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Simultaneous reduction to triangular forms after extension with zeroes

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Abstract

This paper is concerned with pairs of $m \times m$ matrices A, Z for which there exists an invertible $m \times m$ matrix S such that $S^{-1}AS$ is upper triangular and $S^{-1}ZS$ is lower triangular. Such pairs are said to admit simultaneous reduction to complementary triangular forms. In particular, if such a similarity S does not exist for the pair A, Z, the following question is taken into consideration: Does there exist a positive integer r, such that the pair of $(m + r) \times (m + r)$ matrices

$$\begin{pmatrix} A & O \\ O & O_r \end{pmatrix}, \quad \begin{pmatrix} Z & O \\ O & O_r \end{pmatrix}$$

admits simultaneous reduction to complementary triangular forms, so that the pair A, Z obtains the property after extension with zeroes? It is shown that the answer to this question is mixed. An example of a pair is given which obtains the property after extension with one zero. This example is put into contrast with a large number of results that answers the question negatively, i.e., that describes situations where the pair of extended matrices admits simultaneous reduction to complementary triangular forms if and only if the original pair does. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

Each complex square matrix is unitarily similar to an upper (or a lower) triangular matrix. Taking this result – known as Schur's theorem – as a starting point, one may consider a collection of square matrices and study the existence of a similarity which puts all matrices in the collection in triangular form. The following two types of simultaneous similarity to triangular forms which involve a collection of two elements have received considerable attention by several authors. References are given in the sequel.

First, given a pair of $m \times m$ matrices A, Z, the question is of whether there exists an invertible matrix S, such that both $S^{-1}AS$ and $S^{-1}ZS$ are upper triangular. If such a similarity exists, the pair A, Z is said to *admit simultaneous reduction to upper triangular form.*

The second property for a pair of $m \times m$ matrices is the main subject of this paper. A pair of $m \times m$ matrices A, Z is said to *admit simultaneous reduction to complementary triangular forms*, if there exists an invertible $m \times m$ matrix S, such that $S^{-1}AS$ is upper triangular and $S^{-1}ZS$ is lower triangular. This notion originates from mathematical systems theory. It is strongly related to factorizations of rational matrix functions which appear as transfer functions of finite dimensional linear dynamical systems (see [2,3,20] and the references given there).

We shall denote the collection of pairs of $m \times m$ matrices that admit simultaneous reduction to complementary triangular forms by $\mathscr{C}(m)$. The question remains which pairs do and which pairs do not belong to this collection. First of all, there exist pairs which do not belong to this collection. Indeed, if $A = Z$ is a non-diagonalizable matrix, then $(A, Z) \notin \mathscr{C}(m)$. On the other hand, one of the first results on the subject shows that the collection is far from empty. We now state this result.

Theorem 1.1. *Let A and Z be $m \times m$ matrices. If either A or Z is diagonalizable, then the pair A, Z admits simultaneous reduction to complementary triangular forms.*

This theorem first appeared in [2], Theorem 3.4, in terms of complete factorization of rational matrix functions. The result as stated here can be found in [1], Theorem 3.2. An extension of this result is given by Proposition 4.3 below. Throughout this paper, more results on complementary triangular forms are presented. Many of them have already appeared in the literature, some are new.

Most new results in this paper, however, focus on the question of whether a pair of matrices which does not admit simultaneous reduction to complementary triangular forms obtains this property after extension with zeroes. Before we give a formal description, we shall present the origin of this question. A more detailed discussion is given in [20], Ch. 5.

Let A, Z be a pair of finite rank operators acting on an (infinite dimensional) Banach space X. It is not difficult to see that there exist subspaces $M, N \subseteq X$ with $\dim M < \infty$, such that $X = M \oplus N$ and $\operatorname{Ran} A + \operatorname{Ran} Z \subseteq M$ and $N \subseteq \operatorname{Ker} A \cap \operatorname{Ker} Z$. Let A_M and Z_M be the restrictions of A and Z, respectivily, to the subspace M. The preceding conditions imply that with respect to the decomposition $X = M \oplus N$, the finite rank operators can be written as

$$A = \begin{pmatrix} A_M & O \\ O & O_N \end{pmatrix}, \qquad Z = \begin{pmatrix} Z_M & O \\ O & O_N \end{pmatrix}.$$

The finite rank operators A_M and Z_M act on the finite-dimensional space M and we shall identify them with finite matrices after fixing a vector basis in M. Assume that there exists an invertible matrix S_M such that $S_M^{-1} A_M S_M$ is upper triangular and $S_M^{-1} Z_M S_M$ is lower triangular. The operator $S = S_M \oplus I_N$ is invertible on X and the finite rank operators $S^{-1}AS$ and $S^{-1}ZS$ are the direct sum of a triangular matrix and a zero operator.

The preceding observation could provide a definition of complementary triangular forms for pairs of finite rank operators on a Banach space. There is, however, one problem. Given a pair of finite rank operators A, Z, the subspace M is not defined uniquely. For example, its dimension $\dim M$ can be taken arbitrarily large. This ambiguity naturally leads to the following question: If A_1, Z_1 is a pair of $m_1 \times m_1$ matrices, does there exist a non-negative integer m_2, such that the pair $A_1 \oplus O_{m_2}$, $Z_1 \oplus O_{m_2}$ admits simultaneous reduction to complementary triangular forms? If such a non-negative integer m_2 exists, we say that the pair of $m_1 \times m_1$ matrices A_1, Z_1 admits *simultaneous reduction to complementary triangular forms after extension with (m_2) zeroes.*

Recall that $\mathscr{C}(m)$ denotes the collection of all pairs of $m \times m$ matrices that admit simultaneous reduction to complementary triangular forms. It is left to the reader to show that if A_1, Z_1 is a pair of $m_1 \times m_1$ matrices and if m_2 and m_3 are non-negative integers, with $m_2 \leqslant m_3$, then $(A_1 \oplus O_{m_2}, Z_1 \oplus O_{m_2}) \in \mathscr{C}(m_1 + m_2)$ implies $(A_1 \oplus O_{m_3}, Z_1 \oplus O_{m_3}) \in \mathscr{C}(m_1 + m_3)$. For this reason, it is appropriate to consider for a given pair of $m_1 \times m_1$ matrices A_1, Z_1 the infimum

$$\nu(A_1, Z_1) = \inf\{m_2 \in \mathbb{Z}_0^+ \mid (A_1 \oplus O_{m_2}, Z_1 \oplus O_{m_2}) \in \mathscr{C}(m_1 + m_2)\}.$$

The infimum over the empty set is defined as $\inf \emptyset = +\infty$.

It appears that the study of this infimum is difficult. This is due to the fact that the notion of simultaneous reduction to complementary triangular forms is not completely understood. Only partial results have been obtained in this direction, some of which are presented in this paper. References are [3–6,9,17,20]. In this paper, these results are used in the following way. Typically, we start with a pair of $m_1 \times m_1$ matrices A_1, Z_1 of a type for which we have criteria on simultaneous reducibility to complementary triangular forms. We then assume that the pair $A_1 \oplus O_{m_2}, Z_1 \oplus O_{m_2}$ admits simultaneous reduction to complementary triangular forms and try to verify – by means of the known criteria – whether the pair A_1, Z_1 has the same property. In almost all cases, the latter appears to hold. In these cases, we find that $\nu(A_1, Z_1) \in \{0, \infty\}$. Example 2.7 presents a different situation. In this example, a pair of 4×4 matrices does not admit simultaneous reduction to complementary triangular forms, but obtains this property after extension with one zero. In other words, this example provides a pair of matrices A_1, Z_1 which satisfies $\nu(A_1, Z_1) = 1$.

We return to the notion of simultaneous upper triangular form. We shall denote the collection of pairs of $m \times m$ matrices that admit simultaneous reduction to upper triangular form by $\mathscr{U}(m)$. The study of this notion has a long history: Already at the end of the last century, Frobenius [10] noted that a pair of commuting matrices admits simultaneous reduction to upper triangular form. Conversely, it is obvious that if the pair A, Z admits simultaneous reduction to upper triangular form, then the commutator $AZ - ZA$ is nilpotent. McCoy's theorem [16] stated below provides a necessary and sufficient condition.

Theorem 1.2. *A pair of $m \times m$ matrices A, Z admits simultaneous reduction to upper triangular form if and only if for any polynomial $p(\lambda, \mu)$ in the non-commuting variables λ and μ, the matrix $p(A, Z)(AZ - ZA)$ is nilpotent.*

The proof in [16] is quite involved. Elementary proofs of this theorem are found in [8,11], especially the latter is suggested to the reader. The literature concerning simultaneous reduction of pairs of matrices to upper triangular form is extensive (see [7,13,14] and the references given there).

McCoy's theorem shows us that simultaneous reduction to upper triangular form after extension with zeroes is a trivial matter. Let $\mathscr{A}_1$ denote the algebra generated by a pair of $m_1 \times m_1$ matrices A_1, Z_1, and let m_2 be a non-negative integer. We remark that $\phi : \mathscr{A}_1 \oplus \mathbb{C}I_{m_2} \to \mathscr{A}_1$, defined by $\phi(X_1 \oplus \alpha I_{m_2}) = X_1$, is an algebra homomorphism. Let $A = A_1 \oplus O_{m_2}$ and $Z = Z_1 \oplus O_{m_2}$, and assume that $(A, Z) \in \mathscr{U}(m_1 + m_2)$. McCoy's theorem then provides that for each polynomial $p(\lambda, \mu)$ in the non-commuting variables λ and μ, the matrix $p(A, Z)(AZ - ZA)$ is nilpotent. Since ϕ is a homomorphism, we get that $\phi[p(A, Z)(AZ - ZA)] = p(A_1, Z_1)(A_1Z_1 - Z_1A_1)$ is nilpotent, again for each polynomial $p(\lambda, \mu)$ in the non-commuting variables λ and μ. This implies $(A_1, Z_1) \in \mathscr{U}(m_1)$. The converse can be proved as follows: If the $m_1 \times m_1$ matrix S_1 yields the upper triangular pair $S_1^{-1}A_1S_1, S_1^{-1}Z_1S_1$, then the invertible matrix $S_1 \oplus I_{m_2}$ transforms the pair $A_1 \oplus O_{m_2}, Z_1 \oplus O_{m_2}$ to an upper triangular pair. We conclude that if m_2 is a non-negative integer and A_1, Z_1 is a pair of $m_1 \times m_1$ matrices, then $(A_1, Z_1) \in \mathscr{U}(m_1)$ if and only if $(A_1 \oplus O_{m_2}, Z_1 \oplus O_{m_2}) \in \mathscr{U}(m_1 + m_2)$. This cancellation property can also be derived using the general Jordan–Hölder theorem (cf. [18]).

The next proposition gives a non-trivial connection between simultaneous reduction to complementary triangular forms and simultaneous reduction to upper triangular form.

Proposition 1.3. *Let A and Z be $m \times m$ matrices. Then $(A, Z) \in \mathscr{C}(m)$, if and only if there exists a positive definite $m \times m$ matrix H, such that $(A, HZ^*H^{-1}) \in \mathscr{U}(m)$.*

Proof. To prove the only if part, assume that there exists an invertible matrix S, such that $S^{-1}AS$ is upper triangular, and $S^{-1}ZS$ is lower triangular. Note that $S^*Z^*S^{-*}$ is upper triangular. Define $H = SS^*$. Then $S^{-1}AS$ and $S^{-1}HZ^*H^{-1}S$ are upper triangular.

