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When to call a linear system nonnegative

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Abstract

In this paper we will consider discrete time invariant linear systems that allow for an input-state-output representation with a finite dimensional state space, and that have a finite number of inputs and outputs. The basic issue in this paper is when to call these systems nonnegative. An important concept in this respect is that of the most powerful unfalsified model.

Keywords: Linear systems; Behavioral theory; Nonnegativity

1. Introduction

In the sequel our time axis is $T = \mathbb{Z}_+$. Classically, a linear time invariant finite dimensional system behavior is defined by

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t \end{aligned} \qquad (\star)$$

and the system behavior is by definition given by

$$\mathscr{B} := \{(u,y) \in (\mathbb{R}^q)^{\mathbb{Z}_+} \mid \exists x \in (\mathbb{R}^n)^{\mathbb{Z}_+} \text{ such that } (\star) \text{ holds}\}.$$

Here, of course it is understood that $A \in \mathbb{R}^{n\times n}$, for some $n \in \mathbb{Z}_+$, and that the sum of inputs and outputs is $q \in \mathbb{N}$. We invariably take $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let us write $C(sI - A)^{-1}B = \sum_{i=0}^{\infty} M_i s^{-i}$. This defines the impulse response $\{M_0, M_1, \ldots\}$.

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Classically, $\mathscr{B}$ is called nonnegative if $M_i \geqslant 0$, $\forall i \in \mathbb{Z}_+$.

In view of the linear systems put forward by Willems and his co-workers in for instance [1,2] a linear system with time axis $\mathbb{Z}_+$ is a set $\mathscr{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ that can be represented by

$$\mathscr{B} = \{\omega \in (\mathbb{R}^q)^{\mathbb{Z}_+} \mid R(\sigma)\omega = 0\},$$

where $R(s) \in \mathbb{R}^{g\times q}[s]$, the set of polynomial matrices with real valued coefficients, and with g rows and q columns. Here $\sigma^k : \mathbb{R}^{\mathbb{Z}_+} \to \mathbb{R}^{\mathbb{Z}_+}$, $k \in \mathbb{Z}_+$, is defined by $(\sigma^k\omega)(t) := \omega(t+k)$, $\forall t \in \mathbb{Z}_+$.

The Greek letter σ is a mnemonic device for 'shift'.

Basically, we address the following question:

When would we call $\ker R(\sigma)$ nonnegative?

The connection with the classical theory of linear systems is among others that $\ker R(\sigma)$ can be represented by an input-state-output representation:

$$\sigma x = Ax + Bu, \qquad y = Cx + Du,$$

where A, B, C, and D are real valued matrices with a finite number of components, and where, possibly after a permutation of the components in ω we have $\omega = (u, y)$.

Another basic question is the following one:

Suppose the linear system $\mathscr{B}$ is nonnegative is some sense. How can we represent $\mathscr{B}$ in such a way that its nonnegativity is obvious from this representation?

We will only briefly discuss this representation issue in the sequel.

Classically, this representation issue is always phrased as follows: Suppose you have $C(sI - A)^{-1}B = \Sigma M_i s^{-i}$ where all the matrices M_i are nonnegative, when is it possible to find nonnegative matrices $\bar{C}$, $\bar{A}$, and $\bar{B}$ such that $\Sigma M_i s^{-i} = \bar{C}(sI - \bar{A})^{-1}\bar{B}$.

This issue has already been discussed for a considerable period of time. For the state of the art we refer to the beautiful paper [3].

In the sequel we will also rephrase this classical representation problem slightly, basically because an impulse response does not uniquely determine a linear system. Later on we will state precisely what we mean by that.

The rest of this paper consists of the following parts. First of all we give a brief introduction to the behavioral theory of linear systems as developed by Willems in for instance [1,2]. Then we will discuss in the next part the concept of identifiability, and that of the most powerful unfalsified behavior. An important reference in this respect is a paper written by Hey [4].

After we have done so, we will give a definition of nonnegative behaviors. We state and prove some results concerning nonnegative behaviors. We will mostly restrict ourselves to autonomous behaviors with one variable or input–output behaviors with two variables. As such, our paper is only the beginning of the exploration of the notion of nonnegative behavior. Then we continue with a brief section on nonnegative realizations of scalar autonomous behaviors. After we have done so, we discuss in some detail the notions of nonnegative and strongly nonnegative input–output behaviors.

The paper ends with some concluding remarks, and a list of references.

2. A brief introduction to the behavioral theory of linear systems

In the sequel our time axis is $T = \mathbb{Z}_+$. In order to define what a behavior is, we need to introduce shift operators.

Definition 1. $\forall n \in \mathbb{N}$, $\forall k \in \mathbb{Z}_+$, $\sigma_n^k\colon (\mathbb{R}^n)^T \to (\mathbb{R}^n)^T$ is defined by $(\sigma_n^k \omega)(t) := \omega(t+k), \forall t \in \mathbb{Z}_+ = T$. The operators σ_n^k are called shift operators.

In the sequel we will invariably write σ^k instead of σ_n^k as will be always clear from the context on which space σ^k is defined.

We endow all spaces $(\mathbb{R}^n)^T$ with the topology of pointwise convergence.

Definition 2. Let $\forall i \in \mathbb{N}$, $\omega^i \in (\mathbb{R}^n)^T$. Let also $\omega \in (\mathbb{R}^n)^T$. Then we say that ω^i converges to ω if $\forall t \in T$

$$\lim_{i\to\infty} \omega^i(t) = \omega(t).$$

In this case we write $\omega^i \to \omega$. A set $S \subseteq (\mathbb{R}^n)^T$ is closed whenever $\omega^i \in S$ and $\omega^i \to \omega$ implies $\omega \in S$.

We are now ready to define what a behavior $\mathscr{B} \subseteq (\mathbb{R}^q)^T$ is, where $q \in \mathbb{N}$.

Definition 3. $\mathscr{B} \subseteq (\mathbb{R}^n)^T$ is a behavior if:

(1) $\mathscr{B}$ is a linear subspace,

(2) $\mathscr{B}$ is closed,

(3) $\sigma\mathscr{B} \subseteq \mathscr{B}$.

A basic issue is that of representing behaviors.

In order to discuss representation issues we need the following notation.

Notation 1. $\forall g \in \mathbb{N}$, $\forall q \in \mathbb{N}$, $\mathbb{R}^{g\times q}[s]$ is the set of $g \times q$ matrices $R(s) = (R(s)_{ij})$ such that $\forall i, j$

$$R(s)_{ij} \in \mathbb{R}[s],$$

the set of polynomials in the indeterminate s with real coefficients. By $\mathbb{R}^{\cdot \times q}[s]$ we denote $\bigcup_{g \in \mathbb{N}} \mathbb{R}^{g \times q}[s]$.

