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# On the solution of the extended linear complementarity problem

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## Abstract

The extended linear complementarity problem (XLCP) has been introduced in a recent paper by Mangasarian and Pang. In the present research, minimization problems with simple bounds associated to this problem are defined. When the XLCP is solvable, their solutions are global minimizers of the associated problems. Sufficient conditions that guarantee that stationary points of the associated problems are solutions of the XLCP will be proved. These theoretical results support the conjecture that local methods for box constrained optimization applied to the associated problems could be efficient tools for solving the XLCP. © 1998 Elsevier Science Inc. All rights reserved.

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## 1. Introduction

In a recent paper [1] Mangasarian and Pang introduced the extended linear complementarity problem (XLCP). Given  $M, N \in \mathbb{R}^{m \times n}$  and a polyhedral set  $\mathcal{C} \subset \mathbb{R}^m$ , the extended linear complementarity problem associated to  $M, N$  and  $\mathcal{C}$  (XLCP  $(M, N, \mathcal{C})$ ) consists on finding  $x, y \in \mathbb{R}^n$  such that

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$$Mx - Ny \in \mathcal{C}, \quad \langle x, y \rangle = 0, \quad x, y \geq 0. \quad (1)$$

(All along this paper,  $\langle \cdot, \cdot \rangle$  denotes the standard scalar product and  $\|\cdot\|$  denotes the Euclidian norm.) When  $m = n$  and  $\mathcal{C}$  is a single point in  $\mathbb{R}^m$ , this is the so-called horizontal linear complementarity problem. Problems of this type arise in the analysis of resistive piecewise-linear circuits (see Ref. [2]), among other important applications.

The following quadratic (bilinear) program can be associated to Eq. (1)

$$\text{minimize } \langle x, y \rangle \quad \text{subject to} \quad Mx - Ny \in \mathcal{C}, \quad x \geq 0, y \geq 0. \quad (2)$$

Methods for solving the horizontal linear complementarity problem based on Eq. (2) can be found, for example, in Ref. [3,4]. We say that the problem (1) is *feasible* if the feasible region of Eq. (2) is nonempty.

Proposition 2.2 of Ref. [9] relates stationary points of Eq. (2) to solutions of Eq. (1) when the polyhedral set is given in the form  $\mathcal{C} = \{u \in \mathbb{R}^m \mid Au - b \geq 0\}$  ( $A \in \mathbb{R}^{\ell \times m}$ ). This result, which was later generalized in Ref. [5], states that a stationary point of Eq. (2) is necessarily a solution of Eq. (1) if  $MN^T$  is copositive on the cone

$$\mathcal{R} \equiv \{v \in \mathbb{R}^m \mid v = A^T \lambda \text{ for some } \lambda \in \mathbb{R}_+^\ell\}.$$

We will consider two different representations of the polyhedral set  $\mathcal{C}$ . In the first case,  $\mathcal{C}$  is defined by a set of equalities and inequalities. So, we can set, without loss of generality,

$$\mathcal{C} = \{u \in \mathbb{R}^m \mid Au - b - z = 0, \quad z = (z_1, 0), \quad z_1 \in \mathbb{R}^p, z_1 \geq 0\}, \quad (3)$$

where  $A \in \mathbb{R}^{\ell \times m}$ ,  $b \in \mathbb{R}^\ell$ ,  $z \equiv (z_1, 0) \in \mathbb{R}^{\ell+1}$ .

Following the ideas developed in Refs. [6–8], we propose the associated bound-constrained problem given by

$$\begin{aligned} & \text{minimize} \quad \langle x, y \rangle^2 + \rho \|AMx - ANy - b - z\|^2 \\ & \text{subject to} \quad x \geq 0, \quad y \geq 0, \quad z_1 \geq 0, \end{aligned} \quad (4)$$

where  $\rho > 0$  is an arbitrary constant. If  $(x_*, y_*, z_*)$  is a global minimizer of Eq. (4) for which the value of the objective function is zero, then  $(x_*, y_*)$  is a solution of Eq. (1). However, most efficient algorithms for solving bound-constrained problems like Eq. (4) are proved to find stationary points and not global minimizers. Therefore, it is interesting, from a practical point of view, to find sufficient conditions under which stationary points of Eq. (4) are in fact solutions of Eq. (1). It will be proved here that a condition introduced by Gowda [5] for problem (2) is also sufficient for guaranteeing that stationary points of Eq. (4) are solutions of the XLCP. Moreover, we will prove a converse proposition for the case in which  $\mathcal{C}$  is defined by a affine subspace. This includes, of course, the horizontal linear complementarity problem. It is interesting to

observe that, if one uses  $\langle x, y \rangle$  instead of  $\langle x, y \rangle^2$  in the definition of the objective function of Eq. (4) the above mentioned conditions are not sufficient anymore.