The if part is proved as follows: Let H be a positive definite matrix, and S be an invertible matrix, such that $S^{-1}AS$ and $S^{-1}HZ^*H^{-1}S$ are upper triangular. Write $S^*S = R^*R$ with R an upper triangular invertible matrix. Then $U = SR^{-1} = S^{-*}R^*$ is a unitary matrix, and $S = UR$. Note that U^*AU and $U^*HZ^*H^{-1}U$ are also upper triangular. Let $K = U^*HU$. Then K is positive definite, so there exists an invertible upper triangular matrix T, such that $K = TT^*$. Consequently, $TT^*U^*Z^*UT^{-*}T^{-1}$ is upper triangular, and therefore, $T^*U^*Z^*UT^{-*}$ is upper triangular. Taking the adjoint gives that $T^{-1}U^*ZUT$ is lower triangular. Further $T^{-1}U^*AUT$ is upper triangular, and the proposition is proved. □

The notion of simultaneous reduction to upper triangular form allows for a non-trivial characterization (McCoy's theorem); the concept of simultaneous reduction to complementary triangular forms does not (or at least, not yet). One might expect that Proposition 1.3 would translate McCoy's theorem into a workable theorem for the complementary case. Unfortunately, the description of the positive definite matrix in Proposition 1.3 seems as complicated as the notion of complementary triangular forms itself. In fact, simultaneous reduction to upper triangular form on the one hand, and simultaneous reduction to complementary triangular forms on the other hand are quite distinct notions. This is nicely illustrated by Example 2.7, which provides a pair of 4×4 matrices A_1, Z_1 that admits simultaneous reduction to complementary triangular forms after extension with one zero. As before, we could define a homomorphism which maps a pair $A_1 \oplus O_{m_2}, Z_1 \oplus O_{m_2}$ to A_1, Z_1. Example 2.7 shows that the property of complementary triangular forms need not be preserved by this homomorphism, contrary to the property of simultaneous upper triangular form.

As soon as a pair of $m_1 \times m_1$ matrices admits simultaneous reduction to complementary triangular forms only after extension with a number of zeroes, the question remains of how many zeroes are needed. Proposition 2.8 gives a rough estimate from above; it states that for such a pair, at most $8m_1^2 - m_1$ zeroes are required.

We finish this introduction with a few notational conventions used in the paper. Let $\mathbb{Z}_0^+$ ($\mathbb{Z}^+$) denote the set of all (strictly) positive integers and let $\mathbb{C}$ denote the complex plane. All vector spaces are assumed to have complex scalars. Further, $\subseteq$ denotes inclusion where equality may hold, while $\subset$ denotes strict inclusion. If X is a vector space, then dim X denotes its dimension. If T is an $m \times n$ matrix, write Ran $T = \{Tx \mid x \in \mathbb{C}^n\}$ and Ker $T = \{x \mid x \in \mathbb{C}^n, Tx = 0\}$. The rank of a matrix is denoted by rank $T = \dim \operatorname{Ran} T$. Further, $\sigma(T)$ denotes the spectrum (set of eigenvalues) of the matrix T. If X, Y are subspaces in a vector space, then $X + Y = \{x + y \mid x \in X,\ y \in Y\}$. In particular, if $X \cap Y = (0)$, then $X + Y = X \oplus Y$. Let O_n denote the $n \times n$ zero matrix and I_n the $n \times n$ identity matrix. Usually, we identify matrices with their action as a linear operator. This leads to the following convention, which is used several times in this paper. Let B_1 be an $m_1 \times m_1$ matrix and let B_2 be an $m_2 \times m_2$ matrix, and define the $m \times m$ matrix $B = B_1 \oplus B_2$, where $m = m_1 + m_2$. In general, we make the identification that an $n \times n$ matrix acts as a linear operator on $\mathbb{C}^n$. Therefore, B acts on $\mathbb{C}^m$ and with respect to the decomposition $\mathbb{C}^m = \mathbb{C}^{m_1} \oplus \mathbb{C}^{m_2}$ it assumes the form

$$B = \begin{pmatrix} B_1 & O \\ O & B_2 \end{pmatrix}.$$

After making the identifications $\mathbb{C}^{m_1} = \mathbb{C}^{m_1} \oplus (0)$ and $\mathbb{C}^{m_2} = (0) \oplus \mathbb{C}^{m_2}$, we get that B_1 and B_2 denote the restrictions of B to the subspaces $\mathbb{C}^{m_1}$ and $\mathbb{C}^{m_2}$, respectively.

2. Nests and first results

In this section, we shall describe complementary triangular forms in terms of nests of invariant subspaces. This description will be used to derive a number of results on simultaneous reduction to complementary triangular forms after extension with zeroes. Recall that each complex $m \times m$ matrix A can be reduced to upper triangular form by means of a unitary transformation. This fact is known as Schur's theorem; see for example Theorem 5.2.2 in [15] or Theorem 1.9.1 in [12].

Let S be an invertible (e.g. unitary) $m \times m$ matrix, such that $S^{-1}AS$ is an upper triangular matrix. Write the matrix S in the form

$$S = (s_1, \ldots, s_m), \tag{1}$$

i.e., as a row of m linear independent column vectors $s_1, \ldots, s_m$. These vectors define subspaces

$$M_k = \operatorname{span}\{s_1, \ldots, s_k\}, \quad k = 0, \ldots, m, \tag{2}$$

which form the set $\mathcal{M} = \{M_k\}_{k=0}^m$. The set $\mathcal{M}$ is a *nest*, i.e., it is a set that is linearly ordered by inclusion. Furthermore, $\mathcal{M}$ is not properly contained in any other nest of subspaces. Therefore, $\mathcal{M}$ is called a *maximal nest of subspaces*. As a matter of fact, *all* maximal nests of subspaces in $\mathbb{C}^m$ are of the form

$$\mathcal{M} = \{(0) = M_0 \subset M_1 \subset \cdots \subset M_m = \mathbb{C}^m\} = \{M_k\}_{k=0}^m,$$

where $\dim M_k = k$ for $k = 0, \ldots, m$. Finally, the maximal nest of subspaces as defined in Eq. (2) consists of invariant subspaces for A, and is called a *maximal invariant nest of subspaces* for A. Schur's theorem thus says that each $m \times m$ matrix A has a maximal invariant nest of subspaces.

Conversely, let $\mathcal{M} = \{M_k\}_{k=0}^m$ be a maximal invariant nest of subspaces for A. For each $k \in \{1, \ldots, m\}$, there exists a unique complex number α_k, such that

$$(A - \alpha_k)M_k \subseteq M_{k-1}.$$

We define the diagonal vector of A with respect to the maximal invariant nest of subspaces $\mathcal{M}$ as

$$\operatorname{diag}(A; \mathcal{M}) = \alpha = (\alpha_1, \ldots, \alpha_m)^{\mathrm{T}}.$$

Each set of vectors $s_1, \ldots, s_m$ that satisfies Eq. (2) defines an invertible $m \times m$ matrix S as in Eq. (1), such that $S^{-1}AS$ is upper triangular, with $\operatorname{diag}(S^{-1}AS) = \alpha$.

To reformulate the notion of simultaneous reduction to complementary triangular forms in terms of maximal invariant nests of subspaces, we introduce the following terminology. Two maximal nests $\mathcal{M} = \{M_k\}_{k=0}^m$ and $\mathcal{N} = \{N_k\}_{k=0}^m$ in $\mathbb{C}^m$ are called *matching*, if

$$M_k \oplus N_{m-k} = \mathbb{C}^m, \quad k = 0, \ldots, m. \tag{3}$$

An $m \times m$ matrix P is a projection if $P = P^2$. We shall not require P to be self-adjoint, i.e., P need not be a orthogonal projection. A set of projections $\mathcal{P} = \{P_k\}_{k=0}^m$ is called a *maximal nest of projections*, if the sets of subspaces $\{\operatorname{Ran} P_k\}_{k=0}^m$ and $\{\operatorname{Ker} P_k\}_{k=0}^m$ are matching maximal nests. The following simple observation provides alternative descriptions of simultaneous reduction to complementary triangular forms.

Lemma 2.1. *Let A and Z be two $m \times m$ matrices, then the following statements are equivalent.*

1. *The pair A, Z admits simultaneous reduction to complementary triangular forms.*
2. *There exist matching maximal invariant nests $\mathcal{M}$ and $\mathcal{N}$ for A and Z, respectively.*
3. *There exists a maximal nest of projections $\mathcal{P} = \{P_k\}_{k=0}^m$ such that $AP_k = P_kAP_k$ and $P_kZ = P_kZP_k$ for $k = 0, \ldots, m$.*

Proof. To prove that the first statement implies the second one, let S be an invertible $m \times m$ matrix as in Eq. (1), such that $S^{-1}AS$ is upper triangular and $S^{-1}ZS$ is lower triangular. Then the maximal nests of subspaces $\mathcal{M} = \{M_k\}_{k=0}^m$ and $\mathcal{N} = \{N_k\}_{k=0}^m$, defined by

$$M_k = \text{span}\{s_1, \dots, s_k\}, \qquad N_{m-k} = \text{span}\{s_{k+1}, \dots, s_m\}, \quad k = 0, \dots, m$$

are invariant for A and Z respectively, and are matching. Next, we prove that the second statement implies the first one. If $\mathcal{M} = \{M_k\}_{k=0}^m$ and $\mathcal{N} = \{N_k\}_{k=0}^m$ are matching maximal nests of subspaces, invariant for A and Z respectively, then the non-zero vectors $s_1, \dots, s_m$, determined up to multiplicative constants by $s_k \in M_k \cap N_{m-k+1}$, define an invertible $m \times m$ matrix S as in Eq. (1), which reduces A and Z to complementary triangular forms. Here, we used

$$\dim(M_k \cap N_l) = \max\{k + l - m, 0\}.$$

The equivalence between the second and third statement is proved as follows. By definition, two matching maximal nests of subspaces give rise to a maximal nest of projections, and vice versa. In addition, $AP_k = P_kAP_k$ is equivalent to $A(\text{Ran}\, P_k) \subseteq \text{Ran}\, P_k$, and $P_kZ = P_kZP_k$ is equivalent to $Z(\text{Ker}\, P_k) \subseteq \text{Ker}\, P_k$. The proof is finished. □

We make one more remark on complementary triangular forms in terms of maximal invariant nests: Let S denote the invertible $m \times m$ matrix, obtained from the nests of subspaces $\mathcal{M}$ and $\mathcal{N}$ as in the proof of Lemma 2.1; note that S is determined up to multiplication to the right by an invertible diagonal matrix. Assume that

$$\text{diag}(A; \mathcal{M}) = (\alpha_1, \dots, \alpha_m)^{\mathrm{T}} = \alpha, \qquad \text{diag}(Z; \mathcal{N}) = (\zeta_1, \dots, \zeta_m)^{\mathrm{T}} = \zeta.$$

Then

$$\text{diag}(S^{-1}AS) = (\alpha_1, \dots, \alpha_m)^{\mathrm{T}} = \alpha,$$

but

$$\text{diag}(S^{-1}ZS) = (\zeta_m, \dots, \zeta_1)^{\mathrm{T}},$$

i.e., the entries of $\text{diag}(Z; \mathcal{N})$ appear in *reversed order* on the diagonal of the lower triangular matrix $S^{-1}ZS$.

Let $\sigma \subseteq \sigma(A)$ be a non-empty subset of the spectrum of the $m \times m$ complex matrix A and define the linear subspace

$$N_\sigma(A) = \text{span}\{\text{Ker}(A - \alpha)^m \mid \alpha \in \sigma\}.$$

If $\sigma = \{\alpha\}$, we usually write $N_{\{\alpha\}}(A) = N_\alpha(A)$. The following well-known lemma states that an invariant subspace of a matrix admits a decomposition, related to the spectrum of the matrix.

Lemma 2.2. *Let A be a complex $m \times m$ matrix and assume that the spectrum of A is the disjoint union of two non-empty subsets σ_1 and σ_2. Let M be an invariant subspace for A. Then M admits the decomposition*

$$M = M_1 \oplus M_2, \qquad M_i = M \cap N_{\sigma_i}(A), \quad i = 1, 2.$$

A decomposition as in Lemma 2.2 is referred to as a *spectral decomposition* of M, associated with σ_1 and σ_2. For the proof of Lemma 2.2, we refer to Section 2.1 in [12].