The following result is the basic representation theorem of behaviors.

Theorem 1 (see [1,2]). *Let $\mathscr{B} \subseteq (\mathbb{R}^q)^T$ be a behavior. Then there is a matrix $R(s) \in \mathbb{R}^{\cdot \times q}[s]$ such that $\mathscr{B} = \ker R(\sigma) := \{\omega \mid R(\sigma)\omega = 0\}$. One may choose $R(s)$ such that $R(s)$ has full row-rank (over $\mathbb{R}(s)$, the rational functions). Conversely, for all $R(s) \in \mathbb{R}^{\cdot \times q}[s]$ we have that* $\ker R(\sigma)$ *is a behavior.*

The following result is also very useful in the sequel.

Theorem 2 (see [1,2]). *Let $\mathscr{B}_i = \ker R_i$, $i = 1, 2$, where R_1 and R_2 are polynomial matrices. Then $\mathscr{B}_1 \subseteq \mathscr{B}_2$ if and only if there is a polynomial matrix V such that $R_2 = VR_1$.*

In order to make a connection with the classical theory of linear systems we have to introduce some new notions.

Let us call $f(s) \in \mathbb{R}(s)$ proper if there are polynomials $d(s)$, and $n(s)$ with $d(s) \neq 0$ and with degree $n(s) \leqslant$ degree $d(s)$ such that $d(s)f(s) = n(s)$.

In order to introduce the notion of input–output representation of a behavior one further notion is needed.

Definition 4. We call a behavior $\mathscr{B} \subseteq (\mathbb{R}^q)^T$ autonomous if $\mathscr{B}$ is finite dimensional.

The following result can now be stated.

Theorem 3 (see [1,2]). *$\mathscr{B}$ is autonomous if and only if $\mathscr{B} = \ker R(\sigma)$ with $R(s) \in \mathbb{R}^{q \times q}[s]$ such that* $\det R(s) \neq 0$.

We now come to inputs and outputs.

Theorem 4 (see [1,2]). *Let $\mathscr{B}$ be a nonautonomous behavior. Then there is a partitioning of the variables in ω, $\omega = (u, y)$ such that $\mathscr{B}$ can be represented by $P(\sigma)y = Q(\sigma)u$ where $(P(s))^{-1}Q(s)$ only has proper elements.*

In this case we call the variables in u the *inputs*, and the variables in y the *outputs*. We would like to stress the fact that in general this decomposition is not unique.

Now we come, as promised, to the connection with the classical theory of linear systems.

Definition 5. Let $R(s)$, $\bar{R}(s)$, and $\bar{M}(s)$ be polynomial matrices. We say that $\ker R(\sigma)$ allows for a representation $\bar{R}(\sigma)\omega = \bar{M}(\sigma)\zeta$ if $\ker R(\sigma) = \{\omega \mid \exists\zeta$ with $\bar{R}(\sigma)\omega = \bar{M}(\sigma)\zeta\}$.

In this case we call the variables in ζ auxiliary.

One can prove the following, see [1,2].

Theorem 5. *Let* $\bar{\mathscr{B}} := \{(\omega,\zeta) \mid \bar{R}(\sigma)\omega = \bar{M}(\sigma)\zeta\}$ *be a behavior. Then* $\{\omega \mid \exists\zeta$ *with* $(\omega,\zeta) \in \bar{\mathscr{B}}\}$ *is also a behavior.*

Theorem 6 (see [1,2]). 1. *Let* $\mathscr{B}$ *be an autonomous behavior. Then* $\mathscr{B}$ *allows for a representation*

$$\sigma x = Ax, \quad \omega = Cx$$

for some matrices A and C with a finite number of entries. So $\mathscr{B} = \{\omega \mid \exists x$ *with* $\sigma x = Ax,\ \omega = Cx\}$.

2. *Let* $\mathscr{B}$ *be a nonautonomous behavior with input–output representation* $P(\sigma)y = Q(\sigma)u$. *Then, for some matrices A, B, C, and D, all real matrices with a finite number of entries we have, with* $A \in \mathbb{R}^{n\times n}$, $\mathscr{B} = \{(y,u) \mid \exists x \in (\mathbb{R}^n)^T$ *with* $\sigma x = Ax + Bu,\ y = Cx + Du\}$. *So* $\mathscr{B}$ *is represented by:*

$$\sigma x = Ax + Bu, \qquad y = Cx + Du.$$

The variables in x are called state-variables, and the last representation is called an input-state-output representation.

We need one final notion, namely, that of controllability.

Definition 6. We call a behavior $\mathscr{B}$ controllable if for some polynomial matrix $M(s)$ we have $\mathscr{B} = \operatorname{im} M(\sigma) := \{\omega \mid \exists\zeta$ with $\omega = M(\sigma)\zeta\}$.

One can prove the following results.

Theorem 7 (see [1,2,5]). 1. *The behavior* $\mathscr{B} \subseteq (\mathbb{R}^q)^T$ *is controllable if and only if for some* $R(s)$ *with full row-rank, and with the further property that rank* $R(\lambda)$ *is constant,* $\forall\lambda \in \mathbb{C}$ *we have* $\mathscr{B} = \ker R(\sigma)$.

2. *Every behavior* $\mathscr{B}$ *can be written as* $\mathscr{B} = \mathscr{B}^a + \mathscr{B}^c$ *where* $\mathscr{B}^a$ *is autonomous, and* $\mathscr{B}^c$ *is controllable.* $\mathscr{B}^a$ *is not necessarily unique whereas* $\mathscr{B}^c$ *is unique. So, we may call* $\mathscr{B}^c$ *the controllable part of* $\mathscr{B}$.

It is the controllable part that determines the impulse response. In order to make this precise, let $\mathscr{B}$ be given by $P(\sigma)y = Q(\sigma)u$ with $(P(s))^{-1}Q(s)$ proper. Let $(P(s), -Q(s)) = V(s)(\bar{P}(s), -\bar{Q}(s))$ such that rank $(\bar{P}(\lambda), -\bar{Q}(\lambda))$ is constant, $\forall\lambda \in \mathbb{C}$ and such that $P(\lambda)$ is invertible. Then it is not difficult to prove that $\bar{P}(\sigma)y = \bar{Q}(\sigma)u$ determines $\mathscr{B}^c$, the controllable part of $\mathscr{B}$. Let

$(P(s))^{-1}Q(s) = \sum_{i=0}^{\infty} M_i s^{-i}$, then $\{M_0, M_1, \ldots\}$ is by definition the impulse response of $P(\sigma)y = Q(\sigma)u$.

