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Bounds for determinants of matrices associated with classes of arithmetical functions

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Abstract

Let f be an arithmetical function and $S = \{x_1, \ldots, x_n\}$ a set of distinct positive integers. Let $(f(x_i, x_j))$ denote the $n \times n$ matrix having f evaluated at the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j entry and $(f[x_i, x_j])$ denote the $n \times n$ matrix having f evaluated at the least common multiple $[x_i, x_j]$ of x_i and x_j as its i, j entry. In this paper, we show for a certain class of arithmetical functions new bounds for $\det[f(x_i, x_j])$, which improve the results obtained by Bourque and Ligh in 1993. As a corollary, we get new lower bounds for $\det[(x_i, x_j)]$, which improve the results obtained by Rajarama Bhat in 1991. We also show for a certain class of semi-multiplicative function new bounds for $\det(f[x_i, x_j])$, which improve the results obtained by Bourque and Ligh in 1995.

1. Introduction

Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct integers. The $n \times n$ matrix (S) having the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j-entry is called the greatest common divisor (GCD) matrix of S [3–6]. The $n \times n$ matrix (S) having the least common multiple $[x_i, x_j]$ of x_i and x_j as its i, j-entry is called the least common multiple (LCM) matrix on S [2,6,10–12]. A set S is factor-closed if it contains every divisor of x for any $x \in S$. A set S is gcd-closed, if $(x_i, x_j) \in S$ for $1 \leqslant i, j \leqslant n$. Clearly, a factor-closed set is gcd-closed, but not conversely.

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Let f be an arithmetical function and $(f(x_i, x_j))$ denote the $n \times n$ matrix having f evaluated at the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j-entry. Bourque and Ligh in [7,8] showed that for a certain class of arithmetical functions C_s, the matrices $\{(f(x_i, x_j)) \mid f \in C_s\}$ have properties similar to GCD matrices. In fact, they proved [7,8] that if $f \in C_s$, then the $n \times n$ matrix $(f(x_i, x_j))$ is positive definite, and thus

$$\det[f(x_i, x_j)] \leqslant f(x_1) \ldots f(x_n). \tag{1.1}$$

Furthermore,

$$\det[f(x_i, x_j)] \geqslant \prod_{k=1}^{n} (f * \mu)(x_k) \tag{1.2}$$

and the equality in Eq. (1.2) holds if and only if S is factor-closed.

If f is an arithmetical function, let $(f[x_i, x_j])$ denote the $n \times n$ matrix having f evaluated at the least common multiple of x_i and x_j as its i, j-entry. In [9], Bourque and Ligh proved that for a certain class of multiplicative functions the matrix $(f[x_i, x_j])$ has properties which are similar to GCD matrix. In fact, they proved [9] that if f is a multiplicative function and $(1/f) * \mu \in C_s$, then the $n \times n$ matrix $(f[x_i, x_j])$ is positive definite and thus

$$\det(f[x_i, x_j]) \leqslant f(x_1) \cdots f(x_n). \tag{1.3}$$

Furthermore,

$$\det(f[x_i, x_j]) \geqslant \prod_{k=1}^{n} [f(x_k)]^2 \left(\frac{1}{f} * \mu\right)(x_k) \tag{1.4}$$

and the equality in Eq. (1.4) holds if and only if S is factor-closed.

In this paper, we will improve the bounds in Eqs. (1.1)–(1.4) by a new method.

Throughout this paper ϕ and μ will denote Euler's totient function and the Möbius function, respectively, and $f * \mu$ the Dirichlet convolution of f and μ, let f be a real arithmetical function and $S = \{x_1, x_2, \ldots, x_n\}$ be a set of n distinct positive integers.

2. Definitions and notations

Definition 1. An arithmetical function f is said to be multiplicative if $f(mn) = f(m)f(n)$ when m and n are relatively prime.

