ELSEVIER Linear Algebra and its Applications 281 (1998) 59–85

LINEAR ALGEBRA AND ITS APPLICATIONS

Numerical ranges and Poncelet curves

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Received 22 August 1997; accepted 2 March 1998

Submitted by L. Rodman

Abstract

Convex circuits which have the property of circles of The Great Poncelet Theorem are introduced. The circuits are generated as boundaries of numerical ranges of specially constructed matrices. Algebraic and geometric properties of such matrices and their numerical ranges are obtained. These properties lead to a new proof of the Poncelet Theorem. The proposed approach illustrates the fact that complex relations between geometrical figures in a plane may have simple interpretation in a space of higher dimension. © 1998 Elsevier Science Inc. All rights reserved.

1. Introduction

The Great Poncelet Theorem [1], p. 203 states that if two circles (or ellipses) $\mathbf{C}$ and $\mathbf{C}_1$ are such that there is an n-sided polygon inscribed in $\mathbf{C}$ and circumscribed around $\mathbf{C}_1$, then there exist many other such polygons, and one vertex of such a polygon may be chosen on $\mathbf{C}$ arbitrarily. Let us define and construct other curves (not necessarily quadrics) with a similar property.

Definition 1. A convex circuit $\mathbf{K}$ is called *Poncelet curve* of rank n with respect to (w.r.t.) a circle $\mathbf{C}$, if there are infinitely many n-sided polygons which are inscribed in $\mathbf{C}$ and circumscribed around $\mathbf{K}$, and one vertex of such a polygon may be chosen on $\mathbf{C}$ arbitrarily.

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PII: S0024-3795(98)10037-X

The proposed construction of Poncelet curves is based on the notion of the numerical ranges of matrices, [2–4], also called "fields of values" [5]. Square matrices of size n with complex entries are viewed as operators in n-dimensional Hilbert space $\mathscr{H}_n$ with the inner product of vectors $\mathbf{a}$ and $\mathbf{b}$ denoted $\langle \mathbf{a}, \mathbf{b} \rangle$. Let T^* denote the matrix which is the adjoint for matrix T. The identity matrix in $\mathscr{H}_n$ is denoted by I_n or I. The linear span of vectors $\mathbf{a}, \mathbf{b}, \ldots$ is denoted by $\operatorname{span}(\mathbf{a}, \mathbf{b}, \ldots)$.

Definition 2. The numerical range $\Omega(T)$ of an $n \times n$ matrix T is the set of complex numbers $\langle T\mathbf{x}, \mathbf{x} \rangle$ where $\mathbf{x}$ is a vector on unit sphere in $\mathscr{H}_n$:

$$\Omega(T) = \left\{ \langle T\mathbf{x}, \mathbf{x} \rangle : \|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = 1 \right\}.$$

We conclude, straight from this definition, that unitarily similar matrices have the same numerical range, and that the numerical range of a diagonal matrix is the convex hull of the diagonal elements. Combining these two statements, we come to the following proposition.

Proposition 1. *The numerical range of a normal matrix is the convex hull of its eigenvalues* ([2], p. 162, [3]).

Moreover, for any matrix, not necessarily normal, its numerical range is convex. This is known as Toeplitz–Hausdorff Theorem ([3], [5], p. 27). A way to determine the boundary $\partial\Omega(T)$ of the numerical range of an arbitrary matrix T was proposed by Toeplitz (who introduced the numerical range). This way is based on the fact that, for a Hermitian matrix A, the maximum of the quadratic form $\langle A\mathbf{x}, \mathbf{x} \rangle$ on the unit sphere is the largest eigenvalue of A.

Namely, let the points (ξ, η) of $\partial\Omega(T)$ be parametrized by the angles $0 \leqslant \varphi < 2\pi$ between the straight lines of support $\mathbf{L}$ and η-axis (Fig. 1). Then ([6,7]),

$$\begin{aligned} \xi &= \xi(\varphi) = \lambda(\varphi) \cos \varphi - \lambda'(\varphi) \sin \varphi, \\ \eta &= \eta(\varphi) = \lambda(\varphi) \sin \varphi + \lambda'(\varphi) \cos \varphi, \end{aligned} \tag{1}$$

where $\lambda(\varphi)$ is the largest root of the equation

$$\det\left[\mathscr{R}(T\,\mathrm{e}^{-\mathrm{i}\varphi}) - \lambda(\varphi) I\right] = 0. \tag{2}$$

Here, $\mathscr{R}(T\,\mathrm{e}^{-\mathrm{i}\varphi}) = \mathscr{R}(T) \cos \varphi + \mathscr{I}(T) \sin \varphi$, $\mathscr{R}(T) = (T + T^*)/2$ and $\mathscr{I}(T) = (T - T^*)/(2\mathrm{i})$. The derivative $\lambda'(\varphi)$ is determined for the so-called "regular arcs" of $\partial\Omega(T)$. If $(\xi(\varphi), \eta(\varphi))$ is a point of a regular arc, then $\lambda''(\varphi) + \lambda(\varphi)$ is the radius of curvature of the arc at this point. A regular arc of $\partial\Omega(T)$ contains neither corner points nor straight line segments. A common point of $\partial\Omega(T)$ and the line of support $\mathbf{L}$ which forms angle φ with η-axis, is generated by a unit eigenvector of $\mathscr{R}(T\,\mathrm{e}^{-\mathrm{i}\varphi})$ corresponding to $\lambda(\varphi)$.

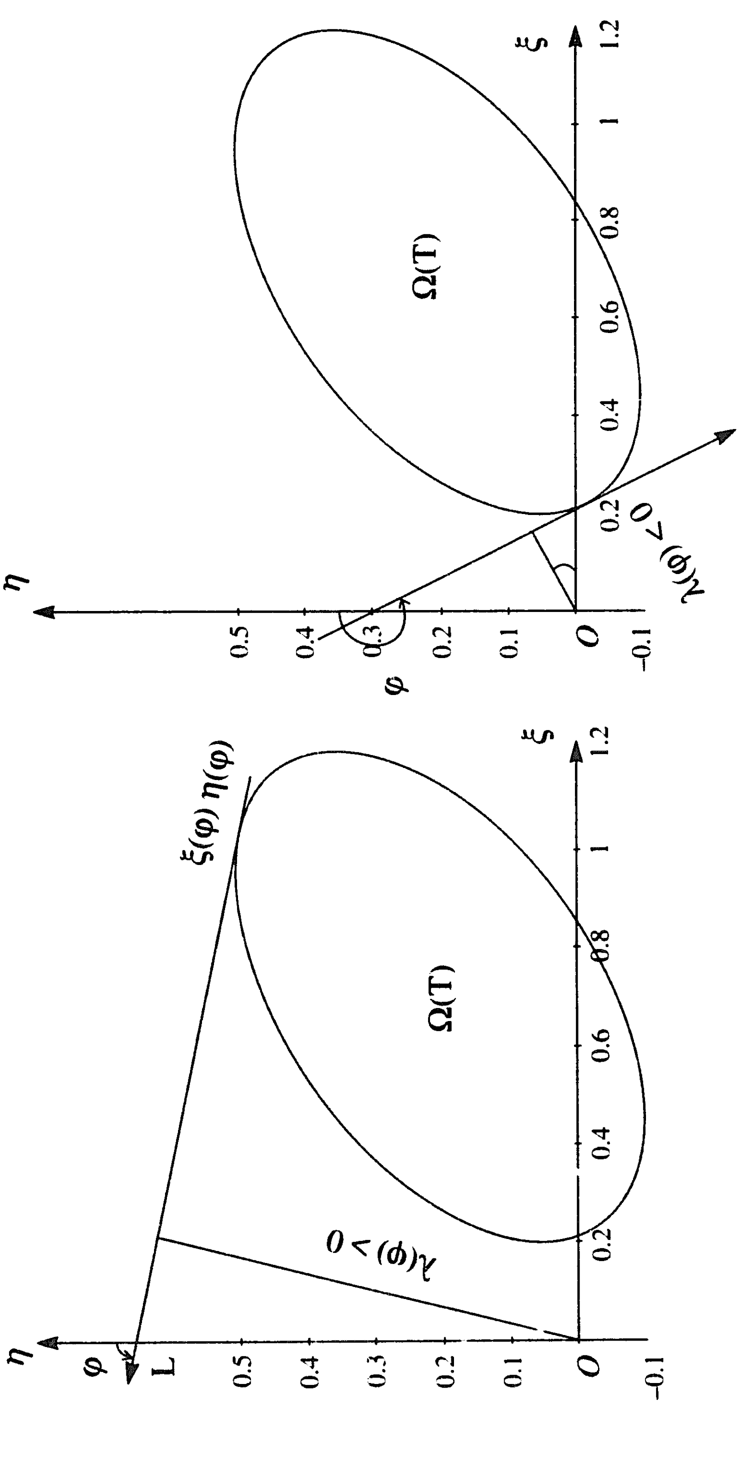

Fig. 1. Determination of the boundary of a numerical range.

Proposition 2. *If a connected subset of $\partial\Omega(T)$ does not contain corner points and, for all points of this subset, $\lambda(\varphi)$ is a* simple *eigenvalue of $\mathscr{R}(T\,e^{-i\varphi})$, then this subset is a regular arc of $\partial\Omega(T)$.*

Proposition 3. *If the boundary of the numerical range of a matrix T contains a corner point $z = \langle T\mathbf{e}, \mathbf{e}\rangle$, then z is a* normal *eigenvalue of T* ([5], pp. 50–51): *$T\mathbf{e} = z\mathbf{e}$ and $T^*\mathbf{e} = \bar{z}\mathbf{e}$.*

For more details on computation of $\partial\Omega(T)$ and properties of numerical ranges, see Ref. [4].

2. Main construction

The idea behind the construction of Poncelet curves is to construct an $n \times n$ matrix T such that its numerical range $\Omega(T)$ is strictly inside the unit circle $\mathbf{C}$, and T has a unitary dilation of size $n+1$. Such a matrix T can be obtained from a unitary matrix U with distinct eigenvalues by deleting a row and a column (not necessarily the corresponding one). If each row of matrix U has at least one nonzero element outside the diagonal, then $\partial\Omega(T)$ is a Poncelet curve w.r.t. the unit circle $\mathbf{C}$. This is proved in the following theorem. For short, we say about such a matrix T that "matrix T admits unitary bordering", or T is a UB-matrix.

Theorem 1. *Let U be an $(n+1)\times(n+1)$ unitary matrix with distinct eigenvalues, $\mathscr{L}$ – an n-dimensional subspace, $n > 1$, which does not contain eigenvectors of U:*

$$\mathscr{L} = \{\mathbf{x} \in \mathscr{H}_{n+1}\colon \langle\mathbf{x}, \mathbf{w}\rangle = 0; \langle\mathbf{e}, \mathbf{w}\rangle \neq 0 \textit{ for any eigenvector } \mathbf{e} \textit{ of } U;\\ \|\mathbf{w}\| = 1\}.$$

Let Q be the orthoprojector $I_{n+1} - \mathbf{w}\mathbf{w}^*$ on $\mathscr{L}$. Then, the matrix $T = QUQ$ viewed as an operator on $\mathscr{L}$ is such that the boundary of its numerical range is a Poncelet curve of rank $n+1$ w.r.t. the unit circle $\mathbf{C}$.

Proof. In accordance with Proposition 1, $\Omega(U)$ is an $(n+1)$-sided convex polygon $\mathbf{P}$ inscribed in $\mathbf{C}$. $\Omega(T)$ is inside $\mathbf{P}$ because U is a dilation of T. Each side of $\mathbf{P}$ joins two eigenvalues of U, say $e^{i\psi_1}$ and $e^{i\psi_2}$ corresponding to eigenvectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$. The side of $\mathbf{P}$ which joins $e^{i\psi_1}$ and $e^{i\psi_2}$ is the numerical range Ω' of U reduced to its invariant subspace $\mathscr{M} = \mathrm{span}(\mathbf{e}^{(1)}, \mathbf{e}^{(2)})$. This subspace is two-dimensional, and therefore each side of $\mathbf{P}$ has a point in common with $\Omega(T)$ – simply because $\dim\mathscr{L} + \dim\mathscr{M} = n+2 > \dim\mathscr{H}_{n+1}$.

