LINEAR ALGEBRA AND ITS APPLICATIONS

ELSEVIER Linear Algebra and its Applications 281 (1998) 137–170

On several types of resolvent matrices of nondegenerate matricial Carathéodory problems

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Received 8 July 1997; received in revised form 3 March 1998; accepted 17 March 1998

Submitted by V. Pták

Abstract

The solution set of a nondegenerate matricial Carathéodory problem can be described with the aid of linear fractional transformations of matrices where the parameter sets are either the matricial Schur class or the class of J_q-nonnegative meromorphic nondegenerate column pairs. This paper is aimed at clarifying the connections between these two different types of parametrizations. © 1998 Elsevier Science Inc. All rights reserved.

AMS classification: 30E05; 47A57

Keywords: Matricial Carathéodory problem; Resolvent matrices; J_q-nonnegative meromorphic nondegenerate column pairs

1. Introduction

At the end of the sixties, V.P. Potapov and his school started a systematic investigation of matricial versions of classical interpolation problems of Carathéodory–Schur–Nevanlinna–Pick type from the point of view of J-theory. Based on a generalization of the classical Schwarz–Pick inequalities, he created

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PII: S0024-3795(98)10034-4

a powerful method for treating such problems, namely the so-called fundamental matrix inequality method. The solution set of such a matricial interpolation problem, which coincides with the solution set of an appropriately constructed matrix inequality, can be described by linear fractional transformations of matrices. The generating matrix-valued functions of which are called resolvent matrices associated with the interpolation problem under consideration. In many cases, resolvent matrices can be chosen as rational $J^{(1)}$–$J^{(2)}$-inner matrix-valued functions where $J^{(1)}$ and $J^{(2)}$ are some signature matrices.

This paper is aimed at studying interrelations between different types of resolvent matrices associated with nondegenerate matricial Carathéodory problems. In this case we will be concerned with the concrete signature matrices

$$J_q = \begin{pmatrix} 0 & -I_q \\ -I_q & 0 \end{pmatrix}$$

and $j_{qq} := \operatorname{diag}(I_q, -I_q)$, where I_q stands for the $q \times q$ identity matrix. The representations of the solution sets of the nondegenerate matricial Carathéodory problems we will consider use different classes of meromorphic matrix-valued functions. We will compare parametrizations which work with so-called J_q-nonnegative meromorphic nondegenerate columns pairs [18] as well as parametrizations where the matricial Schur class is the parameter set [1,9,10]. Using a result due to Simakova [22] we will describe the whole variety of resolvent matrices of both types.

2. Some preliminaries

Throughout this paper, let p and q be positive integers. If $\mathfrak{X}$ is a nonempty set, then we will write $\mathfrak{X}^{p\times q}$ for the set of all $p \times q$ matrices each entry of which belongs to $\mathfrak{X}$. The null matrix that belongs to the set $\mathbb{C}^{p\times q}$ of all $p \times q$ complex matrices will be designated by $0_{p\times q}$, and I_q stands for the identity matrix that belongs to $\mathbb{C}^{q\times q}$. If the size of a null matrix or of an identity matrix is clear, then we will often omit the indexes. A $p \times q$ complex matrix A is called *contractive* (respectively, *strictly contractive*) if $B := I - A^*A$ is nonnegative Hermitian (respectively, positive Hermitian). We will use $\mathbb{K}_{p\times q}$ (respectively, $\mathbb{D}_{p\times q}$) to denote the set of all $p \times q$ contractive (respectively, $p \times q$ strictly contractive) matrices. If $A \in \mathbb{C}^{q\times q}$, then $\operatorname{Re} A$ and $\operatorname{Im} A$ stand for the real part of A and the imaginary part of A, respectively:

$$\operatorname{Re} A := \frac{1}{2}(A + A^*) \quad \text{and} \quad \operatorname{Im} A := \frac{1}{2\mathrm{i}}(A - A^*).$$

If τ is a nonnegative integer or if $\tau = +\infty$, then we will write $\mathbb{N}_{0,\tau}$ for the set of all integers k which satisfy $0 \leqslant k \leqslant \tau$, and for every sequence $(A_k)_{k=0}^{\tau}$ of $p \times q$ complex matrices, we will associate the block Toeplitz matrices

$$S_{k,A} := \begin{pmatrix} A_0 & 0 & \dots & 0 & 0 \\ A_1 & A_0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ A_{k-1} & A_{k-2} & \dots & A_0 & 0 \\ A_k & A_{k-1} & \dots & A_1 & A_0 \end{pmatrix}, \tag{1}$$

$k \in \mathbb{N}_{0,\tau}$. If f is a $p \times q$ matrix-valued function which is holomorphic in a neighborhood of the origin, then we set

$$S_k^{[f]} := S_{k,A} \tag{2}$$

for all nonnegative integers k, where

$$f(z) = \sum_{k=0}^{\infty} A_k z^k \tag{3}$$

is the Taylor representation of f for all complex numbers z which belong to some neighborhood of the origin.

Remark 1. If f is a $q \times q$ matrix-valued function which is holomorphic in some neighborhood of the origin and which satisfies $\det(I + f(0)) \neq 0$, then the Cayley transform $\Omega := (I - f)(I + f)^{-1}$ is a $q \times q$ matrix-valued function which is holomorphic in some neighborhood of the origin, and, for every nonnegative integer n, it is readily checked that $\det(I + S_n^{[f]}) \neq 0$ and $S_n^{[\Omega]} = (I - S_n^{[f]})(I + S_n^{[f]})^{-1}$ (see, e.g., [6], Lemma 1.1.21).

Observe that a similar result can be formulated only by using sequences of $q \times q$ complex matrices (see, e.g., [11], Lemma 1). If τ is a nonnegative integer or if $\tau = +\infty$, and if $(X_k)_{k=0}^{\tau}$ is a sequence of $q \times q$ complex matrices with $\det(I + X_0) \neq 0$, then the unique sequence $(Y_k)_{k=0}^{\tau}$ of $q \times q$ complex matrices which satisfies $S_{k,Y} = (I - S_{k,X})(I + S_{k,X})^{-1}$ for all $k \in \mathbb{N}_{0,\tau}$ is called the Cayley transform of $(X_k)_{k=0}^{\tau}$.

A function $\Omega : G \to \mathbb{C}^{q \times q}$ is said to be a $q \times q$ *Carathéodory function in* G, if Ω is holomorphic in G and if $\operatorname{Re} \Omega(z)$ is nonnegative Hermitian for all $z \in G$. We will use $\mathscr{C}_q(G)$ to denote the set of all $q \times q$ Carathéodory functions in G. If Ω belongs to $\mathscr{C}_q(G)$, then $\det(I + \Omega)$ nowhere vanishes in G and the Cayley transform $S := (I - \Omega)(I + \Omega)^{-1}$ is a so-called $q \times q$ *Schur function in* G (see, e.g., [6], Proposition 2.1.3). A function $f : G \to \mathbb{C}^{p \times q}$ is said to be a $p \times q$ Schur function if f is holomorphic in G and if $f(z)$ is contractive for all $z \in G$. The set of all $p \times q$ Schur functions in G will be designated by $\mathscr{S}_{p \times q}(G)$. Note that a $p \times q$ Schur function in G is called *strictly contractive* if $f(z)$ is strictly contractive for every choice of z in G.

If $(\Gamma_k)_{k=0}^n$ is a given sequence of $q \times q$ complex matrices, then the Carathéodory problem with respect to $(\Gamma_k)_{k=0}^n$ consists of the description of the set $\mathscr{C}_q[\Gamma_0, \Gamma_1, \dots, \Gamma_n]$ of all functions Ω which belong to $\mathscr{C}_q(\mathbb{D})$ and which satisfy

$$\frac{\Omega^{(k)}(0)}{k!} = \Gamma_k \tag{4}$$

for all $k \in \mathbb{N}_{0,n}$, where $\Omega^{(k)}$ denotes the kth derivative of Ω. It is a well-known fact that the set $\mathscr{C}_q[\Gamma_0, \Gamma_1, \dots, \Gamma_n]$ is nonempty if and only if the matrix $T_{n,\Gamma} := \mathrm{Re}\, S_{n,\Gamma}$ is nonnegative Hermitian where $S_{n,\Gamma}$ is defined by Eq. (1). For this reason, a sequence $(\Gamma_k)_{k=0}^n$ of $q \times q$ complex matrices is called a $q \times q$ *Carathéodory sequence* if the matrix $T_{n,\Gamma}$ is nonnegative Hermitian. If $T_{n,\Gamma}$ is even positive Hermitian, then the $q \times q$ Carathéodory sequence $(\Gamma_k)_{k=0}^n$ is said to be *nondegenerate*. If a nondegenerate $q \times q$ Carathéodory sequence is given, then the solution set $\mathscr{C}_q[\Gamma_0, \Gamma_1, \dots, \Gamma_n]$ of the Carathéodory problem with respect to $(\Gamma_k)_{k=0}^n$ can be represented by some linear fractional transformation where the Schur class $\mathscr{S}_{p\times q}(\mathbb{D})$ is the parameter set (see, e.g., [1,9,10]). To explain such representations we introduce notations for linear transformation of matrices. Let $A \in \mathbb{C}^{(p+q)\times(p+q)}$ be given with block partition

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{5}$$

where A_{11} is a $p \times p$ block. Then the set

$$\mathscr{Q}_{A_{21},A_{22}} := \{X \in \mathbb{C}^{p\times q}\colon\ \det(A_{21}X + A_{22}) \neq 0\} \tag{6}$$

is nonempty if and only if rank $(A_{21}, A_{22}) = q$ (see, e.g., [6], Lemma 1.6.1). In this case, the right linear fractional transformation $\mathscr{S}_A^{(p,q)} : \mathscr{Q}_{A_{21},A_{22}} \to \mathbb{C}^{p\times q}$ is defined by

$$\mathscr{S}_A^{(p,q)}(X) := (A_{11}X + A_{12})(A_{21}X + A_{22})^{-1}. \tag{7}$$

Similarly, the set

$$\mathscr{E}_{A_{12},A_{22}} := \{X \in \mathbb{C}^{q\times p}\colon\ \det(XA_{12} + A_{22}) \neq 0\} \tag{8}$$

is nonempty if and only if rank $(A_{12}^*, A_{22}^*) = q$. In this case, the left linear fractional transformation $\mathscr{T}_A^{(p,q)} : \mathscr{E}_{A_{12},A_{22}} \to \mathbb{C}^{q\times p}$ is given by

$$\mathscr{T}_A^{(p,q)}(X) := (XA_{12} + A_{22})^{-1}(XA_{11} + A_{21}). \tag{9}$$

Now let G be a simply connected domain of the extended complex plane $\mathbb{C}_0$, and let A be a $(p+q) \times (p+q)$ matrix-valued function which is meromorphic in G. We will use the block partition (5) of A with $p \times p$ block A_{11}. If the set $\mathscr{Q}$ of all functions $g : G \to \mathbb{C}^{p\times q}$ which are holomorphic in G and for which the function $\det(A_{21}g + A_{22})$ does not identically vanish in G is nonempty, then let

$$\mathscr{S}^{(p,q)}_{[A]}(g) := (A_{11}g + A_{12})(A_{21}g + A_{22})^{-1} \tag{10}$$

for each $g \in \mathscr{G}$. If Y is a nonempty subset of $\mathscr{G}$, then we set

$$\mathscr{S}^{(p,q)}_{[A]}(Y) := \left\{\mathscr{S}^{(p,q)}_{[A]}(g):\ g \in Y\right\}. \tag{11}$$

On the other hand, if the set $\mathscr{E}$ of all functions $h : G \to \mathbb{C}^{q\times p}$ which are holomorphic in G and for which the function $\det(hA_{12} + A_{22})$ does not identically vanish in G is nonempty, then let

$$\mathscr{T}^{(p,q)}_{[A]}(h) := (hA_{12} + A_{22})^{-1}(hA_{11} + A_{21}) \tag{12}$$

for each $h \in \mathscr{E}$. Further, if Y is a nonempty subset of $\mathscr{E}$, then let

$$\mathscr{T}^{(p,q)}_{[A]}(Y) := \left\{\mathscr{T}^{(p,q)}_{[A]}(h):\ h \in Y\right\}. \tag{13}$$

Note that if the representations on the right-hand sides of Eqs. (11) and (12) have removable singularities we will use the symbols $\mathscr{S}^{(p,q)}_{[A]}(g)$ and $\mathscr{T}^{(p,q)}_{[A]}(h)$ for the extended functions.

Before we explain the above mentioned way to state representations of the solution set of the matricial Carathéodory problem by recalling the notion of a resolvent matrix, let us note that if we consider a $2q \times 2q$ matrix-valued function ∇ or Δ, then we will use the $q \times q$ block partitions

$$\nabla = \begin{pmatrix} \nabla_{11} & \nabla_{12} \\ \nabla_{21} & \nabla_{22} \end{pmatrix} \quad \text{and} \quad \Delta = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix}. \tag{14}$$

Definition 2. Let $(\Gamma_k)_{k=0}^n$ be a nondegenerate $q \times q$ Carathéodory sequence. A $2q \times 2q$ matrix-valued function ∇ which is meromorphic in $\mathbb{D}$ is called a *right* (respectively, *left*) *type I resolvent matrix of the Carathéodory problem with respect to* $(\Gamma_k)_{k=0}^n$ if the following three conditions are satisfied:

(i) The function $\det \nabla$ does not identically vanish in $\mathbb{D}$.

(ii) For each $g \in \mathscr{S}_{q\times q}(\mathbb{D})$, the function $\det(\nabla_{21}g + \nabla_{22})$ (respectively, the function $\det(g\nabla_{12} + \nabla_{22})$) does not identically vanish in $\mathbb{D}$.

(iii) $\mathscr{S}^{(q,q)}_{[\nabla]}(\mathscr{S}_{q\times q}(\mathbb{D})) = \mathscr{C}_q[\Gamma_0, \Gamma_1, \ldots, \Gamma_n]$ (respectively, $\mathscr{T}^{(q,q)}_{[\nabla]}(\mathscr{S}_{q\times q}(\mathbb{D})) = \mathscr{C}_q[\Gamma_0, \Gamma_1, \ldots, \Gamma_n]$).

In Section 7 we will state another type of resolvent matrices of a matricial Carathéodory problem, the statement of which is much more complicated, and we will study how these representations of the solution set of this interpolation problem are connected. Further, we will discuss how these types of resolvent matrices can be obtained from an appropriately constructed matricial Schur problem. We will restrict our considerations on the right resolvent matrices. The treatment of the left resolvent matrices is similar and will be done somewhere else.

If $(A_k)_{k=0}^n$ is a sequence of $p \times q$ complex matrices, then the Schur problem with respect to $(A_k)_{k=0}^n$ consists of the description of the set $\mathcal{S}_{p\times q}[A_0, A_1, \ldots, A_n]$ of all functions f which belong to the Schur class $\mathcal{S}_{p\times q}(\mathbb{D})$ and which satisfy $f^{(k)}(0)/k! = A_k$ for all $k \in \mathbb{N}_{0,n}$ where $f^{(k)}$ denotes the kth derivative of f. It is a well-known fact that $\mathcal{S}_{p\times q}[A_0, A_1, \ldots, A_n]$ is non-empty if and only if the block Toeplitz matrix $S_{n,A}$ given by Eq. (1) is contractive. Sequences $(A_k)_{k=0}^n$ of $p \times q$ complex matrices $(A_k)_{k=0}^n$ for which $S_{n,A}$ is contractive are called $p \times q$ *Schur sequences*. If the matrix $S_{n,A}$ is even strictly contractive, then the $p \times q$ Schur sequence $(A_k)_{k=0}^n$ is said to be *nondegenerate*. If $(A_k)_{k=0}^n$ is a given nondegenerate $p \times q$ Schur sequence, then the set $\mathcal{S}_{p\times q}[A_0, A_1, \ldots, A_n]$ can be also represented by some linear fractional transformation where $\mathcal{S}_{p\times q}(\mathbb{D})$ is the parameter set (see, e.g., [2,7], Theorem 6.5, [6], Theorems 3.9.1, 3.10.1, 5.3.1, 5.3.2), i.e., one can find resolvent matrices in the following sense:

Definition 3. Let $(A_k)_{k=0}^n$ be a nondegenerate $p \times q$ Schur sequence.

(a) A $(p+q) \times (p+q)$ matrix-valued function D which is meromorphic in $\mathbb{D}$ is called a *right resolvent matrix of the Schur problem with respect to* $(A_k)_{k=0}^n$ if the following three conditions are satisfied:

(i) The function det D does not identically vanish in $\mathbb{D}$.

(ii) For each $g \in \mathcal{S}_{p\times q}(\mathbb{D})$, the function $\det(D_{21}g + D_{22})$ does not identically vanish in $\mathbb{D}$ where

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \tag{15}$$

is the block partition of D with $p \times p$ block D_{11}.