Definition 2. An arithmetical function f is said to be semi-multiplicative if there exist a non-zero constant c, a positive integer b, and a multiplicative function f' such that for all n, $f(n) = cf'(\frac{n}{b})$ if $b \mid n$, and $f(n) = 0$ if $b \nmid n$.

Definition 3. If f is an arithmetical function, we denote by $1/f$ the arithmetical function defined as follows

$$\frac{1}{f}(m) = \begin{cases} 0 & \text{if } f(m) = 0, \\ \frac{1}{f(m)} & \text{otherwise.} \end{cases}$$

Definition 4. Given any set S of positive integers, define the class of arithmetical function $C_s = \{f \mid (f * \mu)(d) > 0 \text{ whenever } d \mid x \text{ for any } x \in S\}$.

Definition 5. Let T be a set of distinct positive integers. Then the minimal factor-closed set containing T is said to be the factor closure of T, and denoted by $\bar{T}$.

Let $\mathbb{R}$ be the set of real numbers. Then the set of all $m \times n$ matrices over $\mathbb{R}$ is denoted by $M_{m,n}(\mathbb{R})$, and $M_{1,n}(\mathbb{R})$ is abbreviated to $\mathbb{R}^n$. Let $\alpha \in \mathbb{R}^n$. Then the ith component of α is denoted by $\alpha^{(i)}$. Let $\alpha, \beta \in \mathbb{R}^n$. Then the scalar $\alpha \cdot \beta^{\mathrm{T}}$ is said to be the inner product (scalar product), and denoted by $\langle \alpha, \beta \rangle \equiv \alpha \cdot \beta^{\mathrm{T}}$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}^n$. Then the $n \times n$ matrix $G(\alpha_1, \ldots, \alpha_n) = [\langle \alpha_i, \alpha_j \rangle]_{i,j=1}^n$ is said to be the Gram matrix of $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and its determinant, written $\det G(\alpha_1, \ldots, \alpha_n) = g(\alpha_1, \ldots, \alpha_n)$ is said to be the Gramian or Gram determinant of $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

3. Lower bounds for $\det[f(x_i, x_j)]$

Lemma 1. *For any positive integer n, we have $\sum_{d \mid n} (f * \mu)(d) = f(n)$.*

Proof. Let the arithmetical functions I and U be defined for any positive integer m as follows: $I(m) = [1/m], U(m) = 1$, where $[x]$ denotes the greatest integer not greater than x. Since $\mu * U = I$ (see [1]) and $f = f * I$,

$$\begin{aligned} f(n) &= (f * I)(n) = (f * (\mu * U))(n) = ((f * \mu) * U)(n) \\ &= \sum_{d \mid n} (f * \mu)(d) U\left(\frac{n}{d}\right) = \sum_{d \mid n} (f * \mu)(d). \qquad \square \end{aligned}$$

Lemma 2. *Let $\bar{S} = \{y_1, \ldots, y_m\}$ be the factor closure of S. Define $n \times m$ matrix $A = (a_{ij})$ as follows*

$$a_{ij} = \begin{cases} \sqrt{(f * \mu)(y_j)} & \text{if } y_j \mid x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(f(x_i, x_j)) = A \cdot A^{\mathrm{T}}$.

Proof. For $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m$, the (i,j) entry in $A \cdot A^{\mathrm{T}}$ is

$$(A \cdot A^{\mathrm{T}})_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk} = \sum_{\substack{y_k | x_i \\ y_k | x_j}} (\sqrt{(f * \mu)(y_k)})^2 = \sum_{y_k | (x_i, x_j)} (f * \mu)(y_k) = \sum_{d | (x_i, x_j)} (f * \mu)(d).$$

Therefore it follows from Lemma 1 that $(A \cdot A^{\mathrm{T}})_{ij} = f((x_i, x_j))$. The proof is complete. □

Theorem 1. *If $f \in C_s$, then we have*

$$\det[f(x_i, x_j)] \geqslant \prod_{k=1}^{n} \sum_{\substack{d | x_k \\ d \nmid x_t \\ x_t < x_k}} (f * \mu)(d) \tag{3.1}$$

and the equality holds if and only if S is gcd-closed.