Indeed, the latter inequality implies that there exists a nonzero vector $\mathbf{f}$ which belongs to both subspaces $\mathscr{L}$ and $\mathscr{M}$. We may assume that $\|\mathbf{f}\| = 1$. Since $\mathbf{f} \in \mathscr{M}$, the point $\zeta = \langle U\mathbf{f}, \mathbf{f}\rangle$ lies on the side Ω' of $\mathbf{P}$. On the other hand, because $\mathbf{f} \in \mathscr{L}$, we have $Q\mathbf{f} = \mathbf{f}$, and

$$\langle T\mathbf{f}, \mathbf{f}\rangle = \langle U\mathbf{f}, \mathbf{f}\rangle = \zeta$$

i.e., the numerical range $\Omega(T)$ has at least one common point (ζ) with Ω'. Moreover, the side Ω' of $\mathbf{P}$ has exactly *one* common point with $\Omega(T)$. Otherwise, there are two linearly independent unit vectors $\mathbf{f}^{(1)} \in \mathscr{M} \bigcap \mathscr{L}$, $\mathbf{f}^{(2)} \in \mathscr{M} \bigcap \mathscr{L}$, and the entire subspace $\mathscr{M}$ is in $\mathscr{L}$. This, however, contradicts the fact that $\mathscr{L}$ does not contain eigenvectors of U Because of arbitrariness of the side Ω' of $\mathbf{P}$, each side of $\mathbf{P}$ has exactly one common point with $\Omega(T)$, i.e., $\mathbf{P}$ is circumscribed around $\Omega(T)$, and $\Omega(T)$ is strictly inside $\mathbf{C}$.

Matrix U of size $n + 1$ is a unitary dilation of matrix T of size n. Consider the matrix

$$U_\gamma = U\left(I - \mathbf{w}\mathbf{w}^* + \mathrm{e}^{\mathrm{i}\gamma}\,\mathbf{w}\mathbf{w}^*\right).$$

Obviously, $U_\gamma^* U_\gamma = I_{n+1}$ and $QU_\gamma Q = T$, i.e., U_γ is a unitary dilation of T. $\mathbf{P}_\gamma = \Omega(U_\gamma)$ is an $(n + 1)$-sided polygon as well as $\mathbf{P}$: otherwise U_γ has a multiple eigenvalue, the subspace $\mathscr{L}$ must contain an eigenvector of U_γ (because $\dim \mathscr{L} = n$), and $\Omega(T)$ is not strictly inside $\mathbf{C}$, which condradicts what we have proven above. For $0 < |\gamma_1 - \gamma_2| < 2\pi$, the polygons $\mathbf{P}_{\gamma_1}$ and $\mathbf{P}_{\gamma_2}$ have different vertices. Indeed, determinant $\det(U_\gamma) = \mathrm{e}^{\mathrm{i}\gamma}\det(U)$ because $\det(I - \mathbf{w}\mathbf{w}^* + \mathrm{e}^{\mathrm{i}\gamma}\,\mathbf{w}\mathbf{w}^*) = \mathrm{e}^{\mathrm{i}\gamma}$. Therefore, $\det(U_{\gamma_1}) \neq \det(U_{\gamma_2})$ and not all vertices of the polygons $\mathbf{P}_{\gamma_1}$ and $\mathbf{P}_{\gamma_2}$ coincide. However, these polygons cannot differ only in a part of vertices, because these polygons are circumscribed around a convex circuit $\Omega(T)$ (for any point z outside $\Omega(T)$, there are exactly two straight lines of support of $\Omega(T)$ crossing z). Thus, when γ traverses a 2π-long segment, $\mathbf{P}_\gamma$ traverses all possible $(n + 1)$-sided polygons inscribed in $\mathbf{C}$ and circumscribed around $\Omega(T)$, without repetitions, and $\mathbf{P}_{\gamma+2\pi} = \mathbf{P}_\gamma$. Hence, $\partial\Omega(T)$ is a Poncelet curve of rank $n + 1$ w.r.t. $\mathbf{C}$. $\square$

Lemma 1. *If U is an $(n + 1) \times (n + 1)$ unitary matrix with distinct eigenvalues, $Q = I_{n+1} - \mathbf{w}\mathbf{w}^*$, $\|\mathbf{w}\| = 1$, $n > 1$, and $T = QUQ$, then the following statements are equivalent.*

1. *$\Omega(T)$ is strictly inside the unit circle $\mathbf{C}$.*
2. *All eigenvalues of T are strictly inside $\mathbf{C}$.*
3. *$\langle \mathbf{w}, \mathbf{e}\rangle \neq 0$ for any eigenvector $\mathbf{e}$ of U.*
4. *Subspace $\mathscr{L} = Q\mathscr{H}_{n+1}$ contains no eigenvectors of U.*
5. *T does not have normal eigenvalues.*

Remark 1. Statement 5 is true only if T is viewed as an operator on $\mathscr{L}$. Then the origin does not necessarily belong to $\Omega(T)$. If T were an operator on $\mathscr{H}_{n+1}$, then 0 would be a normal eigenvalue, $\mathbf{w}$ – the corresponding eigenvector, and the origin would necessarily be in $\Omega(T)$. This can distort $\Omega(T)$.

Proof. Let us prove that statement 4 implies statement 5 (other parts of the proof are obvious). Let, on the contrary, ζ be a normal eigenvalue of T, i.e., there is a unit vector $\mathbf{f} \in \mathscr{L}$ such that $QU\mathbf{f} = \zeta\mathbf{f}$ and $QU^*\mathbf{f} = \bar{\zeta}\mathbf{f}$, or

$$U\mathbf{f} = \zeta\mathbf{f} + \langle U\mathbf{f}, \mathbf{w}\rangle\mathbf{w},$$
$$U^*\mathbf{f} = \bar{\zeta}\mathbf{f} + \langle U^*\mathbf{f}, \mathbf{w}\rangle\mathbf{w}.$$

Due to statement 4, $\mathbf{f}$ is not an eigenvector of U (and U^*). Therefore, $\langle U^*\mathbf{f}, \mathbf{w}\rangle \neq 0$, and (because $UU^* = I$) $U\mathbf{w}$ is a linear combination of $\mathbf{f}$ and $\mathbf{w}$. Hence, the subspace $\mathscr{L}_1 = \mathrm{span}(\mathbf{f}, \mathbf{w})$ is invariant w.r.t. U, as well as the orthogonal supplement subspace $\mathscr{L}_2$. The latter subspace contains nonzero vectors because $n > 1$. Obviously, $\mathscr{L}_2 \subset \mathscr{L}$, i.e., $\mathscr{L}$ contains eigenvectors of U, which contradicts statement 4. □

Lemma 2. *Let* $e^{i\psi_j}, j = 1, \ldots, n+1$ *be the eigenvalues of* U; *and* $\mathbf{e}^{(j)}$ – *the corresponding eigenvectors. Furthermore, let* $Q = I - \mathbf{w}\mathbf{w}^*$, $\|\mathbf{w}\| = 1$, $\mathbf{w} = \sum_{j=1}^{n+1} w_j \mathbf{e}^{(j)}$, $w_j \neq 0$. *Then an eigenvalue* $e^{i\phi}$ *of* $U_\gamma = U + (e^{i\gamma} - 1)U\mathbf{w}\mathbf{w}^*$, *can be determined from the equation w.r.t.* ϕ

$$\sum_{j=1}^{n+1} |w_j|^2 \cot\frac{\phi - \psi_j}{2} = \cot\frac{\gamma}{2}.$$

In particular, for $\gamma \to 0$ *and the disturbed vertex* $e^{i\psi_k}$,

$$\phi_k = \psi_k + \gamma|w_k|^2 + \frac{\gamma^2}{2}|w_k|^2 \sum_{j \neq k} |w_j|^2 \cot\frac{\psi_k - \psi_j}{2} + o(\gamma^2). \tag{3}$$

Proof. Indeed, the equations $U_\gamma\mathbf{f} = e^{i\phi}\,\mathbf{f}$ and $\mathbf{f} = \sum_{j=1}^{n+1} f_j \mathbf{e}^{(j)}$ lead to

$$(e^{i\gamma} - 1)\langle \mathbf{f}, \mathbf{w}\rangle U\mathbf{w} = \sum_{j=1}^{n+1} (e^{i\phi} - e^{i\psi_j}) f_j \mathbf{e}^{(j)}$$

or

$$\langle \mathbf{f}, \mathbf{w}\rangle w_j \frac{e^{i\gamma} - 1}{e^{i(\phi - \psi_j)} - 1} = f_j$$

or

$$\langle \mathbf{f}, \mathbf{w}\rangle \sum_{j=1}^{n+1} |w_j|^2 \frac{e^{i\gamma} - 1}{e^{i(\phi - \psi_j)} - 1} = \langle \mathbf{f}, \mathbf{w}\rangle,$$

where $\langle \mathbf{f}, \mathbf{w}\rangle \neq 0$ (because of statement of 3 of Lemma 1). The last equation yields the equations for ϕ. □

It follows from Eq. (3):

Corollary. *Let* $\mathbf{e}(\psi)$ *be a unit eigenvector of* U_γ *corresponding* $e^{i\psi}$:

$$U_\gamma \mathbf{e}(\psi) = e^{i\psi}\,\mathbf{e}(\psi).$$

Then the differential of the arclength of the unit circle $\mathbf{C}$

$$d\psi = |\langle \mathbf{w}, \mathbf{e}(\psi)\rangle|^2\, d\gamma,$$

and

$$\int_{\psi_i}^{\psi_{i+1}} \frac{d\psi}{|\langle \mathbf{w}, \mathbf{e}(\psi)\rangle|^2} = \int_0^{2\pi} d\gamma = 2\pi.$$

As per Kolodziej [8] and King [9], the latter equation has the following interpretation: Let us consider transform $\mathbf{R} : \mathbf{C} \to \mathbf{C}$ defined by the condition that $e^{i\psi}$ and $\mathbf{R}\, e^{i\psi} = e^{i\phi}$, $\phi > \psi$ are "neighboring" eigenvalues of an unitary bordering of T. Then the hord, which joins $e^{i\psi}$ and $e^{i\phi}$, is tangent to $\partial\Omega(T)$, and the function on $\mathbf{C}$

$$h(e^{i\psi}) = \frac{1}{(n+1)|\langle \mathbf{w}, \mathbf{e}(\psi)\rangle|^2}$$

defines a measure $\int_A h(e^{i\psi})\, d\psi$ which is invariant w.r.t. $\mathbf{R}$, and such that it makes equivalent the transform $\mathbf{R}$ to the rigid rotation of $\mathbf{C}$ by the angle $2\pi/(n+1)$. There is a simplified way to calculate $h(e^{i\psi})$.

Theorem 2. *Let* U *be an* $(n+1)\times(n+1)$ *unitary matrix with distinct eigenvalues,* $Q = I_{n+1} - \mathbf{w}\mathbf{w}^*$, $\|\mathbf{w}\| = 1$, $T = QUQ$, *and* $\Omega(T)$ *be strictly inside the unit circle* $\mathbf{C}$. *Let, further, transform* $\mathbf{R} : \mathbf{C} \to \mathbf{C}$ *of the unit circle* $\mathbf{C}$ *be defined by the condition that the hord, which joins* $e^{i\psi}$ *and* $\mathbf{R}\, e^{i\psi} = e^{i\phi}$, *is tangent to* $\partial\Omega(T)$, *leaving* $\Omega(T)$ *to the left from this hord. Consider the function on* $\mathbf{C}$

$$h(e^{i\psi}) = \frac{1}{(n+1)|\langle \mathbf{e}(\psi), \mathbf{w}\rangle|^2},$$

where $\mathbf{e}(\psi)$ *is a unit eigenvector of* $U_\gamma = U(I + \mathbf{w}\mathbf{w}^*(e^{i\gamma} - 1))$ *corresponding* $e^{i\psi}$. *Then*:

$$h(e^{i\psi}) = \frac{1}{n+1}\left\{1 + \|(T - e^{i\psi} I_n)^{-1} QU\mathbf{w}\|^2\right\}.$$

Function $h(e^{i\psi})$ *defines the measure*

$$\int_A h(e^{i\psi})\, d\psi$$

which is invariant w.r.t. $\mathbf{R}$, *and for any* ψ

$$\int_{\psi}^{\phi} h(\mathrm{e}^{\mathrm{i}\psi_1})\,\mathrm{d}\psi_1 = \frac{2\pi}{n+1},$$

where $\mathrm{e}^{\mathrm{i}\phi} = \mathbf{R}\,\mathrm{e}^{\mathrm{i}\psi}$.

