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LINEAR ALGEBRA AND ITS APPLICATIONS

# How symmetric can a function be?

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**Abstract**

The symmetric complexity of a polynomial $f$ in $n$ variables is defined as the number of times the fundamental theorem on symmetric functions is applicable. In this paper a sharp upper bound on this measure is derived by a matrix method. © 1998 Elsevier Science Inc. All rights reserved.

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## 1. Introduction

Consider a field $K$ of characteristic 0, and let $R$ be the ring $K[x_1, \ldots, x_n]$ where $n$ is $> 0$.

A *symmetric function* is any element of $R$ invariant under the symmetric group acting as coordinate permutations. Examples are the *elementary symmetric functions*: $a_0 = 1$, $a_i = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_i \leqslant n} x_{j_1} x_{j_2} \ldots x_{j_i}$ $(1 \leqslant i \leqslant n)$; $a_i = 0$ $(i < 0$ or $i > n)$.

The *Fundamental Theorem on Symmetric Functions* [1,2] states that any symmetric function $f$ can be uniquely written as $g(a_1, \ldots, a_n)$ for some $g = g(x_1, \ldots, x_n)$ from $R$, called the *symmetric representation* of $f$.

The symmetric functions are beautiful objects with a large algebraic-combinatorial theory [1]. By the above theorem they are the result of the substitution

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$\sigma : g(x_1, \ldots, x_n) \to g(a_1, \ldots, a_n)$ that maps $R$ isomorphically onto a smaller subring. Inspired by this fact and by the recent interest in fractals and dynamical systems ([3]), one might be curious about those polynomials that are obtained by iterating $\sigma$.

One question to ask in this context is of course, how to find fixpoints (if any). Here however, we shall restrict ourselves to a small numerical example in Section 5 because we wish to look at the matter from the perspective of complexity theory.

**Definition 1.** A polynomial $f$ in $n$ variables is 0-fold symmetric if $f$ is not symmetric; and $k$-fold symmetric with $k > 0$ if $f$ is symmetric and the symmetric representation of $f$ is $(k-1)$-fold symmetric. The number $k$ is called the *symmetric complexity* of $f$.

A $k$-fold symmetric function $f$ possesses a high degree of symmetry indeed, and it seems an interesting complexity problem to find a bound on the natural measure $k$, expressed in the coefficients and exponents of $f$. Such a result will be given in the corollary to Theorem 1.

Equivalently, from an algorithmic point of view, any implementation of the fundamental theorem on symmetric functions can be seen as an abstract machine of which one might ask if and when it will halt on repeated action from a given input. In fact, the problem of this article arose from a study of efficient computer algebra implementations of this theorem.

Our method is based on term orderings and the like, familiar from Groebner basis theory [4]. Thus it is possible to translate the problem into linear algebra, involving the explicit calculation of the spectrum and eigenvectors of a matrix.

As an addendum to this paper, some Maple code can be found on the WWW at http://www.cs.kun.nl/bolke/ksymmaple.

## 2. Notations and generalities

Set $\mathbf{x} = (x_1, \ldots, x_n)$. Let $a_i = a_i(\mathbf{x})$ be defined as above, and $\mathbf{a} = (a_1, \ldots, a_n)$. Stretching notation a bit, we can view $\{c_1, \ldots, c_n\} \to \mathbf{a}(\mathbf{c})$ as a mapping from the unordered lists of length $n$ over $K$ to $K^n$, which is a bijection if $K$ is algebraically closed. Indeed, one has $\sum_{i=0}^{n} a_i(c_1, \ldots, c_n)T^i = \prod_{i=0}^{n}(c_iT + 1)$. Instead of this however we shall consider the simpler mappings $\mathbf{c} \to a(\mathbf{c})$ from $K^n$ to $K^n$ and $a : \mathbf{x} \to a(\mathbf{x})$ from $R$ to $R$.

**Definition 2.** Let $\mathbf{a}^0 = (x_1, \ldots, x_n)$ and for $k > 0$ define $\mathbf{a}^k = (a_1^k, a_2^k, \ldots, a_n^k)$, where $a_i^k = a_i^k(\mathbf{x}) = a_i(a_1^{k-1}, a_2^{k-1}, \ldots, a_n^{k-1})$, $1 \leqslant i \leqslant n$.