One can also prove the following: (see [1,2])

Theorem 8. *Let $P(\sigma)y = Q(\sigma)u$ be an input–output representation of the behavior $\mathscr{B}$. Let $\sigma x = Ax + Bu$, $y = Cx + Du$ be an input-state-output representation of $\mathscr{B}$. Then $(P(s))^{-1}Q(s) = D + C(sI - A)^{-1}B$.*

We will now continue with a brief introduction to the concepts of most powerful unfalsified behavior, and that of identifiability.

3. Most powerful unfalsified behavior and identifiability

Definition 7. Suppose, for some $n \in \mathbb{N}$, that $\omega^1, \omega^2, \ldots, \omega^n$ are elements of $(\mathbb{R}^q)^T$. Then we call $\mathscr{B} \subseteq (\mathbb{R}^q)^T$ the most powerful unfalsified behavior explaining $\{\omega^1, \omega^2, \ldots, \omega^n\}$ if:
1. $\mathscr{B}$ is a behavior.
2. $\forall i: \omega^i \in \mathscr{B}$.
3. Suppose that a behavior $\tilde{\mathscr{B}} \subseteq (\mathbb{R}^q)^T$ is such that

$$\forall i: \omega^i \in \tilde{\mathscr{B}}, \text{ then } \mathscr{B} \subseteq \tilde{\mathscr{B}}.$$

We denote this model by $\mathscr{B}(\omega^1, \omega^2, \ldots, \omega^n)$.

Theorem 9. *$\mathscr{B}(\omega^1, \omega^2, \ldots, \omega^n)$ exists and is the intersection of all behaviors in $(\mathbb{R}^q)^T$ such that $\forall i$: ω^i is an element of that behavior. So $\mathscr{B}(\omega^1, \omega^2, \ldots, \omega^n) = \bigcap\{\mathscr{B} \mid \mathscr{B}$ is a behavior and $\forall i$: $\omega^i \in \mathscr{B}\}$. Furthermore $\mathscr{B}(\omega^1, \omega^2, \ldots, \omega^n) \neq \emptyset$.*

The proof is trivial when we notice that $(\mathbb{R}^q)^T$ itself is a behavior and that the intersection of an arbitrary collection of behaviors is again a behavior.

Now we come to the notion of identifiability.

Definition 8. Let $\mathscr{B} \subseteq (\mathbb{R}^q)^T$ be a behavior. We say that, for $n \in \mathbb{N}$, $\mathscr{B}$ is n-identifiable if there is a collection $\{\omega^1, \omega^2, \ldots, \omega^n\} \subseteq (\mathbb{R}^q)^T$ such that $\mathscr{B} = \mathscr{B}(\omega^1, \omega^2, \ldots, \omega^n)$. We say that $\mathscr{B}$ is identifiable if, for some $n \in \mathbb{N}$, $\mathscr{B}$ is n-identifiable.

The paper written by Hey [4], contains a great deal of results concerning the notions introduced above. Slightly adapting some of the proofs in [4], Hey works with $T = \mathbb{Z}$ and our time axis in $\mathbb{Z}_+$, the following results arise.

Theorem 10. 1. *Every behavior is identifiable.*
2. *Every controllable behavior is 1-identifiable.*
3. *For $q = 1$ every behavior (in $\mathbb{R}^T$) is 1-identifiable.*

Now we have done enough preliminary work to start with a formal definition of nonnegative behaviors.

4. Nonnegative behaviors

We propose the following definition of nonnegative behaviors.

Definition 9. A behavior $\mathscr{B} \subseteq (\mathbb{R}^q)^T$ is nonnegative if, for some $n \in \mathbb{N}$, $\mathscr{B} = \mathscr{B}(\omega^1, \omega^2, \dots, \omega^n)$ with all ω^i nonnegative.

Notice that this definition is not in terms of inputs and outputs, and that it does not refer to any representation of $\mathscr{B}$.

The investigation of this notion in this paper will be restricted to $q = 1$ or $q = 2$. In the latter case, we will mostly discuss controllable behaviors.

We continue with $\mathscr{B} \subseteq (\mathbb{R}^2)^{\mathbb{Z}_+}$ defined by $p(\sigma)y = q(\sigma)u$, with

$$\begin{aligned} p(s) &:= s^n + p_{n-1}s^{n-1} + \dots + p_0 \in \mathbb{R}[s] \\ q(s) &:= q_m s^m + q_{m-1}s^{m-1} + \dots + q_0 \in \mathbb{R}[s] \end{aligned} \tag{$\star\star$}$$

Without loss of generality we may assume that $q_m \neq 0$.

Now we would like to give necessary or sufficient conditions such that $\mathscr{B}$ is nonnegative. To that end we will first derive a useful lemma.

Suppose we have a collection of real $q \times q$ matrices $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ for some $n \in \mathbb{N}$.

For n even we define $m_n := n/2$, and for n odd we define m_n to be the smallest integer greater than $n/2$.

We define $H(\alpha_1, \alpha_2, \dots, \alpha_n)$, where H stands for 'Hankel' as

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{n-1} & \alpha_n \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_n & \\ \vdots & & & & & \\ \alpha_{m_n} & \alpha_{m_n+1} & \dots & \alpha_n & & \end{bmatrix}.$$

When n is even we define the rank of $H(\alpha_1, \alpha_2, \dots, \alpha_n)$ to be the rank of the first $(m_n + 1)q$ columns of it.

When n is odd the rank of $H(\alpha_1, \alpha_2, \dots, \alpha_n)$ is by definition the rank of its first $m_n \cdot q$ columns.

Now it is rather straightforward to prove the following result.

Lemma 1. *Let n be even, and the rank of $H(\alpha_1, \alpha_2, \dots, \alpha_n)$ be equal to $m_n \cdot q$. Then the collection of matrices $\beta \in \mathbb{R}^{q \times q}$ such that rank $H(\alpha_1, \alpha_2, \dots, \alpha_n, \beta) = m_n \cdot q + q$ is the complement of an algebraic variety in $\mathbb{R}^{q \times q}$.*

As an immediate corollary we have:

Corollary 1. $(\mathbb{R}^q)^T$ *is a nonnegative behavior.*

Our first result concerning nonnegative behaviors now reads as follows.