Lemma 2.3. *Let $\mathscr{M} = \{M_k\}_{k=0}^m$ be a maximal invariant nest for the $m \times m$ matrix A and let N be an n-dimensional invariant subspace for A. Write $M_{k,1} = M_k \cap N$ for $k = 0, \dots, m$, and denote the restriction of A to N by A_1. Put*

$$\tau(i) = \min\{k \mid 0 \leqslant k \leqslant m, \dim M_{k,1} = i\}, \quad i = 0, \dots, n.$$

Then the nest $\mathscr{M}_1 = \{M_{\tau(i),1}\}_{i=0}^n$ is a maximal invariant nest in N for A_1.

Proof. Fix $k \in \{0, \dots, m\}$. Since M_k and N are invariant subspaces for A, the same is true for $M_{k,1}$. If $x \in M_{k,1}$, then $A_1 x = Ax \in M_{k,1}$, i.e., $M_{k,1}$ is an invariant subspace for A_1. Further note that $M_{0,1} = (0)$ and $M_{m,1} = N$. In addition it is easily verified that $\dim M_{k,1} - \dim M_{k-1,1} \leqslant 1$. It follows that the integers $\tau(0), \tau(1), \dots, \tau(n)$ are well defined, that $\tau(0) < \tau(1) < \cdots < \tau(n)$ and that $\mathscr{M}_1$ is a maximal invariant nest in N for A_1. $\square$

In the following proposition, we shall make extensive use of the preceding lemmas on nests of subspaces. Moreover, Corollary 2.5 provides a first class of pairs of square matrices A_1, Z_1 for which $\nu(A_1, Z_1) \in \{0, \infty\}$.

Proposition 2.4. *Let A_i and Z_i be $m_i \times m_i$ matrices $(i = 1, 2)$, and write $m = m_1 + m_2$. Define the $m \times m$ matrices $A = A_1 \oplus A_2$ and $Z = Z_1 \oplus Z_2$. Assume that $\sigma(A_1) \cap \sigma(A_2) = \sigma(Z_1) \cap \sigma(Z_2) = \emptyset$. If the pair A, Z admits simultaneous reduction to complementary triangular forms, then so do the pair A_1, Z_1 and the pair A_2, Z_2.*

Proof. We write

$$A = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}, \qquad Z = \begin{pmatrix} Z_1 & O \\ O & Z_2 \end{pmatrix}$$

with respect to the decomposition $\mathbb{C}^m = \mathbb{C}^{m_1} \oplus \mathbb{C}^{m_2}$. Let $\mathscr{M} = \{M_k\}_{k=0}^m$ and $\mathscr{N} = \{N_k\}_{k=0}^m$ be matching maximal invariant nests of subspaces for A and Z, respectively. Since $\sigma(A_1) \cap \sigma(A_2) = \emptyset$, we get

$$\mathbb{C}^{m_i} = \operatorname{span}\{\operatorname{Ker}(A - \alpha)^m \mid \alpha \in \sigma(A_i)\}, \quad i = 1, 2$$

and since $\sigma(Z_1) \cap \sigma(Z_2) = \emptyset$, we get

$$\mathbb{C}^{m_i} = \operatorname{span}\{\operatorname{Ker}(Z - \zeta)^m \mid \zeta \in \sigma(Z_i)\}, \quad i = 1, 2.$$

We may apply Lemma 2.2 to obtain the decompositions

$$M_k = (M_k \cap \mathbb{C}^{m_1}) \oplus (M_k \cap \mathbb{C}^{m_2}),$$

$$N_{m-k} = (N_{m-k} \cap \mathbb{C}^{m_1}) \oplus (N_{m-k} \cap \mathbb{C}^{m_2})$$

for $k = 0, \dots, m$. The matching condition $M_k \oplus N_{m-k} = \mathbb{C}^m$ then gives

$$(M_k \cap \mathbb{C}^{m_i}) \oplus (N_{m-k} \cap \mathbb{C}^{m_i}) = \mathbb{C}^{m_i}, \tag{4}$$

$k = 0, \dots, m$ and $i = 1, 2$. Fix $i \in \{1, 2\}$ and define the integers

$$\tau_i(s) = \min\{k \mid 0 \leqslant k \leqslant m, \dim(M_k \cap \mathbb{C}^{m_i}) = s\}, \quad s = 0, \dots, m_i.$$

Lemma 2.3 provides that $\{M_{\tau_i(s)} \cap \mathbb{C}^{m_i}\}_{s=0}^{m_i}$ is a maximal invariant nest for A_i. The matching condition (4) gives that $\{N_{m-\tau_i(s)} \cap \mathbb{C}^{m_i}\}_{s=0}^{m_i}$ is a maximal invariant nest for Z_i which matches $\{M_{\tau_i(s)} \cap \mathbb{C}^{m_i}\}_{s=0}^{m_i}$. By Lemma 2.1, the pair A_i, Z_i admits simultaneous reduction to complementary triangular forms. □

For a more elaborate proof of Proposition 2.4 in which certain additional details are presented, we refer to [19].

Corollary 2.5. *Let A_1 and Z_1 be invertible $m_1 \times m_1$ matrices, let m_2 be a non-negative integer, and write $m = m_1 + m_2$. If the pair of $m \times m$ matrices $A = A_1 \oplus O_{m_2}$, $Z = Z_1 \oplus O_{m_2}$ admits simultaneous reduction to complementary triangular forms, then so does the pair A_1, Z_1.*

An $m \times m$ matrix is *unicellular or uniserial*, i.e., has a unique maximal invariant nest of subspaces, if and only if it is similar to an $m \times m$ Jordan block. The following theorem shows that for pairs of unicellular matrices A_1, Z_1 the infimum $\nu(A_1, Z_1) \in \{0, \infty\}$.

Theorem 2.6. *Let A_1 and Z_1 be unicellular $m_1 \times m_1$ matrices, m_2 a non-negative integer, and let $m = m_1 + m_2$. Define the $m \times m$ matrices $A = A_1 \oplus O_{m_2}$ and $Z = Z_1 \oplus O_{m_2}$. If the pair A, Z admits simultaneous reduction to complementary triangular forms then so does the pair A_1, Z_1.*

Proof. The proof of the theorem is divided into three parts, corresponding to the following cases:
1. The matrices A_1 and Z_1 are invertible.
2. The matrices A_1 and Z_1 are singular, therefore nilpotent.
3. The matrix A_1 is singular, the matrix Z_1 is invertible. By a symmetry argument, the case A_1 invertible and Z_1 singular is also covered by this part.

Part 1: The matrices A_1 and Z_1 are invertible. Apply Corollary 2.5.

Part 2: The matrices A_1 and Z_1 are nilpotent.

Assume there exist maximal invariant nests $\mathscr{M} = \{M_k\}_{k=0}^{m}$ and $\mathscr{N} = \{N_k\}_{k=0}^{m}$ for A and Z respectively, that are matching. Write $M_{k,i} = M_k \cap \mathbb{C}^{m_i}$ and $N_{k,i} = N_k \cap \mathbb{C}^{m_i}$ for $k = 0, \dots, m$ and $i = 1, 2$. For $s = 0, \dots, m_1$, introduce

$$\pi(s) = \min\{k \mid 0 \leqslant k \leqslant m, \dim M_{k,1} = s\},$$

and

$$\rho(s) = \min\{l \mid 0 \leqslant l \leqslant m, \dim N_{l,1} = s\}.$$

By Lemma 2.3, it follows that $\{M_{\pi(s),1}\}_{s=0}^{m_1}$ and $\{N_{\rho(s),1}\}_{s=0}^{m_1}$ are maximal invariant nests for A_1 and Z_1, respectively. It remains to prove that these nests are matching.

Since A_1 and Z_1 are unicellular nilpotent $m_1 \times m_1$ matrices, there exists a basis $\phi_1, \dots, \phi_{m_1}$ for $\mathbb{C}^{m_1}$, such that

$$A_1\phi_1 = 0, \qquad A_1\phi_s = \phi_{s-1}, \quad s = 2, \dots, m_1.$$

There also exists a basis $\psi_1, \dots, \psi_{m_1}$ for $\mathbb{C}^{m_1}$, such that

$$Z_1\psi_1 = 0, \qquad Z_1\psi_s = \psi_{s-1}, \quad s = 2, \dots, m_1.$$

We now have

$$M_{\pi(s),1} = \operatorname{span}\{\phi_1, \dots, \phi_s\}, \qquad N_{\rho(s),1} = \operatorname{span}\{\psi_1, \dots, \psi_s\}, \quad s = 0, \dots, m_1.$$

Claim. *For $s = 0, \dots, m_1$, the following two identities hold:*

$$M_{\pi(s)} = M_{\pi(s),1} \oplus M_{\pi(s),2}, \qquad N_{\rho(s)} = N_{\rho(s),1} \oplus N_{\rho(s),2}.$$

To prove the first identity, fix $s \in \{1, \dots, m_1\}$, the case $s = 0$ being trivial. Let $x \in M_{\pi(s)}$ and write $x = x_1 + x_2$, where $x_i \in \mathbb{C}^{m_i}$. We need to prove that $x_1 \in M_{\pi(s)}$. If $x_1 = 0$ this is trivial, so assume that $0 \neq x_1 = \gamma_1\phi_1 + \dots + \gamma_p\phi_p$, where $\gamma_1, \dots, \gamma_p$ are complex numbers and $p \in \{1, \dots, m_1\}$ such that $\gamma_p \neq 0$. Then $Ax = A_1x_1 = \gamma_2\phi_1 + \dots + \gamma_p\phi_{p-1}$. On the other hand,

$$AM_{\pi(s)} \subseteq M_{\pi(s-1),1} = \operatorname{span}\{\phi_1, \dots, \phi_{s-1}\}.$$

Therefore $p - 1 \leqslant s - 1$, thus $p \leqslant s$ and hence $x_1 \in M_{\pi(s)}$. The second identity is dealt with similarly, so the claim is proved.

To finish the proof of Part 2, fix $s \in \{0, \dots, m_1\}$ and distinguish two cases:

Case 1: $\pi(s) + \rho(m_1 - s) \leqslant m$. In this case, $M_{\pi(s)} \cap N_{\rho(m_1-s)} = (0)$ and hence $M_{\pi(s),1} \oplus N_{\rho(m_1-s),1} \subseteq \mathbb{C}^{m_1}$. A dimension argument shows that equality holds.

Case 2: $\pi(s) + \rho(m_1 - s) > m$. In this case, $M_{\pi(s)} + N_{\rho(m_1-s)} = \mathbb{C}^m$, and hence $M_{\pi(s),1} + N_{\rho(m_1-s),1} = \mathbb{C}^{m_1}$. A dimension argument shows that $M_{\pi(s),1} \cap N_{\rho(m_1-s),1} = (0)$.

In both cases, it is proved that $M_{\pi(s),1} \oplus N_{\rho(m_1-),1} = \mathbb{C}^{m_1}$.

Part 3: The matrix A_1 is nilpotent, the matrix Z_1 is invertible.

Assume there exist maximal invariant nests $\mathscr{M} = \{M_k\}_{k=0}^{m}$ and $\mathscr{N} = \{N_l\}_{l=0}^{m}$ for A and Z respectively, which are matching. Since A_1 is nilpotent, one may define – as in Part 2 of the proof – a strictly increasing mapping $\pi : \{0,\dots,m_1\} \to \{0,\dots,m\}$ such that $\dim M_{\pi(s),1} = s$, and $M_{\pi(s)} = M_{\pi(s),1} \oplus M_{\pi(s),2}$. Since Z_1 is invertible, it follows by the proof of Proposition 2.4, that $N_k = N_{k,1} \oplus N_{k,2}$ for $k = 0,\dots,m$. In particular ($s = 0,\dots,m_1$),

$$\begin{aligned} M_{\pi(s)} \oplus N_{m-\pi(s)} &= [M_{\pi(s),1} \oplus N_{m-\pi(s),1}] \oplus [M_{\pi(s),2} \oplus N_{m-\pi(s),2}] \\ &= \mathbb{C}^{m_1} \oplus \mathbb{C}^{m_2}. \end{aligned}$$

Consequently, the maximal invariant nests $\{M_{\pi(s),1}\}_{s=0}^{m_1}$ and $\{N_{m-\pi(m_1-s),1}\}_{s=0}^{m_1}$ for A_1 and Z_1 are matching. This finishes the proof of the theorem. □

We conclude this section with two important results which consider the case when extension with zeroes makes a difference for a pair of $m_1 \times m_1$ matrices A_1, Z_1, i.e., in the case when $0 < \nu(A_1, Z_1) < \infty$. The following example provides a pair $m_1 \times m_1$ matrices A_1, Z_1 for which $\nu(A_1, Z_1)$ is a strictly positive integer.