Proof. Consider projections of $\mathbf{e}(\psi)$ on $\mathscr{L} = (I - \mathbf{ww}^*)\mathscr{H}_{n+1}$ and $\mathbf{w}$

$$\mathbf{e}(\psi) = \mathbf{f} + \langle \mathbf{e}(\psi), \mathbf{w}\rangle \mathbf{w}.$$

By the definition of U_γ and $\mathbf{e}(\psi)$,

$$T\mathbf{f} + \langle \mathbf{e}(\psi), \mathbf{w}\rangle \mathrm{e}^{\mathrm{i}\gamma} QU\mathbf{w} = \mathrm{e}^{\mathrm{i}\psi}\mathbf{f}.$$

Hence,

$$\mathbf{f} = -\langle \mathbf{e}(\psi), \mathbf{w}\rangle \mathrm{e}^{\mathrm{i}\gamma}\left(T - \mathrm{e}^{\mathrm{i}\psi} I_n\right)^{-1} QU\mathbf{w}$$

and because $\|\mathbf{e}(\psi)\| = \|\mathbf{w}\| = 1$,

$$|\langle \mathbf{e}(\psi), \mathbf{w}\rangle|^2 \left\{1 + \left\|\left(T - \mathrm{e}^{\mathrm{i}\psi} I_n\right)^{-1} QU\mathbf{w}\right\|^2\right\} = 1.$$

The rest of the proof follows from the corollary of Lemma 2. □

3. Structure of matrices which admit unitary bordering

The following several theorems provide us with criteria for matrices which admit unitary bordering (and cosequently, generate Poncelet curves).

Theorem 3. *Matrix* T *admits unitary bordering, iff* $TT^* = I - \mathbf{uu}^*$ *where* $\mathbf{u}$ *is a nonzero vector.*

Proof. Indeed, let $T = QUQ$, $Q = I_{n+1} - \mathbf{ww}^*$, where $\|\mathbf{w}\| = 1$. Then $TT^* = I_n - QU\mathbf{w}(QU\mathbf{w})^*$. Conversely, let $TT^* = I_n - \mathbf{uu}^*$, where $\mathbf{u} \neq \mathbf{o}$. Then the polar representation of T is $T = (I_n - \mathbf{uu}^*)^{1/2} V$, where V is a unitary matrix in $\mathscr{H}_n$. This implies that

$$T = \sum_{j=1}^{n-1} \mathbf{e}^{(j)}\mathbf{v}^{(j)*} + \theta \mathbf{e}^{(n)}\mathbf{v}^{(n)*},$$

where $\mathbf{v}^{(j)}$ and $\mathbf{e}^{(j)}$, $j = 1, \ldots, n$ are orthonormal systems in $\mathscr{H}_n$, and $0 \leqslant \theta < 1$. Let us dilate $\mathscr{H}_n$ supplementing it with a unit vector $\mathbf{w} \perp \mathscr{H}_n$, and let

$$U = T + \sqrt{1 - \theta^2}\left(\mathbf{w}\mathbf{v}^{(n)*} - \mathbf{e}^{(n)}\mathbf{w}^*\right) + \theta \mathbf{ww}^*.$$

Then $UU^* = I_{n+1}$, i.e., T admits unitary bordering. □

Corollary 1. *If $n \times n$ matrix T admits unitary bordering and $VV^* = I_n$, then TV admits unitary bordering as well.*

Corollary 2. *An $n \times n$ matrix T admits unitary bordering, iff $T = V - \theta \mathbf{e}\mathbf{e}^* V$, where $VV^* = I_n$, $||\mathbf{e}|| = 1$, and $2\mathscr{R}(\theta) > |\theta|^2$.*

Corollary 3. *An $n \times n$ matrix S is similar to a matrix which admits unitary bordering, iff $S = U - \theta \mathbf{f}\mathbf{g}^*$, where U is a matrix which is similar to an unitary matrix ($U = R^{-1}VR$, $VV^* = I_n$), $||R\mathbf{f}|| = 1$, $g = U^* R^* R\mathbf{f}$, and $2\mathscr{R}(\theta) > |\theta|^2$.*

This corollary links UB-matrices and matrices with big spectra [10].

Now consider T in upper triangular form. The following theorem shows that if T admits unitary bordering, then the off-diagonal entries in the triangular form are defined by the eigenvalues of T (with some insignificant arbitrariness).

Theorem 4. *An $n \times n$ matrix T admits unitary bordering, iff its upper triangular form may be presented as follows:*

$$T = \begin{pmatrix} \kappa_1 \sin \theta_1 & -\kappa_1 \cos \theta_1 \cos \theta_2 & -\kappa_1 \cos \theta_1 \sin \theta_2 \cos \theta_3 & \dots & -\kappa_1 \cos \theta_1 \sin \theta_2 \sin \theta_3 \dots \cos \theta_n \\ 0 & \kappa_2 \sin \theta_2 & -\kappa_2 \cos \theta_2 \cos \theta_3 & \dots & -\kappa_2 \cos \theta_2 \sin \theta_3 \dots \cos \theta_n \\ 0 & 0 & \kappa_3 \sin \theta_3 & \dots & -\kappa_3 \cos \theta_3 \sin \theta_4 \dots \cos \theta_n \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \kappa_n \sin \theta_n \end{pmatrix}, \tag{4}$$

where $|\kappa_j| = 1 (j = 1, \dots, n)$.

Proof. If Eq. (4) holds, then the unitary bordering U_T of T may be chosen as follows: the bordering $(n+1)$-st row is

$$\big(\cos \theta_1; \sin \theta_1 \cos \theta_2; \sin \theta_1 \sin \theta_2 \cos \theta_3; \dots; \sin \theta_1 \dots \sin \theta_{n-1} \cos \theta_n; \\ u_{n+1,n+1} e^{i\gamma}\big),$$

where $u_{n+1,n+1} = \sin \theta_1 \dots \sin \theta_{n-1} \sin \theta_n$, and the bordering $(n+1)$-st column:

$$(-\bar{\kappa}_1 \cos \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_n; \\ -\bar{\kappa}_2 \cos \theta_2 \sin \theta_3 \dots \sin \theta_n; \dots; -\bar{\kappa}_n \cos \theta_n; u_{n+1,n+1})^* e^{i\gamma}.$$

Obviously, the converse is also true (within unitary similarity). □

Corollary. *If matrix T_1 is a reduction of matrix T to its invariant subspace, and T admits unitary bordering, then T_1 admits unitary bordering as well.*

The geometric meaning of Eq. (4) is that if matrix T admits unitary bordering then its spectrum defines the angles between eigenvectors of T. Namely, if $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$ are unit eigenvectors of T which correspond to not-equal eigenvalues α_1 and α_2, then

$$|\langle \mathbf{f}^{(1)}, \mathbf{f}^{(2)} \rangle| = \frac{\sqrt{(1-|\alpha_1|^2)(1-|\alpha_2|^2)}}{|1-\alpha_1\bar{\alpha}_2|}. \tag{5}$$

The linear-fractional function $f(z) = (\alpha + z)/(1 + \bar{\alpha}z)$ with any $|\alpha| < 1$ preserves the right-hand side of Eq. (5), i.e.,

$$\frac{\sqrt{(1-|f(\alpha_1)|^2)(1-|f(\alpha_2)|^2)}}{|1-f(\alpha_1)\bar{f}(\alpha_2)|} = \frac{\sqrt{(1-|\alpha_1|^2)(1-|\alpha_2|^2)}}{|1-\alpha_1\bar{\alpha}_2|}.$$

That is the reason why the following theorem takes place.

Theorem 5. *If T admits unitary bordering, then so does matrix*

$$S = (\alpha I + T)(I + \bar{\alpha}T)^{-1},$$

where $|\alpha| < 1$.

Proof. Consider matrix T of the form introduced in Corollary 2 of Theorem 3.

$$T = V - \theta \mathbf{e}\mathbf{e}^* V,$$

where $VV^* = I_n$, $\|\mathbf{e}\| = 1$, and $2\mathscr{R}(\theta) > |\theta|^2$. Noting that if $\rho_2[1 - \rho_1\langle \mathbf{f}, \mathbf{g}\rangle] = \rho_1 \neq 0$, then $(I - \rho_1 \mathbf{f}\mathbf{g}^*)^{-1} = I + \rho_2 \mathbf{f}\mathbf{g}^*$, we have:

$$S = W - \kappa \mathbf{f}\mathbf{f}^* W,$$

where $W = (\alpha I + V)(I + \bar{\alpha}V)^{-1}$, and

$$\mathbf{f} = \frac{(I - \bar{\alpha}W)\mathbf{e}}{\|(I - \bar{\alpha}W)\mathbf{e}\|}, \kappa = \frac{\theta\|\mathbf{e}^*(W - \alpha I)\|^2}{1 - |\alpha|^2 - \bar{\alpha}\theta\mathbf{e}^*(W - \alpha I)\mathbf{e}}.$$

Matrix W, vector $\mathbf{f}$, and κ satisfy the conditions of Corollary 2 of Theorem 3 ($WW^* = I$ and $\|\mathbf{f}\| = 1$ by the definition of W and $\mathbf{f}$, $2\mathscr{R}(\kappa) > |\kappa|^2$ because $\|S\| = 1$). Hence, S admits unitary bordering. □

Remark 2. It is appropriate to remind here that $\Omega(f(T))$ is not $f(\Omega(T))$. One of these ranges may be a circle, while another may be not. If $\Omega(T)$ is a circle with center ζ_1, then ζ_1 is an eigenvalue of T ([11,7]), and $\zeta_2 = f(\zeta_1)$ is an eigenvalue of $f(T)$, whereas $f(\Omega(T))$ is a circle with center $\neq \zeta_2$ (for $\alpha \neq 0$). Therefore it is surprising that either both boundaries of the numerical ranges, $\partial\Omega(T)$ and $\partial\Omega(f(T))$, are Poncelet curves, or neither is.

Example 1. Let T be a two-dimensional truncated shift

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $f(z) = (\alpha + z)/(1 + \bar{\alpha} z)$. Then $\Omega(f(T))$ is a circle with center α, and its circumference is a Poncelet curve of rank 3 w.r.t. $\mathbf{C}$.

Example 2. Let T be a three-dimensional truncated shift

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and $f(z) = (\alpha + z)/(1 + \bar{\alpha} z)$. Then

$$f(T) = \begin{pmatrix} \alpha & 1 - |\alpha|^2 & -\bar{\alpha}(1 - |\alpha|^2) \\ 0 & \alpha & 1 - |\alpha|^2 \\ 0 & 0 & \alpha \end{pmatrix}.$$

$\Omega(f(T))$ is not a circle (as well as any quadric), just a Poncelet curve of rank 4 w.r.t. $\mathbf{C}$.

4. Equations for matrices which admit unitary bordering

For a matrix T that admits unitary bordering U, Eq. (2) can be expressed in terms of eigenvalues of U and components of vector $\mathbf{w}$. This may simplify finding numerical range of T and determining properties of roots of Eq. (2). First, let us formulate a simple lemma.

Lemma 3. *If $a_1 < a_2 < \cdots < a_{n+1}$, and all $b_j > 0, j = 1, \ldots, n+1$, then equation for λ*

$$\sum_{j=1}^{n+1} \frac{b_j}{a_j - \lambda} = 0$$

has n distinct real roots.