(iii) $\mathcal{S}_{[D]}^{(p,q)}(\mathcal{S}_{p\times q}(\mathbb{D})) = \mathcal{S}_{p\times q}[A_0, A_1, \ldots, A_n]$.

(b) A $(q+p) \times (q+p)$ matrix-valued function E which is meromorphic in $\mathbb{D}$ is said to be a *left resolvent matrix of the Schur problem with respect to* $(A_k)_{k=0}^n$ if the following three conditions are fulfilled:

(iv) The function det E does not identically vanish.

(v) For each $g \in \mathcal{S}_{p\times q}(\mathbb{D})$, the function $\det(gE_{12} + E_{22})$ does not identically vanish, where

$$\tilde{D} = \begin{pmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{21} & \tilde{D}_{22} \end{pmatrix}$$

is the block partition of E with $q \times q$ block E_{11}.

(vi) $\mathcal{T}_{[D]}^{(q,p)}(\mathcal{S}_{p\times q}(\mathbb{D})) = \mathcal{S}_{p\times q}[A_0, A_1, \ldots, A_n]$.

The following remark and the following lemma show the way how one can connect type I resolvent matrices of a matricial Carathéodory problem with resolvent matrices of an appropriately constructed matricial Schur problem.

Remark 4. Lemma 3 in [11] in particular shows that

(a) If $(\Gamma_k)_{k=0}^n$ is a nondegenerate $q \times q$ Carathéodory sequence, then the Cayley transform $(A_k)_{k=0}^n$ of $(\Gamma_k)_{k=0}^n$ is a nondegenerate $q \times q$ Schur sequence.

(b) If $(A_k)_{k=0}^n$ is a nondegenerate $q \times q$ Schur sequence, then the Cayley transform $(\Gamma_k)_{k=0}^n$ of $(A_k)_{k=0}^n$ is a nondegenerate $q \times q$ Carathéodory sequence.

Lemma 5. *Let $(\Gamma_k)_{k=0}^n$ be a nondegenerate $q \times q$ Carathéodory sequence, and let $(A_k)_{k=0}^n$ be the Cayley transform of $(\Gamma_k)_{k=0}^n$. Then*

$$\mathscr{S}_{q\times q}[A_0, A_1, \ldots, A_n] = \mathscr{S}^{(q,q)}_{[\tilde{C}_q]}(\mathscr{C}_q[\Gamma_0, \Gamma_1, \ldots, \Gamma_n]) \tag{16}$$

and

$$\mathscr{C}_q[\Gamma_0, \Gamma_1, \ldots, \Gamma_n] = \mathscr{S}^{(q,q)}_{[\tilde{C}_q]}(\mathscr{S}_{q\times q}[A_0, A_1, \ldots, A_n]), \tag{17}$$

where $\tilde{C}_q$ stands for the constant function defined on $\mathbb{D}$ with value

$$C_q := \frac{1}{\sqrt{2}}\begin{pmatrix} -I_q & I_q \\ I_q & I_q \end{pmatrix}. \tag{18}$$

Proof. Use Lemma 1 in [11], Propositions 2.1.2 and 2.1.3 in [6], and Remark 1. □

In the following, we will continue to use the symbol C_q for the matrix given by Eq. (18) as well as for constant function defined on $\mathbb{D}$ with value C_q.

3. Nondegenerate (p, q)-column pairs

In this section, we will study particular pairs of matrices under the view of linear fractional transformations of matrices. This can be considered as a slight generalization of the usual concept of linear fractional transformation of matrices (see, e.g., [21,23,13] and Section 1.6 of [6]).

Definition 6. Let $A \in \mathbb{C}^{p\times q}$, and let $B \in \mathbb{C}^{q\times q}$. Then $[A, B]$ is called a *nondegenerate (p, q)-column pair* if $C := A^*A + B^*B$ is positive Hermitian.

Such pairs of matrices were first used by Orlov [20] who preferred row pairs instead of column pairs. Let us use $\mathscr{P}^{(p,q)}$ to denote the set of all nondegenerate (p, q)-column pairs. Obviously, if $A \in \mathbb{C}^{p\times q}$ and $B \in \mathbb{C}^{q\times q}$, then $[A, B]$ is a nondegenerate (p, q)-column pair if and only if $\det(A^*A + B^*B) \neq 0$, i.e., rank $\begin{pmatrix} A \\ B \end{pmatrix} = q$. If $[A, B] \in \mathscr{P}^{(p,q)}$, then, for every nonsingular $q \times q$ complex matrix Q, we easily see that $[AQ, BQ]$ belongs to $\mathscr{P}^{(p,q)}$ as well.

Definition 7. Nondegenerate (p,q)-column pairs $[A_1, B_1]$ and $[A_2, B_2]$ are said to be *equivalent*, if there is a nonsingular $q \times q$ complex matrix Q such that $A_2 = A_1 Q$ and $B_2 = B_1 Q$. Obviously, in this way an equivalence relation in the set $\mathscr{P}^{(p,q)}$ is generated. If $[A, B] \in \mathscr{P}^{(p,q)}$, then we will write $\langle [A, B] \rangle$ for the equivalence class to which $[A, B]$ belongs.

Obviously, if B is a nonsingular $p \times q$ complex matrix, then, for each $A \in \mathbb{C}^{p \times q}$, $[A, B]$ is a nondegenerate (p,q)-column pair.

Definition 8. A nondegenerate (p,q)-column pair $[A, B]$ is said to be *proper* if det $B \neq 0$.

The set of all proper nondegenerate (p,q)-column pairs will be denoted by $\mathscr{P}_{\square}^{(p,q)}$. If $[A, B]$ is a proper nondegenerate (p,q)-column pair, then every nondegenerate (p,q)-column pair which is equivalent to $[A, B]$ is proper as well. Further, we see that if $[A, B]$ is a proper nondegenerate (p,q)-column pair, then $[AB^{-1}, I_q]$ is a proper nondegenerate (p,q)-column pair which is equivalent to $[A, B]$.

Observe that the mapping $\mathscr{I}_{pq}: \mathbb{C}^{p \times q} \to \langle \mathscr{P}_{\square}^{(p,q)} \rangle$ defined by $\mathscr{I}_{pq}(A) := \langle [A, I_q] \rangle$ is bijective.

Remark 9. Let $A \in \mathbb{C}^{p \times q}$, and let $B \in \mathbb{C}^{q \times q}$. Then there is a positive real number ε such that $\det(B + zI_q) \neq 0$ for each $z \in (0, \varepsilon]$. Hence $([A_n, B_n])_{n \in \mathbb{N}}$ defined by $A_n := A$ and $B_n := B + (\varepsilon/n)I_q$ for all $n \in \mathbb{N}$ is a sequence of proper nondegenerate (p,q)-column pairs which satisfies

$$\lim_{n \to \infty} A_n = A \quad \text{and} \quad \lim_{n \to \infty} B_n = B. \tag{19}$$

In the following, we will continue to use the notations given in Eqs. (6) and (7). Further, we will work with a similar transformation. We will see that some properties of the mapping given by Eq. (7), which is studied, e.g., in [21,23] and Section 1.6 of [6], can be used to verify analogous properties for these similar transformations. To prepare this we introduce the notation

$$[\mathscr{S}_{c,d}] := \left\{ [A, B] \in \mathscr{P}^{(p,q)}: \det(cA + dB) \neq 0 \right\} \tag{20}$$

for arbitrary matrices $c \in \mathbb{C}^{q \times p}$ and $d \in \mathbb{C}^{q \times q}$.

Lemma 10. *Let $c \in \mathbb{C}^{q \times p}$ and $d \in \mathbb{C}^{q \times q}$. Then the set $[\mathscr{S}_{c,d}]$ is nonempty if and only if* $\operatorname{rank}(c, d) = q$.

Proof. First suppose $[\mathscr{S}_{c,d}] \neq \emptyset$. Let $[A, B] \in [\mathscr{S}_{c,d}]$. According to Remark 9, there is a sequence $([A_n, B_n])_{n \in \mathbb{N}}$ of proper nondegenerate (p,q)-column pair such that Eq. (19) holds true. This implies $\lim_{n \to \infty} \det(cA_n + dB_n) = \det(cA + dB) \neq 0$. Hence there is a positive integer m for which

$$\det(cA_mB_m^{-1} + d) = \det(cA_m + dB_m)\ \det(B_m^{-1}) \neq 0.$$

Thus $A_mB_m^{-1} \in \mathscr{D}_{c,d}$, and it follows rank $(c,d) = q$.

Conversely, now assume that rank $(c,d) = q$. Then $\mathscr{D}_{c,d} \neq \emptyset$, i.e., there is a matrix $A \in \mathbb{C}^{p\times q}$ for which $\det(cA + d) \neq 0$ holds. Therefore $[A, I_q] \in [\mathscr{D}_{c,d}]$. □

If a matrix $E \in \mathbb{C}^{(p+q)\times(p+q)}$ is given, then we will work with the block partition

$$E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{21}$$

where a is a $p \times p$-block. Further if $E \in \mathbb{C}^{(p+q)\times(p+q)}$ is such that rank $(c,d) = q$, then we will use the mapping $\mathfrak{s}_E^{(p,q)} : [\mathscr{D}_{c,d}] \to \mathbb{C}^{p\times q}$ given by

$$\mathfrak{s}_E^{(p,q)}([A,B]) := (aA + bB)(cA + dB)^{-1}. \tag{22}$$

Observe that Nudelman [19] discussed convexity properties of certain subsets contained in the image of linear fractional transformations of nondegenerate column pairs.

Lemma 11. *Let E be such that rank $(c,d) = q$ and let $[A,B] \in [\mathscr{D}_{c,d}]$. Then every nondegenerate (p,q)-column pair $[C,D]$ which is equivalent to $[A,B]$ also belongs to $[\mathscr{D}_{c,d}]$ and satisfies*

$$\mathfrak{s}_E^{(p,q)}([C,D]) = \mathfrak{s}_E^{(p,q)}([A,B]).$$

The proof of Lemma 11 is straightforward. We omit the details.

Let $E \in \mathbb{C}^{(p+q)\times(p+q)}$ be such that rank $(c,d) = q$. In view of Lemma 11, then the mapping $\sigma_E^{(p,q)} : \langle[\mathscr{D}_{c,d}]\rangle \to \mathbb{C}^{p\times q}$ given by

$$\sigma_E^{(p,q)}(\langle[A,B]\rangle) := (aA + bB)(cA + dB)^{-1} \tag{23}$$

is well-defined.

Remark 12. Let $E \in \mathbb{C}^{(p+q)\times(p+q)}$ be such that rank $(c,d) = q$, and $A \in \mathbb{C}^{p\times q}$. Then $[A, I_q] \in [\mathscr{D}_{c,d}]$ if and only if $A \in \mathscr{D}_{c,d}$. In this case,

$$\sigma_E^{(p,q)}\big(\langle[A, I_q]\rangle\big) = \mathscr{S}_E^{(p,q)}(A). \tag{24}$$

Proposition 13. *Let $E \in \mathbb{C}^{(p+q)\times(p+q)}$ be such that $\operatorname{rank}(c,d) = q$. Then the mapping $\sigma_E^{(p,q)} : \langle[\mathscr{D}_{c,d}]\rangle \to \mathbb{C}^{p\times q}$ is one-to-one if and only if* $\det E \neq 0$.

Proof. First suppose $\det E = 0$. According to a result due to Potapov [21] (see also [6], Proposition 1.6.2), there are matrices A_1 and A_2 that belong to $\mathscr{D}_{c,d}$ and which satisfy $\mathscr{S}_E^{(p,q)}(A_1) = \mathscr{S}_E^{(p,q)}(A_2)$ as well as $A_1 \neq A_2$. In view of Remark 12 the proper nondegenerate (p,q)-column pairs $[A_1,I_q]$ and $[A_2,I_q]$ then fulfill $\sigma_E^{(p,q)}(\langle[A_1,I_q]\rangle) = \sigma_E^{(p,q)}(\langle[A_2,I_q]\rangle)$ and $\langle[A_1,I_q]\rangle \neq \langle[A_2,I_q]\rangle$. Hence the mapping $\sigma_E^{(p,q)}$ is not one-to-one.

Now assume that $\det E \neq 0$ holds. Let $[A_1,B_1]$ and $[A_2,B_2]$ belong to $[\mathscr{D}_{c,d}]$ and be such that $\sigma_E^{(p,q)}(\langle[A_1,B_1]\rangle) = \sigma_E^{(p,q)}(\langle[A_2,B_2]\rangle)$. Then it is readily checked that

$$E\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}(cA_1 + dB_1)^{-1} = E\begin{pmatrix} A_2 \\ B_2 \end{pmatrix}(cA_2 + dB_2)^{-1}.$$

Then $\det E \neq 0$ implies $A_2 = A_1Q$ and $B_2 = B_1Q$, where $Q := (cA_1 + dB_1)^{-1}(cA_2 + dB_2)$ is obviously nonsingular. Therefore $\langle[A_1,B_1]\rangle = \langle[A_2,B_2]\rangle$. $\square$

Proposition 14. *Let E_1 and E_2 be $(p+q)\times(p+q)$ complex matrices, and let*

$$E_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

be the block partitions where a_1 and a_2 belong to $\mathbb{C}^{p\times p}$. Further, let $E := E_2E_1$, and let (21) *be the block partition of E with $p\times p$-block a. Suppose that rank $(c_1,d_1) = q$ and that rank $(c_2,d_2) = q$. Then*

$$[\mathscr{D}] := \left\{[A,B] \in [\mathscr{D}_{c_1,d_1}]:\ \mathfrak{s}_{E_1}^{(p,q)}([A,B]) \in \mathscr{D}_{c_2,d_2}\right\}$$

is a subset of $[\mathscr{D}_{c,d}]$ which admits the representation

$$[\mathscr{D}] = [\mathscr{D}_{c_1,d_1}] \cap [\mathscr{D}_{c,d}]. \tag{25}$$

If $[\mathscr{D}] \neq \emptyset$, then, for all $[A,B] \in [\mathscr{D}]$,

$$\mathfrak{s}_E^{(p,q)}([A,B]) = \mathscr{S}_{E_2}^{(p,q)}\left(\mathfrak{s}_{E_1}^{(p,q)}([A,B])\right). \tag{26}$$

Proof. If $[A,B] \in [\mathscr{D}_{c_1,d_1}]$, then, in view of $c = c_2a_1 + d_2c_1$ and $d = c_2b_1 + d_2d_1$, we have

$$\det(cA + dB) = \det\left(c_2\left(\mathfrak{s}_{E_1}^{(p,q)}([A,B])\right) + d_2\right)\ \det(c_1A + d_1B).$$

Hence we see that Eq. (25) holds. If $[\mathscr{D}] \neq \emptyset$, then, for each $[A,B] \in [\mathscr{D}]$, we obtain

$$\begin{aligned}\mathscr{S}_{E_2}^{(p,q)}\left(\mathfrak{s}_{E_1}^{(p,q)}([A,B])\right) &= (a_2(a_1A + b_1B) + b_2(c_1A + d_1B)) \\ &\quad (c_2(a_1A + b_1B) + d_2(c_1A + d_1B))^{-1} \\ &= (aA + bB)(cA + dB)^{-1} = \mathfrak{s}_E^{(p,q)}([A,B]). \qquad \square\end{aligned}$$

Remark 15. Let the assumptions of Proposition 14 be satisfied. If $[\mathscr{D}] \neq \emptyset$, then each $\langle [A,B] \rangle \in \langle [\mathscr{D}] \rangle$ fulfills $\sigma_E^{(p,q)}(\langle [A,B] \rangle) = \mathscr{S}_{E_2}^{(p,q)}(\sigma_{E_1}^{(p,q)}(\langle [A,B] \rangle))$.

4. J-Nonnegative nondegenerate (p,q)-column pairs

In this section, we will turn our attention to particular subclasses of nondegenerate (p,q)-column pairs. More precisely, we have in mind such subclasses which are linked with certain signature matrices, i.e., $m \times m$ complex matrices J which satisfy $J^* = J$ and $J^2 = I$. Our investigations are aimed at describing the image of these subclasses under linear fractional transformations generated by J-contractive matrices. If J and j are $m \times m$ signature matrices, then an $m \times m$ complex matrix A is called *J–j-contractive* (respectively, *J–j-unitary*) if $J \geqslant A^* jA$ (respectively, $J = A^* jA$). A J–J-contractive (respectively, J–J-unitary) matrix A is also said to be *J-contractive* (respectively, *J-unitary*). Observe that the matrix C_q given by Eq. (18) is j_{qq}–J_q-unitary.