Example 2.7. This example provides a pair of nilpotent 4×4 matrices A_1, Z_1, that does not admit simultaneous reduction to complementary triangular forms, while the pair of 5×5 matrices $A_1 \oplus 0$, $Z_1 \oplus 0$ does have this property. Indeed, the pair of 4×4 matrices

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad Z_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

does not admit simultaneous reduction to complementary triangular forms (as will be shown later), while the pair of 5×5 matrices $A_1 \oplus 0$, $Z_1 \oplus 0$ is reduced to complementary triangular forms by the invertible 5×5 matrix

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

as follows:

$$S^{-1}(A_1 \oplus 0)S = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad S^{-1}(Z_1 \oplus 0)S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

On the other hand, we will show that $(A_1, Z_1) \notin \mathscr{C}(4)$. Indeed, there exist no matching maximal invariant nests of subspaces for A_1 and Z_1, respectively. A description of complementary triangular forms in terms of maximal invariant nests has been given in Section 2. Note that Z_1 is unicellular, and that its unique maximal invariant nest of subspaces $\mathscr{N} = \{N_k\}_{k=0}^4$ is given by

$$
\begin{aligned}
N_0 &= (0), \\
N_1 &= \operatorname{span}\{e_1\}, \\
N_2 &= \operatorname{span}\{e_1, e_2\}, \\
N_3 &= \operatorname{span}\{e_1, e_2, e_3\}, \\
N_4 &= \mathbb{C}^4.
\end{aligned}
$$

Assume there exists a maximal invariant nest of subspaces $\mathscr{M} = \{M_k\}_{k=0}^4$ for A_1, that matches $\mathscr{N}$. First of all, $M_1 \subseteq \operatorname{Ker} A_1 = \operatorname{span}\{e_3, e_4\}$. Further, to obtain that $M_1 \oplus N_3 = \mathbb{C}^4$, we have to take $M_1 = \operatorname{span}\{e_4 + \alpha e_3\}$ for some complex number α, since $e_3 \in N_3$. Since $A_1(M_2) \subseteq M_1 \cap \operatorname{Ran} A_1 = (0)$, we get $M_2 = \operatorname{Ker} A_1$. Since $A_1(M_3) \subseteq M_2 = \operatorname{Ker} A_1$, it follows that $M_3 = \operatorname{Ker} A_1^2 = \operatorname{span}\{e_1, e_3, e_4\}$. But then $M_3 \cap N_1 = \operatorname{span}\{e_1\} \neq 0$, and a contradiction has been obtained. Therefore, $(A_1, Z_1) \notin \mathscr{C}(4)$. □

In short, Example 2.7 provides a pair of 4×4 matrices A_1, Z_1 for which $\nu(A_1, Z_1) = 1$. Given a pair of $m_1 \times m_1$ matrices A_1, Z_1, the question is, whether anything can be said about the integer $\nu(A_1, Z_1)$ in the case when $0 < \nu(A_1, Z_1) < \infty$. The following proposition provides an upper bound only in terms of m_1. This upper bound is probably never sharp.

Proposition 2.8. *Let A_1, Z_1 be a pair of $m_1 \times m_1$ matrices such that $0 < \rho(A_1, Z_1) < \infty$. In other words, the pair does not admit simultaneous reduction to complementary triangular forms, but obtains this property after extension with a sufficiently large number of zeroes. Then the following estimate holds*:

$$\nu(A_1, Z_1) \leqslant 8m_1^2 - m_1.$$

Proof. Let m_2 be a positive integer and $m = m_1 + m_2$, such that the pair of $m \times m$ matrices $A = A_1 \oplus O_{m_2}$, $Z = Z_1 \oplus O_{m_2}$ admits simultaneous reduction to complementary triangular forms. Let S be an invertible $m \times m$ matrix, such that $S^{-1}AS$ is upper triangular and $S^{-1}ZS$ is lower triangular. Write $S^{-1}AS = (a_1, \ldots, a_m)$, $S^{-1}ZS = (z_1, \ldots, z_m)$ and $S = (s_1, \ldots, s_m)$ as row matrices. For $0 \leqslant k \leqslant m$, let P_k denote the projection with $\operatorname{Ran} P_k = \operatorname{span}\{s_1, \ldots, s_k\}$ and $\operatorname{Ker} P_k = \operatorname{span}\{s_{k+1}, \ldots, s_m\}$. Further, put $\mathscr{P} = \{P_0, \ldots, P_m\}$.

Consider the mapping $d_A : \{0, \ldots, m\} \to \{0, \ldots, \operatorname{rank} A\}$ given by $d_A(k) = \dim \operatorname{span}\{a_1, \ldots, a_k\}$ for $0 \leqslant k \leqslant m$. Then d_A is monotonically increasing, $d_A(0) = 0$ and $d_A(m) = \operatorname{rank} A$. Since $d_A(k) \leqslant d_A(k-1) + 1$ for $1 \leqslant k \leqslant m$, it follows that d_A is surjective. Define the sets of integers

$$\mathscr{M}_j = \{k \mid d_A(k) = j\}, \quad 0 \leqslant j \leqslant \operatorname{rank} A.$$

Define $\mu_j = \max \mathscr{M}_j$ for all $0 \leqslant j \leqslant \operatorname{rank} A$ and $\mu_{-1} = 0$. Then $\mathscr{M}_j = \{\mu_{j-1} + 1, \ldots, \mu_j\}$ for all j. The set of integers

$$\tilde{\mathscr{M}} = \{\mu_{j-1} + 1, \mu_j \mid 0 \leqslant j \leqslant \operatorname{rank} A\},$$

which has $\# \tilde{\mathscr{M}} \leqslant 2 \operatorname{rank} A + 2$ elements, defines a useful nest of subspaces for A. For $\mu \in \tilde{\mathscr{M}}$, define $M_\mu = \operatorname{span}\{s_1, \ldots, s_\mu\}$. Note that

$$\begin{aligned} A\, M_{\mu_j} &= A \operatorname{span}\{s_1, \ldots, s_{\mu_j}\} = S \operatorname{span}\{a_1, \ldots, a_{\mu_j}\} \\ &= S \operatorname{span}\{a_1, \ldots, a_{\mu_{j-1}+1}\} \subseteq \operatorname{span}\{s_1, \ldots, s_{\mu_{j-1}+1}\} = M_{\mu_{j-1}+1}. \end{aligned}$$

If one considers the upper triangular matrix $S^{-1}AS$, it is immediate that for each $0 \leqslant j \leqslant \operatorname{rank} A - 1$, there exists a complex number β_j, such that

$$(A - \beta_j) M_{\mu_j+1} \subseteq M_{\mu_j}.$$

Note that with the nest of subspaces $\{M_\mu \mid \mu \in \tilde{\mathscr{M}}\}$, we may associate a nest of projections in $\mathscr{P}$. Indeed, consider the nest of projections $\mathscr{P}_A \subseteq \mathscr{P}$ with $\{\operatorname{Ran} P \mid P \in \mathscr{P}_A\} = \{M_\mu \mid \mu \in \tilde{\mathscr{M}}\}$.

We have derived that for the matrix A, there exists a nest of projections $\mathscr{P}_A = \{P_0, \ldots, P_n\}$ and complex numbers $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$, such that the following statements hold:

1. $P_0 = O_m$, $P_n = I_m$, $P_{k-1} < P_k$, $1 \leqslant k \leqslant n$.
2. $n \leqslant 2 \operatorname{rank} A + 1$.
3. $(A - \tilde{\alpha}_k)P_k = P_{k-1}(A - \tilde{\alpha}_k)P_k$, $1 \leqslant k \leqslant n$.
4. If $\operatorname{rank}(P_k - P_{k-1}) \geqslant 2$, then $\tilde{\alpha}_k = 0$.

In the same manner, a nest of projections $\mathscr{P}_Z \subseteq \mathscr{P}$ can be obtained for the matrix Z. Indeed, first define a nest of invariant subspaces $\tilde{\mathscr{N}}$ for Z, just as $\tilde{\mathscr{M}}$ has been defined for A. Then define the nest of projections $\mathscr{P}_Z \subseteq \mathscr{P}$, such that $\{\operatorname{Ker} P \mid P \in \mathscr{P}_Z\} = \tilde{\mathscr{N}}$. (We consider kernels of the projections here instead of ranges since $S^{-1}ZS$ is lower triangular.) Next, consider

$$\mathscr{P}_1 = \mathscr{P}_A \cup \mathscr{P}_Z \subseteq \mathscr{P}.$$

Write $\mathscr{P}_1 = \{P_0, P_1, \ldots, P_\nu\}$, with $O_m = P_0 < P_1 < \cdots < P_\nu = I_m$. There exist complex numbers $\alpha_1, \ldots, \alpha_\nu$, and $\zeta_1, \ldots, \zeta_\nu$, such that

$$(A - \alpha_k)P_k = P_{k-1}(A - \alpha_k)P_k, \qquad P_k(Z - \zeta_k) = P_k(Z - \zeta_k)P_{k-1}, \quad 1 \leqslant k \leqslant \nu$$

Also, we have the estimate $\nu \leqslant 1 + 2 \operatorname{rank} A + 2 \operatorname{rank} Z$. Put $\Delta P_k = P_k - P_{k-1}$ for $k = 1, \ldots, \nu$. Define the subspaces

$$M_k = \Delta P_k(\operatorname{Ran} A + \operatorname{Ran} Z), \qquad N_k = \operatorname{Ker}(A\Delta P_k) \cap \operatorname{Ker}\,(Z\Delta P_k),$$
$$k = 1, \ldots, \nu.$$

Then $N_k = \operatorname{Ker}\Delta P_k \oplus \tilde{N}_k$, where $\tilde{N}_k = N_k \cap \operatorname{Ran}\Delta P_k$. Note that $M_k + \tilde{N}_k$ is a subspace in $\operatorname{Ran}\Delta P_k$. Therefore, there exists a subspace $R_k \subseteq \operatorname{Ran}\Delta P_k$, such that $(M_k + \tilde{N}_k) \oplus R_k = \operatorname{Ran}\Delta P_k$. In addition, let $\hat{N}_k \subseteq N_k$, such that $\hat{M}_k \oplus \hat{N}_k = \operatorname{Ran}\Delta P_k$, where $\hat{M}_k = M_k \oplus R_k$. Define

$$M = M_1 \oplus \cdots \oplus M_\nu, \qquad R = R_1 \oplus \cdots \oplus R_\nu,$$

$$N = \tilde{N}_1 \oplus \cdots \oplus \tilde{N}_\nu, \qquad \hat{N} = \hat{N}_1 \oplus \cdots \oplus \hat{N}_\nu.$$

As a matter of fact,

$$N = \bigcap_{k=1}^{\nu} N_k.$$

Indeed, if $x = x_1 + \cdots + x_\nu$ with $x_k \in \tilde{N}_k$, then $m_k = x - x_k \in \operatorname{Ker}\Delta P_k$ and hence $x = x_k + m_k \in \tilde{N}_k + \operatorname{Ker}\Delta P_k$. On the other hand, if $x = x_k + m_k$ with $x_k \in \tilde{N}_k$ and $m_k \in \operatorname{Ker}\Delta P_k$, then $x = (\sum_{k=1}^{\nu}\Delta P_k)x = \sum_{k=1}^{\nu} x_k \in \tilde{N}_1 \oplus \cdots \oplus \tilde{N}_\nu$.