The proof follows from the fact that the left-hand side of this equation changes the sign in the interval $a_j < \lambda < a_{j+1}, j = 1, \ldots, n$.

Theorem 6. *Let $e^{i\psi_j}$ $(j = 1, \ldots, n+1,\ 0 < \psi_1 < \cdots < \psi_{n+1} < 2\pi)$ be distinct eigenvalues of a unitary matrix U; $\mathbf{e}^{(j)}$ – the corresponding eigenvectors of U; $T = QUQ$; $Q = I - \mathbf{ww}^*$; $\mathbf{w} = \sum_{j=1}^{n+1} w_j \mathbf{e}^{(j)}$, where all $w_j \neq 0$, and $\|\mathbf{w}\| = 1$. Then* *the roots λ of Eq.* (2) *for matrix T can be determined from the following equations*:

$$\sum_{j=1}^{n+1} |w_j|^2 \prod_{1 \leqslant l \leqslant n+1, l \neq j} [\cos(\psi_l - \varphi) - \lambda] = 0 \tag{6}$$

or, for almost all $0 \leqslant \varphi < 2\pi$ (*specified below*),

$$\sum_{j=1}^{n+1} \frac{|w_j|^2}{\cos(\psi_j - \varphi) - \lambda} = 0 \tag{7}$$

For any φ, Eq. (6) *has n distinct roots.*

The main points of the proof of this theorem are as follows: Eq. (7) is derived from Eq. (2) assuming U is diagonal. Eqs. (6) and (7) have the same roots if the denominators in Eq. (7) are nonzero. This occurs if, by Lemma 3, all $\cos(\psi_j - \varphi)$ are distinct, i.e., if for any pair ψ_{j_1}, ψ_{j_2}, φ is equal neither to $(\psi_{j_1} + \psi_{j_2})/2$ nor to $(\psi_{j_1} + \psi_{j_2})/2 \pm \pi$. If, on the contrary, say $\varphi = \varphi_0 = (\psi_{j_1} + \psi_{j_2})/2$, then $\cos(\psi_{j_1} - \varphi_0) = \cos(\psi_{j_2} - \varphi_0)$, and there is root $\lambda = \cos((\psi_{j_1} - \psi_{j_2})/2)$ of Eq. (6) for $\varphi = \varphi_0$ such that the j_1-th and j_2-th components in the left-hand side of Eq. (7) annihilate each other in the vicinity of φ_0. The detailed proof is below.

Proof. Let us denote for $0 \leqslant \varphi < 2\pi$,

$$S(\varphi) = \mathscr{R}(T)\cos\varphi + \mathscr{I}(T)\sin\varphi$$

$$R(\varphi) = \mathscr{R}(U)\cos\varphi + \mathscr{I}(U)\sin\varphi$$

and let $\mathbf{y}(\varphi)$ be a unit eigenvector of $S(\varphi)$ corresponding to an eigenvalue $\lambda(\varphi)$:

$$[S(\varphi) - \lambda(\varphi)I_n]\mathbf{y}(\varphi) = \mathbf{o}.$$

Then, considering $\mathbf{y}(\varphi)$ in $\mathscr{H}_{n+1}$, we have: $\mathbf{y}(\varphi) = Q\mathbf{y}(\varphi)$, $S(\varphi) = QR(\varphi)Q$, and

$$[R(\varphi) - \lambda(\varphi)I_{n+1}]\mathbf{y}(\varphi) = \kappa(\varphi)\mathbf{w},$$

where $\kappa(\varphi)$ is a normalization factor so that $||\mathbf{y}(\varphi)|| = 1$. Obviously, for any j,

$$[R(\varphi) - \lambda(\varphi)I_{n+1}]\mathbf{e}^{(j)} = [\cos(\psi_j - \varphi) - \lambda(\varphi)]\mathbf{e}^{(j)}.$$

Hence, if $\lambda(\varphi) \neq \cos(\psi_j - \varphi)$ for all $j = 1, \ldots, n+1$, then

$$\mathbf{y}(\varphi) = \kappa(\varphi)[R(\varphi) - \lambda(\varphi)I_{n+1}]^{-1}\mathbf{w} = \kappa(\varphi)\sum_{j=1}^{n+1} \frac{w_j}{\cos(\psi_j - \varphi) - \lambda}\mathbf{e}^{(j)} \tag{8}$$

and $\lambda = \lambda(\varphi)$ satisfies Eq. (7), because $\langle \mathbf{y}(\varphi), \mathbf{w} \rangle = 0$.

Let us first consider the case when all $\cos(\psi_j - \varphi)$ are distinct. Then, due to Lemma 3, Eq. (7) has n distinct real roots, and these roots, obviously, coincide with the roots of Eq. (6). Let us show that all these roots are eigenvalues of $S(\varphi)$. Indeed, $S(\varphi)$ is an Hermitian operator which acts on n-dimensional space $\mathscr{L} = Q\mathscr{H}_{n+1}$. The eigenvalues of $S(\varphi)$ must be distinct because n linearly independent eigenvectors of $S(\varphi)$ satisfy Eq. (8). Thus the theorem is proved for φ's such that $\cos(\psi_j - \varphi)$ are distinct.

Now suppose that for some $j_1 \neq j_2$,

$$\cos(\psi_{j_1} - \varphi) = \cos(\psi_{j_2} - \varphi). \tag{9}$$

Then either $\varphi = (\psi_{j_1} + \psi_{j_2})/2$, or $\varphi = (\psi_{j_1} + \psi_{j_2})/2 \pm \pi$. Hence, there are only finite number of φ's for which $\cos(\psi_j - \varphi)$ are not distinct. Therefore all eigenvalues of $S(\varphi)$ satisfy Eq. (6) for any φ. It remains to prove that Eq. (6) does not have multiple roots.

It follows from Eq. (9) that there is no j_3, different from j_1 and j_2, such that $\cos(\psi_{j_3} - \varphi) = \cos(\psi_{j_1} - \varphi)$. Indeed, otherwise either $(\psi_{j_3} + \psi_{j_1})/2 = (\psi_{j_1} + \psi_{j_2})/2$ or $(\psi_{j_3} + \psi_{j_1})/2 = (\psi_{j_1} + \psi_{j_2})/2 \pm \pi$. Both cases contradict the conditions for the ψ_j's (they are distinct, and they are between 0 and 2π). Let us determine the behavior of the roots of Eq. (7) in the vicinity of $\varphi_0 = (\psi_{j_1} + \psi_{j_2})/2$, i.e., let

$$\varphi = \frac{\psi_{j_1} + \psi_{j_2}}{2} + \varepsilon.$$

Substituting this φ and

$$\lambda = \cos\frac{\psi_{j_1} - \psi_{j_2}}{2} + \varepsilon\frac{|w_2|^2 - |w_1|^2}{|w_2|^2 + |w_1|^2}\sin\frac{\psi_{j_1} - \psi_{j_2}}{2} + \tilde{\lambda}$$

into Eq. (7), we have for $\varepsilon \to 0$, that $\tilde{\lambda} = o(\varepsilon)$. Therefore,

$$\lambda_1\left(\frac{\psi_{j_1} + \psi_{j_2}}{2}\right) = \cos\frac{\psi_{j_1} - \psi_{j_2}}{2} \tag{10}$$

is a root of Eq. (6) for $\varphi = \varphi_0$.

It may happen, that ψ_{j_1} and ψ_{j_2} is not the only pair of ψ_j's with the mean which is equal to φ_0. Let ψ_{j_3} and $\psi_{j_4}, \ldots, \psi_{j_{2m-1}}$ and $\psi_{j_{2m}}$ be such that

$$\frac{\psi_{j_3} + \psi_{j_4}}{2} = \ldots = \frac{\psi_{j_{2m-1}} + \psi_{j_{2m}}}{2} = \varphi_0. \tag{11}$$

Then, similarly to Eq. (10),

$$\lambda_2(\varphi_0) = \cos\frac{\psi_{j_3} - \psi_{j_4}}{2}$$

$$\lambda_m(\varphi_0) = \cos\frac{\psi_{j_{2m-1}} - \psi_{j_{2m}}}{2}$$

are the roots of Eq. (6) for $\varphi = \varphi_0$. In accordance with Eq. (11) and conditions for ψ_j's, these m roots are distinct. Furthermore, for $\varphi = \varphi_0$, the left-hand side of Eq. (7) has $(n - m + 1)$ different denominators, and consequently, $(n - m)$ distinct roots none of which equals $\lambda_k(\varphi_0)$ $(k = 1, m)$. Thus, for any φ, Eq. (6) has n distinct roots. □

Note. Below, we will not distinguish roots of Eqs. (6) and (7), i.e., we will say "$\lambda(\varphi_0)$ is a root of Eq. (7)" even for the case when some denominators in this equation equal to zero, just because the left-hand side of Eq. (7) $\to 0$ for $\lambda = \lambda(\varphi_0)$ and $\varphi \to \varphi_0$.

Corollary 1. *If $\lambda(\varphi)$ satisfies Eq. (7), $0 \leqslant \varphi < 2\pi$, then so does $\tilde{\lambda}(\varphi) = -\lambda(\varphi + \pi)$.*

Corollary 2. *For any pair j_1, j_2 of nonequal positive integers less than $n + 2$, there exist:*

- *a root $\lambda(\varphi)$ of Eq. (7) which continuously depends on φ, and*

$$\lambda\left(\frac{\psi_{j_1} + \psi_{j_2}}{2}\right) = \cos\frac{\psi_{j_1} - \psi_{j_2}}{2},$$

- *the derivative $\lambda'(\varphi)$ of this root, and*

$$\lambda'\left(\frac{\psi_{j_1} + \psi_{j_2}}{2}\right) = \frac{|w_{j_2}|^2 - |w_{j_1}|^2}{|w_{j_2}|^2 + |w_{j_1}|^2}\sin\frac{\psi_{j_1} - \psi_{j_2}}{2}.$$

Corollary 3. *If matrix T admits unitary bordering and does not have normal eigenvalues, then the entire boundary of $\Omega(T)$ is a regular arc (see Proposition 2).*

Geometric meaning of **w**: Vector **w** defines the points where the sides of the polygon $\Omega(U)$ are tangent to $\partial\Omega(T)$.

Let z_j be the common point of $\partial\Omega(T)$ and the side of $\Omega(U)$ which connects vertices $e^{i\psi_j}$ and $e^{i\psi_{j+1}}$, i.e., $z_j = p_j e^{i\psi_j} + (1 - p_j)e^{i\psi_{j+1}}$, $j = 1, \ldots, n + 1$, $\psi_{n+2} = \psi_1$. Then,

$$p_j = \frac{|e^{i\psi_{j+1}} - z_j|}{|e^{i\psi_{j+1}} - e^{i\psi_j}|} \tag{12}$$

and the components $w_j = \langle \mathbf{w}, \mathbf{e}^{(j)} \rangle$ of vector **w** satisfy the equation

$$w_{j+1} = -w_j\sqrt{\frac{p_j}{1 - p_j}}, (1 \leqslant j \leqslant n) \tag{13}$$

Vectors $\mathbf{f}^{(j)} = \sqrt{p_j}\mathbf{e}^{(j)} + \sqrt{1-p_j}\mathbf{e}^{(j+1)}, (j \leqslant n)$, $\mathbf{f}^{(n+1)} = \sqrt{p_{n+1}}\mathbf{e}^{(n+1)} + (-1)^{n+1}\sqrt{1-p_{n+1}}\mathbf{e}^{(1)}$ are orthogonal to $\mathbf{w}$ and generate the common points z_j of $\partial\Omega(T)$ and the sides of polygon $\Omega(U)$. Hereafter, we require the converse statement.