Definition 16. Let J be a Hermitian $(p+q) \times (p+q)$ complex matrix. A nondegenerate (p,q)-column pair $[A,B]$ is said to be *J-nonnegative* (respectively, *J-positive*), if the matrix

$$\triangle := \begin{pmatrix} A \\ B \end{pmatrix}^* J \begin{pmatrix} A \\ B \end{pmatrix} \tag{27}$$

is nonnegative Hermitian (respectively, positive Hermitian). If $\triangle = 0_{q\times q}$ holds, the nondegenerate (p,q)-column pair $[A,B]$ is called *J-neutral.*

J-nonnegative nondegenerate column pairs were introduced by Orlov [20] in the study of the limit behaviour of the semi-radii of families of nested matrix balls. Furthermore, note that such pairs were used to describe solution sets of matricial interpolation problems by several authors (see, e.g., [3–5, 14–18]).

If J is a Hermitian $(p+q) \times (p+q)$ complex matrix, then we will use the symbol $\mathscr{P}_{J,\geqslant}^{(p,q)}$ (respectively, $\mathscr{P}_{J,>}^{(p,q)}$) to denote the set of all J-nonnegative (respectively, J-positive) nondegenerate (p,q)-column pairs, and we will write $\mathscr{P}_{J,0}^{(p,q)}$ for the set of all J-neutral nondegenerate (p,q)-column pairs.

Remark 17. Let J be a Hermitian $(p+q) \times (p+q)$ complex matrix, and let $[A_1,B_1]$ and $[A_2,B_1]$ be nondegenerate (p,q)-column pairs which are equivalent. Then,

(a) $[A_1,B_1] \in \mathscr{P}_{J,\geqslant}^{(p,q)}$ if and only if $[A_2,B_2] \in \mathscr{P}_{J,\geqslant}^{(p,q)}$.

(b) $[A_1,B_1] \in \mathscr{P}_{J,>}^{(p,q)}$ if and only if $[A_2,B_2] \in \mathscr{P}_{J,>}^{(p,q)}$.

(c) $[A_1,B_1] \in \mathscr{P}_{J,0}^{(p,q)}$ if and only if $[A_2,B_2] \in \mathscr{P}_{J,0}^{(p,q)}$.

Observe that, in view Remark 17, we can write $\langle \mathscr{P}^{(p,q)}_{J,\geqslant} \rangle$ to designate the set of all equivalence classes of J-nonnegative nondegenerate (p,q)-column pairs. Analogously, we will use the notations $\langle \mathscr{P}^{(p,q)}_{J,>} \rangle$ and $\langle \mathscr{P}^{(p,q)}_{J,0} \rangle$. Now we specify the Hermitian matrix J. We will work with the particular $(p+q)\times(p+q)$ signature matrices

$$j_{pq} := \operatorname{diag}(I_p, -I_q) \tag{28}$$

and $-j_{pq}$.

Remark 18. In view of the identity

$$B^*B - A^*A = \begin{pmatrix} A \\ B \end{pmatrix}^* (-j_{pq}) \begin{pmatrix} A \\ B \end{pmatrix}, \tag{29}$$

which holds true for every choice of A in $\mathbb{C}^{p\times q}$ and B in $\mathbb{C}^{q\times q}$, it is readily checked that both sets $\mathscr{P}^{(p,q)}_{-j_{pq},\geqslant}$ and $\mathscr{P}^{(p,q)}_{-j_{pq},>}$ are nonempty.

Lemma 19. (a) *If* $[A,B] \in \mathscr{P}^{(p,q)}_{-j_{pq},\geqslant}$, *then the nondegenerate* (p,q)*-column pair* $[A,B]$ *is necessarily proper and the matrix* $K := AB^{-1}$ *is contractive.*

(b) *If* $[A,B]$ *belongs to* $\mathscr{P}^{(p,q)}_{-j_{pq},>}$, *then* $K := AB^{-1}$ *is strictly contractive.*

(c) *If* $[A,B]$ *belongs to* $\mathscr{P}^{(p,q)}_{-j_{pq},0}$, *then* $K := AB^{-1}$ *satisfies* $K^*K = I_q$.

Proof. Let $[A,B] \in \mathscr{P}^{(p,q)}$. Then $C := A^*A + B^*B$ is positive Hermitian and the equation $B^*B = \frac{1}{2}((B^*B - A^*A) + C)$ is valid. Hence, if $[A,B]$ belongs to $\mathscr{P}^{(p,q)}_{-j_{pq},\geqslant}$, then we get from Eq. (29) that B^*B is positive Hermitian, i.e., that B is nonsingular. Since $I - K^*K = B^{-*}(B^*B - A^*A)B^{-1}$ holds for all $[A,B] \in \mathscr{P}^{(p,q)}_{-j_{pq},\geqslant}$, the proof is finished. $\square$

Remark 20. Identity (29) and part (c) of Lemma 19 show that $\mathscr{P}^{(p,q)}_{-j_{pq},0}$ is nonempty if and only if $p \geqslant q$.

Observe that the set $\mathbb{T}_{p\times q} := \left\{X \in \mathbb{C}^{p\times q}\colon\ X^*X = I_q\right\}$ is also nonempty if and only if $p \geqslant q$.

Lemma 21. *Let* $\mathscr{I}_{pq} : \mathbb{C}^{p\times q} \to \langle \mathscr{P}^{(p,q)}_{\square} \rangle$ *be defined by* $\mathscr{I}_{p,q}(A) := \langle [A, I_q] \rangle$.

(a) $\mathrm{Rstr.}_{\mathbb{K}_{p\times q}}\, \mathscr{I}_{pq}$ *is a bijective mapping of* $\mathbb{K}_{p\times q}$ *onto* $\langle \mathscr{P}^{(p,q)}_{-j_{pq},\geqslant} \rangle$.

(b) $\mathrm{Rstr.}_{\mathbb{D}_{p\times q}}\, \mathscr{I}_{pq}$ *is a bijective mapping of* $\mathbb{D}_{p\times q}$ *onto* $\langle \mathscr{P}^{(p,q)}_{-j_{pq},>} \rangle$.

(c) *If* $p \geqslant q$, *then* $\mathrm{Rstr.}_{\mathbb{T}_{p\times q}}\, \mathscr{I}_{pq}$ *is a bijective mapping of* $\mathbb{T}_{p\times q}$ *onto* $\langle \mathscr{P}^{(p,q)}_{-j_{pq},0} \rangle$.

Proof. For all $K \in \mathbb{C}^{p\times q}$, we have

$$\begin{pmatrix} K \\ I_q \end{pmatrix}^* (-j_{pq}) \begin{pmatrix} K \\ I_q \end{pmatrix} = I_q - K^*K. \tag{30}$$

In view of Remark 17, then it follows

$$\mathscr{I}_{pq}(\mathbb{K}_{p\times q}) \subseteq \langle \mathscr{P}^{(p,q)}_{-j_{pq},\geqslant} \rangle, \qquad \mathscr{I}_{pq}(\mathbb{D}_{p\times q}) \subseteq \langle \mathscr{P}^{(p,q)}_{-j_{pq},>} \rangle$$

and if $p \geqslant q$,

$$\mathscr{I}_{pq}(\mathbb{T}_{p\times q}) \subseteq \langle \mathscr{P}^{(p,q)}_{-j_{pq},0} \rangle.$$

Let $[A,B] \in \mathscr{P}^{(p,q)}_{-j_{pq},\geqslant}$. Then Lemma 19 shows that $K := AB^{-1}$ belongs to $\mathbb{K}_{p\times q}$. Thus $\mathscr{I}_{pq}(K) = \langle [AB^{-1}, I_q] \rangle = \langle [A,B] \rangle$. Hence $\langle \mathscr{P}^{(p,q)}_{-j_{pq},\geqslant} \rangle \subseteq \mathscr{I}_{pq}(\mathbb{K}_{p\times q})$. Analogous arguments yield $\langle \mathscr{P}^{(p,q)}_{-j_{pq},>} \rangle \subseteq \mathscr{I}_{pq}(\mathbb{D}_{p\times q})$ and, if $p \geqslant q$, $\langle \mathscr{P}^{(p,q)}_{-j_{pq},0} \rangle \subseteq \mathscr{I}_{pq}(\mathbb{T}_{p\times q})$. Since the mapping $\mathscr{I}_{pq}$ is obviously bijective the proof is complete. $\square$

Now we turn our attention to the $2q \times 2q$ signature matrices

$$J_q := \begin{pmatrix} 0 & -I_q \\ -I_q & 0 \end{pmatrix} \tag{31}$$

and $-J_q$. We will use $\mathscr{R}_q^{\geqslant}$ (respectively, $\mathscr{R}_q^{>}$) to denote the set of all matrices $A \in \mathbb{C}^{q\times q}$ the real part $\operatorname{Re} A := \frac{1}{2}(A + A^*)$ of which is nonnegative Hermitian (respectively, positive Hermitian). Further, let $\mathscr{R}_q^0 := \{A \in \mathbb{C}^{q\times q}\colon\ \operatorname{Re} A = 0_{q\times q}\}$.

Lemma 22. (a) *If* $[A,B] \in \mathscr{P}^{(q,q)}_{-J_q,\geqslant} \cap \mathscr{P}^{(q,q)}_{\square}$, *then* $R := AB^{-1}$ *belongs to* $\mathscr{R}_q^{\geqslant}$.

(b) *If* $[A,B] \in \mathscr{P}^{(q,q)}_{-J_q,>} \cap \mathscr{P}^{(p,q)}_{\square}$, *then* $R := AB^{-1}$ *belongs to* $\mathscr{R}_q^{>}$.

(c) *If* $[A,B] \in \mathscr{P}^{(q,q)}_{-J_q,0}$, *then* $R := AB^{-1}$ *belongs to* $\mathscr{R}_q^0$.

(d) *For each* $R \in \mathscr{R}_q^{\geqslant}$, *the pair* $[R, I_q]$ *belongs to* $\mathscr{P}^{(q,q)}_{-J_q,\geqslant} \cap \mathscr{P}^{(q,q)}_{\square}$.

(e) *For each* $R \in \mathscr{R}_q^{>}$, *the pair* $[R, I_q]$ *belongs to* $\mathscr{P}^{(q,q)}_{-J_q,>} \cap \mathscr{P}^{(q,q)}_{\square}$.

(f) *For each* $R \in \mathscr{R}_q^0$, *the pair* $[R, I_q]$ *belongs to* $\mathscr{P}^{(q,q)}_{-J_q,0} \cap \mathscr{P}^{(q,q)}_{\square}$.

Proof. Apply the identity

$$\begin{pmatrix} R \\ I_q \end{pmatrix}^* (-J_q) \begin{pmatrix} R \\ I_q \end{pmatrix} = 2 \operatorname{Re} R, \tag{32}$$

which holds true for all $q \times q$ complex matrices R. $\square$

Lemma 23. *Let* $\mathscr{I}_{qq} : \mathbb{C}^{q\times q} \to \langle \mathscr{P}^{q,q)}_{\square} \rangle$ *be defined by* $\mathscr{I}_{qq}(A) := \langle [A, I_q] \rangle$. *Then,*

(a) $\operatorname{Rstr.}_{\mathscr{R}_q^{\geqslant}} \mathscr{I}_{qq}$ *is a bijective mapping of* $\mathscr{R}_q^{\geqslant}$ *onto* $\langle \mathscr{P}^{(q,q)}_{-J_q,\geqslant} \cap \mathscr{P}^{(q,q)}_{\square} \rangle$.

(b) $\operatorname{Rstr.}_{\mathscr{R}_q^{>}} \mathscr{I}_{qq}$ *is a bijective mapping of* $\mathscr{R}_q^{>}$ *onto* $\langle \mathscr{P}^{(q,q)}_{-J_q,>} \cap \mathscr{P}^{(q,q)}_{\square} \rangle$.

(c) $\operatorname{Rstr.}_{\mathscr{R}_q^0} \mathscr{I}_{qq}$ *is a bijective mapping of* $\mathscr{R}_q^0$ *onto* $\langle \mathscr{P}^{(q,q)}_{-J_q,0} \cap \mathscr{P}^{(q,q)}_{\square} \rangle$.

Proof. Use Lemma 22 and Remark 17. □

Remark 24. For all $q \times q$ complex matrices A and B, the identity

$$\begin{pmatrix} A \\ B \end{pmatrix}^* (-J_q) \begin{pmatrix} A \\ B \end{pmatrix} = B^*A + A^*B \tag{33}$$

holds true. In particular $\mathscr{P}^{(q,q)}_{-J_q,\geqslant}$, $\mathscr{P}^{(q,q)}_{-J_q,>}$ and $\mathscr{P}^{(q,q)}_{-J_q,0}$ are nonempty sets.

Now we will continue to consider the mapping $\mathfrak{s}^{(q,q)}_E$ given in Eq. (22) in the case $p = q$ and $E = C_q$.

Lemma 25. (a) $\mathscr{P}^{(q,q)}_{-J_q,\geqslant} \subseteq \left[\mathscr{D}_{(1/\sqrt{2})I_q,(1/\sqrt{2})I_q}\right]$.

(b) *For each* $[A,B] \in \mathscr{P}^{(q,q)}_{-J_q,\geqslant}$, *the inequality* $\det(B+A) \neq 0$ *holds and* $\mathfrak{s}^{(q,q)}_{C_q}([A,B])$ *is a contractive matrix which satisfies*

$$\mathfrak{s}^{(q,q)}_{C_q}([A,B]) = (B-A)(B+A)^{-1}. \tag{34}$$

(c) *If* $[A,B] \in \mathscr{P}^{(q,q)}_{-J_q,>}$, *the* $\mathfrak{s}^{(q,q)}_{C_q}([A,B])$ *is strictly contractive.*

(d) *If* $[A,B] \in \mathscr{P}^{(q,q)}_{-J_q,0}$, *the* $\mathfrak{s}^{(q,q)}_{C_q}([A,B])$ *is unitary.*

(e) *Let* $K \in \mathbb{C}^{q\times q}$. *Then* $[I-K, I+K]$ *is a nondegenerate* (q,q)*-column pair which satisfies*

$$\mathfrak{s}^{(q,q)}_{C_q}([I-K, I+K]) = K. \tag{35}$$

If K *belongs to* $\mathbb{K}_{q\times q}$, *then* $[I-K, I+K]$ *is* $(-J_q)$*-nonnegative.*

(f) *For each* $K \in \mathbb{D}_{q\times q}$, *the nondegenerate* (q,q)*-column pair* $[I-K, I+K]$ *is* $(-J_q)$*-positive.*

(g) *For each* $K \in \mathbb{T}_{q\times q}$, *the nondegenerate* (q,q)*-column pair* $[I-K, I+K]$ *is* $(-J_q)$*-neutral.*

Proof. Let $[A,B] \in \mathscr{P}^{(q,q)}$. Then $C := A^*A + B^*B$ is positive Hermitian. In view of Remark 24, we have

$$(A+B)^*(A+B) = \begin{pmatrix} A \\ B \end{pmatrix}^* (-J_q) \begin{pmatrix} A \\ B \end{pmatrix} + C.$$

Thus, if $[A,B] \in \mathscr{P}^{(q,q)}_{-J_q,\geqslant}$, then $[A,B]$ obviously satisfies $\det(B+A) \neq 0$, $[A,B] \in \left[\mathscr{D}_{(1/\sqrt{2})I_q,(1/\sqrt{2})I_q}\right]$ and Eq. (34). For $Y := \mathfrak{s}^{(q,q)}_{C_q}([A,B])$ we obtain

$$I - Y^*Y = 2(B+A)^{-*}(B^*A + A^*B)(B+A)^{-1}.$$

In view of the identity (33), then parts (a)–(d) are proved. Now let $K \in \mathbb{C}^{q\times q}$. Then $(I-K)^*(I-K)+(I+K)^*(I+K)=2(I+K^*K)$ and

$$\begin{pmatrix} I-K \\ I+K \end{pmatrix}^* (-J_q) \begin{pmatrix} I-K \\ I+K \end{pmatrix} = 2(I-K^*K),$$

and the identity (35) follows easily by straightforward calculation. The rest is plain now. □

Now we will consider the mapping $\sigma_E^{(p,q)}$ defined by Eq. (23) in the case $p=q$ and $E=C_q$.

Theorem 26. (a) $\operatorname{Rstr.}_{\langle \mathscr{P}^{(q,q)}_{-J_q,\geqslant}\rangle}\, \sigma_{C_q}^{(q,q)}$ *is a bijective mapping of* $\langle \mathscr{P}^{(q,q)}_{-J_q,\geqslant}\rangle$ *onto* $\mathbb{K}_{q\times q}$.

(b) $\operatorname{Rstr.}_{\langle \mathscr{P}^{(q,q)}_{-J_q,>}\rangle}\, \sigma_{C_q}^{(q,q)}$ *is a bijective mapping of* $\langle \mathscr{P}^{(q,q)}_{-J_q,>}\rangle$ *onto* $\mathbb{D}_{q\times q}$.

(c) $\operatorname{Rstr.}_{\langle \mathscr{P}^{(q,q)}_{-J_q,0}\rangle}\, \sigma_{C_q}^{(q,q)}$ *is a bijective mapping of* $\langle \mathscr{P}^{(q,q)}_{-J_q,0}\rangle$ *onto* $\mathbb{T}_{q\times q}$.

