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LINEAR ALGEBRA AND ITS APPLICATIONS

A Schur complement inequality for certain P-matrices

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Abstract

Suppose A and B are $n \times n$ matrices over the complex field. An inequality is derived that relates the Schur complement of the Hadamard product of A and B and the Hadamard product of Schur complements of A and B for positive definite matrices. Then an analog is given for the class of tridiagonal totally nonnegative matrices. A similar result is given for classes of Z-matrices where the Hadamard product is replaced by the Fan product.

1. Introduction

Suppose A and B are $n \times n$ matrices over the complex field. In this paper, under the Loewner ordering we prove an inequality relating the Schur complement of the Hadamard product of A and B and the Hadamard product of Schur complements of A and B for positive definite matrices. Then under the entry-wise dominance partial ordering an analog is given for the class of tridiagonal totally nonnegative matrices and a similar result is given for M-matrices and certain other classes of Z-matrices; in the latter case it is necessary for the Hadamard product to be replaced by the Fan product.

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We dedicate this paper to the memory of our teacher and friend, Emilie Haynsworth. Many aspects of Schur complements, such as the quotient theorem and the inertia theorem for hermitian matrices, were initiated and refined by her keen insight. The relevance of these concepts is evidenced by their commonplace use today. To paraphrase a famous anonymous saying, her counsel was worth more than ten thousand words.

2. An inequality for positive-definite matrices

Let A and B be $n \times n$ positive definite matrices over the field of complex numbers. We partition A as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{1}$$

where A_{11} and A_{12} are square of orders k and $n-k$, respectively. In general, the matrix

$$(A/A_{22}) = A_{11} - A_{12}A_{22}^{-1}A_{21} \tag{2}$$

is called the *Schur complement of* A_{22} *in* A, provided A_{22} is invertible. Throughout, we will assume B is partitioned with blocks the same size as those of A in Eq. (1).

The *Hadamard* product of A and B, denoted $A * B$, is the $n \times n$ matrix $(a_{ij}b_{ij})$, and it is well-known (due to Schur [1]) that if A and B are positive definite, then $A * B$ is also positive definite. We write $A \geqslant B$ and this means that $A - B$ is positive semidefinite, or $A - B \geqslant 0$. This partial order is usually called the Loewner ordering.

If A is partitioned as in Eq. (1) for a general $n \times n$ matrix with A_{22} invertible and a_{nn} an invertible element, Crabtree and Haynsworth [2] proved the quotient rule.

$$((A/a_{nn})/(A_{22}/a_{nn})) = (A/A_{22}). \tag{3}$$

First, we state a lemma; we omit its straight-forward proof.

Lemma 1.1. *If A is an $n \times n$ positive definite matrix and B is an $n \times r$ matrix, then $C = A * (BB^*) = 0$ if and only if $B = 0$.*

Now we state our main result.

Theorem 1.2. *Suppose A and B are $n \times n$ positive definite matrices partitioned as in Eq. (1). Then*

$$(A * B/A_{22} * B_{22}) \geqslant (A/A_{22}) * (B/B_{22}). \tag{4}$$

Equality holds in Eq. (4) if and only if A and B are block-diagonal in Eq. (1).

Proof. Let

$$\hat{A} = \begin{bmatrix} A_{12}A_{22}^{-1}A_{21} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

and let $\hat{B}$ be similarly defined. Since A_{22} has order $n-k$, $\hat{A}$ and $\hat{B}$ are positive semidefinite of rank $n-k$. Hence $\hat{A} * \hat{B}$ is positive semidefinite (of rank $\geqslant n-k$), and also $\hat{A} * \hat{B}/A_{22} * B_{22} \geqslant 0$. Now the inequality (4) can be written as

$$(A/A_{22}) * (B_{12}B_{22}^{-1}B_{21}) + (A_{12}A_{22}^{-1}A_{21}) * (B/B_{22}) + (\hat{A} * \hat{B})/(A_{22} * B_{22}) \geqslant 0. \tag{5}$$

But Eq. (5) holds since the sum of positive semidefinite matrices is again positive semidefinite. This concludes the proof of (4).