Proof. Without loss of generality, let $x_1 < x_2 < \cdots < x_n$. Define $S_k = \{d \mid d \in \mathbb{Z}^+, d \mid x_k, d \nmid x_t, t < k\}, 1 \leqslant k \leqslant n$. Then for any $k_1 \neq k_2, 1 \leqslant k_1, k_2 \leqslant n$, we have $S_{k_1} \cap S_{k_2} = \emptyset$ and $S_1 \cup S_2 \cup \cdots \cup S_n = \bar{S}$, where $\bar{S}$ is the factor closure of S. Let $S_k = \{y_{k,1}, \ldots, y_{k,p_k}\} (1 \leqslant k \leqslant n)$ and $m = p_1 + p_2 + \cdots + p_n$, where $y_{k,1} < \cdots < y_{k,p_k}$. For $1 \leqslant j \leqslant m$, let

$$y_j = \begin{cases} y_{1,j} & \text{if } 1 \leqslant j \leqslant p_1; \\ y_{k,t} & \text{if } j = p_1 + \cdots + p_{k-1} + t \ (k \geqslant 2). \end{cases}$$

Then we have that $\bar{S} = \{y_1, y_2, \ldots, y_m\}$. Let the $n \times m$ matrix $A = (a_{ij})$ be defined as in Lemma 2. By Lemma 2, we have

$$\det[f(x_i, x_j)] = \det(A \cdot A^{\mathrm{T}}). \tag{3.2}$$

Now let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ denote the system of row vectors of A. Then

$$\det(A \cdot A^{\mathrm{T}}) = \det G(\alpha_1, \alpha_2, \ldots, \alpha_n). \tag{3.3}$$

Let $\{\beta_1, \beta_2, \ldots, \beta_n\}$ denote the orthogonalization system obtained from $\{\alpha_1, \ldots, \alpha_n\}$ by using the Gram–Schmidt orthogonalization process (see [14]). Thus

$$\det G(\beta_1, \ldots, \beta_n) = \det(\langle \beta_i, \beta_j \rangle) = \prod_{k=1}^{n} \langle \beta_k, \beta_k \rangle. \tag{3.4}$$

It is well known that

$$\det G(\alpha_1, \ldots, \alpha_n) = \det G(\beta_1, \ldots, \beta_n). \tag{3.5}$$

It follows from Eqs. (3.2)–(3.5) that

$$\det[f(x_i, x_j)] = \prod_{k=1}^{n} \langle \beta_k, \beta_k \rangle. \tag{3.6}$$

Since $x_i < x_2 < \cdots < x_n$, then from the definition of the matrix A, it is not difficult to see that

$$(\alpha_1)^{(i)} = \begin{cases} \sqrt{(f * \mu)(y_{1,i})} & \text{if } 1 \leqslant i \leqslant p_1, \\ 0 & \text{if } p_1 < i \leqslant m \end{cases}$$

and for $k \geqslant 2, i > p_1 + \cdots + p_{k-1}$, we have

$$(\alpha_k)^{(i)} = \begin{cases} \sqrt{(f * \mu)(y_{k,t})} & \text{if } i = p_1 + \cdots + p_{k-1} + t \ (1 \leqslant t \leqslant p_k), \\ 0 & \text{if } p_1 + \cdots + p_{k-1} + p_k < i \leqslant m. \end{cases}$$

Therefore for $i = p_i + \cdots + p_{k-1} + t \ (1 \leqslant t \leqslant p_k)$, $k \geqslant 2$, we have $(\beta_k)^{(i)} = \sqrt{(f * \mu)(y_{k,t})}$. Note that $\beta_1 = \alpha_1$. Hence for any $k, 1 \leqslant k \leqslant n$, we have

$$\langle \beta_k, \beta_k \rangle \geqslant \sum_{t=1}^{p_k} \left(\sqrt{(f * \mu)(y_{k,t})} \right)^2 = \sum_{d \in S_k} (f * \mu)(d). \tag{3.7}$$

Then it follows from Eq. (3.6) and Eq. (3.7) that Eq. (3.1) holds.