Theorem 7. *Let n points z_j be on n sides of a convex $(n+1)$-sided polygon $\mathbf{P}$ inscribed in the unit circle. Then there exists a UB-matrix T such that $\Omega(T)$ is inscribed in $\mathbf{P}$, and z_j's are the tangent points. The $(n+1)$-st tangent point z_{n+1} is determined uniquely.*

Proof. $\mathbf{P}$ may be viewed as the numerical range of a unitary matrix U with distinct eigenvalues $\mathrm{e}^{\mathrm{i}\psi_j}$, $j = 1, \ldots, n+1$, corresponding unit eigenvectors $\mathbf{e}^{(j)}$. Points z_j, $j = 1, \ldots, n$, define p_j through Eq. (12) and unit vector $\mathbf{w}$ through Eq. (13). This, in turn, defines orthoprojector $Q = I - \mathbf{w}\mathbf{w}^*$ and matrix $T = QUQ$. The $(n+1)$-st tangent point $z_{n+1} = p_{n+1}\mathrm{e}^{\mathrm{i}\psi_{n+1}} + (1-p_{n+1})\mathrm{e}^{\mathrm{i}\psi_1}$ can be determined from the condition $\mathbf{f}^{(n+1)} \in \operatorname{span}(\mathbf{f}^{(j)}, j = 1, \ldots, n)$. This condition yields the equation for p_{n+1}

$$\prod_{j=1}^{n+1} p_j = \prod_{j=1}^{n+1}(1-p_j) \tag{14}$$

and $w_1 = -w_{n+1}\sqrt{p_{n+1}/(1-p_{n+1})}$. $\square$

Eqs. (12) and (13) link the measure density $h(\mathrm{e}^{\mathrm{i}\psi})$ which is introduced in Corollary of Lemma 2 and the function $\lambda(\varphi)$ which defines the boundary $\partial\Omega(T)$. Namely, let us, again, consider transform $\mathbf{R} : \mathbf{C} \to \mathbf{C}$ defined by the condition that the hord, which joins $\mathrm{e}^{\mathrm{i}\psi}$ and $\mathbf{R}\mathrm{e}^{\mathrm{i}\psi} = \mathrm{e}^{\mathrm{i}\phi}$, $\phi > \psi$, is tangent to $\partial\Omega(T)$. Then it follows from Eqs. (12) and (13) that there is a continuous root $\lambda(\varphi)$ of Eq. (7) such that for $\varphi = (\psi+\phi)/2$ and the tangent point z,

$$\frac{h(\mathrm{e}^{\mathrm{i}\psi})}{h(\mathrm{e}^{\mathrm{i}\phi})} = \frac{|\langle \mathbf{w}, \mathbf{e}(\phi)\rangle|^2}{|\langle \mathbf{w}, \mathbf{e}(\psi)\rangle|^2} = \frac{|\mathrm{e}^{\mathrm{i}\phi} - z|}{|z - \mathrm{e}^{\mathrm{i}\psi}|} = \frac{\sqrt{1-\lambda^2(\varphi)} - \lambda'(\varphi)}{\sqrt{1-\lambda^2(\varphi)} + \lambda'(\varphi)}.$$

Notation. Let a circuit which is defined by Eq. (1) and a differentiable periodic function $\lambda(\varphi) = \lambda(\varphi + 2\pi)$ be denoted by $\mathbf{K}\{\lambda(\varphi)\}$.

Remark 3. The largest root $\lambda_1(\varphi)$ of Eq. (7) generates the circuit $\mathbf{K}\{\lambda_1(\varphi)\}$ which is the boundary of the numerical range of the matrix T. Other roots of Eq. (7) also generate some circuits. Obviously, $\mathbf{K}\{-\lambda(\varphi+\pi)\} = \mathbf{K}\{\lambda(\varphi)\}$. Due to Theorem 6 and its Corollary 1, there are, for an even n, $n/2$ different circuits defined by Eq. (7). For an odd n, in addition to $(n-1)/2$ different circuits, we have a degenerated "circuit" – either two coincided arcs, or a point $\xi = \alpha$, $\eta = \beta$, if $\lambda(\varphi) = \alpha \cos \varphi + \beta \sin \varphi$.

5. Examples

Example 3. Let $n \times n$ matrix T be a truncated shift:

$$T = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

The left-hand side of Eq. (2) for this matrix T

$$D_n = \det\left[\mathscr{R}(Te^{-i\varphi}) - \lambda I\right]$$

satisfies the recurrent equation $D_n = -\lambda D_{n-1} - 0.25D_{n-2}$, where $D_0 = 1$, $D_1 = -\lambda$. These equations yield the roots of D_n, $\lambda_j = \cos j\pi/(n+1)$, where $j = 1, \dots, n$. $\partial\Omega(T) = \mathbf{K}\{\cos\pi/(n+1)\}$ is a circle $\mathbf{C}_1$ with center in origin and radius $\cos\pi/(n+1)$.

The following $(n+1) \times (n+1)$ matrix with an arbitrary real β is a unitary bordering of T:

$$U = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & e^{i(n+1)\beta} \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The numerical range $\Omega(U)$ is an $(n+1)$-sided polygon $\mathbf{P}$ with eigenvalues of U as its vertices:

$$e^{i\beta}; e^{i(\beta+2\pi/(n+1))}; \dots; e^{i(\beta+2\pi n/(n+1))}.$$

This is the obvious case of the Poncelet Theorem for concentric circles $\mathbf{C}$ and $\mathbf{C}_1$, and a convex polygon $\mathbf{P}$ which is inscribed in $\mathbf{C}$ and circumscribed around $\mathbf{C}_1$. Note that other roots of D_n generate the circles $\mathbf{C}_j = \mathbf{K}\{\cos j\pi/(n+1)\}$, $2 \leqslant j \leqslant (n+1)/2$ which are also Poncelet curves. The polygons inscribed in $\mathbf{C}$ and circumscribed around $\mathbf{C}_j$ consist of diagonals of $\mathbf{P}$.

The elements of 3×3, 4×4, and 5×5 matrices can be chosen such that these matrices admit unitary bordering and their numerical ranges are circles with an arbitrary center inside the unit circle. Such a construction is presented in Examples 4–6. It proves the Poncelet Theorem for quadrangles, five- and six-sided polygons (Fuss' equations, [14]).

Example 4 (*Poncelet Theorem for quadrangles*). Let

$$T = \begin{pmatrix} \alpha & \alpha^2 - 1 & -\alpha(1-\alpha^2)^{3/2}/(1+\alpha^2) \\ 0 & \alpha & -(1-\alpha^2)^{3/2}/(1+\alpha^2) \\ 0 & 0 & 2\alpha/(1+\alpha^2) \end{pmatrix}.$$

A unitary bordering of T is

$$U_\gamma = \begin{pmatrix} \alpha & \alpha^2 - 1 & -\alpha(1-\alpha^2)^{3/2}/(1+\alpha^2) & -2\alpha^2\sqrt{1-\alpha^2}/(1+\alpha^2)e^{i\gamma} \\ 0 & \alpha & -(1-\alpha^2)^{3/2}/(1+\alpha^2) & -2\alpha\sqrt{1-\alpha^2}/(1+\alpha^2)e^{i\gamma} \\ 0 & 0 & 2\alpha/(1+\alpha^2) & (\alpha^2-1)/(1+\alpha^2)e^{i\gamma} \\ \sqrt{1-\alpha^2} & \alpha\sqrt{1-\alpha^2} & \alpha^2(1-\alpha^2)/(1+\alpha^2) & 2\alpha^3/(1+\alpha^2)e^{i\gamma} \end{pmatrix}.$$

Eq. (2) for this T is:

$$(\alpha \cos \varphi + r - \lambda)(\alpha \cos \varphi - r - \lambda)\left(\frac{2\alpha}{1+\alpha^2} \cos \varphi - \lambda\right) = 0,$$

where

$$r = \frac{1-\alpha^2}{\sqrt{2(1+\alpha^2)}}.$$

Consequently, $\Omega(T)$ is a circle with center α and radius r. $\Omega(U_\gamma)$ is a quadrangle (for any real γ) inscribed in the unit circle and circumscribed around $\Omega(T)$. Note that all diagonals of these quadrangles cross the point $\xi = 2\alpha/(1+\alpha^2), \eta = 0$.

Example 5 (*Poncelet Theorem for five-sided polygons*). Let

$$T = \begin{pmatrix} \alpha & \alpha^2 - 1 & -\alpha\sqrt{1-\alpha^2}\cos\theta & -\alpha\sqrt{1-\alpha^2}\sin\theta\cos\theta \\ 0 & \alpha & -\sqrt{1-\alpha^2}\cos\theta & -\sqrt{1-\alpha^2}\sin\theta\cos\theta \\ 0 & 0 & \sin\theta & -\cos^2\theta \\ 0 & 0 & 0 & \sin\theta \end{pmatrix}.$$

A unitary bordering of T is

$$U_\gamma = \begin{pmatrix} \alpha & \alpha^2 - 1 & -\alpha\sqrt{1-\alpha^2}\cos\theta & -\alpha\sqrt{1-\alpha^2}\sin\theta\cos\theta & -\alpha\sqrt{1-\alpha^2}\sin^2\theta\, e^{i\gamma} \\ 0 & \alpha & -\sqrt{1-\alpha^2}\cos\theta & -\sqrt{1-\alpha^2}\sin\theta\cos\theta & -\sqrt{1-\alpha^2}\sin^2\theta\, e^{i\gamma} \\ 0 & 0 & \sin\theta & -\cos^2\theta & -\sin\theta\cos\theta e^{i\gamma} \\ 0 & 0 & 0 & \sin\theta & -\cos\theta e^{i\gamma} \\ \sqrt{1-\alpha^2} & \alpha\sqrt{1-\alpha^2} & \alpha^2\cos\theta & \alpha^2\sin\theta\cos\theta & \alpha^2\sin^2\theta\, e^{i\gamma} \end{pmatrix}.$$

If $\alpha^2 = (1 + 2v - 4v^2)/(8v^3)$, $\cos^2\theta = (4v^2 - 1)(2v - 1)$, where $0.5 < v < (1+\sqrt{5})/4$, then Eq. (2) for this T is:

$$(\lambda - \alpha \cos \varphi - r)(\lambda - \alpha \cos \varphi + r)\left(\lambda - \sin \theta \cos \varphi - \frac{\cos^2\theta}{4v}\right)$$
$$\left(\lambda - \sin \theta \cos \varphi + \frac{\cos^2\theta}{4v}\right) = 0,$$

where

$$r = v(1 - \alpha^2) = \frac{(4v^2 - 1)(2v + 1)}{8v^2} > (\sin \theta - \alpha)\cos \varphi + \frac{\cos^2\theta}{4v}.$$

$\Omega(T)$ is a circle ($\mathbf{C}_1$) with center α and radius r. $\Omega(U_\gamma) = \mathbf{P}_\gamma$ is a five-sided polygon which is inscribed in the unit circle $\mathbf{C}$ and circumscribed around $\mathbf{C}_1$ (Fig. 2a). The second root of Eq. (2) for this T, $\lambda_2 = \sin \theta \cos \varphi + \cos^2\theta/4v$, generates the circuit $\mathbf{K}\{\lambda_2\}$ which is a circle ($\mathbf{C}_2$) with center $\sin \theta$ and radius $\cos^2\theta/4v$. It can be verified that $\mathbf{C}_2$ (as well as $\mathbf{C}_1$) is a Poncelet