Theorem 11. *Let the behavior* $\mathscr{B}$ *be given by* $p(\sigma)y = q(\sigma)u$ *where* $p(s)$ *and* $q(s)$ *are as in Eq.* $(\star\star)$, *with* $q_m > 0$. *Let* $\mathscr{B}$ *also be controllable, then* $\mathscr{B}$ *is nonnegative.*

Proof. Using Lemma 1 it is clear that recursively we can construct a pair $(\bar{y}, \bar{u})$, where both sequences are nonnegative, where $\mathscr{B}(\bar{u}) = \mathbb{R}^T$, such that $p(\sigma)\bar{y} = q(\sigma)\bar{u}$.

We contend that $\mathscr{B}(\bar{\omega}) = \mathscr{B}$, where $\bar{\omega} := (\bar{y}, \bar{u})$. Assume to the contrary that this is not the case, then we have that $\mathscr{B}(\bar{\omega})$ is strictly contained in $\mathscr{B}$.

Let $\mathscr{B}(\bar{\omega})$ be equal to $\ker \bar{R}(\sigma)$. Without less of generality we may assume that $\bar{R}(s)$ has full row rank. We know that this rank is at most 2. Assume that it is 2. Then it is easy to see that there is a nonzero polynomial $\bar{r}(s)$ such that $\bar{r}(\sigma)\bar{u} = 0$, but this leads to a contradiction. So rank $\bar{R}(s)$ is equal to one.

Now it follows from Theorem 2 that $(p(s), -q(s)) = v(s)\bar{R}(s)$ for a nonzero polynomial $v(s)$. But this would imply that $\mathscr{B}$ is not controllable, again a contradiction, and we are done with the proof. □

The condition in the previous theorem is not necessary as can be seen from the following example.

Example 1. Let $\mathscr{B} \subseteq (\mathbb{R}^2)^T$ be defined by $(\sigma - 1)y = -u$. It is clear that, starting with $y(0) > 0$, one can recursively construct a nonnegative pair $(\bar{y}, \bar{u}) =: \bar{\omega}$ such that $\mathscr{B}(\bar{\omega}) = \mathscr{B}$.

Now we will try to answer the following questions.

Let $\mathscr{B} \subseteq (\mathbb{R}^2)^T$ be a nonnegative controllable behavior. Is there a nonnegative $\bar{\omega} \in (\mathbb{R}^2)^T$ such that $\mathscr{B}(\bar{\omega}) = \mathscr{B}$?

Let $\mathscr{B} \subseteq \mathbb{R}^T$ be an autonomous nonnegative behavior. Is there a nonnegative $\bar{\omega} \in \mathbb{R}^T$ such that $\mathscr{B}(\bar{\omega}) = \mathscr{B}$?

Theorem 12. 1. *Let* $\mathscr{B} \subseteq (\mathbb{R}^2)^T$ *be a nonnegative controllable behavior. Then there is a nonnegative* $\omega \in (\mathbb{R}^2)^T$ *such that* $\mathscr{B}(\omega) = \mathscr{B}$.

2. *Let* $\mathscr{B} \subseteq \mathbb{R}^T$ *be a nonnegative behavior. Then there is a nonnegative* $\omega \in \mathbb{R}^T$ *such that* $\mathscr{B}(\omega) = \mathscr{B}$.

Proof. (1) As $(\mathbb{R}^2)^T$ is nonnegative, we may assume without loss of generality that $\mathscr{B} = \{(y, u) \mid p(\sigma)y = q(\sigma)u\}$ where $p(s)$ and $q(s)$ are both nonzero and relatively prime, i.e., the greatest common divisor of $p(s)$ and $q(s)$ is equal to one.

As $\mathscr{B}$ is supposed to be nonnegative we know that for some $n \in \mathbb{N}$ and some $\{\omega^1, \omega^2, \ldots, \omega^n\} \subseteq (\mathbb{R}^2_+)^T$ $\mathscr{B} = \mathscr{B}(\omega^1, \omega^2, \ldots, \omega^n)$. It is straightforward to prove that $\mathscr{B}(\omega^1, \omega^2, \ldots, \omega^n) = \mathscr{B}(\omega^1) + \mathscr{B}(\omega^2) + \cdots + \mathscr{B}(\omega^n)$.

As $\mathscr{B}$ is infinite-dimensional, and autonomous behaviors are finite-dimensional, we may assume without loss of generality that $\mathscr{B}(\omega^1) = \{(y, u) \mid \bar{p}(\sigma)y = \bar{q}(\sigma)u\}$ for a pair of polynomials $(\bar{p}(s), \bar{q}(s)) \neq (0,0)$. As $\mathscr{B}(\omega^1) \subseteq \mathscr{B}$ we know because of Theorem 2 that $(p(s), q(s)) = v(s)(\bar{p}(s), \bar{q}(s))$ for some polynomial $v(s) \in \mathbb{R}[s]$. As the pair $(p(s), q(s))$ is relatively prime it follows that $\mathscr{B}(\omega^1) = \mathscr{B}$, and we are done with the proof of the first part.

(2) As $\mathbb{R}^T$ is nonnegative we may assume that $\mathscr{B}$ is autonomous, and hence of the form $\mathscr{B} = \{\omega \in \mathbb{R}^T \mid r(\sigma)\omega = 0\}$ for some nonzero polynomial $r(s)$. By assumption we have for some $n \in \mathbb{N}$ and some $\{\omega^1, \omega^2, \ldots, \omega^n\} \subseteq \mathbb{R}^T$ that $\mathscr{B} = \mathscr{B}(\omega^1, \omega^2, \ldots, \omega^n) = \mathscr{B}(\omega^1) + \mathscr{B}(\omega^2) + \cdots + \mathscr{B}(\omega^n)$.

First we consider $\mathscr{B}(\omega^1 + \alpha\omega^2)$ for $\alpha > 0$. As $\mathscr{B}(\omega^1)$ and $\mathscr{B}(\omega^2)$ are both finite-dimensional, and $\mathscr{B}(\omega^1 + \alpha\omega^2) \subseteq \mathscr{B}(\omega^1) + \mathscr{B}(\omega^2)$ we may assume that for some $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_1 \neq \alpha_2$ we have $\mathscr{B}(\omega^1 + \alpha_1\omega^2) = \mathscr{B}(\omega^1 + \alpha_2\omega^2)$. This follows from Theorem 2.

When $\mathscr{B}(\omega^1 + \alpha\omega^2) = \mathscr{B}(\omega^1) + \mathscr{B}(\omega^2)$ we have $\mathscr{B} = \mathscr{B}(\omega^1 + \alpha\omega^2) + \mathscr{B}(\omega^3) + \cdots + \mathscr{B}(\omega^n)$.