For the case of equality, we apply the lemma to (5). It follows that $A_{12} = B_{12} = 0$ and hence $A_{21} = B_{21} = 0$. Thus equality holds in (4) if and only if both A and B are block-diagonal matrices. □

Suppose A and B are positive definite $n \times n$ matrices and that the respective eigenvalues of A and B are arranged in the same increasing order. Then, if $\det(\cdot)$ denotes the determinant function, it is known that $\det(A * B) \geqslant \det(A)\det(B)$ ([3], p. 311), $\lambda_1(A * B) \geqslant \lambda_1(AB) \geqslant \lambda_1(A)\lambda_1(B)$ ([3], p. 312–315), and $A^{-1} * B^{-1} \geqslant (A * B)^{-1}$ ([4], Theorem 7.7.9a). Further, if $A \geqslant B$, it is well-known ([4], Corollary 7.7.4) that $\det(A) \geqslant \det(B)$, $B^{-1} \geqslant A^{-1}$, and $\lambda_k(A) \geqslant \lambda_k(B)$, $k = 1, 2, \ldots, n$

It follows from the Schur complement form of the inverse [4] that, for $n \geqslant 3$, the inequality given in Theorem 7.7.9a of Ref. [4] is equivalent to: $(A/A_{22})^{-1} * (B/B_{22})^{-1} \geqslant (A * B/A_{22} * B_{22})^{-1}$, or, equivalently, $(A * B/A_{22} * B_{22}) \geqslant [(A/A_{22})^{-1} * (B/B_{22})^{-1}]^{-1}$ for all $(n-k) \times (n-k)$ principal submatrices A_{22} (respectively, B_{22}) of A (respectively, B), $k = 1, \ldots, n-1$. Since $(A/A_{22}) * (B/B_{22}) \geqslant [(A/A_{22})^{-1} * (B/B_{22})^{-1}]^{-1}$ for all such A_{22} and B_{22}, we see that inequality (4) is stronger than Theorem 7.7.9a of Ref. [4].

We mention a related result ([4], Theorem 7.7.8a): the inverse of a principal submatrix of a positive definite matrix is less than or equal to the corresponding principal submatrix of the inverse. In terms of the partitioning (1) this means that $(A_{11} * B_{11})^{-1} \leqslant (A * B/A_{22} * B_{22})^{-1}$ or, equivalently, $(A * B/A_{22} * B_{22}) \leqslant A_{11} * B_{11}$.

Lastly, it is worth mentioning that if $B = A$, our inequality becomes $(A * A/A_{22} * A_{22}) \geqslant (A/A_{22}) * (A/A_{22})$.

Corollary 1.3. *If A and B are positive definite $n \times n$ matrices partitioned as in Eq. (1) then*

(i) $$\frac{\det(A * B)}{\det(A_{22} * B_{22})} \geqslant \frac{\det(A)\det(B)}{\det(A_{22})\det(B_{22})},$$

(ii) $(A/A_{22})^{-1} * (B/B_{22})^{-1} \geqslant [(A/A_{22}) * (B/B_{22})]^{-1}$
$\geqslant (A * B/A_{22} * B_{22})^{-1} \geqslant (A_{11} * B_{11})^{-1}$,

(iii) $\lambda_j(A * B/A_{22} * B_{22}) \geqslant \lambda_j[(A/A_{22}) * (B/B_{22})]$
for $j = 1, 2, \ldots, k$ *where* k *is the order of* A_{11}

In particular,

$$\lambda_1(A_{11} * B_{11}) \geqslant \lambda_1[(A/B) * (A_{22}/B_{22})] \geqslant \lambda_1[(A/A_{22}) * (B/B_{22})]$$
$$\geqslant \lambda_1[(A/A_{22})(B/B_{22})] \geqslant \lambda_1(A/A_{22})\lambda_1(B/B_{22}).$$

Proof. Since inequality (4) holds, we get

$$\det(A * B/A_{22} * B_{22} \geqslant [(A/A_{22}) * (B/B_{22})]$$
$$\geqslant \det(A/A_{22})\det(B/B_{22}).$$

(i) then follows by noting that $\det(A/A_{22}) \geqslant \det(A)/\det(A_{22})$ (Schur's formula). (ii) and (iii) follow immediately from the remarks preceding the statement of the theorem. □

3. An analog for tridiagonal totally nonnegative matrices

A matrix A is called *totally nonnegative* (*totally positive*) if all the minors of A of all orders are nonnegative (positive). We will write the principal minor of A in rows and columns $i_1, \ldots, i_k$ as $A(i_1, \ldots, i_k)$.

In a manner similar to the development of Chapter 1, it is possible to prove that if A, partitioned as in Eq. (1), is an $n \times n$ tridiagonal totally nonnegative matrix, where A_{22} is invertible, then

$$\hat{A} = \begin{bmatrix} A_{12}A_{22}^{-1}A_{21} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{6}$$

is a tridiagonal totally nonnegative matrix.