The $a_i^k$ are called the *iterated elementary symmetric functions* (iesf's.). An interesting fact is given by

**Lemma 1.** *For all $k \geqslant 1$, the iesf's $a_1^k, a_2^k, \ldots, a_n^k$ are algebraically independent.*

**Proof.** Induction w.r.t. $k$. For $k = 1$ this is well-known [2]. Now let $f(y_1, \ldots, y_n)$ be such that $f(a_1^{k+1}, a_2^{k+1}, \ldots, a_n^{k+1}) = 0$ in $R$.

By definition of the $a_i^k$'s, there exists a symmetric polynomial $g(z_1, \ldots, z_n) = f(a_1(\mathbf{z}), a_2(\mathbf{z}), \ldots, a_n(\mathbf{z}))$ with $g(a_1^k, a_2^k, \ldots, a_n^k) = f(a_1^{k+1}, a_2^{k+1}, \ldots, a_n^{k+1}) = 0$; hence $g(z_1, \ldots, z_n) = 0$ by the induction hypothesis. But now we are in the case $k = 1$ again, since $g(z_1, \ldots, z_n) = f(a_1(\mathbf{z}), a_2(\mathbf{z}), \ldots, a_n(\mathbf{z}))$ and it follows that $f(y_1, \ldots, y_n) = 0$. $\square$

A *term* is any monomial $t = x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$. Its *total degree* is $\operatorname{tdeg}(t) = \sum_{j=1}^n i_j$ and the total degree $\operatorname{tdeg}(f)$ of $f \in R$ is $\max_{t \in f} \operatorname{tdeg}(t)$ (which of course is equal to $\operatorname{tdeg}(t)$, any $t$ in $f$ if $f$ is symmetric.)

An *admissible ordering* [4] on the set $T$ of terms in $R$ is a total order on $T$ that satisfies:

$$1 < t \quad \text{and} \quad t < t' \;\Rightarrow\; st < st' \quad \text{for all terms } s, t, t'.$$

The latter property is called *monotonicity* of term multiplication.

An admissible ordering is a well-ordering. Admissible orders abound and have been classified; well-known examples are the lexicographic orders and various total degree orderings like the "grevlex" [4].

For a given ordering, the *leading term* $\operatorname{lt}(f)$ of $f$ is the highest term occurring in $f$.

## 3. Main results

As an admissible ordering on $T$, let us take the lexicographic order with $x_1 > x_2 > \cdots > x_n$.

**Definition 3.** The exponents (column) vector $\operatorname{ev}(t)$ of a term $t = x_1^{i_1} \ldots x_n^{i_n}$ is $\mathbf{i} = (i_1, i_2, \ldots, i_n)^{\mathrm{T}}$.

**Definition 4.** The term $\operatorname{lt}(a_i^k)$ $(i \geqslant 1)$ will be denoted by $t_i^k$, and $\operatorname{ev}(t_i^k)$ by $\mathbf{e}_i^k = (e_{i,1}^k, \ldots, e_{i,n}^k)^{\mathrm{T}}$.

Our first result is the following.

**Theorem 1.** *Let $U$ be the upper triangular all-one matrix and $D$ the (symmetric) lower antitriangular all-one matrix. Then for any $k$-fold symmetric function $f$ and $k \geqslant 1$,*

$$\operatorname{ev}(\operatorname{lt}(f)) \in UD^{k-1}((\mathbb{N} \cup \{0\})^n).$$

A reasonable way to get a grip on the entries of $D^k$ is via the eigenvalues of $D$. The eigenvalue computation will be done in Section 4.1 and the entries of $D^k$ will be estimated in Section 4.2. As a consequence, we have

**Corollary 1.** *Let $f$ be any nonconstant k-fold symmetric polynomial in $n \geqslant 2$ variables. Then the symmetric complexity $k$ is bounded by*

$$\mathrm{tdeg}(f) \geqslant \frac{(2n+1)^{k-1}}{\pi^{k-1}}\{1.149 - 1.048(0.53)^{k-1}\}.$$

**Remark 1.** This bound is fairly precise: it is an approximation of a more complex bound, which is sharp in the sense that it is reached by $f = a_1^k$. This will follow from the proof.