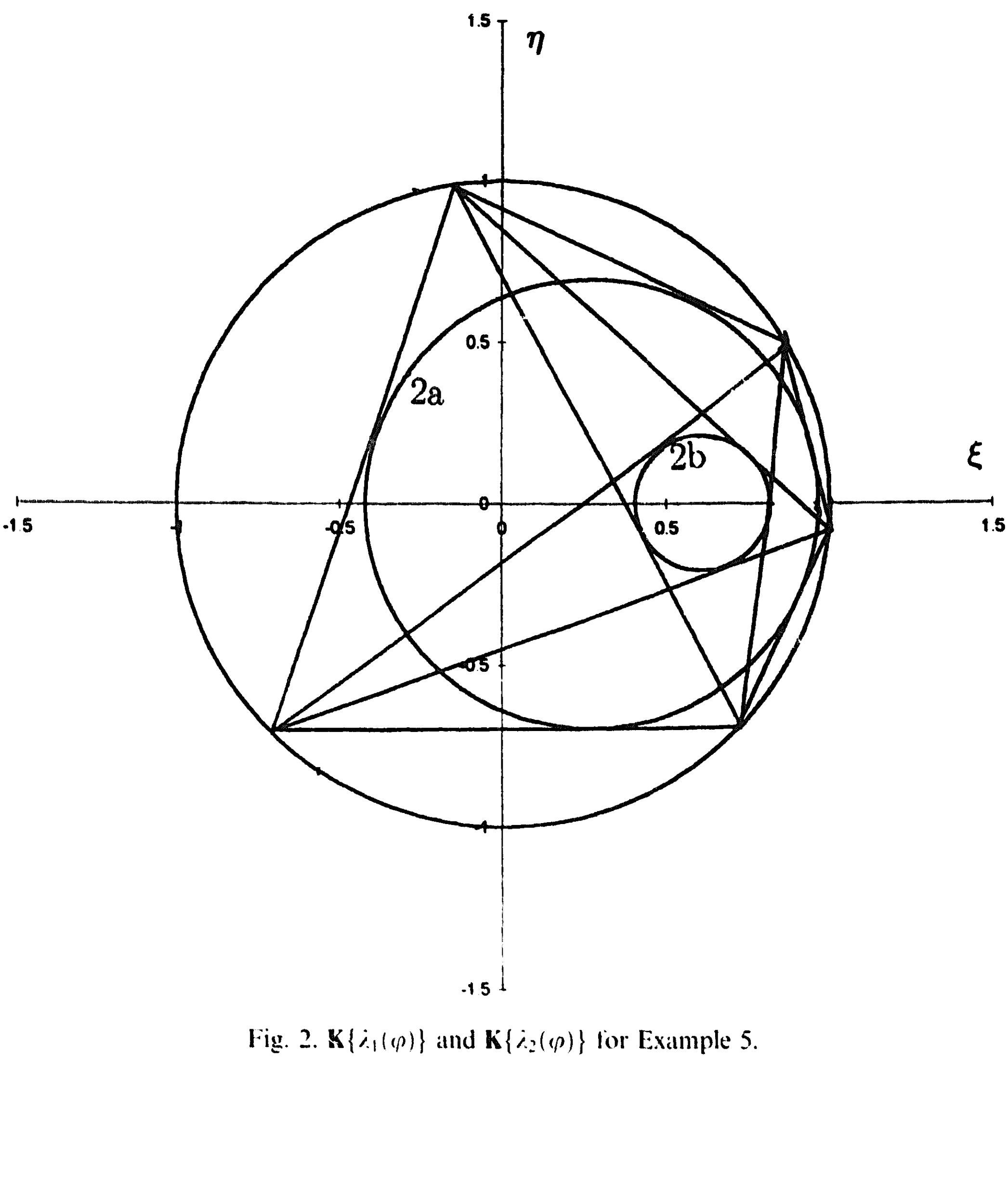

Fig. 2. $\mathbf{K}\{\lambda_1(\varphi)\}$ and $\mathbf{K}\{\lambda_2(\varphi)\}$ for Example 5.

curve of rank 5 w.r.t **C**. Polygons which are inscribed in **C** and circumscribed around $\mathbf{C}_2$ are nonconvex: they consist of five diagonals of $\mathbf{P}_\gamma$ (Fig. 2b).

Example 6 (*Poncelet Theorem for six-sided polygons*). Consider a unitary matrix

$$U = \operatorname{diag}(1, e^{i\psi_1}, e^{i\psi_2}, -1, e^{-i\psi_2}, e^{-i\psi_1}),$$

where $\psi_1 = \arccos((1 + 2\alpha - \alpha^2)/2)$, $\psi_2 = \arccos((-1 + 2\alpha + \alpha^2)/2)$ and α is an arbitrary real parameter between -1 and 1. And let $\mathbf{w} = \kappa^{-1}(w_k)_{k=1}^{k=6}$, where $\kappa = \sqrt{\sum_{j=1}^{6} w_j^2}$ and

$$w_1 = \sqrt[4]{1 + \alpha_1^2 - r_1^2 - 2\alpha_1},$$

$$w_2 = w_6 = -\sqrt[4]{1 + \alpha_1^2 - r_1^2 - 2\alpha_1 \cos\psi_1},$$

$$w_3 = w_5 = \sqrt[4]{1 + \alpha_1^2 - r_1^2 - 2\alpha_1 \cos\psi_2},$$

$$w_4 = -\sqrt[4]{1 + \alpha_1^2 - r_1^2 + 2\alpha_1},$$

$$\alpha_1 = \frac{\cos(\psi_1/2) - \sin(\psi_2/2)}{\cos(\psi_1/2) + \sin(\psi_2/2)},$$

$$r_1 = \frac{2\cos(\psi_1/2)\sin(\psi_2/2)}{\cos(\psi_1/2) + \sin(\psi_2/2)}.$$

Then the numerical range $\Omega(T)$ of matrix $T = (I - \mathbf{ww}^*)U(I - \mathbf{ww}^*)$ is a circle with center α_1 and radius r_1. Obviously, this circle is a Poncelet curve of rank 6 w.r.t. the unit circle **C**. Eq. (2) for this T is:

$$(\lambda - \alpha_1\cos\varphi - r_1)(\lambda - \alpha_1\cos\varphi + r_1)(\lambda - \alpha_2\cos\varphi - r_2)$$
$$(\lambda - \alpha_2\cos\varphi + r_2)(\lambda - \alpha_3\cos\varphi) = 0.$$

where $\alpha_2 = \alpha$, $r_2 = (1 - \alpha^2)/2$, and $\alpha_3 = \sin(\psi_1 + \psi_2)/(\sin\psi_1 + \sin\psi_2)$.

The second and third roots ($\lambda_2 = \alpha_2\cos\varphi + r_2$ and $\lambda_3 = \alpha_3\cos\varphi$) of this equation generate circles which are Poncelet curves also. λ_2 generates the circle $\mathbf{K}\{\lambda_2\}$ with center α_2 and radius r_2 which is Poncelet curve of rank 3 w.r.t **C**. The "circle" $\mathbf{K}\{\lambda_3\}$ degenerates into the point α_3. Any large diagonal (i.e., diagonal which connects a vertex $e^{i\phi_k}$ with the vertex $e^{i\phi_{k+3}}$) of any six-sided polygon $\mathbf{P}_\gamma = \Omega(U_\gamma)$ crosses the point α_3. This means that, *in addition to the*

Pascal's and Brianchon's Theorems for the six-sided polygons $\mathbf{P}_\gamma$, *there is one common point of all large diagonals of these polygons.*

Example 7. Let T and its unitary bordering U_γ be:

$$T = \begin{pmatrix} -1/\sqrt{2} & -1/2 & 1/2\sqrt{2} & 1/4 \\ 0 & -1/\sqrt{2} & -1/2 & -1/2\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/2 \\ 0 & 0 & 0 & 1/\sqrt{2} \end{pmatrix},$$

$$U_\gamma = \begin{pmatrix} -1/\sqrt{2} & -1/2 & 1/2\sqrt{2} & 1/4 & 1/4\,e^{i\gamma} \\ 0 & -1/\sqrt{2} & -1/2 & -1/2\sqrt{2} & -1/2\sqrt{2}\,e^{i\gamma} \\ 0 & 0 & 1/\sqrt{2} & -1/2 & -1/2\,e^{i\gamma} \\ 0 & 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2}\,e^{i\gamma} \\ 1/\sqrt{2} & -1/2 & 1/2\sqrt{2} & 1/4 & 1/4\,e^{i\gamma} \end{pmatrix}.$$

Then the roots of Eq. (2) for this T are $\lambda_{1,2}(\varphi) = \frac{1}{16}\sqrt{25 + 128\cos^2\varphi} \pm 3/16$. Substituting

$$\cos^2\varphi = \frac{2}{9}\left(u - \frac{153}{16}\right)^2 - \frac{25}{128}$$

we have from Eq. (1)

$$\xi^2 = u^2\left(u - \frac{138}{16}\right)\left(u - \frac{168}{16}\right)\Big/\left[128\left(u - \frac{153}{16}\right)^2\right],$$

$$\eta^2 = u^2(u - u_2)(u_1 - u)\Big/\left[128\left(u - \frac{153}{16}\right)^2\right],$$

where $u_1 = (153 + 3\sqrt{153})/16$, $u_2 = (153 - 3\sqrt{153})/16$. For $\mathbf{K}\{\lambda_1(\varphi)\}$, $168/16 \leqslant u \leqslant u_1$, and for $\mathbf{K}\{\lambda_2(\varphi)\}$, $u_2 \leqslant u \leqslant 138/16$.

The five-sided polygon $\Omega(U_\gamma)$ is inscribed in $\mathbf{C}$ and circumscribed around the boundary $\mathbf{K}\{\lambda_1(\varphi)\}$ of the numerical range $\Omega(T)$ (Fig. 3). $\mathbf{K}\{\lambda_2(\varphi)\}$ is not a convex circuit, this curve has returning and double points, and the definition of Poncelet curves is not applicable to this curve. However, the following is true: Let $F_j \in \mathbf{C}$ and $F_jF_{j+1} = \mathbf{L}_j$ be *tangent* to $\mathbf{K}\{\lambda_2(\varphi)\}$, $j = 1, \ldots, 5$. Then, $F_6 = F_1$, i.e., the zig-zag line $\mathbf{L}_1$-$\mathbf{L}_2$-$\mathbf{L}_3$-$\mathbf{L}_4$-$\mathbf{L}_5$ closes. Again, as in Examples above, this fact is not an accident. Theorems 8 and 9 below extend the results of Theorems 1 and 7 for nonconvex polygons and general curves $\mathbf{K}\{\lambda(\varphi)\}$ for the roots $\lambda(\varphi)$ of Eq. (7).

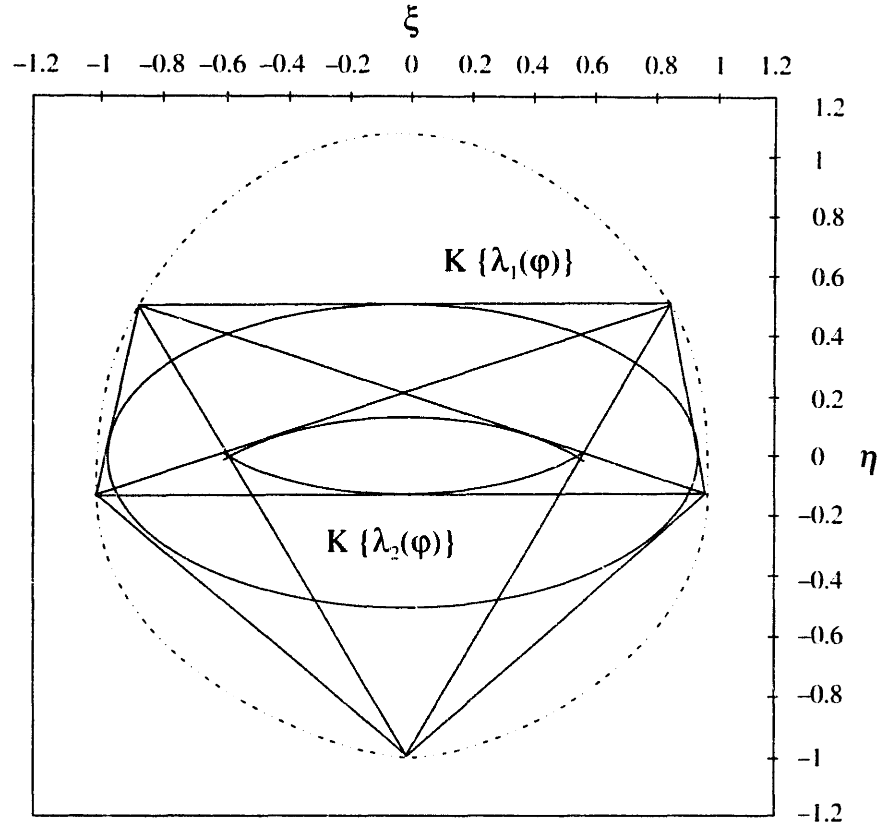

Fig. 3. $\mathbf{K}\{\lambda_1(\varphi)\}$ and $\mathbf{K}\{\lambda_2(\varphi)\}$ for Example 7.