Proof. Since the matrix C_q is nonsingular we see from Proposition 13 that is sufficient to show that the equations

$$\sigma_{C_q}^{(q,q)}\left(\langle \mathscr{P}^{(q,q)}_{-J_q,\geqslant}\rangle\right) = \mathbb{K}_{q\times q}, \qquad \sigma_{C_q}^{(q,q)}\left(\langle \mathscr{P}^{(q,q)}_{-J_q,>}\rangle\right) = \mathbb{D}_{q\times q},$$
$$\sigma_{C_q}^{(q,q)}\left(\langle \mathscr{P}^{(q,q)}_{-J_q,0}\rangle\right) = \mathbb{T}_{q\times q}$$

are valid. However, these identities follow immediately from Lemma 25. □

Corollary 27. *Let* $[A,B] \in \mathscr{P}^{(q,q)}_{-J_q,\geqslant}$. *Then* $K := \mathfrak{s}_{C_q}^{(q,q)}([A,B])$ *is a contractive matrix which satisfies* $\langle [A,B]\rangle = \langle [I-K, I+K]\rangle$.

Proof. Use Lemma 25 and Theorem 26. □

Lemma 28. (a) *Let* $[A,B] \in \mathscr{P}^{(q,q)}_{-J_q,\geqslant}$. *Then* $[A,B]$ *is proper if and only if* $K := \mathfrak{s}_{C_q}^{(q,q)}([A,B])$ *satisfies* $\det(I+K) \neq 0$.

(b) *Each* $(-J_q)$*-positive nondegenerate* (q,q)*-column pair is necessarily proper.*

Proof. (a) Use Corollary 27.

(b) Each strictly contractive matrix G satisfies necessarily $\det(I+G) \neq 0$. In view of part (c) of Lemma 25 and Corollary 27, thus the assertion stated in (b) follows. □

Theorem 29. $\operatorname{Rstr.}_{\langle \mathscr{P}^{(q,q)}_{-J_q,\geqslant} \cap \mathscr{P}^{(q,q)}_{\square}\rangle}\, \sigma_{C_q}^{(q,q)}$ *is a bijective mapping of* $\langle \mathscr{P}^{(q,q)}_{-J_q,\geqslant} \cap \mathscr{P}^{(q,q)}_{\blacksquare}\rangle$ *onto* $\mathbb{K}^{\blacksquare}_{q\times q} := \{K \in \mathbb{K}_{q\times q} \colon \det(I+K) \neq 0\}$.

Proof. The matrix C_q is nonsingular. Thus, in view of Proposition 13 and part (a) of Lemma 28 it is sufficient to show that

$$\mathbb{K}^{\blacksquare}_{q\times q} \subseteq \sigma^{(q,q)}_{C_q}\left(\langle \mathscr{P}^{(q,q)}_{-J_q,\geqslant} \cap \mathscr{P}^{(q,q)}_{\square}\rangle\right)$$

holds. However, this is clear since part (e) of Lemma 25 yields that, for each $K \in \mathbb{K}^{\blacksquare}_{q\times q}$, the pair $[I-K, I+K]$ belongs to $\mathscr{P}^{(q,q)}_{-J_q,\geqslant} \cap \mathscr{P}^{q,q}_{\square}$ and satisfies Eq. (35). □

Remark 30. Let G be a J_q-contractive $2q \times 2q$ complex matrix. Since $C_q J_q C_q = j_{qq}$ holds true, it is readily checked that $H := GC_q$ is a j_{qq}–J_q-contractive matrix. If

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \tag{36}$$

is the $q \times q$ block partition of G, then one can easily see from parts (a) and (b) of Lemma 8 in [8], that $\det(G_{21} + G_{22}) \neq 0$ and that $F = (G_{21} + G_{22})^{-1}(G_{22} - G_{21})$ belongs to $\mathbb{K}_{q\times q}$. If G is J_q-unitary, then F is unitary.

In the following, we will continue to use the $q \times q$ block partition (36) if a $2q \times 2q$ complex matrix G is given.

Theorem 31. *Let G be a J_q-contractive $2q \times 2q$ complex matrix. Then*

$$\sigma^{(q,q)}_{G}\left(\langle \mathscr{P}^{(q,q)}_{-J_q,\geqslant} \cap [\mathscr{D}_{G_{21},G_{22}}]\rangle\right) \subseteq \mathscr{R}^{\geqslant}_{q}, \tag{37}$$

$$\mathscr{P}^{(q,q)}_{-J_q,>} \subseteq [\mathscr{D}_{G_{21},G_{22}}] \text{ and}$$

$$\sigma^{(q,q)}_{G}\left(\langle \mathscr{P}^{(q,q)}_{-J_q,>}\rangle\right) \subseteq \mathscr{R}^{>}_{q}. \tag{38}$$

Proof. Let $[A,B] \in \mathscr{P}^{(q,q)}_{-J_q,\geqslant} \cap [\mathscr{D}_{G_{21},G_{22}}]$. Then $\det(G_{21}A + G_{22}B) \neq 0$ and $Y := \sigma^{(q,q)}_{G}(\langle [A,B]\rangle)$ satisfies

$$\begin{pmatrix} Y \\ I_q \end{pmatrix} = G\begin{pmatrix} A \\ B \end{pmatrix}(G_{21}A + G_{22}B)^{-1}.$$

This implies

$$\begin{aligned} Y + Y^* &= \begin{pmatrix} Y \\ I_q \end{pmatrix}^*(-J_q)\begin{pmatrix} Y \\ I_q \end{pmatrix} \\ &= (G_{21}A + G_{22}B)^{-*}\begin{pmatrix} A \\ B \end{pmatrix}^* G^*(-J_q)G\begin{pmatrix} A \\ B \end{pmatrix}(G_{21}A + G_{22}B)^{-1}. \end{aligned} \tag{39}$$

Because G is J_q-contractive, thus we obtain

$$\operatorname{Re} Y \geqslant \frac{1}{2}(G_{21}A + G_{22}B)^{-*}\begin{pmatrix} A \\ B \end{pmatrix}^*(-J_q)\begin{pmatrix} A \\ B \end{pmatrix}(G_{21}A + G_{22}B)^{-1}. \tag{40}$$

Since the nondegenerate (q,q)-column pair $[A,B]$ is $(-J_q)$-nonnegative, we then get $Y \in \mathscr{R}_q^{\geqslant}$. Hence Eq. (37) is verified. Now we assume that $[A,B]$ belongs to $\mathscr{P}^{(q,q)}_{-J_q,>}$. From Lemma 25 we see that $\det(B+A) \neq 0$ holds and that $K := (B-A)(B+A)^{-1}$ belongs to $\mathbb{D}_{q\times q}$. On the other hand, we know from Remark 30 that the matrix $H := GC_q$ is j_{qq}-J_q-contractive. Let

$$H := \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \tag{41}$$

be the $q \times q$ block partition of H. Then part (c) of Lemma 8 in [8] shows $\det(H_{21}K + H_{22}) \neq 0$. An easy calculation yields

$$H_{21}K + H_{22} = \sqrt{2}(G_{21}A + G_{22}B)(B+A)^{-1}. \tag{42}$$

Thus $\det(G_{21}A + G_{22}B) \neq 0$, i.e., $[A,B] \in [\mathscr{D}_{G_{21},G_{22}}]$ holds true. Hence $\mathscr{P}^{(q,q)}_{-J_q,>} \subseteq [\mathscr{D}_{G_{21},G_{22}}]$ is proved. Since the nondegenerate (q,q)-column pair $[A,B]$ is $(-J_q)$-positive, the inequality (40) implies $Y \in \mathscr{R}_q^{>}$. Consequently, Eq. (38) is proved as well. □

Corollary 32. *Let G be a J_q-contractive matrix. Then $\mathscr{R}_q^{>} \subseteq \mathscr{D}_{G_{21},G_{22}}$. The mapping $\mathscr{S}_G^{(q,q)} : \mathscr{D}_{G_{21},G_{22}} \to \mathbb{C}^{q\times q}$ defined by (5)–(7) satisfies $\mathscr{S}_G^{(q,q)}(\mathscr{R}_q^{\geqslant} \cap \mathscr{D}_{G_{21},G_{22}}) \subseteq \mathscr{R}_q^{\geqslant}$ and $\mathscr{S}_G^{(q,q)}(\mathscr{R}_q^{>}) \subseteq \mathscr{R}_q^{>}$.*

Proof. Let $R \in \mathscr{R}_q^{\geqslant} \cap \mathscr{D}_{G_{21},G_{22}}$. From part (d) of Lemma 22 and Remark 12 then we know that $[R, I_q] \in \mathscr{P}^{(q,q)}_{-J_q,\geqslant} \cap [\mathscr{D}_{G_{21},G_{22}}]$ and $\mathscr{S}_G^{(q,q)}(R) = \sigma_G^{(q,q)}(\langle [R, I_q] \rangle)$. Thus Theorem 31 provides $\mathscr{S}_G^{(q,q)}(R) \in \mathscr{R}_q^{\geqslant}$. Now let $R \in \mathscr{R}_q^{>}$. Part (e) of Lemma 22 then yields that $[R, I_q]$ belongs to $\mathscr{P}^{(q,q)}_{-J_q,>}$. Theorem 31 thus implies $[R, I_q] \in [\mathscr{D}_{G_{21},G_{22}}]$, i.e., $R \in \mathscr{D}_{G_{21},G_{22}}$, and finally $\mathscr{S}_G^{(q,q)}(R) \in \mathscr{R}_q^{>}$. □

Lemma 33. *Let G be a J_q-contractive matrix, and let $H := GC_q$. The following statements are equivalent*:

(i) $\mathscr{P}^{(q,q)}_{-J_q,\geqslant} \subseteq [\mathscr{D}_{G_{21},G_{22}}]$.
(ii) $g := (G_{21} + G_{22})^{-1}(G_{22} - G_{21})$ *is strictly contractive.*
(iii) $\mathbb{K}_{q\times q} \subseteq \mathscr{D}_{H_{21},H_{22}}$.
(iv) $h := H_{22}^{-1}H_{21}$ *is strictly contractive.*

Proof. Using Lemma 3 in [12] and Remark 30 we easily get that $G_{21} + G_{22}$ and H_{22} are nonsingular matrices and that (iii) holds true if and only if (iv) is valid. Since $g = h$ can be easily verified, the equivalence of (ii) and (iv) is trivial.

According to Lemma 25, for every pair $[A,B] \in \mathscr{P}^{(q,q)}_{-J_q,\geqslant}$, we have $\det(B+A) \neq 0$ and the matrix $K := (B-A)(B+A)^{-1}$ is contractive. Moreover, the identity (42) is fulfilled. Thus we see that (iii) implies $\det(G_{21}A + G_{22}B) \neq 0$ and consequently (i). It remains to check that (iii) is necessary for (i). Assume that (i) is fulfilled, and that K is an arbitrary element of $\mathbb{K}_{q\times q}$. By virtue of Theorem 26, then there is a unique $\langle [A,B] \rangle \in \langle \mathscr{P}^{(q,q)}_{-J_q,\geqslant} \rangle$ such that $\sigma^{(q,q)}_{C_q}(\langle [A,B] \rangle) = K$. Thus Lemma 25 shows $\det(B+A) \neq 0$ and $K = (B-A)(B+A)^{-1}$. Because of (i) then $\det(G_{21}A + G_{22}B) \neq 0$ is valid. Hence the identity (42) yields $K \in \mathscr{D}_{H_{21},H_{22}}$, i.e., (iii) holds true. $\square$

Remark 34. Let G be a J_q-unitary matrix. From Remark 30 and Lemma 33 one can easily see that there exists a pair $[A,B] \in \mathscr{P}^{(q,q)}_{-J_q,\geqslant}$ which does not belong to $[\mathscr{D}_{G_{21},G_{22}}]$.

Lemma 35. *If G is a J_q-unitary matrix, then* $\sigma^{(q,q)}_G\left(\langle \mathscr{P}^{(q,q)}_{-J_q,0} \cap [\mathscr{D}_{G_{21},G_{22}}] \rangle \right) \subseteq \mathscr{R}^0_q$.

Proof. Let $\langle [A,B] \rangle \in \mathscr{P}^{(q,q)}_{-J_q,0} \cap [\mathscr{D}_{G_{21},G_{22}}]$ and set $Y := \sigma^{(q,q)}_G(\langle [A,B] \rangle)$. Using the identity (39) we easily obtain $\operatorname{Re} Y = 0_{q\times q}$. $\square$

In the following, let

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \tag{43}$$

be $q \times q$ block partitions if $2q \times 2q$ complex matrices U or V are given. Observe that a result due to Potapov [21] (see also Theorem 1.6.1 in [6]) shows that if U is a j_{qq}-contractive matrix, then $\mathbb{K}_{q\times q} \subseteq \mathscr{D}_{U_{21},U_{22}}$ and $\mathscr{S}^{(q,q)}_U(\mathbb{K}_{q\times q}) \subseteq \mathbb{K}_{q\times q}$.

Lemma 36. *Let U be a j_{qq}-contractive matrix for which $\mathscr{S}^{(q,q)}_U(\mathbb{K}_{q\times q}) \subseteq \mathbb{D}_{q\times q}$ is satisfied. Then $V := C_q U$ is a j_{qq}–J_q-contractive matrix which fulfills $\mathbb{K}_{q\times q} \subseteq \mathscr{D}_{V_{21},V_{22}}$ and $\mathscr{S}^{(q,q)}_V(\mathbb{K}_{q\times q}) \subseteq \mathscr{R}^>_q$.*

Proof. Because C_q is J_q–j_{qq}-unitary, we immediately see that V is a j_{qq}–J_q-contractive matrix. Let $K \in \mathbb{K}_{q\times q}$. By assumption, then $Y := \mathscr{S}^{(q,q)}_U(\mathbb{K}_{q\times q})$ belongs to $\mathbb{D}_{q\times q}$. Since the matrix C_q is obviously j_{qq}–J_q-unitary, then part (c) of Lemma 8 in [8] yields that $X := \mathscr{S}^{(q,q)}_{C_q}(Y)$ is a well-defined matrix which belongs to $\mathscr{R}^>_q$. Using a further result on linear fractional transformations of matrices due to Potapov [21] (see also [6], Proposition 1.6.3), we finally obtain $K \in \mathscr{D}_{V_{21},V_{22}}$ and $X = \mathscr{S}^{(q,q)}_{C_q}(\mathscr{S}^{(q,q)}_U(K)) = \mathscr{S}^{(q,q)}_V(K)$. $\square$

Theorem 37. *Let U be a j_{qq}-contractive matrix for which $\mathscr{S}^{(q,q)}_U(\mathbb{K}_{q\times q}) \subseteq \mathbb{D}_{q\times q}$ is satisfied. Then $G := C_q U C_q$ is a J_q-contractive matrix which fulfills $\mathscr{P}^{(q,q)}_{-J_q,\geqslant} \subseteq [\mathscr{D}_{G_{21},G_{22}}]$ and $\sigma^{(q,q)}_G(\langle \mathscr{P}^{(q,q)}_{-J_q,\geqslant} \rangle) \subseteq \mathscr{R}^>_q$.*

Proof. Since C_q is a matrix which is J_q–j_{qq}-unitary and j_{qq}–J_q-unitary, it is readily checked that G is J_q-contractive. Let $[A, B] \in \mathscr{P}^{(q,q)}_{-J_q,\geqslant}$. Then Lemma 25 shows that $[A,B] \in [\mathscr{D}_{G_{21},G_{22}}]$ and that $K := \mathfrak{s}^{(q,q)}_{C_q}([A,B])$ belongs to $\mathbb{K}_{q\times q}$. If we set $V := C_qU$, then $G = VC_q$, and Lemma 36 provides $K \in \mathscr{D}_{V_{21},V_{22}}$ and $\mathscr{S}^{(q,q)}_V(K) \in \mathscr{R}^{>}_q$. Since Proposition 14 yields

$$\sigma^{(q,q)}_G(\langle[A,B]\rangle) = \mathfrak{s}^{(q,q)}_G([A,B]) = \mathscr{S}^{(q,q)}_V\left(\mathfrak{s}^{(q,q)}_{C_q}([A,B])\right) = \mathscr{S}^{(q,q)}_V(K),$$

the proof is complete. □

Remark 38. Let U be a j_{qq}-contractive matrix for which $\mathscr{S}^{(q,q)}_U(\mathbb{K}_{q\times q}) \subseteq \mathbb{D}_{q\times q}$ is satisfied. From Lemma 36 then it is clear that $V := C_qU$ is a j_{qq}–J_q-contractive matrix which fulfills $\mathbb{K}_{q\times q} \subseteq \mathscr{D}_{V_{21},V_{22}}$ and, in view of Lemma 3 in [12], the matrix $V_{22}^{-1}V_{21}$ belongs to $\mathbb{D}_{q\times q}$. Furthermore, Theorem 37 and Lemma 33 show that $G := C_qUC_q$ is a J_q-contractive matrix for which $(G_{21} + G_{22})^{-1}(G_{22} - G_{21})$ is strictly contractive.