Sublemma. *With the above notations, if S is gcd-closed, then*

$$\beta_1 = \left(\sqrt{(f * \mu)(y_{1,1})}, \ldots, \sqrt{(f * \mu)(y_{1,p_1})}, 0, \ldots, 0 \right),$$

and for $k \geqslant 2$, we have

$$\beta_k = \left(\underbrace{0, \ldots, 0}_{p_1 + \cdots + p_{k-1}}, \sqrt{(f * \mu)(y_{k,1})}, \ldots, \sqrt{(f * \mu)(y_{k,p_k})}, 0, \ldots, 0 \right).$$

Proof of the Sublemma. Obviously the sublemma is true for β_1 (since $\beta_1 = \alpha_1$). Since S is gcd-closed, $(x_2, x_1) = x_1$. Thus

$$\alpha_2 = \left(\sqrt{(f * \mu)(y_{1,1})}, \ldots, \sqrt{(f * \mu)(y_1, p_1)}, \sqrt{(f * \mu)(y_{2,1})}, \ldots, \right.$$
$$\left. \sqrt{(f * \mu)(y_{2,p_2})}, 0, \ldots, 0 \right).$$

Therefore $\langle \alpha_2, \beta_1 \rangle = \langle \beta_1, \beta_1 \rangle$. Then

$$\beta_2 = \alpha_2 - \frac{\langle \alpha_2, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}\beta_1 = \alpha_2 - \beta_1$$
$$= \left(\underbrace{0, \ldots, 0}_{p_1}, \sqrt{(f * \mu)(y_{2,1})}, \ldots, \sqrt{(f * \mu)(y_{2,p_2})}, 0, \ldots, 0 \right).$$

So the sublemma is true for β_2. Assume that the sublemma is true for β_l, $1 \leqslant l \leqslant k-1$. Now consider β_k. Obviously, for $1 \leqslant i \leqslant p_1$, we have

$$\left(\alpha_k - \frac{\langle \alpha_k, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}\beta_1 \right)^{(i)} = 0.$$

We claim that for each $e, 2 \leqslant e \leqslant k-1$, we have that

$$\left(\alpha_k - \frac{\langle \alpha_k, \beta_e \rangle}{\langle \beta_e, \beta_e \rangle}\beta_e \right)^{(i)} = 0 \quad \text{for } p_1 + \cdots + p_{e-1} < i < p_1 + \cdots + p_e.$$

Now let $2 \leqslant e \leqslant k-1$. If $(x_k, x_e) = x_e$. Then $x_e | x_k$. So $y_{e,i} | x_k$ for each $1 \leqslant i \leqslant p_e$. Thus $\langle \alpha_k, \beta_e \rangle = \langle \beta_e, \beta_e \rangle$. Hence

$$\left(\alpha_k - \frac{\langle \alpha_k, \beta_e \rangle}{\langle \beta_e, \beta_e \rangle}\beta_e \right)^{(i)} = (\alpha_k - \beta_e)^{(i)} = 0$$
$$\text{for } p_1 + \cdots + p_{e-1} < i < p_1 + \cdots + p_e.$$