Define $\hat{M} = M \oplus R$, then $\operatorname{Ran} A + \operatorname{Ran} Z \subseteq \hat{M}$. Further,

$$\hat{N} \subseteq N = \bigcap_{k=1}^{\nu} [\operatorname{Ker}(A\Delta P_k) \cap \operatorname{Ker}(Z\Delta P_k)]$$

implies $\operatorname{Ker} A \cap \operatorname{Ker} Z \supseteq \hat{N}$. Also, $\hat{M} \oplus \hat{N} = \mathbb{C}^m$.

Since $A\Delta P_k = \sum_{j=1}^{k-1} \Delta P_j A \Delta P_k + \alpha_k \Delta P_k$, it follows that

$$A\hat{M}_k \subseteq \operatorname{Ran}(A\Delta P_k) \subseteq \sum_{j=1}^{k-1} \hat{M}_j \oplus \ \operatorname{Ran}(\alpha_k \mathrm{Delta} P_k).$$

In addition, $Z\Delta P_k = \sum_{j=k+1}^{\nu} \Delta P_j Z \Delta P_k + \zeta_k \Delta P_k$ implies

$$Z\hat{M}_k \subseteq \operatorname{Ran}(Z\Delta P_k) \subseteq \sum_{j=k+1}^{\nu} \hat{M}_j \oplus \operatorname{Ran}(\zeta_k \Delta P_k).$$

With respect to the decomposition $\mathbb{C}^m = \hat{M}_1 \oplus \cdots \hat{M}_\nu \oplus \hat{N}$, the matrices A and Z assume the forms

$$A = \begin{pmatrix} \alpha_1 \hat{I}_1 & * & * & \cdots & * & O \\ O & \alpha_2 \hat{I}_2 & * & \cdots & * & O \\ O & O & \alpha_3 \hat{I}_3 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & * & O \\ \vdots & & & \ddots & \alpha_\nu \hat{I}_\nu & O \\ O & \cdots & \cdots & \cdots & O & O_{\hat{N}} \end{pmatrix}$$

and

$$Z = \begin{pmatrix} \zeta_1 \hat{I}_1 & O & O & \cdots & \cdots & O \\ * & \zeta_2 \hat{I}_2 & O & & & \vdots \\ * & * & \zeta_3 \hat{I}_3 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ * & * & \cdots & * & \zeta_\nu \hat{I}_\nu & O \\ O & O & \cdots & O & O & O_{\hat{N}} \end{pmatrix}.$$

Here $\hat{I}_k$ denotes the identity matrix on the subspace $\hat{M}_k$. We shall estimate the dimension of $\hat{M}$ from above. First of all,

$$\dim M = \sum_{k=1}^{\nu} \dim M_k \leqslant \nu \dim(\operatorname{Ran} A + \operatorname{Ran} Z).$$

Further,

$$\operatorname{codim} N \leqslant \sum_{k=1}^{\nu} \operatorname{codim} N_k \leqslant \nu \operatorname{codim}(\operatorname{Ker} A \cap \operatorname{Ker} Z).$$

Then $(M + N) \oplus R = \mathbb{C}^m$ implies $\dim R \leqslant \operatorname{codim} N$. Therefore,

$$\dim \hat{M} = \dim M + \dim R \leqslant \nu\,(\dim[\operatorname{Ran} A + \operatorname{Ran} Z] \\ + \operatorname{codim}(\operatorname{Ker} A \cap \operatorname{Ker} Z)).$$

If $\operatorname{rank} A = \operatorname{rank} Z = m_1$, then Corollary 2.5 provides that $\nu(A_1, Z_1) = m_1$ and we are ready. In all other cases, we get $\operatorname{rank} A + \operatorname{rank} Z \leqslant 2m_1 - 1$. Together with $\nu \leqslant 1 + 2 \operatorname{rank} A + 2 \operatorname{rank} Z$, we obtain the estimate in the proposition. The proposition is proved. □

3. Companion matrices

In this section, we discuss complementary triangular forms after extension with zeroes for pairs of companion matrices. Recall that a *first companion* matrix is of the form (see for example [15])

$$C_a = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & 1 & \\ & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{m-2} & -a_{m-1} \end{pmatrix}, \tag{5}$$

where $a_0, \ldots, a_{m-1}$ are complex numbers. Note that there is a one-to-one correspondence between monic polynomials of degree m and first companion matrices, given by the equation

$$\det(\lambda - C_a) = a_0 + a_1\lambda + \cdots + a_{m-1}\lambda^{m-1} + \lambda^m.$$

The generalized eigenvectors of a first companion matrix are determined by the corresponding eigenvalues as follows. If α is an eigenvalue for the $m \times m$ first companion matrix C_a with algebraic multiplicity n, then the vectors

$$x_1(\alpha) = (\ 1, \alpha, \ldots, \alpha^{m-1}\ )^{\mathrm{T}},$$

$$x_k(\alpha) = \frac{1}{(k-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^{k-1} x_1(\alpha), \quad k = 2, \ldots, n$$

satisfy $(A - \alpha)x_1(\alpha) = 0$ and $(A - \alpha)x_k(\alpha) = x_{k-1}(\alpha)$ for $k = 2, \ldots, n$. In other words, the vectors $x_1(\alpha), \ldots, x_n(\alpha)$ form a Jordan chain of C_a.

At this point, we also introduce some general terminology concerning matrices. A vector $\beta = (\beta_1, \ldots, \beta_m)^{\mathrm{T}}$ is called a *spectral vector* for an $m \times m$ matrix B, if $\beta_1, \ldots, \beta_m$ are the eigenvalues of B, counted according to their algebraic multiplicities. If $T = (\ T_{ij})_{i,j=1}^m$ denotes a complex $m \times m$ matrix, then $\mathrm{diag}(T) = (T_{11}, \ldots, T_{mm})^{\mathrm{T}}$ denotes the *diagonal vector* of T. In particular, the diagonal vector of an upper or lower triangular matrix is a spectral vector for that matrix.

A pair of $m \times m$ matrices A, Z admits *simultaneous reduction to complementary triangular forms with diagonals*

$$\alpha = (\alpha_1, \ldots, \alpha_m)^{\mathrm{T}}, \quad \zeta = (\zeta_1, \ldots, \zeta_m)^{\mathrm{T}} \tag{6}$$

if there exists an invertible $m \times m$ matrix S, such that $S^{-1}AS$ is upper triangular, $S^{-1}ZS$ is lower triangular, and

$$\mathrm{diag}(S^{-1}AS) = \alpha, \qquad \mathrm{diag}(S^{-1}ZS) = (\zeta_m, \ldots, \zeta_1)^{\mathrm{T}}.$$

The *reversed identity* or *rotation matrix* R, defined by $Re_k = e_{m-k+1}$ for $k = 1, \ldots, m$, transforms upper triangular matrices into lower triangular matrices and vice versa, i.e., T is an upper triangular $m \times m$ matrix if and only if $R^{-1}TR$ is a lower triangular $m \times m$ matrix. Also, $R^{-1} = R$. Using this matrix, it is immediate that the pair A, Z admits simultaneous reduction to complementary triangular forms with diagonals α and ζ if and only if the pair Z, A admits simultaneous reduction to complementary triangular forms with diagonals ζ and α.

The following theorem, which describes simultaneous reduction to complementary triangular forms for pairs of first companion matrices, is taken from [3], Theorem 3.2.

Theorem 3.1. *Let A and Z be first companion $m \times m$ matrices. Then the pair A, Z admits simultaneous reduction to complementary triangular forms with diagonal* *vectors α and ζ as in Eq.* (6) *if and only if these vectors are spectral vectors for A and Z respectively, and satisfy*

$$\alpha_k \neq \zeta_l, \quad k + l \leqslant m.$$

We now state the main theorem of this section.

Theorem 3.2. *Let A_1 and Z_1 be first companion $m_1 \times m_1$ matrices, m_2 a non-negative integer, and define $m = m_1 + m_2$. Consider the $m \times m$ matrices $A = A_1 \oplus O_{m_2}$ and $Z = Z_1 \oplus O_{m_2}$. Assume that the pair A, Z admits simultaneous reduction to complementary triangular forms with diagonals*

$$(\alpha_1, \dots, \alpha_m)^{\mathrm{T}}, \quad (\zeta_1, \dots, \zeta_m)^{\mathrm{T}}. \tag{7}$$

Then there exist strictly increasing mappings $\pi, \rho : \{1, \dots, m_1\} \to \{1, \dots, m\}$, such that the pair A_1, Z_1 admits simultaneous reduction to complementary triangular forms with diagonals

$$(\alpha_{\pi(1)}, \dots, \alpha_{\pi(m_1)})^{\mathrm{T}}, \quad (\zeta_{\rho(1)}, \dots, \zeta_{\rho(m_1)})^{\mathrm{T}}. \tag{8}$$

Theorem 3.2 shows that for pairs of first companion $m_1 \times m_1$ matrices, the infimum $\nu(A_1, Z_1) \in \{0, \infty\}$. The transpose of a first companion matrix is called a *second companion matrix*, while a third companion matrix is obtained from a first companion matrix after transformation by means of the reversed identity. Finally, a *fourth companion matrix* is the transpose of a third companion matrix. In this manner, it is not difficult to see that Theorem 3.2 carries over to pairs of second, pairs of third and pairs of fourth companion matrices. Before proving Theorem 3.2, we present three lemmas.

Lemma 3.3. *Let A be an $m \times m$ matrix and let $\mathcal{M} = \{M_k\}_{k=0}^{m}$ be a maximal invariant nest for A with* $\operatorname{diag}(A; \mathcal{M}) = (\alpha_1, \dots, \alpha_m)^{\mathrm{T}}$. *Let $\alpha \in \sigma(A)$ and let N be an n-dimensional invariant subspace of A in* $\operatorname{Ker}(A - \alpha)^m$. *Define*

$$\tau(i) = \min\{k \mid 1 \leqslant k \leqslant m, \dim(M_k \cap N) = i\}, \quad i = 1, \dots, n,$$

then $\alpha_{\tau(i)} = \alpha$.

Proof. Fix $i \in \{1, \dots, n\}$, and assume that $\alpha_{\tau(i)} = \hat{\alpha} \neq \alpha$. By definition, $\tau(i) \geqslant 1$, and $M_{\tau(i)-1} \cap N \subset M_{\tau(i)} \cap N$. Further recall that $(A - \hat{\alpha})M_{\tau(i)} \subseteq M_{\tau(i)-1}$. Consequently,

$$(A - \hat{\alpha})\bigl(M_{\tau(i)} \cap N\bigr) \subseteq M_{\tau(i)-} \cap N \subset M_{\tau(i)} \cap N.$$

On the other hand, the restriction of $A - \hat{\alpha}$ to the subspace $M_{\tau(i)} \cap N \subseteq N_\alpha(A)$ is invertible. A contradiction is obtained and the lemma is proved. □

Lemma 3.4. *Let m_1 be a positive integer, m_2 be a non-negative integer, and let $m = m_1 + m_2$. Let $\pi, \rho : \{1, \dots, m_1\} \to \{1, \dots, m\}$* and *$\tau, \sigma : \{1, \dots, m_2\} \to$*

$\{1, \ldots, m\}$ be strictly increasing mappings, such that $\pi(s) \neq \tau(i)$ and $\rho(s) \neq \sigma(i)$ for all $s = 1, \ldots, m_1$ and $i = 1, \ldots, m_2$. Then $\sigma \leqslant \tau$ implies that $\pi \leqslant \rho$.

