Heat Kernel and Riemannian Manifolds

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Basic philosophy of shape matching

(1) $\mathcal{O}$: a space of all (or enough number of) shapes

(2) To compare (dis)similarity of shapes, usually we construct some invariants (or signature)

$I : \mathcal{O} \times \mathcal{O} \to (V, d)$ where (V, d) is appropriate metric space

Example) diameter, bending invariant signature, invariant histograms,…

(3) If two shapes X and Y are similar enough, $d(X, Y)$ will be small

(in particular, $I(X) = I(Y)$ whenever $X = Y$)

Basic philosophy of shape matching

(4) But, what is the meaning of "X and Y are similar enough" ?

We need a metric D : $\mathcal{O}\times\mathcal{O}\to\mathbb{R}^{+}$ between pairs of shapes

(5) Candidates: Hasudorff, Gromov-Hausdorff, or Gromov-Wasserstein,…

(6) OBSERVATION!! : heat kernel can be used to construct NEW metric D : $\mathcal{O}\times\mathcal{O}\to\mathbb{R}^{+}$

What is “good” metric D?

Def) In order to say the invariant $I : \mathcal{O} \to V$ is quantitatively stable under D, we require that

$$D(X,Y) \geq d(I(X), I(Y)) \text{ for all shapes } X, Y \in \mathcal{O}$$

(1) small $D(X,Y)$ implies small $d(I(X), I(Y))$

(2) $d(I(X), I(Y))$ is a “lower Bound” for $D(X,Y)$

Heat Kernel

From now on, let's consider the family of Compact Riemannian manifolds without boundary, denote $\mathfrak{R}$

MOTIVATION) the heat kernel $k_X(t,x,y)$ contains all the information about shape

Varadhan's Lemma) For any $X\in\mathfrak{R}$, $\lim_{t\to 0}(-4t\ln K_X(t,x,y))=d_X^2(x,y)\ \forall x,y\in X$

Here $d_X(x,y)$ is the geodesic distance bwteen x and y on X

G.Berard, G.Besson, and S.Gallot's way

(1) we can embed Riemannian manifolds into the Hilbert space l^2. Then, by using the Hausdorff distance between subsets of l^2, we can construct metric between Riemannian manifolds.

(2) we can express heat kernel $k(t,x,y)=\sum_{i=0}^{\infty} e^{-\lambda_i t}\varphi_i(x)\varphi_i(y)$

Here, $0=\lambda_0 \leq \lambda_1 \leq \lambda_2 \cdots \nearrow \infty$ are the eigenvalues of Laplacian and $\{\varphi_i\}_{i=0}^{\infty}$ is an $L^2(M)$-orthonormal basis of the Laplacian.

(3)Given an n-dimensional compact Riemannian manifold M and an orthonormal basis

$a=\{\varphi_i\}_{i=0}^{\infty}$, one can defines a family of maps $I_t^a : M \to l^2$ by

$I_t^a(x)=\sqrt{Vol(M)}\,\{e^{-\lambda_j t/2}\varphi_j(x)\}_{j\geq 1}$ (note: invariant under scaling of the metric)

G.Berard, G.Besson, and S.Gallot's way

(4) Let HD denote the Hausdorff distance between subsets of l^2

Def) Given two Riemannian manifolds M and M', we define a family of distances

$$d_t(M,M')=\max\{\sup_a \inf_{a'} HD(I_t^a(M), I_t^{a'}(M')), \sup_{a'} \inf_a HD(I_t^{a'}(M'), I_t^a(M))\}$$

A.Kasue and H.Komura's way

(1) Def) Given two Riemannian manifolds M and N, two maps $f: M \to N$ and $h: N \to M$ are said to be $\varepsilon-$*spectral approxamatins* if they satisfy

$$e^{-(t+1/t)}\,|k_M(t,x,y) - k_N(t,f(x),f(y))| < \varepsilon \text{ and}$$

$$e^{-(t+1/t)}\,|k_M(t,h(x),h(y)) - k_N(t,x,y)| < \varepsilon$$

(2) the spectral distance $SD(M,N)$ is defined to be the infimum of the numbers $\varepsilon > 0$ so that they admit ε-spectral approximations.

(3) note that its construction is similar to Gromov-Hausdorff

Spectral Gromov-Wasserstein distance

(1) Def) Define $\Gamma_{M,N,t}^{spec}: M\times N\times M\times N\to\mathbb{R}^{+}$ by

$$(x,y,x',y')\mapsto Vol(M)k_M(t,x,x')-Vol(N)k_N(t,y,y')$$

(2)Def) For Riemannin manifolds M and N, and $p\in[1,\infty]$ let

$$d_{GW,p}^{spec}(M,N)=\inf_{\mu\in\mathcal{M}(\mu_M,\mu_N)}\sup_{t>0}c^2(t)\left|\Gamma_{M,N,t}^{spec}\right|_{L^p(\mu\otimes\mu)}$$

Where $c(t)=e^{-(t+1/t)}$

Goal of this project

(1)ambitious one- compare those three distances. Which one is better? Which one is more practically useful?

(2) humble one-computing distances between some classic Riemannian manifolds