In some cases, the polyhedral set  $\mathcal{C}$  can be defined in a parametric form, instead of the implicit form (3). In other words,  $\mathcal{C}$  can be viewed as the image of a simple cone of  $\mathbb{R}^s$ , namely

$$\mathcal{C} = \{w \in \mathbb{R}^m \mid w = Lz + q, z = (z_1, z_2), z_1 \in \mathbb{R}^r, z_1 \geq 0, z_2 \in \mathbb{R}^{s-r}\}. \quad (5)$$

This possibility is also considered in the present research. In fact, corresponding to Eq. (5) we define the following associated bound-constrained problem:

$$\begin{aligned} & \text{minimize} && \langle x, y \rangle^2 + \rho \|Mx - Ny - Lz - q\|^2 \\ & \text{subject to} && x \geq 0, \quad y \geq 0, \quad z_1 \geq 0. \end{aligned} \quad (6)$$

Analogously to the case (3), we can use box-constraint solvers for solving Eq. (6). Consequently, we are also presenting a theorem that states sufficient conditions under which stationary points of Eq. (6) are solutions of Eq. (1).

## 2. Main results

The Hadamard (componentwise) product between two vectors will be denoted  $x \circ y$ . So,

$$x \circ y = (x_1 y_1, \dots, x_n y_n) \quad \forall x, y \in \mathbb{R}^n.$$

We also denote, for all  $x \in \mathbb{R}^n$ ,

$$x^+ = (\max\{0, x_1\}, \dots, \max\{0, x_n\}), \quad x^- = x - x^+.$$

Given the polyhedral set  $\mathcal{C} \subset \mathbb{R}^m$ , its recession cone will be denoted  $0^+ \mathcal{C}$ . Therefore, if  $\mathcal{C}$  is defined by Eq. (3), we have that

$$0^+ \mathcal{C} = \{u \in \mathbb{R}^m \mid Au = \delta, \delta = (\delta_1, 0), \delta_1 \geq 0 \in \mathbb{R}^r\}$$

and

$$(0^+ \mathcal{C})^* = \{v \in \mathbb{R}^m \mid v = A^T \lambda, \lambda = (\lambda_1, \lambda_2), \lambda_1 \geq 0 \in \mathbb{R}^r\}.$$

On the other hand, if  $\mathcal{C}$  is defined by Eq. (5),

$$0^+ \mathcal{C} = \{u \in \mathbb{R}^m \mid u = L\delta, \delta = (\delta_1, \delta_2), \delta_1 \geq 0 \in \mathbb{R}^r\}$$

and

$$(0^+ \mathcal{C})^* = \{v \in \mathbb{R}^m \mid L^T v = \lambda, \lambda = (\lambda_1, 0), \lambda_1 \geq 0 \in \mathbb{R}^r\}.$$

Let us denote  $C_i$  the  $i$ th column of a matrix  $C$ . Given  $M, N \in \mathbb{R}^{m \times n}$ , we say that  $\{\bar{M}, \bar{N}\}$  is a column rearrangement of  $\{M, N\}$  if for each index  $i$ ,  $\bar{M}_i = M_i$  and  $\bar{N}_i = N_i$  or  $\bar{M}_i = N_i$  and  $\bar{N}_i = M_i$ .

The following definition was introduced by Gowda in Ref. [5].

**Definition 2.1.** Given matrices  $M, N \in \mathbb{R}^{m \times n}$  and a polyhedral set  $\mathcal{C} \subset \mathbb{R}^m$ , we say that  $\{M, N\}$  has the extended row-sufficiency property with respect to  $\mathcal{C}$  if the following condition holds:

$$\left. \begin{array}{l} w \in (0^+ \mathcal{C})^* \\ u = M^T w, \quad v = N^T w \\ u \circ v \leq 0 \end{array} \right\} \Rightarrow u \circ v = 0. \quad (7)$$

Note that this property is invariant under column rearrangements.