6. Extension of main construction to non-convex polygons

Theorem 8. *Let U be an unitary matrix with distinct eigenvalues $e^{i\psi_j}$ and corresponding eigenvectors $\mathbf{e}^{(j)}, j = 1, \ldots, n+1, 0 < \psi_1 < \cdots < \psi_{n+1} < 2\pi$. And let $T = QUQ, Q = I - \mathbf{ww}^*, \mathbf{w} = \sum_{j=1}^{n+1} w_j \mathbf{e}^{(j)}$, where all $w_j \neq 0$ and $\|\mathbf{w}\| = 1$. For any positive integer $k \leqslant (n+1)/2$, there exists a continuous root $\lambda(\varphi)$ of Eq. (7) such that any straight line $\mathbf{L}_j^{(k)}$ which connects vertices $e^{i\psi_j}$ and $e^{i\psi_{j+k}}$ is tangent to the circuit $\mathbf{K}\{\lambda(\varphi)\} = \mathbf{K}_k$. Here, for $m \geqslant 2$, $\psi_{n+m} = \psi_{m-1} + 2\pi$. If the curve $\mathbf{K}_k$ degenerates into one point, then $\mathbf{L}_j^{(k)}$ passes through this point. If $\mathbf{K}_k$ is a convex circuit, then $\mathbf{K}_k$ is a Poncelet curve of a rank m w.r.t. $\mathbf{C}$, where*

$$m = \frac{n+1}{\text{Greatest Common Divisor}(n+1, k)}.$$

Proof. In accordance with Corollary 2 of Theorem 6, there exists a continuous root $\lambda(\varphi)$ of Eq. (7) such that

$$\lambda\left(\frac{\psi_j + \psi_{j+k}}{2}\right) = \cos\frac{\psi_j - \psi_{j+k}}{2}.$$

It follows from Eq. (1) that, for $\varphi = (\psi_j + \psi_{j+k})/2$, the point $(\xi(\varphi), \eta(\varphi))$ lies on $\mathbf{L}_j^{(k)}$. Furthermore, differentiating Eq. (1) w.r.t. φ, we have:

$$\begin{aligned} \xi'(\varphi) &= -\Big[\lambda(\varphi) + \lambda''(\varphi)\Big] \sin \varphi, \\ \eta'(\varphi) &= \Big[\lambda(\varphi) + \lambda''(\varphi)\Big] \cos \varphi. \end{aligned} \tag{15}$$

Hence, if $\lambda''(\varphi) + \lambda(\varphi) \neq 0$, then $\mathrm{d}\eta/\mathrm{d}\xi = -\cot\varphi$ and $\mathbf{L}_j^{(k)}$ is tangent to $\mathbf{K}_k$ at point $(\xi(\varphi), \eta(\varphi))$. A simple analysis of Eq. (15) shows that the following three cases exhaust all possibilities:

1. $\lambda''(\varphi) + \lambda(\varphi) > 0$ for all $0 \leqslant \varphi < 2\pi$. This means that there is a convex area inside the curve $\mathbf{K}\{\lambda(\varphi)\}$. We have, particularly, this case for the maximum root of Eq. (7), and for this root, $k = 1$, $\mathbf{K}_1$ is the boundary $\partial\Omega(T)$, and $\mathbf{L}_j^{(1)}$ are the sides of the polygon $\mathbf{P} = \Omega(U)$. For other continuous roots of Eq. (7), $k > 1$ and $\mathbf{L}_j^{(k)}$ are diagonals of $\mathbf{P}$.
2. $\lambda''(\varphi) + \lambda(\varphi) = 0$ for a finite number of φ's, denoted $\tilde{\varphi}$. For these $\tilde{\varphi}$'s, the curve $\mathbf{K}_k$ may have singular points. Still, there are straight tangent lines even in the singular points, and $\mathrm{d}\eta/\mathrm{d}\xi \to -\cot\tilde{\varphi}$ for $\varphi \to \tilde{\varphi}$.
3. $\lambda''(\varphi) + \lambda(\varphi) = 0$ for all $0 \leqslant \varphi < 2\pi$. Then $\lambda(\varphi) = \alpha \cos\varphi + \beta \sin\varphi$, and $\mathbf{K}\{\lambda(\varphi)\}$ degenerates into the point (α, β).

It remains to prove that line $\mathbf{L}_i^{(k)}$ with $i \neq j$ is a tangent to the same curve $\mathbf{K}_k$. For that, we will use the construction utilized in the proof of Theorem 1. Let $U_\gamma = U + (\mathrm{e}^{\mathrm{i}\gamma} - 1)U\mathbf{w}\mathbf{w}^*$. As is noted in the proof of Theorem 1, if γ traverses from 0 to 2π, then the polygon $\mathbf{P}_\gamma = \Omega(U_\gamma)$ traverses without repetition all possible polygons inscribed in $\mathbf{C}$ and circumscribed around $\Omega(T)$, starting from the polygon $\mathbf{P}_0 = \Omega(U)$ and coming back to this polygon because $\mathbf{P}_{2\pi} = \mathbf{P}_0$. In accordance with Lemma 2, the dependence of the arguments $\phi_j(\gamma)$ of the vertices of $\mathbf{P}_\gamma$ on γ, $0 < \gamma < 2\pi$, $\phi_j(0) = \psi_j$, is continuous, and $\phi_j(2\pi) = \psi_{j+1}$. Due to Corollary 2 of Theorem 6 and Eq. (15), $\mathbf{L}_i^{(k)}$ is tangent to $\mathbf{K}_k$.

Corollary. *Let a polygon $\tilde{\mathbf{P}}$ be inscribed in $\mathbf{C}$ and circumscribed around a circuit $\mathbf{K}\{\lambda(\varphi)\}$. And let $\mathbf{P}$ be the convex hull of $\tilde{\mathbf{P}}$. Then the number of "missed vertices" of $\mathbf{P}$ for the sequential vertices of $\tilde{\mathbf{P}}$ is constant.*

The following generalization of Theorem 7 is left to be proven by the reader.

Theorem 9. *Let n points z_j be on n sides of an $(n+1)$-sided polygon $\mathbf{P}$ (not necessarily convex) with vertices $\mathrm{e}^{\mathrm{i}\psi_j}$. Then there exists a UB-matrix T and a continuous root $\lambda(\varphi)$ of Eq. (7) for these T and ψ_j's such that $\mathbf{K}\{\lambda(\varphi)\}$ is inscribed in $\mathbf{P}$, and z_j's are the tangent points. The $(n+1)$-st tangent point is determined by Eqs. (12) and (14).*

Remark 4. Any equation of the form of Eq. (7) corresponds to a UB-matrix and a set of circuits described in Theorem 8. Indeed, let

$$\sum_{j=1}^{n+1} \frac{q_j}{\cos(\psi_j - \varphi) - \lambda} = 0,$$

where $q_j > 0$ and ψ_j's are distinct parameters from $(0, 2\pi)$. Then a unitary matrix U of size $n+1$ is defined by its eigenvalues $e^{i\psi_j}$. The unit vector $\mathbf{w}$ is defined by the condition

$$\frac{w_{j+1}}{w_j} = -\sqrt{\frac{q_{j+1}}{q_j}}$$

and the UB-matrix

$$T = (I - \mathbf{w}\mathbf{w}^*)U(I - \mathbf{w}\mathbf{w}^*).$$

Let $\lambda_1(\varphi) > \cdots > \lambda_m(\varphi)$, $m \leqslant (n+1)/2$ be the first m largest roots of the equation for λ. If $\mathbf{K}_k = \mathbf{K}\{\lambda_k(\varphi)\}$ is a convex circuit, then $\mathbf{K}_k$ is a Poncelet curve.

7. Application of matrices which admit unitary bordering

7.1. Solution of American Mathematical Monthly Problem 10542 (1996)

Problem (Proposed by Jean Anglesio, Garches, France). Let C be the circumcircle of a triangle $A_oB_oC_o$ and J the incircle. It is known that, for each point A on C, there is a triangle ABC having C for circumcircle and J for incircle. Show that the locus of the centroid G of triangle ABC is a circle that is traversed three times by G as A traverses C once, and determine the center and radius of this circle.

Solution (*Its essential part repeats the proof of Theorem* 1): Let for $-1 < \alpha < 1$,

$$T = \begin{pmatrix} \alpha & \alpha^2 - 1 \\ 0 & \alpha \end{pmatrix}.$$

It is easy to verify that $\Omega(T) = \mathbf{J}$ is a circle with center α and radius $(1-\alpha^2)/2$. The following matrix U_γ with any real γ is an one-dimensional unitary dilation of T.

$$U_\gamma = \begin{pmatrix} \alpha & \alpha^2 - 1 & -\alpha\sqrt{1-\alpha^2}\, e^{i\gamma} \\ 0 & \alpha & -\sqrt{1-\alpha^2}\, e^{i\gamma} \\ \sqrt{1-\alpha^2} & \alpha\sqrt{1-\alpha^2} & \alpha^2\, e^{i\gamma} \end{pmatrix}.$$

$\Omega(U_\gamma)$ is a triangle which is inscribed in unit circle $\mathbf{C}$ and circumscribed around $\mathbf{J}$. For $0 < |\gamma_1 - \gamma_2| < 2\pi$, the triangles $\Omega(U_{\gamma_1})$ and $\Omega(U_{\gamma_2})$ have different vertices because the determinants $\det(U_{\gamma_1}) \neq \det(U_{\gamma_2})$. When γ traverses a 2π-long segment, triangles $\Omega(U_\gamma)$ traverse all possible triangles inscribed in $\mathbf{C}$ and circumscribed around $\mathbf{J}$, because $\Omega(U_{\gamma+2\pi}) = \Omega(U_\gamma)$.

The centroid G_γ of triangle $\Omega(U_\gamma)$ is one third of the trace of U_γ,

$$G_\gamma = \frac{2}{3}\alpha + \frac{1}{3}\alpha^2 e^{i\gamma}.$$

Hence, the locus of this centroid is a circle with the center $(2/3)\alpha$ and radius $(1/3)\alpha^2$.

Let $A_oB_oC_o = \Omega(U_{\gamma_o})$ be a triangle with circumcircle **C** and incircle **J**. Then, the circle **C** is broken into three arcs A_oB_o, B_oC_o and C_oA_o such that for any triangle $ABC \neq A_oB_oC_o$ with the circumcircle **C** and incircle **J**, each of these arcs contains one vertex of ABC. On the other hand, a location of vertex A defines the locations of B and C. Therefore, when A traverses from A_o to B_o, we will have all possible different triangles with circumcircle **C** and incircle **J**. This means that γ traverses from γ_o to $\gamma_o + 2\pi$, and the centroid G_γ makes an entire revolution. The same occurs when A traverses from B_o to C_o and from C_o to A_o, i.e., the centroid makes three revolutions when A makes one.

For an arbitrary circumcircle of radius R, we obtain the results by homogeneous stretching of the plane: the distance between the centers O of the circumcircle and o of the incircle is $d = \alpha R$, the radius of the incircle is $r = R \times (1 - \alpha^2)/2 = (R^2 - d^2)/(2R)$ [12], p. 324, and the center of the centroid locus divides the segment Oo by $2 : 1$. The radius of the centroid locus is $\rho = \alpha^2 R/3$, or

$$\rho = \frac{d^2}{3R} = \frac{1}{3}(R - 2r)$$

7.2. A proof of the great Poncelet theorem

The usual difficulty in a proof of the Poncelet Theorem is the proof of the closedness of the zig-zag line. In the known proofs, the circles/ellipses are given. Here, on the contrary, the closedness of the zig-zag line with any starting point on **C** is obvious. The difficulty is shifted to the determination of a UB-matrix T such that its numerical range (or the curve $\mathbf{K}\{\lambda(\varphi)\}$ for a continuous root $\lambda(\varphi)$ of Eq. (7)) is the given circle or ellipse. This is a kind of "uniqueness" theorem: it is easy to construct a Poncelet curve which has common tangent points with the given circle inscribed in the given polygon. The main point is to prove that this Poncelet curve coincides with the circle.