Assume now that $\mathscr{B}(\omega^1 + \alpha_1\omega^2) \cap \mathscr{B}(\omega^1) \neq \mathscr{B}(\omega^1)$.

As $\omega^1 + \alpha_1\omega^2 \in \mathscr{B}(\omega^1 + \alpha_1\omega^2)$ and also $\omega^1 + \alpha_2\omega^2 \in \mathscr{B}(\omega^1 + \alpha_1\omega^2)$ we find that $\omega^1 \in \mathscr{B}(\omega^1 + \alpha_1\omega^2)$.

So $\omega^1 \in \mathscr{B}(\omega^1 + \alpha_1\omega^2) \cap \mathscr{B}(\omega^1) \neq \mathscr{B}(\omega^1)$, a contradiction. Therefore we have $\mathscr{B}(\omega^1 + \alpha_1\omega^2) \cap \mathscr{B}(\omega^1) = \mathscr{B}(\omega^1)$. Similarly, we prove that $\mathscr{B}(\omega^1 + \alpha_1\omega^2) \cap \mathscr{B}(\omega^2) = \mathscr{B}(\omega^2)$. Therefore we have $\mathscr{B}(\omega^1 + \alpha_1\omega^2) = \mathscr{B}(\omega^1) + \mathscr{B}(\omega^2)$.

It is now straightforward to complete the proof by induction.

By means of an example we will prove that in general one does not have the following: $\mathscr{B}(\omega^1 + \omega^2) = \mathscr{B}(\omega^1) + \mathscr{B}(\omega^2)$ when $\omega^1 \geqslant 0$ and $\omega^2 \geqslant 0$.

Example 2.

$$\begin{aligned}
&\omega^1 := (1, 0, 0, \ldots) \in \mathbb{R}^T_+; \quad \mathscr{B}(\omega^1) = \{\omega \mid \sigma\omega = 0\},\\
&\omega^2 := (0, 1, 1, \ldots) \in \mathbb{R}^T_+; \quad \mathscr{B}(\omega^2) = \{\omega \mid (\sigma^2 - \sigma)\omega = 0\},\\
&\mathscr{B}(\omega^1) + \mathscr{B}(\omega^2) = \mathscr{B}(\omega^2) \neq \mathscr{B}(\omega^1 + \omega^2) = \{\omega \mid (\sigma - 1)\omega = 0\}.
\end{aligned}$$

Actually, we proved a bit more in the previous theorem.

Having a look at the proof, we also have the following: Every controllable behavior $\mathscr{B}$ in $(\mathbb{R}^2)^T$ with $\mathscr{B} = \mathscr{B}(\omega^1, \omega^2, \ldots, \omega^n)$ is equal to $\mathscr{B}(\omega^i)$ for some $i \in \{1, 2, \ldots, n\}$.

As a corollary of the previous theorem we also have:

Corollary 2. *Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be two nonnegative autonomous behaviors in $\mathbb{R}^T$ then $\mathscr{B}_1 + \mathscr{B}_2$ is also a nonnegative autonomous behavior.*

As a conjecture we would like to formulate:

Conjecture 1. The sum of a finite number of nonnegative behaviors is nonnegative.

Unfortunately, we were not able to give a proof or refutation of this conjecture.

By means of an example we show that the intersection of two nonnegative behaviors need not be nonnegative. But before we do that we state and prove a useful result.

Theorem 13. *Let* $\bar{\mathscr{B}} := \{(\omega, \zeta) \mid \bar{R}(\sigma)\omega = \bar{M}(\sigma)\zeta\}$. *Let* $\mathscr{B} := \{\omega \mid \exists\zeta$ *with* $\bar{R}(\sigma)\omega = \bar{M}(\sigma)\zeta\}$. *Then the following is true. When* $\bar{\mathscr{B}}$ *is nonnegative, so is* $\mathscr{B}$.

Proof. Because of Theorem 5 we know that for some polynomial matrix $R(s)$ we have $\mathscr{B} = \ker R(\sigma)$, so $\mathscr{B}$ is a behavior. Let $\bar{\mathscr{B}} = \mathscr{B}((\omega^1, \zeta^1), \ldots (\omega^n, \zeta^n))$ with (ω^i, ζ^i) nonnegative $\forall i \in \{1, 2, \ldots, n\}$. Trivially we have that $\mathscr{B}(\omega^1, \omega^2, \ldots, \omega^n) \subseteq \mathscr{B}$. Assume to the contrary that $\mathscr{B}(\omega^1, \omega^2, \ldots, \omega^n)$ is strictly contained in $\mathscr{B}$. Then there is an element $(\bar{\omega}, \bar{\zeta}) \in \bar{\mathscr{B}}$ such that $\bar{\omega} \notin \mathscr{B}(\omega^1, \omega^2, \ldots, \omega^n)$. We also have $\forall i \in \{1, 2, \ldots, n\}$:

$$(\omega^i, \zeta^i) \in \{(\omega, \zeta) \mid (\omega, \zeta) \in \bar{\mathscr{B}}, \omega \in \mathscr{B}(\omega^1, \omega^2, \ldots, \omega^n)\},$$

whereas this latter set is strictly contained in $\bar{\mathscr{B}}$.

But now we arrived at a contradiction, and we are done with the proof. $\square$

Now we come to the promised example.

Example 3. $\mathscr{B}_1 := \{(y, u) \in (\mathbb{R}^2)^T \mid (\sigma + 1)y = u\}$. Because of Theorem 11 this behavior is nonnegative.

$$\mathscr{B}_2 := \{(y, u) \in (\mathbb{R}^2)^T \mid (\sigma - \tfrac{1}{2})u = 0\}.$$

As $\{u \in \mathbb{R}^T \mid (\sigma - \frac{1}{2})u = 0\}$ and $\mathbb{R}^T$ are nonnegative we have that $\mathscr{B}_2$ is nonnegative.

Now $\{y \mid \exists u$ with $(y, u) \in \mathscr{B}_1 \cap \mathscr{B}_2\}$ is equal to $\{y \in \mathbb{R}^T \mid (\sigma^2 + \frac{1}{2}\sigma - \frac{1}{2})y = 0\}$ as a straightforward calculation shows. With $p(s) = s^2 + \frac{1}{2}s - \frac{1}{2}$ we have that $p(-1) = p(\frac{1}{2}) = 0$. Therefore all elements of $\{y \mid (\sigma^2 + \frac{1}{2}\sigma - \frac{1}{2})y = 0\}$ are of the form $\alpha y^1 + \beta y^2$ with $\alpha, \beta \in \mathbb{R}$ and $y^1(t) := (-1)^t$ and $y^2(t) := (\frac{1}{2})^t$. Now it is easy to see that $\{y \mid (\sigma^2 + \frac{1}{2}\sigma - \frac{1}{2})y = 0\}$ is not nonnegative and therefore, because of Theorem 13, we have that $\mathscr{B}_1 \cap \mathscr{B}_2$ is not nonnegative.