Proof. The mappings π and ρ are completely determined by the mappings τ and σ respectively. In fact,

$$\pi(s) = \begin{cases} s, & 1 \leqslant s \leqslant \tau(1) - 1, \\ s + 1, & \tau(1) \leqslant s \leqslant \tau(2) - 2, \\ \vdots & \vdots \\ s + m_2, & \tau(m_2) - m_2 + 1 \leqslant s \leqslant m_1, \end{cases}$$

i.e., $\pi(s) = s + i$, if $\tau(i) - i + 1 \leqslant s \leqslant \tau(i+1) - i - 1$. For convenience, we write $\tau(0) = 0$ and $\tau(m_2 + 1) = m + 1$. In addition,

$$\rho(t) = \begin{cases} t, & 1 \leqslant t \leqslant \sigma(1) - 1, \\ t + 1, & \sigma(1) \leqslant t \leqslant \sigma(2) - 2, \\ \vdots & \vdots \\ t + m_2, & \sigma(m_2) - m_2 + 1 \leqslant t \leqslant m_1, \end{cases}$$

thus $\rho(s) = s + j$, if $\sigma(j) - j + 1 \leqslant s \leqslant \sigma(j+1) - j - 1$. Again, we write $\sigma(0) = 0$ and $\sigma(m_2 + 1) = m + 1$. Fix $s \in \{1, \ldots, m_1\}$. To show that $\pi(s) \leqslant \rho(s)$, let $i, j \in \{0, \ldots, m_2\}$ such that

$$\tau(i) - i + 1 \leqslant s \leqslant \tau(i+1) - i - 1, \quad \sigma(j) - j + 1 \leqslant s \leqslant \sigma(j+1) - j - 1.$$

We prove $i \leqslant j$: If we assume that $i \geqslant j + 1$, then, using $\sigma \leqslant \tau$ in the first inequality, we get

$$\begin{aligned} &\sigma(i) - i + 1 \leqslant \tau(i) - i + 1 \leqslant s \leqslant \sigma(j+1) - j - 1 < \sigma(j+1) - j + 1) + 1 \\ &\leqslant \sigma(i) - i + 1, \end{aligned}$$

a contradiction. Further, $i \leqslant j$ implies that $\pi(s) = s + i \leqslant s + j = \rho(s)$. □

Lemma 3.5. *Let B_1 be an $m_1 \times m_1$ matrix,* dim Ker $B_1 = 1$ *and $B = B_1 \oplus O_{m_2}$. If $M \subseteq N_0(B)$ is an invariant subspace for B and* Ker $B_1 \not\subseteq BM$, *then $M \subseteq$* Ker *B.*

Proof. Let dim $N_0(B_1) = n$. Since dim Ker $B_1 = 1$, there exist a basis $y_1, \ldots, y_n$ for $N_0(B_1)$, such that $B_1 y_1 = 0$ and $B_1 y_{k+1} = y_k$ for $k = 1, \ldots, n-1$. Let $0 \neq x \in M$, and assume that $x \notin \operatorname{Ker} B$. Then there exists $p \in \{2, \ldots, n\}$ and complex numbers $\xi_1, \ldots, \xi_p$, with $\xi_p \neq 0$, and $u \in \mathbb{C}^{m_2}$, such that $x = \xi_1 y_1 + \cdots + \xi_p y_p + u$. Then $B^{p-1} x = \xi_p y_1 \in BM$, which contradicts the assumption $\operatorname{span}\{y_1\} = \operatorname{Ker} B_1 \not\subseteq BM$. □

Proof of Theorem 3.2. We will actually prove a slight generalization of the result as stated in the theorem. In the proof we shall consider, for a given complex number γ, the $m \times m$ matrices $A = A_1 \oplus \gamma I_{m_2}$ and $Z = Z_1 \oplus \gamma I_{m_2}$. The theorem corresponds to the case when $\gamma = 0$. The proof consists of three parts, dealing with the following cases:
1. $\gamma \notin \sigma(A_1) \cup \sigma(Z_1)$.
2. $\gamma \notin \sigma(A_1)$, $\gamma \in \sigma(Z_1)$. By a symmetry argument, the case $\gamma \in \sigma(A_1)$, $\gamma \notin \sigma(Z_1)$ is also covered here.
3. $\gamma \in \sigma(A_1) \cap \sigma(Z_1)$.

Part 1: $\gamma \notin \sigma(A_1) \cup \sigma(Z_1)$. Apply Proposition 2.4.

Part 2: $\gamma \notin \sigma(A_1)$, $\gamma \in \sigma(Z_1)$.

Assume that the pair A, Z admits simultaneous reduction to complementary triangular forms, by means of an $m \times m$ similarity S, with diagonals α and ζ as in Eq. (7). Let the maximal invariant nests (defined by S) for A and Z be $\mathcal{M} = \{M_k\}_{k=0}^m$ and $\mathcal{N} = \{N_l\}_{l=0}^m$, respectively.

We have to prove that the pair A_1, Z_1 admits simultaneous reduction to complementary triangular forms with diagonals as in Eq. (8). By Theorem 3.1, it suffices to find strictly increasing mappings $\pi, \rho : \{1, \ldots, m_1\} \to \{1, \ldots, m\}$ such that the vectors in Eq. (8) are spectral vectors for A_1 and Z_1, and such that

$$\alpha_{\pi(s)} \neq \zeta_{\rho(t)}, \quad s + t \leqslant m_1. \tag{9}$$

To define π and ρ, we will analyse the diagonals α and ζ. More specifically, we will determine restrictions on the positions of the eigenvalues on the diagonals. The first claim below concerns the eigenvalues different from γ.

Claim 1. *If* $\alpha_k = \zeta_l \neq \gamma$, *then* $k + l > m$.

To prove the claim, assume that $\alpha_k = \zeta_l = \beta \neq \gamma$. Then $\beta \in \sigma(A_1) \cap \sigma(Z_1)$ and

$$\operatorname{Ker}(\beta - A_1) = \operatorname{Ker}(\beta - Z_1) = \operatorname{span}\{x_1(\beta)\},$$

since A_1 and Z_1 are first companion matrices. Because $\beta \neq \gamma$, it holds that

$$\begin{aligned}\operatorname{Ker}(\beta - A) &= \operatorname{Ker}(\beta - Z)\\ &= \operatorname{span}(\ 1,\ \beta,\ \ldots,\ \beta^{m_1-1},\ 0,\ \ldots,\ 0\ )^{\mathrm{T}} = L \subseteq \mathbb{C}^m.\end{aligned}$$

On the other hand, $L \subseteq M_k$, since $\beta = \alpha_k$, and $L \subseteq N_l$, since $\beta = \zeta_l$. Therefore $(0) \neq L \subseteq M_k \cap N_l$, so $k + l > m$. The claim is proved.

To obtain restrictions on the positions of the eigenvalues equal to γ, the generalized eigenspaces $N_\gamma(A)$ and $N_\gamma(Z)$ are studied. Note that $N_\gamma(A) = N_\gamma(A_1) \oplus \mathbb{C}^{m_2} = \mathbb{C}^{m_2}$, and $N_\gamma(Z) = N_\gamma(Z_1) \oplus \mathbb{C}^{m_2}$; so in particular, $N_\gamma(A) \subseteq N_\gamma(Z)$. Note that $\dim N_\gamma(Z_1) = q$ is a strictly positive integer and that $\dim N_\gamma(Z) = q + m_2$. Define

$$\tau(i) = \min\{k \mid 1 \leqslant k \leqslant m, \dim(M_k \cap N_\gamma(A)) = i\}, \quad i = 1, \dots, m_2,$$

$$\sigma(j) = \min\{l \mid 1 \leqslant l \leqslant m, \dim(N_l \cap N_\gamma(Z)) = j\}, \quad j = 1, \dots, q + m_2.$$

Claim 2. *It is immediate from Lemma* 3.3 *that*

$$\alpha_{\tau(i)} = \gamma, \quad i = 1, \dots, m_2,$$

and

$$\zeta_{\sigma(j)} = \gamma, \quad j = 1, \dots, q + m_2. \tag{10}$$

Claim 3. *If $i + j > q + m_2$, then $\tau(i) + \sigma(j) > m$.*

To prove the claim, assume that $\tau(i) + \sigma(j) \leqslant m$. Then $M_{\tau(i)} \cap N_{\sigma(j)} = (0)$, and hence

$$(M_{\tau(i)} \cap N_\gamma(A)) \oplus (N_{\sigma(j)} \cap N_\gamma(Z)) \subseteq N_\gamma(A) + N_\gamma(Z) = N_\gamma(Z).$$

A dimension argument shows that $i + j \leqslant q + m_2$ and the claim is proved.

As a consequence of Claim 3, the following inequalities hold:

$$\tau(i) + \sigma(q + m_2 - i + 1) > m, \quad i = 1, \dots, m_2. \tag{11}$$

It will be convenient to use the following notation:

$$\hat{\zeta}_l = \zeta_{m-l+1}, \quad l = 1, \dots, m, \tag{12}$$

and

$$\hat{\sigma}(j) = m - \sigma(q + m_2 - j + 1) + 1, \quad j = 1, \dots, q + m_2. \tag{13}$$

Expressions (10) and (11) are rewritten according to Eqs. (12) and (13) as follows:

$$\hat{\zeta}_{\hat{\sigma}(j)} = \gamma, \quad j = 1, \dots, q + m_2, \tag{14}$$

and

$$\hat{\sigma}(i) \leqslant \tau(i), \quad i = 1, \dots, m_2. \tag{15}$$

Define the strictly increasing mappings $\pi, \hat{\rho} : \{1, \dots, m_1\} \to \{1, \dots, m\}$, such that $\pi(s) \neq \tau(i)$ and $\hat{\rho}(s) \neq \hat{\sigma}(i)$ for all $s = 1, \dots, m_1$ and $i = 1, \dots, m_2$. By Lemma 3.4, inequality (15) implies that $\pi \leqslant \hat{\rho}$. Define the strictly increasing mapping $\rho : \{1, \dots, m_1\} \to \{1, \dots, m\}$ by

$$\rho(t) = m - \hat{\rho}(m_1 - t + 1) + 1, \quad t = 1, \dots, m_1.$$

This equation and $\pi \leqslant \hat{\rho}$ together imply that

$$\pi(s) + \rho(m_1 - s + 1) \leqslant m + 1, \quad s = 1, \dots, m_1. \tag{16}$$

Note that the vectors in Eq. (8) are indeed spectral vectors for A_1 and Z_1, as they are obtained from the spectral vectors for A and Z by omitting m_2

eigenvalues γ. (Consider Claim 2 and the definition of π and ρ.) To prove that the spectral vectors in Eq. (8) satisfy the ordering condition (9), assume that $\alpha_{\pi(s)} = \zeta_{\rho(t)} = \beta$. Then $\beta \neq \gamma$, since $\gamma \notin \sigma(A_1)$. It follows from Claim 1 that $\pi(s) + \rho(t) > m$. Then Eq. (16) implies that $\rho(m_1 - s + 1) \leqslant m + 1 - \pi(s) \leqslant \rho(t)$, so $m_1 - s + 1 \leqslant t$ (ρ is strictly increasing) or $s + t > m_1$. Part 2 of the theorem is proved.

Part 3: $\gamma \in \sigma(A_1) \cap \sigma(Z_1)$. Assume that the pair A, Z admits simultaneous reduction to complementary triangular forms with diagonals α and ζ as in Eq. (7). Notation will be consistent with Part 2 of the proof, unless explicitly stated otherwise.

As in Part 2, the main course of the proof of Part 3 will be as follows: Restrictions on the diagonals α and ζ, based on the matching condition on the maximal invariant nests $\mathscr{M}$ and $\mathscr{N}$ for A and Z respectively, are used to define strictly increasing mappings $\pi, \rho : \{1, \ldots, m_1\} \to \{1, \ldots, m\}$, such that the vectors Eq. (8) are spectral vectors for A_1 and Z_1, and such that condition (9) is satisfied.

First of all, Claim 1 in Part 2 of the proof remains valid. The restrictions on the positions of the eigenvalues equal to γ on the diagonals α and ζ need more attention. Note that both $p = \dim N_\gamma(A_1)$ and $q = \dim N_\gamma(Z_1)$ are strictly positive integers and $\dim N_\gamma(A) = p + m_2$, $\dim N_\gamma(Z) = q + m_2$. By symmetry, we may assume without loss of generality that $p \leqslant q$. Since A_1 and Z_1 are first companion matrices, it follows that $N_\gamma(A) \subseteq N_\gamma(Z)$.