Here, we shall take our partial order to be entry-wise domination; that is, we write $A \geqslant B$ to mean $A - B$ is entry-wise nonnegative (and $B \geqslant 0$ means that B is entry-wise nonnegative). It follows that we obtain the following analog of Theorem 1.2.

Theorem 2.1. *If A and B are $n \times n$ tridiagonal totally nonnegative matrices partitioned as in Eq.* (1), *where A_{22} and B_{22} are invertible, then*

$$(A * B/A_{22} * B_{22}) \geqslant (A/A_{22}) * (B/B_{22}). \tag{7}$$

Example 2.2. In [5], the first author proved that if

$$A = \begin{bmatrix} 8 & 12 & 13.05 \\ 4 & 7 & 8 \\ 0 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 21 & 24 & 14 \\ 90.3 & 105 & 70 \end{bmatrix},$$

then $A * B$ is not totally nonnegative, although A and B are. Using the partitioning defined by $A_{11} = [8]$ and $B_{11} = [1]$, we find that $(A * B/A_{22} * B_{22}) = -10$ while $(A/A_{22}) * (B/B_{22}) = (2/25)(1/50) = (1/25)^2$.

It is clear we cannot hope to improve Theorem 2.1.

4. An analog for Z-matrices

In this section we investigate whether an analog to the inequality given in Section 1 holds for Z-matrices, i.e., square matrices whose off-diagonal entries are nonpositive. As in the totally nonnegative case we shall take our partial order to be entry-wise domination. First, observe that the analog to the inequality given in Section 1 does not hold under Hadamard product. To see this, consider the M-matrices

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ -1 & 0 & 0 & 2 \end{bmatrix}$$

and $B = A^T$, the transpose of A. With the partitioning defined by letting A_{22} be the lower right 2×2 principal submatrix, we see that

$$(A * B/A_{22} * B_{22}) - (A/A_{22}) * (B/B_{22}) = \begin{bmatrix} 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 \end{bmatrix} \ngeq 0.$$

Moreover, M-matrices (and Z-matrices for that matter) are clearly not closed under Hadamard product. For these reasons, we will instead consider the Fan product [7] of two Z-matrices.

Definition 3.1. Let A and B be Z-matrices. The Fan product $C = A \otimes B$ is defined as follows: $c_{ii} = a_{ii}b_{ii}$ for all i and $c_{ij} = -a_{ij}b_{ij}$ for $i \neq j$.

Recall [3] that the *comparison matrix* $M(A) = [m_{ij}]$ of a given complex matrix $A = (a_{ij})$ is defined by $m_{ij} = |a_{ij}|$ if $i = j$ and $m_{ij} = -|a_{ij}|$ if $i \neq j$ and that A is an *H-matrix* if $M(A)$ is an (invertible) M-matrix. Thus, the Fan product of two Z-matrices is really the comparison matrix of their Hadamard product. Further, we note that in [7] it was shown that invertible M-matrices are closed under Fan product and this was shown to be true for general M-matrices in [8]. It then follows that H-matrices are closed under Hadamard product.

For a square matrix B we let $\rho(B)$ denote the spectral radius of B and, for $k = 1, \ldots, n$, let $\rho_k(B)$ denote the maximum spectral radius of all $k \times k$ principal submatrices of B. It is well-known [6] that, if $B \geqslant 0$, then $0 \leqslant \rho_1(B) \leqslant \cdots \leqslant \rho_{n-1}(B) \leqslant \rho_n(B) = \rho(B)$ and that the latter inequality is strict if B is irreducible. Based on this fact Fiedler and the first author [9] made the following definition.

Definition 3.2. For $s = 0, \ldots, n$, let L_s denote the real $n \times n$ matrices which have the form $A = tI - B$, where $B \geqslant 0$ and $\rho_s(B) \leqslant t < \rho_{s+1}(B)$ (here, for completeness, $\rho_0(B) = -\infty$ and $\rho_{n+1}(B) = +\infty$).

We see that these matrices form a partition of the Z-matrices and that the familiar (invertible) M-matrices (matrices of the form $A = tI - B$ where $B \geqslant 0$ and $t > \rho(B)$) are properly contained in the class L_n. In fact, L_n is precisely the class of general M-matrices (matrices of the form $A = tI - B$ where $B \geqslant 0$ and $t \geqslant \rho(B)$). Following Fiedler and Pták [10] we let K denote the invertible M-matrices and K_0 the general M-matrices. Since a Z-matrix is in L_k only if all of its principal minors of order k or less are in K_0 [9], we see that if A is in L_s and B is in L_t, then $A \otimes B$ is in L_m where $m \geqslant \min\{s, t\}$.