In the sequel of this section we continue with autonomous behaviors in $\mathbb{R}^T$ i.e., behaviors of the form $\{\omega \in \mathbb{R}^T \mid r(\sigma)\omega = 0\}$ for a nonzero polynomial $r(s)$. Let $r(s) := s^n + r_{n-1}s^{n-1} + \cdots + r_0 \in \mathbb{R}[s]$.

Associated with $r(s)$ we define the following linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$(y_0, y_1, \ldots, y_{(n-1)}) \xrightarrow{P} (y_1, y_2, \ldots, y_{(n-1)}, y_n)$$

where $y_n := -r_{n-1}y_{n-1} - r_{n-2}y_{n-2} - \cdots - r_0 y_0$. We are now able to state and prove the following result.

Theorem 14. *$\mathscr{B} := \{\omega \mid r(\sigma)\omega = 0\}$ is a nonnegative behavior if and only if there is a closed convex cone $K \subseteq \mathbb{R}^n_+$ with nonempty interior,* int K, *such that $PK \subseteq K$.*

Proof. Let us assume that $\mathscr{B}$ is nonnegative. Hence, by Theorem 12, there is a nonnegative $y \in \mathbb{R}^T$ such that $\mathscr{B}(y) = \mathscr{B}$. We define $y^i := (y_i, y_{i+1}, \ldots, y_{i+n-1})$, $i \in \mathbb{Z}_+$.

We contend that $\{y^0, y^1, \ldots, y^{n-1}\}$ is a linearly independent collection in $\mathbb{R}^n_+$. Notice that by construction we have $Py^i = y^{i+1}$, $\forall i \in \mathbb{Z}_+$. Assume to the contrary that for some numbers $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\} \subseteq \mathbb{R}$, not all zero, we have $\alpha_0 y^0 + \alpha_1 y^1 + \cdots + \alpha_{n-1}y^{n-1} = 0$. Then we have $\forall k \in \mathbb{Z}_+$, $\alpha_0 P^k y^0 + \alpha_1 P^k y^1 + \cdots + \alpha_{n-1}P^k y^{n-1} = 0$. And hence we have: $(\alpha_0 + \alpha_1\sigma + \alpha_2\sigma^2 + \cdots + \alpha_{n-1}\sigma^{n-1})\, y = 0$. But his contradicts the assumption that $\mathscr{B}(y) = \ker r(\sigma)$. Let now K be the smallest closed convex cone in $\mathbb{R}^n_+$ such that for all $i \in \mathbb{Z}_+$ we have $y^i \in K$.

Then we have by construction that $PK \subseteq K$ and also because of the linear independence of $\{y^0, y^1, \ldots, y^{n-1}\}$ that int $K \neq \emptyset$.

Assume now that $PK \subseteq K$ where K is a closed convex cone in $\mathbb{R}^n_+$ with nonempty interior. We will now prove that $\ker r(\sigma)$ is nonnegative. To that end we take $\omega^1 \in \mathbb{R}^{\mathbb{Z}_+}$ such that $(\omega^1_0, \omega^1_1, \ldots, \omega^1_{n-1}) \in \operatorname{int} K$, and such that $\omega^1 \in \ker r(\sigma)$. This can be done by recursively defining $(\omega^1_k, \omega^1_{k+1}, \ldots, \omega^1_{k+n-1}) = P(\omega^1_{k-1}, \omega^1_k, \ldots, \omega^1_{k+n-2})$, $k \in \mathbb{N}$. Because of Theorem 12, we may take a sequence $\omega^2 \in \mathbb{R}^T$ such that $\mathscr{B}(\omega^2) = \ker r(\sigma)$. For all $\alpha > 0$ we define $\omega^\alpha := \omega^1 + \alpha\omega^2$. By construction we have $\mathscr{B}(\omega^\alpha) \subseteq \ker r(\sigma)$, for all $\alpha > 0$. By construction there is a number $\delta > 0$ such that $\forall \alpha \in (0, \delta)$ we have $(\omega^\alpha_0, \omega^\alpha_1, \ldots, \omega^\alpha_{n-1}) \in \operatorname{int} K$. As $\ker r(\sigma)$ is finite dimensional, there are numbers α_1 and α_2 with $\alpha_1 \neq \alpha_2$, both contained in $(0, \delta)$ such that $\mathscr{B}(\omega^1 + \alpha_1\omega^2) = \mathscr{B}(\omega^1 + \alpha_2\omega^2) \subseteq \ker r(\sigma)$.

By linearity we have that $\omega^1 \in \mathscr{B}(\omega^1 + \alpha_1\omega^2)$ and hence $\mathscr{B}(\omega^1 + \alpha_1\omega^2) = \ker r(\sigma)$. As by construction $\omega^1 + \alpha_1\omega^2$ is nonnegative, we are done. $\square$

It is now a good moment to say something about nonnegative realizations.

5. Nonnegative realizations

We continue our investigation of autonomous behaviors in $\mathbb{R}^T$. Let $\mathscr{B}$ be such a behavior, i.e.,

$$\mathscr{B} = \ker r(\sigma) \quad \text{where } r(s) := s^n + r_{n-1}s^{n-1} + \cdots r_0 \in \mathbb{R}[s].$$

Because of Theorem 6 we know that we may represent $\mathscr{B}$ as $\sigma x = Ax$, $y = Cx$ for some matrices A and B with a finite number of all real entries. So, $\mathscr{B} = \{y \in \mathbb{R}^T \mid \exists x \text{ with } \sigma x = Ax, y = Cx\}$. We say that $\mathscr{B}$ is nonnegative-realizable if A and C can be chosen in such a way that $A \geqslant 0$, $C \geqslant 0$, i.e., such that all entries of A and C are nonnegative.

Recall that a cone $K \subseteq \mathbb{R}^n$ is polyhedral if there are vectors $k^1, k^2, \ldots, k^m$ in $\mathbb{R}^n$ such that $k \in K$ if and only if $k = \sum_{i=1}^m \lambda_i k^i$ for nonnegative numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$.