For the proof, the following simple statement is required.

Lemma 4a. *For any circle* $\mathbf{C}_1$ *inscribed in an* $(n+1)$*-sided polygon* **P** *with vertices* Z_j*, the tangent points* z_j $(j = 1, \ldots, n+1,\ z_j \in (Z_j, Z_{j+1}),\ Z_{n+2} = Z_1)$ *are such that*

$$\prod_{j=1}^{n+1} |Z_{j+1} - z_j| = \prod_{j=1}^{n+1} |z_j - Z_j|.$$

If vertices Z_j are on the unit circle, then Eq. (14) *holds with p_j defined by Eq.* (12).

The proof yields from the obvious equation $|z_{j+1} - Z_{j+1}| = |Z_{j+1} - z_j|$ $(j = 1, \ldots, n+1; z_{n+2} = z_1)$.

Theorem 10a (The great Poncelet theorem for circles). *Let an $(n+1)$-sided polygon $\tilde{\mathbf{P}}$ be inscribed in the unit circle $\mathbf{C}$ and circumscribed around a circle $\tilde{\mathbf{C}}$. Then $\tilde{\mathbf{C}}$ is a Poncelet curve of rank $(n+1)$ w.r.t $\mathbf{C}$.*

Proof. Without loss of generality, we may assume that the center α of $\tilde{\mathbf{C}}$ is real and positive. Then, $\tilde{\mathbf{C}} = \mathbf{K}\{\alpha \cos \varphi + r\}$, where r is the radius of $\tilde{\mathbf{C}}$. Let $e^{i\psi_j}$ $(j = 1, \ldots, n+1)$ be the sequential vertices of $\tilde{\mathbf{P}}$ $(\psi_j < \psi_{j+1} < \psi_j + \pi)$, and z_j, $j = 1, \ldots, n+1$, – the common tangent points of $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{P}}$. These points define the numbers p_j by Eq. (12). Due to Lemma 4, the numbers p_j satisfy Eq. (14). In accordance with Theorem 9, there exists a UB-matrix T and a continuous root $\lambda(\varphi)$ of Eq. (7) for the ψ_j's and z_j's (or w_j's by Eqs. (12) and (13) such that $\tilde{\mathbf{K}} = \mathbf{K}\{\lambda(\varphi)\}$ is inscribed in $\tilde{\mathbf{P}}$, and z_j's are the tangent points.

In other words, $\tilde{\mathbf{K}}$ has the same common points with $\tilde{\mathbf{P}}$ as the circle $\tilde{\mathbf{C}}$ has, i.e., the root $\lambda(\varphi)$ of Eq. (7) is equal to $\alpha \cos \varphi + r = \cos\left((\psi_{j+1} - \psi_j)/2\right)$ for $\varphi = (\psi_{j+1} + \psi_j)/2$, $j = 1, \ldots, n+1$, $\psi_{n+2} = \psi_1 + 2\pi k$, where $2\pi(k-1) < \psi_{n+1} - \psi_1 < 2\pi k$. To prove the theorem, it should be shown that the circuit $\tilde{\mathbf{K}}$ coincides with the circle $\tilde{\mathbf{C}}$, i.e., that $\lambda(\varphi)$ is equal to $\alpha \cos \varphi + r$ for all φ.

Consider the left-hand side of Eq. (7) with $\lambda = \alpha \cos \varphi + r$:

$$\Sigma(\varphi) = \sum_{j=1}^{n+1} \frac{|w_j|^2}{\cos(\psi_j - \varphi) - (\alpha \cos \varphi + r)}. \tag{16}$$

For $|\psi_j - \varphi| < \pi/2$, $\cos(\psi_j - \varphi) - \alpha \cos \varphi$ obviously is the distance from α to the straight line which crosses $e^{i\psi_j}$ and forms angle φ with the η-axis. Therefore, the $(j+1)$-st denominator of Eq. (16) is equal to zero only for either $\varphi = \varphi_1 + 2\pi m$ or $\varphi = \varphi_2 + 2\pi m$, where m is any integer, and

$$\varphi_1 = (\psi_j + \psi_{j+1})/2, \qquad \varphi_2 = (\psi_{j+1} + \psi_{j+2})/2.$$

For $\varphi \to \varphi_1$, as well as for $\varphi \to \varphi_2$, $\Sigma(\varphi) \to 0$. Hence, $\Sigma(\varphi)$ is bounded in the vicinities of denominators' zeros, and moreover, $\Sigma(\varphi)$ is bounded for all real φ.

On the other hand, it is easy to see that if φ is complex, $\varphi = a + ib$, then the magnitude of any of the denominators of Eq. (16) is $\geqslant (1-\alpha)|\sinh b|$. Hence, $\Sigma(\varphi)$ is bounded in the entire complex plane of φ. By the Cauchy–Liouville Theorem, $\Sigma(\varphi)$ is constant. Namely, because of its behavior in the vicinities of zeros of the denominators, $\Sigma(\varphi) = 0$ for all φ. Thus, $\lambda(\varphi) = \alpha \cos \varphi + r$ satisfies Eq. (7) for all φ, and $\tilde{\mathbf{K}} = \tilde{\mathbf{C}}$.

Remark 5. It follows from the construction of $\mathbf{w}$ that the ratio $\kappa = \sqrt{1+\alpha^2-r^2-2\alpha\cos\psi_j}/|w_j|^2$ does not depend on j. Therefore, for any j, k,

$$|w_j|^4\cos\psi_k - |w_k|^4\cos\psi_j = \frac{1+\alpha^2-r^2}{2\alpha}\left(|w_j|^4-|w_k|^4\right).$$

Obviously, 2κ is the perimeter of the polygon $\tilde{\mathbf{P}}$. The latter equation yields the following proposition.

Proposition 4. *Let $\lambda_1(\varphi)$ and $\lambda_2(\varphi)$ be roots of Eq.* (7) *for a UB-matrix T, $\mathbf{K}\{\lambda_1(\varphi)\}$ and $\mathbf{K}\{\lambda_2(\varphi)\}$ be circles with centers, real α_1, α_2, and radii r_1, r_2, respectively. Then*

$$\frac{1+\alpha_1^2-r_1^2}{\alpha_1} = \frac{1+\alpha_2^2-r_2^2}{\alpha_2}.$$

This equation means that $\mathbf{K}\{\lambda_1(\varphi)\}$, $\mathbf{K}\{\lambda_2(\varphi)\}$, and the unit circle $\mathbf{C}$ are from one pencil [13]: they have a common "imaginary hord" with "imaginary ends" $\xi = (1+\alpha^2-r^2)/(2\alpha)$, $\eta = \pm\mathrm{i}\sqrt{x^2-1}$. In particular, if $\mathbf{K}\{\lambda_1(\varphi)\}$ is concentric with $\mathbf{C}$ (i.e., $\alpha_1 = 0$), then $\mathbf{K}\{\lambda_2(\varphi)\}$ is concentric with $\mathbf{C}$ as well.

Theorem 10b (The great Poncelet theorem for a circle and ellipse). *Let an $(n+1)$-sided polygon $\tilde{\mathbf{P}}$ be inscribed in the unit circle $\mathbf{C}$ and circumscribed around an ellipse $\tilde{\mathbf{C}}$. Then $\tilde{\mathbf{C}}$ is a Poncelet curve of rank $(n+1)$ w.r.t $\mathbf{C}$.*

Proof. Ellipse $\tilde{\mathbf{C}}$ may be taken with axes which are parallel to ξ- and η-axis: $(\xi-\xi_0)^2/a^2 + (\eta-\eta_0)^2/b^2 = 1$. Then, $\tilde{\mathbf{C}} = \tilde{\mathbf{K}}\{\lambda(\varphi)\}$ for $\lambda(\varphi) = \xi_0\cos\varphi + \eta_0\sin\varphi + \sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}$. Let again, $\mathrm{e}^{\mathrm{i}\psi_j}$ $(j = 1,\dots,n+1)$ be the sequential vertices of $\tilde{\mathbf{P}}$ $(\psi_j < \psi_{j+1} < \psi_j + \pi)$, and z_j, $j = 1,\dots,n+1$, - the common tangent points of $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{P}}$. These points define the numbers p_j by Eq. (12). In order to apply Theorem 7 for construction of the required UB-matrix T, we should know that p_j's satisfy Eq. (14). This is proved in the following.

Lemma 4b. *For any ellipse $\mathbf{C}_1$ inscribed in an $(n+1)$-sided polygon $\mathbf{P}$ with vertices Z_j, the tangent points z_j $(j = 1,\dots,n+1,\ z_j \in (Z_j, Z_{j+1}),\ Z_{n+2} = Z_1)$ are such that*

$$\prod_{j=1}^{n+1}|Z_{j+1}-z_j| = \prod_{j=1}^{n+1}|z_j-Z_j|.$$

If vertices Z_j are on the unit circle, then Eq. (14) *holds with p_j defined by Eq.* (12).

Proof of Lemma 4b. It repeats the consideration by King ([9], Fig. 1.9): There exists a linear map $\mathbf{A}$ which transforms ellipse $\mathbf{C}_1$ into a circle $\mathbf{C}_2$. This map preserves the ratio $|Z_{j+1} - z_j|/|z_j - Z_j|$:

$$\frac{|Z_{j+1} - z_j|}{|z_j - Z_j|} = \frac{|\mathbf{A}Z_{j+1} - \mathbf{A}z_j|}{|\mathbf{A}z_j - \mathbf{A}Z_j|}$$

Equation $\prod_{j=1}^{n+1} |\mathbf{A}Z_{j+1} - \mathbf{A}z_j| = \prod_{j=1}^{n+1} |\mathbf{A}z_j - \mathbf{A}Z_j|$ takes place because of Lemma 4a. Hence,

$$\prod_{j=1}^{n+1} \frac{|Z_{j+1} - z_j|}{|z_j - Z_j|} = 1. \qquad \square$$

The rest of the proof of the theorem is similar to the proof of Theorem 10a.

Acknowledgements

I am grateful to Chandler Davis and Sol Khozioski for stimulating discussions on these topics which took place a long time ago. It is my pleasure to thank my friends and colleagues for calling to my attention links with related problems and pointing out to me the corresponding references.

References

[1] M. Berger, Geometry 2, Springer, Berlin, 1987.
[2] P.R. Halmos, Finite-dimensional Vector Spaces, Springer, New York, 1974.
[3] P.R. Halmos, A Glimpse into Hilbert Space, Lectures on Modern Mathematics, vol. 1, Wiley, New York, 1963, pp. 1–22.
[4] K.E. Gustafson, D.K. Rao, Numerical Range, Springer, Berlin, 1996.
[5] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge Univ. Press, New York, 1991.
[6] Von R. Kippenhahn, Über den Wertevorrat einer matrix, Math. Nachr. 6, pp. 193–228, 1951 & 1952.
[7] B. Mirman, About the numerical range of a finite dimensional operator (in Russian), Latv. Mat. Yezhegodnik, Riga, 1966, pp. 187–197.
[8] R. Kolodziej, The rotation number of some transformation related to billiards in an ellipse, Studia Mathematica, T.LXXXL, 1985, pp. 293–302.
[9] J.L. King, Three problems in search of a measure, The American Mathematical Monthly 101 (7) (1994) 609–628.
[10] J.A. Holbrook, Interpenetration of Ellipsoids and the Polynomial Bound of a Matrix, Linear Algebra Appl. 229 (1995) 151–166.
[11] W.F. Donoghue, On the numerical range of a bounded operator, Michigan Mathematical Journal 4 (3) (1957).
[12] M. Berger, Geometry 1, Springer, Berlin, 1987.
[13] I.M. Yaglom, Geometric Transformations III, Mathematical Association of America, 1973.
[14] H.J.M. Bos, C. Kers, F. Oort, D.W. Raven, Poncelet's closure theorem, Exposition. Math. 5 (1987) 289–364.