Let us give an outline first. The idea is very simple and consists of three steps.

(i) If $k$ increases, one observes that the iesf's $a_i^k$ grow very quickly in "size" (explicit calculation of the complete $a_i^k$'s in Maple, say, leads to considerable memory problems). To measure this size, we consider the highest terms $t_i^k$ of $a_i^k$ in the chosen admissible ordering, and derive a recursion for the exponents occurring in this terms.

(ii) Next, for a given $f$ of symmetric complexity $k$ we shall show that some term $t_i^k$ actually occurs in $f$ as $\mathrm{lt}(f)$. This proves Theorem 1 and shows that $k$ is bounded as a function of $\mathrm{lt}(f)$.

(iii) Finally, we shall be able to estimate the exponents occurring in $\mathrm{lt}(f)$; this is the technical part of Section 4.

### *3.1. Proof of Theorem 1*

We shall start with a recursion for $t_i^k$.

**Lemma 2.** (a) $t_i^k = t_n^{k-1} t_{n-1}^{k-1} \dots t_{n-i+1}^{k-1}$ $(k > 1)$. (b) *If* $p > q$, $t_p^k > t_q^k$ $(k \geqslant 1)$.

**Proof.** For $k = 1$, statement (b) holds. Indeed, $t_i^1 = \mathrm{lt}(a_i) = x_1 x_2 \dots x_i$. Also, (a) holds trivially. Now if for any $k$ (a) and (b) are true, then by definition one has $a_i^{k+1} = \sum_{1 \leqslant j_1 < j_2 < \dots < j_i \leqslant n} a_{j_1}^k a_{j_2}^k \dots a_{j_i}^k$. All coefficients are positive, so no terms cancel. By the monotonicity property, $\mathrm{lt}(a_{j_1}^k a_{j_2}^k \dots a_{j_i}^k) = \mathrm{lt}(a_{j_1}^k)\mathrm{lt}(a_{j_2}^k)\dots\mathrm{lt}(a_{j_i}^k) = t_{j_1}^k t_{j_2}^k \dots t_{j_i}^k$. Since (b) holds and, again, by monotonicity, this is maximal if $j_i = n$, $j_{i-1} = n-1, \dots, j_1 = n-i+1$. This proves (a) for index $k+1$. But then, if $p > q$ one has $t_p^{k+1} > t_q^{k+1}$ since the r.h.s. divides the l.h.s. Hence (b) holds as well. $\square$

Now consider $\mathrm{ev}(t_i^k) = \mathbf{e}_i^k = (e_{i,1}^k, \ldots, e_{i,n}^k)^\mathrm{T}$ (see Definition 4). Define $E_k$ to be the matrix having the $\mathbf{e}_i^k$'s as its columns. One has $\mathbf{e}_i^1 = (1, 1, \ldots, 1, 1, 0, \ldots, 0)^\mathrm{T}$ ($i$ ones), so $E_1 = U$.

**Lemma 3.** (a) *Let $t = x_1^{i_1} \ldots x_n^{i_n}$ be any term; then for all $k \geqslant 1$ the exponents vector of* $\mathrm{lt}(t(a_1^k, \ldots, a_n^k))$ *equals* $E_k\mathbf{i}$.

(b) *Let $D$ be the symmetric matrix with ones below and on the antidiagonal and zeroes above. Let $U$ be the upper triangular all-one matrix. Then $E_k = UD^{k-1}$. Hence $E_k$ is nonsingular and for $k \geqslant 1$, $1 \leqslant a \leqslant n$ one has* $\mathbf{e}_{i,a}^k = \sum_{j=i}^n (D^{k-1})_{j,a}$.

**Proof.** By monotonicity, $\mathrm{lt}((a_1^k)^{i_1} \ldots (a_n^k)^{i_n}) = (t_1^k)^{i_1} \ldots (t_n^k)^{i_n}$, the exponents vector of which is $E_k\mathbf{i}$ by linearity. This proves part (a).