If $(x_k, x_e) = x_\gamma$, $\gamma < e$. Then $y_{e,i} \nmid x_k$ for all $1 \leqslant i \leqslant p_e$. Otherwise, there exists $i, 1 \leqslant i \leqslant p_e$, such that $y_{e,i} \mid x_k$. So $y_{e,i} \mid x_k$. We can deduce that $e = \gamma$. It is a contradiction. Therefore $(\alpha_k)^{(i)} = 0$ for $p_1 + \cdots + p_{e-1} < i < p_1 + \cdots + p_e$ and thus $\langle \alpha_k, \beta_e \rangle = 0$. Then

$$\left(\alpha_k - \frac{\langle \alpha_k, \beta_e \rangle}{\langle \beta_e, \beta_e \rangle}\beta_e \right)^{(i)} = (\alpha_k)^{(i)} = 0 \quad \text{for } p_1 + \cdots + p_{e-1} < i < p_1 + \cdots + p_e.$$

The proof of the claim is complete. Therefore it follows from the inductive hypotheses and the claim that

$$\beta_k = \alpha_k - \frac{\langle \alpha_k, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}\beta_1 - \cdots - \frac{\langle \alpha_k, \beta_{k-1} \rangle}{\langle \beta_{k-1}, \beta_{k-1} \rangle}\beta_{k-1}$$
$$= \left(\underbrace{0, \ldots, 0}_{p_1 + \cdots + p_k}, \sqrt{(f * \mu)(y_{k,1})}, \ldots, \sqrt{(f * \mu)(y_{k,p_k})}, 0, \ldots, 0 \right).$$

This completes the proof of the sublemma. $\square$

Now we continue to complete the proof of Theorem 1. Clearly, it follows from the sublemma that if S is gcd-closed, then the equality in (Eq. (3.1)) holds.

Conversely, if S is not gcd-closed, then for some $1 \leqslant i, j \leqslant n$, $(x_i, x_j) \notin S$. Let $a = \min\{i \in \mathbb{Z}^+ \mid (x_i, x_j) \notin S, 1 \leqslant j < i \leqslant n\}$ and $b = \min\{j \in \mathbb{Z}^+ \mid (x_a, x_j) \notin S, 1 \leqslant j < a\}$. Then $a \geqslant 2$ and $\{x_1, \ldots, x_{a-1}\}$ is gcd-closed. Otherwise, there exist $1 \leqslant j < i \leqslant a-1$ such that $(x_i, x_j) \notin S$. This contradicts to the minimality of a. In the same way as in the sublemma, we have

$$\beta_1 = \left(\sqrt{(f*\mu)(y_{1,1})}, \ldots, \sqrt{(f*\mu)(y_{1,p_1})}, 0, \ldots, 0\right),$$

$$\beta_2 = \left(\underbrace{0, \ldots, 0}_{p_1}, \sqrt{(f*\mu)(y_{2,1})}, \ldots, \sqrt{(f*\mu)(y_{2,p_2})}, 0, \ldots, 0\right),$$

$$\cdots\cdots\cdots$$

$$\beta_{a-1} = \left(\underbrace{0, \ldots, 0}_{p_1+\cdots+p_{k-2}}, \sqrt{(f*\mu)(y_{\alpha-1,1})}, \ldots, \sqrt{(f*\mu)(y_{\alpha-1,p_{\alpha-1}})}, 0, \ldots, 0\right).$$

Since $(x_a, x_b) \notin S$, then there exist d and c, where $1 \leqslant d \leqslant n$, $1 \leqslant c < p_d$, such that $(x_a, x_b) = y_{d,c}$. Thus $y_{d,c} < x_c$. Clearly $d \leqslant b$. If $d < b$. Then $(x_a, x_d) \in S$. Let $(x_a, x_d) = x_l$, $l \leqslant d$. Note that $y_{d,c} | (x_a, x_d)$. Thus $y_{d,c} | x_l$. It can be deduced that $d = l$. Namely $(x_a, x_d) = x_d$. So $x_d | x_a$. Similarly we have $x_d | x_b$. Thus $x_d | (x_a, x_b)$. So $x_d | y_{d,c}$. Then $y_{d,c} = x_d$ and thus $c = p_d$. It is a contradiction. Therefore $d = b$. Thus $(x_a, x_b) = y_{b,c}$ for some $1 \leqslant c < p_b$. So $(\alpha_a)^{(p_1+\cdots+p_{b-1}+c)} = \sqrt{(f*\mu)(y_{b,c})}$. On the other hand, since $x_b \nmid x_a$, we have $(\alpha_a)^{(p_1+\cdots+p_{b-1}+p_b)} = 0$. Thus $0 < \langle \alpha_a, \beta_b \rangle < \langle \beta_b, \beta_b \rangle$. Hence for $i = p_1 + \cdots + p_{b-1} + c$ we have