Theorem 39. *Let W be a J_q–j_{qq}-unitary matrix, and let*

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \tag{44}$$

be the $q \times q$ block partition of W. Then $\mathscr{P}^{(q,q)}_{-J_q,\geqslant} \subseteq [\mathscr{D}_{W_{21},W_{22}}]$ and $\sigma^{(q,q)}_W$ is an injective mapping which satisfies

$$\sigma^{(q,q)}_W\left(\langle\mathscr{P}^{(q,q)}_{-J_q,\geqslant}\rangle\right) = \mathbb{K}_{q\times q},$$

and

$$\sigma^{(q,q)}_W\left(\langle\mathscr{P}^{(q,q)}_{-J_q,>}\rangle\right) = \mathbb{D}_{q\times q}, \qquad \sigma^{(q,q)}_W\left(\langle\mathscr{P}^{(q,q)}_{-J_q,0}\rangle\right) = \mathbb{T}_{q\times q}. \tag{45}$$

Proof. Each J_q–j_{qq}-unitary matrix is necessarily nonsingular. Thus Proposition 13 shows that $\sigma^{(q,q)}_W$ is injective. It is readily checked that $U := WC_q$ is a j_{qq}-unitary matrix which fulfills $W = UC_q$. From Lemma 25 we know that $\mathscr{P}^{(q,q)}_{-J_q,\geqslant}$ is a subset of $\left[\mathscr{D}_{(1/\sqrt{2})I_q,(1/\sqrt{2})I_q}\right]$. Further, we see from Theorem 26 that

$$\sigma^{(q,q)}_{C_q}\left(\langle\mathscr{P}^{(q,q)}_{-J_q,\geqslant}\rangle\right) = \mathbb{K}_{q\times q}, \qquad \sigma^{(q,q)}_{C_q}\left(\langle\mathscr{P}^{(q,q)}_{-J_q,>}\rangle\right) = \mathbb{D}_{q\times q} \quad \text{and}$$
$$\sigma^{(q,q)}_{C_q}\left(\langle\mathscr{P}^{(q,q)}_{-J_q,0}\rangle\right) = \mathbb{T}_{q\times q}.$$

Theorems 1.6.1 and 1.6.2 in [6] and their proofs show that $\mathbb{K}_{q\times q} \subseteq \mathscr{D}_{U_{21},U_{22}}$,

$$\mathscr{S}^{(q,q)}_U(\mathbb{K}_{q\times q}) = \mathbb{K}_{q\times q}, \qquad \mathscr{S}^{(q,q)}_U(\mathbb{D}_{q\times q}) = \mathbb{D}_{q\times q} \quad \text{and}$$
$$\mathscr{S}^{(q,q)}_U(\mathbb{T}_{q\times q}) = \mathbb{T}_{q\times q}.$$

Hence the application of Proposition 14 provides $\mathscr{P}^{(q,q)}_{-J_q} \subseteq [\mathscr{D}_{W_{21},W_{22}}]$ and

$$\sigma_W^{(q,q)}\left(\langle\mathscr{P}^{(q,q)}_{-J_q,\geqslant}\rangle\right) = \mathfrak{s}_W^{(q,q)}\left(\mathscr{P}^{(q,q)}_{-J_q,\geqslant}\right) = \mathscr{S}_U^{(q,q)}\left(\mathfrak{s}_{C_q}^{(q,q)}\left(\mathscr{P}^{(q,q)}_{-J_q,\geqslant}\right)\right)$$
$$= \mathscr{S}_U^{(q,q)}\left(\sigma_{C_q}^{(q,q)}\left(\langle\mathscr{P}^{(q,q)}_{-J_q,\geqslant}\rangle\right)\right) = \mathscr{S}_U^{(q,q)}(\mathbb{K}_{q\times q}) = \mathbb{K}_{q\times q}$$

and, analogously, the identities stated in Eq. (45). □

5. Meromorphic nondegenerate (p,q)-column pairs

The goal of this section is to generalize the results obtained in Section 3 to meromorphic matrix-valued functions.

Let G be a simply connected domain of the extended complex domain $\mathbb{C}_0 := \mathbb{C} \cup \{\infty\}$. The symbol $\mathscr{M}(G)$ (respectively, $\mathscr{H}(G)$) will be used to denote the set of all complex-valued functions which are meromorphic (respectively, holomorphic) in G. If $A \in [\mathscr{M}(G)]^{p\times q}$, then we will write $\mathbb{H}_A$ for the set of all points of analyticity of A (in G).

Definition 40. Let $A \in [\mathscr{M}(G)]^{p\times q}$, and let $B \in [\mathscr{M}(G)]^{q\times q}$. Then $[A,B]$ is called a *meromorphic nondegenerate (p,q)-column pair* in G, if the set $\mathscr{N}_{[A,B]}$ of all $z \in \mathbb{H}_A \cap \mathbb{H}_B$ for which $\det(A^*(z)A(z) + B^*(z)B(z)) = 0$ holds true is a discrete subset of G. A meromorphic nondegenerate (p,q)-column pair $[A,B]$ in G for which A belongs to $[\mathscr{H}(G)]^{p\times q}$ and B belongs to $[\mathscr{H}(G)]^{q\times q}$ is said to be a *holomorphic nondegenerate (p,q)-column pair in G*. A holomorphic nondegenerate (p,q)-column pair $[A,B]$ in G is called *complete* if $\mathscr{N}_{[A,B]} = \emptyset$.

We will continue to use the notation $\mathscr{N}_{[A,B]}$ given in Definition 40. Furthermore, let (p,q)-$\mathscr{M}\mathscr{P}(G)$ (respectively, (p,q)-$\mathscr{H}\mathscr{P}(G)$) the set of all meromorphic (respectively, holomorphic) nondegenerate (p,q)-column pairs in G. The set of all complete holomorphic nondegenerate (p,q)-column pairs will be denoted by (p,q)-$\mathscr{V}\mathscr{H}\mathscr{P}(G)$.

Remark 41. Let $A \in [\mathscr{M}(G)]^{p\times q}$ and $B \in [\mathscr{M}(G)]^{q\times q}$. If $\det B$ does not identically vanish in G, then $[A,B]$ belongs to (p,q)-$\mathscr{M}\mathscr{P}(G)$.

Remark 42. Let $A \in [\mathscr{H}(G)]^{p\times q}$ and $B \in [\mathscr{H}(G)]^{q\times q}$. If $\det B$ does not identically vanish in G, then $[A,B]$ belongs to (p,q)-$\mathscr{H}\mathscr{P}(G)$. If $\det B$ nowhere vanishes in G, then $[A,B] \in \mathscr{V}\mathscr{H}\mathscr{P}(G)$.

Lemma 43. *Let* $A \in [\mathscr{M}(G)]^{p\times q}$ *and* $B \in [\mathscr{M}(G)]^{q\times q}$. *Then the following statements are equivalent.*

(i) $[A,B] \in (p,q)$-$\mathscr{M}\mathscr{P}(G)$.

(ii) *There is a* $z_0 \in \mathbb{H}_A \cap \mathbb{H}_B$ *such that* $\det(A^*(z_0)A(z_0) + B^*(z_0)B(z_0)) \neq 0$.

Proof. One can easily see that (ii) is necessary for (i). To show that (ii) is sufficient for (i) we assume that (ii) is satisfied. If we set $C := \begin{pmatrix} A \\ B \end{pmatrix}$, then we see that $\operatorname{rank} C(z_0) = q$. Hence we can find p rows such that if we cancel these p rows of C, then we get a $q \times q$ matrix-valued function X which is meromorphic in G and which fulfills $\det X(z_0) \neq 0$. Therefore the set $\mathcal{N}_{\det X} := \{z \in \mathbb{H}_X : \det X(z) = 0\}$ is a discrete subset of G. For all $z \in (\mathbb{H}_A \cap \mathbb{H}_B) \setminus \mathcal{N}_{\det X}$, then $\det X(z) \neq 0$, i.e., $\operatorname{rank} C(z) = q$ holds true. Consequently, $\mathcal{N}_{[A,B]} \subseteq (G \setminus (\mathbb{H}_A \cap \mathbb{H}_B)) \cup \mathcal{N}_{\det X}$. Thus we can conclude that (i) is valid. □

Remark 44. (a) Let $Q \in [\mathcal{M}(G)]^{q \times q}$ be such that $\det Q$ does not identically vanish in G, and $[A, B] \in (p, q)\text{-}\mathcal{MP}(G)$. Then it is readily checked that $[AQ, BQ]$ also belongs to $(p, q)\text{-}\mathcal{MP}(G)$.

(b) Let $Q \in [\mathcal{H}(G)]^{q \times q}$ and $[A, B] \in (p, q)\text{-}\mathcal{NHP}(G)$. If $\det Q$ nowhere vanishes in G, then $[AQ, BQ]$ belongs to $(p, q)\text{-}\mathcal{NHP}(G)$.

Definition 45. Pairs $[A_1, B_1]$ and $[A_2, B_2]$ which belong to $(p, q)\text{-}\mathcal{MP}(G)$ are said to be *equivalent* if there exists a function $Q \in [\mathcal{M}(G)]^{q \times q}$ for which the following two statements are satisfied.

(i) The function $\det Q$ does not identically vanish in G.

(ii) The identities $A_2 = A_1 Q$ and $B_2 = B_1 Q$ are fulfilled.

One can easily see that the relation given in Definition 45 is a equivalence relation on $(p, q)\text{-}\mathcal{MP}(G)$. If $[A, B] \in (p, q)\text{-}\mathcal{MP}(G)$, then we will write $\langle [A, B] \rangle$ for the equivalence class of all pairs $[C, D] \in (p, q)\text{-}\mathcal{MP}(G)$ which are equivalent to $[A, B]$. Further, we set

$$\langle (p, q)\text{-}\mathcal{MP}(G) \rangle := \{\langle [A, B] \rangle : [A, B] \in (p, q)\text{-}\mathcal{MP}(G)\}.$$

Definition 46. A meromorphic nondegenerate (p, q)-column pair $[A, B]$ in G is said to be *proper* if the function $\det B$ does not identically vanish in G.

Remark 47. If $[A, B]$ is a proper meromorphic nondegenerate (p, q)-column pair in G, then every meromorphic nondegenerate (p, q)-column pair in G which is equivalent to $[A, B]$ is proper as well.

In view of Remark 47 it makes sense to speak of equivalence classes of proper meromorphic nondegenerate (p, q)-column pairs in G. We will write $(p, q)\text{-}\mathcal{MP}_{\square}(G)$ for the set of all proper meromorphic nondegenerate (p, q)-column pairs in G. Moreover, then $\langle (p, q)\text{-}\mathcal{MP}_{\square}(G) \rangle$ designates the set of all equivalence classes $\langle [A, B] \rangle$ for which $[A, B]$ belongs to $(p, q)\text{-}\mathcal{MP}_{\square}(G)$.

Remark 48. Let $[A,B]$ be a proper meromorphic nondegenerate (p,q)-column pair in G. Then $[AB^{-1},I_q]$ is a proper meromorphic nondegenerate (p,q)-column pair which is equivalent to $[A,B]$.

Lemma 49. *The mapping* $\mathscr{I}_{pq;G}: [\mathscr{M}(G)]^{p\times q} \to \langle (p,q)\text{-}\mathscr{M}\mathscr{P}_{\square}(G)\rangle$ *given by* $\mathscr{I}_{pq;G}(A) := \langle [A,I_q]\rangle$ *is bijective.*

Proof. The injectivity of $\mathscr{I}_{pq,G}$ is obvious. If $[A,B]$ is an arbitrary element of $(p,q)\text{-}\mathscr{M}\mathscr{P}_{\square}(G)$, then we know from Remark 48 that $\mathscr{I}_{pq;G}(AB^{-1}) = \langle [A,B]\rangle$. Hence $\mathscr{I}_{pq;G}$ is surjective as well. □

In the following, if a $(p+q)\times(p+q)$ matrix-valued function W is given, then we will work with the block partition (44) of W where W_{11} is a $p\times p$ block. If $W\in[\mathscr{M}(G)]^{(p+q)\times(p+q)}$ and if $[A,B]\in(p,q)\text{-}\mathscr{M}\mathscr{P}(G)$, then let

$$\mathscr{M}_{W;[A,B]} := \{z\in \mathbb{H}_W\cap\mathbb{H}_A\cap\mathbb{H}_B\colon [A(z),B(z)]\notin[\mathscr{D}_{W_{21}(z),W_{22}(z)}]\}. \tag{46}$$

The set $\mathscr{M}_{W;[A,B]}$ is a discrete subset of G if and only if the function $\det(W_{21}A+W_{22}B)$ does not identically vanish in G. In this case, we set $\mathfrak{s}^{(p,q)}_{W;[A,B]} := (W_{11}A+W_{12}B)(W_{21}A+W_{22}B)^{-1}$, i.e., $\mathfrak{s}^{(p,q)}_{W;[A,B]}$ is the unique function which belongs to $[\mathscr{M}(G)]^{p\times q}$ and which admits the representation

$$\mathfrak{s}^{(p,q)}_{W;[A,B]}(z) = (W_{11}(z)A(z)+W_{12}(z)B(z))(W_{21}(z)A(z)+W_{22}(z)B(z))^{-1} \tag{47}$$

for all $z\in\mathbb{H}_W\cap\mathbb{H}_A\cap\mathbb{H}_B\setminus\mathscr{M}_{W;[A,B]}$.

Lemma 50. *Let* $W\in[\mathscr{M}(G)]^{(p+q)\times(p+q)}$ *and* $[A_1,B_1]\in(p,q)\text{-}\mathscr{M}\mathscr{P}(G)$ *be such that* $\mathscr{M}_{W;[A_1,B_1]}$ *is a discrete subset of G. Further, let* $[A_2,B_2]\in(p,q)\text{-}\mathscr{M}\mathscr{P}(G)$ *be such that* $\langle[A_1,B_1]\rangle=\langle[A_2,B_2]\rangle$. *Then* $\mathscr{M}_{W;[A_2,B_2]}$ *is also a discrete subset of G and* $\mathfrak{s}^{(p,q)}_{W;[A_1,B_1]}=\mathfrak{s}^{(p,q)}_{W;[A_2,B_2]}$ *holds true.*

Proof. By assumption there is a $Q\in[\mathscr{M}(G)]^{q\times q}$ such that $\det Q$ does not identically vanish in G and that the identities $A_2=A_1Q$ and $B_2=B_1Q$ are satisfied. Let

$$\begin{aligned}H := {}&\mathbb{H}_W\cap\big((\mathbb{H}_{A_1}\cap\mathbb{H}_{B_1})\setminus\mathscr{N}_{[A_1,B_1]}\big)\cap\big((\mathbb{H}_{A_2}\cap\mathbb{H}_{B_2})\setminus\mathscr{N}_{[A_2,B_2]}\big)\\ &\cap\{z\in\mathbb{H}_Q\colon \det Q(z)\neq 0\}.\end{aligned}$$

Then $G\setminus H$ and $N := \{z\in H\colon [A_1(z),B_1(z)]\notin[\mathscr{D}_{W_{21}(z),W_{22}(z)}]\}$ are obviously discrete subsets of G. For each $z\in H\setminus N$, the nondegenerate (p,q)-column pairs $[A_1(z),B_1(z)]$ and $[A_2(z),B_2(z)]$ are equivalent, and $[A_1(z),B_1(z)]$ belongs to $[\mathscr{D}_{W_{21}(z),W_{22}(z)}]$. Applying Lemma 11 we obtain that, for all $z\in H\setminus N$, the pair $[A_2(z),B_2(z)]$ belongs to $[\mathscr{D}_{W_{21}(z),W_{22}(z)}]$ as well, and that

$$\mathfrak{s}^{(p,q)}_{W(z)}([A_1(z),B_1(z)]) = \mathfrak{s}^{(p,q)}_{W(z)}([A_2(z),B_2(z)]) \tag{48}$$

is satisfied. From $H \setminus N \subseteq G \setminus \mathcal{M}_{W,[A_2,B_2]}$ and $G \setminus (H \setminus N) = (G \setminus H) \cup N$ we get $\mathcal{M}_{W;[A_2,B_2]} \subseteq (G \setminus H) \cup N$, where $(G \setminus H) \cup N$ is a discrete subset of G. Hence $\mathcal{M}_{W,[A_2,B_2]}$ is a discrete subset of G. Thus we see from Eq. (48) that $\mathfrak{s}^{(p,q)}_{W;[A_1,B_1]}(z) = \mathfrak{s}^{(p,q)}_{W;[A_2,B_2]}(z)$ is fulfilled for all $z \in H \setminus N$. Consequently, the functions $\mathfrak{s}^{(p,q)}_{W;[A_1,B_1]}$ and $\mathfrak{s}^{(p,q)}_{W;[A_2,B_2]}$ coincide. $\square$

Let $W \in [\mathcal{M}(G)]^{(p+q)\times(p+q)}$. Then, in view of Lemma 50 we will use $\langle(\mathscr{D}_{W_{21},W_{22}})\rangle$ to denote that set of all $\langle[A,B]\rangle \in \langle(p,q)\text{-}\mathcal{MP}(G)\rangle$ for which the set $\mathcal{M}_{W,[A,B]}$ given by Eq. (46) is a discrete subset of G. Further, for each $\langle[A,B]\rangle \in \langle(\mathscr{D}_{W_{21},W_{22}})\rangle$, let $\sigma^{(p,q)}_{W;\langle[A,B]\rangle}$ be that $[\mathcal{M}(G)]^{p\times q}$-function which is given by

$$\sigma^{(p,q)}_{W;\langle[A,B]\rangle}(z) = (W_{11}A + W_{12}B)(W_{21}A + W_{22}B)^{-1}. \tag{49}$$

6. *J*-Nonnegative meromorphic nondegenerate (p,q)-column pairs

Now we will extend the results of Section 4 to meromorphic matrix-valued functions.