For part (b), note that statement (a) of Lemma 2 can be written as: $\mathbf{e}_i^k = \mathbf{e}_n^{k-1} + \mathbf{e}_{n-1}^{k-1} + \cdots + \mathbf{e}_{n-i+1}^{k-1}$, which is equivalent to $E_k = E_{k-1}D$. So $E_k = E_1 D^{k-1} = UD^{k-1}$. $\square$

This is step i of the outline. In Section 4.1 we shall find an explicit solution to this recursion.

Now consider step ii. Suppose that $f$ is not constant and $k$-fold symmetric, $k \geqslant 1$. We wish to prove that some $t_i^k$ actually occurs in $f$.

By definition, there exists $f_k \in R$ such that $f_k(a_1^k, \ldots, a_n^k) = f$ (though we shall not need it, note that $f_k$ is unique by Lemma 1). Let $t = x_1^{i_1} \ldots x_n^{i_n}$ be a term of the polynomial $f_k(\mathbf{x})$ such that $\tau =_{\mathrm{Def}} \mathrm{lt}((a_1^k)^{i_1}, \ldots, (a_n^k)^{i_n})$ is *maximal* in the term ordering. By Lemma 3, $\mathrm{ev}(\tau) = E_k(i_1, \ldots, i_n)^\mathrm{T} = E_k\mathbf{i}$.

First note that $\tau$ is unique. Indeed, suppose that besides $t$ there is another term $s = x_1^{j_1} \ldots x_n^{j_n}$ yielding the same $\tau$, then by Lemma 3 one would have $E_k\mathbf{i} = E_k\mathbf{j}$ (with $\mathbf{j} = (j_1, \ldots, j_n)^\mathrm{T}$); hence $E_k(\mathbf{i} - \mathbf{j}) = \mathbf{0}^\mathrm{T}$. But $E_k$ was nonsingular so $\mathbf{i} = \mathbf{j}$ and $s = t$.

Also, $\tau$ does not cancel when $f_k(a_1^k, \ldots, a_n^k)$ is expanded to $f$. Otherwise, there would be some term $s$ in $f_k$ and a term $\sigma$ from $s(a_1^k, \ldots, a_n^k)$ such that $\tau = \sigma$. (Note: all these terms are in $R$, i.e. of the form $x_1^{p_1} \ldots x_n^{p_n}$.) Then however, $\sigma \leqslant \mathrm{lt}(s(\mathbf{a}^k)) < \mathrm{lt}(t(\mathbf{a}^k))$. This contradicts the unicity of $t$ and the maximality of $\tau$.

We conclude that $\tau = \mathrm{lt}(f)$. This shows what we wanted, namely that some $t_i^k$ occurs in $t$, hence in $f$. $\square$

In fact we have implicitly proven Theorem 1.

## 4. The size of the exponents

How good is Theorem 1? In order to answer this question let us give an estimate of the entries of powers of $D$. As stated in the introduction, a good way to do this is by looking at the eigenvalues.

A matrix like $D$ has been used before by Raney [6] in a completely different context. In his paper [5], he considers a matrix $Q_n$ equivalent to $D$ by reversal of the order of the coordinates, and the dominant eigenvalue and its eigenvector are computed. Though his result is in principle applicable to estimate $\mathrm{tdeg}(f)$ asymptotically, we shall by a different procedure be able to find the complete spectrum of $D$ with its eigenvector basis, obtain a more precise estimate for $\mathrm{tdeg}(f)$, and, moreover, derive an explicit formula in closed form for $D^k$.