The following easily verified lemma is due to Watford [11].

Lemma 3.3. *Let A and B be $n \times n$ Z-matrices partitioned as in Eq. (1) with $A \geqslant B$ and B_{22} (and hence A_{22}) in K. Then, $A/A_{22} \geqslant B/B_{22}$.*

We now show that the inequality given in Section 1 holds for Fan products of Z-matrices provided neither matrix is in L_0 or L_1.

Theorem 3.4. *For $2 \leqslant s,\ t \leqslant n$, let A and B be $n \times n$ L_s and L_t matrices, respectively, partitioned as in Eq. (1) with A_{22} and B_{22} in K. Then,*

$$(A \otimes B/A_{22} \otimes B_{22}) \geqslant A/A_{22} \otimes (B/B_{22}). \tag{8}$$

Proof. First, assume A_{22} and B_{22} are of order 1. Then,

$$\begin{aligned} E &= A \otimes B/A_{22} \otimes B_{22} = A \otimes B/a_{nn}b_{nn} \\ &= A_{11} \otimes B_{11} - (A_{12} * B_{12})(a_{nn}b_{nn})^{-1}(A_{21} * B_{21}) \end{aligned}$$

(here, * denotes the Hadamard product) and

$$\begin{aligned} F &= (A/A_{22}) \otimes (B/B_{22}) = (A/a_{nn}) \otimes (B/b_{nn}) \\ &= (A_{11} - A_{12}a_{nn}^{-1}A_{21}) \otimes (B_{11} - B_{12}b_{nn}^{-1}B_{21}) \end{aligned}$$

For $1 \leqslant i \leqslant n-1$,

$$
\begin{aligned}
f_{ii} &= \left(a_{ii} - \frac{a_{in}a_{ni}}{a_{nn}}\right)\left(b_{ii} - \frac{b_{in}b_{ni}}{b_{nn}}\right) \\
&= a_{ii}b_{ii} - \frac{a_{in}b_{in}a_{ni}b_{ni}}{a_{nn}b_{nn}} + \frac{a_{in}a_{ni}}{a_{nn}}\left(\frac{b_{in}b_{ni}}{b_{nn}} - b_{ii}\right) + \frac{b_{in}b_{ni}}{b_{nn}}\left(\frac{a_{in}a_{ni}}{a_{nn}} - a_{ii}\right) \\
&\leqslant a_{ii}b_{ii} - \frac{a_{in}b_{in}a_{ni}b_{ni}}{a_{nn}b_{nn}} = e_{ii}
\end{aligned}
$$

since all 2×2 principal minors of A and B are nonnegative.

For $1 \leqslant i, j \leqslant n-1$ with $i \neq j$,

$$
\begin{aligned}
f_{ij} &= -\left(a_{ij} - \frac{a_{in}a_{nj}}{a_{nn}}\right)\left(b_{ij} - \frac{b_{in}b_{nj}}{b_{nn}}\right) \\
&= -a_{ij}b_{ij} - \frac{a_{in}b_{in}a_{ni}b_{ni}}{a_{nn}b_{nn}} + a_{ij}\frac{b_{in}b_{nj}}{b_{nn}} + b_{ij}\frac{a_{in}a_{nj}}{a_{nn}} \\
&\leqslant -a_{ij}b_{ij} - \frac{a_{in}b_{in}a_{ni}b_{ni}}{a_{nn}b_{nn}} = e_{ij}.
\end{aligned}
$$

Thus, the result holds if the order of A_{22} and B_{22} is 1.

$A_{22} \otimes B_{22}/a_{nn}b_{nn}$ is the (2, 2)-block of $A \otimes B/a_{nn}b_{nn}$. Similarly $A_{22}/a_{nn}(B_{22}/b_{nn})$ is the (2, 2)-block of $A/a_{nn}(B/b_{nn})$ and thus $(A_{22}/a_{nn}) \otimes (B_{22}/b_{nn})$ is the (2, 2) block of $(A/a_{nn}) \times (B/b_{nn})$. Thus, $A \otimes B/a_{nn}b_{nn} \geqslant (A/a_{nn}) \otimes (B/b_{nn})$ by the first part of the proof. So

$$
\begin{aligned}
A \otimes B/A_{22} \otimes B_{22}) &= (A \otimes B/a_{nn}b_{nn})/(A_{22} \otimes B_{22}/a_{nn}b_{nn}) \\
&\geqslant (A/a_{nn}) \otimes (B/b_{nn})/(A_{22}/a_{nn}) \otimes (B_{22}/b_{nn}) \\
&\quad \text{(by the lemma)} \\
&\geqslant ((A/a_{nn})/(A_{22}/a_{nn})) \otimes ((B/b_{nn})/(B_{22}/b_{nn})) \\
&\quad \text{(by the first part of the proof)} \\
&= (A/A_{22}) \otimes (B/B_{22})
\end{aligned}
$$

which completes the proof. □

Corollary 3.5. *If A and B are invertible M-matrices partitioned as in Eq.* (1), *then Eq.* (8) *holds.*