**Theorem 2.1.** Assume that the pair  $\{M, N\}$  has the extended row-sufficiency property with respect to  $\mathcal{C}$  defined by Eq. (3). Then for each  $f \in \mathbb{R}^m$  and every column rearrangement  $\{\bar{M}, \bar{N}\}$  such that Eqs. (1)–(3) is feasible, every stationary point of

$$\begin{aligned} & \text{minimize} \quad \langle x, y \rangle^2 + \rho \|A\bar{M}x - A\bar{N}y - b - Af - z\|^2 \\ & \text{subject to} \quad x \geq 0, \quad y \geq 0, \quad z_1 \geq 0, \end{aligned}$$

is a solution of  $\text{XLCP}(\bar{M}, \bar{N}, \mathcal{C} + f)$ .

**Proof.** Since the extended row-sufficiency property is invariant under column rearrangements and  $\mathcal{C} + f = \{u \in \mathbb{R}^m \mid Au - b - Af - z = 0, z = (z_1, 0), z_1 \geq 0\}$ , then without loss of generality, it is enough to prove the desired result for the problem  $\text{XLCP}(M, N, \mathcal{C})$ .

Let us call

$$u_* = AMx_* - ANy_* - b - z_*. \quad (8)$$

The first-order optimality conditions of Eq. (4) are:

$$2\rho M^T(A^T u_*) + 2\langle y_*, x_* \rangle y_* - \mu_1 = 0, \quad (9)$$

$$-2\rho N^T(A^T u_*) + 2\langle x_*, y_* \rangle x_* - \mu_2 = 0, \quad (10)$$

$$-2\rho[u_*]_i - [\mu_3]_i = 0, \quad i = 1, \dots, p, \quad (11)$$

$$\langle x_*, \mu_1 \rangle = 0, \quad (12)$$

$$\langle y_*, \mu_2 \rangle = 0, \quad (13)$$

$$\langle [z_1]_*, \mu_3 \rangle = 0, \quad (14)$$

$$x_* \geq 0, \quad y_* \geq 0, \quad [z_1]_* \geq 0, \quad \mu_1 \geq 0, \quad \mu_2 \geq 0, \quad \mu_3 \geq 0. \quad (15)$$

By Eqs. (9)–(15), we have that

$$4\rho^2[M^T(A^T(-u_*))]_i[N^T(A^T(-u_*))]_i = -4\langle x_*, y_* \rangle^2[y_*]_i[x_*]_i - [\mu_1]_i[\mu_2]_i \leq 0, \quad (16)$$

for all  $i = 1, \dots, n$  and

$$-[u_*]_i = \frac{[\mu_3]_i}{2\rho} \geq 0, \quad i = 1, \dots, p. \quad (17)$$

This implies that  $A^T(-u_*) \in (0^+ \mathcal{C})^*$ . Since  $\{M, N\}$  has the extended row-sufficiency property with respect to  $\mathcal{C}$ , then Eq. (16) and Eq. (17) imply that

$$[M^T A^T(-u_*)]_i [N^T A^T(-u_*)]_i = 0 \quad \text{for all } i = 1, \dots, n.$$

Therefore, by Eq. (16) we obtain

$$[y_*]_i [x_*]_i = 0 \quad \text{for all } i = 1, \dots, n. \quad (18)$$

Now, by Eq. (18), Eqs. (9)–(15) remain

$$2\rho(AM)^T u_* - \mu_1 = 0, \quad (19)$$

$$-2\rho(AN)^T u_* - \mu_2 = 0, \quad (20)$$

$$-2\rho u_* - \mu_3 = 0, \quad (21)$$

$$\langle x_*, \mu_1 \rangle = 0, \quad (22)$$

$$\langle y_*, \mu_2 \rangle = 0, \quad (23)$$

$$\langle [z_1]_*, \mu_3 \rangle = 0, \quad (24)$$

$$x_* \geq 0, \quad y_* \geq 0, \quad [z_1]_* \geq 0, \quad \mu_1 \geq 0, \quad \mu_2 \geq 0, \quad \mu_3 \geq 0. \quad (25)$$

But Eqs. (19)–(25) are necessary and sufficient conditions for global minimizers of the following convex quadratic minimization problem:

$$\begin{aligned} & \text{minimize} && \rho \|AMx - ANy - b - z\|^2 \\ & \text{subject to} && x \geq 0, y \geq 0, z_1 \geq 0. \end{aligned} \quad (26)$$

Since the XLCP is feasible, it turns out that  $(x_*, y_*, z_*)$  is a global solution of Eq. (26) with minimum value zero, that is

$$AMx_* - ANy_* - b - z_* = 0. \quad (27)$$

Therefore, by Eq. (18) and Eq. (27),  $(x_*, y_*)$  is a solution of the extended linear complementarity problem.  $\square$

The following theorem is a converse result for Theorem 2.1 in the case in which  $\mathcal{C}$  is an affine subspace. Gowda [5] proved that a similar converse property hold for general  $\mathcal{C}$  in the case of formulation (2).