### 4.1. The eigenvectors of D

For $p = 1, 2, \ldots, n$, let us define the following quantities:

$$w_p = -\exp\left(\frac{-2p\pi i}{2n+1}\right);$$

$$\alpha_p = w_p + w_p^{-1} = -2\cos\left(\frac{2p\pi}{2n+1}\right); \qquad V_p = w_p - w_p^{-1} = 2i\sin\left(\frac{2p\pi}{2n+1}\right);$$

$$\lambda_p = 4\cos^2\left(\frac{p\pi}{2n+1}\right); \qquad \mu_p = (-1)^n/\,2\cos\left(\frac{2pn\pi}{2n+1}\right);$$

$$x_m^p = 2(-1)^{m+1}\frac{\sin(2pm\pi/2n+1)}{\sqrt{2n+1}} \qquad (m = 1, 2\ldots, n);$$

$$\mathbf{x}^p = (x_1^p, x_2^p, \ldots, x_n^p).$$

These numbers satisfy the relations:

$$\lambda_p = 2 + \alpha_p; \qquad V_p^2 = \alpha_p^2 - 4; \qquad w_p^{2n+1} = -1; \qquad \mu_p = \frac{1}{(w_p^n + w_p^{-n})};$$

$$\mu_p^{-2} = \lambda_p; \qquad x_m^p = \frac{(w_p^m - w_p^{-m})}{\mathrm{i}\sqrt{2n+1}} \qquad (m = 1, 2, \ldots, n);$$

also, $w_p = (\alpha_p + V_p)/2$ and $w_p^{-1} = (\alpha_p - V_p)/2$ are the roots of $X^2 - \alpha_p X + 1 = 0$.

Let $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{n} x_i \bar{y}_i$ be the standard Hermitian inner product. It is elementary to verify that the $\mathbf{x}_p$ are perpendicular of length 1. Now one has the following.

**Proposition 1.** *The vectors $\mathbf{x}_p$ form an orthonormal basis upon which the matrix $D$ assumes a diagonal form $\Delta = \mathrm{Diag}(\mu_1, \mu_2, \ldots, \mu_n)$.*

**Proof.** Since the proof is fairly standard, let us just outline it. Instead of working with $D$ (like in [5]) we shall first diagonalize $D^{-2}$, which is much easier.

Indeed, one readily verifies that the inverse of $D$ is the matrix with ones on the antidiagonal, $-1$'s just above it, and zeroes elsewhere. Next, its square $D^{-2}$ is seen to be tridiagonal: $(D^{-2})_{i,i} = 2$ $(i < n)$; $(D^{-2})_{n,n} = 1$; $(D^{-2})_{i,j} = -1$ $(|i-j| = 1)$.

Tridiagonal matrices have been studied extensively in the theory of orthogonal polynomials [6] and the numerical theory of parabolic differential equations.

$D^{-2}$, being symmetric, can be diagonalized on a real orthonormal basis. Let $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ be an eigenvector of $D^{-2}$ with eigenvalue $\lambda$. Put $\mathbf{z} = \mathbf{z}(\alpha)$, again with $\alpha = 2 - \lambda$. W.l.o.g, let $z_1 = 1$ and $z_0 =_{\mathrm{Def}} 0$. Then $(D^{-2} - \lambda I)\mathbf{z}^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}}$ amounts to the recursion

$$z_0 = 0; \qquad z_1 = 1; \qquad z_m = \alpha z_{m-1} - z_{m-2} \quad (1 < m \leqslant n);$$
$$-z_{n-1} + (\alpha - 1)z_n = 0$$

(the latter being the characteristic equation).

**Remark 2.** This is the familiar recursion of the Tchebycev polynomials $T_{m-1}(x)$ in $x = \alpha/2$, though these have initial values $T_0 = 1$, $T_1 = x$. In fact it is not difficult to prove that

$$z_m = \frac{((\alpha/2)T_m(\alpha/2) - T_{m-1}(\alpha/2))}{((\alpha/2)^2 - 1)}.$$

Let $V = \sqrt{\alpha^2 - 4}$ and $w = (\alpha + V)/2$, $w' = (\alpha - V)/2$, the roots of $X^2 - \alpha X + 1 = 0$. If $w = w', \alpha = \pm 2$; but then $z_m = (\pm 1)^{m-1} m$, $-z_{n-1} + (\alpha - 1)z_n \neq 0$, and there are no eigenvalues. So suppose $w \neq w'$.