$$(\beta_a)^{(i)} = \left(\alpha_a - \frac{\langle \alpha_a, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}\beta_1 - \frac{\langle \alpha_a, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle}\beta_2 - \cdots - \frac{\langle \alpha_a, \beta_{a-1} \rangle}{\langle \beta_{a-1}, \beta_{a-1} \rangle}\beta_{a-1}\right)^{(i)}$$

$$= \left(\alpha_a - \frac{\langle \alpha_a, \beta_b \rangle}{\langle \beta_b, \beta_b \rangle}\beta_b\right)^{(i)} = \left(1 - \frac{\langle \alpha_a, \beta_b \rangle}{\langle \beta_b, \beta_b \rangle}\right)\sqrt{(f*\mu)(y_{b,c})}.$$

Therefore

$$\langle \beta_a, \beta_a \rangle \geqslant \left(1 - \frac{\langle \alpha_a, \beta_b \rangle}{\langle \beta_b, \beta_b \rangle}\right)^2 (f*\mu)(y_{b,c}) + \sum_{k=1}^{p_a} (f*\mu)(y_{a,k})$$

$$> \sum_{k=1}^{p_a} (f*\mu)(y_{a,k}) = \sum_{d \in S_a} (f*\mu)(d). \tag{3.8}$$

It follows from (Eqs. (3.6)–(3.8)) that

$$\det[f(x_i, x_j)] > \prod_{k=1}^{n} \sum_{d \in S_k} (f*\mu)(d).$$

Therefore, the equality in (Eq. (3.1)) holds implies that S is gcd-closed. The proof of the Theorem 1 is complete.

Remark. (Eq. (3.1)) improves (Eq. (1.2)).

Corollary 1. *We have that*

$$\det[(x_i, x_j)] \geqslant \prod_{k=1}^{n} \sum_{\substack{d \mid x_k \\ d \nmid x_t \\ x_t < x_k}} \varphi(d) \tag{3.9}$$

and the equality holds if and only if S is gcd-closed.

Proof. Let $f = N$, where $N(n) = n$ for any $n \in \mathbb{Z}^+$. Note that $N * \mu = \varphi$ and $\varphi(d) > 0$ for any $d \in \mathbb{Z}^+$. Then the result follows from Theorem 1. □

Remark. Eq. (3.9) improves Theorem 10 in [16].

4. Lower bounds for $\det(f[x_i, x_j])$

Lemma 3 ([17]). *The arithmetical function f is a semi-multiplicative function if and only if for any positive integers m and n,*

$$f(m)f(n) = f((m,n))f([m,n]).$$

Obviously, if f is multiplicative, then f is semi-multiplicative.

Theorem 2. *Let f be a semi-multiplicative function. If $1/f \in C_s$ then we have*

$$\det(f[x_i, x_j]) \geqslant \prod_{k=1}^{n} f[x_k]^2 \sum_{\substack{d \mid x_k \\ d \nmid x_t \\ x_t < x_k}} \left(\frac{1}{f} * \mu\right)(d), \tag{4.1}$$

and the equality holds if and only if S is gcd-closed.