Definition 51. Let J be a Hermitian $(p+q)\times(p+q)$ complex matrix and $[A,B] \in (p,q)\text{-}\mathcal{MP}(G)$.

(a) The pair $[A,B]$ is called *J-nonnegative* if the matrix

$$C(z) := \begin{pmatrix} A(z) \\ B(z) \end{pmatrix}^* J \begin{pmatrix} A(z) \\ B(z) \end{pmatrix} \tag{50}$$

is nonnegative Hermitian for all $z \in \mathbb{H}_A \cap \mathbb{H}_B$.

(b) The pair $[A,B]$ is said to be *J-positive* if there is a discrete subset $\mathscr{D}_{[A,B]}$ of $\mathbb{H}_A \cap \mathbb{H}_B$ such that $C(z)$ is positive Hermitian for all $z \in \mathbb{H}_A \cap \mathbb{H}_B \setminus \mathscr{D}_{[A,B]}$.

(c) The pair $[A,B]$ is called *J-neutral* if $C(z) = 0_{q\times q}$ for each $z \in \mathbb{H}_A \cap \mathbb{H}_B$.

Let us use $(p,q)\text{-}\mathscr{P}_{J,\geqslant}(G)$ (respectively, $(p,q)\text{-}\mathscr{P}_{J,>}(G)$) to denote the set of all J-nonnegative (respectively, J-positive) meromorphic nondegenerate (p,q)-column pairs in G. Further, we will write $(p,q)\text{-}\mathscr{P}_{J,0}(G)$ for the set of all pairs which belong to $(p,q)\text{-}\mathcal{MP}(G)$ and which are J-neutral.

Remark 52. Let J be a Hermitian $(p+q)\times(p+q)$ complex matrix, and let $[A_1,B_1] \in (p,q)\text{-}\mathcal{MP}(G)$ and $[A_2,B_2] \in (p,q)\text{-}\mathcal{MP}(G)$ be such that $\langle[A_1,B_1]\rangle = \langle[A_2,B_2]\rangle$. Then it is readily checked that the following equivalences hold true.

(a) $[A_1,B_1] \in (p,q)\text{-}\mathscr{P}_{J,\geqslant}(G)$ if and only if $[A_2,B_2] \in (p,q)\text{-}\mathscr{P}_{J,\geqslant}(G)$.

(b) $[A_1,B_1] \in (p,q)\text{-}\mathscr{P}_{J,>}(G)$ if and only if $[A_2,B_2] \in (p,q)\text{-}\mathscr{P}_{J,>}(G)$.

(c) $[A_1,B_1] \in (p,q)\text{-}\mathscr{P}_{J,0}(G)$ if and only if $[A_2,B_2] \in (p,q)\text{-}\mathscr{P}_{J,0}(G)$.

In view of Remark 52 it is meaningful to introduce the notation

$$\langle (p,q)\text{-}\mathscr{P}_{J,\geqslant}(G)\rangle := \{\langle [A,B]\rangle :\ [A,B] \in (p,q)\text{-}\mathscr{P}_{J,\geqslant}(G)\}.$$

Furthermore, we will use the notations $\langle (p,q)\text{-}\mathscr{P}_{J,>}(G)\rangle$ and $\langle (p,q)\text{-}\mathscr{P}_{J,0}(G)\rangle$, which are analogously defined.

Now we again specify the Hermitian matrix J. We will turn our attention to the $2q \times 2q$ signature matrix J_q which is given by Eq. (31).

Proposition 53. (a) *For each* $[A,B] \in (q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(G)$, *the equivalence class* $\langle [A,B]\rangle$ *belongs to* $\langle (\mathscr{D}_{(1/\sqrt{2})I_q,(1/\sqrt{2})I_q})\rangle$, *and* $\sigma^{(q,q)}_{C_q;\langle [A,B]\rangle}$ *is a function which belongs to the Schur class* $\mathscr{S}_{q\times q}(G)$ *and admits the representation* $\sigma^{(q,q)}_{C_q;\langle [A,B]\rangle} = (B-A)(B+A)^{-1}$.

(b) *If* S *is an arbitrary function which belongs to* $\mathscr{S}_{q\times q}(G)$, *then* $[I-S, I+S] \in (q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(G) \cap (q,q)\text{-}\mathscr{V}\mathscr{H}\mathscr{P}(G)$ *and*

$$\sigma^{(q,q)}_{C_q;\langle [I-S,I+S]\rangle} = S. \tag{51}$$

(c) *For each* $[A,B] \in (q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(G)$, *the pair* $[I - \sigma^{(q,q)}_{C_q;\langle [A,B]\rangle}, I + \sigma^{(q,q)}_{C_q;\langle [A,B]\rangle}]$ *belongs to* $(q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(G) \cap (q,q)\text{-}\mathscr{V}\mathscr{H}\mathscr{P}(G)$, *and the identity*

$$\langle [A,B]\rangle = \left\langle \left[I - \sigma^{(q,q)}_{C_q;\langle [A,B]\rangle}, I + \sigma^{(q,q)}_{C_q;\langle [A,B]\rangle}\right]\right\rangle$$

holds true.

Proof. (a) Let $[A,B] \in (q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(G)$. From Lemma 25 then we see that, for all $z \in \mathscr{F}_{[A,B]} := (\mathbb{H}_A \cap \mathbb{H}_B) \setminus \mathscr{N}_{[A,B]}$, the pair $[A(z), B(z)]$ belongs to $[\mathscr{D}_{(1/\sqrt{2})I_q,(1/\sqrt{2})I_q}]$ and satisfies $\det(B(z)+A(z)) \neq 0$, and the function $f : \mathscr{F}_{[A,B]} \to \mathbb{C}^{q\times q}$ given by $f(z) := (B(z)-A(z))(B(z)+A(z))^{-1}$ satisfies $f(z) \in \mathbb{K}_{q\times q}$ for all $z \in \mathscr{F}_{[A,B]}$. Since $G \setminus \mathscr{F}_{[A,B]}$ is a discrete subset of G, Riemann's theorem on removable isolated singularities of a bounded meromorphic function shows that f can be extended to a function that belongs to $\mathscr{S}_{q\times q}(G)$. Since $\sigma^{(q,q)}_{C_q;\langle [A,B]\rangle}(z) = f(z)$ is obviously fulfilled for all $z \in \mathscr{F}_{[A,B]}$, part (a) is verified.

(b) Let $S \in \mathscr{S}_{q\times q}(G)$. According to part (e) of Lemma 25, for all $z \in G$ the nondegenerate (q,q)-column pair $[I-S(z), I+S(z)]$ belongs to $(q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(G) \cap (q,q)\text{-}\mathscr{V}\mathscr{H}\mathscr{P}(G)$ and satisfies the equation $\mathfrak{s}^{(q,q)}_{C_q}([I-S(z), I+S(z)]) = S(z)$. According to (a), part (b) is also proved.

(c) Let $[A,B] \in (q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(G)$. In view of the proof of part (a), for all $z \in \mathscr{F}_{[A,B]}$, we have $\det(B(z)+A(z)) \neq 0$ and

$$\begin{pmatrix} I - \sigma^{(q,q)}_{C_q;\langle [A,B]\rangle}(z) \\ I + \sigma^{(q,q)}_{C_q;\langle [A,B]\rangle}(z) \end{pmatrix} = \begin{pmatrix} A(z) \\ B(z) \end{pmatrix} Q(z),$$

where $Q := 2(B + A)^{-1}$. Thus the assertion stated in part (c) follows from (a) and (b). $\square$

Theorem 54. *By the formula*

$$\Xi(\langle[A, B]\rangle) := \sigma^{(q,q)}_{C_q;\langle[A,B]\rangle} \tag{52}$$

a well-defined and bijective mapping $\Xi : \langle(q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(G)\rangle \to \mathscr{S}_{q\times q}(G)$ *is given.*

Proof. In view of parts (a) and (b) of Proposition 53, we see that the mapping Ξ is well-defined and surjective. On the other hand, the application of part (c) of Proposition 53 provides that Ξ is one-to-one. $\square$

Remark 55. Let $\Xi : \langle(q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(G)\rangle \to \mathscr{S}_{q\times q}(G)$ be defined by Eq. (52). Then part (b) of Proposition 53 yields immediately that the inverse mapping $\Xi^{[-1]}$ of Ξ admits the representation $\Xi^{[-1]}(g) = \langle[I - g, I + g]\rangle$ for all $g \in \mathscr{S}_{q\times q}(G)$.

Lemma 56. *Let* $[A, B] \in (q,q)\text{-}\mathscr{P}_{-J_q,>}(G)$. *Then,*

(a) *The meromorphic nondegenerate* (q,q)*-column pair* $[A, B]$ *is necessarily proper.*

(b) *The function* $\Omega := AB^{-1}$ *belongs to the Carathéodory class* $\mathscr{C}_q(G)$. *For all* $z \in G$, *the matrix* $\Omega(z)$ *belongs to* $\mathscr{K}_{(1/\sqrt{2})I_q,(1/\sqrt{2})I_q}$, *and its real part* Re $\Omega(z)$ *is positive Hermitian.*

(c) *The* $q \times q$ *Schur function* $\sigma^{(q,q)}_{C_q;\langle[A,B]\rangle}$ *is strictly contractive and admits the representation* $\sigma^{(q,q)}_{C_q;\langle[A,B]\rangle} = \mathscr{S}^{(q,q)}_{[C_q]}(\Omega)$.

Proof. By assumption there is a discrete subset $\mathscr{L}_{[A,B]}$ of $\mathbb{H}_A \cap \mathbb{H}_B$ such that the matrix $C(z)$ given by Eq. (50) is positive Hermitian for all $z \in \mathbb{H}_A \cap \mathbb{H}_B \setminus \mathscr{L}_{[A,B]}$. Let $\mathscr{F} := \mathbb{H}_A \cap \mathbb{H}_B \setminus (\mathscr{L}_{[A,B]} \cup \mathscr{N}_{[A,B]})$. Part (b) of Lemma 28 then shows that $[A(z), B(z)]$ is a proper nondegenerate (q,q)-column pair for each $z \in \mathscr{F}$. Hence Ω is a well-defined function which belongs to $[\mathscr{M}(G)]^{q\times q}$ and, in view of part (b) of Lemma 22, which satisfies $\Omega(z) \in \mathscr{R}^{>}_q$ for all $z \in \mathscr{F}$. Since $G \setminus \mathscr{F}$ is a discrete subset of G, we can conclude that Ω belongs to $\mathscr{C}_q(G)$ (see, e.g., [6], Lemma 2.1.9). Moreover, we see from Proposition 2.1.3 and part (b) of Lemma 1.1.13 in [6] that $\Omega(z) \in \mathscr{R}^{>}_q$ and $\det(I + \Omega(z) \neq 0$ are satisfied for all $z \in G$. Since $\langle[A, B]\rangle = \langle[\Omega, I_q]\rangle$ is valid, it follows

$$\sigma^{(q,q)}_{C_q;\langle[A,B]\rangle} = \sigma^{(q,q)}_{C_q;\langle[\Omega,I_q]\rangle} = \mathscr{S}^{(q,q)}_{[C_q]}(\Omega). \tag{53}$$

Part (c) of Proposition 2.1.3 in [6] yields that the $q \times q$ Schur function $\sigma^{(q,q)}_{C_q;\langle[A,B]\rangle}$ in G is even strictly contractive. $\square$

Remark 57. Part (f) of Lemma 25 and part (b) of Lemma 28 show that, for every strictly contractive $q \times q$ Schur function S in G, the pair $[I - S, I + S]$ is a proper and $(-J_q)$-positive holomorphic nondegenerate (q,q)-column pair in G.

Lemma 58. (a) *If* $[A, B]$ *belongs to* (q,q)-$\mathscr{P}_{-J_q,0}(G)$, *then the function* $\sigma^{(q,q)}_{C_q;\langle [A,B]\rangle}$ *is constant with unitary value.*

(b) *If* $S : G \to \mathbb{C}^{q \times q}$ *is a constant function with unitary value, then* $[I - S, I + S]$ *belongs to* (q,q)-$\mathscr{P}_{-J_q,0}(G)$.

Proof. (a) Let $[A, B]$ belong to (q,q)-$\mathscr{P}_{-J_q,0}(G)$. For each $z \in (\mathbb{H}_A \cap \mathbb{H}_B) \setminus \mathcal{N}_{[A,B]}$, then $[A(z), B(z)]$ is a $(-J_q)$-neutral nondegenerate (q,q)-column pair and, by virtue of part (c) of Theorem 26, $\sigma^{(q,q)}_{C_q;\langle [A,B]\rangle}(z)$ is unitary. From part (a) of Proposition 53 we know that $\sigma^{(q,q)}_{C_q;\langle [A,B]\rangle}$ belongs to $\mathscr{S}_{q\times q}(G)$. Thus Lemma 2.1.4 in [6] implies that $\sigma^{(q,q)}_{C_q;\langle [A,B]\rangle}$ is a constant function (with unitary value).

(b) Use part (g) of Lemma 25. □

Remark 59. Let $[A, B]$ belongs to (q,q)-$\mathscr{P}_{-J_q,\geqslant}(G)$. Then part (c) of Proposition 53 shows that $[A, B]$ is proper if and only if $\det(I + \sigma^{(q,q)}_{C_q;\langle [A,B]\rangle})$ does not identically vanish in G. Since $\sigma^{(q,q)}_{C_q;\langle [A,B]\rangle}$ belongs to $\mathscr{S}_{q\times q}(G)$ (see part (a) of Proposition 53), Lemma 2.1.7 in [6] provides then that $[A, B]$ is proper if and only if $\det(I + \sigma^{(q,q)}_{C_q;\langle [A,B]\rangle})$ nowhere vanishes in G.

Theorem 60. *Let* $\Xi : \langle (q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(G)\rangle \to \mathscr{S}_{q\times q}(G)$ *be defined by* (52). *Then,*

(a) $\mathrm{Rstr.}_{\langle (q,q)\text{-}\mathscr{P}_{-J_q,>}(G)\rangle}\, \Xi$ *is a bijective mapping of* $\langle (q,q)\text{-}\mathscr{P}_{-J_q,>}(G)\rangle$ *onto the set of all strictly contractive* $q \times q$ *Schur functions in* G.

(b) $\mathrm{Rstr.}_{\langle (q,q)\text{-}\mathscr{P}_{-J_q,0}(G)\rangle}\, \Xi$ *is a bijective mapping of* $\langle (q,q)\text{-}\mathscr{P}_{-J_q,0}(G)\rangle$ *onto the set of all constant* $q \times q$ *Schur functions in* G *with unitary value.*

Proof. Theorem 54 shows that Ξ is one-to-one. Then the assertion stated in part (a) follows from part (c) of Lemma 56, Remark 57 and part (b) of Proposition 53. Finally, part (b) is an immediate consequence of Lemma 58 and part (b) of Proposition 53. □

Theorem 61. *Let* $\Xi : \langle (q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(G)\rangle \to \mathscr{S}_{q\times q}(G)$ *be defined by* (54), *and let* (q,q)-$\mathscr{P}^{\square}_{-J_q,\geqslant}(G)$ *be the set of all proper pairs which belong to* (q,q)-$\mathscr{P}_{-J_q,\geqslant}(G)$. *Then* $\mathrm{Rstr.}_{\langle (q,q)\text{-}\mathscr{P}^{\square}_{-J_q,\geqslant}(G)\rangle}\, \Xi$ *is a bijective mapping of* $\langle (q,q)\text{-}\mathscr{P}^{\square}_{-J_q,\geqslant}(G)\rangle$ *onto the set of all* $q \times q$ *Schur functions* S *in* G *for which* $\det(I + S)$ *nowhere vanishes in* G.