Solving the recursion by standard techniques yields $z_m = (w^m - w'^{-m})/V$; $1 \leqslant m \leqslant n$. By some easy calculations, the eigenvalue equation $-z_{n-1} + (\alpha - 1)z_n = 0$ reduces to $w^{2n+1} = -1$ (where $w \neq -1$ since $w \neq w'$). From this, $w = -\exp(\frac{-2p\pi i}{2n+1})$, $p = 1, 2, \ldots, n$. From now on we shall take this $p$ as an index (i.e., use $\alpha_p$, $\lambda_p$, $\mu_p$, $w_p$, $V_p$, $z^p_m$, $\mathbf{z}^p$, $x^p_m$, $\mathbf{x}^p$).

The numbers and vectors $\alpha_p$, $\lambda_p$, $\mu_p$, $w_p$, $V_p$, $x^p_m$, $\mathbf{x}^p$ are in fact those defined earlier. Normalization of $V_p \mathbf{z}^p$ yields the $p$th eigenvector $\mathbf{x}^p$ as

$$x^p_m = 2(-1)^{m+1} \frac{\sin(2pm\pi/2n+1)}{\sqrt{2n+1}}.$$

Similarly, one finds the formulas for $\alpha_p, \lambda_p$ etc.

The $\mathbf{x}^p(\alpha)$ form an orthogonal eigenbasis over which the symmetric matrix $D^{-2}$ diagonalizes. But in fact by an easy calculation, $D^{-1}(\mathbf{x}^p)^{\mathrm{T}} = \mu_p^{-1}(\mathbf{x}^p)^{\mathrm{T}}$; hence $D^{-1}$ and $D$ diagonalize as well. This ends the proof. □

Note that the eigenvalues $\mu_p$ of $D$ are all different and $\max_p |\mu_p| = |\mu_n| = 1/2 \cos(n\pi/(2n+1))$. Also, sign $\mu_p = (-1)^{n+p}$ (consider $pn$ mod $2n+1$ for $p$ odd and $p$ even).

**Corollary 2.** *The (nonnegative integral) entries of $D^k$ are given in closed form by the formula*

$$(D^k)_{i,j} = \sum_{p=1}^{n} (-1)^{i+j+(n+p)k} \frac{\sin(2pi\pi/(2n+1))\,\sin(2pj\pi/(2n+1))}{(2n+1)2^{k-2}\cos^k(p\pi/(2n+1))}.$$

**Proof.** $D^k$ can be written as $S\Delta^k S^{\mathrm{T}}$ with $\Delta = \mathrm{Diag}(\mu_1, \mu_2, \ldots, \mu_n)$ and $S$ the orthogonal basis transformation matrix having the $(\mathbf{x}^p)^{\mathrm{T}}$'s as its columns. Thus, $(D^k)_{i,j} = \sum_{p=1}^{n} \mu_p^k x_i^p x_j^p$. □

This also is the explicit solution of the recursion for the exponents vectors $\mathbf{e}_i^k$.

**Remark 3.** The following very nice graph-theoretic argument to find the eigenvalues of the matrix $D$ was communicated by Blokhuis et al. [7].

Let $N = (-1)^n D^{-1}$. We can write $N = A - B$, where both $A$ and $B$ are 0–1 matrices (and $A$ and $B$ are zero wherever $N$ is zero). With

$$P = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

we see that $P$ is the adjacency matrix of a path of length $2n$. Each eigenvector $u$ of $N$ with eigenvalue $\theta$ yields an antisymmetric eigenvector

$$\begin{pmatrix} u \\ -u \end{pmatrix}$$

of $P$ with eigenvalue $\theta$, and conversely. But the antisymmetric eigenvectors of $P$ are precisely those that can be extended to eigenvectors of a $(2n+1)$-cycle by defining it to be zero on the additional point. It follows that the eigenvalues are $\theta = 2\cos(2\pi j/(2n+1))$, where $1 \leqslant j \leqslant n$ from which those of $D$ follow. □