Define

$$\tau(i) = \min\{k \mid 1 \leqslant k \leqslant m, \dim(M_k \cap N_\gamma(A)) = i\}, \quad i = 1, \ldots, p + m_2,$$

$$\sigma(j) = \min\{l \mid 1 \leqslant l \leqslant m, \dim(N_l \cap N_\gamma(Z)) = j\}, \quad j = 1, \ldots, q + m_2.$$

Claim 4. *The following is an immediate consequence of Lemma* 3.3.

$$\alpha_{\tau(i)} = \gamma, \quad i = 1, \ldots, p + m_2,$$

and

$$\zeta_{\sigma(j)} = \gamma, \quad j = 1, \ldots, q + m_2.$$

Since A_1 and Z_1 are first companion matrices, $L = \mathrm{Ker}(\gamma - A_1) = \mathrm{Ker}(\gamma - Z_1) = \mathrm{span}\{x_1(\gamma)\}$. Define the integers

$$\tau_* = \min\{k \mid 1 \leqslant k \leqslant m, L \subseteq M_k\}$$

and

$$\sigma_* = \min\{l \mid 1 \leqslant l \leqslant m\}, L \subseteq N_l\}.$$

There exists $i_* \in \{1, \ldots, p + m_2\}$, such that $\tau_* = \tau(i_*)$. Indeed, by definition of τ_*, we get $x_1(\gamma) \notin M_{\tau_* - 1}$ and $x_1(\gamma) \in M_{\tau_*}$. Therefore, $M_{\tau_*} = M_{\tau_* - 1} \oplus \mathrm{span}\{x_1(\gamma)\}$.

It follows that $(A-\gamma)M_{\tau_*} \subseteq M_{\tau_*-1}$, so $\alpha_{\tau_*} = \gamma$. This gives $\tau_* \in \{\tau(1), \dots, \tau(p+m_2)\}$.

Similarly, it is shown that there exists $j_* \in \{1, \dots, q+m_2\}$, such that $\sigma_* = \sigma(j_*)$.

Apply Lemma 3.5 to $B_1 = A_1 - \gamma$, $B = A - \gamma$, $M = M_{\tau(i_*)} \cap N_\gamma(A)$, and to $B_1 = Z_1 - \gamma$, $B = Z - \gamma$, $M = N_{\sigma(j_*)} \cap N_\gamma(Z)$, to obtain

$$M_{\tau(i_*)} \cap N_\gamma(A), N_{\sigma(j_*)} \cap N_\gamma(Z) \subseteq L \oplus \mathbb{C}^{m_2}, \tag{17}$$

and by a dimension argument, $i_*, j_* \leqslant m_2 + 1$. In addition, $L \subseteq M_{\tau(i_*)} \cap N_{\sigma(j_*)}$ implies that

$$\tau(i_*) + \sigma(j_*) > m. \tag{18}$$

Claim 5. *If $j \leqslant j_*$ and $i + j > p + m_2$, then*

$$\tau(i) + \sigma(j) > m.$$

The claim is proved as follows: Assume that $j \leqslant j_*$, then

$$N_{\sigma(j)} \cap N_\gamma(Z) \subseteq N_{\sigma(j_*)} \cap N_\gamma(Z) \subseteq L \oplus \mathbb{C}^{m_2} \subseteq N_\gamma(A).$$

If in addition, $\tau(i) + \sigma(j) \leqslant m$, it follows that

$$\big(M_{\tau(i)} \cap N_\gamma(A)\big) \oplus \big(N_{\sigma(j)} \cap N_\gamma(Z)\big) \subseteq N_\gamma(A)$$

and a dimension argument provides $i + j \leqslant p + m_2$. The claim is proved.

In particular, Claim 5 implies that

$$\tau(p + m_2 - j + 1) + \sigma(j) > m, \quad j = 1, \dots, j_*. \tag{19}$$

Claim 6. *If $i \leqslant i_*$, $j \leqslant j_*$ and $i + j > m_2 + 1$, then*

$$\tau(i) + \sigma(j) > m.$$

To prove the claim, let $i \leqslant i_*$ and $j \leqslant j_*$. If we assume that $\tau(i) + \sigma(j) \leqslant m$, then

$$\begin{aligned}(M_{\tau(i)} \cap N_\gamma(A)) \oplus (N_{\sigma(j)} \cap N_\gamma(Z)) &\subseteq (M_{\tau(i_*)} \cap N_\gamma(A)) + (N_{\sigma(j_*)} \cap N_\gamma(Z)) \\ &\subseteq L \oplus \mathbb{C}^{m_2}\end{aligned}$$

and a dimension argument provides $i + j \leqslant m_2 + 1$. The claim is proved.

Claim 7. *There exists a pair κ, λ such that*

$$\kappa \in \{0, \dots, i_* - 1\}, \quad \lambda \in \{0, \dots, j_* - 1\}, \tag{20}$$

$$\kappa + \lambda \leqslant m_2, \tag{21}$$

$$\tau(\kappa+1)+\sigma(\lambda+1)>m. \tag{22}$$

To prove the claim, we consider two cases.

Case 1 ($i_*+j_*\leqslant m_2+2$). In this case, put $\kappa=i_*-1$ and $\lambda=j_*-1$. Then it is immediate that Eqs. (20) and (21) are satisfied. Further Eq. (18) implies Eq. (22).

Case 2 ($i_*+j_*>m_2+2$). Define the integer $d=i_*+j_*-m_2>2$. In this case, define $\kappa=i_*-1$ and $\lambda=j_*-d+1$. Then it is easily verified that Eqs. (20) and (21) are satisfied. Since $(\kappa+1)+(\lambda+1)=m_2+2$, Claim 6 implies Eq. (22). The claim is proved.

In the proof of Claim 7, the integers κ and λ were defined as follows (put $d=\max\{i_*+j_*-m_2,2\}$)

$$\kappa=i_*-1,\quad \lambda=j_*-d+1.$$

In general, there may exist other pairs of integers κ,λ, which also satisfy the conditions of Claim 7. For the proof, it suffices to consider only this pair of integers.

Note that Claim 3 in Part 2 remains valid, and that we also obtain Eq. (11). As in Part 2, the notation introduced by the Eqs. (12) and (13) is used to rewrite Eqs. (10) and (11) as Eqs. (14) and (15).

Further, Eq. (19) can be rewritten as

$$\hat{\sigma}(q+m_2-j+1)\leqslant\tau(p+m_2-+1),\quad j=1,\dots,j_* \tag{23}$$

and Eq. (22) as

$$\hat{\sigma}(q+m_2-\lambda)\leqslant\tau(\kappa+1). \tag{24}$$

Define the strictly increasing mappings $\tilde{\tau},\tilde{\sigma}:\{1,\dots,m_2\}\to\{1,\dots,m\}$ as follows:

$$\tilde{\tau}(i)=\begin{cases}\tau(i), & i=1,\dots,m_2-\lambda,\\ \tau(p+i), & i=m_2-\lambda+1,\dots,m_2\end{cases} \tag{25}$$

and

$$\tilde{\sigma}(j)=\begin{cases}\hat{\sigma}(j), & j=1,\dots,\kappa,\\ \hat{\sigma}(q+j), & j=\kappa+1,\dots,m_2.\end{cases} \tag{26}$$

Note that the p integers

$$\tau(m_2-\lambda+1),\dots,\tau(p+m_2-\lambda) \tag{27}$$

are not in the range of $\tilde{\tau}$, and that the q integers

$$\hat{\sigma}(\kappa+1),\dots,\hat{\sigma}(q+\kappa) \tag{28}$$

are not in the range of $\tilde{\sigma}$.

Claim 8. $\tilde{\sigma} \leqslant \tilde{\tau}$.

First, let $1 \leqslant i \leqslant \kappa$. Since $\kappa \leqslant i_* - 1 \leqslant m_2$, Eq. (15) implies that $\tilde{\sigma}(i) = \hat{\sigma}(i) \leqslant \tau(i) = \tilde{\tau}(i)$.

Second, let $\kappa + 1 \leqslant i \leqslant m_2 - \lambda$. Then Eq. (24) implies $\tilde{\sigma}(i) = \hat{\sigma}(q+i) \leqslant \hat{\sigma}(q+m_2-\lambda) \leqslant \tau(\kappa+1) \leqslant \tau(i) = \tilde{\tau}(i)$.

Finally, let $m_2 - \lambda + 1 \leqslant i \leqslant m_2$. Then Eq. (23) implies $\tilde{\sigma}(i) = \hat{\sigma}(q+i) \leqslant \tau(p+i) = \tilde{\tau}(i)$, and the claim is proved.

Define the strictly increasing mappings $\pi, \tilde{\rho} : \{1,\dots,m_1\} \to \{1,\dots,m\}$, such that $\pi(s) \neq \tilde{\tau}(i)$ and $\tilde{\rho}(s) \neq \tilde{\sigma}(i)$ for all $s = 1,\dots,m_1$ and $i = 1,\dots,m_2$. By Lemma 3.4, the inequality in Claim 8 implies that $\pi \leqslant \tilde{\rho}$. Define the strictly increasing mapping $\rho : \{1,\dots,m_1\} \to \{1,\dots,m\}$ by

$$\rho(t) = m - \tilde{\rho}(m_1 - t + 1) + 1, \quad t = 1,\dots,m_1.$$

This equation and $\pi \leqslant \tilde{\rho}$ imply Eq. (16).

The vectors in Eq. (8) are spectral vectors for A_1 and Z_1 respectively, for the same reason as described in Part 2 of the proof. The integers in Eq. (27) are in the range of π. They indicate the positions of the eigenvalue γ in the spectral vector of A_1 given in Eq. (8) as follows: If $\pi(s)$ is one of the integers Eq. (27), then $\alpha_{\pi(s)} = \gamma$. The integers Eq. (28) are in the range of $\tilde{\rho}$. For that reason, the q integers

$$\sigma(m_2-\kappa+1),\dots,\sigma(q+m_2-\kappa) \tag{29}$$

are in the range of ρ and indicate the positions of the eigenvalues equal to γ in the spectral vector of Z_1 in Eq. (8).

We need to prove that condition (9) is satisfied for the spectral vectors of A_1 and Z_1, as given in Eq. (8).

If $\alpha_{\pi(s)} = \zeta_{\rho(t)} \neq \gamma$, then by the same argument as given in Part 2, it follows that $s + t > m_2$.

If $\alpha_{\pi(s)} = \zeta_{\rho(t)} = \gamma$, then $\pi(s)$ is one of the integers (27) and $\rho(t)$ is one of the integers (29). Therefore,

$$\pi(s) + \rho(t) \geqslant \tau(m_2 - \lambda + 1) + \sigma(m_2 - \kappa + 1) \geqslant \tau(\kappa+1) + \sigma(\lambda+1) > m.$$

Apply Eq. (16) as in Part 2, to obtain that $s + t > m_1$. This finally finishes the proof of the theorem. □

The reversed identity R transforms the first companion matrix C_a, defined in Eq. (5), to the *third companion matrix*

$$\hat{C}_a = R^{-1}C_aR = \begin{pmatrix} -a_{m-1} & -a_{m-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}.$$

We quote Theorem 3.2 in [6], which deals with pairs of matrices consisting of a first companion matrix and a third companion matrix.

Theorem 3.6. *Let A be a first companion $m \times m$ matrix and Z be a third companion $m \times m$ matrix. Then the pair A, Z admits simultaneous reduction to complementary triangular forms with diagonal vectors α and ζ as in Eq.* (6) *if and only if these vectors are spectral vectors for A and Z respectively, and satisfy*

$$\alpha_k\zeta_l \neq 1, \quad k + l \leqslant m.$$

For the proof of Theorem 3.7, see [20].