**Theorem 2.2.** Assume that  $\mathcal{C} = \{u \in \mathbb{R}^m \mid Au = b\} \neq \emptyset$  and that for each  $f \in \mathbb{R}^m$  and every column rearrangement  $\{\bar{M}, \bar{N}\}$  such that  $XLCP(\bar{M}, \bar{N}, \mathcal{C} + f)$  is feasible, every stationary point of

$$\begin{aligned} & \text{minimize} \quad \langle x, y \rangle^2 + \rho \|A\bar{M}x - A\bar{N}y - b - Af\|^2 \\ & \text{subject to} \quad x \geq 0, y \geq 0 \end{aligned}$$

solves  $XLCP(\bar{M}, \bar{N}, \mathcal{C} + f)$ . Then the pair  $\{M, N\}$  has the extended row-sufficiency property with respect to  $\mathcal{C}$ .

**Proof.** Assume that the extended row-sufficiency property of  $\{M, N\}$  with respect to  $\mathcal{C}$  does not hold. Without loss of generality (we can interchange columns  $M_j$  and  $N_j$  and work with a column rearrangement of  $\{M, N\}$ ), we can ensure that there exists a nonnull  $r_* \in \mathbb{R}^\ell$  and an index  $j \in \{1, \dots, n\}$  such that

$$0 \neq [M^T A^T r_*]_i \geq 0 \quad \text{and} \quad 0 \neq [N^T A^T r_*]_i \leq 0$$

for all  $i = 1, \dots, n$ , but

$$[M^T r_*]_j [N^T r_*]_j < 0.$$

Let us define

$$\mathcal{B} = \{t \in \mathbb{R}^\ell \mid t = AMx - ANy, x \geq 0, y \geq 0\}.$$

Let us call  $u_* = AM\bar{x} - AN\bar{y}$  the orthogonal projection of  $r_*$  on  $\mathcal{B}$ . By the convexity of  $\mathcal{B}$ , we have that

$$\langle r_* - u_*, AMx - ANy - (AM\bar{x} - AN\bar{y}) \rangle \leq 0 \tag{28}$$

for all  $x, y \geq 0$ . Therefore,

$$\sum_{i=1}^n ([M^T A^T r_*]_i - [M^T A^T u_*]_i) [x - \bar{x}]_i - \sum_{i=1}^n ([N^T A^T r_*]_i - [N^T A^T u_*]_i) [y - \bar{y}]_i \leq 0$$

for all  $x, y \geq 0$ . This implies that

$$0 \leq [M^T A^T r_*]_i \leq [M^T A^T u_*]_i, \quad 0 \geq [N^T A^T r_*]_i \geq [N^T A^T u_*]_i \tag{29}$$

for all  $i = 1, \dots, n$  and

$$[M^T u_*]_j [N^T u_*]_j < 0.$$

So,

$$[M^T A^T (-u_*)] \leq 0, \quad [N^T A^T (-u_*)] \geq 0.$$

Now, define

$$\beta = \rho^2 \langle (N^T A^T(-u_*))^+, (M^T A^T(-u_*))^+ \rangle > 0,$$

$$x_* = \rho \frac{(N^T A^T(-u_*))^+}{\beta^{\frac{1}{3}}}, \quad \mu_1 = 0, \quad (30)$$

$$y_* = \rho \frac{(M^T A^T(-u_*))^+}{\beta^{\frac{1}{3}}}, \quad \text{and} \quad \mu_2 = 0. \quad (31)$$

Clearly,  $\langle x_*, y_* \rangle > 0$ . In addition,

$$2\rho M^T A^T(-u_*) + 2\langle y_*, x_* \rangle y_* - \mu_1 = 0, \quad (32)$$

$$-2\rho N^T A^T(-u_*) + 2\langle x_*, y_* \rangle x_* - \mu_2 = 0, \quad (33)$$

$$\langle x_*, \mu_1 \rangle = 0, \quad (34)$$

$$\langle y_*, \mu_2 \rangle = 0, \quad (35)$$

$$x_* \geq 0, \quad y_* \geq 0, \quad \mu_1 \geq 0, \quad \mu_2 \geq 0. \quad (36)$$