Proof. Use Theorem 54, Remark 59 and part (b) of Proposition 53. □

7. Various types of resolvent matrices

In this section, we will specify the simply connected domain G. We will consider matrix-valued functions which are meromorphic in the open unit disc $\mathbb{D} := \{z \in \mathbb{C}: |z| < 1\}$. The main goal of this section is to "translate" different types of (right) resolvent matrices of the matricial Carathéodory problem. Hereby, if a function $W \in [\mathscr{M}(G)]^{2q\times 2q}$ with $q \times q$ block partition (44) and an equivalence class $\langle [A,B] \rangle \in \langle (\mathscr{D}_{W_{21},W_{22}}) \rangle$ are given, then we will again use the symbol $\sigma^{(q,q)}_{W;\langle [A,B] \rangle}$ for the function introduced by Eq. (51). Furthermore, we observe that the matrix C_q given by Eq. (18) satisfies $C_q^2 = I_{2q}, C_q^* = C_q$ and $C_q j_{qq} C_q = J_q$.

Remember that type I resolvent matrices for nondegenerate Carathéodory problems were introduced in Definition 2.

Definition 62. Let $(\Gamma_k)_{k=0}^n$ be a nondegenerate $q \times q$ Carathéodory sequence. A $2q \times 2q$ complex matrix-valued function $\triangle$ is called a *right type II resolvent matrix of the Carathéodory problem with respect to* $(\Gamma_k)_{k=0}^n$ if the following three conditions are satisfied.

(i) The function $\det \triangle$ does not identically vanish in $\mathbb{D}$.

(ii) If Eq. (14) is the $q \times q$ block partition of $\triangle$, then

$$\langle (q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(\mathbb{D}) \rangle \subseteq \langle (\mathscr{D}_{\triangle_{21},\triangle_{22}}) \rangle. \tag{54}$$

(iii) The mapping $\Phi_\triangle : \langle (q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(\mathbb{D}) \rangle \to [\mathscr{M}(\mathbb{D})]^{q\times q}$ given by

$$\Phi_\triangle(\langle [A,B] \rangle) := \sigma^{(q,q)}_{\triangle;\langle [A,B] \rangle} \tag{55}$$

fulfills

$$\Phi_\triangle\big(\langle (q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(\mathbb{D}) \rangle\big) = \mathscr{C}_q[\Gamma_0, \Gamma_1, \ldots, \Gamma_n]. \tag{56}$$

Recall the fact which was already stated in Remark 4 that the Cayley transform of a nondegenerate $q \times q$ Carathéodory sequence is a nondegenerate $q \times q$ Schur sequence.

Now we are able to prove the main theorem of this section. It describes interrelations between different types of resolvent matrices connected with nondegenerate Carathéodory or Schur interpolation problems.

Theorem 63. *Let $(\Gamma_k)_{k=0}^n$ be a nondegenerate $q \times q$ Carathéodory sequence, and let $(A_k)_{k=0}^n$ be the Cayley transform of $(\Gamma_k)_{k=0}^n$. Further, let $D, \triangle$ and $\bigtriangledown$ be $2q \times 2q$ complex matrix-valued functions which are meromorphic in $\mathbb{D}$ and which satisfy the identities*

$$\triangle = C_q D C_q \quad \text{and} \quad \bigtriangledown = C_q D. \tag{57}$$

Then the following statements are equivalent.

(i) ∇ *is a right type I resolvent matrix of the Carathéodory problem with respect to* $(\Gamma_k)_{k=0}^n$.

(ii) Δ *is a right type II resolvent matrix of the Carathéodory problem with respect to* $(\Gamma_k)_{k=0}^n$.

(iii) D *is a right resolvent matrix of the Schur problem with respect to* $(A_k)_{k=0}^n$.

Proof. First we observe that Lemma 5 shows that Eqs. (16) and (17) hold true. We know from Theorem 54 that the mapping $\Xi : \langle (q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(\mathbb{D})\rangle \to \mathscr{S}_{q\times q}(\mathbb{D})$ defined by Eq. (52) is bijective. Further, because of $C_q^2 = I_{2q}$ we get from Eq. (59) that

$$\nabla C_q = \Delta \quad \text{and} \quad D = C_q \, \Delta \, C_q \tag{58}$$

are satisfied. Let ∇, Δ and D be partitioned into $q \times q$ blocks via Eqs. (14) and (15).

(i) $\Rightarrow$ (ii): Let (i) be satisfied. Then it follows from Eq. (57) that $\det \nabla$ does not identically vanish in $\mathbb{D}$. Moreover, we see from (i) and Theorem 54 that, for each pair $[A,B] \in (q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(\mathbb{D})$, the function $\Omega := \mathscr{S}^{(q,q)}_{[\nabla]}(\Xi(\langle [A,B]\rangle))$ belongs to $\mathscr{C}_q[\Gamma_0,\Gamma_1,\ldots,\Gamma_n]$ and, conversely, for each $\psi \in \mathscr{C}_q[\Gamma_0,\Gamma_1,\ldots,\Gamma_n]$, there is a pair $[\tilde{A},\tilde{B}] \in (q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(\mathbb{D})$ for which the representation $\psi = \mathscr{S}^{(q,q)}_{[\nabla]}\big(\Xi(\langle [\tilde{A},\tilde{B}]\rangle)\big)$ holds true. Thus, in order to show that (ii) is valid it remains to verify that

$$\langle [A,B]\rangle \in \langle (\mathscr{D}_{\Delta_{21},\Delta_{22}})\rangle \tag{59}$$

and

$$\mathscr{S}^{(q,q)}_{[\nabla]}(\Xi(\langle [A,B]\rangle)) = \sigma^{(q,q)}_{\Delta;\langle [A,B]\rangle} \tag{60}$$

are fulfilled for all pairs $[A,B]$ which belong to $(q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(\mathbb{D})$. Let $[A,B] \in (q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(\mathbb{D})$. Then part (a) of Proposition 53 yields that $\langle [A,B]\rangle \in \langle (\mathscr{D}_{(1/\sqrt{2})I_q,(1/\sqrt{2})I_q})\rangle$ and that $g := \Xi(\langle [A,B]\rangle)$ belongs to $\mathscr{S}_{q\times q}(\mathbb{D})$. Lemma 25 and Proposition 53 provide that

$$\det(A(z) + B(z)) \neq 0, \tag{61}$$

i.e.,

$$[A(z), B(z)] \in \left[\mathscr{D}_{\frac{1}{\sqrt{2}}I_q,\frac{1}{\sqrt{2}}I_q}\right], \tag{62}$$

and

$$g(z) = (B(z) - A(z))(B(z) + A(z))^{-1} = \mathfrak{s}^{(q,q)}_{C_q}([A(z), B(z)]) \tag{63}$$

are satisfied for all $z \in (\mathbb{H}_A \cap \mathbb{H}_B) \setminus \mathscr{N}_{[A,B]}$ where the right-hand side of Eq. (63) is given by Eq. (22). From the first identity in Eq. (58) we infer

$$\Delta_{21}(z)A(z)+\Delta_{22}(z)B(z)=\frac{1}{\sqrt{2}}((\nabla_{22}(z)-\nabla_{21}(z))A(z)$$
$$+(\nabla_{22}(z)+\nabla_{21}(z))B(z))$$
$$=\frac{1}{\sqrt{2}}(\nabla_{21}g(z)-\nabla_{22}(z))(B(z)+A(z))$$

and consequently

$$\det(\Delta_{21}(z)A(z)+\Delta_{22}(z)B(z))=\det(\nabla_{21}(z)g(z)+\nabla_{22}(z))\,\det(B(z)+A(z)) \tag{64}$$

for each $z\in(\mathbb{H}_\Delta\cap\mathbb{H}_\nabla\cap\mathbb{H}_A\cap\mathbb{H}_B)\setminus\mathcal{N}_{[A,B]}$. By virtue of (i), the function $\det(\nabla_{21}g+\nabla_{22})$ does not identically vanish in $\mathbb{D}$. Hence there is a discrete subset $\mathscr{F}$ of $\mathbb{D}$ such that $\mathbb{D}\setminus\mathscr{F}\subseteq\mathbb{H}_\nabla$ and

$$\det(\nabla_{21}(z)g(z)+\nabla_{22}(z))\neq 0 \tag{65}$$

are fulfilled for all $z\in\mathbb{D}\setminus\mathscr{F}$. Therefore it follows from Eqs. (61) and (64) that the set $\mathscr{M}_{\Delta:[A,B]}$ given by Eq. (46) is a subset of $(\mathbb{D}\setminus((\mathbb{H}_A\cap\mathbb{H}_B)\setminus\mathcal{N}_{[A,B]}))\cup(\mathbb{D}\setminus\mathscr{H}_\Delta)\cup\mathscr{F}$. Hence $\mathscr{M}_{\Delta:[A,B]}$ is a discrete subset of $\mathbb{D}$, i.e., Eq. (59) holds true. We see from Eqs. (65) and (63) that $g(z)\in\mathscr{Q}_{\nabla_{21}(z),\nabla_{22}(z)}$ and

$$\mathscr{S}^{(q,q)}_{\nabla(z)}(g(z))=\mathscr{S}^{(q,q)}_{\nabla(z)}\left(\mathfrak{s}^{(q,q)}_{C_q}([A(z),B(z)])\right) \tag{66}$$

are satisfied for all $z\in\mathscr{G}:=(\mathbb{D}\setminus\mathscr{F})\cap((\mathbb{H}_A\cap\mathbb{H}_B)\setminus\mathcal{N}_{[A,B]})$. Applying Proposition 14 and using the first identity in Eq. (58) we then get

$$\mathscr{S}^{(q,q)}_{\nabla(z)}(g(z))=\left(\mathfrak{s}^{(q,q)}_{\Delta(z)}([A(z),B(z)])\right)=\sigma^{(q,q)}_{\Delta:([A,B])}(z)$$

for each $z\in\mathscr{H}:=(\mathbb{H}_\Delta\cap\mathscr{G})\setminus\mathscr{M}_{\Delta:[A,B]}$. Since $\mathbb{D}\setminus\mathscr{H}$ is a discrete subset of $\mathbb{D}$ the identity (62) is also proved.

(ii) ⇒ (iii): Now suppose that (ii) is fulfilled. From the second identity in Eq. (58) then one can easily see that $\det D$ does not identically vanish in $\mathbb{D}$. Furthermore, we know that Eq. (54) holds true and that the mapping $\Phi_\Delta:\langle(q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(\mathbb{D})\rangle\to[\mathscr{M}(\mathbb{D})]^{q\times q}$ given by Eq. (55) fulfills Eq. (56). Applying Lemma 5 we can conclude

$$\mathscr{S}^{(q,q)}_{[C_q]}\left(\Phi_\Delta\left(\langle(q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(\mathbb{D})\rangle\right)\right)=\mathscr{S}_{q\times q}[A_0,A_1,\ldots,A_n]$$

By virtue of Theorem 54, then

$$\mathscr{S}^{(q,q)}_{[C_q]}\left(\Phi_\Delta\left(\Xi^{[-1]}(\mathscr{S}_{q\times q}(\mathbb{D}))\right)\right)=\mathscr{S}_{q\times q}[A_0,A_1,\ldots,A_n]$$

follows. Thus, in order to check that (iii) is valid it remains to show that, for each $f\in\mathscr{S}_{q\times q}(\mathbb{D})$, the function $\det(D_{21}f+D_{22})$ does not identically vanish in $\mathbb{D}$ and that

$$\mathscr{S}^{(q,q)}_{[C_q]}\left(\Phi_\triangle\left(\Xi^{[-1]}(f)\right)\right) = \mathscr{S}^{(q,q)}_{[D]}(f) \tag{67}$$

holds true. Let $f \in \mathscr{S}_{q\times q}(\mathbb{D})$. In view of Remark 55, we have

$$\Xi^{[-1]}(f) = \langle [I-f, I+f]\rangle \in \langle (q,q)\text{-}\mathscr{P}_{-J_q,\geqslant}(\mathbb{D})\rangle. \tag{68}$$

Then we obtain from Eq. (54) that $\langle [I-f, I+f]\rangle \in \langle (\mathscr{D}_{\triangle_{21},\triangle_{22}})\rangle$. Hence $\mathscr{M}_{\triangle;[I-f,I+f]}$ is a discrete subset of $\mathbb{D}$, and

$$\det(\triangle_{21}(z)(I-f(z)) + \triangle_{22}(z)(I+f(z))) \neq 0$$

holds for all $z \in \mathbb{H}_\triangle \setminus \mathscr{M}_{\triangle;[I-f,I+f]}$. Setting $W := \triangle C_q$ and using the $q\times q$ block partition (44) of W, then we obtain

$$\det(W_{21}(z)f(z) + W_{22}(z)) \neq 0 \tag{69}$$

and therefore $f(z) \in \mathscr{D}_{W_{21}(z),W_{22}(z)}$ for all $z \in \mathbb{H}_\triangle \setminus \mathscr{M}_{\triangle;[I-f,I+f]}$. Hence the identities

$$\begin{aligned}\Phi_\triangle(\langle [I-f, I+f]\rangle) &= (\triangle_{11}(I-f) + \triangle_{12}(I+f))(\triangle_{21}(I-f) \\ &\qquad + \triangle_{22}(I.+f))^{-1} \\ &= ((\triangle_{12}-\triangle_{11})f + \triangle_{11} + \triangle_{12})((\triangle_{22}-\triangle_{21})f + \triangle_{21} + \triangle_{22})^{-1} \\ &= (W_{11}f + W_{12})(W_{21}f + W_{22})^{-1} = \mathscr{S}^{(q,q)}_{[W]}(f)\end{aligned} \tag{70}$$

hold true with exception of a discrete subset of $\mathbb{D}$. Thus we see from Eqs. (56) and (68) that $\Omega := \mathscr{S}^{(q,q)}_{[W]}(f)$ belongs to $\mathscr{C}_q(\mathbb{D})$. Consequently $\det(I+\Omega)$ nowhere vanishes in $\mathbb{D}$ (see, e.g., [6], Proposition 2.1.3), i.e., $\Omega(z)$ belongs to $\mathscr{D}_{(1/\sqrt{2})I_q,(1/\sqrt{2})I_q}$ for all $z \in \mathbb{D}$. In view of $D = C_qW$ and Eq. (69) we infer from

$$\sqrt{2}(D_{21}f + D_{22}) = (W_{11}+W_{21})f + (W_{12}+W_{22}) = (I+\Omega)(W_{21}f + W_{22})$$

that $\det(D_{21}f + D_{22})$ does not identically vanish in $\mathbb{D}$ and that

$$\mathscr{S}^{(q,q)}_{[C_q]}\left(\mathscr{S}^{(q,q)}_{[W]}(f)\right) = \mathscr{S}^{(q,q)}_{[C_qW]}(f) = \mathscr{S}^{(q,q)}_{[D]}(f) \tag{71}$$

holds (see, e.g., [6], Proposition 1.6.3). Thus from Eqs. (68), (70) and (71) it follows Eq. (67).

(iii) $\Rightarrow$ (i): Now suppose (iii). In view of Eq. (57), the function $\det \nabla$ does not identically vanish in $\mathbb{D}$. Further, since (iii) holds we have

$$\mathscr{S}^{(q,q)}_{[D]}(\mathscr{S}_{q\times q}(\mathbb{D})) = \mathscr{S}_{q\times q}[A_0, A_1, \ldots, A_n], \tag{72}$$

and, for all $f \in \mathscr{S}_{q\times q}(\mathbb{D})$, the function $\det(D_{21}f + D_{22})$ does not identically vanish in $\mathbb{D}$. Let $f \in \mathscr{S}_{q\times q}(\mathbb{D})$. Then Eq. (72) implies that $h := \mathscr{S}^{(q,q)}_{[D]}(f)$ belongs to $\mathscr{S}_{q\times q}[A_0, A_1, \ldots, A_n]$. Since $(A_k)_{k=0}^n$ is a nondegenerate $q\times q$ Schur sequence, we have

$$\operatorname{rank}(I - h^*(z)h(z)) = \operatorname{rank}(I - h^*(0)h(0)) = \operatorname{rank}(I - A_0^*A_0) = q$$

for all $z \in \mathbb{D}$ (see, e.g., [6], Lemma 2.1.5). Thus the function $\det(I + h)$ nowhere vanishes in $\mathbb{D}$, i.e., $h(z)$ belongs to $\mathscr{D}_{(1/\sqrt{2})I_q,(1/\sqrt{2})I_q}$ for all $z \in \mathbb{D}$. Taking into account

$$\nabla_{21} f + \nabla_{22} = (D_{11} + D_{21})f + D_{12} + D_{22} = (I + h)(D_{21}f + D_{22})$$

we see from (iii) that $\det(\nabla_{21}f + \nabla_{22})$ does not identically vanish in $\mathbb{D}$. Moreover, in view of Eq. (57), we have

$$\mathscr{S}^{(q,q)}_{[C_q]}\left(\mathscr{S}^{(q,q)}_{[D]}(f)\right) = \mathscr{S}^{(q,q)}_{[C_qD]}(f) = \mathscr{S}^{(q,q)}_{[\nabla]}(f)$$

(see, e.g., [6], Proposition 1.6.3). Finally, then (iii) and Lemma 5 imply

$$\begin{aligned}\mathscr{S}^{(q,q)}_{[\nabla]}(\mathscr{S}_{q\times q}(\mathbb{D})) &= \mathscr{S}^{(q,q)}_{[C_q]}\left(\mathscr{S}^{(q,q)}_{[D]}(\mathscr{S}_{q\times q}(\mathbb{D}))\right)\\ &= \mathscr{S}^{(q,q)}_{[C_q]}(\mathscr{S}_{q\times q}[A_0, A_1, \ldots, A_n]) = \mathscr{C}_q[\Gamma_0, \Gamma_1, \ldots, \Gamma_n]. \qquad \square\end{aligned}$$

Simakova [21] studied the image of the Schur class $\mathscr{S}_{p\times q}(\mathbb{D})$ under linear fractional transformations the generating matrix-valued function of which is meromorphic. In particular, her results enable a complete description of the whole variety of all resolvent matrices associated with a nondegenerate Schur problem. Theorem 63 provides the key for analogous descriptions of the sets of resolvent matrices associated with nondegenerate Carathéodory problems.