Theorem 3.7. *Let A_1 be a first companion $m_1 \times m_1$ matrix, and Z_1 be a third companion $m_1 \times m_1$ matrix. Define the matrices $A = A_1 \oplus O_{m_2}$ and $Z = Z_1 \oplus O_{m_2}$, and let $m = m_1 + m_2$. Assume that the pair A, Z admits simultaneous reduction to complementary triangular forms with diagonals α and ζ as in Eq.* (7). *Then there exist strictly increasing mappings $\pi, \rho : \{1, \ldots, m_1\} \to \{1, \ldots, m\}$, such that the pair A_1, Z_1 admits simultaneous reduction to complementary triangular forms with diagonals as in Eq.* (8).

We have not dealt with all pairs of companion matrices. For example, we have not described complementary triangular forms after extension with zeroes for pair of matrices, consisting of a first and a second companion matrix (or, after using the reversed identity, a third and a fourth companion matrix). Proposition 4.3 in [17] deals with such pairs, but it seems difficult to make use of that result here. Pairs consisting of a first and a fourth companion matrix (or, by taking transposes, a second and a third companion matrix) also appear to be hard to handle.

4. Nilpotent and Jordan matrices

We conclude this paper with a section which provides a survey of results that are proved elsewhere [20]. In this section, a number of results on complementary triangular forms is presented which use the notion of the so-called spectral polynomial of a matrix. Also, simultaneous reduction to complementary triangular forms after extension with zeroes is discussed for pairs of nilpotent and Jordan matrices.

As in [20], we introduce the following type of polynomial: Let B be an $m \times m$ matrix and let the mutually distinct eigenvalues of B be denoted by $\beta_1, \dots, \beta_s$. Define the *spectral polynomial* of B by

$$p_B(\lambda) = (\lambda - \beta_1) \cdots (\lambda - \beta_s). \tag{30}$$

This polynomial is the monic polynomial of minimal degree, vanishing on the spectrum of B. Note that the matrix $p_B(B)$ is nilpotent and that $p_B(B) = O_m$ if and only if B is diagonalizabl . As a matter of fact, the subspace $\operatorname{Ker} p_B(B)$ is the linear span of all eigenvectors of B.

First, we turn to simultaneous reduction to complementary triangular forms for pairs of nilpotent matrices. The following theorem is a generalization of Lemmas 1.1 and 1.2 in [4].

Theorem 4.1. *Let A and Z be nilpotent $m \times m$ matrices, $Z \neq O_m$. Assume that*

$$\operatorname{Ker} A \subseteq \operatorname{Ker} Z + \operatorname{Ran} Z \tag{31}$$

and

$$\operatorname{Ker} A \cap \operatorname{Ran} A \subseteq \operatorname{Ran} Z. \tag{32}$$

Then the pair A, Z does not admit simultaneous reduction to complementary triangular forms.

In Theorem 4.1, the condition $Z \neq O_m$ and Eq. (31) imply that $A \neq O_m$.

If $p(\lambda)$ and $q(\lambda)$ are polynomials, and A and Z are $m \times m$ matrices, then it is not difficult to see that if the pair A, Z admits simultaneous reduction to complementary triangular forms, then the pair $p(A)$, $q(Z)$ has the same property. This argument provides the following corollary to Theorem 4.1.

Corollary 4.2. *Let A and Z be non-diagonalizable $m \times m$ matrices. If*

$$\operatorname{Ker} p_A(A) \subseteq \operatorname{Ker} p_Z(Z) + \operatorname{Ran} p_Z(Z)$$

and

$$\operatorname{Ker} p_A(A) \cap \operatorname{Ran} p_A(A) \subseteq \operatorname{Ran} p_Z(Z),$$

then the pair A, Z does not admit simultaneous reduction to complementary triangular forms.

Corollary 4.2 gives a *necessary* condition for simultaneous reduction to complementary triangular forms. Proposition 4.3 below provides a *sufficient* condition.

Proposition 4.3. *Let A and Z be $m \times m$ matrices. If either*

$$\operatorname{Ker} p_A(A) + \operatorname{Ker} p_Z(Z) = \mathbb{C}^m \tag{33}$$

or

$$\operatorname{Ran} p_A(A) \cap \operatorname{Ran} p_Z(Z) = (0), \tag{34}$$

then the pair A, Z admits simultaneous reduction to complementary triangular forms.

The subspace $\operatorname{Ker} p_A(A) + \operatorname{Ker} p_Z(Z)$ in Eq. (33) is the linear span of the eigenvectors of A and the eigenvectors of Z. Since the eigenvectors of a diagonalizable matrix span the whole space, it follows that Theorem 4.1 is contained in Proposition 4.3.

Corollary 4.2 and Proposition 4.3 lead to necessary and sufficient conditions for simultaneous reduction to complementary triangular forms on a special class of matrices; the almost diagonalizable matrices. An $m \times m$ matrix B is called *almost diagonalizable*, if rank $p_B(B) = 1$. In other words, the Jordan canonical form of an almost diagonalizable square matrix contains one Jordan block of size two, and all other blocks are of size one. The following theorem specifies Corollary 4.2 and Proposition 4.3 for almost diagonalizable matrices.

Theorem 4.4. *Let A and Z be almost diagonalizable $m \times m$ matrices. Then the following are equivalent:*

1. *The pair A, Z admits simultaneous reduction to complementary triangular forms.*
2. $\operatorname{Ker} p_A(A) \neq \operatorname{Ker} p_Z(Z)$ *or* $\operatorname{Ran} p_A(A) \neq \operatorname{Ran} p_Z(Z)$.
3. *$p_A(A)$ is not a scalar multiple of $p_Z(Z)$.*

The equivalence between the second and the third statement is contained in [5], Theorem 1.4. A somewhat different characterization of simultaneous reduction to complementary triangular forms for pairs of almost diagonalizable matrices has been given in Theorem 2.2 in [17], which extends the main result in [9]. The following result states that for pairs A_1, Z_1 that contain an almost diagonalizable matrix, we get $\nu(A_1, Z_1) \in \{0, \infty\}$.

Theorem 4.5. *Let Z_1 be an almost diagonalizable $m_1 \times m_1$ matrix, and let A_1 be any $m_1 \times m_1$ matrix. Let m_2 be a non-negative integer, and define $m = m_1 + m_2$. If the the pair of $m \times m$ matrices $A = A_1 \oplus O_{m_2}$, $Z = Z_1 \oplus O_{m_2}$ admits simultaneous reduction to complementary triangular forms, then so does the pair A_1, Z_1.*

In [5], pairs of nilpotent matrices called *sharply upper triangular matrices* have been studied. The following result is an extension of Theorem 4.1 from this paper. Note that an important class of sharply upper triangular matrices is the class of non-zero strictly upper triangular Toeplitz matrices.

Proposition 4.6. *Let* $1 \leqslant \alpha, \omega \leqslant m_1 - 1$, *and let* A_{12} *be an invertible upper triangular* $(m_1 - \alpha) \times (m_1 - \alpha)$ *matrix and* Z_{12} *be an invertible upper triangular* $(m_1 - \omega) \times (m_1 - \omega)$ *matrix. Define the* $m_1 \times m_1$ *matrices*

$$A_1 = \begin{pmatrix} O & A_{12} \\ O_\alpha & O \end{pmatrix}, \qquad Z_1 = \begin{pmatrix} O & Z_{12} \\ O_\omega & O \end{pmatrix}.$$

Let m_2 *be any non-negative integer,* $m = m_1 + m_2$, *and consider the* $m \times m$ *matrices* $A = A_1 \oplus O_{m_2}$ *and* $Z = Z_1 \oplus O_{m_2}$. *Then the pair* A, Z *admits simultaneous reduction to complementary triangular forms, if and only if* $\alpha + \omega > m_1$, α *does not divide* ω, *and* ω *does not divide* α.

Proof. By a symmetry argument, we may assume without loss of generality, that $\alpha \leqslant \omega$. First, we prove the only if part. Note that A and Z are non-zero nilpotent $m \times m$ matrices. If $\alpha + \omega \leqslant m_1$, then Eqs. (31) and (32) hold and Theorem 4.1 implies that the pair A, Z does not admit simultaneous reduction to complementary triangular forms. Second, assume that $\alpha = \omega$. Then $\operatorname{Ker} A = \operatorname{Ker} Z$, and $\operatorname{Ran} A = \operatorname{Ran} Z$. Again, Theorem 4.1 implies that the pair A, Z does not admit simultaneous reduction to complementary triangular forms. The case when ω is a multiple of α is reduced to the case $\alpha = \omega$, by taking an appropriate power of A.

To prove the if part, note that the if part of Theorem 4.1 in [4] provides that the pair A_1, Z_1 admits simultaneous reduction to complementary triangular forms, and hence does the pair A, Z. □

Proposition 4.6 implies the following theorem, which states that also for pairs of sharply upper triangular matrices A_1, Z_1, we have $\nu(A_1, Z_1) \in \{0, \infty\}$.

Theorem 4.7. *Let* A_1 *and* Z_1 *be nilpotent matrices as in Proposition* 4.6, m_2 *a non-negative integer, and let* $A = A_1 \oplus O_{m_2}$ *and* $Z = Z_1 \oplus O_{m_2}$. *If the pair* A, Z *admits simultaneous reduction to complementary triangular forms, then so does the pair* A_1, Z_1.

We will now discuss pairs of non-derogatory Jordan Matrices. Recall that a matrix B is called *non-derogatory*, if each eigenvalue $\beta \in \sigma(B)$ has geometric multiplicity $\dim \operatorname{Ker}(B - \beta) = 1$. If α is a complex number, then the $n \times n$ matrix

$$J(\alpha,n)=\begin{pmatrix}\alpha & 1 & 0 & \cdots & 0\\ 0 & \alpha & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ \vdots & & \ddots & \alpha & 1\\ 0 & \cdots & \cdots & 0 & \alpha\end{pmatrix}$$

denotes the upper triangular $n\times n$ Jordan block with eigenvalue α. A matrix of the form $J(\alpha_1,n_1)\oplus\cdots\oplus J(\alpha_s,n_s)$ is called a *Jordan matrix*. This Jordan matrix is non-derogatory if and only if the eigenvalues $\alpha_1,\dots,\alpha_s$ are mutually distinct. Consider the non-derogatory Jordan $m\times m$ matrices

$$J_\alpha=J(\alpha_1,k_1)\oplus\cdots\oplus J(\alpha_s,k_s),\quad J_\zeta=J(\zeta_1,l_1)\oplus\cdots\oplus J(\zeta_t,l_t) \tag{35}$$

with $k_1,\dots,k_s$ and $l_1,\dots,l_t$ nonzero positive integers, such that $k_1+\cdots+k_s=l_1+\cdots+l_t=m$, with $\alpha_1,\dots,\alpha_s$ the distinct eigenvalues for J_α, and $\zeta_1,\dots,\zeta_t$ the distinct eigenvalues for J_ζ. For $1\leqslant\rho\leqslant s$ and $1\leqslant\sigma\leqslant t$, we say that the Jordan blocks $J(\alpha_\rho,k_\rho)$ and $J(\zeta_\sigma,l_\sigma)$ have a *diagonal overlap on more than one position*, if the set

$$\left\{1+\sum_{i=1}^{\rho-1}k_i,\dots,\sum_{i=1}^{\rho}k_i\right\}\bigcap\left\{1+\sum_{j=1}^{\sigma-1}l_j,\dots,\sum_{j=1}^{\sigma}l_j\right\}$$

contains more than one element. We now state Theorem 4.1 in [5].

Theorem 4.8. *Let J_α and J_ζ be non-derogatory Jordan matrices. Then the pair J_α, J_ζ admits simultaneous reduction to complementary triangular forms, if and only if J_α and J_ζ contain no Jordan blocks, that have an overlap on more than one diagonal position.*

Also for pairs of non-derogatory Jordan matrices A_1,Z_1, we find $\nu(A_1,Z_1)\in\{0,\infty\}$.

Theorem 4.9. *Let A_1 and Z_1 be non-derogatory $m_1\times m_1$ Jordan matrices, let m_2 be a non-negative integer, and define the matrices $A=A_1\oplus O_{m_2}$ and $Z=Z_1\oplus O_{m_2}$. If the pair A, Z admits simultaneous reduction to complementary triangular forms, then so does the pair A_1, Z_1.*

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