Harmonic Soft Maps and the Laplace-Beltrami Operator

Overview Presentation

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Introduction

- Project goal is to discover and understand the landscape of ideas surrounding harmonic maps and the Laplace-Beltrami operator while exploring related applications.
- In particular
  - The variational formulation of the Laplace-Beltrami operator
  - The generalization of Harmonic Maps from functions between manifolds to soft maps
  - The interplay between these ideas

Laplace-Beltrami Operator

Throughout this presentation M_S and M_T will denote two smooth, compact Riemannian manifolds with gradient operators ∇_S, ∇_T, respectively.

Definition

Let g_S denote the metric for M_S, then the Laplace-Beltrami Operator at a point $x \in M_S$ is given by $\Delta_{g_S} u \equiv \text{div} \cdot \text{grad} u$.

- The Laplace-Beltrami Operator on a Riemannian Manifold is determined by the metric.
- The converse direction holds too, by intermediate construction of the heat kernel

The Heat Equation

Consider a heat diffusion process on a manifold. The goal is to find the amount of heat at each location of the manifold $u(x,t) : M_S \times \mathbb{R}^+ \to \mathbb{R}$ for all $t > 0$, given an initial distribution $u(x,0)$.

Definition

$$\Delta_{g_S} u(x,t) = -\frac{\partial u(x,t)}{\partial t}$$

Solving the Heat Equation

Intuitively, suppose we had a function $K : M_S \times M_S \times \mathbb{R}^+ \to \mathbb{R}$ which given two points $x, y \in M_S$ encodes the proportion of heat at position y from position x after a time Δt. Then,

Fact

$$u(x,t) = \int_{M_S} K(x,y,t)\, u(y,0)\, dy$$

What would such a K look like?

The Heat Kernel

Definition

Let λ_n, ϕ_n be the eigenvalues and eigenfunctions of Δ_{g_S}. The heat kernel $K(x, y, t) \in C^\infty(M_S \times M_S \times \mathbb{R}^+)$ is given by

$$K(x, y, t) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \phi_n(x) \phi_n(y)$$

- Analogous to Fourier analysis on the manifold
- Contains all of the information regarding the structure of the manifold
  - can be used to determine the metric

Soft Maps Between Surfaces [SNBBcG]

Definition

A *Soft Map* from M_S to M_T is a function assigning to each point $x \in M_S$ a probability distribution μ_x over M_T.

- Let $\mathcal{U} \subseteq M_T$ then $\mu_x(\mathcal{U})$ encodes the probability that x maps into $\mathcal{U}$. In particular it is required that:
  - $\mu_x(\mathcal{U}) \in [0, 1]$
  - $\mu_x(M_T) = 1$
- Soft maps generalize point to point maps.

Harmonic Maps [SGB]

Definition

The *Dirichlet Energy* of a map $\phi : M_S \to M_T$ is given by

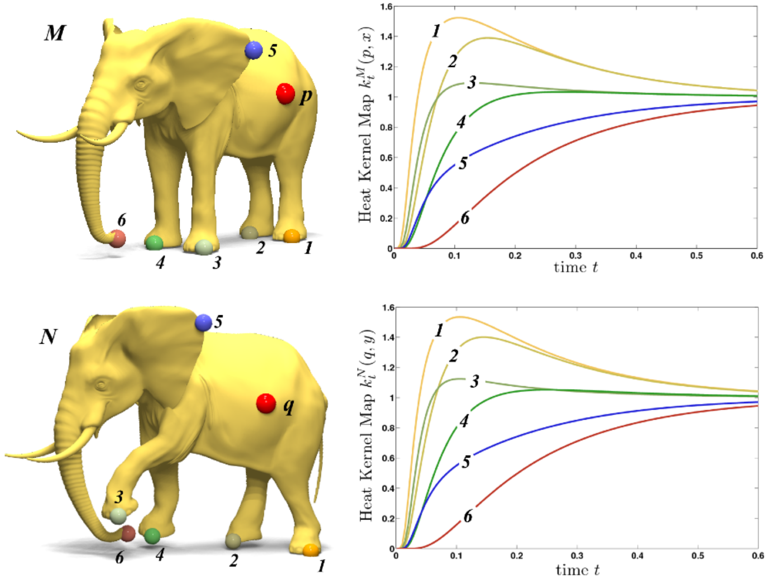
$$\mathcal{E}_D(\phi) \equiv \int_{M_S} \|\nabla_S \phi(x)\|^2 \, dx$$

- Can think of this as a measure of the "intrinsic stretching" of ϕ.
- Maps which extremize this functional are known as *harmonic maps.*
- Project goal: Investigate and understand how to generalize this notion to soft maps.

One Point Isometric Matching by Heat Kernel [OMMG]

- Gives a method to recover an isometry when given only a single pair of corresponding points between two geodesically identical meshes
- Works by exploiting the invariance of the heat kernel under isometry
  - Given a pair of isometric manifolds M, N and a pair of points $p \in M$, $q \in M$ such that for some isometry $\psi : M \to N$, $\psi(p) = q$. One can associate to every point on $x \in M$ a signature $k_t^M(p, x)$ and likewise for any $y \in N$ a signature $k_t^N(q, y)$.
  - Related points can be identified by matching signatures
  - Both the initial correspondence and isometry can be found efficiently in practice
- Very general. That is, it does not depend genus of the surface, or dimension

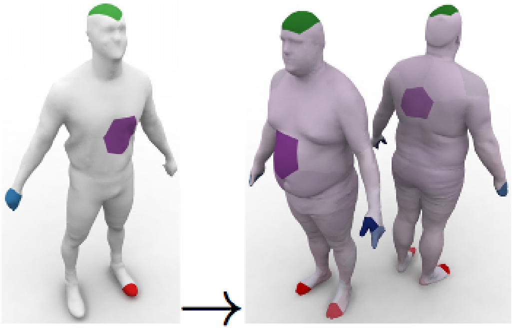
One Point Isometric Matching by Heat Kernel [OMMG]

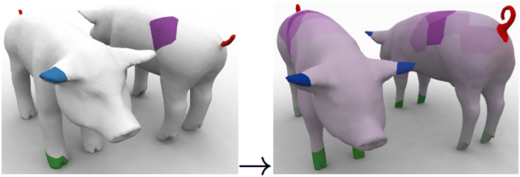


Applications of Soft Maps [SNBBcG]

- Models M_1, M_2 are descritized into patches corresponding to geodesic voroni cells
- A descriptor $\phi_s(U) \in \mathbb{R}^k$ is associated with each patch $U \subset M_s$, $s \in \{1, 2\}$ that encodes some information about its geometry.
- A solver numerically finds a soft correspondence A which minimizes the energy $E(A) = E_\phi(A) + \lambda E_{cont}(A) + \beta E_s(A)$ where
  - E_ϕ ensures the corresponding patches have similar geometry
  - E_{cont} attempts to keep local patches local in the correspondence by applying a cost based on *GEMD*
  - E_s is a sharpness term that tries to mediate tradeoff between E_ϕ and E_{cont} when there is an exact symmetry.

Applications of Soft Maps [SNBBcG]





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