Immediately, we have the following result for comparison matrices of Hadamard products of H-matrices.

Corollary 3.6. *If A and B are H-matrices partitioned as in Eq.* (1), *then*

$$M(A * B)/M(A_{22} * B_{22}) \geqslant M[(M(A)/M(A_{22})) * (M(B)/M(B_{22}))]. \tag{9}$$

For an M-matrix $A = tI - B$ where $B \geqslant 0$ and $t \geqslant \rho(B), q(A) = t - \rho(B)$ is the "minimum" eigenvalue of A in the sense that it lies farthest to the left in the complex plane. If A and B are in K with $A \geqslant B$, it is well-known ([10], Theorem 4.6) that $\det(A) \geqslant \det(B) > 0, 0 \leqslant A^{-1} \leqslant B^{-1}$, and $q(A) \geqslant q(B)$. Further, it was shown in Corollary 5.7.4.1 of [3] that, if A and B are in K, then $A^{-1} * B^{-1} \geqslant (A \otimes B)^{-1}$ and $q(A \otimes B) \geqslant q(A)q(B)$. Lastly, it was shown by the second author in [8] that $\det(A)\det(B) = \det(AB) \leqslant \det(A \otimes B)$ for general M-matrices A and B.

Combining these facts together with Theorem 3.4 and applying Schur's formula, we have the M-matrix analog of Corollary 1.3.

Corollary 3.7. *If A and B are invertible M-matrices partitioned as in Eq.* (1), *then*

(i) $\dfrac{\det(A \otimes B)}{\det(A_{22} \otimes B_{22})} \geqslant \dfrac{\det(A)\det(B)}{\det(A_{22})\det(B_{22})}$,

(ii) $(A/A_{22})^{-1} \otimes (B/B_{22})^{-1} \geqslant [(A/A_{22}) \otimes (B/B_{22})]^{-1}$
$\geqslant (A \otimes B/A_{22} \otimes B_{22})^{-1} \geqslant (A_{11} \otimes B_{11})^{-1}$,

(iii) $q(A_{11} \otimes B_{11}) \geqslant q(A \otimes B/A_{22} \otimes B_{22})$
$\geqslant q[(A/A_{22}) \otimes (B/B_{22})] \geqslant q(A/A_{22})q(B/B_{22})$.

We note that the theorem does not hold for all Z-matrices.

Example 3.8. Consider the L_1 matrices

$$A = \begin{bmatrix} 2 & -1 & -2 \\ -5 & 2 & -4 \\ -2 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & -3 \\ -6 & 2 & -5 \\ -2 & -1 & 2 \end{bmatrix}.$$

Using the partition defined by $A_{22} = [2] = B_{22}$, we find that

$$A/A_{22} = \begin{bmatrix} 0 & -2 \\ -9 & 0 \end{bmatrix}, \quad B/B_{22} = \begin{bmatrix} -1 & -\frac{5}{2} \\ -11 & -\frac{1}{2} \end{bmatrix}, \quad A \otimes B = \begin{bmatrix} 4 & -1 & -6 \\ -30 & 4 & -20 \\ -4 & -1 & 4 \end{bmatrix}.$$

Thus,

$$A \otimes B/A_{22} \otimes B_{22} = \begin{bmatrix} -2 & -\frac{5}{2} \\ -50 & -1 \end{bmatrix} \not\geqslant \begin{bmatrix} 0 & -5 \\ -99 & 0 \end{bmatrix} = (A/A_{22}) \otimes (B/B_{22}).$$

In closing, we mention that the analogous inequality to Eq. (4) does not hold for Hadamard products of inverse M-matrices (with the partial order being entry-wise dominance). This can be shown by considering the example in p. 360 of [3] and taking Schur complements with respect to the principal submatrix whose index set is $\{1, 2, 4, 5\}$.

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We mention that the positive definite case of our main result was proved independently by B.Y. Wang and F. Zhang [12].

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