### 4.2. *The final estimate*

In order to prove Corollary 1, we have to estimate the total degree of $\mathrm{lt}(f)$. Let $\mathbf{j}$ be the all-one column. By Theorem 1, there exists some nonzero column vector $\mathbf{i}$ over $\mathbb{N} \cup \{0\}$ such that $\mathrm{lt}(f) = \langle UD^{k-1}\mathbf{i}, \mathbf{j}\rangle = \langle D\mathbf{i}, D^{k-2}U^T\mathbf{j}\rangle = \langle D\mathbf{i}, D^{k-2}(1,2,\ldots,n)^{\mathrm{T}}\rangle$. Note that $D\mathbf{i}$ has at least one positive entry, namely the $n$th. Hence,

$$\begin{aligned}\mathrm{tdeg}\,f &\geqslant \sum_{q=1}^{n} q(D^{k-2})_{n,q} \\ &= \sum_{q=1}^{n} q \sum_{p=1}^{n} (-1)^{n+q+(n+p)t} \frac{\sin(2pn\pi/(2n+1))\,\sin(2pq\pi/(2n+1))}{(2n+1)2^{t-2}\cos^t(p\pi/(2n+1))}\end{aligned}$$

by Corollary 2, putting $k-2=t$. Equality occurs if $\mathbf{i}=(1,0,\ldots,0)^{\mathrm{T}}$: e.g., if $f=a_1^k$.

The summation over the index $q$ can easily, though tediously, be calculated explicitly (e.g, using the complex form of the sine or by computer algebra). The double sum then reduces to

$$\frac{(-1)^{nt}}{(2n+1)2^t}\sum_{p=1}^{n}(-1)^{pt}\frac{\sin(2pn\pi/(2n+1))^2}{\cos^{t+2}(p\pi/(2n+1))}.$$

Let $H$ be the largest ($n$th) term; we shall see that it dominates. By Taylor expansion around $\pi/2$ one has, for some $|\theta|\leqslant 1$,

$$\sin\left(\frac{n\pi}{2n+1}\right)=1-\left(\frac{\pi}{2(2n+1)}\right)^2\frac{\theta}{2!}\geqslant\frac{19}{20}.\quad(n\geqslant 2)$$

Similarly, $\cos(n\pi/2n+1)\leqslant\pi/2(2n+1)$. Thus,

$$H\geqslant\frac{4(2n+1)^{t+1}}{\pi^{t+2}}\left(\frac{19}{20}\right)^2.$$

Since $\cos x\geqslant 1-(2x/\pi)$ on $[0,\pi/2]$, one has $\cos(p\pi/(2n+1))\geqslant(2(n-p)+1)/(2n+1)$. Also, $\sin(2pn\pi/(2n+1))^2\leqslant 1$. Hence, the sum of the absolute values of the first $n-1$ terms is not more than

$$\sum_{r=1}^{n-1}\frac{(2n+1)^{t+1}}{2^t(2r+1)^{t+2}},$$

where $r=n-p$. However,

$$\sum_{r=1}^{n-1}\frac{1}{(2r+1)^{t+2}}\leqslant\frac{1}{3^{t+2}}+\int_1^{\infty}\frac{\mathrm{d}x}{(2x+1)^{t+2}}\leqslant\frac{1}{3^{t+1}},$$

which yields the upper bound $\frac{(2n+1)^{t+1}}{2^t3^{t+1}}$ for the small terms.

Combining this with our estimate for $H$ we finally find

$$\mathrm{tdeg}(f)\geqslant\frac{4(2n+1)^{t+1}}{\pi^{t+2}}\left(\frac{19}{20}\right)^2-\frac{(2n+1)^{t+1}}{2^t3^{t+1}}$$

from which Corollary 1 follows immediately. □

## 5. An example of a "fixpoint polynomial"

In the introduction we mentioned the fixpoints of the iteration $(x_1,\ldots,x_n)\to(a_1,\ldots,a_n)$. An amusing and perhaps intriguing numerical example for $n=4$ is the following:

$$(-T+1)(-1.324717957T+1)(0.7548776668T+1)$$
$$\times(0.5698402906T+1)\approx 1-0.9999999994T$$
$$-1.324717957T^2+0.7548776668T^3+0.5698402912T^4$$

The relevant equations were solved in the obvious way using Maple, by first constructing a Groebner basis of the ideal $I(x_1+x_2+x_3+x_4-x_1, x_1x_2+x_1x_3+x_2x_3+x_1x_4+x_2x_4+x_3x_4-x_2, x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4-x_3, x_1x_2x_3x_4-x_4)$.

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