Proof. Let $g = 1/f$ and the matrix $D = \mathrm{diag}(f(x_1), \ldots, f(x_n))$. Since f is semi-multiplicative, it follows form Lemma 3 that $(f[x_i, x_j]) = D(g(x_i, x_j))D$. Therefore

$$\det(f[x_i, x_j]) = \prod_{k=1}^{n} [f(x_k)]^2 \ \det[g(x_i, x_j)]. \tag{4.2}$$

Note that $g \in C_s$. Thus, from Theorem 1 applied to the matrix $(g(x_i, x_j))$. We have that

$$\det[g(x_i,x_j)] \geqslant \prod_{k=1}^{n} \sum_{\substack{d|x_k \\ d\nmid x_t \\ x_t<x_k}} (g*\mu)(d), \tag{4.3}$$

and the equality holds if and only if S is gcd-closed. Then it follows from Eqs. (4.2) and (4.3) that (Eq. (4.1)) holds. Obviously, the equality in (Eq. (4.1)) holds if and only if the equality in (Eq. (4.3)) holds. Therefore the equality in (Eq. (4.1)) holds if and only if S is gcd-closed. The proof is complete. □

Lemma 4. *Let f be an arithmetical function. If $(1/f)*\mu \in C_s$, then $(1/f) \in C_s$.*

Proof. Suppose that $(1/f)*\mu \in C_s$. Let $x \in S$ and $d|x$. It follows from Lemma 1 that

$$((1/f)*\mu)(d) = \sum_{d_1|d} \left(\frac{1}{f}*\mu*\mu\right)(d_1). \tag{4.4}$$

Since $d_1|d, d|x$, namely $d_1|x$. Thus the condition $(1/f)*\mu \in C_s$, implies that $((1/f)*\mu*\mu)(d_1) > 0$. Therefore it follows from Eq. (4.4) that $((1/f)*\mu)(d) > 0$. So $1/f \in C_s$. The proof is complete. □

Corollary 2. *Let f be a semi-multiplicative function. If $(1/f)*\mu \in C_s$. Then the inequality* (4.1) *holds, and the equlity in Eq.* (4.1) *holds if and only if S is gcd-closed.*

Proof. The result follows from Theorem 2 and Lemma 4. □

Since a multiplicative function is a semi-multiplicative function, then we have the following.

Corollary 3. *Let f be a multiplicative function. If $1/f \in C_s$, then the inequality* (4.1) *holds, and the equality in Eq.* (4.1) *holds if and only if S is gcd-closed.*

Remark. Under the conditions of Corollary 3, Bourque and Ligh in [9] proved that the matrix $(f[x_i,x_j])$ is positive definite. But if the condition $1/f \in C_s$ does not hold, then the matrix $(f[x_i,x_j])$ is not necessarily postive definite, it may not even be non-singular. In fact, we proved in [12] that for positive integer $n \geqslant 8$, there exists a gcd-closed set S of n distinct positve integers, such that the LCM matrix $([x_i,x_j])$ defined on S is singular.

Corollary 4. *Let f be a multiplicative function. If $(1/f)*\mu \in C_s$, then the inequality* (4.1) *holds, and the equlaity in Eq.* (4.1) *holds if and only if S is gcd-closed.*

Proof. The result follows from Lemma 4 and Collorary 3. □

Remark. The inequality in Corollary 4 improves Eq. (1.4).

5. Upper bounds for $\det[f(x_i, x_j)]$ and $\det(f[x_i, x_j])$

Li in [15] proved for the GCD matrix $[(x_i, x_j)]$ the inequality

$$\det[(x_i, x_j)] \leqslant x_1 x_2 \cdots x_n - \frac{n!}{2}.$$

Here we show for the matrix $[f(x_i, x_j)]$ a weaker inequality.

Lemma 5 (Fischer's inequality [13]). *Suppose that*

$$P = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

is a positive definite matrix that is partitioned so that A and C are square and non-empty, where $B^* = \bar{B}^{\mathrm{T}}$, $\bar{B}$ and $\bar{B}^{\mathrm{T}}$ are the component-wise conjugate of B and the transpose of $\bar{B}$ respectively. Then $\det P \leqslant (\det A)(\det C)$.