Again P is the linear operator associated with $r(s)$. We are now able to formulate and prove the following result.

Theorem 15. *Let $\mathscr{B} = \ker r(\sigma)$. Then $\mathscr{B}$ is nonnegative realizable if and only if there is a polyhedral cone $K \subseteq \mathbb{R}^n_+$ with a nonempty interior such that $PK \subseteq K$.*

Proof. Assume that $m \in \mathbb{N}$, $A \in \mathbb{R}^{m\times m}_+$ and $C \in \mathbb{R}^{1\times m}_+$ are such that $\mathscr{B} = \{y \mid \exists x \text{ with } \sigma x = Ax, y = Cx\}$. We define

$$K := \{(k_1, k_2, \ldots, k_n) \in \mathbb{R}^n \mid \exists x \in \mathbb{R}^m_+ \text{ such that } k_i = CA^{i-1}x\}.$$

By construction K is polyhedral and contained in $\mathbb{R}^n_+$. It is also clear that $PK \subseteq K$.

As $\deg r(s) = n$ we know that there is a sequence $\bar{\omega} \in \mathscr{B}$ such that $\{(\bar{\omega}_0, \bar{\omega}_1, \ldots, \bar{\omega}_{n-1}), \ldots (\bar{\omega}_{n-1}, \bar{\omega}_n, \ldots, \bar{\omega}_{2n-2})\}$ is a collection of n linearly independent vectors in $\mathbb{R}^n$.

But we also have $\bar{\omega}_i = CA^{i-1}\bar{x}$ for some $\bar{x} \in \mathbb{R}^m$. Therefore

$$\operatorname{rank}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

is equal to n.

When we denote this latter matrix by M we have $K = \{k \in \mathbb{R}^n_+ \mid \exists x \in \mathbb{R}^m_+ \text{ with } k = Mx\}$, so K has a nonempty interior.

Assume now that for some polyhedral cone $K \subseteq \mathbb{R}^n_+$ with nonempty interior we have $PK \subseteq K$. Let $K = \{k \in \mathbb{R}^n_+ \mid k = Mx \text{ for some } x \in \mathbb{R}^m_+\}$. Now $PK \subseteq K \subseteq \mathbb{R}^n_+$ is equivalent to the existence of a square nonnegative matrix A with $PM = MA$. We define $C := (1, 0, \ldots, 0)M$, the first row of M.

By definition of P we have $y \in \mathscr{B}$ if and only if there is a vector $x \in \mathbb{R}^n$ with $y(t) = (1, 0, \ldots, 0)P^t x$, $\forall t \in \mathbb{Z}_+$. As K has nonempty interior we know that rank M is equal to n and hence we know that $x = M\alpha$ for some vector α. We now calculate

$$\begin{aligned} P^t x = P^t M\alpha = P^{t-1}PM\alpha &= P^{t-1}MA\alpha = P^{t-2}PMA\alpha \\ = P^{t-2}MA^2\alpha = &\cdots = MA^t\alpha. \end{aligned}$$

So $y(t) = CA^t\alpha$ and we are done with the proof. □

By means of an example we will show that not every nonnegative autonomous system is nonnegative realizable.

Example 4. Let $r(s) := (s - \frac{1}{2})(s - \frac{1}{2}\mathrm{e}^{2\mathrm{i}})(s - \frac{1}{2}\mathrm{e}^{-2\mathrm{i}})$ where i is the imaginary unit. Some easy calculations show that ker $r(\sigma)$ is nonnegative. Using exercise 1.5.17 from [6], a result due to Barker, one can show that ker $r(\sigma)$ is not nonnegative realizable.

At the end of this paper we will consider nonnegative input–output systems. Here we make a distinction between inputs and outputs of a behavior. We will restrict our attention to single input-single output behaviors.

6. Nonnegative input–output systems

Let us define

$$p(s) := s^n + p_{n-1}s^{n-1} + \cdots + p_0 \in \mathbb{R}[s]$$
$$q(s) := q_m s^m + \cdots + q_0 \in \mathbb{R}[s].$$

We assume $q_m \neq 0$ and $m \leqslant n$. We further define $\mathscr{B} := \{(y, u) \mid p(\sigma)y = q(\sigma)u\}$. The variables in u are called inputs, and those in y are called outputs.

By definition the impulse response of $\mathscr{B}$ is $\{\alpha_0, \alpha_1, \ldots\} \subseteq \mathbb{R}$ where $q(s) = p(s)(\sum_{i=0}^{\infty} \alpha_i s^{-i})$. Further we define the following.

Definition 10. 1. $\mathscr{B}$ is input–output nonnegative if for all $u \in (\mathbb{R}_+)^T$ there is a $y \in (\mathbb{R}_+)^T$ such that $p(\sigma)y = q(\sigma)u$.

2. $\mathscr{B}$ is called strongly input–output nonnegative if $\forall u \in (\mathbb{R}_+)^T$ there is a convex cone $K_u \subseteq \mathbb{R}^n_+$ with nonempty interior such that for all $k \in K_u$ there is a $y \in (\mathbb{R}_+)^T$ with $p(\sigma)y = q(\sigma)u$ and $(y(0), y(1), \ldots, y(n-1)) = k$.

In order to prove the next result we introduce the following. $e \in (\mathbb{R}_+)^T$ is defined by $e(0) = 1$ and $e(t) = 0$, $\forall t \in \mathbb{N}$. We further define $\sigma^{-1}: \mathbb{R}^T \to \mathbb{R}^T$ as follows:

$$\forall \omega \in \mathbb{R}^T: (\sigma^{-1}\omega)(0) := 0$$
$$(\sigma^{-1}\omega)(t) := \omega(t-1), \quad \forall t \in \mathbb{N}.$$

Recursively, we define $\sigma^{-k}: \mathbb{R}^T \to \mathbb{R}^T$, $k \in \mathbb{N}$, $k \geqslant 2$ as follows:

$$\forall \omega \in \mathbb{R}^T: \sigma^{-k}\omega := \sigma^{-1}(\sigma^{-k+1}\omega).$$

Theorem 16. *$\mathscr{B}$ is input–output nonnegative if the impulse response of $\mathscr{B}$ is nonnegative. If $\mathscr{B}$ is input–output nonnegative and $\alpha_0 = 0$ then the impulse response is nonnegative.*

Proof. We recall that $p(s)(\sum_{i=0}^{\infty} \alpha_i s^{-i}) = q(s)$. It is easy to see that $\forall \ell, k \in \mathbb{Z}_+$ we have: $\sigma^{\ell} \cdot \sigma^{-k} = \sigma^{\ell-k}$. We certainly do not have that $\sigma^{-k} \cdot \sigma^{\ell} = \sigma^{\ell-k}$! It now follows easily that

$$p(\sigma)(\sigma^{-k}\alpha) = q(\sigma)(\sigma^{-k}e) \quad \forall k \in \mathbb{Z}_+ \quad \text{where}$$

$$\alpha := (\alpha_0, \alpha_1, \ldots) = \left(\sum_{i=0}^{\infty} \alpha_i \sigma^{-i} e\right).$$

Suppose now that α is nonnegative. Take a nonnegative $u \in \mathbb{R}^T$, so $u = \sum_{i=0}^{\infty} u_i(\sigma^{-i}e)$.