Since  $\mathcal{C}$  is nonempty, there exists  $f_*$  satisfying

$$A(Mx_* - Ny_* + M\bar{x} - N\bar{y} - f_*) = b.$$

$$Af_* = AMx_* - ANy_* + u_* - b.$$

Thus,

$$\begin{aligned} AMx - ANy - Af_* &= AMx - ANy - AMx_* + ANy_* - u_* + b \\ &= AM(x - \bar{x} - x_*) - AN(y - \bar{y} - y_*) + b. \end{aligned}$$

Now, taking  $\tilde{x} = x_* + \bar{x} \geq 0$ ,  $\tilde{y} = y_* + \bar{y} \geq 0$ , we see that  $AM\tilde{x} - AN\tilde{y} - Af_* = b$ , so that the problem XLCP( $M, N, \mathcal{C} + f_*$ ) is feasible.

Summing up, using Eqs. (30)–(36), we see that  $(x_*, y_*)$  is a stationary point of

$$\begin{aligned} \text{minimize} \quad & \langle x, y \rangle^2 + \rho \|AMx - ANy - b - Af_*\|^2 \\ \text{subject to} \quad & x \geq 0, \quad y \geq 0. \end{aligned}$$

However, we saw that  $(x_*, y_*)$  is infeasible and not complementary for the feasible problem XLCP( $M, N, \mathcal{C} + f_*$ ). This completes the proof.  $\square$

**Remark.** The bound-constrained problem (4) can be interpreted as a single external penalty subproblem associated to the minimization of  $\langle x, y \rangle^2$  subject to  $Mx - Ny \in \mathcal{C}$ ,  $x \geq 0$ ,  $y \geq 0$ . Therefore, Theorem 2.1 states a single sufficient

condition under which any stationary point of any of those subproblems, independently of the penalty parameter, is a global minimizer of the constrained original problem. On the other hand, the external penalty subproblem associated to Eq. (2) and Eq. (3) is

$$\begin{aligned} & \text{minimize} \quad \langle x, y \rangle + \rho \|AMx - ANy - b - z\|^2 \\ & \text{subject to} \quad x \geq 0, y \geq 0, z_1 \geq 0. \end{aligned} \tag{37}$$

However, stationary points of Eq. (37) are not necessarily solutions of Eq. (1) under the general conditions of Theorem 2.2. For example, consider the example given in Ref. [8], in which  $n = 1$ ,  $\mathcal{C} = \{(x, y) \in \mathbb{R}_+^2 \mid 1 - y = 0\}$ . Here  $A = 1$ ,  $M = 0$  and  $N = 1$ . Clearly the extended row-sufficiency condition is satisfied. However, for  $\rho = 1$ ,  $(2, 0)$  is a stationary point of Eq. (37) but it is not a solution of Eq. (1).

Now we analyze the stationary points of Eq. (6), connected with definition Eq. (5) of  $\mathcal{C}$ . Let us recall that a particular case of Eq. (5) corresponds to the case in which  $z \geq 0$ . This case is known as “generalized linear complementarity problem” in the literature. Moreover, when the matrix  $L$  is null, the XLCP with the definition Eq. (5) is the horizontal linear complementarity problem. Assume that  $l_1, \dots, l_s \in \mathbb{R}^m$  are the columns of the  $m \times s$  matrix  $L$ .

**Theorem 2.3.** *Assume that the pair  $\{M, N\}$  has the extended row-sufficiency property with respect to  $\mathcal{C}$  defined by Eq. (5). Then for each  $f \in \mathbb{R}^m$  and every column rearrangement  $\bar{M}, \bar{N}$  such that  $\text{XLCP}(\bar{M}, \bar{N}, \mathcal{C} + f)$  is feasible, every stationary point of Eq. (6) is a solution of the  $\text{XLCP}(\bar{M}, \bar{N}, \mathcal{C} + f)$ .*