Lemma 64. *Let U be a j_{pq}-unitary matrix, and let the block partition of U with $p \times p$ block U_{11} be given in Eq.* (43). *For each $g \in \mathscr{S}_{p\times q}(\mathbb{D})$, then the function* $\det(U_{21}g + U_{22})$ *nowhere vanishes in* $\mathbb{D}$. *Moreover,* $\mathscr{S}^{(p,q)}_{[U]}(\mathscr{S}_{p\times q}(\mathbb{D})) = \mathscr{S}_{p\times q}(\mathbb{D})$.

Proof. From Theorem 1.6.1 in [6] we immediately see that, for each $g \in \mathscr{S}_{p\times q}(\mathbb{D})$, the function $\det(U_{21}g + U_{22})$ does not vanish in $\mathbb{D}$, and that $\mathscr{S}^{(p,q)}_{[U]}(\mathscr{S}_{p\times q}(\mathbb{D})) \subseteq \mathscr{S}_{p\times q}(\mathbb{D})$. Obviously, $\det U \neq 0$. Moreover, the matrix U^{-1} admits the representation $U^{-1} = j_{pq}U^*j_{pq}$ and is j_{pq}-unitary as well. Therefore, $\mathscr{S}^{(p,q)}_{[U^{-1}]}(\mathscr{S}_{p\times q}(\mathbb{D})) \subseteq \mathscr{S}_{p\times q}(\mathbb{D})$. Hence it follows

$$\begin{aligned}\mathscr{S}_{p\times q}(\mathbb{D}) &= \mathscr{S}^{(p,q)}_{[I_{p+q}]}(\mathscr{S}_{p\times q}(\mathbb{D})) = \mathscr{S}^{(p,q)}_{[UU^{-1}]}(\mathscr{S}_{p\times q}(\mathbb{D}))\\ &= \mathscr{S}^{(p,q)}_{[U]}\left(\mathscr{S}^{(p,q)}_{[U^{-1}]}(\mathscr{S}_{p\times q}(\mathbb{D}))\right) \subseteq \mathscr{S}^{(p,q)}_{[U]}(\mathscr{S}_{p\times q}(\mathbb{D}))\end{aligned}$$

(see, e.g., [6], Proposition 1.6.3). $\square$

Theorem 65. *Let $(\Gamma_k)_{k=1}^n$ be a nondegenerate $q \times q$ Carathéodory sequence, and let ∇ be a right type I resolvent matrix of the Carathéodory problem with respect to $(\Gamma_k)_{k=0}^n$.*

(a) *If ρ is a complex-valued function which is meromorphic in $\mathbb{D}$ and which does not identically vanish in $\mathbb{D}$, and if U is a j_{qq}-unitary matrix, then $\nabla^{\square} := \rho \nabla U$ is also a right type I resolvent matrix of the Carathéodory problem with respect to $(\Gamma_k)_{k=0}^{n}$.*

(b) *If $\nabla^{\square}$ is an arbitrary right type I resolvent matrix of the Carathéodory problem with respect to $(\Gamma_k)_{k=0}^{n}$, then there are a complex-valued function ρ which is meromorphic in $\mathbb{D}$ and which does not identically vanish in $\mathbb{D}$ and a j_{qq}-unitary matrix U such that $\nabla^{\square} = \rho \nabla U$.*

Proof. (a) Let ρ be a complex-valued function which is meromorphic in $\mathbb{D}$ and which does not identically vanish in $\mathbb{D}$, and let U be a j_{qq}-unitary matrix. Set $W := \rho U$. We will use the $q \times q$ block partitions (43) and (44) of U and W. Let $g \in \mathcal{S}_{q\times q}(\mathbb{D})$. Then we know from Lemma 64 that the functions $\det(U_{21}g + U_{22})$ and $\det(W_{21}g + W_{22})$ do not identically vanish in $\mathbb{D}$ and that $\mathcal{S}_{[U]}^{(q,q)}(\mathcal{S}_{q\times q}(\mathbb{D})) = \mathcal{S}_{q\times q}(\mathbb{D})$. Hence

$$\begin{aligned}\mathcal{S}_{[W]}^{(q,q)}(\mathcal{S}_{q\times q}(\mathbb{D})) &= \mathcal{S}_{[\rho I_{2q}]}^{(q,q)}\left(\mathcal{S}_{[U]}^{(q,q)}(\mathcal{S}_{q\times q}(\mathbb{D}))\right)\\ &= \mathcal{S}_{[\rho I_{2q}]}^{(q,q)}(\mathcal{S}_{q\times q}(\mathbb{D})) = \mathcal{S}_{q\times q}(\mathbb{D}).\end{aligned} \tag{73}$$

In particular, $f := \mathcal{S}_{[W]}^{(q,q)}(g)$ belongs to $\mathcal{S}_{q\times q}(\mathbb{D})$. Let

$$\nabla^{\square} = \begin{pmatrix} \nabla_{11}^{\square} & \nabla_{12}^{\square} \\ \nabla_{21}^{\square} & \nabla_{22}^{\square} \end{pmatrix}$$

be the $q \times q$ block partition of $\nabla^{\square}$. From $\nabla^{\square} = \nabla W$ then we get

$$\begin{aligned}\nabla_{21}^{\square} g + \nabla_{22}^{\square} &= \nabla_{21}(W_{11}g + W_{12}) + \nabla_{22}(W_{21}g + W_{22})\\ &= (\nabla_{21}f + \nabla_{22})(W_{21}g + W_{22}).\end{aligned} \tag{74}$$

Since ∇ is a right type I resolvent matrix of the Carathéodory problem with respect to $(\Gamma_k)_{k=0}^{n}$, the function $\det(\nabla_{21}f + \nabla_{22})$ does not identically vanish in $\mathbb{D}$. Hence Eq. (74) shows that $\det(\nabla_{21}^{\square}g + \nabla_{22}^{\square})$ does not identically vanish in $\mathbb{D}$ as well. Using Eq. (73) then we obtain

$$\begin{aligned}\mathcal{S}_{[\nabla^{\square}]}^{(q,q)}(\mathcal{S}_{q\times q}(\mathbb{D})) &= \mathcal{S}_{[\nabla]}^{(q,q)}\left(\mathcal{S}_{[W]}^{(q,q)}(\mathcal{S}_{q\times q}(\mathbb{D}))\right)\\ &= \mathcal{S}_{[\nabla]}^{(q,q)}(\mathcal{S}_{q\times q}(\mathbb{D})) = \mathcal{C}_q[\Gamma_0, \Gamma_1, \ldots, \Gamma_n].\end{aligned}$$

(b) Now let $\nabla^{\square}$ be an arbitrary right type I resolvent matrix of the Carathéodory problem with respect to $(\Gamma_k)_{k=0}^{n}$. We know from Remark 4 that the Cayley transform $(A_k)_{k=0}^{n}$ of $(\Gamma_k)_{k=0}^{n}$ is a nondegenerate $q \times q$ Schur sequence. We see from Theorem 63 that $D := C_q\nabla$ and $D^{\square} := C_q\nabla^{\square}$ are right resolvent matrices of the Schur problem with respect to $(A_k)_{k=0}^{n}$. Applying a result due to Simakova ([22], p. 169) we get that there are a complex-valued function

ρ which is meromorphic in $\mathbb{D}$ and which does not identically vanish in $\mathbb{D}$ and a j_{qq}-unitary matrix U such that $D^{\square} = \rho D U$. Thus the assertion stated in (b) immediately follows from $C_q^2 = I_{2q}$. $\square$

Theorem 66. *Let $(\Gamma_k)_{k=0}^n$ be a nondegenerate $q \times q$ Carathéodory sequence, and let $\triangle$ be a right type I resolvent matrix of the Carathéodory problem with respect to $(\Gamma_k)_{k=0}^n$.*

(a) *If ρ is a complex-valued function which is meromorphic in $\mathbb{D}$ and which does not identically vanish in $\mathbb{D}$, and if V is a J_q-unitary matrix, then $\triangle^{\square} := \rho \triangle V$ is a right type II resolvent matrix of the Carathéodory problem with respect to $(\Gamma_k)_{k=0}^n$.*

(b) *If $\triangle^{\square}$ is an arbitrary right type II resolvent matrix of the Carathéodory problem with respect to $(\Gamma_k)_{k=0}^n$, then there are a complex-valued function ρ which is meromorphic in $\mathbb{D}$ and which does not identically vanish in $\mathbb{D}$ and a J_q-unitary matrix V such that $\triangle^{\square} = \rho \triangle V$.*

Proof. According to Remark 4, the Cayley transform $(A_k)_{k=0}^n$ of $(\Gamma_k)_{k=0}^n$ is a nondegenerate $q \times q$ Schur sequence. Further, we know from Theorem 63 that $D := C_q \triangle C_q$ is a right resolvent matrix of the Schur problem with respect to $(A_k)_{k=0}^n$.

(a) Let ρ be a complex-valued function which is meromorphic in $\mathbb{D}$ and which does not identically vanish in $\mathbb{D}$, and let V be a J_q-unitary matrix. It is readily checked that then $U := C_q V C_q$ is a j_{qq}-unitary matrix. Applying the result due to Simakova ([22], p. 169) mentioned already above we get that $D^{\square} := \rho D U$ is a right resolvent matrix of the Schur problem with respect to $(A_k)_{k=0}^n$. In view of $C_q D^{\square} C_q = \rho C_q D C_q V = \rho \triangle V = \triangle^{\square}$ the application of Theorem 63 completes the proof of part (a).

(b) Now let $\triangle^{\square}$ be an arbitrary right type II resolvent matrix of the Carathéodory problem with respect to $(\Gamma_k)_{k=0}^n$. Then we see from Theorem 63 that $D^{\square} := C_q \triangle^{\square} C_q$ is a right resolvent matrix of the Schur problem with respect to $(A_k)_{k=0}^n$. Applying again Simakova's result ([22], p. 169) we obtain $D^{\square} = \rho D U$ where ρ is some complex-valued function which is meromorphic in $\mathbb{D}$ and which does not identically vanish in $\mathbb{D}$ and where U is some j_{qq}-unitary matrix. Then $V := C_q U C_q$ is a J_q-unitary matrix, and we finally have

$$\triangle^{\square} = C_q D^{\square} C_q = \rho C_q D U C_q = \rho C_q D C_q V = \rho \triangle V. \qquad \square$$

References

[1] D.Z. Arov, M.G. Krein, Problems of the search of the minimum of entropy in indeterminate extension problems (in Russian), Funktsional. Anal. i Prilozhen. 15 (2) (1981) 61–64.

[2] D.Z. Arov, M.G. Krein, On computation of entropy functionals and their minimums in indeterminate extension problems, Acta Sci. Math. (Szeged) 45 (1983) 33–50.

[3] J.M. Djukarev, Multiplicative and additive classes of analytic matrix functions of Stieltjes type and associated interpolation problems, Part II (in Russian), Teor. Funkcii, Funkcional. Anal. i Prilozen. 38 (1982) 40–48.
[4] J.M. Djukarev, V.E. Katsnelson, Multiplicative and additive classes of analytic matrix functions of Stieltjes type and associated interpolation problems, Part I (in Russian), Teor. Funkcii, Funkcional. Anal. i Prilozen. 36 (1981) 13–27.
[5] J.M. Djukarev, V.E. Katsnelson, Multiplicative and additive classes of analytic matrix functions of Stieltjes type and associated interpolation problems, Part III (in Russian), Teor. Funkcii, Funkcional. Anal. i Prilozen. 41 (1984) 64–70.
[6] V.K. Dubovoj, B. Fritzsche, B. Kirstein, Matricial Version of the Classical Schur Problem, in: Teubner-Texte zur Mathematik, vol. 129, Teubner, Stuttgart, 1992.
[7] H. Dym, J contractive Matrix Functions, Reproducing Kernel Hilbert Spaces and Interpolation, in: CBMS Regional Conf. Ser. Math. 71, Amer. Math. Soc., Providence, RI, 1989.
[8] B. Fritzsche, S. Fuchs, B. Kirstein, On an inverse problem of entropy optimization and related questions, Optimization 26 (1992) 359–384.
[9] B. Fritzsche, B. Kirstein, An extension problem for non-negative Hermitian block Toeplitz matrices, Part IV, Math. Nachr. 143 (1989) 329–354.
[10] B. Fritzsche, B. Kirstein, An extension problem for non-negative Hermitian block Toeplitz matrices, Part V, Math. Nachr. 144 (1989) 283–308.
[11] B. Fritzsche, B. Kirstein, V. Krug, Carathéodory sequence parametrizations of Potapov-normalized full-rank J_q-elementary factors, Linear Algebra Appl. 214 (1995) 145–185.
[12] B. Fritzsche, B. Kirstein, V. Krug, On Arov-normalized j_{qq}-J_q-elementary factors, Linear Algebra Appl., 267 (1997) 11–40.
[13] L. Harris, Linear fractional transformations of circular domains in operator spaces, Indiana Univ. Math. J. 41 (1992) 125–147.
[14] L.A. Ivanchenko, L.A. Sakhnovich, Operator approach to the investigation of interpolation problems (in Russian), deposited to Ukr. NIINTI, No. 701, Uk-85, 1985.
[15] L.A. Ivanchenko, L.A. Sakhnovich, Operator identities in the theory of interpolation problems (in Russian), Izv. Akad. Nauk Armyan. SSR Ser. Mat. 22 (3) (1987) 298–308.
[16] L.A. Ivanchenko, L.A. Sakhnovich, Operator approach to Potapov scheme (in Russian), Ukrainian Math. J. 39 (5) (1987) 573–578.
[17] L.A. Ivanchenko, L.A. Sakhnovich, An operator approach to the Potapov scheme for the solution of interpolation problems, in: I. Gohberg, L.A. Sakhnovich (Eds.), Matrix and Operator Valued Functions, The Vladimir Petrovich Potapov Memorial Volume, Operator Theory: Advances and Applications, vol. 72, Birkhäuser, Basel, 1994, pp. 48–86.
[18] I.V. Kovalishina, Analytic theory of a class of interpolation problems (in Russian), Izv. Akad. Nauk SSR Ser. Mat. 47 (1983) 455–497.
[19] A.A. Nudelman, Some properties of linear-fractional transformations and the harmonic mean of matrix functions, in: I. Gohberg, L.A. Sakhnovich, Matrix and Operator Valued Functions, The Vladimir Petrovich Potapov Memorial Volume, Operator Theory: Advances and Applications, vol. 72, Birkhäuser, Basel, 1994, pp. 171–184.
[20] S.A. Orlov, Nested matrix balls, analytically depending on a parameter, and theorems on invariance of ranks of radii of limit matrix balls (in Russian), Izv. Akad. Nauk SSR Ser. Mat. 40 (1976) 593–644.
[21] V.P. Potapov, Linear fractional transformations of matrices (in Russian), in: V.A. Marcenko, Studies in the Theory of Operators and Their Applications, Naukova Dumka, Kiev, 1975, pp. 75–97; English transl. in: Amer. Math. Soc. Transl. (2) 138 (1988) 21–35.
[21] L.A. Simakova, On plus-matrix valued functions of bounded characteristics (in Russian), Matem. Issled. (Kishinev) 9 (2) (1974) 149–171.
[23] N. Young, Linear fractional transformations in rings and modules, Linear Algebra Appl. 56 (1984) 251–290.