \gamma'$ such that $|a_{kk}| > R_k(A)$.*

Proof. Let $r_i = \frac{R_i(A)}{|a_{ii}|}$, $i = 1, \ldots, n$ and $r = (r_1, \ldots, r_n)^{\mathrm{T}}$. Then from Lemma 4.1 we have $(|D| - |L|)r = |U|e$, and thus

$$\tilde{A}r = (|D| - |L| - |U|)r = |U|(e - r) \geqslant 0$$

and the latter is $\neq 0$ if and only if there is $k \in \gamma'$ such that $1 > r_k$, $|a_{ii}| > R_k(A)$. Let T be the diagonal matrix with the r_i in the diagonal. Then $\tilde{A}r \neq 0$ if and only if AT is a c.d.d matrix since A is irreducible. From Theorem 3.3 we conclude the theorem holds. □

5. Nonsingularity of a generalized Nekrasov matrix

In this section we apply the previous theorems to give some criteria of nonsingularity for a generalized Nekrasov matrix, which answer the question posed in Section 1.

Lemma 5.1. *A g.d.d. matrix is nonsingular.*

Proof. Well-known. □

The following is one of our main results in the paper.

Theorem 5.2. *Let $A \in \mathbb{C}^{nn}$ be a generalized Nekrasov matrix. Then A is a nonsingular matrix provided one of the following conditions holds*:

(1) *A satisfies the chain condition*;

(2) *A is an irreducible matrix and there exists an integer $k \in \gamma'$ such that $|a_{kk}| > R_k(A)$*;

(3) *For each $i \in \alpha_A$*

$$|a_{ii}| > \sum_{j=1}^{i-1} |a_{ij}| \frac{R_j(A)}{|a_{jj}|}.$$

Proof. The result follows from Theorems 4.6 and 4.8 and Corollary 4.7 and Lemma 5.1. □

Remark. If there is a permutation matrix P such that $B = PAP^{\mathrm{T}}$ is a generalized Nekrasov matrix and satisfies one of the conditions (1)–(3) of Theorem 5.2, then it is easy to see that A is nonsingular.

Remark. It is easy to observe that conditions (1) and (2) of Theorem 5.2 are similar to Eqs. (1.4) and (1.5) respectively, each of conditions (1) and (3) of Theorem 5.2 is weaker than condition Eq. (1.8). The following example illustrates that Theorem 5.2 is a generalization of the mentioned criteria in Section 1 for nonsingularity. Let

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 4 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

Then A is a generalized Nekrasov matrix with $\alpha_A = \{2\}$. Clearly, $2 \rightarrow^{p} 3$. Hence A is nonsingular from Theorem 5.2 (1). Clearly we may also apply Theorem 5.2 (2) to illustrate that A is nonsingular because A is irreducible and $\gamma' = \{3\}$.

Now we consider the location of eigenvalues according to Theorem 5.2.

Corollary 5.3. *Let $A \in \mathbb{C}^{nn}$ be a generalized Nekrasov matrix with all positive diagonal elements. If A satisfies one of conditions* (1)–(3) *in Theorem 5.2, then all eigenvalues of A are located in the open right half-plane.*

Proof. By the same proof as Theorem 2 of [1] one can deduce that the corollary holds. □

The following corollaries are interesting, which provide the characterization of a nonsingular M-matrix for a (irreducible) generalized Nekrasov Z-matrix.

Corollary 5.4. *Let $A \in \mathbb{R}^{nn}$ be a generalized Nekrasov Z-matrix. Then A is a nonsingular M-matrix if and only if A satisfies the chain condition.*

Proof. Follows immediately from Theorem 4.6. □

Corollary 5.5. *Let $A \in \mathbb{R}^{nn}$ be an irreducible generalized Nekrasov Z-matrix. Then A is a nonsingular M-matrix if and only if* $\gamma' \cap \alpha'_A \neq \emptyset$.

Proof. Follows immediately from Theorem 4.8. □

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