In the following, let $a = \min\{x \mid x \in S\}$ and $b = \min\{x \mid x \in S \backslash \{a\}\}$.

Theorem 3. *If $f \in C_s$ and $f(a) \leqslant f(b)$. Then* $(n \geqslant 2)$

$$\det[f(x_i, x_j)] \leqslant \left(1 - \frac{f(a)}{f(b)}\right) \cdot \prod_{k=1}^{n} f(x_k). \tag{5.1}$$

Proof. Since $f \in C_s$, then for any $x \in S$, $f(x) = \sum_{d \mid x} (f * \mu)(d) > 0$. We proceed by induction on n. Without loss of generality, let $a = x_1 < b = x_2 < x_3 < \cdots < x_n$.

For the case $n = 2$, we have that

$$\det[f(x_i, x_j)] = f(x_1) f(x_2) - f(x_1)^2 = f(x_1) f(x_2) \left(1 - \frac{f(x_1)}{f(x_2)}\right).$$

Thus Eq. (5.1) holds.

Suppose Eq. (5.1) holds for the case $n - 1$ $(n \geqslant 3)$. Now we consider the case n.

Let $S' = \{x_1, x_2, \ldots, x_{n-1}\}$ and $(S') = (f(x_i, x_j))_{i,j=1}^{n-1}$. Then

$$(f(x_i, x_j)) = \begin{pmatrix} (S') & N^{\mathrm{T}} \\ N & f(x_n) \end{pmatrix},$$

where $N = (f((x_n, x_1)), \dots, f((x_n, x_{n-1})))$. Since $(f(x_i, x_j))$ is positive definite, it follows from Lemma 5 that

$$\det[f(x_i, x_j)] \leqslant \det[(S')]f(x_n). \tag{5.2}$$

From the induction hypothesis and Eq. (5.2), it then follows that

$$\begin{aligned}\det[f(x_i, x_j)] &\leqslant \det[(S')]f(x_n) \leqslant \left(1 - \frac{f(x_1)}{f(x_2)}\right)\prod_{k=1}^{n-1} f(x_k)f(x_n)\\ &= \left(1 - \frac{f(a)}{f(b)}\right)\prod_{k=1}^{n} f(x_k).\end{aligned}$$

This completes the proof. □

Remark. Eq. (5.1) improves Eq. (1.1).

Theorem 4. *If f is semi-multiplicative and $(1/f) \in C_s$ and $f(a) \geqslant f(b)$, then*

$$\det(f[x_i, x_j]) \leqslant \left(1 - \frac{f(b)}{f(a)}\right)\prod_{k=1}^{n} f(x_k). \tag{5.3}$$

Proof. Let $g = 1/f$. As in the proof of Theorem 2, we can deduce that

$$\det(f[x_i, x_j]) = \prod_{k=1}^{n} [f(x_k)]^2 \det(g(x_i, x_j)). \tag{5.4}$$

Since $g \in C_s$ and $g(a) \leqslant g(b)$, it follows from Theorem 3 that

$$\det[g(x_i, x_j)] \leqslant \left(1 - \frac{g(a)}{g(b)}\right)\prod_{k=1}^{n} g(x_k). \tag{5.5}$$

Therefore it follows from Eq. (5.4) and Eq. (5.5) that Eq. (5.3) holds. This completes the proof. □

Corollary 5. *If f is a multiplicative function and $(1/f) * \mu \in C_s$ and $f(a) \geqslant f(b)$, then we have*

$$\det(f[x_i, x_j]) \leqslant \left(1 - \frac{f(b)}{f(a)}\right)\prod_{k=1}^{n} f(x_k). \tag{5.6}$$

Proof. The result follows from Lemma 4 and Theorem 4. □

Remark. Eq. (5.6) improves Eq. (1.3).

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