**Proof.** Since the extended row-sufficiency property is invariant under column rearrangements and  $\mathcal{C} + f = \{u \in \mathbb{R}^m \mid u = Lz + q + f, z_1 \geq 0\}$ , it is sufficient to prove the property for the  $\text{XLCP}(M, N, \mathcal{C})$ . Let us call

$$u_* = Mx_* - Ny_* - Lz_* - q \tag{38}$$

The first-order optimality conditions of Eq. (6) are:

$$2\rho M^T u_* + 2\langle x_*, y_* \rangle y_* - \mu_1 = 0, \tag{39}$$

$$-2\rho N^T u_* + 2\langle x_*, y_* \rangle x_* - \mu_2 = 0, \tag{40}$$

$$-2\rho \langle l_i, u_* \rangle - [\mu_3]_i = 0, \quad i = 1, \dots, v, \tag{41}$$

$$\langle l_i, u_* \rangle = 0, \quad i = v + 1, \dots, s, \tag{42}$$

$$\langle x_*, \mu_1 \rangle = 0, \tag{43}$$

$$\langle y_*, \mu_2 \rangle = 0, \tag{44}$$

$$\langle [z_1]_*, \mu_3 \rangle = 0, \quad (45)$$

$$x_* \geq 0, \quad y_* \geq 0, \quad [z_*]_1 \geq 0, \quad \mu_1 \geq 0, \quad \mu_2 \geq 0. \quad (46)$$

By Eq. (41) and Eq. (42) and the form of  $0^+ \mathcal{C}$  in the case (5) we have that  $-u_* \in (0^+ \mathcal{C})^*$ . Now, by Eqs. (39)–(46),

$$4\rho^2 [M^T(-u_*)]_i [N^T(-u_*)]_i = -4\langle x_*, y_* \rangle^2 [y_*]_i [x_*]_i - [\mu_1]_i [\mu_2]_i \leq 0 \quad (47)$$

for all  $i = 1, \dots, n$ . By the hypotheses of the theorem, we obtain

$$[y_*]_i [x_*]_i = 0 \quad \text{for all } i = 1, \dots, n. \quad (48)$$

Now, using the arguments of Theorem 2.2, the desired result follows straightforwardly.  $\square$

As we mentioned above, the horizontal linear complementarity problem is the particular case of Eq. (5) that corresponds to  $L = 0$ . Therefore, the corresponding associated problem is

$$\begin{aligned} & \text{minimize} && \langle x, y \rangle^2 + \rho \|Mx - Ny - q\|^2 \\ & \text{subject to} && x \geq 0, \quad y \geq 0, \end{aligned} \quad (49)$$

and we obtain the sufficient condition given by the following corollary, which generalizes Theorem 1 of Ref. [8]. In fact, this condition is the one given in Ref. [1] for proving that stationary points of the bilinear program associated to the horizontal linear complementarity problem are solutions of this problem. Moreover, applying Theorem 2.2 to the horizontal linear complementarity case, we also obtain a converse result for that problem. Both results are condensed in the following corollary.

**Corollary 2.4.** *Let  $(x_*, y_*)$  be a stationary point of Eq. (49), assume that the complementarity problem is feasible and the pair  $(M, N)$  is row-sufficient. Then  $(x_*, y_*)$  is a solution of the horizontal linear complementarity problem. Moreover, if for all the choices of  $q \in \mathbb{R}^n$  and column rearrangements that make the horizontal linear complementarity problem feasible, every stationary point of Eq. (49) is a solution, then the pair  $\{M, N\}$  is row-sufficient.*

### 3. Conclusions

The consequences of the results proved in this paper are mainly practical. Solving the XLCP using Eq. (2) involves the consideration of a quadratic programming problem while the solution by means of Eq. (4) or Eq. (6) requires the minimization of a (quartic) function with positivity constraints. From the point of view of global minimization, Eq. (2) can be reduced to

the minimization of a quadratic with positivity constraints. However, the conditions under which stationary points of this problem are global minimizers are stronger than the conditions under which stationary points of Eq. (2) are solutions of the XLCP. On the other hand, the conditions guaranteeing that stationary points of Eq. (2) are global solutions also guarantee that stationary points of the quartic reformulation solve the XLCP. The fact that the “bounded quartic” reformulation is better than the “bounded quadratic” reformulation is perhaps surprising and has been exploited for other problems in Ref. [6–10], among others. See, also, the survey [11].

The reformulation of complementarity, variational inequalities and equilibrium problems as simple optimization problems is an area of current intense research. More than 30 communications on this subject were presented in a specific cluster in the recent International Symposium of Mathematical Programming held in Lausanne (1997). Alternative bound constrained and unconstrained reformulations of the XLCP were also introduced recently in Ref. [12]. The appeal of our reformulations lies on the polynomial (fourth degree) structure of the objective function, which suggests that much research trying to exploit this very particular algebraic form is deserved.

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