We now have $p(\sigma)(\sum_{i=0}^{\infty} u_i(\sigma^{-i}\alpha)) = q(\sigma)u$. Therefore $\mathscr{B}$ is input–output nonnegative.

For the rest of the proof we have to work a bit harder. We now assume that $\forall u \geqslant 0\ \exists y \geqslant 0$ with $p(\sigma)y = q(\sigma)u$. In order to prove that α is nonnegative we represent $\mathscr{B}$ by $\sigma x = Ax + bu$, $y = cx + du$, where c is a row-vector, and b is a column-vector and $d \in \mathbb{R}$.

We recall that $\alpha_0 = d$ and $\alpha_i = cA^{i-1}b$, $\forall i \in \mathbb{N}$. Notice that we now assume $\alpha_0 = d = 0$.

Now we will prove that $cb \geqslant 0$. In order to prove that we take a $\lambda \in \mathbb{R}_+$ such that $\lambda > \max\{\|A\|_2, 1\}$.

Further we take $u \in \mathbb{R}^T$ defined by $u(t) := \lambda^{t^2} \forall t \in \mathbb{Z}_+$. By assumption we know that there is a vector x_0 such that with $y(t) = cA^t x_0 + cA^{t-1}bu(0) + \cdots + cbu(t-1) + du(t)$, $t \in \mathbb{Z}_+$, we have $\forall t \in \mathbb{Z}_+$, $y(t) \geqslant 0$. Now we will prove that $y(t)/u(t-1)$ converges to cb as t goes to infinity. This, of course, will imply that $cb \geqslant 0$. Now recall that $\|A^t\|_2 \leqslant (\|A\|_2)^t$. Let us write $\delta := \|A\|_2$.

First we consider, for $k \geqslant 1$, $(\delta^{t-k-1}\lambda^{k^2})/(\delta^{t-k}\lambda^{(k-1)^2}) = \delta^{-1}\lambda^{2k-1} = \lambda/\delta\lambda^{2k-2} \geqslant 1$. Therefore, in order to estimate the norm of $y(t)/u(t-1)$ it is sufficient to consider $(t-1)\delta\lambda^{-2t+3}$. As λ is greater than one this term converges to zero as $t \to \infty$. Hence, as $cA^t x_0/u(t-1) \to 0$, we have that $cb \geqslant 0$.

Let us now consider cAb. With λ as above we now define $u(t) := \lambda^{t^2}$ when t is even and $u(t) = 0$ when t is odd.

For even $t \in \mathbb{Z}_+$ we have:

$$y(t) = cA^t x_0 + \cdots + cAbu(t-2),$$

where x_0, depending on u, is such that, $\forall t \in \mathbb{Z}_+$, $y(t) \geqslant 0$.

It is now rather easy to prove that $y(t)/u(t-2)$ converges to cAb when $t \to \infty$ and t is even.

In order to prove that $cA^2b \geqslant 0$ we define $u(t) = \lambda^{t^2}$ when t is a multiple of 3, $t \geqslant 1$ and $u(t) = 0$ for all other $t \in \mathbb{Z}_+$. Proceeding as before we easily prove that $cA^2b \geqslant 0$.

An induction argument completes the proof. □

Our next result reads as follows.

Theorem 17. *$\mathscr{B}$ is strongly input–output nonnegative if and only if $\mathscr{B}$ is input–output nonnegative and $\{y \mid p(\sigma)y = 0\}$ is nonnegative.*

Proof. Let $\mathscr{B}$ be strongly input–output nonnegative. Then certainly $\mathscr{B}$ is input–output nonnegative. Because of Theorem 14 we now immediately have that $\{y \mid p(\sigma)y = 0\}$ is a nonnegative autonomous behavior. Based on Theorem 14 it is also straightforward to prove that $\mathscr{B}$ is strongly input–output nonnegative whenever $\mathscr{B}$ is input–output nonnegative and $\{y \mid p(\sigma)y = 0\}$ is also nonnegative.

Notice that the input–output nonnegativity of $p(\sigma)y = q(\sigma)u$ only depends on its controllable behavior, whereas the stronger notion does not only depend on the controllable subbehavior.

Let us again consider $p(\sigma)y = q(\sigma)u$.

Definition 11. $p(\sigma)y = q(\sigma)u$ is nonnegative input-state-output realizable if $\mathscr{B} = \{(y, u) \mid p(\sigma)y = q(\sigma)u\}$ can be represented by $\sigma x = Ax + Bu$, $y = Cx + Du$, where all matrices are nonnegative.

It is almost immediate that in case $p(\sigma)y = q(\sigma)u$ is nonnegative input-state realizable we have that $p(\sigma)y = q(\sigma)u$ is strongly nonnegative.

We conjecture the following.

Conjecture 2. When $p(\sigma)y = q(\sigma)u$ is strongly input–output nonnegative and $p(\sigma)y = 0$ can be realized in a nonnegative way, then $p(\sigma)y = q(\sigma)u$ is nonnegative input-state-output realizable.

In this paper, however, we will not address the issue of realizability any further. Again we refer to [3] for the state of the art of nonnegative realization theory.

7. Concluding remarks

In the previous pages we made a beginning with the development of the basic notions of nonnegative behaviors in the framework of Willems. We briefly touched nonnegative realization theory, and matrices leaving cones invariant, a subject that is extremely important for nonnegative realization theory. For this 'matricial' theory we refer to the classic written by Berman and Plemmons [6].

A good marriage between Willems' behavioral theory and nonnegative realizations should be the subject of a next paper. We have chosen to stress in this paper various notions of nonnegative behavior.

It is our opinion that a modern treatment of linear systems theory cannot be called so when the behavioral theory is not present in one way or the other. In the same vein the theory of nonnegative linear systems should be embedded in the behavioral framework. As such the present paper is only a beginning.

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